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(BANACH SPACES-BASED MIXED FINITE ELEMENT METHODS FOR COUPLED DIFFUSION PROBLEMS AND RELATED MODELS)

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Banach Spaces-Based Mixed Finite Element Methods for Coupled Diffusion Problems and Related Models

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Abstract

In this thesis, new Banach spaces-based mixed finite element methods are explored to address coupled diffusion problems and related models in continuous mechanics. The focus is on numerical analysis and simulation of the stress-assisted diffusion problem and the chemotaxis-Navier-Stokes problem.

First, we introduce and analyze mixed variational formulations based on Banach spaces for the nearly incompressible linear elasticity problem and the Stokes problem. This approach is motivated by the similarities between the variational formulations of these models with respect to those obtained for the stress-assisted diffusion problem, which will be subsequently studied. To avoid the imposition of weak symmetry on the Cauchy stress tensor, we reformulate the problems in terms of the pseudostress tensor. We apply integration by parts formulas appropriate for the Banach spaces used, resulting in continuous schemes for both models. We employ the Babuška-Brezzi theory in Banach spaces and generalize classic results to establish that the obtained formulations are well-posed within these spaces.

Next, we address the system of partial differential equations describing the diffusion of a solute in an elastic material. The elasticity model, whose momentum equation includes a source term dependent on diffusion, is reformulated using the non-symmetric pseudostress tensor and the deformation of the solid as unknowns of the mixed scheme. The diffusion equation, with the diffusivity function and source term depending on the stress and strain tensor of the solid, respectively, is approached using a primal formulation with concentration as the unknown. Dirichlet boundary conditions are considered for both equations.

As a natural continuation of the above, a fully-mixed approach based on Banach spaces is proposed and analyzed, generating a new finite element method for the coupled stress-assisted diffusion problem to be solved numerically. We introduce two mixed schemes for the diffusion problem, using diffusion flux as an additional variable, and for the second, we also consider the concentration gradient as an unknown.

Finally, we introduce and analyze a fully-mixed method based on Banach spaces to numerically solve the stationary chemotaxis-Navier-Stokes problem. This coupled and nonlinear model represents the biological process driven by cellular movements induced by an external or internal chemical signal within an incompressible fluid. In addition to the velocity and pressure of the fluid, the velocity gradient and the Bernoulli-type stress tensor are introduced as additional variables, allowing the fluid pressure to be eliminated from the equations and calculated by post-processing after solving the system. In turn, in addition to the cellular density and the concentration of the chemical signal, the pseudostresses associated with these last variables and their corresponding gradients are introduced as additional unknowns. The resulting continuous formulation, set in a Banach framework, consists of a coupled

system of three saddle point problems, each perturbed with trilinear forms dependent on the data and the unknowns of the other two problems.

The continuous formulations resulting from each of the schemes are approached through a fixed-point strategy. Therefore, the Babuška-Brezzi theory in Banach spaces allows us to establish that the operators associated with each of the problems are well-stated. In turn, the classic Banach fixed-point theorem, in conjunction with assumptions of small data, results in the existence and uniqueness of the solution at a continuous level. Then, on arbitrary finite element subspaces, we establish Galerkin schemes corresponding to each of the problems. Assuming that the mentioned subspaces are inf-sup stable, Brouwer's theorem allows us to establish the existence of solutions at the discrete level. Additionally, for the scheme associated with the stationary chemotaxis-Navier-Stokes problem, Banach's fixed-point theorem also allows establishing the uniqueness of such discrete solution. We obtain Céa's estimates corresponding to each scheme, and once the finite element subspaces are particularized, the approximation properties allow us to establish the corresponding convergence rates. Finally, numerical experiments confirm these rates and illustrate the good performance of our methods.

Resumen

En esta tesis, se exploran nuevos métodos de elementos finitos mixtos basados en espacios de Banach para abordar problemas de difusión acoplada y modelos relacionados en mecánica de medios continuos. El enfoque se centra en el análisis numérico y simulación de los problemas de difusión asistida por esfuerzos y chemotaxis-Navier-Stokes.

Primero, introducimos y analizamos formulaciones variacionales mixtas basadas en espacios de Banach para el problema de elasticidad lineal casi incompresible y el problema de Stokes. Este enfoque esta motivado por las similitudes entre las formulaciones variacionales de estos modelos con respecto a las obtenidas para el problema de difusión asistida por esfuerzo, el cual sera estudiado subsecuentemente. Con el fin de evadir la imposición de simetría débil sobre el tensor de esfuerzos de Cauchy, reformulamos los problemas en términos del tensor de pseudoesfuerzos. Aplicamos fórmulas de integración por partes acordes a los espacios de Banach utilizados y obteniendo como resultado los esquemas continuos para ambos modelos. Empleamos la teoría de Babuška-Brezzi en espacios de Banach y generalizamos resultados clásicos para establecer que las formulaciones obtenidas estén bien planteadas dentro de estos espacios.

A continuación, abordamos el sistema de ecuaciones diferenciales parciales que describen la difusión de un soluto en un material elástico. El modelo de elasticidad, inicialmente definido de acuerdo a la relación constitutiva de la ley de Hooke, cuya ecuación de momentum incluye un término fuente dependiente de la difusión, es reformulado usando el tensor de psudoesfuerzos no simétrico y la deformación del solido como incógnitas del esquema mixto. La ecuación de difusión, con función de difusividad y termino fuente dependiendo del tensor de esfuerzos y deformación del sólido, respectivamente, es abordada usando una formulación primal con la concentración como incógnita. Para ambas ecuaciones son consideradas condiciones de contorno Dirichlet.

Como continuación natural de lo anterior, se plantea y analiza un enfoque completamente mixto basado en espacios de Banach, generando un nuevo método de elementos finitos para el problema acoplado de difusión asistido por esfuerzo a ser resuelto numéricamente. Introducimos dos esquemas mixtos para el problema de difusión, empleando al flujo de difusión como variable adicional, y para el segundo, consideramos además el gradiente de la concentración como incógnita.

Finalmente, introducimos y analizamos un método completamente mixto basado en espacios de Banach para resolver numéricamente el problema de chemotaxis-Navier-Stokes en estado estacionario. Este modelo acoplado y no lineal representa el proceso biológico dado por movimientos celulares conducidos por una señal química externa o interna dentro de un fluido incompresible. Además de la velocidad y presión del fluido, el gradiente de la velocidad y el tensor de esfuerzos de tipo Bernoulli se

introducen como variables adicionales, lo que permite eliminar la presión del fluido de las ecuaciones y calcularse mediante un postproceso tras resolver el sistema. A su vez, además de la densidad celular y la concentración de la señal química, los psudoesfuerzos asociados a estas últimas variables y sus correspondientes gradientes son introducidos como incógnitas adicionales. La formulación continua resultante, establecida en un marco Banach, consiste en un sistema acoplado de tres problemas de punto silla, cada uno perturbado con formas trilineales dependientes de los datos y de las incógnitas de los otros dos problemas.

Las formulaciones continuas resultantes de cada uno de los esquemas son abordadas mediante una estrategia de punto fijo, por lo cual, la teoría de Babŭzka-Brezzi en espacios de Banach permite establecer que los operadores asociados a cada uno de los problemas están bien planteados. Por su parte, el clásico teorema de punto fijo de Banach en conjunto con suposiciones de datos pequeños da como resultado la existencia y unicidad de solución a nivel continuo. Luego, sobre subespacios de elementos finitos arbitrarios, establecemos esquemas de Galerkin correspondientes a cada uno de los problemas. Asumiendo que los subspespacios mencionados son inf-sup estables, con lo cual el teorema de Brouwer permite establecer la existencia de solución a nivel discreto. Adicionalmente, para el esquema asociado al problema estacionario de chemotaxis-Navier-Stokes, el teorema de punto fijo de Banach permite además establecer unicidad de dicha solución discreta. Obtenemos estimaciones de Céa correspondiente a cada esquema, y una vez particularizados los subespacios de elementos finitos, las propiedades de aproximación permiten establecer las correspondientes tasas de convergencia. Finalmente, experimentos numéricos confirman dichas tasas e ilustran el buen desempeño de nuestros métodos.

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Cristian Andrés Inzunza Domínguez

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Introduction

Nonlinear and coupled models are common in continuous mechanics. In these models, the equations often include coefficients, source terms, or arbitrary terms that depend on the variables of other equations. This complexity significantly increases the difficulty of the corresponding analyses compared to simpler linear problems. This feature represents a challenge both in the formulation and in the resolution of such problems.

To address this difficulty, various approaches have been proposed. One of them involves incorporating additional penalized redundant Galerkin terms into the original formulations, resulting in the so-called augmented methods. Some examples include [7] and [8] for coupled flow and transport problems, [3] and [31] for Boussinesq equations, [20] and [22] for Navier-Stokes equations, and [46] and [47] for stress-assisted diffusion. Although these extra terms allow for the reestablishment of a Hilbertian framework for the models, simplifying their analysis, it is important to note that the incorporation of such terms introduces greater complexity into the discrete schemes and associated computational implementations, which could be mitigated through a proper analysis of the original variational formulations not augmented. It is also important to emphasize that in some models, such as the coupled Darcy and heat equations [52], generating an augmented method is not possible, leading to the necessity of a Banach space-based approach.

Therefore, this thesis aims to develop new finite element methods based on Banach spaces to solve problems in continuum mechanics, with a main focus on using the mixed approach to address problems related to the mechanics of fluids and solids. Generalizing common tools in analysis within Hilbertian frameworks to a Banach context, facing the challenges that this will entail. We specifically focus on the numerical analysis of coupled models, addressing challenges posed by the stress-assisted diffusion problem and the chemotaxis-Navier-Stokes problem, for which we will derive mainly fully-mixed variational formulations, establishing the existence, uniqueness, stability, and regularity of the solutions, and highlighting under what conditions these are held.

The following sections of this thesis will focus on the detailed presentation of the models we will work with, exploring some of their most significant applications, and providing the corresponding references. In addition, the general organization of the thesis will be described, detailing the mathematical and numerical approach we will employ to address each of the proposed models.

Model Problems

This thesis focuses on coupled problems in two directions: solid mechanics and fluid mechanics. Regarding solid mechanics, our primary emphasis lies on the analysis of a diffusion-deformation problem, wherein stress acts as a coupling variable, commonly recognized as stress-assisted diffusion problems [2, 73]. This models the diffusion of a solute in an elastic material occupying the domain Ω and is described by the following system of partial differential equations:

$$\begin{aligned} \boldsymbol{\rho} &= \mathcal{C}\big(\mathbf{e}(\mathbf{u})\big) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\rho}) = \mathbf{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \\ \widetilde{\boldsymbol{\sigma}} &= \widetilde{\vartheta}(\boldsymbol{\rho})\nabla\phi \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) = g(\mathbf{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{1}$$

where the Lamé constants λ and $\mu > 0$ characterize the material properties. Additionally, ϕ represents the local species concentration, $\widetilde{\boldsymbol{\sigma}}$ is the diffusive flux, and $\widetilde{\vartheta} : \mathbb{R} \to \mathbb{R}$ is a tensorial diffusivity function. Finally, $\mathbf{f} : \mathrm{R} \to \mathbf{R}$ is a vector field of body forces, and $g : \mathbf{R} \to \mathrm{R}$ represents an additional source term that depends on the solid displacement $\mathbf{u}$, and $\mathbf{u}_D$ is the Dirichlet datum for $\mathbf{u}$. The system (1) describes the constitutive relationships inherent to linear elastic materials, the conservation of linear momentum, the constitutive description of diffusion flows, and the mass transport of the diffusive substance, respectively. It is also assumed that the diffusion time scales are much slower than those of elastic wave propagation, justifying the static nature of the system. We note that the effects of stress-assisted diffusion constitute the main mechanism in many applied problems, including diffusion of boron and arsenic in silicon, hydrogen diffusion in metals, aluminum interconnect voiding in integrated circuits, stress-induced migration in iron, sorption in fiber-reinforced polymeric materials, drying of liquid paint layers and gels, penetration of solutes and anisotropy in cardiac dynamics, among others.

We emphasize that the main challenge in analyzing this model lies in the stress dependence on the diffusivity tensor $\widetilde{\vartheta}$. Thus, to address this problem without making the regularity assumption put forth in previous works, it is necessary to develop tools within the framework of Banach spaces. In order to achieve this, Chapter 1 provides several preliminary results. Additionally, Chapters 2 and 3 offer both continuous and discrete analysis for mixed-primal and fully-mixed formulations, respectively.

In the context of fluid dynamics, we introduce the chemotaxis-Navier-Stokes problem. This model aims to find the velocity $\mathbf{u}$ and pressure p of an incompressible fluid in a region Ω, as well as the cell density η and chemical concentration signal φ. These variables must satisfy the following system of differential equations:

$$\begin{aligned} -\nu\,\Delta\mathbf{u} + \lambda\,(\nabla\mathbf{u})\,\mathbf{u} + \nabla p - \eta\,\nabla f &= \mathbf{f} && \text{in} \quad \Omega\,, \\ \mathbf{div}(\mathbf{u}) &= 0 && \text{in} \quad \Omega\,, \\ \textstyle\int_\Omega p &= 0\,, \\ -\,k_\eta\Delta\eta + \mu\,\mathbf{div}\big(\eta\nabla\varphi\big) + \mathbf{u}\cdot\nabla\eta &= f_\eta && \text{in} \quad \Omega\,, \\ -\,k_\varphi\Delta\varphi + \gamma\eta\varphi + \mathbf{u}\cdot\nabla\varphi &= f_\varphi && \text{in} \quad \Omega\,, \\ \mathbf{u} = \mathbf{u}_D\,, \quad \eta = \eta_D \quad \text{and} \quad \varphi &= \varphi_D && \text{on} \quad \Gamma\,. \end{aligned} \tag{2}$$

Here, f, $\mathbf{f}$, f_η, and f_φ are given functions belonging to appropriate function spaces, while ν, λ, k_η, k_φ, μ, and γ are positive constants representing fluid viscosity, fluid density, cell diffusion constant, chemical diffusion constant, chemotactic coefficient, and chemical signal consumption rate, respectively. Additionally, $\mathbf{u}_D$, η_D, and φ_D are the corresponding Dirichlet boundary data, and $\mathbf{u}D$ satisfies the compatibility condition $\int \Gamma\mathbf{u}_D \cdot \boldsymbol{\nu} = 0$.

The chemotaxis-Navier-Stokes equations have a crucial role in understanding how cells move in response to chemical signals and how they influence the surrounding fluid flow. These equations are used in various biological and medical processes, such as the development of multicellular organisms and the spread of cancer [75, 59, 76]. Although numerical methods have been developed [26, 36, 38], there is still a need for a mixed approach using Banach spaces. This approach can provide a more efficient and conservative formulation for these coupled and nonlinear systems, offering new possibilities for both fundamental research and medical applications.

In Chapter 4, we aim to address the aforementioned gap and expand the utilization of Banach space-based approaches in studying the continuous and discrete formulation of the chemotaxis-Navier-Stokes problem. We introduce and analyze a fully-mixed finite element method for this model.

Outline of the thesis

The structure of this thesis is as follows. In Chapter 1, we introduce mixed variational formulations in Banach spaces for the nearly incompressible linear elasticity and Stokes models, addressing nonlinear coupled problems in continuum mechanics. We employ a pseudostress-based approach and apply the Babuška-Brezzi theory in Banach spaces. The outcomes encompass the construction of a tensorial operator and the generalization of classical estimates for the tensor deviator. The results of this chapter were published in

[48] G.N. Gatica and C. Inzunza, *On the well-posedness of Banach spaces-based mixed formulations for the nearly incompressible Navier-Lamé and Stokes equations.* Computers & Mathematics with Applications, vol. 102, pp. 87–94, (2021).

In Chapter 2, we address the stress-assisted diffusion of a solute in an elastic material using a variational approach based on Banach spaces, employing a mixed-primal finite element method. The initial elasticity model, defined by Hooke's law, is reformulated using non-symmetric stress and displacement as unknowns in the mixed scheme. The diffusion equation, with diffusivity function and source term dependent on stress and displacement, is cast in primal form concerning the unknown concentration. The dependence of the diffusion coefficient and subsequent analysis suggest seeking unknowns in appropriate Lebesgue spaces. The coupled formulation is transformed into an equivalent fixed-point equation, utilizing the classical Banach fixed-point theorem and Babuška-Brezzi theory alongside the Lax-Milgram theorem to establish the uniqueness of the solution. Discrete analysis and Brouwer's theorem ensure the existence of a Galerkin solution. The contents of this chapter can be found in

[49] G.N. Gatica, C. Inzunza and F.A. Sequeira, *A pseudostress-based mixed-primal finite element method for stress-assisted diffusion problems in Banach spaces.* Journal of Scientific Computing, vol. 92, article: 103, (2022).

In Chapter 3, we propose fully-mixed approaches for the previous work. The nonlinear dependence of the elastic variables on the diffusion coefficient and its source term, along with the nonlinear dependence of concentration on the elastic source term, they suggest looking for unknowns in suitable Lebesgue spaces for continuous and discrete analysis. We reformulate the coupled systems as equivalent fixed-point equations, demonstrating the uniqueness of the solution using the classical Banach fixed-point theorem and Babuška-Brezzi theory. We tackle the Galerkin scheme and employ Brouwer's theorem to ensure discrete solutions. The contents of this chapter are covered in

[50] G. N. Gatica, C. Inzunza and F.A. Sequeira, *New Banach spaces-based fully-mixed finite element methods for pseudostress-assisted diffusion problems.* Applied Numerical Mathematics, vol. 193, pp. 148-178, (2023).

In Chapter 4, we present a fully-mixed finite element method based on Banach spaces to numerically solve the steady-state chemotaxis-Navier-Stokes problem. We introduce variables such as the velocity gradient and the stress tensor, removing pressure from the equations. We also used unknowns for stress associated with cell density and chemical signal gradient. After using a fixed-point approach the Banach and Babuška-Brezzi theorems allow us to guarantee the existence and uniqueness of solution under small data constraints. In Galerkin's scheme, we apply the Brouwer and Banach theorems, deriving a priori error estimates, even for the post-processed calculated pressure. We introduce finite element subspaces that guarantee stability and local conservation of momentum, defined in terms of Raviart-Thomas spaces and piecewise polynomials, and provide convergence rates. In addition, other properties of the Raviart-Thomas interpolator are demonstrated, which were necessary for establishing discrete inf-sup conditions. The content of this chapter resulted in the following article:

[23] G. N. Caucao, E. Colmenares, G.N. Gatica and C. Inzunza, *A Banach spaces-based fully-mixed finite element method for the stationary chemotaxis-Navier-Stokes problem.* Computers & Mathematics with Applications, vol. 145, pp. 65-89, (2023).

Throughout the chapters 2-4, we provide a priori error estimates and convergence rates for specific finite element subspaces that satisfy the discrete inf-sup conditions. In addition, we include numerical experiments to validate the accuracy of the schemes and to illustrate the properties of the models. All implementations were carried out using FreeFem++ [58] and Matlab [63]. Post-processing and visualization were performed using Paraview [1].

Preliminary notations

Throughout the thesis, Ω is a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, whose outward unit normal at its boundary Γ is denoted $\mathbf{n}$. Additionally, in the Chapters 1-3, Ω is assumed to be star-shaped with respect to a ball. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$, with $t \in [1,+\infty)$, and Sobolev spaces $\mathrm{W}^{\ell,t}(\Omega)$ and $\mathrm{W}_0^{\ell,t}(\Omega)$, with $\ell \geq 0$, whose corresponding norms and seminorms, either for the scalar, vector, or tensorial version, are denoted by $\|\cdot\|_{0,t;\Omega}$, $\|\cdot\|_{\ell,t;\Omega}$, and $|\cdot|_{\ell,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and that when $t = 2$, we simply write $\mathrm{H}^\ell(\Omega)$ instead of $\mathrm{W}^{\ell,2}(\Omega)$, with its norm and seminorm denoted by $\|\cdot\|_{\ell,\Omega}$ and $|\cdot|_{\ell,\Omega}$, respectively. Now, letting t, $t' \in (1,+\infty)$ conjugate to each other, that is such that $1/t + 1/t' = 1$, we let $\mathrm{W}^{1/t',t}(\Gamma)$ and $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the trace space of $\mathrm{W}^{1,t}(\Omega)$ and its dual, respectively, and denote the duality pairing between them by $\langle\cdot,\cdot\rangle$. In particular, when $t = t' = 2$, we simply write $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$ instead of $\mathrm{W}^{1/2,2}(\Gamma)$ and $\mathrm{W}^{-1/2,2}(\Gamma)$, respectively. Also, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be its vector and tensorial counterparts. Furthermore, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \qquad \mathbf{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \qquad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,n}, \qquad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \qquad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij},$$

$$\text{and} \qquad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ stands for the identity tensor of $\mathbb{R} := \mathrm{R}^{n\times n}$. On the other hand, for each t, $j \in [1,+\infty)$ such that $t \geq j$, we introduce the Banach spaces

$$\mathbf{H}(\mathbf{div}_t;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\Big\}, \tag{3}$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\Big\}, \tag{4}$$

and

$$\mathbf{H}^t(\mathrm{div}_j;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbf{L}^t(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathrm{L}^j(\Omega)\Big\}, \tag{5}$$

which are endowed with the natural norms

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathbf{div}_t;\Omega), \tag{6}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega), \tag{7}$$

and

$$\|\boldsymbol{\tau}\|_{t,\mathrm{div}_j;\Omega} := \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,j;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_j;\Omega). \tag{8}$$

We recall from [44, eq. (1.43), Section 1.3.4] (see also [21, Section 4.1] and [29, Section 3.1]), that for each $t \in \begin{cases} (1,+\infty) & \text{if } n = 2 \\ [6/5,+\infty) & \text{if } n = 3 \end{cases}$ there holds

$$\langle \boldsymbol{\tau}\cdot\mathbf{n}, v\rangle = \int_\Omega \Big\{\boldsymbol{\tau}\cdot\nabla v + v\,\mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},v) \in \mathbf{H}(\mathbf{div}_t;\Omega)\times \mathrm{H}^1(\Omega), \tag{9}$$

and analogously

$$\langle \boldsymbol{\tau}\,\mathbf{n}, \boldsymbol{v}\rangle = \int_\Omega \Big\{\boldsymbol{\tau} : \nabla \boldsymbol{v} + \boldsymbol{v}\cdot\mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega)\times \mathbf{H}^1(\Omega), \tag{10}$$

where $\langle\cdot,\cdot\rangle$ denotes in (9) (resp. (10)) the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ (resp. $\mathbf{H}^{1/2}(\Gamma)$) and $\mathrm{H}^{-1/2}(\Gamma)$ (resp. $\mathbf{H}^{-1/2}(\Gamma)$). In turn, given t, $t' \in (1,+\infty)$ conjugate to each other, there also holds (cf. [41, Corollary B.57])

$$\langle \boldsymbol{\tau}\cdot\mathbf{n}, v\rangle = \int_\Omega \Big\{\boldsymbol{\tau}\cdot\nabla v + v\,\mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},v) \in \mathbf{H}^t(\mathbf{div}_t;\Omega)\times \mathrm{W}^{1,t'}(\Omega), \tag{11}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathrm{W}^{-1/t,t}(\Gamma)$ and $\mathrm{W}^{1/t,t'}(\Gamma)$.

Introducción

Los modelos no lineales y acoplados son frecuentes en la mecánica de medios continuos. En estos modelos, las ecuaciones suelen incluir coeficientes, términos fuente o términos arbitrarios que dependen de las variables de otras ecuaciones. Esta complejidad incrementa notablemente la dificultad de los análisis correspondientes, en comparación con problemas lineales más simples. Esta característica representa un desafío tanto en la formulación como en la resolución de dichos problemas.

Para abordar esta dificultad, se han propuesto diversos enfoques. Uno de ellos consiste en incorporar términos de Galerkin redundantes penalizados adicionales a las formulaciones originales, lo que resulta en los llamados métodos aumentados. Algunos ejemplos incluyen [7] y [8] para problemas acoplados de flujo y transporte, [3] y [31] para ecuaciones de Boussinesq, [20] y [22] para ecuaciones de Navier-Stokes, y en [46] y [47] para la difusión asistida por estrés. Si bien estos términos adicionales permiten restablecer un marco Hilbertiano para los modelos, simplificando su análisis, es importante señalar que la incorporación de tales términos introduce una mayor complejidad en los esquemas discretos y las implementaciones computacionales asociadas, lo cual se podría mitigar mediante un análisis adecuado de las formulaciones variacionales originales no aumentadas. A su vez, es importante destacar que en algunos modelos, como las ecuaciones acopladas de Darcy y calor [52], no es posible generar un método aumentado, lo que hace inevitable un enfoque basado en espacios de Banach.

Por lo tanto, esta tesis tiene como objetivo desarrollar nuevos métodos de elementos finitos basados en espacios de Banach para resolver problemas en la mecánica de medios continuos, con un enfoque principal en utilizar el enfoque mixto para abordar problemas relacionados con la mecánica de fluidos y sólidos. Generalizando herramientas comunes en el análisis en marcos Hilbertianos a un contexto Banach, afrontando las dificultades que ello implicará. Nos centramos específicamente en el análisis numérico de modelos acoplados, abordando desafíos planteados por el problema de difusión asistida por estrés y el problema de chemotaxis-Navier-Stokes, para los cuales derivaremos formulaciones variacionales principalmente completamente mixtas, estableciendo existencia, unicidad, estabilidad y regularidad de las soluciones, y destacando bajo que condiciones se tienen estas.

Las siguientes secciones de esta tesis se centrarán en la presentación detallada de los modelos con los que trabajaremos, explorando algunas de sus aplicaciones más significativas y proporcionando las referencias correspondientes. Además, se describirá la organización general de la tesis, detallando el enfoque matemático y numérico que emplearemos para abordar cada uno de los modelos propuestos.

Problemas Modelo

Esta tesis se centra en problemas acoplados en dos direcciones: la mecánica de sólidos y la mecánica de fluidos. En lo que respecta a la mecánica de sólidos, nuestro principal énfasis se centra en el análisis de un problema de difusión-deformación, mientras que el estrés actúa como una variable de acoplamiento, comúnmente conocido como problema de difusión asistida por estrés [2, 73]. Esto modela la difusión de un soluto en un material elástico que ocupa el dominio Ω y está descrito por el siguiente sistema de ecuaciones en derivadas parciales:

$$\begin{aligned} \boldsymbol{\rho} \,=\, \mathcal{C}\big(\mathbf{e}(\mathbf{u})\big) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\rho}) \,=\, \mathbf{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \\ \widetilde{\boldsymbol{\sigma}} = \widetilde{\vartheta}(\boldsymbol{\rho})\nabla\phi \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) = g(\mathbf{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{12}$$

donde $\Gamma := \partial\Omega$, $\boldsymbol{\rho}$ es el tensor de Cauchy del sólido, $\mathbf{u}$ es el campo de desplazamiento, $\mathbf{e}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathtt{t}})$ es el tensor de deformación infinitesimal (gradiente simetrizado de desplazamientos), y $\mathcal{C}$ representa el operador lineal que define la ley de Hooke (cf. ecuación (2.36) en [44]), es decir,

$$\mathcal{C}(\boldsymbol{\tau}) := \lambda\,\mathrm{tr}\,(\boldsymbol{\tau})\,\mathbb{I} \,+\, 2\mu\,\boldsymbol{\tau} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{R}\,,$$

de modo que

$$\boldsymbol{\rho} \,=\, \lambda\,\mathrm{tr}\,\big(\mathbf{e}(\mathbf{u})\big)\,\mathbb{I} \,+\, 2\mu\,\mathbf{e}(\mathbf{u})\,, \tag{13}$$

donde las constantes de Lamé λ y $\mu > 0$ (módulos de dilatación y cizallamiento) caracterizan las propiedades del material. Además, ϕ representa la concentración local de especies, $\widetilde{\boldsymbol{\sigma}}$ es el flujo difusivo, y $\widetilde{\vartheta} : \mathbb{R} \to \mathbb{R}$ es una función tensorial de difusividad. Finalmente, $\mathbf{f} : \mathrm{R} \to \mathbf{R}$ es un campo vectorial de cargas en el cuerpo (que depende de la concentración de especies), $g : \mathbf{R} \to \mathrm{R}$ representa un término fuente adicional que depende del desplazamiento sólido $\mathbf{u}$, y $\mathbf{u}_D$ es el dato de Dirichlet para $\mathbf{u}$. El sistema (12) describe las relaciones constitutivas inherentes a materiales elásticos lineales, la conservación del momento lineal, la descripción constitutiva de los flujos de difusión y el transporte de masa de la sustancia difusiva, respectivamente. También se asume que las escalas de tiempo de difusión son mucho más lentas que las de la propagación de ondas elásticas, lo que justifica la naturaleza estática del sistema (cf. [67]). Cabe destacar que los efectos de la difusión asistida por estrés constituyen el mecanismo principal en muchos problemas aplicados [5, 27], que incluyen la difusión de boro y arsénico en silicio, la difusión de hidrógeno en metales, la formación de huecos en interconexiones de aluminio en circuitos integrados, la migración inducida por estrés en hierro, la sorción en materiales poliméricos reforzados con fibra, el secado de capas de pintura líquida, la penetración de geles y solutos, y la anisotropía en la dinámica cardíaca, entre otros [5, 27, 69, 70, 77, 82, 87].

Destacamos que el principal desafío en el análisis de este modelo reside en la dependencia del tensor de difusividad $\widetilde{\vartheta}$ sobre el tensor de Cauchy del sólido. Por lo tanto, para el análisis matemático correspondiente y para eludir la suposición de regularidad planteada en trabajos anteriores [46, 47], se vuelve imperativo desarrollar herramientas que nos permitan abordar el problema en el marco de los espacios de Banach. Para lograr esto, se proporcionan varios resultados preliminares en el Capítulo 1. Luego, en los Capítulos 2 y 3, llevamos a cabo análisis continuos y discretos para formulaciones mixtas-primal y totalmente mixtas, respectivamente.

En el contexto de mecánica de fluidos, presentamos el problema de chemotaxis-Navier-Stokes. Este modelo tiene como objetivo encontrar la velocidad $\mathbf{u}$ y la presión p de un fluido incompresible que ocupa una región Ω, además de la densidad celular η y la señal de concentración química φ. Estas variables deben satisfacer el siguiente sistema acoplado de ecuaciones diferenciales:

$$\begin{aligned}
-\,\nu\,\Delta\mathbf{u} \,+\, \lambda\,(\nabla\mathbf{u})\,\mathbf{u} \,+\, \nabla p \,-\, \eta\,\nabla f &= \mathbf{f} && \text{in} \quad \Omega\,,\\
\mathbf{div}(\mathbf{u}) &= 0 && \text{in} \quad \Omega\,,\\
\textstyle\int_\Omega p &= 0\,, && \\
-\,k_\eta\Delta\eta + \mu\,\mathbf{div}\big(\eta\nabla\varphi\big) \,+\, \mathbf{u}\cdot\nabla\eta &= f_\eta && \text{in} \quad \Omega\,,\\
-\,k_\varphi\Delta\varphi \,+\, \gamma\eta\varphi \,+\, \mathbf{u}\cdot\nabla\varphi &= f_\varphi && \text{in} \quad \Omega\,,\\
\mathbf{u} \,=\, \mathbf{u}_D\,, \quad \eta \,=\, \eta_D \quad \text{and} \quad \varphi &= \varphi_D && \text{on} \quad \Gamma\,,
\end{aligned} \tag{14}$$

donde f, $\mathbf{f}$, f_η y f_φ son funciones dadas que pertenecen a espacios adecuados que se indicarán más adelante, mientras que ν, λ, κ_η, κ_φ, μ y γ son constantes positivas que representan la viscosidad del fluido, la densidad del fluido, la constante de difusión celular, la constante de difusión química, el coeficiente quimiotáctico y la tasa de consumo de la señal química, respectivamente. A su vez, $\mathbf{u}_D$, η_D y φ_D son los correspondientes datos de Dirichlet y además $\mathbf{u}_D$ satisface la condición de compatibilidad $\int_\Gamma \mathbf{u}_D\cdot\boldsymbol{\nu} = 0$.

Las ecuaciones de chemotaxis-Navier-Stokes desempeñan un papel fundamental en la comprensión del desplazamiento directo de las células mediante señales químicas y su consecuente influencia en el flujo de fluido circundante. Estos modelos encuentran aplicaciones en diversos procesos biológicos y médicos, que van desde el desarrollo de organismos multicelulares hasta la propagación del cáncer [75, 59, 76]. A pesar de la existencia de métodos numéricos [26, 36, 38], aún falta un enfoque mixto dentro de los espacios de Banach, que podría proporcionar una formulación más eficiente y conservativa para estos sistemas acoplados y no lineales. Este enfoque tiene el potencial de abrir oportunidades innovadoras tanto en la investigación fundamental como en las aplicaciones médicas.

Dentro del contexto de la discusión anterior, el Capítulo 4 tiene como objetivo abordar tanto la brecha mencionada como ampliar aún más las posibilidades de aplicar enfoques basados en espacios de Banach para el estudio de la formulación continua y discreta del problema de chemotaxis-Navier-Stokes. Aquí, se introduce y analiza un método de elementos finitos completamente mixto para este modelo.

Organización de la tesis

La estructura de esta tesis es la siguiente. En el Capítulo 1, introducimos formulaciones variacionales mixtas en espacios de Banach para modelos de elasticidad lineal casi incompresible y la ecuación de Stokes. Utilizamos un enfoque basado en el pseudoesfuerzo y aplicamos la teoría de Babuška-Brezzi en espacios de Banach. Los resultados incluyen la construcción de un operador tensorial y la generalización de estimaciones clásicas para el desviador de tensores. Los resultados de este capítulo se publicaron en

[48] G.N. Gatica and C. Inzunza, *On the well-posedness of Banach spaces-based mixed formulations for the nearly incompressible Navier-Lamé and Stokes equations.* Computers & Mathematics with Applications, vol. 102, pp. 87–94, (2021).

En el Capítulo 2, abordamos la difusión asistida por tensión de un soluto en un material elástico utilizando un enfoque variacional basado en espacios de Banach, empleando un método de elementos finitos primarios mixtos. El modelo de elasticidad inicial, definido por la ley de Hooke, se reformula utilizando tensiones y desplazamientos asimétricos como incógnitas en el esquema mixto. La ecuación de difusión, cuya función de difusividad y término fuente dependen de la tensión y el desplazamiento, se formula en forma primaria con respecto a la concentración desconocida. La dependencia del coeficiente de difusión y el análisis posterior sugieren buscar incógnitas en espacios de Lebesgue apropiados. La formulación acoplada se transforma en una ecuación de punto fijo equivalente, utilizando el teorema clásico del punto fijo de Banach y la teoría de Babuška-Brezzi junto con el teorema de Lax-Milgram para establecer la unicidad de la solución. El análisis discreto y el teorema de Brouwer aseguran la existencia de una solución de Galerkin. El contenido de este capítulo se puede encontrar en

[49] G.N. Gatica, C. Inzunza and F.A. Sequeira, *A pseudostress-based mixed-primal finite element method for stress-assisted diffusion problems in Banach spaces.* Journal of Scientific Computing, vol. 92, article: 103, (2022).

En el Capítulo 3, proponemos enfoques completamente mixtos para el trabajo anterior. La dependencia no lineal de las variables elásticas del coeficiente de difusión y su término fuente, junto con la dependencia no lineal de la concentración del término fuente elástico, sugieren buscar incógnitas en espacios de Lebesgue adecuados para análisis continuo y discreto. Reformulamos los sistemas acoplados como ecuaciones de punto fijo equivalentes, demostrando la unicidad de la solución utilizando el teorema clásico del punto fijo de Banach y la teoría de Babuška-Brezzi. Abordamos el esquema de Galerkin y empleamos el teorema de Brouwer para garantizar soluciones discretas. El contenido de este capítulo está cubierto en

[50] G. N. Gatica, C. Inzunza and F.A. Sequeira, *New Banach spaces-based fully-mixed finite element methods for pseudostress-assisted diffusion problems.* Applied Numerical Mathematics, vol. 193, pp. 148-178, (2023).

En el Capítulo 4, presentamos un método de elementos finitos completamente mixto basado en espacios de Banach para resolver numéricamente el problema estacionario de chemotaxis-Navier-Stokes. Introducimos variables como el gradiente de velocidad y el tensor de esfuerzo, eliminando la presión de las ecuaciones. También utilizamos incógnitas para el esfuerzo asociado con la densidad celular y el gradiente de señal química. Después de aplicar un enfoque de punto fijo, los teoremas de Banach y Babuška-Brezzi nos permiten garantizar la existencia y unicidad de la solución bajo restricciones de datos pequeños. En el esquema de Galerkin, aplicamos los teoremas de Brouwer y Banach, derivando estimaciones de error a priori, incluso para la presión calculada postprocesada. Introducimos subespacios de elementos finitos que garantizan la estabilidad y la conservación local del momento, definidos en términos de espacios de Raviart-Thomas y polinomios por partes, y proporcionamos tasas de convergencia. Además, se demuestran otras propiedades del interpolador de Raviart-Thomas, que fueron necesarias para establecer condiciones inf-sup discretas. El contenido de este capítulo dio lugar al siguiente artículo:

[23] G. N. Caucao, E. Colmenares, G.N. Gatica and C. Inzunza, *A Banach spaces-based fully-mixed finite element method for the stationary chemotaxis-Navier-Stokes problem.* Computers & Mathematics with Applications, vol. 145, pp. 65-89, (2023).

A lo largo de los capítulos 2-4, proporcionamos estimaciones de error a priori y tasas de convergencia para subespacios específicos de elementos finitos que cumplen con las condiciones inf-sup discretas. Además, incluimos experimentos numéricos para validar la precisión de los esquemas e ilustrar las propiedades de los modelos. Todas las implementaciones se realizaron utilizando FreeFem++ [58] y Matlab [63]. El postprocesamiento y la visualización se realizaron utilizando Paraview [1].

CHAPTER 1

On the well-posedness of Banach spaces-based mixed formulations for the nearly incompressible Navier-Lamé and Stokes equations

1.1 Introduction

In many nonlinear models in continuum mechanics, specially in coupled ones, the coefficients, source terms, or arbitrary terms of each equation depend on the unknowns from the other equations involved, which certainly makes the corresponding analyses much more cumbersome than for simple linear problems. Indeed, one of the main challenges that one often encounters there refers to the fact that the natural spaces to which the unknowns belong force the respective variational formulations to be posed in terms of Banach spaces instead of Hilbert ones. In order to overcome this, in some cases one may resort to the incorporation of augmented terms, as done for instance in [7] and [8] for coupled flow-transport problems, in [3] and [31] for the Boussinesq equations, in [20] and [22] for the Navier-Stokes equations, or in [46] and [47] for stress-assisted diffusion, thanks to which one recovers Hilbertian frameworks for the models, which are much easier to analyse. Nevertheless, while showing this and other advantages as well, the augmentation procedure adds further complexity to the problems, mainly affecting the associated discrete schemes and the respective computational implementations, which could be avoided if proper analyses are developed for the original non-augmented variational formulations. Needless to mention, in some models the augmentation is not even possible, as for the coupled Darcy and heat equations, and hence a Banach framework becomes unavoidable in these cases (see, e.g. [52]).

As a matter of illustration of the above, let us briefly recall that the model from [46] and [47] consists of a system of partial differential equations governing the diffusion of a solute interacting with the motion of an elastic solid occupying a bounded domain Ω with boundary Γ. In particular, the respective diffusion coefficient ϑ depends on the Cauchy stress tensor $\boldsymbol{\sigma}$ of the solid, so that the diffusive flux $\mathbf{p}$ and the diffusion equation become

$$\mathbf{p} := \vartheta(\boldsymbol{\sigma})\,\nabla\phi \quad \text{and} \quad -\operatorname{div}(\mathbf{p}) = g(\mathbf{u}) \quad \text{in} \quad \Omega\,, \tag{1.1}$$

respectively, where ϕ is the solute concentration, ∇ and div are the usual gradient and divergence operators, respectively, and g is a source term depending on the displacement $\mathbf{u}$ of the solid. Then, dividing the first equation of (1.1) by $\vartheta(\boldsymbol{\sigma})$, which is assumed to be strictly positive, multiplying by a test vector $\mathbf{q}$ associated with the unknown $\mathbf{p}$, formally integrating by parts, and assuming for simplicity that ϕ vanishes on Γ, one obtains

$$\int_\Omega \frac{1}{\vartheta(\boldsymbol{\sigma})}\,\mathbf{p}\cdot\mathbf{q} \,+\, \int_\Omega \phi\,\mathrm{div}(\mathbf{q}) \,=\, 0\,. \tag{1.2}$$

In turn, denoting by ψ a test function associated with ϕ, the second equation of (1.1) yields

$$\int_\Omega \psi\,\mathrm{div}(\mathbf{p}) \,=\, -\int_\Omega \psi\,g(\mathbf{u})\,. \tag{1.3}$$

Thus, because of the terms $\vartheta(\boldsymbol{\sigma})$ and $g(\mathbf{u})$, with $\boldsymbol{\sigma}$ and $\mathbf{u}$ coming from the elasticity model, one can employ fixed point arguments to analyse the solvability of (1.2) - (1.3). A similar procedure is applied to the linear elasticity equation, whose source term depends on ϕ. As a consequence, and in order to derive, in particular, a continuity property of the fixed-point operator for the stress-assisted diffusion problem, most likely one will have to deal, among others, with the following expression arising from the first term of (1.2)

$$\int_\Omega \left\{\frac{\vartheta(\boldsymbol{\tau})-\vartheta(\boldsymbol{\zeta})}{\vartheta(\boldsymbol{\tau})\vartheta(\boldsymbol{\zeta})}\right\}\mathbf{p}\cdot\mathbf{q}\,, \tag{1.4}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$ are generic tensors belonging to the same space where $\boldsymbol{\sigma}$ lives. In this case, if ϑ is assumed to be bounded from below and satisfy a Lipschitz-continuity property, the Cauchy-Schwarz and Hölder inequalities allow to conclude that the above expression can be controlled only if $\boldsymbol{\tau}-\boldsymbol{\zeta}$, $\mathbf{p}$, and $\mathbf{q}$, belong to particular Lebesgue spaces. This simple example illustrates that, even if $\boldsymbol{\sigma}$ and $\mathbf{u}$ are solutions of a linear elasticity problem, for which the solvability via Hilbert spaces is already well-established, when this equation is coupled with (1.1), the fixed-point argumentation requires that the analysis of the former be performed within a suitable Banach spaces framework. Same conclusions arise if linear elasticity is coupled with other equations, if other model, as Stokes in [7], is employed, or if similar coupled problems are considered.

According to the above discussion, the initial purpose of this work is to introduce and analyse a Banach spaces-based mixed variational formulation for linear elasticity, particularly for the nearly incompressible case, which is of much more interest in applications. Additionally, and because of the similarities between the resulting continuous formulations, we also include the Stokes system in our discussion. In this way, the rest of the chapter is organized as follows. In Section 1.2 we introduce both models of interest and use a suitable integration by parts formula to derive their mixed variational formulations. Some preliminary results, namely the well-posedness of Banach spaces-based primal formulations for the Stokes and Poisson equations, a suitable operator mapping a tensor Lebesgue space into itself, and a generalization to arbitrary Lebesgue spaces of a key inequality for the Hilbertian analysis of linear elasticity, are stated in Section 1.3. Finally, the well-posedness of the formulations from Section 1.2 are established in Section 1.4.

1.2 The models and their mixed formulations

In this section we define our models of interest and derive their corresponding Banach spaces-based mixed formulations. In what follows, Ω is a bounded Lipschitz-continuous domain of R^n, $n\in\{2,3\}$, which is star shaped with respect to a ball, and whose outward normal at Γ is denoted by $\boldsymbol{\nu}$.

1.2.1 Nearly incompressible linear elasticity

The aim of the linear elasticity model is to determine the displacement $\mathbf{u}$ and the Cauchy stress tensor $\boldsymbol{\rho}$ of a linear elastic material occupying the region Ω, under the action of external forces. More precisely, given a volume force $\mathbf{f}$ and a Dirichlet datum $\mathbf{u}_D$, we seek a symmetric tensor field $\boldsymbol{\rho}$ and a vector field $\mathbf{u}$ satisfying the constitutive relation given by Hooke's law, the corresponding momentum balance, and a Dirichlet boundary condition on Γ, that is

$$\begin{aligned} \boldsymbol{\rho} \,&=\, 2\mu\,\mathbf{e}(\mathbf{u}) \,+\, \lambda\,\mathrm{tr}\,(\mathbf{e}(\mathbf{u}))\,\mathbb{I} \quad \text{in} \quad \Omega\,, \\ \mathbf{div}(\boldsymbol{\rho}) \,&=\, -\mathbf{f} \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{1.5}$$

where $\mathbf{e}(\mathbf{u}) := \frac{1}{2}\big(\nabla\,\mathbf{u} + (\nabla\,\mathbf{u})^{\mathtt{t}}\big)$ is the strain tensor of small deformations, λ, $\mu > 0$ denote the corresponding Lamé constants, and $\mathbf{div}$ stands for the operator div acting along the rows of each tensor. We are particularly interested in the nearly incompressible case, which reduces to assume from now on that λ is sufficiently large. In addition, in order to avoid the symmetry of $\boldsymbol{\rho}$, we reformulate (1.5) in terms of the non-symmetric pseudostress tensor $\boldsymbol{\sigma}$ introduced in [45]. More precisely, according to the analysis provided in [45, Section 2.1], we know that (1.5) is equivalent to the Navier-Lamé equations, which are given by

$$\begin{aligned} \boldsymbol{\sigma} \,&=\, \mu\,\nabla\mathbf{u} \,+\, \big(\lambda+\mu\big)\,\mathrm{tr}\,(\nabla\mathbf{u})\,\mathbb{I} \quad \text{in} \quad \Omega\,, \\ \mathbf{div}(\boldsymbol{\sigma}) \,&=\, -\mathbf{f} \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{1.6}$$

Hence, applying matrix trace to the first equation of (1.6), we can express $\mathrm{tr}\,(\nabla\mathbf{u})$ in terms of $\mathrm{tr}\,(\boldsymbol{\sigma})$ (cf. [45, eq. (2.3)]), so that the former is eliminated and (1.6) is rewritten, equivalently, as

$$\begin{aligned} \frac{1}{\mu}\boldsymbol{\sigma}^{\mathrm{d}} \,+\, \frac{1}{n\big(n\lambda+(n+1)\mu\big)}\,\mathrm{tr}\,(\boldsymbol{\sigma})\,\mathbb{I} \,&=\, \nabla\mathbf{u} \quad \text{in} \quad \Omega\,, \\ \mathbf{div}(\boldsymbol{\sigma}) \,&=\, -\mathbf{f} \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{1.7}$$

Note that the original Cauchy stress tensor $\boldsymbol{\rho}$ can be recovered in terms of the pseudostress $\boldsymbol{\sigma}$ through the postprocessing formula (cf. [45, eq. (2.14)])

$$\boldsymbol{\rho} \,=\, \boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathtt{t}} \,-\, \frac{(\lambda+2\mu)}{\big(n\lambda+(n+1)\mu\big)}\mathrm{tr}\,(\boldsymbol{\sigma})\,\mathbb{I}\,. \tag{1.8}$$

Next, in order to set the Banach spaces-based variational formulation of (1.7), we need a couple of further concepts and tools. Indeed, we first introduce for each $t \in (1,+\infty)$ the Banach space

$$\mathbb{H}^t(\mathbf{div}_t;\Omega) \,:=\, \Big\{\boldsymbol{\tau} \in \mathbb{L}^t(\Omega): \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\Big\}\,, \tag{1.9}$$

which is endowed with the natural norm defined as

$$\|\boldsymbol{\tau}\|_{t,\mathbf{div}_t;\Omega} \,:=\, \|\boldsymbol{\tau}\|_{0,t;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega)\,. \tag{1.10}$$

Note that $\mathbb{H}^2(\mathbf{div}_2;\Omega)$ is the usual Hilbert space $\mathbb{H}(\mathbf{div};\Omega)$. Then, given t, $t' \in (1,+\infty)$ conjugate to each other, we invoke the integration by parts formula (cf. [41, Corollary B. 57])

$$\langle\boldsymbol{\tau}\,\boldsymbol{\nu},\mathbf{v}\rangle_\Gamma \,=\, \int_\Omega\Big\{\boldsymbol{\tau}:\nabla\mathbf{v} \,+\, \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbb{H}^t(\mathbf{div}_t;\Omega)\times\mathbf{W}^{1,t'}(\Omega)\,, \tag{1.11}$$

where $\langle \cdot, \cdot \rangle_\Gamma$ stands for the duality pairing between $\mathbf{W}^{-1/t,t}(\Gamma)$ and $\mathbf{W}^{1/t,t'}(\Gamma)$. Finally, we observe that for each $t \in (1, +\infty)$ there holds

$$\mathbb{H}^t(\mathbf{div}_t;\Omega) \,=\, \mathbb{H}^t_0(\mathbf{div}_t;\Omega) \,\oplus\, \mathrm{R}\,\mathbb{I}\,, \tag{1.12}$$

where

$$\mathbb{H}^t_0(\mathbf{div}_t;\Omega) \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega) : \quad \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}) \,=\, 0 \Big\}\,. \tag{1.13}$$

Equivalently, each $\boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega)$ can be decomposed, uniquely, as

$$\begin{aligned} \boldsymbol{\tau} \,&=\, \boldsymbol{\tau}_0 \,+\, d\,\mathbb{I}\,, \quad \text{with} \quad \boldsymbol{\tau}_0 \in \mathbb{H}^t_0(\mathbf{div}_t;\Omega) \\ \text{and} \qquad d &:= \frac{1}{n|\Omega|} \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}) \in \mathrm{R}\,. \end{aligned} \tag{1.14}$$

Now, given r, $s \in (1, +\infty)$ conjugate to each other, we assume that $\mathbf{f} \in \mathbf{L}^r(\Omega)$ and $\mathbf{u}_D \in \mathbf{W}^{1/s,r}(\Gamma)$, and initially look for $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}^r(\mathbf{div}_r;\Omega) \times \mathbf{W}^{1,r}(\Omega)$ as the solution of (1.7). In this way, multiplying the first equation of (1.7) by a test tensor $\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div}_s;\Omega)$, applying (1.11) with $t = s$ and $t' = r$, and using the Dirichlet boundary condition for $\mathbf{u}$, we find that

$$\frac{1}{\mu} \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \,+\, \frac{1}{n\big(n\lambda + (n+1)\mu\big)} \int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma})\,\mathrm{tr}\,(\boldsymbol{\tau}) \,+\, \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \,=\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D \rangle_\Gamma\,, \tag{1.15}$$

whereas the second equation of (1.7) tested against $\mathbf{v} \in \mathbf{L}^s(\Omega)$ becomes

$$\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) \,=\, -\int_\Omega \mathbf{f} \cdot \mathbf{v}\,. \tag{1.16}$$

In turn, taking $\boldsymbol{\tau} = \mathbb{I}$ in (1.15), it follows that

$$\frac{1}{\big(n\lambda + (n+1)\mu\big)} \int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}) \,=\, \int_\Gamma \mathbf{u}_D \cdot \boldsymbol{\nu}\,,$$

from which, along with (1.14), we deduce that

$$\begin{aligned} \boldsymbol{\sigma} \,&=\, \boldsymbol{\sigma}_0 \,+\, c\,\mathbb{I}\,, \quad \text{with} \quad \boldsymbol{\sigma}_0 \in \mathbb{H}^r_0(\mathbf{div}_r;\Omega) \\ \text{and} \qquad c &:= \frac{\big(n\lambda + (n+1)\mu\big)}{n|\Omega|} \int_\Gamma \mathbf{u}_D \cdot \boldsymbol{\nu} \in \mathrm{R}\,. \end{aligned} \tag{1.17}$$

Regarding the explicit knowledge of the unknown $\boldsymbol{\sigma}$, the foregoing equation shows that it only remains to find its $\mathbb{H}^r_0(\mathbf{div}_r;\Omega)$-component $\boldsymbol{\sigma}_0$. Hence, replacing $\boldsymbol{\sigma} \,=\, \boldsymbol{\sigma}_0 \,+\, c\,\mathbb{I}$ back into (1.15), redenoting $\boldsymbol{\sigma}_0$ simply by $\boldsymbol{\sigma}$, noting that the testing of the resulting (1.15) against $\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div}_s;\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}^s_0(\mathbf{div}_s;\Omega)$, and placing this new equation jointly with (1.16), we arrive at the following mixed variational formulation of (1.7): Find $(\boldsymbol{\sigma}, \mathbf{u}) \in X_2 \times M_1$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}_1(\boldsymbol{\tau}, \mathbf{u}) \,&=\, F(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in X_1\,, \\ \mathbf{b}_2(\boldsymbol{\sigma}, \mathbf{v}) \,&=\, G(\mathbf{v}) \qquad \forall\, \mathbf{v} \in M_2\,, \end{aligned} \tag{1.18}$$

where

$$X_2 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega)\,, \quad M_1 := \mathbf{L}^r(\Omega)\,, \quad X_1 := \mathbb{H}_0^s(\mathbf{div}_s;\Omega)\,, \quad \text{and} \quad M_2 := \mathbf{L}^s(\Omega)\,, \tag{1.19}$$

and the bilinear forms $\mathbf{a} : X_2 \times X_1 \to \mathrm{R}$ and $\mathbf{b}_i : X_i \times M_i \to \mathrm{R}$, $i \in \{1,2\}$, and the functionals $F \in X_1'$ and $G \in M_2'$, are defined, respectively, as

$$\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau}) := \frac{1}{\mu}\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{n\big(n\lambda + (n+1)\mu\big)}\int_\Omega \operatorname{tr}(\boldsymbol{\zeta})\operatorname{tr}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau}) \in X_2 \times X_1\,, \tag{1.20}$$

$$\mathbf{b}_i(\boldsymbol{\tau},\mathbf{v}) := \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau},\mathbf{v}) \in X_i \times M_i\,, \tag{1.21}$$

$$F(\boldsymbol{\tau}) := \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D\rangle_\Gamma \qquad \forall\,\boldsymbol{\tau} \in X_1\,, \tag{1.22}$$

and

$$G(\mathbf{v}) := -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v} \in M_2\,. \tag{1.23}$$

We remark here that the above notations for the spaces involved have been chosen for convenience of the definitions of $\mathbf{b}_1$ and $\mathbf{b}_2$.

1.2.2 The Stokes system

The goal of this model is to determine the pseudostress tensor $\boldsymbol{\sigma}$, the velocity $\mathbf{u}$, and the pressure p of a steady flow occupying the region Ω, under the action of external forces. More precisely, given a volume force $\mathbf{f}$ and a Dirichlet datum $\mathbf{u}_D$, we now seek a tensor field $\boldsymbol{\sigma}$, a vector field $\mathbf{u}$, and a scalar field p such that

$$\begin{gathered} \boldsymbol{\sigma} = 2\mu\nabla\mathbf{u} - p\mathbb{I} \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\boldsymbol{\sigma}) = -\mathbf{f} \quad \text{in} \quad \Omega\,, \\ \operatorname{div}(\mathbf{u}) = 0 \quad \text{in} \quad \Omega\,, \quad \int_\Omega p = 0\,, \quad \text{and} \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \end{gathered} \tag{1.24}$$

where μ is the kinematic viscosity, and, as required by the incompressibility equation $\operatorname{div}(\mathbf{u}) = 0$, the datum $\mathbf{u}_D$ satisfies the compatibility condition $\int_\Gamma \mathbf{u}_D\cdot\boldsymbol{\nu} = 0$. Then, proceeding exactly as in [53, Section 2.1], we can show that (1.24) can be rewritten as

$$\begin{gathered} \boldsymbol{\sigma} = 2\mu\nabla\mathbf{u} - p\mathbb{I} \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\boldsymbol{\sigma}) = -\mathbf{f} \quad \text{in} \quad \Omega\,, \\ p + \frac{1}{n}\operatorname{tr}(\boldsymbol{\sigma}) = 0 \quad \text{in} \quad \Omega\,, \quad \int_\Omega p = 0\,, \quad \text{and} \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \end{gathered} \tag{1.25}$$

from which, eliminating the pressure p, which can calculated later on by the postprocessing formula $p = -\frac{1}{n}\operatorname{tr}(\boldsymbol{\sigma})$, we arrive at the equivalent system

$$\begin{gathered} \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathrm{d}} = \nabla\mathbf{u} \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\boldsymbol{\sigma}) = -\mathbf{f} \quad \text{in} \quad \Omega\,, \\ \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0\,, \quad \text{and} \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,. \end{gathered} \tag{1.26}$$

In this way, assuming that $\mathbf{f} \in \mathbf{L}^r(\Omega)$ and $\mathbf{u}_D \in \mathbf{W}^{1/s,r}(\Gamma)$, and proceeding analogously to the derivation of (1.18), we obtain the following mixed variational formulation of (1.26): Find $(\boldsymbol{\sigma}, \mathbf{u}) \in X_2 \times M_1$ such that

$$\begin{aligned} \widetilde{\mathbf{a}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}_1(\boldsymbol{\tau}, \mathbf{u}) &= F(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in X_1\,, \\ \mathbf{b}_2(\boldsymbol{\sigma}, \mathbf{v}) &= G(\mathbf{v}) \qquad \forall\, \mathbf{v} \in M_2\,, \end{aligned} \tag{1.27}$$

where the spaces X_2, M_1, X_1 and M_2, the bilinear forms $\mathbf{b}_i : X_i \times M_i \to \mathrm{R}$, $i \in \{1, 2\}$, and the functionals F and G are those given by (1.19), (1.21), (1.22), and (1.23), whereas the bilinear form $\widetilde{\mathbf{a}} : X_2 \times X_1 \to \mathrm{R}$ is defined as

$$\widetilde{\mathbf{a}}(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \frac{1}{2\mu} \int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \qquad \forall\, (\boldsymbol{\zeta}, \boldsymbol{\tau}) \in X_2 \times X_1\,. \tag{1.28}$$

Later on in Section 1.4 we prove the well-posedness of the mixed variational formulations (1.18) and (1.27), for which we establish below some results that will be employed in the respective proofs.

1.3 Some preliminary results

We begin by considering a Banach spaces-based primal formulation for a slight generalization of the Stokes system (1.24) with viscosity $\mu = 1/2$ and null Dirichlet boundary condition, which, given r, $s \in (1, +\infty)$ conjugate to each other, $\mathbf{g} \in \mathbb{L}^r(\Omega)$, and $\mathbf{f} \in \mathbf{L}^r(\Omega)$, consists of seeking a pair $(\mathbf{u}, p) \in \mathbf{W}^{1,r}(\Omega) \times \mathrm{L}^r(\Omega)$ such that

$$\begin{gathered} \mathbf{div}(\nabla \mathbf{u} - p\,\mathbb{I} - \mathbf{g}) \;=\; -\mathbf{f} \quad \text{in} \quad \Omega\,, \\ \operatorname{div}(\mathbf{u}) = 0 \quad \text{in} \quad \Omega, \qquad \int_\Omega p = 0, \quad \text{and} \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma\,. \end{gathered} \tag{1.29}$$

Note that the above mentioned generalization refers to the incorporation of a further datum $\mathbf{g}$ within the divergence operator, whose purpose, rather mathematical than physical, has to do with the introduction in Lemma 1.2 of a key operator for our analysis, and particularly with the verification of its divergence free property (cf. (1.35)).

Then, applying (1.11) with $\boldsymbol{\tau} := \nabla \mathbf{u} - p\,\mathbb{I} - \mathbf{g} \in \mathbb{H}^r(\mathbf{div}_r; \Omega)$ and $\mathbf{v} \in \mathbf{W}_0^{1,s}(\Omega)$, and performing some minor algebraic rearrangements, the testing of the first equation of (1.29) becomes

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \operatorname{div}(\mathbf{v}) \;=\; \int_\Omega \mathbf{g} : \nabla \mathbf{v} + \int_\Omega \mathbf{f} \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in \mathbf{W}_0^{1,s}(\Omega)\,. \tag{1.30}$$

In turn, it is easy to see, thanks to the homogeneous Dirichlet boundary condition satisfied by $\mathbf{u}$, that testing the incompressibility equation $\operatorname{div}(\mathbf{u}) = 0$ in Ω against $q \in \mathrm{L}^s(\Omega)$ is equivalent to doing it against $q \in \mathrm{L}_0^s(\Omega)$. Consequently, the weak formulation of (1.29) reduces to: Find $(\mathbf{u}, p) \in \mathbf{W}_0^{1,r}(\Omega) \times \mathrm{L}_0^r(\Omega)$ such that

$$\begin{aligned} \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \operatorname{div}(\mathbf{v}) &= F(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{W}_0^{1,s}(\Omega)\,, \\ \int_\Omega q \operatorname{div}(\mathbf{u}) &= 0 \qquad\quad \forall\, q \in \mathrm{L}_0^s(\Omega)\,, \end{aligned} \tag{1.31}$$

where the functional $F \in \mathbf{W}^{-1,r}(\Omega) := \mathbf{W}_0^{1,s}(\Omega)'$ is defined as

$$F(\mathbf{v}) := \int_\Omega \mathbf{g} : \nabla \mathbf{v} + \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{W}_0^{1,s}(\Omega)\,. \tag{1.32}$$

We now establish, as a consequence of a more general result from [71], the well-posedness of (1.31), even irrespective of the particular form of F given by (1.32).

Theorem 1.1. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let r, $s \in (1,+\infty)$ conjugate to each other. Then, there exists $\delta > 0$ such that for each $r \in \big(\frac{2n}{n+1} - \delta, \frac{2n}{n-1} + \delta\big)$, and for each $F \in \mathbf{W}^{-1,r}(\Omega)$, there exists a unique pair $(\mathbf{u}, p) \in \mathbf{W}_0^{1,r}(\Omega) \times \mathrm{L}_0^r(\Omega)$ solution to (1.31). Moreover, there exists a positive constant c_r, such that*

$$\|\mathbf{u}\|_{1,r;\Omega} \,+\, \|p\|_{0,r;\Omega} \,\leq\, c_r\, \|F\|_{-1,r;\Omega}\,. \tag{1.33}$$

Proof. We first assume that $\Omega \subset \mathrm{R}^2$. Then, taking the local parameters $\alpha = -1$ and $q = 2$ in [71, Corollary 1.7], we deduce, according to [71, eq. (1.47)], that there exists $\epsilon \in (0, \frac{1}{2}]$ such that for each $F \in \mathbf{W}^{-1,r}(\Omega)$ the problem (1.31) has a unique solution $(\mathbf{u}, p) \in \mathbf{W}_0^{1,r}(\Omega) \times \mathrm{L}_0^r(\Omega)$ satisfying (1.33) whenever the point $\big(\alpha - \frac{1}{r} + 2, \frac{1}{r}\big) = \big(1 - \frac{1}{r}, \frac{1}{r}\big)$ belongs to the two-dimensional region specified by [71, Figure 1]. More precisely, the latter means either

i) $0 \,<\, 1 - \dfrac{1}{r} \,<\, \dfrac{1}{2} + \epsilon \quad$ and $\quad 0 \,<\, \dfrac{1}{r} \,<\, \dfrac{3}{2} - \dfrac{1}{r} + \epsilon$, or

ii) $\dfrac{1}{2} + \epsilon \,\leq\, 1 - \dfrac{1}{r} \,<\, 1 \quad$ and $\quad \dfrac{1}{2} - \dfrac{1}{r} - \epsilon \,<\, \dfrac{1}{r} \,<\, \dfrac{3}{2} - \dfrac{1}{r} + \epsilon$.

Then, solving these inequalities, one obtains $r \in \big(\frac{4}{3} - \epsilon_1, \frac{2}{1-2\epsilon}\big)$, with $\epsilon_1 := \frac{8\epsilon}{9+6\epsilon}$, and $r \in \big[\frac{2}{1-2\epsilon}, 4 + \epsilon_2\big)$, with $\epsilon_2 := \frac{8\epsilon}{1-2\epsilon}$, as solutions of i) and ii), respectively, so that the final feasible range for r is the interval $\big(\frac{4}{3} - \epsilon_1, 4 + \epsilon_2\big)$. In this way, observing now that $\epsilon_1 < \epsilon < \epsilon_2$, we arrive at the indicated range for r (cf. [71, eq. (1.52)]) with $\delta = \epsilon_1$. In turn, the case $\Omega \subset \mathrm{R}^3$ proceeds analogously by imposing now the point $\big(1 - \frac{1}{r}, \frac{1}{r}\big)$ to belong to the two-dimensional region specified by [71, Figure 2]. We omit further details. □

We stress here that when F is given by (1.32), the a priori estimate (1.33) becomes

$$\|u\|_{1,r;\Omega} \,+\, \|p\|_{0,r;\Omega} \,\leq\, c_r \Big\{ \|\mathbf{g}\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,r;\Omega} \Big\}\,. \tag{1.34}$$

The following result, which constitutes an extension of [52, Lemma 2.3] to the present tensor context, makes use of Theorem 1.1 to introduce a suitable operator mapping $\mathbb{L}^t(\Omega)$ into itself for each t in the range specified by this theorem.

Lemma 1.2. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let t, $t' \in (1,+\infty)$ conjugate to each other with t satisfying the range given by Theorem 1.1. Then, there exists a linear and bounded operator $D_t : \mathbb{L}^t(\Omega) \to \mathbb{L}^t(\Omega)$ such that*

$$\mathbf{div}\big(D_t(\boldsymbol{\tau})\big) \,=\, \mathbf{0} \quad in \quad \Omega\,, \tag{1.35}$$

and

$$\int_\Omega \mathrm{tr}\left(D_t(\boldsymbol{\tau})\right) \,=\, \int_\Omega \mathrm{tr}\left(\boldsymbol{\tau}\right), \tag{1.36}$$

for all $\boldsymbol{\tau} \in \mathbb{L}^t(\Omega)$. *In addition, for each* $\boldsymbol{\zeta} \in \mathbb{L}^{t'}(\Omega)$ *such that* $\mathbf{div}(\boldsymbol{\zeta}) \,=\, \mathbf{0}$ *in* Ω*, there holds*

$$\int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \left(D_t(\boldsymbol{\tau})\right)^{\mathtt{d}} \,=\, \int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{L}^t(\Omega)\,. \tag{1.37}$$

Proof. Given $\boldsymbol{\tau} \in \mathbb{L}^t(\Omega)$, we let $(\mathbf{u}, p) \in \mathbf{W}^{1,t}(\Omega) \times \mathrm{L}^t(\Omega)$ be the unique solution, guaranteed by Theorem 1.1, of the Stokes problem (1.29) with $r = t$, $\mathbf{g} = \boldsymbol{\tau}$ and $\mathbf{f} = \mathbf{0}$, that is

$$\begin{gathered} \mathbf{div}(\nabla \mathbf{u} - p\,\mathbb{I} - \boldsymbol{\tau}) = \mathbf{0} \quad \text{in} \quad \Omega\,, \\ \mathrm{div}(\mathbf{u}) = 0 \quad \text{in} \quad \Omega, \qquad \int_\Omega p = 0, \quad \text{and} \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma\,, \end{gathered} \tag{1.38}$$

whose weak formulation is given by (1.31) and (1.32). Note that the functional $F \in \mathbf{W}^{-1,t}(\Omega) = \mathbf{W}_0^{1,t'}(\Omega)'$ (cf. (1.32)) reduces in this case to $F(\mathbf{v}) := \int_\Omega \boldsymbol{\tau} : \nabla \mathbf{v}$ for all $\mathbf{v} \in \mathbf{W}_0^{1,t'}(\Omega)$. It follows, in virtue of the continuous dependence result (1.34), that $\|\mathbf{u}\|_{1,t;\Omega} + \|p\|_{0,t;\Omega} \,\le\, c_t\, \|\boldsymbol{\tau}\|_{0,t;\Omega}$, so that, defining

$$D_t(\boldsymbol{\tau}) \,:=\, \boldsymbol{\tau} - (\nabla \mathbf{u} - p\,\mathbb{I}) \in \mathbb{L}^t(\Omega)\,, \tag{1.39}$$

we see that D_t is linear and bounded, namely

$$\|D_t(\boldsymbol{\tau})\|_{0,t;\Omega} \,\le\, (1 + n^{1/t}\, c_t)\, \|\boldsymbol{\tau}\|_{0,t;\Omega}\,, \tag{1.40}$$

which implies $\|D_t\| \,\le\, (1 + n^{1/t}\, c_t)$, and clearly $D_t(\boldsymbol{\tau})$ is divergence free in Ω. In addition, since $\mathrm{tr}\,(\nabla \mathbf{u}) = \mathrm{div}(\mathbf{u}) = 0$ and $\int_\Omega p = 0$, we readily deduce from (1.39) that for each $\boldsymbol{\tau} \in \mathbb{L}^t(\Omega)$ there holds

$$\int_\Omega \mathrm{tr}\left(D_t(\boldsymbol{\tau})\right) \,=\, \int_\Omega \mathrm{tr}\left(\boldsymbol{\tau}\right) \,+\, n\int_\Omega p \,=\, \int_\Omega \mathrm{tr}\left(\boldsymbol{\tau}\right),$$

which proves (1.36). Furthermore, using again that $\mathrm{tr}\,(\nabla \mathbf{u}) = 0$, we have that $\left(D_t(\boldsymbol{\tau})\right)^{\mathtt{d}} = \boldsymbol{\tau}^{\mathtt{d}} - \nabla \mathbf{u}$, and hence, given $\boldsymbol{\zeta} \in \mathbb{L}^{t'}(\Omega)$ such that $\mathbf{div}(\boldsymbol{\zeta}) \,=\, 0$ in Ω, and applying (1.11) to $\boldsymbol{\zeta} \in \mathbb{H}^{t'}(\mathbf{div}_{t'}; \Omega)$ and $\mathbf{u} \in \mathbf{W}_0^{1,t}(\Omega)$, we deduce that

$$\int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \nabla \mathbf{u} \,=\, \int_\Omega \boldsymbol{\zeta} : \nabla \mathbf{u} \,=\, 0\,,$$

which yields (1.37) and ends the proof. □

On the other hand, for each $t \in (1, +\infty)$ we introduce the subspace of $\mathrm{L}^t(\Omega)$ given by

$$\mathrm{L}_0^t(\Omega) \,:=\, \left\{ v \in \mathrm{L}^t(\Omega) : \quad \int_\Omega v = 0 \right\}. \tag{1.41}$$

Then, we have from [41, Lemma B.69] (see [16] for the original reference, or [40]) the following result.

Lemma 1.3. *Let* Ω *be a bounded domain of* R^n*,* $n \ge 2$*, which is star-shaped with respect to a ball. Then, for each* $t \in (1, +\infty)$ *the operator* $\mathrm{div} : \mathbf{W}_0^{1,t}(\Omega) \to \mathrm{L}_0^t(\Omega)$ *is surjective.*

Thanks to Lemma 1.3 and the open mapping theorem (cf. [4, Theorem 7.7]), we readily deduce that, given $t \in (1, +\infty)$, there exists a constant $C_{\mathtt{t}} > 0$, such that for every $v \in \mathrm{L}^t_0(\Omega)$ there exists $\mathbf{z}_v \in \mathbf{W}^{1,t}_0(\Omega)$ satisfying

$$\operatorname{div}(\mathbf{z}_v) \,=\, v \quad \text{and} \quad \|\mathbf{z}_v\|_{1,t;\Omega} \,\le\, C_{\mathtt{t}}\, \|v\|_{0,t;\Omega}\,. \tag{1.42}$$

We now employ Lemma 1.3, and particularly (1.42), to provide a generalization from $r = 2$ to any $r \in (1, +\infty)$ of the inequality stated in [17, Chapter IV, Proposition 3.1] (see also [44, Lemma 2.3]), namely

$$\|\boldsymbol{\tau}\|_{0,\Omega} \,\le\, C \left\{ \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega} \right\} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^2_0(\mathbf{div}_2;\Omega)\,,$$

which plays a key role in the solvability analysis of the classical Hilbertian dual-mixed variational formulation of linear elasticity (cf. [17, Chapter IV, Section IV.3], [44, Section 2.4.3]). More precisely, we have the following result.

Lemma 1.4. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2, 3\}$, which is star-shaped with respect to a ball, and let $r \in (1, +\infty)$. Then, there exist positive constants $\widetilde{C}_{\mathtt{r}}$ and $\widehat{C}_{\mathtt{r}}$ such that*

$$\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,r;\Omega} \,\le\, \widetilde{C}_r \left\{ \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,r;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \right\} \tag{1.43}$$

and

$$\|\boldsymbol{\tau}\|_{0,r;\Omega} \,\le\, \widehat{C}_r \left\{ \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,r;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \right\} \tag{1.44}$$

for all $\boldsymbol{\tau} \in \mathbb{H}^r_0(\mathbf{div}_r;\Omega)$.

Proof. Given r, $s \in (1, +\infty)$ conjugate to each other, we first recall that the dual of $\mathrm{L}^s(\Omega)$ is identified with $\mathrm{L}^r(\Omega)$. Then, given $\boldsymbol{\tau} \in \mathbb{H}^r_0(\mathbf{div}_r;\Omega)$, which yields $\mathrm{tr}\,(\boldsymbol{\tau}) \in \mathrm{L}^r_0(\Omega)$, we apply the associated duality argument and the fact that $\mathrm{L}^s(\Omega) = \mathrm{L}^s_0(\Omega) \oplus \mathrm{R}$, to observe that

$$\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,r;\Omega} \,=\, \sup_{\substack{v \in \mathrm{L}^s(\Omega)\\ v \neq 0}} \frac{\displaystyle\int_\Omega v\,\mathrm{tr}\,(\boldsymbol{\tau})}{\|v\|_{0,s;\Omega}} \,=\, \sup_{\substack{v \in \mathrm{L}^s_0(\Omega)\\ v \neq 0}} \frac{\displaystyle\int_\Omega v\,\mathrm{tr}\,(\boldsymbol{\tau})}{\|v\|_{0,s;\Omega}}\,. \tag{1.45}$$

Next, given $v \in \mathrm{L}^s_0(\Omega)$, $v \neq 0$, we make use of (1.42) (with $t = s$) and proceed analogously to the proof of [17, Chapter IV, Proposition 3.1] to estimate $\int_\Omega v\,\mathrm{tr}\,(\boldsymbol{\tau})$. Indeed, recalling that $\operatorname{div}(\mathbf{z}_v) = \mathrm{tr}\,(\nabla \mathbf{z}_v)$, utilizing the definition and properties of the deviatoric tensors, and then integrating by parts according to (1.11) with $\boldsymbol{\tau} \in \mathbb{H}^r_0(\mathbf{div}_r;\Omega)$ and $\mathbf{z}_v \in \mathbf{W}^{1,s}_0(\Omega)$, we find that

$$\begin{aligned}
\int_\Omega v\,\mathrm{tr}\,(\boldsymbol{\tau}) \,&=\, \int_\Omega \operatorname{div}(\mathbf{z}_v)\,\mathrm{tr}\,(\boldsymbol{\tau}) \,=\, \int_\Omega \mathrm{tr}\,(\nabla \mathbf{z}_v)\,\boldsymbol{\tau} : \mathbb{I} \\
&=\, \int_\Omega \boldsymbol{\tau} : \mathrm{tr}\,(\nabla \mathbf{z}_v)\,\mathbb{I} \,=\, n \int_\Omega \boldsymbol{\tau} : \big(\nabla \mathbf{z}_v - (\nabla \mathbf{z}_v)^{\mathtt{d}}\big) \\
&=\, n \int_\Omega \boldsymbol{\tau} : \nabla \mathbf{z}_v - n \int_\Omega \boldsymbol{\tau}^{\mathtt{d}} : \nabla \mathbf{z}_v \\
&=\, -n \int_\Omega \mathbf{z}_v \cdot \operatorname{div}(\boldsymbol{\tau}) - n \int_\Omega \boldsymbol{\tau}^{\mathtt{d}} : \nabla \mathbf{z}_v\,,
\end{aligned}$$

from which, employing Hölder's inequality and (1.42), we obtain

$$\begin{aligned} \left| \int_\Omega v \,\mathrm{tr}\,(\boldsymbol{\tau}) \right| \,&\le\, n \, \|\mathbf{z}_v\|_{1,s;\Omega} \left\{ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,r;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \right\} \\ &\le\, n \, C_s \, \|v\|_{0,s;\Omega} \left\{ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,r;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \right\}. \end{aligned} \tag{1.46}$$

In this way, replacing (1.46) back into (1.45) we arrive at (1.43) with $\widetilde{C}_r := n\, C_s$. Furthermore, using the triangle inequality and the fact that $\|\mathrm{tr}\,(\boldsymbol{\tau})\,\mathbb{I}\|^r_{0,r;\Omega} = n\|\mathrm{tr}\,(\boldsymbol{\tau})\|^r_{0,r;\Omega}$, we get

$$\|\boldsymbol{\tau}\|_{0,r;\Omega} \,\le\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,r;\Omega} + \frac{1}{n}\, \|\mathrm{tr}\,(\boldsymbol{\tau})\,\mathbb{I}\|_{0,r;\Omega} = \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,r;\Omega} + n^{1/r-1}\, \|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,r;\Omega}\,,$$

which, along with (1.43), implies (1.44) with $\widehat{C}_r := 1 + n^{1/r}\, C_s$. □

We end this section with a Banach spaces-based primal formulation for the vector Poisson equation, which, given $r,\, s \in (1,+\infty)$ conjugate to each other, $\mathbf{g} \in \mathbb{L}^r(\Omega)$, and $\mathbf{f} \in \mathbf{L}^r(\Omega)$, consists of seeking $\mathbf{u} \in \mathbf{W}^{1,r}(\Omega)$ such that

$$\mathbf{div}(\nabla \mathbf{u} - \mathbf{g}) \,=\, -\mathbf{f} \quad \text{in} \quad \Omega\,, \qquad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma\,. \tag{1.47}$$

Then, proceeding similarly as for (1.29), that is applying (1.11) with $\boldsymbol{\tau} := \nabla \mathbf{u} - \mathbf{g} \in \mathbb{H}^r(\mathbf{div}_r;\Omega)$ and $\mathbf{w} \in \mathbf{W}^{1,s}_0(\Omega)$, we arrive at the following weak formulation of (1.47): Find $\mathbf{u} \in \mathbf{W}^{1,r}_0(\Omega)$ such that

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{w} \,=\, F(\mathbf{w}) \qquad \forall\, \mathbf{w} \in \mathbf{W}^{1,s}_0(\Omega)\,, \tag{1.48}$$

where $F \in \mathbf{W}^{-1,r}(\Omega) := \mathbf{W}^{1,s}_0(\Omega)'$ is defined as in (1.32), that is

$$F(\mathbf{w}) := \int_\Omega \mathbf{g} : \nabla \mathbf{w} + \int_\Omega \mathbf{f} \cdot \mathbf{w} \qquad \forall\, \mathbf{w} \in \mathbf{W}^{1,s}_0(\Omega)\,. \tag{1.49}$$

We establish next the analogue of Theorem 1.1 for the vector Poisson equation, which arises in this case as a straightforward consequence of more general results provided in [60]. We remark in advance that the arguments of the proof are very similar to those from Theorem 1.1, whereas the resulting ranges for r are exactly the same. In addition, we stress that while [60] addresses the scalar Poisson equation, the analysis and results certainly applies to the present version as well. Actually, there is no intrinsic difference between both versions, so that we provide below some details just for sake of clearness.

Theorem 1.5. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let $r,\, s \in (1,+\infty)$ conjugate to each other. Then, there exists $\delta > 0$ such that for each $r \in \big(\frac{2n}{n+1} - \delta, \frac{2n}{n-1} + \delta\big)$, and for each $F \in \mathbf{W}^{-1,r}(\Omega)$, there exists a unique $\mathbf{u} \in \mathbf{W}^{1,r}_0(\Omega)$ solution to (1.48). Moreover, there exists a positive constant $\bar{c}_r$, such that*

$$\|\mathbf{u}\|_{1,r;\Omega} \,\le\, \bar{c}_r\, \|F\|_{-1,r;\Omega}\,. \tag{1.50}$$

Proof. We first assume that $\Omega \subset \mathrm{R}^3$. Then, taking the local parameter $\alpha = 1$ in [60, Theorem 1.1], we deduce that there exists $\epsilon \in (0,1]$ such that for each $F \in \mathbf{W}^{-1,r}(\Omega)$ the problem (1.48) has a unique solution $\mathbf{u} \in \mathbf{W}^{1,r}_0(\Omega)$ satisfying (1.50) whenever:

i) $1 < r \le \kappa$ and $\dfrac{3}{r} - 1 - \epsilon < 1 < 1 + \dfrac{1}{r}$, or

ii) $\kappa < r < \kappa'$ and $\dfrac{1}{r} < 1 < 1 + \dfrac{1}{r}$, or

iii) $\kappa' \le r < +\infty$ and $\dfrac{1}{r} < 1 < \dfrac{3}{r} + \epsilon$,

where $\kappa = \frac{2}{1+\epsilon}$ and $\kappa' = \frac{2}{1-\epsilon}$. Then, in order to guarantee that at least one of the above is accomplished, one simply solves the three inequalities on the right hand-side, which gives

$$\frac{3}{2} - \epsilon_1 < r < 3 + \epsilon_2 \quad \text{with} \quad \epsilon_1 := \frac{3\epsilon}{2(2+\epsilon)} \quad \text{and} \quad \epsilon_2 := \frac{3\epsilon}{1-\epsilon}\,.$$

Hence, noticing that $\epsilon_1 < \epsilon < \epsilon_2$, we obtain the indicated range for r with $\delta = \epsilon_1$. The case $\Omega \subset \mathrm{R}^2$ proceeds analogously by taking now $\alpha = 1$ in [60, Theorem 1.3]. Further details are omitted. □

1.4 The main results

In this section we apply the Babuška-Brezzi theory in Banach spaces and the results from Section 1.3 to prove the unique solvability and continuous dependence result for each one of the mixed variational formulations (1.18) and (1.27). For sake of completeness and clearness, we follow [12, Theorem 2.1, Corollary 2.1, Section 2.1] to state below the main theorem concerning the aforementioned theory.

Theorem 1.6. *Let X_1, X_2, M_1, and M_2 be real reflexive Banach spaces, and let $a : X_2 \times X_1 \to \mathrm{R}$ and $b_i : X_i \times M_i \to \mathrm{R}$, $i \in \{1,2\}$, be bounded bilinear forms with boundedness constants given by $\|a\|$ and $\|b_i\|$, $i \in \{1,2\}$, respectively. In addition, for each $i \in \{1,2\}$, let $\mathcal{K}_i$ be the kernel of the operator induced by b_i, that is*

$$\mathcal{K}_i := \Big\{ \tau \in X_i : \quad b_i(\tau, v) = 0 \quad \forall\, v \in M_i \Big\}.$$

Assume that

i) *there exists $\alpha > 0$ such that*

$$\sup_{\substack{\tau \in \mathcal{K}_1 \\ \tau \neq 0}} \frac{a(\zeta,\tau)}{\|\tau\|_{X_1}} \ge \alpha\, \|\zeta\|_{X_2} \qquad \forall\, \zeta \in \mathcal{K}_2\,,$$

ii) *there holds*

$$\sup_{\zeta \in \mathcal{K}_2} a(\zeta,\tau) > 0 \qquad \forall\, \tau \in \mathcal{K}_1\,,\ \tau \neq 0\,,$$

iii) *for each $i \in \{1,2\}$ there exists $\beta_i > 0$ such that*

$$\sup_{\substack{\zeta \in X_i \\ \zeta \neq 0}} \frac{b_i(\zeta, v)}{\|\zeta\|_{X_i}} \ge \beta_i\, \|v\|_{M_i} \qquad \forall\, v \in M_i\,.$$

Then, for each $(F,G) \in X_1' \times M_2'$ *there exists a unique* $(\sigma,u) \in X_2 \times M_1$ *such that*

$$\begin{aligned} a(\sigma,\tau) + b_1(\tau,u) &= F(\tau) \qquad \forall\, \tau \in X_1\,, \\ b_2(\sigma,v) &= G(v) \qquad \forall\, v \in M_2\,, \end{aligned} \tag{1.51}$$

and the following a priori estimates hold:

$$\begin{aligned} \|\sigma\|_{X_2} &\le \frac{1}{\alpha}\,\|F\|_{X_1'} + \frac{1}{\beta_2}\left(1 + \frac{\|a\|}{\alpha}\right)\|G\|_{M_2'}\,, \\ \|u\|_{M_1} &\le \frac{1}{\beta_1}\left(1 + \frac{\|a\|}{\alpha}\right)\|F\|_{X_1'} + \frac{\|a\|}{\beta_1\,\beta_2}\left(1 + \frac{\|a\|}{\alpha}\right)\|G\|_{M_2'}\,. \end{aligned} \tag{1.52}$$

Moreover, i)*,* ii)*, and* iii) *are also necessary conditions for the well-posedness of* (1.51)*.*

We begin by providing a tensor version of [52, Lemma 2.2]. Indeed, given t, $t' \in (1,+\infty)$ conjugate to each other, we define for each $\boldsymbol{\tau} \in \mathbb{L}^t(\Omega)$

$$\boldsymbol{\mathcal{J}}_t(\boldsymbol{\tau}) := \begin{cases} \|\boldsymbol{\tau}\|^{t-2}\,\boldsymbol{\tau} & \text{if } \boldsymbol{\tau} \ne \mathbf{0}\,, \\ \mathbf{0} & \text{otherwise}\,, \end{cases} \tag{1.53}$$

and observe, after simple algebraic computations, that

$$\boldsymbol{\tau}_{t'} := \boldsymbol{\mathcal{J}}_t(\boldsymbol{\tau}) \in \mathbb{L}^{t'}(\Omega) \quad \text{if and only if} \quad \boldsymbol{\tau} = \boldsymbol{\mathcal{J}}_{t'}(\boldsymbol{\tau}_{t'}), \quad \text{and} \tag{1.54}$$

$$\int_\Omega \boldsymbol{\tau} : \boldsymbol{\tau}_{t'} = \|\boldsymbol{\tau}\|_{0,t;\Omega}^t = \|\boldsymbol{\tau}_{t'}\|_{0,t';\Omega}^{t'} = \|\boldsymbol{\tau}\|_{0,t;\Omega}\,\|\boldsymbol{\tau}_{t'}\|_{0,t';\Omega}\,. \tag{1.55}$$

Next, for each $i \in \{1,2\}$ we let $K_i \subset X_i$ be the kernel of the bilinear form $\mathbf{b}_i$, which, according to the definition of the spaces involved (cf. (1.19)), and b_i (cf. (1.21)), yields

$$K_i := \Big\{\boldsymbol{\tau} \in X_i : \quad \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0}\Big\}\,. \tag{1.56}$$

Then, the inf-sup conditions required for the bilinear form $\mathbf{a}$ (cf. (1.20)) are established as follows.

Lemma 1.7. *Assume that* r *and* s *satisfy the range specified by Theorem* 1.1*. Then, there exist positive constants* M *and* α *such that for each* $\lambda > M$ *there hold*

$$\sup_{\substack{\boldsymbol{\tau}\in K_1 \\ \boldsymbol{\tau}\ne\mathbf{0}}} \frac{\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \ge \alpha\,\|\boldsymbol{\zeta}\|_{X_2} \qquad \forall\,\boldsymbol{\zeta} \in K_2\,, \tag{1.57}$$

and

$$\sup_{\boldsymbol{\zeta}\in K_2} \mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau}) > 0 \qquad \forall\,\boldsymbol{\tau} \in K_1\,,\ \boldsymbol{\tau} \ne \mathbf{0}\,. \tag{1.58}$$

Proof. We first observe that for each pair $(\boldsymbol{\zeta},\boldsymbol{\tau}) \in X_2 \times X_1 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega) \times \mathbb{H}_0^s(\mathbf{div}_s;\Omega)$ there holds

$$\left|\int_\Omega \mathrm{tr}\,(\boldsymbol{\zeta})\,\mathrm{tr}\,(\boldsymbol{\tau})\right| \le n^{1/r}\,\|\mathrm{tr}\,(\boldsymbol{\zeta})\|_{0,r;\Omega}\,\|\boldsymbol{\tau}\|_{0,s;\Omega}\,, \tag{1.59}$$

which follows from simple applications of the Hölder and triangle inequalities, the latter in $\mathrm{L}^s(\Omega)$ and the former in $\mathrm{L}^r(\Omega) \times \mathrm{L}^s(\Omega)$ and $\mathbf{R} \times \mathbf{R}$. Now, let $\boldsymbol{\zeta} \in K_2$, that is $\boldsymbol{\zeta} \in X_2 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega)$ and $\mathbf{div}(\boldsymbol{\zeta}) = \mathbf{0}$, and assume that $\boldsymbol{\zeta} \neq \mathbf{0}$. Then, bearing in mind the definition of $\mathbf{a}$ (cf. (1.20)), and employing (2.62) and (1.43) (cf. Lemma 1.4), we readily find that

$$\sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \frac{1}{\mu} \sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}}}{\|\boldsymbol{\tau}\|_{X_1}} \,-\, \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda + (n+1)\mu\big)}\, \|\boldsymbol{\zeta}^{\mathtt{d}}\|_{0,r;\Omega}\,. \tag{1.60}$$

In turn, letting $\boldsymbol{\zeta}_s := \boldsymbol{\mathcal{J}}_r(\boldsymbol{\zeta}^{\mathtt{d}}) \in \mathbb{L}^s(\Omega)$ as defined in (1.53), we clearly have $\operatorname{tr}(\boldsymbol{\zeta}_s) = 0$, and thus, thanks to Lemma 1.2, it follows that $D_s(\boldsymbol{\zeta}_s)$ belongs to K_1. Next, using (1.37) and (1.55), we get

$$\int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \big(D_s(\boldsymbol{\zeta}_s)\big)^{\mathtt{d}} \,=\, \int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\zeta}_s^{\mathtt{d}} \,=\, \int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\zeta}_s \,=\, \|\boldsymbol{\zeta}^{\mathtt{d}}\|_{0,r;\Omega}\, \|\boldsymbol{\zeta}_s\|_{0,s;\Omega}\,,$$

and hence, noting that $\|D_s(\boldsymbol{\zeta}_s)\|_{X_1} = \|D_s(\boldsymbol{\zeta}_s)\|_{0,s;\Omega}$, and employing the boundedness of D_s (cf. (1.40)), we deduce that

$$\sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}}}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \big(D_s(\boldsymbol{\zeta}_s)\big)^{\mathtt{d}}}{\|D_s(\boldsymbol{\zeta}_s)\|_{X_1}} \,=\, \frac{\|\boldsymbol{\zeta}^{\mathtt{d}}\|_{0,r;\Omega}\, \|\boldsymbol{\zeta}_s\|_{0,s;\Omega}}{\|D_s(\boldsymbol{\zeta}_s)\|_{0,s;\Omega}} \,\geq\, \frac{1}{\|D_s\|}\, \|\boldsymbol{\zeta}^{\mathtt{d}}\|_{0,r;\Omega}\,. \tag{1.61}$$

In this way, replacing the foregoing estimate back into (1.60), we arrive at

$$\sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \left\{ \frac{1}{\mu\|D_s\|} \,-\, \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda + (n+1)\mu\big)} \right\} \|\boldsymbol{\zeta}^{\mathtt{d}}\|_{0,r;\Omega}\,, \tag{1.62}$$

from which, choosing λ sufficiently large such that

$$\frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda + (n+1)\mu\big)} \,<\, \frac{1}{2\mu\|D_s\|}\,,$$

that is

$$\lambda \,>\, M_s \,:=\, \frac{\mu}{n^{1+1/s}}\, \max\Big\{2\|D_s\|\widetilde{C}_r - n^{1/s}(n+1), 0\Big\}\,,$$

and applying (1.44), we conclude (1.57) with $\alpha := \frac{1}{2\mu\|D_s\|\widetilde{C}_r}$. On the other hand, given now $\boldsymbol{\tau} \in K_1$, $\boldsymbol{\tau} \neq \mathbf{0}$, we proceed analogously as above, but exchanging the roles of $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$, and obtain

$$\sup_{\boldsymbol{\zeta}\in K_2} \mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau}) \,\geq\, \sup_{\substack{\boldsymbol{\zeta}\in K_2\\ \boldsymbol{\zeta}\neq\mathbf{0}}} \frac{\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\zeta}\|_{X_2}} \,\geq\, \frac{1}{2\mu\|D_r\|\widehat{C}_s}\, \|\boldsymbol{\tau}\|_{X_1} \,>\, 0 \tag{1.63}$$

for $\lambda \,>\, M_r \,:=\, \frac{\mu}{n^{1+1/r}}\, \max\Big\{2\|D_r\|\widetilde{C}_s - n^{1/r}(n+1), 0\Big\}$, which proves (1.58). Finally, the proof is completed by choosing $M := \max\big\{M_s, M_r\big\}$. □

We stress here that, constituting the bilinear form $\widetilde{\mathbf{a}}$ a key part of $\mathbf{a}$, some arguments employed in the proof of Lemma 1.7 allow us to establish next the inf-sup conditions required for the former.

Lemma 1.8. *Assume that r and s satisfy the range specified by Theorem 1.1. Then, there exists a positive constant $\widetilde{\alpha}$ such that*

$$\sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\widetilde{\mathbf{a}}(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \alpha\,\|\boldsymbol{\zeta}\|_{X_2} \qquad \forall\,\boldsymbol{\zeta}\in K_2\,. \tag{1.64}$$

In addition, there holds

$$\sup_{\boldsymbol{\zeta}\in K_2} \widetilde{\mathbf{a}}(\boldsymbol{\zeta},\boldsymbol{\tau}) \,>\, 0 \qquad \forall\,\boldsymbol{\tau}\in K_1\,,\ \boldsymbol{\tau}\neq\mathbf{0}\,. \tag{1.65}$$

Proof. It follows straightforwardly from the definition of $\widetilde{\mathbf{a}}$ (cf. (1.28)), and the inequalities (2.66), and (1.44), that for each $\boldsymbol{\zeta}\in K_2$, $\boldsymbol{\zeta}\neq\mathbf{0}$, there holds

$$\sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\widetilde{\mathbf{a}}(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \,=\, \frac{1}{2\mu}\sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}}}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \frac{1}{2\mu\|D_s\|\widehat{C}_r}\,\|\boldsymbol{\zeta}\|_{X_2}\,, \tag{1.66}$$

which yields (1.64) with $\widetilde{\alpha} := \frac{1}{2\mu\|D_s\|\widehat{C}_r}$. In addition, given $\boldsymbol{\tau}\in K_1$, $\boldsymbol{\tau}\neq\mathbf{0}$, and proceeding analogously to the derivation of (1.63), that is exchanging the roles of $\boldsymbol{\zeta}$ and $\boldsymbol{\tau}$ and using (1.66), we easily find that

$$\sup_{\boldsymbol{\zeta}\in K_2} \widetilde{\mathbf{a}}(\boldsymbol{\zeta},\boldsymbol{\tau}) \,\geq\, \sup_{\substack{\boldsymbol{\zeta}\in K_2\\ \boldsymbol{\zeta}\neq\mathbf{0}}} \frac{\widetilde{\mathbf{a}}(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\zeta}\|_{X_2}} \,\geq\, \frac{1}{2\mu\|D_r\|\widehat{C}_s}\,\|\boldsymbol{\tau}\|_{X_1} \,>\, 0\,, \tag{1.67}$$

which shows (1.65) and ends the proof. □

It only remains to verify the inf-sup conditions for the bilinear forms $\mathbf{b}_i$, $i\in\{1,2\}$, which we address in what follows.

Lemma 1.9. *Assume that r and s satisfy the range specified by Theorem 1.5. Then, there exist positive constants β_1, β_2 such that for each $i\in\{1,2\}$ there hold*

$$\sup_{\substack{\boldsymbol{\zeta}\in X_i\\ \boldsymbol{\zeta}\neq\mathbf{0}}} \frac{\mathbf{b}_i(\boldsymbol{\zeta},\mathbf{v})}{\|\boldsymbol{\zeta}\|_{X_i}} \,\geq\, \beta_i\,\|\mathbf{v}\|_{M_i} \qquad \forall\,\mathbf{v}\in M_i\,. \tag{1.68}$$

Proof. Having $\mathbf{b}_1$ and $\mathbf{b}_2$ the same algebraic structure (cf. (1.21)), and being the pairs (X_1,M_1) and (X_2,M_2) one obtained from the other by exchanging r and s, we now proceed to show (1.68) only for $i=2$ since the proof for $i=1$ is completely analogous. In this way, given $\mathbf{v}\in M_2 := \mathbf{L}^s(\Omega)$, we let $\mathcal{J}_s$ be the vector version of $\boldsymbol{\mathcal{J}}_s$ (cf. (1.53)), and set $\mathbf{v}_r := \mathcal{J}_s(\mathbf{v})\in\mathbf{L}^r(\Omega)$, for which, similarly to (1.54) and (1.55), there hold

$$\mathbf{v} \,=\, \mathcal{J}_r(\mathbf{v}_r)\,, \quad \text{and} \quad \int_\Omega \mathbf{v}\cdot\mathbf{v}_r \,=\, \|\mathbf{v}\|_{0,s;\Omega}^s \,=\, \|\mathbf{v}_r\|_{0,r;\Omega}^r \,=\, \|\mathbf{v}\|_{0,s;\Omega}\,\|\mathbf{v}_r\|_{0,r;\Omega}\,. \tag{1.69}$$

Then, we let $\mathbf{z}\in\mathbf{W}_0^{1,r}(\Omega)$ be the unique solution, guaranteed by Theorem 1.5, of the vector Poisson equation (1.47) with $\mathbf{g}=\mathbf{0}$ and $\mathbf{f}=-\mathbf{v}_r$, that is

$$\Delta\mathbf{z} \,=\, \mathbf{v}_r \quad \text{in} \quad \Omega, \qquad \mathbf{z}=0 \quad \text{on} \quad \Gamma\,,$$

whose weak formulation is given by (1.48) and (1.49) with $F(\mathbf{w}) := -\displaystyle\int_\Omega \mathbf{v}_r \cdot \mathbf{w}$ for all $\mathbf{w} \in \mathbf{W}_0^{1,s}(\Omega)$. It follows that $\|F\|_{-1,r} \leq \|\mathbf{v}_r\|_{0,r;\Omega}$, and thus the continuous dependence result (1.50) yields

$$\|\mathbf{z}\|_{1,r;\Omega} \leq \bar{c}_r \, \|\mathbf{v}_r\|_{0,r;\Omega} \,. \tag{1.70}$$

Next, we observe that $\mathbf{div}\big(\nabla \mathbf{z}\big) = \mathbf{v}_r$ in Ω, which proves that $\nabla \mathbf{z} \in \mathbb{H}^r(\mathbf{div}_r;\Omega)$, and let $\widehat{\boldsymbol{\zeta}}$ be the $\mathbb{H}_0^r(\mathbf{div}_r;\Omega)$-component (cf. (1.12)) of $\nabla \mathbf{z}$. In this way, utilizing (1.70) and noting that $\mathbf{div}(\widehat{\boldsymbol{\zeta}}) = \mathbf{v}_r$, we deduce that

$$\|\widehat{\boldsymbol{\zeta}}\|_{X_2} = \|\widehat{\boldsymbol{\zeta}}\|_{0,r;\Omega} + \|\mathbf{div}(\widehat{\boldsymbol{\zeta}})\|_{0,r;\Omega} \leq |\mathbf{z}|_{1,r;\Omega} + \|\mathbf{v}_r\|_{0,r;\Omega} \leq \big(1 + \bar{c}_r\big) \, \|\mathbf{v}_r\|_{0,r;\Omega} \,.$$

Finally, bearing in mind the definition of $\mathbf{b}_2$ (cf. (1.21)), and employing (1.69) and the foregoing inequality, we conclude that

$$\sup_{\substack{\boldsymbol{\zeta}\in X_2 \\ \boldsymbol{\zeta}\neq \mathbf{0}}} \frac{\mathbf{b}_2(\boldsymbol{\zeta},\mathbf{v})}{\|\boldsymbol{\zeta}\|_{X_2}} \geq \frac{\mathbf{b}_2(\widehat{\boldsymbol{\zeta}},\mathbf{v})}{\|\widehat{\boldsymbol{\zeta}}\|_{X_2}} = \frac{\displaystyle\int_\Omega \mathbf{v}\cdot\mathbf{v}_r}{\|\widehat{\boldsymbol{\zeta}}\|_{X_2}} \geq \frac{1}{\big(1+\bar{c}_r\big)} \, \|\mathbf{v}\|_{0,s;\Omega} \,, \tag{1.71}$$

which gives (1.68) for $i = 2$ with $\beta_2 := \big(1 + \bar{c}_r\big)^{-1}$. □

Regarding the assumptions on r and its conjugate s, we remark here that $\big[\frac{2n}{n+1}, \frac{2n}{n-1}\big]$ constitutes the largest subset of $\big(\frac{2n}{n+1} - \delta, \frac{2n}{n-1} + \delta\big)$ guaranteeing that both indexes lie simultaneously within it.

We are now in position to establish below the announced well-posedness of (1.18) and (1.27).

Theorem 1.10. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, which is star shaped with respect to a ball, and let r, $s \in (1,+\infty)$ conjugate to each other such that they satisfy the range specified by Theorem 1.1 (which coincides with that of Theorem 1.5). Then, there exists a positive constant M such that for each $\lambda > M$ and for each pair $(\mathbf{f},\mathbf{u}_D) \in \mathbf{L}^r(\Omega) \times \mathbf{W}^{1/s,r}(\Gamma)$, there exists a unique solution $(\boldsymbol{\sigma},\mathbf{u}) \in X_2 \times M_1 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega) \times \mathrm{L}^r(\Omega)$ to (1.18). Moreover, there exists a positive constant C, independent of the data and the solution, such that*

$$\|\boldsymbol{\sigma}\|_{r,\mathbf{div}_r;\Omega} + \|\mathbf{u}\|_{0,r;\Omega} \leq C \Big\{ \|\mathbf{f}\|_{0r;\Omega} + \|\mathbf{u}_D\|_{1/s,r;\Gamma} \Big\} .$$

Proof. It follows from Lemmas 1.7 and 1.9, along with a straightforward application of Theorem 1.6. □

Theorem 1.11. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, which is star shaped with respect to a ball, and let r, $s \in (1,+\infty)$ conjugate to each other such that they satisfy the range specified by Theorem 1.5 (which coincides with that of Theorem 1.1). Then, for each pair $(\mathbf{f},\mathbf{u}_D) \in \mathbf{L}^r(\Omega) \times \mathbf{W}^{1/s,r}(\Gamma)$, there exists a unique solution $(\boldsymbol{\sigma},\mathbf{u}) \in X_2 \times M_1 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega) \times \mathrm{L}^r(\Omega)$ to (1.27). Moreover, there exists a positive constant C, independent of the data and the solution, such that*

$$\|\boldsymbol{\sigma}\|_{r,\mathbf{div}_r;\Omega} + \|\mathbf{u}\|_{0,r;\Omega} \leq C \Big\{ \|\mathbf{f}\|_{0r;\Omega} + \|\mathbf{u}_D\|_{1/s,r;\Gamma} \Big\} .$$

Proof. It follows from Lemmas 1.8 and 1.9, along with a straightforward application of Theorem 1.6. □

We end the chapter by announcing that the extension of the present analysis to the discrete setting of a Banach spaces-based mixed formulation for the stress-assisted diffusion problem studied in [46] and [47], will be reported in the next chapter. In particular, it will be shown in this case that the feasible ranges for r and its conjugate s are given by the intervals $(2, \frac{2n}{n-1}]$ and $[\frac{2n}{n+1}, 2)$, respectively. Several numerical experiments illustrating the performance of the resulting method will also be included there.

CHAPTER 2

A pseudostress-based mixed-primal finite element method for stress-assisted diffusion problems in Banach spaces

2.1 Introduction

The so-called stress-assisted diffusion models, which refer to diffusion processes in deformable solids, are present in diverse applications, which include, among others, diffusion of boron and arsenic in silicon [69], voiding of aluminum conductor lines in integrated circuits [87], sorption in polymers [77], damage to electrodes in lithium-ion batteries [5], and anisotropy of cardiac dynamics [27]. The usual assumptions in most of these models are, on one hand, that the solid follows an elastic regime, and on the other hand, that the diffusion obeys a Fickean law enriched with further contributions arising from local effects by exerted stresses. Mathematically, this second hypothesis means that the respective diffusion coefficient is a continuous function depending precisely on the stress, which acts then as a coupling variable.

While many contributions to the modelling of stress-assisted (and even strain-assisted) diffusion problems are available in the literature, the same can not be said of the corresponding mathematical and numerical analyses of them, which are rather scarce. Indeed, for the first of the latter issues we can mention the recent works [67], [81], and [42], which deal with a general local-global well-posedness theory for static and transient problems via a primal formulation, homogenization of concentration - electric potential systems, and multiscale analysis of the deterioration of binder in electrodes, respectively. In turn, concerning the second of those issues, and up to our knowledge, we can only refer to [46] and [47], where mixed-primal and fully-mixed finite element methods have been introduced and analysed to numerically solve the stationary problem describing the diffusion of a solute into an elastic material. This diffusion-deformation model is represented by the linear elasticity equations along with a diffusion equation whose function of diffusivity depends on the Cauchy stress of the solid. Further interactions between them are given by the corresponding source terms, which depend on the concentration and the displacement, respectively. In other words, the diffusing species affects the behavior of the solid, whereas the displacement of the latter influences the solute concentration, both through the corresponding external forces, thus yielding a two-way coupled system.

Regarding further details on [46] and [47], we first notice that the approach in [46] follows the usual methodology for the dual-mixed formulation of the linear elasticity problem (cf. [17], [44]), so that the symmetry of the Cauchy stress is imposed weakly through the incorporation of the tensor of solid rotations as the corresponding Lagrange multiplier. In contrast, a primal formulation is employed for the diffusion equation. The well-posedness of the resulting coupled variational formulation is addressed by means of a fixed-point strategy and by applying the Lax-Milgram lemma, the Babuška-Brezzi theory, Sobolev embedding theorems, and suitable regularity estimates. In this way, the Schauder and Banach fixed-point theorems allow establishing existence and uniqueness of continuous solution, respectively. An analogue reasoning is applied to analyse the associated Galerkin scheme and an augmented version of it (for the elasticity equations only), thus deriving existence of discrete solutions, as well as corresponding a priori error estimates and rates of convergence, by employing the Brouwer theorem and a Strang-type lemma.

In turn, while keeping the same dual-mixed scheme for the elasticity equations, an augmented mixed formulation instead of the primal one from [46] is utilized in [47] for the diffusion equation. In addition, similarly to previous works (see, e.g. [51]), the concentration gradient and the diffusive flux are introduced as further unknowns for a more suitable treatment of the nonlinearity arising from the stress-dependent diffusivity. The rest of the continuous and discrete analyses in [47] follows by applying basically the same theoretical tools utilized in [46]. In particular, we highlight that two families of finite element subspaces yielding stable Galerkin schemes are proposed, namely either PEERS or Arnold-Falk-Winther elements for elasticity, and Raviart-Thomas and piecewise polynomials for the mixed formulation of the diffusion equation. We end our discussion on [46] and [47] by pointing out that a significant drawback of their approaches is given by the use of a regularity result for the uncoupled elasticity problem (cf. [46, Theorem 2.4]), which is valid only for convex domains in 2D. In this regard, we remark that the need of this result arises from the handling of the stress-dependent diffusion term when trying to prove a Lipschitz-continuity property of one of the components of the continuous fixed-point operator.

According to the above discussion, and in order to overcome the aforementioned drawback, we have recently realized that the required Lipschitz-continuity property can be established, without any regularity nor convexity assumptions for the linear elasticity problem, by previously restating the whole coupled variational formulation in terms of suitable Lebesgue and Sobolev-type Banach spaces. Moreover, the continuous and discrete analyses can be carried out in this case without employing any augmentation procedure, thus simplifying the computational complexity of the resulting discrete scheme. The purpose of the present work is precisely to introduce and analyse, at the continuous and discrete levels, this new Banach spaces-based formulation for the stress-assisted diffusion problem studied in [46] and [47]. In doing so, we will resort to some results provided in our recent related works [48] and [52]. Moreover, because of greater interest in applications, we consider the nearly incompressible case in linear elasticity, and for sake of further simplicity of its analysis, we adopt a pseudostress-based approach instead of the usual stress-based one.

The rest of the chapter is organized as follows. Required notations and basic definitions are collected at the end of this introductory section. In Section 2.2 we introduce the stress-assisted diffusion model, reformulate the elasticity problem in terms of the non-symmetric pseudostress tensor, and rewrite the diffusivity coefficient in terms of the latter. The continuous formulation is derived in Section 2.3, and its solvability is studied by means of a fixed-point strategy that arises after decoupling the model into the elasticity and diffusion problems. In turn, the well-posedness of each one of the latter is deduced by applying the Babuška-Brezzi theory in Banach spaces and the classical Lax-Milgram theorem, respectively, whereas the unique solvability of the whole coupled model is concluded thanks to the Banach fixed-point theorem. In Section 2.4 we consider arbitrary finite element subspaces, assume that they satisfy suitable stability conditions, and employ the discrete version of the fixed-point strategy introduced in Section 2.3 to analyse the solvability of the associated Galerkin scheme. In this way, and along with the corresponding versions of the theoretical tools employed in Section 2.3, a straightforward application of Brouwer's theorem allows us to conclude the existence of discrete solution. An a priori error estimate in the form of Cea's estimate is also derived here. Next, in Section 2.5 we restrict ourselves to the 2D case and introduce specific finite element subspaces satisfying the theoretical hypotheses that were assumed in Section 2.4. Actually, the latter refer only to a couple of discrete inf-sup conditions for the elasticity equation since any finite element subspace will work for the diffusion model. The lack of a required boundedness property for a particular projector involved stop us from extending the analysis from Section 2.5 to the 3D case. Finally, several numerical results illustrating the performance of the method and confirming the theoretical rates of convergence provided in Section 2.5, are reported in Section 2.6.

2.2 The model problem

The stress-assisted diffusion problem studied in [46] and [47], which models the diffusion of a solute into an elastic material occupying the domain Ω, is described by the following system of partial differential equations:

$$\begin{aligned} \boldsymbol{\rho} \,=\, \mathcal{C}\big(\mathbf{e}(\mathbf{u})\big) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\rho}) \,=\, \mathbf{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \\ \widetilde{\boldsymbol{\sigma}} = \widetilde{\vartheta}(\boldsymbol{\rho})\nabla\phi \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) = g(\mathbf{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\rho}$ is the Cauchy solid stress, $\mathbf{u}$ is the displacement field, $\mathbf{e}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathtt{t}})$ is the infinitesimal strain tensor (symmetrised gradient of displacements), and $\mathcal{C}$ stands for the linear operator defining the Hooke law (cf. [44, eq. (2.36)]), that is

$$\mathcal{C}(\boldsymbol{\tau}) \,:=\, \lambda\,\mathrm{tr}\,(\boldsymbol{\tau})\,\mathbb{I} \,+\, 2\mu\,\boldsymbol{\tau} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{R}\,,$$

so that

$$\boldsymbol{\rho} \,=\, \lambda\,\mathrm{tr}\,\big(\mathbf{e}(\mathbf{u})\big)\,\mathbb{I} \,+\, 2\mu\,\mathbf{e}(\mathbf{u})\,, \tag{2.2}$$

with the Lamé constants λ, $\mu > 0$ (dilation and shear moduli) characterizing the properties of the material. In turn, ϕ represents the local concentration of species, $\widetilde{\boldsymbol{\sigma}}$ is the diffusive flux, and $\widetilde{\vartheta} : \mathbb{R} \to \mathbb{R}$ is a tensorial diffusivity function. Finally, $\mathbf{f} : \mathrm{R} \to \mathbf{R}$ is a vector field of body loads (which depends on the species concentration), $g : \mathbf{R} \to \mathrm{R}$ denotes an additional source term depending on the solid displacement $\mathbf{u}$, and $\mathbf{u}_D$ is the Dirichlet datum for $\mathbf{u}$, which belongs to a suitable trace space to be identified later on. Specific requirements on $\mathbf{f}$ and g will be given below. We note that system (2.1) describes the constitutive relations inherent to linear elastic materials, conservation of linear momentum, the constitutive description of diffusive fluxes, and the mass transport of the diffusive substance, respectively. It also assumes that diffusive time scales are much lower than those of the elastic wave propagation, justifying the static character of the system (cf. [67]).

On the other hand, in this work we are particularly interested in the nearly incompressible case, which reduces to assume from now on that λ is sufficiently large. In addition, in order to avoid the weak imposition of the symmetry of $\boldsymbol{\rho}$, we now reformulate (2.1) in terms of the non-symmetric pseudostress tensor $\boldsymbol{\sigma}$ introduced in [45]. More precisely, according to the analysis provided in [45, Section 2.1], we know that the first row of (2.1) is equivalent to

$$\boldsymbol{\sigma} \,=\, \widehat{\mathcal{C}}\big(\nabla\mathbf{u}\big) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \tag{2.3}$$

where

$$\widehat{\mathcal{C}}(\boldsymbol{\tau}) \,:=\, (\lambda+\mu)\,\mathrm{tr}\,(\boldsymbol{\tau})\mathbb{I} \,+\, \mu\boldsymbol{\tau} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{R}\,,$$

so that

$$\boldsymbol{\sigma} \,=\, (\lambda+\mu)\,\mathrm{tr}\,\big(\nabla\mathbf{u}\big)\mathbb{I} \,+\, \mu\nabla\mathbf{u}\,. \tag{2.4}$$

In this way, applying matrix trace to (2.4), we find that (cf. [45, eq. (2.3)])

$$\mathrm{tr}\,(\nabla\mathbf{u}) \,=\, \frac{1}{n\lambda+(n+1)\mu}\,\mathrm{tr}\,(\boldsymbol{\sigma})\,, \tag{2.5}$$

and hence (2.3) is rewritten, equivalently, as

$$\nabla\mathbf{u} \,=\, \widehat{\mathcal{C}}^{-1}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\sigma}) \,=\, \mathbf{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \mathbf{u} \,=\, \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \tag{2.6}$$

where

$$\widehat{\mathcal{C}}^{-1}(\boldsymbol{\tau}) \,:=\, \frac{1}{\mu}\,\boldsymbol{\tau}^{\mathrm{d}} \,+\, \frac{1}{n\big(n\lambda+(n+1)\mu\big)}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{R}\,. \tag{2.7}$$

In turn, it also follows from (2.4) that

$$\boldsymbol{\sigma}+\boldsymbol{\sigma}^{\mathtt{t}} \,=\, 2(\lambda+\mu)\,\mathrm{tr}\,(\nabla\mathbf{u})\,\mathbb{I} \,+\, 2\mu\,\mathbf{e}(\mathbf{u})\,,$$

which yields

$$2\mu\,\mathbf{e}(\mathbf{u}) \,=\, \boldsymbol{\sigma}+\boldsymbol{\sigma}^{\mathtt{t}} \,-\, 2(\lambda+\mu)\,\mathrm{tr}\,(\nabla\mathbf{u})\,\mathbb{I}\,, \tag{2.8}$$

and thus, noting that $\mathrm{tr}\,\big(\mathbf{e}(\mathbf{u})\big) = \mathrm{tr}\,(\nabla\mathbf{u})$, we deduce from (2.2), along with (2.5) and (2.8), that the original Cauchy stress tensor $\boldsymbol{\rho}$ can be expressed in terms of the pseudostress $\boldsymbol{\sigma}$ through the formula

$$\boldsymbol{\rho} \,=\, \widetilde{\mathcal{C}}(\boldsymbol{\sigma})\,, \tag{2.9}$$

where

$$\widetilde{\mathcal{C}}(\boldsymbol{\tau}) \,:=\, \boldsymbol{\tau}+\boldsymbol{\tau}^{\mathtt{t}} \,-\, \frac{\lambda+2\mu}{n\lambda+(n+1)\mu}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{R}\,. \tag{2.10}$$

Consequently, we can recast the original stress-dependent diffusivity $\widetilde{\vartheta}(\boldsymbol{\rho})$ as a pseudostress-dependent diffusivity

$$\vartheta(\boldsymbol{\sigma}) \,:=\, \widetilde{\vartheta}\big(\widetilde{\mathcal{C}}(\boldsymbol{\sigma})\big)\,. \tag{2.11}$$

In this way, we finally obtain that the model (2.1) can be restated, equivalently, as

$$\begin{gathered}\nabla\boldsymbol{u} \,=\, \widehat{\mathcal{C}}^{-1}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\sigma}) \,=\, \mathbf{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,,\\ \widetilde{\boldsymbol{\sigma}} \,=\, \vartheta(\boldsymbol{\sigma})\,\nabla\phi \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \,=\, g(\mathbf{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,.\end{gathered} \tag{2.12}$$

Throughout this work, we suppose that $\widetilde{\vartheta}$ is of class C^1 and uniformly positive definite, meaning the latter that there exists $\vartheta_0 > 0$ such that

$$\widetilde{\vartheta}(\boldsymbol{\tau})\mathbf{w}\cdot\mathbf{w} \,\geq\, \vartheta_0\,|\mathbf{w}|^2 \quad \forall\,\mathbf{w}\in\mathbf{R}\,, \quad \forall\,\boldsymbol{\tau}\in\mathbb{R}\,. \tag{2.13}$$

We also require uniform boundedness and Lipschitz continuity of $\widetilde{\vartheta}$, that is that there exist positive constants ϑ_1, ϑ_2 and $L_{\widetilde{\vartheta}}$, such that

$$\vartheta_1 \,\leq\, |\widetilde{\vartheta}(\boldsymbol{\tau})| \,\leq\, \vartheta_2 \quad \text{and} \quad |\widetilde{\vartheta}(\boldsymbol{\tau}) - \widetilde{\vartheta}(\boldsymbol{\zeta})| \,\leq\, L_{\widetilde{\vartheta}}\,|\boldsymbol{\tau}-\boldsymbol{\zeta}| \quad \forall\,\boldsymbol{\tau},\boldsymbol{\zeta}\in\mathbb{R}\,. \tag{2.14}$$

Then, it is easily seen that (2.13) and the uniform boundedness of $\widetilde{\vartheta}$ are transferred to ϑ (cf. (2.11)) with the same constants ϑ_0, ϑ_1, and ϑ_2, whereas, being $\widetilde{\mathcal{C}}$ clearly of class C^1 (cf. (2.10)) and noticing that $|\widetilde{\mathcal{C}}(\boldsymbol{\tau}) - \widetilde{\mathcal{C}}(\boldsymbol{\zeta})| \,\leq\, (2+n)\,|\boldsymbol{\tau}-\boldsymbol{\zeta}|$ for all $\boldsymbol{\tau}$, $\boldsymbol{\zeta}\in\mathbb{R}$, the same conclusion is valid for the smoothness and Lipschitz-continuity of ϑ, the latter with constant $L_\vartheta = (2+n)L_{\widetilde{\theta}}$. We summarize the above as follows

$$\vartheta(\boldsymbol{\tau})\mathbf{w}\cdot\mathbf{w} \,\geq\, \vartheta_0\,|\mathbf{w}|^2 \quad \forall\,\mathbf{w}\in\mathbf{R}\,, \quad \forall\,\boldsymbol{\tau}\in\mathbb{R}\,, \tag{2.15}$$

$$\vartheta_1 \,\leq\, |\vartheta(\boldsymbol{\tau})| \,\leq\, \vartheta_2 \quad \text{and} \quad |\vartheta(\boldsymbol{\tau}) - \vartheta(\boldsymbol{\zeta})| \,\leq\, L_\vartheta\,|\boldsymbol{\tau}-\boldsymbol{\zeta}| \quad \forall\,\boldsymbol{\tau},\boldsymbol{\zeta}\in\mathbb{R}\,. \tag{2.16}$$

Similar hypotheses to those of $\widetilde{\vartheta}$, and hence of ϑ, are assumed on the source functions $\mathbf{f}$ and g, which means that there exist positive constants f_1, f_2, L_f, g_1, g_2 and L_g, such that

$$f_1 \,\leq\, |\mathbf{f}(s)| \,\leq\, f_2\,, \quad |\mathbf{f}(s)-\mathbf{f}(t)| \,\leq\, L_f\,|s-t| \quad \forall\,s,t\in\mathrm{R}\,, \tag{2.17}$$

$$g_1 \,\leq\, |g(\mathbf{w})| \,\leq\, g_2\,, \quad \text{and} \quad |g(\mathbf{v})-g(\mathbf{w})| \,\leq\, L_g\,|\mathbf{v}-\mathbf{w}| \quad \forall\,\mathbf{v},\mathbf{w}\in\mathbf{R}\,. \tag{2.18}$$

We find it important to remark in advance that the analysis to be developed in what follows could be extended, under suitable minor modifications, to nonlinear elasticity, say for instance to the hyperelastic model arising from the Hooke law when the Lame constants are actually nonlinear coefficients depending on $|\mathbf{e}(\mathbf{u})^{\mathtt{d}}|$ (cf. [9, eq. (2.6)]). In this case, however, we would need to stay with the original stress dependent diffusivity $\widetilde{\vartheta}(\boldsymbol{\rho})$ since the aforementioned nonlinearity would not allow us to employ a pseudostress. In addition, in order to derive the corresponding mixed formulation, the strain rate tensor and the rotation would need to be introduced as auxiliary unknowns as well. In turn, keeping the elasticity model as in (2.1), our present approach could also be applied to handle Neumann boundary conditions, but again using only the Cauchy stress since it is with this tensor that this condition makes sense. Thus, the same observation is valid for mixed-boundary conditions. Summarizing, the present use of the pseudostress $\boldsymbol{\sigma}$ and the consequent advantage of avoiding to impose the weak symmetry of the Cauchy stress tensor, is possible thanks to the linear character of the Hooke law and the fact that we are considering Dirichlet boundary conditions.

On the other hand, irrespective of whether one uses the stress $\boldsymbol{\rho}$ or the pseudostress $\boldsymbol{\sigma}$, we emphasize here that the major benefit of employing a mixed formulation instead of a displacement-based primal scheme, lies on the fact that the former provides direct discrete approximations of those tensors, and hence of the corresponding diffusivity functions. The latter approach, on the contrary, would need to apply numerical differentiation, with the consequent loss of accuracy that this procedure implies, in order to obtain not as good discrete approximations of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$, and thus of $\widetilde{\vartheta}(\boldsymbol{\rho})$ and $\vartheta(\boldsymbol{\sigma})$. In other words, the fact that one is dealing with a stress-assisted diffusion problem, and not with a displacement-assisted one, is determinant for the present choice of the mixed method.

2.3 The continuous formulation

In this section we introduce a suitable Banach spaces-based variational formulation for (2.12), and then analyse its solvability by means of a fixed-point strategy.

2.3.1 The mixed-primal formulation

We begin by noticing, as suggested by the Dirichlet boundary condition satisfied by the concentration ϕ, that the appropriate trial and test space reduces in this case to

$$\mathrm{H}_0^1(\Omega) = \Big\{ \psi \in \mathrm{H}^1(\Omega) : \quad \psi = 0 \quad \text{on} \quad \Gamma \Big\} .$$

Thus, performing the usual integration by parts procedure in $\mathrm{H}^1(\Omega)$, the primal formulation for the diffusion equation becomes: find $\phi \in \mathrm{H}_0^1(\Omega)$ such that

$$A_{\boldsymbol{\sigma}}(\phi, \psi) \;=\; G_{\mathbf{u}}(\psi) \qquad \forall\, \psi \in \mathrm{H}_0^1(\Omega) , \tag{2.19}$$

where, given $\boldsymbol{\zeta}$ and $\mathbf{w}$ lying, respectively, in the same spaces where $\boldsymbol{\sigma}$ and $\mathbf{u}$ will be sought,

$$A_{\boldsymbol{\zeta}}(\phi, \psi) \;:=\; \int_\Omega \vartheta(\boldsymbol{\zeta}) \nabla\phi \cdot \nabla\psi \qquad \forall\, \phi,\, \psi \in \mathrm{H}_0^1(\Omega) , \tag{2.20}$$

and

$$G_{\mathbf{w}}(\psi) \;:=\; \int_\Omega g(\mathbf{w})\, \psi \qquad \forall\, \psi \in \mathrm{H}_0^1(\Omega) . \tag{2.21}$$

Next, before proceeding with the elasticity equations, we remark that in order to study the continuity property of the diffusivity function ϑ within the definition of the bilinear form A (cf. (2.20)), which will be required for the solvability analysis of the fixed-point operator equation to be proposed afterwards, we need to be able to control the expression

$$\int_\Omega (\vartheta(\boldsymbol{\tau}) - \vartheta(\boldsymbol{\zeta}))\, \nabla\phi \cdot \nabla\psi , \tag{2.22}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$ are generic tensors belonging to the same space in which we will seek the unknown $\boldsymbol{\sigma}$. In this regard, and employing the Lipschitz-continuity property of ϑ (cf. (2.16)), straightforward applications of the Cauchy-Schwarz and Hölder inequalities yield

$$\left| \int_\Omega (\vartheta(\boldsymbol{\tau}) - \vartheta(\boldsymbol{\zeta}))\, \nabla\phi \cdot \nabla\psi \right| \;\le\; L_\vartheta\, \|\boldsymbol{\tau} - \boldsymbol{\zeta}\|_{0,2p;\Omega}\, \|\nabla\phi\|_{0,2q;\Omega}\, \|\nabla\psi\|_{0,\Omega} , \tag{2.23}$$

where $p,\, q \in (1, +\infty)$ are conjugate to each other, which makes sense for $\boldsymbol{\tau},\, \boldsymbol{\zeta} \in \mathbb{L}^{2p}(\Omega)$ and $\nabla\psi \in \mathbf{L}^{2q}(\Omega)$. In this way, the above leads us to initially look for $\boldsymbol{\sigma}$ in the space $\mathbb{L}^r(\Omega)$, with $r := 2p$. The specific choice of r will be discussed later on, so that meanwhile we consider a generic r and let $s \in (1, +\infty)$ be its respective conjugate. In turn, a suitable bounding of the expression $\|\nabla\phi\|_{0,2q;\Omega}$ in (2.23) for a particular ϕ will also be explained subsequently by means of a regularity argument.

Having set the above preliminary choice for the space to which $\boldsymbol{\sigma}$ belongs, it follows now from (2.7) and the first equation of (2.12) that $\mathbf{u}$ should be initially sought in $\mathbf{W}^{1,r}(\Omega)$. Thus, in order to derive the variational formulation of the elasticity equations, we need to invoke a suitable integration by parts formula. Indeed, we first introduce for each $t \in (1,+\infty)$ the Banach space

$$\mathbb{H}^t(\mathbf{div}_t;\Omega) \,:=\, \Big\{\boldsymbol{\tau} \in \mathbb{L}^t(\Omega): \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\Big\}, \tag{2.24}$$

which is endowed with the natural norm defined as

$$\|\boldsymbol{\tau}\|_{t,\mathbf{div}_t;\Omega} \,:=\, \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega)\,. \tag{2.25}$$

Then, given t, $t' \in (1,+\infty)$ conjugate to each other, there holds (cf. [41, Corollary B. 57])

$$\langle \boldsymbol{\tau\nu}, \mathbf{v}\rangle_\Gamma \,=\, \int_\Omega \Big\{\boldsymbol{\tau} : \nabla\mathbf{v} \,+\, \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbb{H}^t\,(\mathbf{div}_t;\Omega)\times\mathbf{W}^{1,t'}(\Omega)\,, \tag{2.26}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $\mathbf{W}^{-1/t,t}(\Gamma)$ and $\mathbf{W}^{1/t,t'}(\Gamma)$. Moreover, thanks to the surjectivity of the trace operator $\gamma_{0,t'} : \mathbf{W}^{1,t'}(\Omega) \longrightarrow \mathbf{W}^{1/t,t'}(\Gamma)$, a straightforward application of the open mapping theorem and (2.26) yield the existence of a constant $C_{t'} > 0$ such that

$$\|\boldsymbol{\tau\nu}\|_{-1/t,t;\Gamma} \,\le\, C_{t'}\,\|\boldsymbol{\tau}\|_{t,\mathbf{div}_t;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}^t\,(\mathbf{div}_t;\Omega)\;. \tag{2.27}$$

Now, applying (2.26) with $t = s$ and $t' = r$ to $\mathbf{u} \in \mathbf{W}^{1,r}(\Omega)$ and $\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div}_s;\Omega)$, and using the Dirichlet boundary condition satisfied by $\mathbf{u}$, for which we assume from now on that $\mathbf{u}_D \in \mathbf{W}^{1/s,r}(\Gamma)$, we find that

$$\int_\Omega \boldsymbol{\tau} : \nabla\mathbf{u} \,=\, -\int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, \langle\boldsymbol{\tau\nu},\mathbf{u}_D\rangle_\Gamma\,, \tag{2.28}$$

so that, according to (2.7), the testing of the first equation of (2.12) against $\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div}_s;\Omega)$ gives

$$\frac{1}{\mu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{n\big(n\lambda+(n+1)\mu\big)}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma})\,\mathrm{tr}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = \langle\boldsymbol{\tau\nu},\boldsymbol{u}_D\rangle_\Gamma\;. \tag{2.29}$$

It follows from the third term on the left-hand side of (2.29) that actually it suffices to look for $\mathbf{u}$ in $\mathbf{L}^r(\Omega)$. Furthermore, testing the second equation of (2.12), also named equilibrium equation, against $\mathbf{v} \in \mathbf{L}^s(\Omega)$, we obtain

$$\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \,=\, -\int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v}\,, \tag{2.30}$$

which makes sense for $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^r(\Omega)$, and hence $\boldsymbol{\sigma}$ is sought from now in $\mathbb{H}^r(\mathbf{div}_r;\Omega)$. To be more precise about the latter, we notice that for each $t \in (1,+\infty)$ there holds the decomposition

$$\mathbb{H}^t(\mathbf{div}_t;\Omega) \,=\, \mathbb{H}^t_0(\mathbf{div}_t;\Omega) \,\oplus\, \mathrm{R}\mathbb{I}\,,$$

where

$$\mathbb{H}^t_0(\mathbf{div}_t;\Omega) \,:=\, \Big\{\boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega): \quad \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}) = 0\Big\}.$$

Equivalently, each $\boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega)$ can be decomposed, uniquely, as

$$\boldsymbol{\tau} \,=\, \boldsymbol{\tau}_0 + d\,\mathbb{I}, \quad \text{with} \quad \boldsymbol{\tau}_0 \in \mathbb{H}^t_0(\mathbf{div}_t;\Omega) \quad \text{and} \quad d := \frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \in \mathrm{R}\,. \tag{2.31}$$

In this way, taking $\boldsymbol{\tau} = \mathbb{I}$ in (2.29) we get

$$\frac{1}{n\lambda+(n+1)\mu}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \,=\, \int_\Gamma \mathbf{u}_D\cdot\nu\,,$$

from which, along with an application of (2.31) to $t = r$ and $\boldsymbol{\tau} = \boldsymbol{\sigma} \in \mathbb{H}^r(\mathbf{div}_r;\Omega)$, we deduce that

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c\,\mathbb{I}\,, \quad \text{with} \quad \boldsymbol{\sigma}_0 \in \mathbb{H}^r_0(\mathbf{div}_r;\Omega) \quad \text{and} \quad c := \frac{n\lambda + (n+1)\mu}{n|\Omega|}\int_\Gamma \mathbf{u}_D\cdot\boldsymbol{\nu} \in \mathrm{R}\,. \tag{2.32}$$

The above shows that, in order to attain the full explicit knowledge of the unknown $\boldsymbol{\sigma}$, it only remains to find its $\mathbb{H}^r_0(\mathbf{div}_r;\Omega)$-component $\boldsymbol{\sigma}_0$. Therefore, replacing $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c\,\mathbb{I}$ back into (2.29), redenoting $\boldsymbol{\sigma}_0$ simply by $\boldsymbol{\sigma}$, replacing $\vartheta(\boldsymbol{\sigma})$ by $\vartheta(\boldsymbol{\sigma} + c\mathbb{I})$ in the diffusion equation, noting that the testing of the resulting (2.29) against $\boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div}_s;\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}^s_0(\mathbf{div}_s;\Omega)$, and placing this new equation jointly with (2.30), we arrive at the following mixed variational formulation of the first row of (2.12): Find $(\boldsymbol{\sigma},\mathbf{u}) \in X_2 \times M_1$ such that

$$\begin{aligned} a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b_1(\boldsymbol{\tau},\mathbf{u}) &= G(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in X_1, \\ b_2(\boldsymbol{\sigma},\mathbf{v}) &= F_\phi(\mathbf{v}) \qquad \forall\,\mathbf{v} \in M_2, \end{aligned} \tag{2.33}$$

where

$$X_2 := \mathbb{H}^r_0(\mathbf{div}_r;\Omega)\,, \quad M_1 := \mathbf{L}^r(\Omega)\,, \quad X_1 := \mathbb{H}^s_0(\mathbf{div}_s;\Omega) \quad \text{and} \quad M_2 := \mathbf{L}^s(\Omega)\,, \tag{2.34}$$

and the bilinear forms $a : X_2 \times X_1 \to \mathrm{R}$ and $b_i : X_i \times M_i \to \mathrm{R}$, $i \in \{1,2\}$, and the functionals $F_\phi \in M_2'$ and $G \in X_1'$, are defined, respectively, as

$$a(\boldsymbol{\zeta},\boldsymbol{\tau}) := \frac{1}{\mu}\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{n\big(n\lambda + (n+1)\mu\big)}\int_\Omega \mathrm{tr}\,(\boldsymbol{\zeta})\,\mathrm{tr}\,(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau}) \in X_2 \times X_1\,, \tag{2.35}$$

$$b_i(\boldsymbol{\tau},\mathbf{v}) := \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau},\mathbf{v}) \in X_i \times M_i\,, \tag{2.36}$$

$$G(\boldsymbol{\tau}) := \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u}_D\rangle_\Gamma, \qquad \forall\,\boldsymbol{\tau} \in X_1, \tag{2.37}$$

$$F_\phi(\mathbf{v}) := -\int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v} \qquad \forall\,\mathbf{v} \in M_2. \tag{2.38}$$

In this way, the mixed-primal formulation of (2.12) reduces to (2.33) and (2.19), that is: Find $(\boldsymbol{\sigma},\mathbf{u},\phi) \in X_2 \times M_1 \times \mathrm{H}^1_0(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b_1(\boldsymbol{\tau},\mathbf{u}) &= G(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in X_1\,, \\ b_2(\boldsymbol{\sigma},\mathbf{v}) &= F_\phi(\mathbf{v}) \qquad \forall\,\mathbf{v} \in M_2\,, \\ A_{\boldsymbol{\sigma}}(\phi,\psi) &= G_{\mathbf{u}}(\psi) \qquad \forall\,\psi \in \mathrm{H}^1_0(\Omega)\,. \end{aligned} \tag{2.39}$$

2.3.2 Fixed-point approach

In this section we follow a similar approach to those employed in previous works, e.g. in [8], [30], [46], and [52], and make use of the decoupled variational formulations (2.33) and (2.19) to introduce a fixed-point strategy for the solvability analysis of (2.39). Indeed, we first let $\mathbf{S} : \mathrm{H}^1_0(\Omega) \to X_2 \times M_1$ be the operator defined for each $\varphi \in \mathrm{H}^1_0(\Omega)$ as $\mathbf{S}(\varphi) := (\widetilde{\boldsymbol{\sigma}},\widetilde{\mathbf{u}})$, where $(\widetilde{\boldsymbol{\sigma}},\widetilde{\mathbf{u}}) \in X_2 \times M_1$ is the unique solution (to be confirmed below) of (2.33) with φ instead of ϕ, that is

$$\begin{aligned} a(\widetilde{\boldsymbol{\sigma}},\boldsymbol{\tau}) + b_1(\boldsymbol{\tau},\widetilde{\mathbf{u}}) &= G(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in X_1, \\ b_2(\widetilde{\boldsymbol{\sigma}},\mathbf{v}) &= F_\varphi(\mathbf{v}) \qquad \forall\,\mathbf{v} \in M_2\,. \end{aligned} \tag{2.40}$$

In turn, we let $\widetilde{\mathrm{S}} : X_2 \times M_1 \to \mathrm{H}_0^1(\Omega)$ be the operator defined for each $(\boldsymbol{\zeta}, \mathbf{w}) \in X_2 \times M_1$ as $\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \mathbf{w}) := \widetilde{\phi}$, where $\widetilde{\phi} \in \mathrm{H}_0^1(\Omega)$ is the unique solution (to be confirmed below as well) of (2.19) with $(\boldsymbol{\zeta}, \mathbf{w})$ instead $(\boldsymbol{\sigma}, \mathbf{u})$, that is

$$A_{\boldsymbol{\zeta}}(\widetilde{\phi}, \psi) \,=\, G_{\mathbf{w}}(\psi) \qquad \forall\, \psi \in \mathrm{H}_0^1(\Omega)\,. \tag{2.41}$$

Thus, we define the operator $T : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ as

$$T(\varphi) \,:=\widetilde{\mathrm{S}}\big(\mathbf{S}(\varphi)\big) \qquad \forall\, \varphi \in\, \mathrm{H}_0^1(\Omega)\,, \tag{2.42}$$

and notice that solving (2.39) is equivalent to seeking a fixed point of T, that is $\phi \in \mathrm{H}_0^1(\Omega)$ such that

$$T(\phi) \,=\, \phi\,. \tag{2.43}$$

2.3.3 Well-posedness of the uncoupled problems

Some preliminary results

We begin with the Babuška-Brezzi theorem in Banach spaces.

Theorem 2.1. *Let H_1, H_2, Q_1 and Q_2 be real reflexive Banach spaces, and let $a : H_2 \times H_1 \to \mathrm{R}$ and $b_i : H_i \times Q_i \to \mathrm{R}$, $i \in \{1,2\}$, be bounded bilinear forms with boundedness constants given by $\|a\|$ and $\|b_i\|$, $i \in \{1,2\}$, respectively. In addition, for each $i \in \{1,2\}$, let K_i be the kernel of the operator induced by b_i, that is*

$$K_i \,:=\, \Big\{ \tau \in H_i : \quad b_i(\tau, v) = 0 \quad \forall\, v \in Q_i \Big\}\,.$$

Assume that

i) *there exists $\alpha > 0$ such that*

$$\sup_{\substack{\tau \in K_1 \\ \tau \neq 0}} \frac{a(\zeta, \tau)}{\|\tau\|_{H_1}} \,\geq\, \alpha\, \|\zeta\|_{H_2} \qquad \forall\, \zeta \in K_2\,,$$

ii) *there holds*

$$\sup_{\zeta \in K_2} a(\zeta, \tau) \,>\, 0 \qquad \forall\, \tau \in K_1\,,\ \tau \neq 0\,,$$

iii) *for each $i \in \{1,2\}$ there exists $\beta_i > 0$ such that*

$$\sup_{\substack{\tau \in H_i \\ \tau \neq 0}} \frac{b_i(\tau, v)}{\|\tau\|_{H_i}} \,\geq\, \beta_i\, \|v\|_{Q_i} \qquad \forall\, v \in Q_i\,.$$

Then, for each $(F, G) \in H_1' \times Q_2'$ there exists a unique $(\sigma, u) \in H_2 \times Q_1$ such that

$$\begin{aligned} a(\sigma, \tau) + b_1(\tau, u) &= F(\tau) \qquad \forall\, \tau \in H_1\,, \\ b_2(\sigma, v) &= G(v) \qquad \forall\, v \in Q_2\,, \end{aligned} \tag{2.44}$$

and the following a priori estimates hold:

$$\begin{aligned} \|\sigma\|_{H_2} &\leq \frac{1}{\alpha}\,\|F\|_{H_1'} + \frac{1}{\beta_2}\left(1+\frac{\|a\|}{\alpha}\right)\|G\|_{Q_2'}\,, \\ \|u\|_{Q_1} &\leq \frac{1}{\beta_1}\left(1+\frac{\|a\|}{\alpha}\right)\|F\|_{H_1'} + \frac{\|a\|}{\beta_1\beta_2}\left(1+\frac{\|a\|}{\alpha}\right)\|G\|_{Q_2'}\,. \end{aligned} \tag{2.45}$$

Moreover, i), ii), *and* iii) *are also necessary conditions for the well-posedness of* (2.44).

Proof. See [12, Theorem 2.1, Corollary 2.1, Section 2.1] for details. □

The results provided by the following two lemmas, which are originally stated and proved in Lemmas 1.2 and 1.4, will serve to establish the well-posedness of (2.33) for a given ϕ (equivalently the well-definedness of the operator $\mathbf{S}$).

The first lemma introduces a suitable linear operator mapping $\mathbb{L}^t(\Omega)$ into itself for a range of t.

Lemma 2.2. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let t, $t' \in (1,+\infty)$ conjugate to each other with t satisfying the range specified by* Theorem 1.1*. Then, there exists a linear and bounded operator $D_t : \mathbb{L}^t(\Omega) \to \mathbb{L}^t(\Omega)$ such that*

$$\mathbf{div}\big(D_t(\boldsymbol{\tau})\big) \,=\, \mathbf{0} \quad in \quad \Omega\,, \tag{2.46}$$

and

$$\int_\Omega \mathrm{tr}\,\big(D_t(\boldsymbol{\tau})\big) \,=\, \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau})\,, \tag{2.47}$$

for all $\boldsymbol{\tau} \in \mathbb{L}^t(\Omega)$. In addition, for each $\boldsymbol{\zeta} \in \mathbb{L}^{t'}(\Omega)$ such that $\mathbf{div}(\boldsymbol{\zeta}) = \mathbf{0}$ in Ω, there holds

$$\int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \big(D_t(\boldsymbol{\tau})\big)^{\mathtt{d}} \,=\, \int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{L}^t(\Omega)\,. \tag{2.48}$$

For later use, we remark in advance here that a particular case in which both t and t' satisfy the range specified by Theorem 1.1 is when they lie in $[\frac{2n}{n+1}, \frac{2n}{n-1}]$, $n \in \{2,3\}$. More precisely, it is easy to see that t belongs to this closed interval if and only if t' does as well.

The second lemma announced previously generalizes from $t = 2$ to any $t \in (1,+\infty)$ the inequality stated in [17, Chapter IV, Proposition 3.1] (see also [44, Lemma 2.3]), which is employed for the solvability analysis of the Hilbertian dual-mixed formulation of linear elasticity.

Lemma 2.3. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, which is star-shaped with respect to a ball, and let $t \in (1,+\infty)$. Then, there exist positive constants $\widetilde{C}_{\mathtt{t}}$ and $\widehat{C}_{\mathtt{t}}$ such that*

$$\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,t;\Omega} \,\leq\, \widetilde{C}_t \left\{ \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,t;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \right\} \tag{2.49}$$

and

$$\|\boldsymbol{\tau}\|_{0,t;\Omega} \,\leq\, \widehat{C}_t \left\{ \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,t;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \right\} \tag{2.50}$$

for all $\boldsymbol{\tau} \in \mathbb{H}_0^t(\mathbf{div}_t;\Omega)$.

We remark here that the proof of Lemma 2.3 (cf. Lemma 1.4) makes use of the surjectivity of the operator $\mathrm{div} : \mathbf{W}^{1,t}(\Omega) \to \mathrm{L}_0^t(\Omega)$ (cf. [41, Lemma B.69]), which, in turn, requires that Ω be star-shaped with respect to a ball. This fact explains the necessity of this geometric hypothesis.

Well-definedness of the operator S

In what follows we employ some of the preliminary results provided in Section 2.3.3, along with Theorem 2.1, to prove that the operator $\mathbf{S}$ (cf. (2.40)) is well-defined. We begin by checking that the bilinear forms and linear functionals involved are all bounded. Indeed, we first observe from (2.35) that a can be rewritten as

$$a(\boldsymbol{\zeta},\boldsymbol{\tau}) \,=\, \frac{1}{\mu}\int_\Omega \boldsymbol{\zeta}:\boldsymbol{\tau} - \frac{\lambda+\mu}{\mu(n\lambda+(n+1)\mu)}\int_\Omega \mathrm{tr}\,(\boldsymbol{\zeta})\,\mathrm{tr}\,(\boldsymbol{\tau})\,,$$

from which, noting that $\frac{\lambda+\mu}{n\lambda+(n+1)\mu} < \frac{1}{n}$, and employing, thanks to the triangle and Hölder inequalities, that for each $t \in (1,+\infty)$ there holds

$$\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,t;\Omega} \,\leq\, n^{1/t'}\|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{L}^t(\Omega)\,, \tag{2.51}$$

we find, using again Hölder's inequality, that

$$\begin{aligned} |a(\boldsymbol{\zeta},\boldsymbol{\tau})| \,&\leq\, \frac{1}{\mu}\|\boldsymbol{\zeta}\|_{0,r;\Omega}\,\|\boldsymbol{\tau}\|_{0,s;\Omega} + \frac{1}{n\mu}\|\mathrm{tr}\,(\boldsymbol{\zeta})\|_{0,r;\Omega}\,\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,s;\Omega} \\ &\leq\, \frac{2}{\mu}\|\boldsymbol{\zeta}\|_{0,r;\Omega}\,\|\boldsymbol{\tau}\|_{0,s;\Omega} \,\leq\, \frac{2}{\mu}\|\boldsymbol{\zeta}\|_{X_2}\,\|\boldsymbol{\tau}\|_{X_1} \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau})\in X_2\times X_1\,. \end{aligned} \tag{2.52}$$

In turn, invoking once more the aforementioned inequality, it follows from (2.36) that

$$|b_1(\boldsymbol{\tau},\mathbf{v})| \,\leq\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,s;\Omega}\,\|\mathbf{v}\|_{0,r;\Omega} \,\leq\, \|\boldsymbol{\tau}\|_{\mathrm{div}_s,s;\Omega}\,\|\mathbf{v}\|_{0,r;\Omega} \quad \forall\,(\boldsymbol{\tau},\mathbf{v})\in X_1\times M_2\,, \tag{2.53}$$

and similarly

$$|b_2(\boldsymbol{\tau},\mathbf{v})| \,\leq\, \|\boldsymbol{\tau}\|_{\mathrm{div}_r,r;\Omega}\,\|\mathbf{v}\|_{0,s;\Omega} \quad \forall\,(\boldsymbol{\tau},\mathbf{v})\in X_2\times M_1\,. \tag{2.54}$$

In addition, bearing in mind the upper bound for $\mathbf{f}$ (cf. (2.17)) and the estimate (2.27), we deduce from (2.37) and (2.38), respectively, that

$$|G(\boldsymbol{\tau})| \,\leq\, C_r\,\|\mathbf{u}_D\|_{1/s,r;\Gamma}\,\|\boldsymbol{\tau}\|_{X_1} \qquad \forall\,\boldsymbol{\tau}\in X_1\,, \tag{2.55}$$

and, for each $\phi\in\mathrm{H}^1_0(\Omega)$,

$$|F_\phi(\mathbf{v})| \,\leq\, |\Omega|^{1/r}\,f_2\,\|\mathbf{v}\|_{0,s;\Omega} \qquad \forall\,\mathbf{v}\in M_2\,. \tag{2.56}$$

In this way, and as a straightforward consequence of (2.52) - (2.56), we conclude that a, b_1, b_2, G and F_ϕ are all bounded with respective constants satisfying

$$\|a\| \leq \frac{2}{\mu}\,, \quad \|b_1\|\,,\ \|b_2\| \leq 1\,, \quad \|G\| \leq C_r\,\|\mathbf{u}_D\|_{1/s,r;\Gamma}\,, \quad \text{and} \quad \|F_\phi\| \leq |\Omega|^{1/r}\,f_2\,. \tag{2.57}$$

Next, we let $\mathcal{K}_i$, $i\in\{1,2\}$, be the kernel of the bilinear form b_i, $i\in\{1,2\}$ (cf. (2.36)), that is

$$\mathcal{K}_i \,:=\, \Big\{\boldsymbol{\tau}\in X_i: \quad b_i(\boldsymbol{\tau},\mathbf{v}) = 0 \quad \forall\,\mathbf{v}\in M_i\Big\}\,,$$

which, according to the definitions of X_1, X_2 and b_i (cf. (2.36)), yields

$$\mathcal{K}_1 \,=\, \Big\{\boldsymbol{\tau}\in\mathbb{H}^s_0(\mathbf{div}_s;\Omega): \quad \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0}\Big\} \tag{2.58}$$

and

$$\mathcal{K}_2 \,=\, \Big\{ \boldsymbol{\zeta} \in \mathbb{H}_0^r(\mathbf{div}_r;\Omega) : \quad \mathbf{div}(\boldsymbol{\zeta}) = 0 \Big\} . \tag{2.59}$$

The continuous inf-sup conditions required for the bilinear forms a (cf. (2.35)) and b_i (cf. (2.36)), $i \in \{1,2\}$, are established next. While these results were already stated and proved in Lemmas 1.7 and 1.9 by following similar approaches to those employed in [52, Lemmas 2.6 and 2.7], we provide them again here for sake of completeness of our presentation.

Lemma 2.4. *Assume that r and s satisfy the particular range specified by* Theorem 1.1*, that is r, $s \in [\frac{2n}{n+1}, \frac{2n}{n-1}]$. Then, there exist positive constants M and α such that for each $\lambda > M$ there hold*

$$\sup_{\substack{\boldsymbol{\tau}\in\mathcal{K}_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \alpha\,\|\boldsymbol{\zeta}\|_{X_2} \qquad \forall\,\boldsymbol{\zeta}\in\mathcal{K}_2\,, \tag{2.60}$$

and

$$\sup_{\boldsymbol{\zeta}\in\mathcal{K}_2} a(\boldsymbol{\zeta},\boldsymbol{\tau}) \,>\, 0 \qquad \forall\,\boldsymbol{\tau}\in\mathcal{K}_1\,,\ \boldsymbol{\tau}\neq\mathbf{0}\,. \tag{2.61}$$

Proof. We begin by noticing, thanks to Hölder's inequality and (2.51), that for each pair $(\boldsymbol{\zeta},\boldsymbol{\tau}) \in X_2 \times X_1 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega) \times \mathbb{H}_0^s(\mathbf{div}_s;\Omega)$ there holds

$$\left| \int_\Omega \mathrm{tr}\,(\boldsymbol{\zeta})\,\mathrm{tr}\,(\boldsymbol{\tau}) \right| \,\leq\, n^{1/r}\,\|\mathrm{tr}\,(\boldsymbol{\zeta})\|_{0,r;\Omega}\,\|\boldsymbol{\tau}\|_{0,s;\Omega}\,. \tag{2.62}$$

Now, we consider $\boldsymbol{\zeta} \in \mathcal{K}_2$, that is $\boldsymbol{\zeta} \in X_2 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega)$ and $\mathbf{div}(\boldsymbol{\zeta}) = \mathbf{0}$, such that $\boldsymbol{\zeta} \neq \mathbf{0}$. Then, according to the definition of a (cf. (2.35)) and the estimates (2.62) and (2.49) (cf. Lemma 2.2), we obtain

$$\sup_{\substack{\boldsymbol{\tau}\in\mathcal{K}_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \frac{1}{\mu} \sup_{\substack{\boldsymbol{\tau}\in\mathcal{K}_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}}{\|\boldsymbol{\tau}\|_{X_1}} \,-\, \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda + (n+1)\mu\big)}\,\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,. \tag{2.63}$$

Next, in order to derive a lower bound for the supremum on the right hand side of (2.63), we let

$$\boldsymbol{\zeta}_s \,:=\, \begin{cases} |\boldsymbol{\zeta}^{\mathrm{d}}|^{r-2}\,\boldsymbol{\zeta}^{\mathrm{d}} & \text{if} \quad \boldsymbol{\zeta}^{\mathrm{d}} \neq \mathbf{0}, \\ \mathbf{0} & \text{if} \quad \boldsymbol{\zeta}^{\mathrm{d}} = \mathbf{0}, \end{cases} \tag{2.64}$$

and observe that $\boldsymbol{\zeta}_s \in \mathbb{L}^s(\Omega)$ and

$$\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\zeta}_s \,=\, \|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}^r \,=\, \|\boldsymbol{\zeta}_s\|_{0,s;\Omega}^s \,=\, \|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,\|\boldsymbol{\zeta}_s\|_{0,s;\Omega}\,. \tag{2.65}$$

In addition, it is clear that $\mathrm{tr}\,(\boldsymbol{\zeta}_s) = 0$, and thus, thanks to Lemma 2.2, it follows that $D_s(\boldsymbol{\zeta}_s)$ belongs to $\mathcal{K}_1$. Moreover, using (2.48) and (2.65), we find that

$$\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \big(D_s(\boldsymbol{\zeta}_s)\big)^{\mathrm{d}} \,=\, \int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\zeta}_s^{\mathrm{d}} \,=\, \int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\zeta}_s \,=\, \|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,\|\boldsymbol{\zeta}_s\|_{0,s;\Omega}\,,$$

and hence, noting that $\|D_s(\boldsymbol{\zeta}_s)\|_{X_1} = \|D_s(\boldsymbol{\zeta}_s)\|_{0,s;\Omega}$, and invoking the boundedness of D_s (cf. Lemma 2.2), we deduce that

$$\sup_{\substack{\boldsymbol{\tau}\in\mathcal{K}_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \big(D_s(\boldsymbol{\zeta}_s)\big)^{\mathrm{d}}}{\|D_s(\boldsymbol{\zeta}_s)\|_{X_1}} \,=\, \frac{\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,\|\boldsymbol{\zeta}_s\|_{0,s;\Omega}}{\|D_s(\boldsymbol{\zeta}_s)\|_{0,s;\Omega}} \,\geq\, \frac{1}{\|D_s\|}\,\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,. \tag{2.66}$$

Consequently, replacing (2.66) back into (2.63), we get

$$\sup_{\substack{\boldsymbol{\tau}\in\mathcal{K}_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \,\geq\, \left\{\frac{1}{\mu\|D_s\|} \,-\, \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda+(n+1)\mu\big)}\right\} \|\boldsymbol{\zeta}^{\mathtt{d}}\|_{0,r;\Omega}\,, \tag{2.67}$$

from which, choosing λ sufficiently large such that

$$\frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda+(n+1)\mu\big)} \,<\, \frac{1}{2\mu\|D_s\|}\,,$$

which reduces to

$$\lambda \,>\, M_s \,:=\, \frac{\mu}{n^{1+1/s}}\,\max\left\{2\|D_s\|\widetilde{C}_r - n^{1/s}(n+1), 0\right\},$$

and applying (2.50) to $\boldsymbol{\zeta}$, we arrive at (2.60) with $\alpha := \frac{1}{2\,\mu\,\|D_s\|\,\widehat{C}_r}$. On the other hand, given now $\boldsymbol{\tau}\in\mathcal{K}_1$, $\boldsymbol{\tau}\neq\mathbf{0}$, we exchange the roles of $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$ in the above analysis, so that we obtain

$$\sup_{\boldsymbol{\zeta}\in\mathcal{K}_2} a(\boldsymbol{\zeta},\boldsymbol{\tau}) \,\geq\, \sup_{\substack{\boldsymbol{\zeta}\in\mathcal{K}_2\\ \boldsymbol{\zeta}\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\zeta}\|_{X_2}} \,\geq\, \frac{1}{2\,\mu\,\|D_r\|\,\widehat{C}_s}\,\|\boldsymbol{\tau}\|_{X_1} \,>\, 0 \tag{2.68}$$

for

$$\lambda \,>\, M_r \,:=\, \frac{\mu}{n^{1+1/r}}\,\max\left\{2\|D_r\|\widetilde{C}_s - n^{1/r}(n+1), 0\right\},$$

which shows (2.61). In this way, the proof is completed by choosing $M := \max\left\{M_s, M_r\right\}$. □

From now on we assume that the Lamé parameter λ is such that

$$\lambda \,>\, M\,,$$

with M defined at the end of the foregoing proof.

Lemma 2.5. *Assume that r and s satisfy the particular range specified by* Theorem 1.1, *that is, r, $s\in[\frac{2n}{n+1},\frac{2n}{n-1}]$. Then, there exist positive constants β_1, β_2 such that for each $i\in\{1,2\}$ there hold*

$$\sup_{\substack{\boldsymbol{\zeta}\in X_i\\ \boldsymbol{\zeta}\neq\mathbf{0}}} \frac{b_i(\boldsymbol{\zeta},\mathbf{v})}{\|\boldsymbol{\zeta}\|_{X_i}} \,\geq\, \beta_i\,\|\mathbf{v}\|_{M_i} \qquad \forall\,\mathbf{v}\in M_i\,. \tag{2.69}$$

Proof. Since b_1 and b_2 have the same algebraic structure (cf. (2.36)), and the pairs (X_1, M_1) and (X_2, M_2) are obtained from each other by exchanging r and s, it suffices to show (2.69) for either $i=1$ or $i=2$. We proceed here with $i=2$, for which, given $\mathbf{v}\in M_2 := \mathbf{L}^s(\Omega)$, we first set

$$\mathbf{v}_r \,:=\, \begin{cases} |\mathbf{v}|^{s-2}\,\mathbf{v} & \text{if} \quad \mathbf{v}\neq\mathbf{0}\,,\\ \quad\mathbf{0} & \text{if} \quad \mathbf{v}=\mathbf{0}\,. \end{cases} \tag{2.70}$$

It follows that $\mathbf{v}_r\in\mathbf{L}^r(\Omega)$, and similarly to (2.65), there holds

$$\int_\Omega \mathbf{v}\cdot\mathbf{v}_r \,=\, \|\mathbf{v}\|_{0,s;\Omega}^s \,=\, \|\mathbf{v}_r\|_{0,r;\Omega}^r \,=\, \|\mathbf{v}\|_{0,s;\Omega}\,\|\mathbf{v}_r\|_{0,r;\Omega}\,. \tag{2.71}$$

Next, we let $\mathbf{z} \in \mathbf{W}_0^{1,r}(\Omega)$ be the unique solution, guaranteed by Theorem 1.5, of the vector Poisson equation (1.47) with $\mathbf{f} = \mathbf{0}$ and $\mathbf{g} = \mathbf{v}_r$, that is

$$\Delta \mathbf{z} = \mathbf{v}_r \quad \text{in} \quad \Omega\,, \qquad \mathbf{z} = \mathbf{0} \quad \text{on} \quad \Gamma\,,$$

whose weak formulation reduces to: Find $\mathbf{z} \in \mathbf{W}_0^{1,r}(\Omega)$ such that

$$\int_\Omega \nabla \mathbf{z} \cdot \nabla \mathbf{w} = -\int_\Omega \mathbf{v}_r \cdot \mathbf{w} \qquad \forall\, \mathbf{w} \in \mathbf{W}_0^{1,s}(\Omega)\,.$$

Note that the corresponding continuous dependence result establishes the existence of a positive constant $\bar{c}_r$ such that

$$\|\mathbf{z}\|_{1,r;\Omega} \,\leq\, \bar{c}_r\, \|\mathbf{v}_r\|_{0,r;\Omega}\,. \tag{2.72}$$

Furthermore, we observe that $\mathbf{div}\big(\nabla \mathbf{z}\big) = \mathbf{v}_r$ in Ω, which proves that $\nabla \mathbf{z} \in \mathbb{H}^r(\mathbf{div}_r;\Omega)$, and hence we let $\widehat{\boldsymbol{\zeta}}$ be the $\mathbb{H}_0^r(\mathbf{div}_r;\Omega)$-component of $\nabla \mathbf{z}$. Thus, employing (2.72) and noting that $\mathbf{div}(\widehat{\boldsymbol{\zeta}}) = \mathbf{v}_r$, we obtain

$$\|\widehat{\boldsymbol{\zeta}}\|_{X_2} \,=\, \|\widehat{\boldsymbol{\zeta}}\|_{0,r;\Omega} \,+\, \|\mathbf{div}(\widehat{\boldsymbol{\zeta}})\|_{0,r;\Omega} \,\leq\, |\mathbf{z}|_{1,r;\Omega} \,+\, \|\mathbf{v}_r\|_{0,r;\Omega} \,\leq\, \big(1 + \bar{c}_r\big)\, \|\mathbf{v}_r\|_{0,r;\Omega}\,.$$

Finally, bearing in mind the definition of b_2 (cf. (2.36), $i = 2$), and making use of (2.71) and the foregoing inequality, we conclude that

$$\sup_{\substack{\boldsymbol{\zeta}\in X_2\\ \boldsymbol{\zeta}\neq\mathbf{0}}} \frac{b_2(\boldsymbol{\zeta},\mathbf{v})}{\|\boldsymbol{\zeta}\|_{X_2}} \,\geq\, \frac{b_2(\widehat{\boldsymbol{\zeta}},\mathbf{v})}{\|\widehat{\boldsymbol{\zeta}}\|_{X_2}} \,=\, \frac{\displaystyle\int_\Omega \mathbf{v}\cdot\mathbf{v}_r}{\|\widehat{\boldsymbol{\zeta}}\|_{X_2}} \,\geq\, \frac{1}{1+\bar{c}_r}\, \|\mathbf{v}\|_{0,s;\Omega}\,, \tag{2.73}$$

which proves (2.69) for $i = 2$ with $\beta_2 := \big(1+\bar{c}_r\big)^{-1}$. □

For the rest of the chapter we assume meanwhile that r and s lie in the range stipulated in Lemmas 2.4 and 2.5, that is

$$r,\, s \,\in\, \Big[\frac{2n}{n+1}, \frac{2n}{n-1}\Big]\,. \tag{2.74}$$

The following result establishes that the operator $\mathbf{S}$ (cf. (2.9)) is well defined.

Lemma 2.6. *For each $\varphi \in \mathrm{H}_0^1(\Omega)$ there exists a unique $\mathbf{S}(\varphi) = \big(\mathbf{S}_1(\varphi), \mathbf{S}_2(\varphi)\big) := (\widetilde{\boldsymbol{\sigma}}, \widetilde{\mathbf{u}}) \in X_2 \times M_1$ solution to (2.40). Moreover, there hold*

$$\begin{aligned} \|\mathbf{S}_1(\varphi)\|_{X_2} \,&=\, \|\widetilde{\boldsymbol{\sigma}}\|_{X_2} \,\leq\, \frac{C_r}{\alpha}\, \|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\beta_2}\left(1 + \frac{2}{\alpha\mu}\right) f_2\,, \quad \textit{and} \\ \|\mathbf{S}_2(\varphi)\|_{M_1} \,&=\, \|\widetilde{\mathbf{u}}\|_{M_1} \,\leq\, \frac{C_r}{\beta_1}\left(1 + \frac{2}{\alpha\mu}\right) \|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu\beta_1\beta_2}\left(1 + \frac{2}{\alpha\mu}\right) f_2\,. \end{aligned} \tag{2.75}$$

Proof. Thanks to the fact that X_1, X_2, M_1 and M_2 are all reflexive Banach spaces, along with the boundedness of all the forms and functionals involved, and the inf-sup conditions provided by Lemmas 2.4 and 2.5, the proof reduces to a direct application of Theorem 2.1. In particular, the a priori estimates (2.75) follow from (2.45) and (2.57). □

Well-definedness of operator $\widetilde{\mathrm{S}}$

In this section we use the classical Lax-Milgram lemma to prove that $\widetilde{\mathrm{S}}$ (cf. (2.41)) is well defined. In fact, we first notice from (2.20) and (2.16) that, given $\boldsymbol{\zeta} \in X_2$, there holds

$$A_{\boldsymbol{\zeta}}(\phi,\varphi) \,\leq\, \vartheta_2\,\|\phi\|_{1,\Omega}\,\|\varphi\|_{1,\Omega} \qquad \forall\,\phi,\varphi\in \mathrm{H}^1_0(\Omega)\,, \tag{2.76}$$

which says that $A_{\boldsymbol{\zeta}}$ is bounded independently of $\boldsymbol{\zeta}$ with

$$\|A_{\boldsymbol{\zeta}}\| \,\leq\, \vartheta_2\,. \tag{2.77}$$

In turn, using now that ϑ is uniformly positive definite (cf. (2.15)), and denoting by c_p the constant of the Poincaré inequality in $\mathrm{H}^1_0(\Omega)$, which says that $\|\phi\|_{1,\Omega} \,\leq\, c_p\,|\phi|_{1,\Omega} \quad \forall\,\phi\in\mathrm{H}^1_0(\Omega)$, we deduce that

$$A_{\boldsymbol{\zeta}}(\phi,\phi) \,=\, \int_\Omega \vartheta(\boldsymbol{\zeta})\,\nabla\phi\cdot\nabla\phi \,\geq\, \widetilde{\alpha}\,\|\phi\|^2_{1,\Omega} \quad \forall\,\phi\in\mathrm{H}^1_0(\Omega)\,, \tag{2.78}$$

where

$$\widetilde{\alpha} \,:=\, \frac{\vartheta_0}{c_p^2}\,, \tag{2.79}$$

thus establishing the $\mathrm{H}^1_0(\Omega)$-ellipticity of $A_{\boldsymbol{\zeta}}$ independently of $\boldsymbol{\zeta}$ as well. Furthermore, given $\mathbf{w}\in M_1$, and bearing in mind (2.21), we employ the upper bound of g (cf. (2.18)) and the Cauchy-Schwarz inequality to arrive at

$$|G_{\mathbf{w}}(\psi)| \,\leq\, |\Omega|^{1/2}\,g_2\,\|\psi\|_{0,\Omega} \qquad \forall\,\psi\in\mathrm{H}^1_0(\Omega)\,, \tag{2.80}$$

which yields $G_{\mathbf{w}}\in\mathrm{H}^1_0(\Omega)'$ with $\|G_{\mathbf{w}}\| \,\leq\, |\Omega|^{1/2}\,g_2$.

Consequently, we are in a position to state that the operator $\widetilde{\mathrm{S}}$ is well-defined.

Lemma 2.7. *For each $(\boldsymbol{\zeta},\mathbf{w}) \in X_2\times M_1$ there exists a unique $\widetilde{\mathrm{S}}(\boldsymbol{\zeta},\mathbf{w}) := \widetilde{\phi}\in\mathrm{H}^1_0(\Omega)$ solution to (2.41). Moreover, there holds*

$$\|\widetilde{\mathrm{S}}(\boldsymbol{\zeta},\mathbf{w})\|_{1,\Omega} \,=\, \|\widetilde{\phi}\|_{1,\Omega} \,\leq\, \widetilde{\mathtt{r}} \,:=\, \frac{1}{\widetilde{\alpha}}\,|\Omega|^{1/2}\,g_2\,. \tag{2.81}$$

Proof. Thanks to the previous analysis, it is a straightforward application of Lax-Milgram's lemma (cf. [44, Theorem 1.1]). □

2.3.4 Solvability of the fixed-point equation

In this section we address the solvability analysis of the fixed-point equation (2.43). For this purpose, the hypotheses of the Banach fixed-point theorem are verified in what follows. We begin by defining the ball

$$W \,:=\, \Big\{\phi\in\mathrm{H}^1_0(\Omega): \quad \|\phi\|_{1,\Omega} \,\leq\, \widetilde{\mathtt{r}}\Big\}\,, \tag{2.82}$$

where $\widetilde{\mathtt{r}} > 0$ is the constant specified in (2.81). The following result states that T maps W into itself.

Lemma 2.8. *There holds $T(W)\subseteq W$.*

Proof. It follows directly from the definition of T (cf. (2.42)) and the a priori estimate for the operator $\widetilde{\mathrm{S}}$ provided by (2.81). ☐

The next goal is to establish the continuity of T, for which we previously prove the corresponding properties of $\mathbf{S}$ and $\widetilde{\mathrm{S}}$. We begin with the one of $\mathbf{S}$.

Lemma 2.9. *There exists a positive constant $C_{\mathbf{S}}$, depending only on μ, α, β_1, β_2, and the norm of the continuous injection $i_r : \mathrm{H}^1(\Omega) \to \mathrm{L}^r(\Omega)$, such that*

$$\|\mathbf{S}(\phi) - \mathbf{S}(\varphi)\|_{X_2\times M_1} \,\leq\, C_{\mathbf{S}}\, L_f\, \|\phi - \varphi\|_{1,\Omega} \qquad \forall\, \phi,\, \varphi \in \mathrm{H}^1_0(\Omega)\,. \tag{2.83}$$

Proof. Given φ, $\psi \in \mathrm{H}^1_0(\Omega)$, we let $\mathbf{S}(\varphi) := (\widetilde{\boldsymbol{\sigma}}, \widetilde{\mathbf{u}}) \in X_2 \times M_1$ and $\mathbf{S}(\psi) := (\bar{\boldsymbol{\sigma}}, \bar{\mathbf{u}}) \in X_2 \times M_1$, which satisfy (2.40) with φ itself and with $\varphi = \psi$, respectively. Then, subtracting the corresponding equations of these systems, we obtain

$$\begin{aligned} a(\widetilde{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}, \boldsymbol{\tau}) \,+\, b_1(\boldsymbol{\tau}, \widetilde{\mathbf{u}} - \bar{\mathbf{u}}) &\;=\; 0 && \forall\, \boldsymbol{\tau} \in X_1\,,\\ b_2(\widetilde{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}, \mathbf{v}) &\;=\; (F_\varphi - F_\psi)(\mathbf{v}) && \forall\, \mathbf{v} \in M_2\,, \end{aligned} \tag{2.84}$$

which says, thanks to the analysis and results from Section 2.3.3, particularly the inf-sup conditions satisfied by a, b_1 and b_2, along with Theorem 2.1, that $(\widetilde{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}, \widetilde{\mathbf{u}} - \bar{\mathbf{u}}) \in X_2 \times M_1$ is the unique solution of (2.40) with G given by the null functional and F_φ replaced by $F_\varphi - F_\psi$. Next, having in mind the definitions of F_φ and F_ψ (cf. (2.38)), employing the Lipschitz-continuity of $\mathbf{f}$ (cf. (2.17)), applying Hölder's inequality, and invoking the continuous injection $i_r : \mathrm{H}^1(\Omega) \to \mathrm{L}^r(\Omega)$, which is valid in particular for $r \in [\frac{2n}{n+1}, \frac{2n}{n-1}]$, we readily find that

$$|(F_\varphi - F_\psi)(\mathbf{v})| \,\leq\, L_f\, \|\varphi - \psi\|_{0,r;\Omega}\, \|\mathbf{v}\|_{0,s;\Omega} \,\leq\, L_f\, \|i_r\|\, \|\varphi - \psi\|_{1,\Omega}\, \|\mathbf{v}\|_{0,s;\Omega} \quad \forall\, v \in M_2\,, \tag{2.85}$$

which implies $\|F_\varphi - F_\psi\|_{M_2'} \,\leq\, L_f\, \|i_r\|\, \|\varphi - \psi\|_{1,\Omega}$. In this way, this latter inequality and the abstract estimate (2.45) applied to problem (2.84), yield (2.83) and end the proof. ☐

On the other hand, in order to establish a continuity property for $\widetilde{\mathrm{S}}$, we follow the approach of diverse previous works (see, e.g. [7], [30], [46], [47], and [52]), and introduce a regularity assumption on the solutions of the problem defining this operator. More precisely, from now on we suppose that there exists $\varepsilon \geq \frac{n}{r}$ and a constant $C_\varepsilon > 0$, such that

(**RA**) for each $(\boldsymbol{\zeta}, \mathbf{w}) \in X_2 \times M_1$ there holds $\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \mathbf{w}) = \widetilde{\phi} \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^{1+\varepsilon}(\Omega)$ and

$$\|\widetilde{\phi}\|_{1+\varepsilon,\Omega} \,\leq\, C_\varepsilon\, g_2\,. \tag{2.86}$$

The reason of the aforementioned lower bound of ε is clarified within the proof of the next lemma, which provides the Lipschitz-continuity of the operator $\widetilde{\mathrm{S}}$. In connection to this, and to be employed in the aforementioned proof as well, we recall now, thanks to the embedding between fractional Sobolev spaces, that for each $\varepsilon < \frac{n}{2}$ there holds $\mathrm{H}^\varepsilon(\Omega) \subset \mathrm{L}^{\varepsilon^*}(\Omega)$, with continuous injection

$$i_\varepsilon : \mathrm{H}^\varepsilon(\Omega) \longrightarrow \mathrm{L}^{\varepsilon^*}(\Omega)\,, \quad \text{where} \quad \varepsilon^* = \frac{2n}{n - 2\varepsilon}\,. \tag{2.87}$$

In this regard, we notice that the indicated lower and upper bounds for the additional regularity ε, which turn out to require that $\varepsilon \in [\frac{n}{r}, \frac{n}{2})$, are compatible if and only if $r > 2$, which is coherent with the fact that initially (cf. (2.23)) $r = 2p$, with $p \in (1, +\infty)$. Then, intersecting this constraint with the one stated previously in (2.74), we deduce that the feasible range for r becomes

$$r \in \Big(2, \frac{2n}{n-1}\Big] = \begin{cases} (2,4] & \text{if } n = 2\,, \\ (2,3] & \text{if } n = 3\,, \end{cases} \tag{2.88}$$

which we assume from now on. As a consequence, the range for the conjugate s of r is

$$s \in \Big[\frac{2n}{n+1}, 2\Big) = \begin{cases} \big[\frac{4}{3}, 2\big) & \text{if } n = 2\,, \\ \big[\frac{3}{2}, 2\big) & \text{if } n = 3\,. \end{cases} \tag{2.89}$$

Lemma 2.10. *There exists a positive constant $C_{\widetilde{\mathrm{S}}}$, depending only on $\widetilde{\alpha}$, the norm of the continuous injection $i_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^s(\Omega)$, $|\Omega|$, r, ε, $\|i_\varepsilon\|$ (cf. (2.87)), and C_ε (cf. (2.86)), such that*

$$\begin{aligned} &\|\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \mathbf{w}) - \widetilde{\mathrm{S}}(\boldsymbol{\tau}, \mathbf{v})\|_{1,\Omega} \\ &\quad \le C_{\widetilde{\mathrm{S}}} \big\{ L_g + L_\vartheta\, g_2 \big\} \|(\boldsymbol{\zeta}, \mathbf{w}) - (\boldsymbol{\tau}, \mathbf{v})\|_{X_2 \times M_1} \quad \forall\, (\boldsymbol{\zeta}, \mathbf{w}),\, (\boldsymbol{\tau}, \mathbf{v}) \in X_2 \times M_1\,. \end{aligned} \tag{2.90}$$

Proof. Given $(\boldsymbol{\zeta}, \mathbf{w})$, $(\boldsymbol{\tau}, \mathbf{v}) \in X_2 \times M_1$, we let $\widetilde{\phi} := \widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \mathbf{w})$ and $\widetilde{\varphi} := \widetilde{\mathrm{S}}(\boldsymbol{\tau}, \mathbf{v})$, which means, according to the definition of $\widetilde{\mathrm{S}}$ (cf. (2.41)), that $\widetilde{\phi}$ and $\widetilde{\varphi}$ are the unique elements in $\mathrm{H}^1_0(\Omega)$ such that

$$A_{\boldsymbol{\zeta}}(\widetilde{\phi}, \psi) = G_{\mathbf{w}}(\psi) \qquad \forall\, \psi \in \mathrm{H}^1_0(\Omega)\,, \tag{2.91}$$

and

$$A_{\boldsymbol{\tau}}(\widetilde{\varphi}, \psi) = G_{\mathbf{v}}(\psi) \qquad \forall\, \psi \in \mathrm{H}^1_0(\Omega)\,. \tag{2.92}$$

Thus, applying the $\mathrm{H}^1_0(\Omega)$-ellipticity of $A_{\boldsymbol{\zeta}}$, adding and subtracting $A_{\boldsymbol{\tau}}(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi})$, and then employing (2.91) and (2.92), we first obtain

$$\widetilde{\alpha}\, \|\widetilde{\phi} - \widetilde{\varphi}\|^2_{1,\Omega} \le A_{\boldsymbol{\zeta}}(\widetilde{\phi} - \widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi}) = (A_{\boldsymbol{\tau}} - A_{\boldsymbol{\zeta}})(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi}) + (G_{\mathbf{w}} - G_{\mathbf{v}})(\widetilde{\phi} - \widetilde{\varphi})\,. \tag{2.93}$$

Next, using the Lipschitz-continuity of g (cf. (2.18)), applying Hölder's inequality, and invoking the continuous injection $i_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^s(\Omega)$, which is also valid for the present range of s, we find that

$$\begin{aligned} |(G_{\mathbf{w}} - G_{\mathbf{v}})(\widetilde{\phi} - \widetilde{\varphi})| &\le \int_\Omega |g(\mathbf{w}) - g(\mathbf{v})|\, |\widetilde{\phi} - \widetilde{\varphi}| \le L_g \int_\Omega |\mathbf{w} - \mathbf{v}|\, |\widetilde{\phi} - \widetilde{\varphi}| \\ &\le L_g\, \|\mathbf{w} - \mathbf{v}\|_{0,r;\Omega}\, \|\widetilde{\phi} - \widetilde{\varphi}\|_{0,s;\Omega} \le L_g\, \|i_s\|\, \|\mathbf{w} - \mathbf{v}\|_{0,r;\Omega}\, \|\widetilde{\phi} - \widetilde{\varphi}\|_{1,\Omega}\,. \end{aligned} \tag{2.94}$$

In turn, employing now the Lipschitz-continuity of ϑ (cf. (2.16)), and making use again of Hölder's inequality, we get

$$\begin{aligned} |(A_{\boldsymbol{\tau}} - A_{\boldsymbol{\zeta}})(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi})| &= \Big| \int_\Omega \big(\vartheta(\boldsymbol{\tau}) - \vartheta(\boldsymbol{\zeta})\big)\, \nabla\widetilde{\varphi} \cdot \nabla(\widetilde{\phi} - \widetilde{\varphi}) \Big| \\ &\le L_\vartheta\, \|\boldsymbol{\tau} - \boldsymbol{\zeta}\|_{0,2q;\Omega}\, \|\nabla\widetilde{\varphi}\|_{0,2p;\Omega}\, \|\widetilde{\phi} - \widetilde{\varphi}\|_{1,\Omega} \end{aligned} \tag{2.95}$$

where p, $q \in (1, +\infty)$ are conjugate to each other. Now, choosing p such that $2p = \varepsilon^*$ (cf. (2.87)), we get $2q = \frac{n}{\varepsilon}$, which, according to the range stipulated for ε, yields $2q \le r$, so that the norm of the embedding of the respective Lebesgue spaces is given by $C_{r,\varepsilon} := |\Omega|^{\frac{r\varepsilon - n}{rn}}$. In this way, using additionally the continuity of i_ε (cf. (2.87)) along with the regularity assumption (2.86), the estimate (2.95) becomes

$$\begin{aligned} |(A_{\boldsymbol{\tau}} - A_{\boldsymbol{\zeta}})(\widetilde{\varphi}, \widetilde{\phi} - \widetilde{\varphi})| &\le L_\vartheta\, C_{r,\varepsilon}\, \|\boldsymbol{\tau} - \boldsymbol{\zeta}\|_{0,r;\Omega}\, \|i_\varepsilon\|\, \|\nabla\widetilde{\varphi}\|_{\varepsilon,\Omega}\, \|\widetilde{\phi} - \widetilde{\varphi}\|_{1,\Omega} \\ &\le L_\vartheta\, C_{r,\varepsilon}\, \|i_\varepsilon\|\, C_\varepsilon\, g_2\, \|\boldsymbol{\tau} - \boldsymbol{\zeta}\|_{0,r;\Omega}\, \|\widetilde{\phi} - \widetilde{\varphi}\|_{1,\Omega}\,. \end{aligned} \tag{2.96}$$

Finally, replacing the resulting estimates from (2.94) and (2.96) back into (2.93), simplifying $\|\widetilde{\phi} - \widetilde{\varphi}\|_{1,\Omega}$ on both sides, and dividing by $\widetilde{\alpha}$, we arrive at (2.90) and finish the proof. □

We are now in a position to establish the Lipschitz-continuity of the fixed point operator T. More precisely, we have the following result.

Lemma 2.11. *There exists a positive constant C_T, depending only on $C_{\mathbf{S}}$ and $C_{\widetilde{\mathrm{S}}}$, such that*

$$\|T(\phi) - T(\varphi)\|_{1,\Omega} \le C_T\, L_f \left\{L_g + L_\vartheta\, g_2\right\} \|\phi - \varphi\|_{1,\Omega} \qquad \forall\, \phi,\, \varphi \in \mathrm{H}^1_0(\Omega)\,. \tag{2.97}$$

Proof. Given ϕ, $\varphi \in \mathrm{H}^1_0(\Omega)$, and bearing in mind the definition of T (cf. (2.42)), straightforward applications of Lemmas 2.10 and 2.9 yield

$$\begin{aligned} \|T(\phi) - T(\varphi)\|_{1,\Omega} &\le C_{\widetilde{\mathrm{S}}} \left\{L_g + L_\vartheta\, g_2\right\} \|\mathbf{S}(\phi) - \mathbf{S}(\varphi)\|_{X_2 \times M_1} \\ &\le C_{\widetilde{\mathrm{S}}}\, C_{\mathbf{S}}\, L_f \left\{L_g + L_\vartheta\, g_2\right\} \|\phi - \varphi\|_{1,\Omega}\,, \end{aligned}$$

which yields (2.97) with $C_T := C_{\mathbf{S}}\, C_{\widetilde{\mathrm{S}}}$. □

Consequently, the main result of this section is stated as follows.

Theorem 2.12. *Assume the regularity assumption* (**RA**) *(cf.* (2.86)*) and that the data L_f, L_g, L_ϑ and g_2 are sufficiently small so that*

$$C_T\, L_f \left\{L_g + L_\vartheta\, g_2\right\} < 1\,. \tag{2.98}$$

Then, the coupled problem (2.39) *has a unique solution $(\boldsymbol{\sigma}, \mathbf{u}, \phi) \in X_2 \times M_1 \times \mathrm{H}^1_0(\Omega)$, with $\phi \in W$ (cf.* (2.82)*). Moreover, there hold*

$$\begin{aligned} \|\boldsymbol{\sigma}\|_{X_2} &\le \frac{C_r}{\alpha}\, \|\mathbf{u}_D\|_{1/s,r;\Gamma} + \frac{|\Omega|^{1/r}}{\beta_2} \left(1 + \frac{2}{\alpha\mu}\right) f_2\,, \quad \textit{and} \\ \|\mathbf{u}\|_{M_1} &\le \frac{C_r}{\beta_1} \left(1 + \frac{2}{\alpha\mu}\right) \|\mathbf{u}_D\|_{1/s,r;\Gamma} + \frac{2|\Omega|^{1/r}}{\mu\beta_1\beta_2} \left(1 + \frac{2}{\alpha\mu}\right) f_2\,. \end{aligned} \tag{2.99}$$

Proof. Thanks to Lemmas 2.8 and 2.11, and the assumption (2.98), the existence of a unique $\phi \in W$ solution to (2.43), and hence, equivalently, the existence of a unique $(\boldsymbol{\sigma}, \mathbf{u}, \phi) \in X_2 \times M_1 \times \mathrm{H}^1_0(\Omega)$ solution to (2.39), is merely an application of the Banach fixed point Theorem. In addition, the fact that $(\boldsymbol{\sigma}, \mathbf{u}) = \mathbf{S}(\phi)$ along with the a priori estimates provided by (2.75), yield (2.99) and conclude the proof. □

At this point we remark that the assumption (2.98) is clearly satisfied if the first row of (2.1) is given by the linear elasticity equations, in whose case $\mathbf{f}$ does not depend on ϕ, and hence obviously the Lipschitz constant L_f is 0. In this way, the closest we get to this model, the more feasible (2.98) becomes. Certainly, this feasibility also increases if g and ϑ get closer to constant functions, thus making L_g and L_ϑ smaller.

2.4 The Galerkin scheme

In this section we introduce the Galerkin scheme of the mixed-primal formulation (2.39), and analyse its solvability by employing a discrete version of the fixed point strategy developed in Section 2.3.2. For this purpose, we begin by considering arbitrary finite element subspaces $X_{2,h} \subseteq X_2$, $M_{1,h} \subseteq M_1$, $X_{1,h} \subseteq X_1$, $M_{2,h} \subseteq X_2$, and $\mathrm{H}_h \subseteq \mathrm{H}_0^1(\Omega)$, whose specific choices satisfying all the required stability conditions will be introduced later on in Section 2.5. In this way, the Galerkin scheme associated with (2.39) reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in X_{2,h} \times M_{1,h}$ and $\phi_h \in \mathrm{H}_h$ such that

$$
\begin{aligned}
a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_1(\boldsymbol{\tau}_h, \mathbf{u}_h) &= G(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in X_{1,h}\,, \\
b_2(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= F_{\phi_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in M_{2,h}\,, \\
A_{\boldsymbol{\sigma}_h}(\phi_h, \psi_h) &= G_{\mathbf{u}_h}(\psi_h) && \forall\, \psi_h \in \mathrm{H}_h\,.
\end{aligned}
\tag{2.100}
$$

2.4.1 The discrete fixed point strategy

Here we adopt the discrete analogue of the fixed point strategy introduced in Section 2.3.2 to analyse the solvability of (2.100). According to it, we now let $\mathbf{S}_h : \mathrm{H}_h \to X_{h,2} \times M_{h,1}$ be the operator defined for each $\varphi_h \in \mathrm{H}_h$ as $\mathbf{S}_h(\varphi_h) := (\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\mathbf{u}}_h)$, where $(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\mathbf{u}}_h) \in X_{2,h} \times M_{1,h}$ is the unique solution (to be confirmed below) of the first two equations of (2.100) with φ_h instead of ϕ_h, that is

$$
\begin{aligned}
a(\widetilde{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + b_1(\boldsymbol{\tau}_h, \widetilde{\mathbf{u}}_h) &= G(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in X_{1,h}\,, \\
b_2(\widetilde{\boldsymbol{\sigma}}_h, \mathbf{v}_h) &= F_{\varphi_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in M_{2,h}\,.
\end{aligned}
\tag{2.101}
$$

In addition, we also let $\widetilde{\mathrm{S}}_h : X_{2,h} \times M_{1,h} \to \mathrm{H}_h$ be the operator defined for each $(\boldsymbol{\zeta}_h, \mathbf{w}_h) \in X_{2,h} \times M_{1,h}$ as $\widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \mathbf{w}_h) := \widetilde{\phi}_h$, where $\widetilde{\phi}_h \in \mathrm{H}_h$ is the unique solution of the last equation of (2.100) with $(\boldsymbol{\zeta}_h, \mathbf{w}_h)$ instead of $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$, that is

$$
A_{\boldsymbol{\zeta}_h}(\widetilde{\phi}_h, \psi_h) \,=\, G_{\mathbf{w}_h}(\psi_h) \quad \forall\, \psi_h \in \mathrm{H}_h\,. \tag{2.102}
$$

Then, we define the operator $T_h : \mathrm{H}_h \to \mathrm{H}_h$ as

$$
T_h(\varphi_h) \,:=\, \widetilde{\mathrm{S}}_h\big(\mathbf{S}_h(\varphi_h)\big) \quad \forall\, \varphi_h \in \mathrm{H}_h\,, \tag{2.103}
$$

and realise that solving (2.100) is equivalent to seeking a fixed point of T_h, that is $\phi_h \in \mathrm{H}_h$ such that

$$
T_h(\phi_h) \,=\, \phi_h\,. \tag{2.104}
$$

2.4.2 Well-posedness of the operators $\mathbf{S}_h$ and $\widetilde{\mathrm{S}}_h$

We now apply the discrete versions of Theorem 2.1 and Lax-Milgram's lemma to show that the discrete operators $\mathbf{S}_h$ and $\widetilde{\mathrm{S}}_h$ are well defined, equivalently that problems (2.101) and (2.102) are well-posed. For this purpose, we now let $\mathcal{K}_{1,h}$ and $\mathcal{K}_{2,h}$ be the discrete kernels of the operators induced by the bilinear forms b_1 and b_2, respectively, that is

$$\mathcal{K}_{1,h} := \Big\{ \boldsymbol{\tau}_h \in X_{1,h} : \quad b_1(\boldsymbol{\tau}_h, \mathbf{v}_h) = 0 \quad \forall\, \mathbf{v}_h \in M_{1,h} \Big\}, \tag{2.105}$$

$$\mathcal{K}_{2,h} := \Big\{ \boldsymbol{\zeta}_h \in X_{2,h} : \quad b_2(\boldsymbol{\zeta}_h, \mathbf{v}_h) = 0 \quad \forall\, \mathbf{v}_h \in M_{2,h} \Big\}. \tag{2.106}$$

Next, we introduce some hypotheses involving the arbitrary spaces $X_{2,h}$, $M_{1,h}$, $X_{1,h}$, and $M_{2,h}$, as well as $\mathcal{K}_{1,h}$ and $\mathcal{K}_{2,h}$. More precisely, from now on we assume the following:

(**H.1**) there exists a constant $\alpha_{\mathrm{d}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathcal{K}_{1,h}\\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{X_1}} \,\geq\, \alpha_{\mathrm{d}}\, \|\boldsymbol{\sigma}_h\|_{X_2} \qquad \forall\, \boldsymbol{\sigma}_h \in \mathcal{K}_{2,h}\,, \quad \text{and}$$

$$\sup_{\boldsymbol{\zeta}_h \in \mathcal{K}_{2,h}} a(\boldsymbol{\zeta}_h, \boldsymbol{\tau}_h) \,>\, 0 \qquad \forall\, \boldsymbol{\tau}_h \in \mathcal{K}_{1,h},\ \boldsymbol{\tau}_h \neq \mathbf{0}\,.$$

(**H.2**) there exist constants $\beta_{1,\mathrm{d}}$, $\beta_{2,\mathrm{d}} > 0$, independent of h, such that for each $i \in \{1, 2\}$ there holds

$$\sup_{\substack{\boldsymbol{\tau}_h \in X_{i,h}\\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{b_i(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{X_i}} \,\geq\, \beta_{i,\mathrm{d}}\, \|\mathbf{v}_h\|_{M_i} \qquad \forall\, \mathbf{v}_h \in M_{i,h}\,.$$

Specific finite element subspaces satisfying (**H.1**) and (**H.2**) will be defined later on in Section 2.5.2. Thus, as a straightforward consequence of these assumptions, we obtain the following result.

Lemma 2.13. *For each $\varphi_h \in \mathrm{H}_h$ there exists a unique $\mathbf{S}_h(\varphi_h) = \big(\mathbf{S}_{1,h}(\varphi_h), \mathbf{S}_{2,h}(\varphi_h)\big) := (\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\mathbf{u}}_h) \in X_{2,h} \times M_{1,h}$ solution to (2.101). Moreover, there hold*

$$\begin{aligned}
\|\mathbf{S}_{1,h}(\varphi_h)\|_{X_2} = \|\widetilde{\boldsymbol{\sigma}}_h\|_{X_2} \,&\leq\, \frac{C_r}{\alpha_{\mathrm{d}}}\, \|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\beta_{2,\mathrm{d}}} \left(1 + \frac{2}{\alpha_{\mathrm{d}}\,\mu}\right) f_2\,, \quad \textit{and}\\
\|\mathbf{S}_{2,h}(\varphi_h)\|_{M_1} = \|\widetilde{\mathbf{u}}_h\|_{M_1} \,&\leq\, \frac{C_r}{\beta_{1,\mathrm{d}}} \left(1 + \frac{2}{\alpha_{\mathrm{d}}\,\mu}\right) \|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu\beta_{1,\mathrm{d}}\,\beta_{2,\mathrm{d}}} \left(1 + \frac{2}{\alpha_{\mathrm{d}}\,\mu}\right) f_2\,.
\end{aligned} \tag{2.107}$$

Proof. Invoking (**H.1**) and (**H.2**), the proof reduces to a direct application of the discrete version of Theorem 2.1 (see, e.g. [12, Corollary 2.2]). In particular, the a priori estimates given by (2.107) follow from the discrete analogue of (2.75). □

Having proved that $\mathbf{S}_h$ is well-defined, we now establish the same property for $\widetilde{\mathrm{S}}_h$ with an arbitrary finite element subspace H_h of $\mathrm{H}^1(\Omega)$.

Lemma 2.14. *For each $(\boldsymbol{\zeta}_h, \mathbf{w}_h) \in X_{2,h} \times M_{1,h}$ there exists a unique $\widetilde{\mathrm{S}}(\boldsymbol{\zeta}_h, \mathbf{w}_h) := \widetilde{\phi}_h \in \mathrm{H}_h$ solution to (2.102). Moreover, with the same constant $\widetilde{\mathtt{r}}$ introduced in Lemma 2.7, there holds*

$$\|\widetilde{\mathrm{S}}(\boldsymbol{\zeta}_h, \mathbf{w}_h)\|_{1,\Omega} = \|\widetilde{\phi}_h\|_{1,\Omega} \,\leq\, \widetilde{\mathtt{r}}\,. \tag{2.108}$$

Proof. It suffices to note that the bilinear form $A_{\boldsymbol{\zeta}_h}$ is H_h-elliptic with the same constant $\widetilde{\alpha}$ given by (2.79), and that $G_{\mathbf{w}_h}$ restricted to H_h belongs to H'_h with $\|G_{\mathbf{w}_h}\| \le |\Omega|^{1/2} g_2$ (cf. (2.80)). In this way, the proof is a direct application of Lax-Milgram's lemma. □

2.4.3 Discrete solvability analysis

Having proved that the discrete operators $\mathbf{S}_h$, $\widetilde{\mathrm{S}}_h$, and hence T_h, are all well defined, we now address the solvability of the corresponding fixed point equation (2.104). To this end, and similarly to (2.82), we first introduce the discrete ball

$$W_h := \left\{ \phi_h \in \mathrm{H}_h : \quad \|\phi_h\|_{1,\Omega} \le \widetilde{\mathtt{r}} \right\}, \tag{2.109}$$

where $\widetilde{\mathtt{r}} > 0$ is the constant specified in (2.81), that is $\widetilde{\mathtt{r}} := \dfrac{1}{\widetilde{\alpha}} |\Omega|^{1/2} g_2$, and establish the discrete analogue of Lemma 2.8.

Lemma 2.15. *There holds $T_h(W_h) \subseteq W_h$.*

Proof. Similarly to the proof of Lemma 2.8, it follows from the definition of T_h (cf. (2.103)) and the a priori estimate for the operator $\widetilde{\mathrm{S}}_h$ provided by (2.108). □

Next, we aim to state the continuity of the operators $\mathbf{S}_h$, $\widetilde{\mathrm{S}}_h$, and T_h. We begin with $\mathbf{S}_h$ by proceeding analogously to the proof of Lemma 2.9. Indeed, considering the Galerkin scheme associated with (2.84), the inf-sup conditions provided by (**H.1**) and (**H.2**), the continuous injection $i_r : \mathrm{H}^1(\Omega) \to \mathrm{L}^r(\Omega)$, and the discrete version of the abstract estimate (2.45) (cf. [12, Corollary 2.2]), we readily deduce that there exists a positive constant $C_{\mathbf{S},\mathrm{d}}$, depending only on μ, α_d, $\beta_{1,\mathrm{d}}$, $\beta_{2,\mathrm{d}}$, and $\|i_r\|$, and hence independent of h, such that

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\varphi_h)\|_{X_2 \times M_1} \le C_{\mathbf{S},\mathrm{d}}\, L_f\, \|\phi_h - \varphi_h\|_{1,\Omega} \qquad \forall\, \phi,\, \varphi \in \mathrm{H}_h\,. \tag{2.110}$$

In turn, for the continuity of $\widetilde{\mathrm{S}}_h$ we slightly modify the reasoning of the proof of Lemma 2.10. In fact, instead of the regularity assumption (**RA**), which is certainly not applicable in the present discrete context, we just employ an $\mathrm{L}^{2q} - \mathrm{L}^{2p} - \mathrm{L}^2$ argument to derive the discrete version of (2.90), where $p,\, q \in (1, +\infty)$ conjugate to each other, are chosen such that $2q = r$. Note that this is a feasible choice since, as stipulated in (2.88), there holds $r > 2$, which yields $r^* := 2p = \frac{2r}{r-2}$. In this way, given $(\boldsymbol{\zeta}_h, \mathbf{w}_h)$, $(\boldsymbol{\tau}_h, \mathbf{v}_h) \in X_{2,h} \times M_{1,h}$, and denoting $\widetilde{\phi}_h = \widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \mathbf{w}_h) \in \mathrm{H}_h$ and $\widetilde{\varphi}_h = \widetilde{\mathrm{S}}_h(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathrm{H}_h$, the discrete analogue of (2.95) becomes

$$|(A_{\boldsymbol{\tau}_h} - A_{\boldsymbol{\zeta}_h})(\widetilde{\varphi}_h, \widetilde{\phi}_h - \widetilde{\varphi}_h)| \le L_\vartheta\, \|\boldsymbol{\tau}_h - \boldsymbol{\zeta}_h\|_{0,r;\Omega}\, \|\nabla \widetilde{\varphi}_h\|_{0,r^*;\Omega}\, \|\widetilde{\phi}_h - \widetilde{\varphi}_h\|_{1,\Omega}\,, \tag{2.111}$$

which, along with the discrete versions of (2.93) and (2.94), imply the existence of a positive constant $C_{\widetilde{\mathrm{S}},\mathrm{d}}$, depending only on $\widetilde{\alpha}$ and the norm of the continuous injection $i_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^s(\Omega)$, and hence independent of h, such that

$$\begin{aligned} &\|\widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \mathbf{w}_h) - \widetilde{\mathrm{S}}_h(\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{1,\Omega} \\ &\qquad \le C_{\widetilde{\mathrm{S}},\mathrm{d}} \left\{ L_g + L_\vartheta\, \|\nabla \widetilde{\mathrm{S}}_h(\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{0,r^*;\Omega} \right\} \|(\boldsymbol{\zeta}_h, \mathbf{w}_h) - (\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{X_2 \times M_1} \end{aligned} \tag{2.112}$$

for all $(\boldsymbol{\zeta}_h, \mathbf{w}_h)$, $(\boldsymbol{\tau}_h, \mathbf{v}_h) \in X_{2,h} \times M_{1,h}$.

In this way, recalling the definition of T_h (cf. (2.103)), and employing the estimates (2.110) and (2.112), we conclude that

$$\|T_h(\phi_h) - T_h(\varphi_h)\|_{1,\Omega} \,\le\, C_{T,\mathrm{d}}\, L_f \left\{ L_g + L_\vartheta \, \|\nabla T_h(\varphi_h)\|_{0,r^*;\Omega} \right\} \|\phi_h - \varphi_h\|_{1,\Omega} \qquad \forall\, \phi_h,\, \varphi_h \in \mathrm{H}_h\,, \tag{2.113}$$

with the positive constant $C_{T,\mathrm{d}} := C_{\mathbf{S},\mathrm{d}}\, C_{\widetilde{\mathrm{S}},\mathrm{d}}$. Regarding the estimate (2.113), we emphasize here that, while it proves the continuity of T_h, the lack of control of the term $\|\nabla T_h(\varphi_h)\|_{0,r^*;\Omega}$ does not allow us to conclude Lipschitz-continuity and hence nor contractivity of this operator. Consequently, we are able to establish next only the existence of a fixed point of T_h.

Theorem 2.16. *The Galerkin scheme (2.100) has at least one solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h) \in X_{2,h} \times M_{1,h} \times \mathrm{H}_h$, with $\phi_h \in W_h$ (cf. (2.109)). Moreover, there hold*

$$\begin{aligned} \|\boldsymbol{\sigma}_h\|_{X_2} \,&\le\, \frac{C_r}{\alpha_\mathrm{d}}\, \|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\beta_{2,\mathrm{d}}} \left(1 + \frac{2}{\alpha_\mathrm{d}\, \mu}\right) f_2\,, \quad \textit{and} \\ \|\mathbf{u}_h\|_{M_1} \,&\le\, \frac{C_r}{\beta_{1,\mathrm{d}}} \left(1 + \frac{2}{\alpha_\mathrm{d}\, \mu}\right) \|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu\beta_{1,\mathrm{d}}\, \beta_{2,\mathrm{d}}} \left(1 + \frac{2}{\alpha_\mathrm{d}\, \mu}\right) f_2\,. \end{aligned} \tag{2.114}$$

Proof. Thanks to Lemma 2.15 and the continuity of T_h (cf. (2.113)), and bearing in mind the equivalence between (2.100) and (2.104), a straightforward application of Brouwer's theorem (cf. [28, Theorem 9.9-2]) implies the first conclusion of this theorem. In turn, the fact that $(\boldsymbol{\sigma}_h, \mathbf{u}_h) = \mathbf{S}_h(\phi_h)$ and the a priori estimates from (2.107) yield (2.114), thus completing the proof. □

2.4.4 A priori error analysis

We now aim to derive an a priori error estimate for the Galerkin scheme (2.100) with arbitrary finite element subspaces satisfying the hypotheses introduced in Section 2.4.2. In other words, we are interested in establishing a Céa estimate for the global error

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{M_1} \,+\, \|\phi - \phi_h\|_{1,\Omega}\,,$$

where $(\boldsymbol{\sigma}, \mathbf{u}, \phi) \in X_2 \times M_1 \times \mathrm{H}^1_0(\Omega)$ and $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h) \in X_{2,h} \times M_{1,h} \times \mathrm{H}_h$ are the unique solutions of (2.39) and (2.100) , respectively, with $\phi \in W$ (cf. (2.82)) and $\phi_h \in W_h$ (cf. (2.109)). For this purpose, and in order to employ suitable Strang estimates, we rewrite (2.39) and (2.100) as the following pairs of corresponding continuous and discrete formulations

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b_1(\boldsymbol{\tau}, \mathbf{u}) \;&=\; G(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in X_1\,, \\ b_2(\boldsymbol{\sigma}, \mathbf{v}) \;&=\; F_\phi(\mathbf{v}) \qquad \forall\, \mathbf{v} \in M_2\,, \\ a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_1(\boldsymbol{\tau}_h, \mathbf{u}_h) \;&=\; G(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in X_{1,h}, \\ b_2(\boldsymbol{\sigma}_h, \mathbf{v}_h) \;&=\; F_{\phi_h}(\mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in M_{2,h}, \end{aligned} \tag{2.115}$$

and

$$\begin{aligned} A_{\boldsymbol{\sigma}}(\phi, \psi) \;&=\; G_{\mathbf{u}}(\psi) \qquad \forall\, \psi \in \mathrm{H}^1_0(\Omega)\,, \\ A_{\boldsymbol{\sigma}_h}(\phi_h, \psi_h) \;&=\; G_{\mathbf{u}_h}(\psi_h) \qquad \forall\, \psi_h \in \mathrm{H}_h\,. \end{aligned} \tag{2.116}$$

In what follows, given a subspace Z_h of a generic Banach space $(Z, \|\cdot\|_Z)$, we set for each $z \in Z$

$$\operatorname{dist}(z, Z_h) := \inf_{z_h \in Z_h} \|z - z_h\|_Z .$$

Then, applying the Strang a priori error estimate from [12, Proposition 2.1, Corollary 2.3, and Theorem 2.3] to the context given by (2.115), we deduce that there exists a positive constant $\widehat{C}_{ST}$, depending only on $\alpha_{\rm d}$, $\beta_{1,\rm d}$, $\beta_{1,\rm d}$, $\|a\|$, $\|b_1\|$, and $\|b_2\|$, where $\|a\| \le \frac{2}{\mu}$ and $\|b_1\|$, $\|b_2\| \le 1$ (cf. (2.57)), such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} + \|\mathbf{u} - \mathbf{u}_h\|_{M_1} \le \widehat{C}_{ST} \left\{ \operatorname{dist}(\boldsymbol{\sigma}, X_{2,h}) + \operatorname{dist}(\mathbf{u}, M_{1,h}) + \|F_\phi - F_{\phi_h}\|_{M'_{2,h}} \right\}. \tag{2.117}$$

Then, proceeding as for the derivation of (2.85) (cf. proof of Lemma 2.9), we readily find that

$$\|F_\phi - F_{\phi_h}\|_{M'_{2,h}} \le L_f \, \|i_r\| \, \|\phi - \phi_h\|_{1,\Omega} , \tag{2.118}$$

which, replaced back into (2.117), gives

$$\begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} + \|\mathbf{u} - \mathbf{u}_h\|_{M_1} \\ &\quad \le \widehat{C}_{ST} \left\{ \operatorname{dist}(\boldsymbol{\sigma}, X_{2,h}) + \operatorname{dist}(\mathbf{u}, M_{1,h}) + L_f \, \|i_r\| \, \|\phi - \phi_h\|_{1,\Omega} \right\}. \end{aligned} \tag{2.119}$$

On the other hand, applying now the classical first Strang Lemma for elliptic variational problems (cf. [41, Lemma 2.27]) to the context given by (2.116), and then adding and subtracting ϕ to the first components of the expressions involving $A_{\boldsymbol{\sigma}}$ and $A_{\boldsymbol{\sigma}_h}$ in the corresponding consistent term, and finally employing the boundedness of these bilinear forms (cf. (2.76) - (2.77)), we arrive at

$$\|\phi - \phi_h\|_{1,\Omega} \le \widetilde{C}_{ST} \left\{ \operatorname{dist}(\phi, \mathrm{H}_h) + \|G_{\mathbf{u}} - G_{\mathbf{u}_h}\|_{\mathrm{H}'_h} + \|A_{\boldsymbol{\sigma}}(\phi, \cdot) - A_{\boldsymbol{\sigma}_h}(\phi, \cdot)\|_{\mathrm{H}'_h} \right\}, \tag{2.120}$$

where $\widetilde{C}_{ST}$ is a positive constant depending only on $\widetilde{\alpha}$ (cf. (2.78) - (2.79)) and the upper bound ϑ_2 of $\|A_{\boldsymbol{\sigma}_h}\|$ (cf. (2.76) - (2.77)). Next, proceeding exactly as for the derivations of (2.94) and (2.96), we find that for each $\varphi_h \in \mathrm{H}_h$ there hold

$$|(G_{\mathbf{u}} - G_{\mathbf{u}_h})(\varphi_h)| \le L_g \, \|i_s\| \, \|\mathbf{u} - \mathbf{u}_h\|_{0,r;\Omega} \, \|\varphi_h\|_{1,\Omega}$$

and

$$|A_{\boldsymbol{\sigma}}(\phi, \varphi_h) - A_{\boldsymbol{\sigma}_h}(\phi, \varphi_h)| \le L_\vartheta \, C_{r,\varepsilon} \, \|i_\varepsilon\| \, C_\varepsilon \, g_2 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,r;\Omega} \, \|\varphi_h\|_{1,\Omega} ,$$

respectively, from which it follows that

$$\|G_{\mathbf{u}} - G_{\mathbf{u}_h}\|_{\mathrm{H}'_h} \le L_g \, \|i_s\| \, \|\mathbf{u} - \mathbf{u}_h\|_{M_1} \tag{2.121}$$

and

$$\|A_{\boldsymbol{\sigma}}(\phi, \cdot) - A_{\boldsymbol{\sigma}_h}(\phi, \cdot)\|_{\mathrm{H}'_h} \le L_\vartheta \, C_{r,\varepsilon} \, \|i_\varepsilon\| \, C_\varepsilon \, g_2 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} . \tag{2.122}$$

In this way, replacing (2.121) and (2.122) back into (2.120), we conclude that

$$\|\phi - \phi_h\|_{1,\Omega} \le \widetilde{C}_{ST} \left\{ \operatorname{dist}(\phi, \mathrm{H}_h) + L_g \, \|i_s\| \, \|\mathbf{u} - \mathbf{u}_h\|_{M_1} + L_\vartheta \, C_{r,\varepsilon} \, \|i_\varepsilon\| \, C_\varepsilon \, g_2 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} \right\}. \tag{2.123}$$

In turn, using the foregoing bound in (2.119), and performing some algebraic arrangements, we get

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} + \|\mathbf{u} - \mathbf{u}_h\|_{M_1} \,\le\, & \bar{C}_0 \Big\{ \mathrm{dist}\,(\boldsymbol{\sigma}, X_{2,h}) + \mathrm{dist}\,(\mathbf{u}, M_{1,h}) + \mathrm{dist}\,(\phi, \mathrm{H}_h) \Big\} \\ & + \bar{C}_1\, L_f\, L_g\, \|\mathbf{u} - \mathbf{u}_h\|_{M_1} \,+\, \bar{C}_2\, L_f\, L_\vartheta\, g_2\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2}\,, \end{aligned} \tag{2.124}$$

where $\bar{C}_0 := \widehat{C}_{ST}\, \max\big\{1, L_f\, \|i_r\|\, \widetilde{C}_{ST}\big\}$, and $\bar{C}_1$ and $\bar{C}_2$ are positive constants depending only on $\widehat{C}_{ST}$, $\widetilde{C}_{ST}$, $\|i_r\|$, $\|i_s\|$, $\|i_\varepsilon\|$, $C_{r,\varepsilon}$, and C_ε.

According to the previous analysis, we are now in a position to establish the announced Céa estimate.

Theorem 2.17. *Assume that the data satisfy*

$$\bar{C}_1\, L_f\, L_g \,\le\, \frac{1}{2} \quad \textit{and} \quad \bar{C}_2\, L_f\, L_\vartheta\, g_2 \,\le\, \frac{1}{2}\,. \tag{2.125}$$

Then, there exists a positive constant C, independent of h, such that

$$\begin{aligned} & \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} + \|\mathbf{u} - \mathbf{u}_h\|_{M_1} + \|\phi - \phi_h\|_{1,\Omega} \\ & \quad \le\; C \Big\{ \mathrm{dist}\,(\boldsymbol{\sigma}, X_{2,h}) + \mathrm{dist}\,(\mathbf{u}, M_{1,h}) + \mathrm{dist}\,(\phi, \mathrm{H}_h) \Big\}\,. \end{aligned} \tag{2.126}$$

Proof. It suffices to employ the assumptions from (2.125) in (2.124), and then combine the resulting estimate with (2.123). □

2.5 Specific finite element subspaces

We now restrict our analysis to the 2D case and define specific finite element subspaces $X_{2,h} \subseteq X_2$, $M_{2,h} \subseteq M_2$, $X_{1,h} \subseteq X_1$, $M_{1,h} \subseteq M_1$, and $\mathrm{H}_h \subseteq \mathrm{H}^1_0(\Omega)$, satisfying the abstract hypotheses (**H.1**) and (**H.2**) that were introduced in Section 2.4.2 in order to guarantee the well-posedness of the Galerkin scheme (2.100).

2.5.1 Preliminaries

We begin by letting $\big\{\mathcal{T}_h\big\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$, which are made of triangles K of diameters h_K, and define the meshsize $h := \max\big\{h_K : \quad K \in \mathcal{T}_h\big\}$, which also serves as the index of $\mathcal{T}_h$. Then, given an integer $k \ge 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_k(K)$ be the space of polynomials defined on K of degree $\le k$, and denote its vector version by $\mathbf{P}_k(K)$. In addition, we let $\mathbf{RT}_k(K) = \mathbf{P}_k(K) \oplus \mathrm{P}_k(K)\mathbf{x}$ be the local Raviart-Thomas space defined on K of order k, where $\mathbf{x}$ stands for a generic vector in R^2, and denote by $\mathbb{RT}_k(K)$ its corresponding tensor counterpart, that is, letting $\boldsymbol{\tau}_i$ be the i-th row of a tensor $\boldsymbol{\tau}$, we set

$$\mathbb{RT}_k(K) \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}; K) : \quad \boldsymbol{\tau}_i \in \mathbf{RT}_k(K) \quad \forall\, i \in \big\{1, 2\big\} \Big\}\,.$$

In turn, we let $\mathbf{P}_k(\mathcal{T}_h)$ and $\mathbb{RT}_k(\mathcal{T}_h)$ be the corresponding global versions of $\mathbf{P}_k(K)$ and $\mathbb{RT}_k(K)$, respectively, that is

$$\mathbf{P}_k(\mathcal{T}_h) \,:=\, \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h \Big\}\,,$$

and

$$\mathbb{RT}_k(\mathcal{T}_h) := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall K \in \mathcal{T}_h \Big\}.$$

We stress here that for each $t \in [1,+\infty]$ there hold $\mathbf{P}_k(\mathcal{T}_h) \subseteq \mathbf{L}^t(\Omega)$ and $\mathbb{RT}_k(\mathcal{T}_h) \subseteq \mathbb{H}^t(\mathbf{div}_t;\Omega)$ (cf. (2.24)), which is implicitly utilized below in Section 2.5.2 to define the announced specific finite element subspaces. Some useful properties concerning $\mathbf{P}_k(\mathcal{T}_h)$ and $\mathbb{RT}_k(\mathcal{T}_h)$ are needed first. For this purpose, we now introduce for each $t \in (1,+\infty)$ the space

$$\mathbb{H}_t := \Big\{ \boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega) : \quad \boldsymbol{\tau}|_K \in \mathbb{W}^{1,t}(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

and let $\Pi_h^k : \mathbb{H}_t \to \mathbb{RT}_k(\mathcal{T}_h)$ be the global Raviart-Thomas interpolation operator (cf. [14, Section 2.5]). Then, we recall from [14, Proposition 2.5.2 and eq. (2.5.27)] that the commuting diagram property states that

$$\mathbf{div}(\Pi_h^k(\boldsymbol{\tau})) = \mathcal{P}_h^k(\mathbf{div}(\boldsymbol{\tau})) \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_t\,, \tag{2.127}$$

where $\mathcal{P}_h^k : \mathbf{L}^1(\Omega) \to \mathbf{P}_k(\mathcal{T}_h)$ is the usual orthogonal projector with respect to the $\mathbf{L}^2(\Omega)$-inner product, that is given $\mathbf{w} \in \mathbf{L}^1(\Omega)$, $\mathcal{P}_h^k(\mathbf{w})$ is the unique element in $\mathbf{P}_k(\mathcal{T}_h)$ satisfying

$$\int_\Omega \mathcal{P}_h^k(\mathbf{w}) \cdot \mathbf{v}_h = \int_\Omega \mathbf{w} \cdot \mathbf{v}_h \qquad \forall\, \mathbf{v}_h \in \mathbf{P}_k(\mathcal{T}_h)\,. \tag{2.128}$$

Regarding the approximation properties of $\mathcal{P}_h^k$ and Π_h^k in the present context of the Banach spaces $\mathbf{L}^t(\Omega)$ and $\mathbb{H}_0^t(\mathbf{div}_t;\Omega)$, we remark that they follow in the usual way by employing now the $\mathbf{W}^{m,t}$ version of the Deny-Lions Lemma (cf. [41, Lemma B.67] with integer $m \ge 0$ and $t \in (1,+\infty)$, the associated scaling estimates (cf. [41, Lemma 1.101]), and the regularity of $\{\mathcal{T}_h\}_{h>0}$. Indeed, one deduces the existence of positive constants C_1, C_2, independent of h, such that for integers l and m verifying $0 \le l \le k+1$ and $0 \le m \le l$, there hold

$$|\mathbf{w} - \mathcal{P}_h^k(\mathbf{w})|_{m,t;\Omega} \le C_1\, h^{l-m}\, |\mathbf{w}|_{l,t;\Omega} \qquad \forall\, \mathbf{w} \in \mathbf{W}^{l,t}(\Omega)\,, \tag{2.129}$$

and

$$|\mathbf{div}(\boldsymbol{\tau}) - \mathbf{div}(\Pi_h^k(\boldsymbol{\tau}))|_{m,t;\Omega} \le C_1\, h^{l-m}\, |\mathbf{div}(\boldsymbol{\tau})|_{l,t;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{l,t}(\Omega) \text{ with } \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,t}(\Omega)\,, \tag{2.130}$$

whereas for integers l and m verifying $1 \le l \le k+1$ and $0 \le m \le l$, there holds

$$|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})|_{m,t;\Omega} \le C_2\, h^{l-m}\, |\boldsymbol{\tau}|_{l,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{l,t}(\Omega)\,. \tag{2.131}$$

Note that actually (2.130) follows from (2.127) and a direct application of (2.129) to $\mathbf{w} = \mathbf{div}(\boldsymbol{\tau})$. Also, we highlight that (2.129) is first derived for $1 \le l \le k+1$, and then using only the scaling estimates one proves the stability of $\mathcal{P}_h^k$, that is the existence of a positive constant c, independent of h, such that

$$\|\mathcal{P}_h^k(\mathbf{w})\|_{0,t;\Omega} \le c\, \|\mathbf{w}\|_{0,t;\Omega} \qquad \forall\, \mathbf{w} \in \mathbf{L}^t(\Omega)\,. \tag{2.132}$$

In turn, employing the triangle inequality and (2.131) with $l = 1$ and $m = 0$, we conclude the boundedness of $\Pi_h^k : \mathbb{W}^{1,t}(\Omega) \to \mathbb{L}^t(\Omega)$, which means that there exists a positive constant C, independent of h, such that

$$\|\Pi_h^k(\boldsymbol{\tau})\|_{0,t;\Omega} \le C\, \|\boldsymbol{\tau}\|_{1,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{W}^{1,t}(\Omega)\,. \tag{2.133}$$

Finally, taking in particular $m = 0$ in (2.131) and (2.130), we readily find that there exists a positive constant C_3, independent of h, such that for $1 \le l \le k+1$ there holds

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{t,\mathrm{div}_t;\Omega} \le C_3\, h^l \Big\{ |\boldsymbol{\tau}|_{l,t;\Omega} + |\mathbf{div}(\boldsymbol{\tau})|_{l,t;\Omega} \Big\} \tag{2.134}$$

for all $\boldsymbol{\tau} \in \mathbb{W}^{l,t}(\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,t}(\Omega)$.

2.5.2 The finite element subspaces

Appropriate finite element subspaces approximating the unknowns of the pseudostress-based mixed variational formulation for the elasticity problem are defined as follows

$$\begin{aligned}
X_{2,h} &:= X_2 \cap \mathbb{RT}_k(\mathcal{T}_h) := \Big\{ \boldsymbol{\zeta}_h \in \mathbb{H}_0^r(\mathbf{div}_r;\Omega) : \quad \boldsymbol{\zeta}_h|_K \in \mathbb{RT}_k(K) \quad \forall K \in \mathcal{T}_h \Big\}, \\
M_{2,h} &:= M_2 \cap \mathbf{P}_k(\mathcal{T}_h) := \Big\{ \mathbf{v}_h \in \mathbf{L}^s(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h \Big\}, \\
X_{1,h} &:= X_1 \cap \mathbb{RT}_k(\mathcal{T}_h) := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_0^s(\mathbf{div}_s;\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall K \in \mathcal{T}_h \Big\}, \\
M_{1,h} &:= M_1 \cap \mathbf{P}_k(\mathcal{T}_h) := \Big\{ \mathbf{v}_h \in \mathbf{L}^r(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h \Big\}.
\end{aligned} \tag{2.135}$$

In turn, the unknown of the diffusion problem is approximated by Lagrange finite elements of degree $\le k+1$, that is

$$\mathrm{H}_h := \Big\{ \psi_h \in \mathcal{C}(\Omega) \cap \mathrm{H}_0^1(\Omega) : \quad \psi_h|_K \in \mathrm{P}_{k+1}(K) \quad \forall\, K \in \mathcal{T}_h \Big\}. \tag{2.136}$$

Regarding the definitions in (2.135) we stress that, while the pairs $\big(X_{2,h}, M_{2,h}\big)$ and $\big(X_{1,h}, M_{1,h}\big)$ are topologically different, they do coincide algebraically, and hence the stiffness matrices associated to the bilinear forms b_1 and b_2 are exactly the same. Moreover, since $\mathbf{div}(X_{i,h}) \subseteq M_{i,h}$, $i \in \{1, 2\}$, it follows that the corresponding discrete kernels of the bilinear forms b_1 and b_2 coincide as well, and that they are given by the space

$$\mathcal{K}_{h,0}^k := \Big\{ \boldsymbol{\tau}_h \in \mathcal{K}_h^k : \quad \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}_h) = 0 \Big\}, \tag{2.137}$$

where

$$\mathcal{K}_h^k := \Big\{ \boldsymbol{\tau}_h \in \mathbb{RT}_k(\mathcal{T}_h) : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \Big\}. \tag{2.138}$$

Moreover, similarly as derived for the vector version in [39, Lemma 2.1] (see also [52, Lemma 4.1] for a slight variant of it), one can show that

$$\mathcal{K}_h^k = \mathbf{curl}\big(\mathbf{P}_{k+1,0}(\mathcal{T}_h)\big), \tag{2.139}$$

where

$$\mathbf{P}_{k+1,0}(\mathcal{T}_h) := \Big\{ \boldsymbol{\phi}_h \in \mathbf{H}^1(\Omega) : \quad \boldsymbol{\phi}_h|_K \in \mathbf{P}_{k+1}(K) \quad \forall\, K \in \mathcal{T}_h, \quad \int_\Omega \boldsymbol{\phi}_h = \mathbf{0} \Big\},$$

and **curl** is the usual curl operator acting component-wise.

Now, we let $\Theta_h^k : \mathbb{L}^1(\Omega) \to \mathcal{K}_h^k$ be the $\mathrm{L}^2(\Omega)$-orthogonal projector, that is, given $\boldsymbol{\zeta} \in \mathbb{L}^1(\Omega)$, $\Theta_h^k(\boldsymbol{\zeta})$ is the unique element in $\mathcal{K}_h^k$ satisfying

$$\int_\Omega \Theta_h^k(\boldsymbol{\zeta}) : \boldsymbol{\tau}_h \;=\; \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau}_h \qquad \forall\, \boldsymbol{\tau}_h \in \mathcal{K}_h^k\,. \tag{2.140}$$

Then, proceeding analogously to the vector version in [39, Theorem 3.1] (see also [52, Lemma 4.2] for a slight variant of it), and employing now (2.139), it can be proved in the present tensor version that for each $t \in (1, +\infty)$ and for each integer $k \geq 0$, there exist positive constants C_t^k and $\bar{C}_t^k$, independent of h, such that, defining

$$c_t^k \;:=\; \begin{cases} C_t^k & \text{if } \Omega \text{ is convex}\,, \\ \bar{C}_t^k \left\{-\log(h)\right\}^{|1-2/t|} & \text{if } \Omega \text{ is non-convex and } k = 0\,, \\ \bar{C}_t^k & \text{if } \Omega \text{ is non-convex and } k \geq 1\,, \end{cases} \tag{2.141}$$

there holds

$$\|\Theta_h^k(\boldsymbol{\tau})\|_{0,t;\Omega} \;\leq\; c_t^k\, \|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega)\,, \tag{2.142}$$

where

$$\widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega) \;:=\; \Big\{ \boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) = 0 \quad \text{in} \quad \Omega \Big\}\,. \tag{2.143}$$

Whether the boundedness property (2.142) is satisfied or not in 3D is, up to our knowledge, still an open problem, and this fact is precisely the reason why we have restricted the analysis in the present Section 2.5 to the 2D case.

2.5.3 The discrete inf-sup conditions for $\mathbf{S}_h$

In this section we show that the specific finite element subspaces introduced in Section 2.5.2 (cf. (2.135)) verify the hypotheses (**H.1**) and (**H.2**). To this end, we first introduce the deviatoric of $\mathcal{K}_h^k$, that is

$$\mathcal{K}_h^{k,\mathtt{d}} \;:=\; \Big\{ \boldsymbol{\tau}_h^{\mathtt{d}} : \quad \boldsymbol{\tau}_h \in \mathcal{K}_h^k \Big\}\,, \tag{2.144}$$

and let $\Theta_h^{k,\mathtt{d}} : \mathbb{L}^1(\Omega) \to \mathcal{K}_h^{k,\mathtt{d}}$ be the projector defined for each $\boldsymbol{\tau} \in \mathbb{L}^1(\Omega)$ as

$$\int_\Omega \Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau}) : \boldsymbol{\zeta}_h \;=\; \int_\Omega \boldsymbol{\tau} : \boldsymbol{\zeta}_h \qquad \forall\, \boldsymbol{\zeta}_h \in \mathcal{K}_h^{k,\mathtt{d}}\,. \tag{2.145}$$

Then, we have the following identity relating $\Theta_h^{k,\mathtt{d}}$ and Θ_h^k.

Lemma 2.18. *There holds*

$$\Theta_h^{k,\mathtt{d}}\big(\Theta_h^k(\boldsymbol{\tau})\big) \;=\; \big(\Theta_h^k(\boldsymbol{\tau})\big)^{\mathtt{d}} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{L}^1(\Omega)\,. \tag{2.146}$$

Proof. Given $\boldsymbol{\tau} \in \mathbb{L}^1(\Omega)$, it follows from (2.144) and (2.145) that for each $\boldsymbol{\tau}_h \in \mathcal{K}_h^k$ there holds

$$\int_\Omega \Theta_h^{k,\mathtt{d}}\big(\Theta_h^k(\boldsymbol{\tau})\big) : \boldsymbol{\tau}_h^{\mathtt{d}} \;=\; \int_\Omega \Theta_h^k(\boldsymbol{\tau}) : \boldsymbol{\tau}_h^{\mathtt{d}} \;=\; \int_\Omega \big(\Theta_h^k(\boldsymbol{\tau})\big)^{\mathtt{d}} : \boldsymbol{\tau}_h^{\mathtt{d}}\,.$$

Hence, since both $\Theta_h^{k,\mathtt{d}}\big(\Theta_h^k(\boldsymbol{\tau})\big)$ and $\big(\Theta_h^k(\boldsymbol{\tau})\big)^{\mathtt{d}}$ belong to $\mathcal{K}_h^{k,\mathtt{d}}$, the identity (2.146) is concluded. $\square$

We suppose from now on that the operators Θ_h^k satisfy the following asymptotic property: for each $t \in (1,+\infty)$ and for each integer $k \ge 0$ there exists $h_t^k > 0$ such that

$$||| \, \mathrm{I} - \Theta_h^k \, |||_t \; := \sup_{\substack{\boldsymbol{\tau} \in \widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\|\boldsymbol{\tau} - \Theta_h^k(\boldsymbol{\tau})\|_{0,t;\Omega}}{\|\boldsymbol{\tau}\|_{0,t;\Omega}} \; < \; 1 \qquad \forall \, h \, \le h_t^k \,. \tag{2.147}$$

Numerical evidences supporting this assumption are provided later on in Section 2.6.

As a consequence of Lemma 2.18 and (2.147), we are able to provide next the $\mathbb{L}^t(\Omega)$-stability of $\Theta_h^{k,\mathtt{d}}$ when restricted to $\widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega)$.

Lemma 2.19. *For each $t \in (1,+\infty)$ and for each integer $k \ge 0$, there exists a positive constant $c_t^{k,\mathtt{d}}$ such that*

$$\|\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau})\|_{0,t;\Omega} \; \le \; c_t^{k,\mathtt{d}} \, \|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall \, \boldsymbol{\tau} \in \widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega) \,, \quad \forall \, h \le h_t^k \,. \tag{2.148}$$

Proof. Given $\boldsymbol{\tau} \in \widetilde{\mathbb{H}}^t(\mathbf{div}_t;\Omega)$, we first observe, thanks to the idempotence property of $\mathrm{I} - \Theta_h^k$, that

$$\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau}) - \Theta_h^{k,\mathtt{d}}\big(\Theta_h^k(\boldsymbol{\tau})\big) = \Theta_h^{k,\mathtt{d}}\big((\mathrm{I} - \Theta_h^k)(\boldsymbol{\tau})\big) \; = \; \Theta_h^{k,\mathtt{d}}\big((\mathrm{I} - \Theta_h^k)^m(\boldsymbol{\tau})\big) \qquad \forall \, m \in \mathbb{N} \,,$$

from which it follows that

$$\|\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau}) - \Theta_h^{k,\mathtt{d}}\big(\Theta_h^k(\boldsymbol{\tau})\big)\|_{0,t;\Omega} \; \le \; \|\Theta_h^{k,\mathtt{d}}\|_t \, ||| \, \mathrm{I} - \Theta_h^k \, |||_t^m \, \|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall \, m \in \mathbb{N} \,, \tag{2.149}$$

where

$$\|\Theta_h^{k,\mathtt{d}}\|_t \; := \sup_{\substack{\boldsymbol{\tau} \in \mathbb{L}^1(\Omega) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\|\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau})\|_{0,t;\Omega}}{\|\boldsymbol{\tau}\|_{0,t;\Omega}} \,.$$

In this way, invoking (2.147), taking $\lim\limits_{m \to +\infty}$ in (2.149), and employing (2.146) (cf. Lemma 2.18), we conclude that

$$\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau}) \; = \; \big(\Theta_h^k(\boldsymbol{\tau})\big)^{\mathtt{d}} \qquad \forall \, h \le h_t^k \,. \tag{2.150}$$

On the other hand, simple algebraic computations and (2.51) give

$$\|\mathrm{tr}\,(\boldsymbol{\tau})\,\mathbb{I}\|_{0,t;\Omega} \; = \; n^{1/t} \, \|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,t;\Omega} \; \le \; n \, \|\boldsymbol{\tau}\|_{0,t;\Omega} \,,$$

which readily implies

$$\|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,t;\Omega} \; \le \; 2 \, \|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall \, \boldsymbol{\tau} \in \mathbb{L}^t(\Omega) \,. \tag{2.151}$$

Hence, employing (2.151) and (2.142), we deduce from (2.150) that

$$\|\Theta_h^{k,\mathtt{d}}(\boldsymbol{\tau})\|_{0,t;\Omega} \; \le \; 2 \, c_t^k \, \|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall \, h \le h_t^k \,, \tag{2.152}$$

which constitutes the required inequality (2.148) with $c_t^{k,\mathtt{d}} = 2\,c_t^k$. □

Having proved Lemma 2.19, we proceed in what follows to establish the discrete analogues of Lemmas 2.4 and 2.5, for which we suitably adapt their respective proofs to the present context. We begin with the discrete inf-sup conditions for a.

Lemma 2.20. *Assume that r and s satisfy the final ranges specified by (2.88) and (2.89), that is $r \in \big(2, \frac{2n}{n-1}\big]$ and $s \in \big[\frac{2n}{n+1}, 2\big)$. Then, there exist positive constants M_{d} and α_{d} such that for each $\lambda > M_{\mathrm{d}}$ and for each $h \le h_0 := \min\big\{h_r^k, h_s^k\big\}$, there hold*

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathcal{K}_{h,0}^k\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{X_1}} \,\ge\, \alpha_{\mathrm{d}}\,\|\boldsymbol{\zeta}_h\|_{X_2} \qquad \forall\,\boldsymbol{\zeta}_h\in\mathcal{K}_{h,0}^k\,, \tag{2.153}$$

and

$$\sup_{\boldsymbol{\zeta}_h\in\mathcal{K}_{h,0}^k} a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h) \,>\, 0 \qquad \forall\,\boldsymbol{\tau}_h\in\mathcal{K}_{h,0}^k,\ \boldsymbol{\tau}_h\neq\mathbf{0}\,. \tag{2.154}$$

Proof. Similarly to the proof of Lemmas 2.4, we first observe that, given $\boldsymbol{\zeta}_h\in\mathcal{K}_{h,0}^k$, there holds the discrete analogue of (2.63), namely

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathcal{K}_{h,0}^k\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{X_1}} \,\ge\, \frac{1}{\mu}\sup_{\substack{\boldsymbol{\tau}_h\in\mathcal{K}_{h,0}^k\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}}:\boldsymbol{\tau}_h^{\mathtt{d}}}{\|\boldsymbol{\tau}_h\|_{X_1}} \,-\, \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda+(n+1)\mu\big)}\,\|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,r;\Omega}\,, \tag{2.155}$$

whence the rest of the proof reduces to get a suitable lower bound for the supremum on the right hand side of (2.155). To this end, we proceed as in (2.64) and set

$$\boldsymbol{\zeta}_{h,s} \,:=\, \begin{cases} |\boldsymbol{\zeta}_h^{\mathtt{d}}|^{r-2}\,\boldsymbol{\zeta}_h^{\mathtt{d}} & \text{if} \quad \boldsymbol{\zeta}_h^{\mathtt{d}}\neq\mathbf{0},\\ \mathbf{0} & \text{if} \quad \boldsymbol{\zeta}_h^{\mathtt{d}}=\mathbf{0}\,, \end{cases} \tag{2.156}$$

which belongs to $\mathbb{L}^s(\Omega)$ and satisfies (cf. (2.65))

$$\int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}}:\boldsymbol{\zeta}_{h,s} \,=\, \|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,r;\Omega}^r \,=\, \|\boldsymbol{\zeta}_{h,s}\|_{0,s;\Omega}^s \,=\, \|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,r;\Omega}\,\|\boldsymbol{\zeta}_{h,s}\|_{0,s;\Omega}\,. \tag{2.157}$$

Then, we recall the definition of the operator D_s (cf. Lemma 2.2) and let $\widetilde{\boldsymbol{\tau}}_h\in\mathcal{K}_h^k$ (cf. (2.138)) such that $\widetilde{\boldsymbol{\tau}}_h^{\mathtt{d}} = \Theta_h^{k,\mathtt{d}}(D_s(\boldsymbol{\zeta}_{h,s}))\in\mathcal{K}_h^{k,\mathtt{d}}$ (cf. (2.144)). In this way, defining the constant

$$c_h \,:=\, \frac{1}{n|\Omega|}\int_\Omega \operatorname{tr}(\widetilde{\boldsymbol{\tau}}_h)\in\mathrm{R}\,,$$

it follows that $\widetilde{\boldsymbol{\tau}}_h - c_h\mathbb{I}\in\mathcal{K}_{h,0}^k$, and hence

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathcal{K}_{h,0}^k\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}}:\boldsymbol{\tau}_h^{\mathtt{d}}}{\|\boldsymbol{\tau}_h\|_{X_1}} \,\ge\, \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}}:(\widetilde{\boldsymbol{\tau}}_h-c_h\mathbb{I})^{\mathtt{d}}}{\|\widetilde{\boldsymbol{\tau}}_h-c_h\mathbb{I}\|_{0,s;\Omega}} \,=\, \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}}:\widetilde{\boldsymbol{\tau}}_h^{\mathtt{d}}}{\|\widetilde{\boldsymbol{\tau}}_h-c_h\mathbb{I}\|_{0,s;\Omega}}\,. \tag{2.158}$$

Now, employing the characterization of $\Theta_h^{k,\mathtt{d}}$ (cf. (2.145)), the identity (2.48) satisfied by D_s, and (2.157), we find that

$$\begin{aligned} \int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}}:\widetilde{\boldsymbol{\tau}}_h^{\mathtt{d}} \,&=\, \int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}}:\Theta_h^{k,\mathtt{d}}(D_s(\boldsymbol{\zeta}_{h,s})) \,=\, \int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}}:D_s(\boldsymbol{\zeta}_{h,s})\\ &=\, \int_\Omega \boldsymbol{\zeta}_h^{\mathtt{d}}:\boldsymbol{\zeta}_{h,s} \,=\, \|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,r;\Omega}\,\|\boldsymbol{\zeta}_{h,s}\|_{0,s;\Omega}\,. \end{aligned} \tag{2.159}$$

In turn, applying (2.50) (cf. Lemma 2.3) to $\widetilde{\boldsymbol{\tau}}_h - c_h\mathbb{I}$, and making use of the boundedness of $\Theta_h^{k,\mathtt{d}}$ (cf. (2.148)) and D_s (cf. Lemma 2.2), we get

$$\begin{aligned} \|\widetilde{\boldsymbol{\tau}}_h - c_h\mathbb{I}\|_{0,s;\Omega} \,\le\, \widehat{C}_s\,\|\widetilde{\boldsymbol{\tau}}_h^{\mathtt{d}}\|_{0,s;\Omega} \,=\, \widehat{C}_s\,\|\Theta_h^{k,\mathtt{d}}(D_s(\boldsymbol{\zeta}_{h,s}))\|_{0,s;\Omega} \\ \le\, \widehat{C}_s\,c_s^{k,\mathtt{d}}\,\|D_s\|\,\|\boldsymbol{\zeta}_{h,s}\|_{0,s;\Omega} \qquad \forall\, h \le h_s^k\,. \end{aligned} \tag{2.160}$$

Therefore, replacing (2.159) and (2.160) back into (2.158), and then the resulting estimate in (2.155), we arrive at

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathcal{K}_{h,0}^k\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{X_1}} \,\ge\, \left\{\frac{1}{\mu\,\widehat{C}_s\,c_s^{k,\mathtt{d}}\|D_s\|} \,-\, \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda+(n+1)\mu\big)}\right\}\|\boldsymbol{\zeta}_h^{\mathtt{d}}\|_{0,r;\Omega} \qquad \forall\, h\le h_s^k\,, \tag{2.161}$$

from which, choosing λ sufficiently large such that

$$\frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda+(n+1)\mu\big)} \,<\, \frac{1}{2\,\mu\,\widehat{C}_s\,c_s^{k,\mathtt{d}}\,\|D_s\|}\,,$$

that is

$$\lambda \,>\, M_{s,\mathrm{d}} := \frac{\mu}{n^{1+1/s}}\max\left\{2\,\mu\,\widehat{C}_s\,\widetilde{C}_r\,c_s^{k,\mathtt{d}}\,\|D_s\| - n^{1/s}(n+1), 0\right\}\,,$$

and applying (2.50) to $\boldsymbol{\zeta}_h$, we conclude (2.153), with $\alpha_{\mathrm{d}} := \frac{1}{2\,\mu\,\widehat{C}_s\,\widetilde{C}_r\,c_s^{k,\mathtt{d}}\,\|D_s\|}$, for each $h\le h_s^k$. Similarly, given $\boldsymbol{\tau}_h\in\mathcal{K}_{h,0}^k$, $\boldsymbol{\tau}_h\neq 0$, we proceed analogously as above, but exchanging the roles of $\boldsymbol{\tau}_h$ and $\boldsymbol{\zeta}_h$, and obtain

$$\sup_{\boldsymbol{\zeta}_h\in\mathcal{K}_{h,0}^k} a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h) \,\ge\, \sup_{\substack{\boldsymbol{\zeta}_h\in\mathcal{K}_{h,0}^k\\ \boldsymbol{\zeta}_h\neq\mathbf{0}}} \frac{a(\boldsymbol{\zeta}_h,\boldsymbol{\tau}_h)}{\|\boldsymbol{\zeta}_h\|_{X_2}} \,\ge\, \frac{1}{2\,\mu\,\widehat{C}_r\,\widehat{C}_s\,c_r^{k,\mathtt{d}}\,\|D_r\|}\,\|\boldsymbol{\tau}_h\|_{X_1} \,>\, 0 \qquad \forall\, h\le h_r^k\,, \tag{2.162}$$

for

$$\lambda \,>\, M_{r,\mathrm{d}} \,:=\, \frac{\mu}{n^{1+1/r}}\max\left\{2\,\mu\,\widehat{C}_r\,\widetilde{C}_s\,c_r^{k,\mathtt{d}}\,\|D_r\| - n^{1/r}(n+1), 0\right\}\,,$$

which proves (2.154) for each $h\le h_r^k$. Finally, defining $M_{\mathtt{d}} := \max\{M_{s,\mathtt{d}}, M_{r,\mathtt{d}}\}$, the proof is completed. □

The discrete inf-sup conditions for the bilinear forms b_i, $i\in\{1,2\}$, are provided next.

Lemma 2.21. *Assume that r and s satisfy the final ranges specified by (2.88) and (2.89), that is $r\in\big(2,\frac{2n}{n-1}\big]$ and $s\in\big[\frac{2n}{n+1},2\big)$. Then, there exist positive constants $\beta_{1,\mathrm{d}}$, $\beta_{2,\mathrm{d}}$, independent of h, such that for each $i\in\{1,2\}$ there holds*

$$\sup_{\substack{\boldsymbol{\tau}_h\in X_{i,h}\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{b_i(\boldsymbol{\tau}_h,\mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{X_i}} \,\ge\, \beta_{i,\mathrm{d}}\,\|\mathbf{v}_h\|_{M_i} \qquad \forall\,\mathbf{v}_h\in M_{i,h}\,. \tag{2.163}$$

Proof. We adapt the proof of Lemma 2.5 to show (2.163) only for $i=2$ since the case $i=1$ is analogous. Indeed, given $\mathbf{v}_h\in M_{2,h}\subseteq M_2=\mathbf{L}^s(\Omega)$, we follow (2.70) and define first

$$\mathbf{v}_{h,r} \,:=\, \begin{cases} |\mathbf{v}_h|^{s-2}\,\mathbf{v}_h & \text{if} \quad \mathbf{v}_h\neq\mathbf{0}\,,\\ \mathbf{0} & \text{if} \quad \mathbf{v}_h=\mathbf{0}\,, \end{cases} \tag{2.164}$$

which belongs to $\mathbf{L}^r(\Omega)$ and, as in (2.71), satisfies

$$\int_\Omega \mathbf{v}_h \cdot \mathbf{v}_{h,r} \,=\, \|\mathbf{v}_h\|_{0,s;\Omega}^s \,=\, \|\mathbf{v}_{h,r}\|_{0,r;\Omega}^r \,=\, \|\mathbf{v}_h\|_{0,s;\Omega}\,\|\mathbf{v}_{h,r}\|_{0,r;\Omega}\,. \tag{2.165}$$

Next, proceeding similarly to the proof of [52, Lemma 4.5], we let $\mathcal{O}$ be a bounded convex polygonal domain containing $\bar\Omega$, and introduce

$$\mathbf{g} \,=\, \begin{cases} \mathbf{v}_{h,r} & \text{in} \quad \Omega\,, \\ \mathbf{0} & \text{on} \quad \mathcal{O}\backslash\bar\Omega\,, \end{cases} \tag{2.166}$$

which is clearly seen to belong to $\mathbf{L}^r(\mathcal{O})$ with $\|\mathbf{g}\|_{0,r;\mathcal{O}} \,=\, \|\mathbf{v}_{h,r}\|_{0,r;\Omega}$. Then, applying the elliptic regularity result provided in [43, Corollary 1], we deduce that there exists a unique $\mathbf{z} \in \mathbf{W}^{2,r}(\mathcal{O}) \cap \mathbf{W}_0^{1,r}(\mathcal{O})$ solution of

$$\Delta\mathbf{z} \,=\, \mathbf{g} \quad \text{in} \quad \mathcal{O}\,, \qquad \mathbf{z} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{O}\,, \tag{2.167}$$

and that there exists a positive constant $C_{\mathtt{reg}}$, depending only on $\mathcal{O}$, such that

$$\|\mathbf{z}\|_{2,r;\mathcal{O}} \,\le\, C_{\mathtt{reg}}\,\|\mathbf{v}_{h,r}\|_{0,r;\Omega}\,. \tag{2.168}$$

In this way, defining now $\boldsymbol{\zeta} := \nabla\mathbf{z}|_\Omega \in \mathbf{W}^{1,r}(\Omega)$, it follows from (2.166), (2.167), and (2.168) that

$$\mathbf{div}(\boldsymbol{\zeta}) \,=\, \mathbf{v}_{h,r} \quad \text{in} \quad \Omega \quad \text{and} \quad \|\boldsymbol{\zeta}\|_{1,r;\Omega} \,\le\, C_{\mathtt{reg}}\,\|\mathbf{v}_{h,r}\|_{0,r;\Omega}\,. \tag{2.169}$$

Thus, letting $\boldsymbol{\zeta}_h$ be the $\mathbb{H}_0^r(\mathbf{div}_r;\Omega)$-component (cf. (2.31)) of $\Pi_h^k(\boldsymbol{\zeta})$, and employing the commuting diagram property (2.127) and the identity from (2.169), we observe that

$$\mathbf{div}(\boldsymbol{\zeta}_h) \,=\, \mathbf{div}(\Pi_h^k(\boldsymbol{\zeta})) \,=\, \mathcal{P}_h^k(\mathbf{div}(\boldsymbol{\zeta})) \,=\, \mathcal{P}_h^k(\mathbf{v}_{h,r}) \quad \text{in} \quad \Omega\,, \tag{2.170}$$

so that, applying the stability estimate of $\mathcal{P}_h^k$ (cf. (2.132)), it follows that

$$\|\mathbf{div}(\boldsymbol{\zeta}_h)\|_{0,r;\Omega} \,\le\, c\,\|\mathbf{v}_{h,r}\|_{0,r;\Omega}\,. \tag{2.171}$$

On the other hand, according to (2.31) and the notations introduced there, and using the triangle and Holder inequality, and (2.51), it is easy to show that for each $t \in (1,+\infty)$ there holds

$$\|\boldsymbol{\tau}_0\|_{0,t;\Omega} \,\le\, 2\,\|\boldsymbol{\tau}\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega)\,. \tag{2.172}$$

Hence, employing now (2.172), the stability estimate of Π_h^k (cf. (2.133)), and the inequality from (2.169), we find that

$$\|\boldsymbol{\zeta}_h\|_{0,r;\Omega} \,\le\, 2\,\|\Pi_h^k(\boldsymbol{\zeta})\|_{0,r;\Omega} \,\le\, 2\,C\,\|\boldsymbol{\zeta}\|_{1,r;\Omega} \,\le\, 2\,C\,C_{\mathtt{reg}}\,\|\mathbf{v}_{h,r}\|_{0,r;\Omega}\,, \tag{2.173}$$

which, jointly with (2.171), yield the existence of a positive constant $\widehat{C}$, independent of h, such that (cf. (2.25))

$$\|\boldsymbol{\zeta}_h\|_{X_2} \,=\, \|\boldsymbol{\zeta}_h\|_{0,r;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\zeta}_h)\|_{0,r;\Omega} \,\le\, \widehat{C}\,\|\mathbf{v}_{h,r}\|_{0,r;\Omega}\,. \tag{2.174}$$

Finally, bearing in mind (2.170), (2.128), (2.165), and (2.174), we obtain

$$\begin{aligned} \sup_{\substack{\boldsymbol{\tau}_h \in X_{2,h} \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{b_2(\boldsymbol{\tau}_h,\mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{X_2}} \,&\ge\, \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\zeta}_h)}{\|\boldsymbol{\zeta}_h\|_{X_2}} \,=\, \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathcal{P}_h^k(\mathbf{v}_{h,r})}{\|\boldsymbol{\zeta}_h\|_{X_2}} \\ &=\, \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{v}_{h,r}}{\|\boldsymbol{\zeta}_h\|_{X_2}} \,=\, \frac{\|\mathbf{v}_h\|_{0,s;\Omega}\,\|\mathbf{v}_{h,r}\|_{0,r;\Omega}}{\|\boldsymbol{\zeta}_h\|_{X_2}} \,\ge\, \frac{1}{\widehat{C}}\,\|\mathbf{v}_h\|_{M_2}\,, \end{aligned}$$

which yields (2.163) for $i = 2$ with $\beta_{2,\mathrm{d}} := \frac{1}{\widehat{C}}$. □

2.5.4 The rates of convergence

The rates of convergence of the Galerkin scheme (2.100) with the specific finite element subspaces introduced in Section 2.5.2 are provided next. To this end, we first collect the approximation properties of $X_{2,h}$ and $M_{1,h}$ (cf. (2.135)), which follow from (2.134) (for $t = r$) and (2.129) (for $m = 0$ and $t = r$), respectively, along with interpolation estimates of Sobolev spaces. More precisely, they are given as follows:

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists $C > 0$, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{W}^{l,r}(\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\tau}, X_{2,h}) := \inf_{\boldsymbol{\tau}_h \in X_{2,h}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{r,\operatorname{div}_r;\Omega} \le C\, h^l \left\{ \|\boldsymbol{\tau}\|_{l,r;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{l,r;\Omega} \right\}.$$

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}(\mathbf{v}, M_{1,h}) := \inf_{\mathbf{v}_h \in M_{1,h}} \|\mathbf{v} - \mathbf{v}_h\|_{0,r;\Omega} \le C\, h^l\, \|\mathbf{v}\|_{l,r;\Omega}.$$

In turn, the approximation property of H_h, which makes use of interpolation estimates of Sobolev spaces as well, is stated as indicated below (cf. [41, Corollary 1.109]):

$(\mathbf{AP}_h^{\phi})$ there exists $C > 0$, independent of h, such that for each $l \in (0, k+1]$, and for each $\varphi \in \mathrm{H}^{l+1}(\Omega)$, there holds

$$\operatorname{dist}(\varphi, \mathrm{H}_h) := \inf_{\varphi_h \in \mathrm{H}_h} \|\varphi - \varphi_h\|_{1,\Omega} \le C\, h^l\, \|\varphi\|_{l+1,\Omega}.$$

Consequently, we can state the following main theorem.

Theorem 2.22. *Let $(\boldsymbol{\sigma}, \mathbf{u}, \phi) \in X_2 \times M_1 \times \mathrm{H}_0^1(\Omega)$ be the unique solution of (2.39) with $\phi \in W$ (cf. (2.82)), and let $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h) \in X_{2,h} \times M_{1,h} \times \mathrm{H}_h$ be a solution of (2.100) with $\phi_h \in W_h$ (cf. (2.109)), whose existences are guaranteed by Theorems 2.12 and 2.16, respectively. Assume that (2.125) (cf. Theorem 2.17) holds, and that there exists $l \in [1, k+1]$ such that $\boldsymbol{\sigma} \in \mathbb{W}^{l,r}(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,r}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{l,r}(\Omega)$ and $\phi \in \mathrm{H}^{l+1}(\Omega)$. Then there exists a constant $C > 0$, independent of h, such that*

$$\begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{X_2} + \|\mathbf{u} - \mathbf{u}_h\|_{M_1} + \|\phi - \phi_h\|_{1,\Omega} \\ &\qquad \le C\, h^l \left\{ \|\boldsymbol{\sigma}\|_{l,r;\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,r;\Omega} + \|\mathbf{u}\|_{l,r;\Omega} + \|\phi\|_{l+1,\Omega} \right\}. \end{aligned}$$

Proof. It follows directly from the Céa estimate (2.126) and the above approximation properties. □

2.6 Numerical results

In this section we report numerical experiments illustrating the performance of the Galerkin scheme (2.100) with the specific finite element spaces defined in (2.135), and confirming the theoretical rates of convergence provided by Theorem 2.22. We begin by recalling that part of the analysis developed in Section 2.5, namely the one referring to the discrete inf-sup conditions for the bilinear form a, depends on the hypothesis (2.147), which establishes an asymptotic behavior of the operators Θ_h^k. However, since proving this assumption has remained elusive, in what follows we present numerical evidence supporting its eventual validity. To this end, we now consider the convex and non-convex domains given by $\Omega_S := (0,1)^2$ and $\Omega_L := (-1,1)^2 \setminus [0,1]^2$, respectively, and let $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and $\boldsymbol{\tau}_3$ be the tensor fields defined for each $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega_S \cup \Omega_L$ as:

$$\boldsymbol{\tau}_1 \;:=\; \mathbf{curl}\begin{pmatrix} \exp(-x_1^2 - x_2^2) \\ \exp(-x_1 x_2) \end{pmatrix} \;=\; \begin{pmatrix} -2x_2\exp(-x_1^2 - x_2^2) & 2x_1\exp(-x_1^2 - x_2^2) \\ -x_1\exp(-x_1 x_2) & x_2\exp(-x_1 x_2) \end{pmatrix},$$

$$\boldsymbol{\tau}_2 \;:=\; \mathbf{curl}\begin{pmatrix} \pi^{-1}\sin(\pi x_1)\cos(\pi x_2) \\ \pi^{-1}\cos(\pi x_1)\sin(\pi x_2) \end{pmatrix} \;=\; \begin{pmatrix} -\sin(\pi x_1)\sin(\pi x_2) & -\cos(\pi x_1)\cos(\pi x_2) \\ \cos(\pi x_1)\cos(\pi x_2) & \sin(\pi x_1)\sin(\pi x_2) \end{pmatrix}$$

and

$$\boldsymbol{\tau}_3 \;:=\; \mathbf{curl}\left(\frac{1}{3}\Big\{(x_1-2)^2 + (x_2-2)^2\Big\}^{3/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \;=\; \sqrt{(x_1-2)^2 + (x_2-2)^2}\begin{pmatrix} x_2 - 2 & 2 - x_1 \\ x_2 - 2 & 2 - x_1 \end{pmatrix},$$

which are clearly all divergence-free. Then, for $p = \frac{4}{3}$, $k \in \{0,1\}$, and five regular triangulations $\mathcal{T}_h$ of Ω_S and Ω_L, respectively, we compute the expressions

$$\mathsf{c}_{h,S}^k(\boldsymbol{\tau}) \;:=\; \frac{\|\boldsymbol{\tau} - \Theta_h^k(\boldsymbol{\tau})\|_{0,p;\Omega_S}}{\|\boldsymbol{\tau}\|_{0,p;\Omega_S}} \quad \text{and} \quad \mathsf{c}_{h,L}^k(\boldsymbol{\tau}) \;:=\; \frac{\|\boldsymbol{\tau} - \Theta_h^k(\boldsymbol{\tau})\|_{0,p;\Omega_L}}{\|\boldsymbol{\tau}\|_{0,p;\Omega_L}} \quad \forall\, \boldsymbol{\tau} \in \big\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3\big\},$$

which are displayed below in Table 2.1. We notice there that these values remain not only below 1, as demanded by (2.147), but they actually approach 0 as the meshsize h tends to 0, thus satisfying by far the required upper bound. The same behavior was observed in many other examples, and hence we actually have no evidence casting doubt on the validity of this assumption. However, irrespective of the above, we stress that (2.147) is employed in the proof of Lemma 2.19 only to obtain the identity (2.150), and hence, even in the unlikely case that (2.147) does not hold, one might still have the chance to arrive to (2.150) through another argument.

Next, we consider the finite element subspaces defined in (2.135) with $k \in \{0,1,2\}$, to illustrate the performance of the mixed-primal finite element scheme (2.100) and confirm the rates of convergence provided by Theorem 2.22, through three numerical examples. We begin by noticing that the total number of degrees of freedom (or unknowns) of (2.100) is given for $n = 2$ by

$$\begin{aligned} N := \big\{\text{number of nodes of } \mathcal{T}_h\big\} \,+\, \big(2(k+1)+k\big) \times \big\{\text{number of edges of } \mathcal{T}_h\big\} \\ + \big(2k(k+1) + (k+1)(k+2) + \tfrac{1}{2}k(k-1)\big) \times \big\{\text{number of elements of } \mathcal{T}_h\big\} \,+\, 1\,, \end{aligned}$$

whereas for $n = 3$ it becomes

$$\begin{aligned} N := \big\{\text{number of nodes of } \mathcal{T}_h\big\} \,+\, k \times \big\{\text{number of edges of } \mathcal{T}_h\big\} \\ +\, \big(2k^2 + 4k + 3\big) \times \big\{\text{number of faces of } \mathcal{T}_h\big\} \\ +\, \big(\tfrac{13k^3 + 42k^2 + 53k + 18}{6}\big) \times \big\{\text{number of elements of } \mathcal{T}_h\big\} \,+\, 1\,. \end{aligned}$$

Now, regarding the resolution itself of (2.100), we remark that the null integral mean condition for the traces of tensors in the space $X_{2,h}$ (cf. (2.135)) is imposed via a real Lagrange multiplier, and that the

	Ω_S			Ω_L		
$\boldsymbol{\tau}$	h	$\mathtt{c}^0_{h,S}(\boldsymbol{\tau})$	$\mathtt{c}^1_{h,S}(\boldsymbol{\tau})$	h	$\mathtt{c}^0_{h,L}(\boldsymbol{\tau})$	$\mathtt{c}^1_{h,L}(\boldsymbol{\tau})$
$\boldsymbol{\tau}_1$	0.1414	6.91e-02	1.58e-03	0.1414	8.43e-02	2.21e-03
	0.0707	3.49e-02	4.00e-04	0.0471	2.85e-02	2.50e-04
	0.0471	2.33e-02	1.78e-04	0.0283	1.71e-02	9.05e-05
	0.0202	1.00e-02	3.29e-05	0.0202	1.22e-02	4.63e-05
	0.0109	5.40e-03	9.56e-06	0.0177	1.07e-02	3.54e-05
$\boldsymbol{\tau}_2$	0.1414	1.52e-01	8.71e-03	0.1414	1.53e-01	8.79e-03
	0.0707	7.66e-02	2.21e-03	0.0471	5.11e-02	9.88e-04
	0.0471	5.11e-02	9.85e-04	0.0283	3.07e-02	3.56e-04
	0.0202	2.19e-02	1.82e-04	0.0202	2.19e-02	1.82e-04
	0.0109	1.18e-02	5.28e-05	0.0177	1.92e-02	1.39e-04
$\boldsymbol{\tau}_3$	0.1414	2.07e-02	1.63e-04	0.1414	1.44e-02	7.89e-05
	0.0707	1.04e-02	4.08e-05	0.0471	4.81e-03	8.80e-06
	0.0471	6.90e-03	1.82e-05	0.0283	2.89e-03	3.17e-06
	0.0202	2.96e-03	3.34e-06	0.0202	2.06e-03	1.62e-06
	0.0109	1.59e-03	9.68e-07	0.0177	1.81e-03	1.24e-06

Table 2.1: Numerical evidence eventually supporting (2.147).

nonlinear algebraic systems obtained are solved following the discrete fixed-point strategy suggested by (2.104), whose computational implementation is given by a C^{++} code. We take as initial guess the trivial solution, and remark in advance that for each one of the examples to be reported below, three iterations are required to achieve a tolerance of 10^{-6}.

Furthermore, given r as specified in (2.88), we introduce the individual errors:

$$\mathsf{e}(\boldsymbol{\sigma}) \;:=\; \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{r,\mathbf{div}_r;\Omega}\,, \qquad \mathsf{e}(\mathbf{u}) \;:=\; \|\mathbf{u} - \mathbf{u}_h\|_{0,r;\Omega}\,,$$

$$\mathsf{e}(\phi) \;:=\; \|\phi - \phi_h\|_{1,\Omega} \qquad \text{and} \qquad \mathsf{e}(\boldsymbol{\rho}) \;:=\; \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,r;\Omega}\,,$$

where, according to (2.9) and (2.32), $\boldsymbol{\rho}_h$ is computed as:

$$\boldsymbol{\rho}_h \;:=\; \boldsymbol{\sigma}_h \,+\, \boldsymbol{\sigma}_h^{\mathrm{t}} \,-\, \left(\frac{\lambda+2\mu}{n\lambda+(n+1)\mu}\,\mathrm{tr}(\boldsymbol{\sigma}_h) \,-\, \frac{n\lambda+2\mu}{n|\Omega|}\int_\Gamma \mathbf{u}_D\cdot\boldsymbol{\nu}\right)\mathbb{I}\,. \tag{2.175}$$

In this way, the respective experimental rates of convergence are defined as:

$$\mathtt{r}(*) \;:=\; \frac{\log(\mathsf{e}(*)\,/\,\mathsf{e}'(*))}{\log(h\,/\,h')} \qquad \forall\, * \in \big\{\boldsymbol{\sigma}, \mathbf{u}, \phi, \boldsymbol{\rho}\big\}\,,$$

where $\mathsf{e}(*)$ and $\mathsf{e}'(*)$ denote errors computed on two consecutive meshes of sizes h and h', respectively.

In what follows we proceed to report on the numerical experiments obtained. The first example uses a smooth manufactured solution to illustrate that the optimal rates of convergence of our method are indeed attained in this case. The second one considers a singular solution to confirm that precisely the lack of smoothness directly affects the order of convergence. Finally, and while, as shown in Section 2.5, the discrete analysis using the specific finite element subspaces introduced in Section 2.5.2 has been guaranteed only in 2D, the third example illustrates the applicability of the method to a three-dimensional problem as well. In each case we let E and ν be the Young modulus and Poisson ratio, respectively, of the isotropic linear elastic solid occupying the region Ω, so that the corresponding Lamé parameters are given by:

$$\mu \;:=\; \frac{E}{2(1+\nu)} \qquad \text{and} \qquad \lambda \;:=\; \frac{E\nu}{(1+\nu)(1-2\nu)}\,. \tag{2.176}$$

While the three examples to be reported here are manufactured ones aiming to illustrate the aforementioned objectives, we stress that the Galerkin scheme (2.100) can certainly be applied as well to problems coming from applications, such as those reported in [46] and [47]. Indeed, because of the singularities and complex geometries that some of them might involve, we plan to consider these applied examples in a forthcoming work addressing the a posteriori error analysis of the present mixed-primal finite element method. As it is well-known, adaptive strategies based on a posteriori error indicators have shown to be very suitable for handling those situations.

Example 1. We consider the very same example from [46, Example 1, Section 5], which means that we let $\Omega = (0,1)^2$, and adequately manufacture the data so that the exact solution of (2.1) is given by the smooth functions

$$\mathbf{u}(\mathbf{x}) \;=\; \begin{pmatrix} \frac{1}{20}\sin(\pi x_1)\cos(\pi x_2) + \frac{1}{2\lambda}x_1^2 \\ \\ \frac{1}{20}\cos(\pi x_1)\sin(\pi x_2) + \frac{1}{2\lambda}x_2^2 \end{pmatrix} \quad \text{and} \quad \phi(\mathbf{x}) \;=\; x_1x_2(x_1-1)(x_2-1)\,,$$

for all $\mathbf{x} := (x_1,x_2)^{\mathtt{t}} \in \Omega$, whereas the body load, the diffusive source, and the tensorial diffusivity, are given, respectively, by

$$\mathbf{f}(\phi) \;=\; \begin{pmatrix} \frac{1}{10}\cos^2(\phi) \\ \\ -\frac{1}{10}\sin(\phi) \end{pmatrix}, \quad g(\mathbf{u}) \;=\; \frac{1}{10}\left(1 + \frac{1}{1+|\mathbf{u}|}\right) \quad \text{and} \quad \widetilde{\vartheta}(\boldsymbol{\rho}) \;=\; \mathbb{I} + \frac{1}{10}\boldsymbol{\rho}^2\,.$$

It is important to remark here that the second and fifth equations of (2.12) actually include additional explicit source terms that are added to $\mathbf{f}(\phi)$ and $g(\mathbf{u})$, respectively. However, yielding only slight modifications of the functionals F_ϕ and $G_{\mathbf{u}}$ in (2.39), this fact does not compromise the continuous and discrete analyses. In addition, we take Young's modulus $E = 10^3$ and Poisson's ratio $\nu = 0.4$, which, according to (2.176), implies that $\mu = 357.1429$ and $\lambda = 1428.5714$. Thus, in Tables 2.2 and 2.3 we summarize the convergence history of the Galerkin scheme (2.100) with $r = 3$ and $r = 4$, respectively. In particular, we stress that the optimal order of convergence $O(h^{k+1})$ predicted by Theorem 2.22 is attained by all the unknowns. Some components and magnitudes of the discrete solutions are displayed in Figure 2.1. Furthermore, in order to compare our discrete scheme (2.100) with those proposed in [46], beyond the fact that they all confirm their theoretical rates of convergence, we first point out that the unknowns $\boldsymbol{\sigma}$ from the present chapter and [46] do not coincide, and hence their numerical approximations $\boldsymbol{\sigma}_h$ and associated errors are not comparable. Actually, the stress $\boldsymbol{\sigma}$ from [46] corresponds to our $\boldsymbol{\rho}$, whose discrete approximation $\boldsymbol{\rho}_h$ (cf. (2.175)) lies only in $\mathbb{L}^r(\Omega)$. Consequently, we extract from Tables 2.2 and 2.3, and [46, Table 1], the necessary information to display in Figures 2.2 and 2.3 the error history for the unknowns $\mathbf{u}$ and ϕ only. The methods from [46] are referred to as "PEERS-Lagrange scheme with $k = 0$", "Augmented scheme with $k = 0$", and "Augmented scheme with $k = 1$", whereas those regarding (2.100) are named "Pseudostress-based scheme with $k = 0$" and "Pseudostress-based scheme with $k = 1$", additionally indicating for the latter the value of r (and hence of s) with which the corresponding norms are defined. Nevertheless, we

k	h	N	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\rho}$)	r($\boldsymbol{\rho}$)
	0.0333	19982	5.26e+01	−−	9.33e-04	−−	5.09e-03	−−	2.07e+01	−−
	0.0270	30342	4.26e+01	1.00	7.54e-04	1.02	4.19e-03	0.93	1.68e+01	1.00
	0.0217	46830	3.43e+01	1.00	6.05e-04	1.01	3.40e-03	0.96	1.35e+01	1.00
0	0.0185	64478	2.92e+01	1.00	5.14e-04	1.01	2.90e-03	0.98	1.15e+01	1.00
	0.0164	82230	2.59e+01	1.00	4.55e-04	1.01	2.57e-03	1.00	1.02e+01	1.00
	0.0139	114482	2.19e+01	1.00	3.85e-04	1.00	2.18e-03	1.00	8.63e+00	1.00
	0.0122	148422	1.92e+01	1.00	3.38e-04	1.00	1.91e-03	1.00	7.58e+00	1.00
	0.0333	65162	6.67e-01	−−	1.21e-05	−−	9.84e-05	−−	2.39e-01	−−
	0.0270	99014	4.38e-01	2.00	7.93e-06	2.01	6.49e-05	1.99	1.58e-01	1.99
1	0.0217	152906	2.84e-01	2.00	5.13e-06	2.00	4.27e-05	1.92	1.02e-01	1.99
	0.0185	210602	2.06e-01	2.00	3.72e-06	2.00	3.11e-05	1.99	7.42e-02	1.99
	0.0164	268646	1.61e-01	2.00	2.91e-06	2.00	2.35e-05	2.28	5.82e-02	1.99
	0.0333	135542	5.74e-03	−−	1.06e-07	−−	1.39e-06	−−	1.97e-03	−−
	0.0270	206018	3.06e-03	3.00	5.63e-08	3.00	7.48e-07	2.94	1.05e-03	3.00
2	0.0217	318230	1.59e-03	3.00	2.93e-08	3.00	3.90e-07	3.00	5.48e-04	3.00
	0.0185	438374	9.85e-04	3.00	1.81e-08	3.00	2.41e-07	3.00	3.39e-04	3.00
	0.0164	559250	6.83e-04	3.00	1.26e-08	3.00	1.67e-07	2.99	2.35e-04	3.00

Table 2.2: History of convergence for Example 1 with $r = 3$.

k	h	N	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(ϕ)	r(ϕ)	e($\boldsymbol{\rho}$)	r($\boldsymbol{\rho}$)
	0.0333	19982	5.48e+01	−−	9.73e-04	−−	5.09e-03	−−	2.15e+01	−−
	0.0270	30342	4.44e+01	1.00	7.86e-04	1.02	4.19e-03	0.93	1.75e+01	1.00
	0.0217	46830	3.57e+01	1.00	6.30e-04	1.01	3.40e-03	0.96	1.41e+01	1.00
0	0.0185	64478	3.04e+01	1.00	5.36e-04	1.01	2.90e-03	0.98	1.20e+01	1.00
	0.0164	82230	2.70e+01	1.00	4.74e-04	1.01	2.57e-03	1.00	1.06e+01	1.00
	0.0139	114482	2.28e+01	1.00	4.02e-04	1.00	2.18e-03	1.00	8.99e+00	1.00
	0.0122	148422	2.00e+01	1.00	3.53e-04	1.00	1.91e-03	1.00	7.89e+00	1.00
	0.0333	65162	7.08e-01	−−	1.28e-05	−−	9.84e-05	−−	2.55e-01	−−
	0.0270	99014	4.66e-01	2.00	8.43e-06	2.01	6.49e-05	1.99	1.68e-01	1.99
1	0.0217	152906	3.01e-01	2.00	5.45e-06	2.00	4.27e-05	1.92	1.09e-01	1.99
	0.0185	210602	2.19e-01	2.00	3.95e-06	2.00	3.11e-05	1.99	7.92e-02	1.99
	0.0164	268646	1.71e-01	2.00	3.10e-06	2.00	2.35e-05	2.28	6.21e-02	1.99
	0.0333	135542	6.21e-03	−−	1.14e-07	−−	1.39e-06	−−	2.17e-03	−−
	0.0270	206018	3.31e-03	3.00	6.09e-08	3.00	7.48e-07	2.94	1.16e-03	3.00
2	0.0217	318230	1.72e-03	3.00	3.17e-08	3.00	3.90e-07	3.00	6.02e-04	3.00
	0.0185	438374	1.07e-03	3.00	1.96e-08	3.00	2.41e-07	3.00	3.72e-04	3.00
	0.0164	559250	7.39e-04	3.00	1.36e-08	3.00	1.67e-07	2.99	2.58e-04	3.00

Table 2.3: History of convergence for Example 1 with $r = 4$.

observe from Figures 2.2 and 2.3 that, at least for this example, there is almost no difference between the curves obtained with $r = 3$ and $r = 4$ for both values of k. Finally, according to the aforementioned figures, and based on the comparison between schemes that use the same polynomial degree k, we infer that in general (2.100) requires fewer degrees of freedom than the methods from [46] to achieve a given accuracy. This fact is particularly notorious for the unknown $\mathbf{u}$ with $k \in \{0, 1\}$, and specially with $k = 1$, whereas for ϕ it is observed only with $k = 0$ since with $k = 1$ the respective curves are very close to each other and therefore no substantial difference is noticed.

Example 2. We let Ω be the L-shaped (and hence non-convex) domain given by $(-1, 1)^2 \setminus [0, 1]^2$, and, again, suitably perturb the definition of the functionals F_ϕ and $G_{\mathbf{u}}$, so that, letting $\theta := \arctan\left(\frac{x_2}{x_1}\right)$,

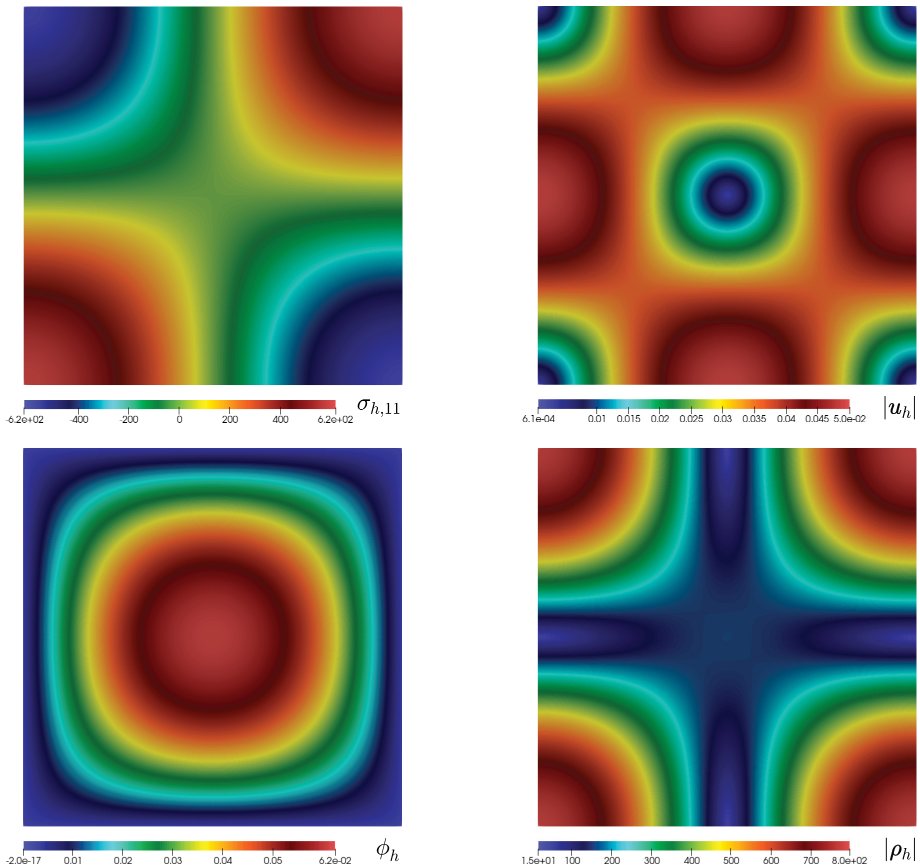

Figure 2.1: Some components and magnitudes of the solution of Example 1 with $k = 2$ and $N = 559250$

the exact solution of (2.1) reduces to:

$$\mathbf{u}(\mathbf{x}) \;=\; \begin{pmatrix} |\mathbf{x}|^{2/3}\sin(\theta) \\ -|\mathbf{x}|^{2/3}\cos(\theta) \end{pmatrix} \qquad \text{and} \qquad \phi(\mathbf{x}) \;=\; e^{x_2}\left(x_1 - \frac{1}{2}\right)^3,$$

for all $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$, whereas the body load, the diffusive source, and the tensorial diffusivity, are given, respectively, by

$$\mathbf{f}(\phi) \;=\; \begin{pmatrix} \frac{1}{40}\phi \\ \\ \frac{1}{40}\phi(1-\phi) \end{pmatrix}, \quad g(\mathbf{u}) \;=\; -|\mathbf{u}| \quad \text{and} \quad \widetilde{\vartheta}(\boldsymbol{\rho}) \;=\; \left(1 + \frac{1}{10}\left(1 + |\boldsymbol{\rho}|^2\right)^{-1/2}\right)\mathbb{I}\,.$$

In addition, we take $E = 100$ and $\nu = 0.4999$, whence the resulting Lamé parameters are given in this case (cf. (2.176)) by $\mu = 33.3356$ and $\lambda = 166644.4430$. Due to the singularity of the vector field $\mathbf{u}$ at the origin, in this example we do not expect to attain the theoretical orders of convergence guaranteed by Theorem 2.22. In fact, in Tables 2.4 and 2.5 we display the corresponding convergence history with $r = 3$ and $r = 4$, respectively, from which we realize that suboptimal, and even negative experimental rates of convergence are obtained. In turn, it is interesting to observe in this case that, differently from Example 1, these rates change not only with k but also with r, which must be certainly connected to the $\mathbf{W}^{l,r}(\Omega)$-regularity of the solution, most likely with a non-integer l depending on r. For instance, this was obtained for the regularity result of the Poisson problem in a non-convex domain,

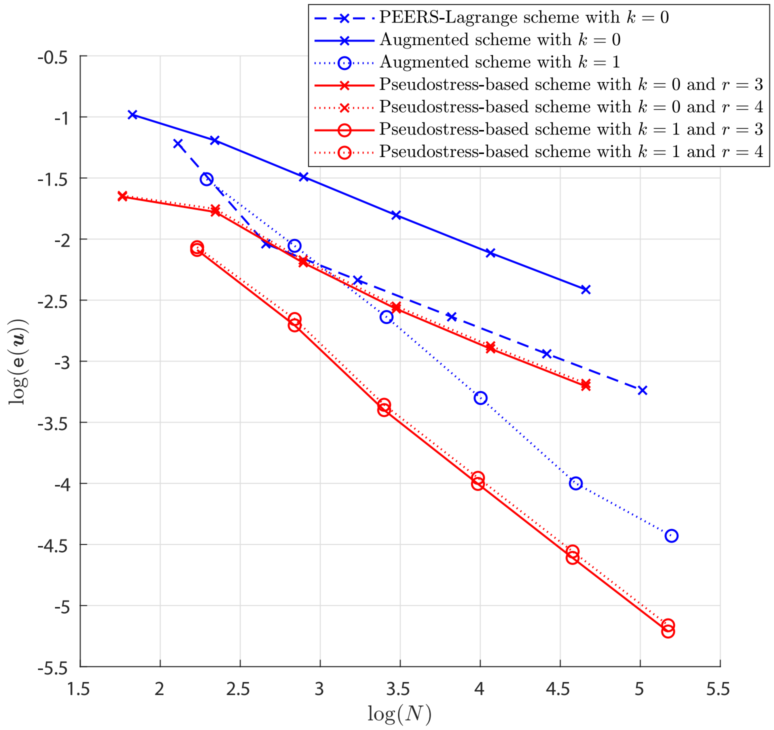

Figure 2.2: Example 1, $\log(\mathtt{e}(\mathbf{u}))$ vs. $\log(N)$ for the present scheme (2.100) and those from [46].

with homogeneous Neumann boundary conditions, and source term in $\mathrm{L}^r(\Omega)$ (see [52, Lemma B.1] for details). Anyhow, the usual way of recovering optimal rates of convergence in these cases is by applying an adaptive strategy based on a posteriori error estimates. This is precisely the subject of an undergoing work to be communicated in a forthcoming contribution.

Example 3. Finally, and while not supported by the theory, we consider the three-dimensional domain $\Omega = (0,1)^3$, and choose the data so that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) \;=\; e^{x_1+x_2+x_3}\begin{pmatrix}\sin(\pi x_1)\\ \sin(\pi x_2)\\ \sin(\pi x_3)\end{pmatrix} \quad \text{and} \quad \phi(\mathbf{x}) \;=\; -64x_1x_2x_3(x_1-1)(x_2-1)(x_3-1)\,,$$

for all $\mathbf{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \Omega$. In addition, the body load, the diffusive source, and the tensorial diffusivity are given, respectively, by

$$\mathbf{f}(\phi) \;=\; \begin{pmatrix}\phi\\ 1-\phi\\ \phi\end{pmatrix}, \quad g(\mathbf{u}) \;=\; x_1+x_2+x_3 \quad \text{and} \quad \widetilde{\vartheta}(\boldsymbol{\rho}) \;=\; \mathbb{I} + \frac{1}{10}\boldsymbol{\rho}^2\,.$$

As for Example 2, we take again $E = 100$ and $\nu = 0.4999$, which yields $\mu = 33.3356$ and $\lambda = 166644.4430$. In addition, we employ the software TetGen (cf. [78]) to generate triangulations of Ω made of tetrahedrons. In this way, in Table 2.6 we present the convergence history of (2.100) with

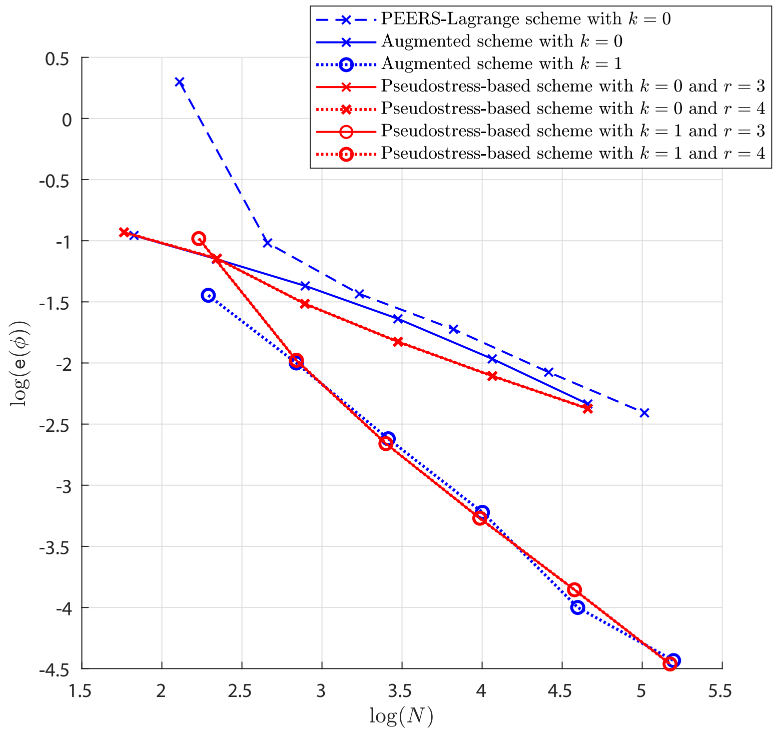

Figure 2.3: Example 1, $\log(\mathtt{e}(\phi))$ vs. $\log(N)$ for the present scheme (2.100) and those from [46].

k	h	N	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$
	0.0566	20927	4.00e+02	$--$	1.98e-02	$--$	2.72e-01	$--$	6.99e+00	$--$
	0.0471	30062	4.51e+02	-0.66	1.65e-02	1.00	2.26e-01	1.00	6.57e+00	0.33
	0.0372	48110	5.27e+02	-0.66	1.30e-02	1.00	1.79e-01	1.00	6.08e+00	0.33
0	0.0321	64418	5.80e+02	-0.66	1.12e-02	1.00	1.54e-01	1.00	5.79e+00	0.33
	0.0283	83102	6.31e+02	-0.66	9.90e-03	1.00	1.36e-01	1.00	5.55e+00	0.33
	0.0240	115583	7.04e+02	-0.66	8.39e-03	1.00	1.15e-01	1.00	5.25e+00	0.33
	0.0208	153410	7.74e+02	-0.66	7.28e-03	1.00	9.99e-02	1.00	5.01e+00	0.33
	0.0566	68102	3.98e+02	$--$	7.30e-04	$--$	2.77e-03	$--$	4.48e+00	$--$
	0.0471	97922	4.49e+02	-0.66	5.75e-04	1.31	1.93e-03	2.00	4.22e+00	0.33
1	0.0372	156866	5.25e+02	-0.66	4.22e-04	1.31	1.20e-03	2.00	3.90e+00	0.33
	0.0321	210146	5.78e+02	-0.66	3.48e-04	1.31	8.96e-04	2.00	3.71e+00	0.33
	0.0283	271202	6.29e+02	-0.66	2.95e-04	1.31	6.94e-04	2.00	3.56e+00	0.33
	0.0566	141527	4.08e+02	$--$	2.43e-04	$--$	2.07e-05	$--$	3.53e+00	$--$
	0.0471	203582	4.60e+02	-0.66	1.91e-04	1.31	1.46e-05	1.94	3.32e+00	0.33
2	0.0372	326270	5.38e+02	-0.66	1.40e-04	1.31	9.85e-06	1.65	3.07e+00	0.33
	0.0321	437186	5.93e+02	-0.66	1.16e-04	1.31	7.95e-06	1.46	2.92e+00	0.33
	0.0283	564302	6.46e+02	-0.66	9.78e-05	1.31	6.67e-06	1.37	2.80e+00	0.33

Table 2.4: History of convergence for Example 2 with $r = 3$.

$k \in \{0, 1, 2\}$ and $r = 3$, from which we observe that the same orders from 2D (cf. Example 1) are attained in all these cases. This fact suggests, in coherence with the remark at the end of Section 2.5.2, that only some technical issues might be stopping us from extending the theoretical analysis to the 3D case. Finally, some components of the approximate solution are depicted in Figure 2.4.

k	h	N	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$
0	0.0566	20927	7.01e+02	−−	1.91e-02	−−	2.72e-01	−−	1.09e+01	−−
	0.0471	30062	8.15e+02	-0.82	1.60e-02	0.99	2.26e-01	1.00	1.05e+01	0.16
	0.0372	48110	9.90e+02	-0.83	1.26e-02	0.99	1.79e-01	1.00	1.01e+01	0.16
	0.0321	64418	1.12e+03	-0.83	1.09e-02	0.99	1.54e-01	1.00	9.89e+00	0.16
	0.0283	83102	1.24e+03	-0.83	9.61e-03	0.99	1.36e-01	1.00	9.69e+00	0.16
	0.0240	115583	1.43e+03	-0.83	8.15e-03	1.00	1.15e-01	1.00	9.43e+00	0.16
	0.0208	153410	1.60e+03	-0.83	7.08e-03	1.00	9.99e-02	1.00	9.21e+00	0.17
1	0.0566	68102	7.35e+02	−−	1.17e-03	−−	2.77e-03	−−	8.14e+00	−−
	0.0471	97922	8.54e+02	-0.83	9.46e-04	1.15	1.93e-03	2.00	7.90e+00	0.17
	0.0372	156866	1.04e+03	-0.83	7.20e-04	1.16	1.20e-03	2.00	7.59e+00	0.17
	0.0321	210146	1.17e+03	-0.83	6.07e-04	1.16	8.96e-04	2.00	7.41e+00	0.17
	0.0283	271202	1.30e+03	-0.83	5.24e-04	1.16	6.94e-04	2.00	7.26e+00	0.17
2	0.0566	141527	7.90e+02	−−	4.22e-04	−−	2.07e-05	−−	7.03e+00	−−
	0.0471	203582	9.19e+02	-0.83	3.41e-04	1.16	1.46e-05	1.94	6.82e+00	0.17
	0.0372	326270	1.12e+03	-0.83	2.59e-04	1.16	9.85e-06	1.65	6.56e+00	0.17
	0.0321	437186	1.26e+03	-0.83	2.19e-04	1.16	7.95e-06	1.46	6.40e+00	0.17
	0.0283	564302	1.40e+03	-0.83	1.89e-04	1.16	6.67e-06	1.37	6.26e+00	0.17

Table 2.5: History of convergence for Example 2 with $r = 4$.

k	h	N	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$
0	0.3542	9487	4.59e+06	−−	1.47e+03	−−	9.06e-01	−−	2.56e+06	−−
	0.3130	13844	4.05e+06	1.01	1.26e+03	1.25	8.01e-01	1.00	2.25e+06	1.04
	0.2804	18203	3.61e+06	1.03	1.13e+03	0.99	7.16e-01	1.02	1.97e+06	1.22
	0.2657	22188	3.43e+06	0.99	1.07e+03	1.07	6.78e-01	1.01	1.86e+06	1.04
	0.2519	26479	3.25e+06	1.00	9.99e+02	1.27	6.42e-01	1.02	1.76e+06	1.00
	0.1832	82222	2.35e+06	1.01	7.33e+02	0.97	4.41e-01	1.18	1.25e+06	1.08
	0.1475	152258	1.88e+06	1.04	5.81e+02	1.08	3.53e-01	1.03	9.98e+05	1.05
1	0.3542	40854	3.03e+05	−−	4.52e+01	−−	1.10e-01	−−	1.53e+05	−−
	0.3130	59920	2.38e+05	1.94	3.54e+01	1.99	8.71e-02	1.92	1.19e+05	2.04
	0.2804	78912	1.91e+05	2.01	2.81e+01	2.07	7.01e-02	1.97	9.60e+04	1.97
	0.2657	96220	1.71e+05	2.01	2.53e+01	1.99	6.28e-02	2.04	8.61e+04	2.02
	0.2519	114932	1.54e+05	2.00	2.27e+01	2.03	5.65e-02	2.01	7.75e+04	2.00
2	0.3542	106570	1.67e+04	−−	1.84e+00	−−	7.93e-03	−−	8.75e+03	−−
	0.3130	156755	1.14e+04	3.07	1.27e+00	2.97	5.47e-03	3.01	5.99e+03	3.07
	0.2804	206621	8.18e+03	3.04	9.14e-01	3.00	3.91e-03	3.04	4.30e+03	3.01
	0.2657	251985	6.96e+03	2.98	7.78e-01	2.99	3.33e-03	3.00	3.66e+03	3.00
	0.2519	301137	5.92e+03	3.05	6.62e-01	3.05	2.84e-03	3.00	3.12e+03	3.01

Table 2.6: History of convergence for Example 3 with $r = 3$.

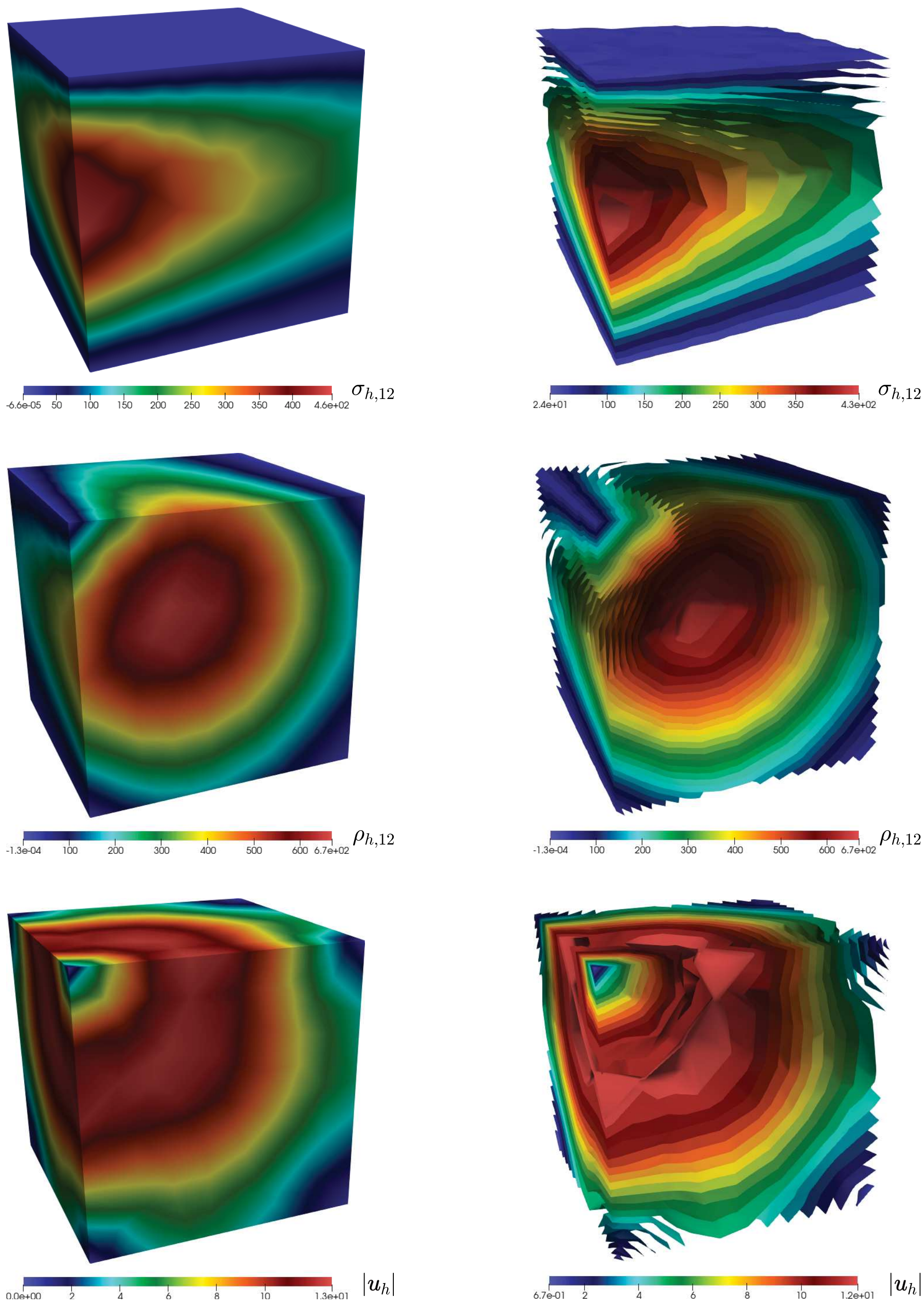

Figure 2.4: Some components and norms of the solution of Example 3 obtained with $k = 2$ and $N = 301137$ degrees of freedom. Surface (left) and contours (right).

CHAPTER 3

New Banach spaces–based fully–mixed finite element methods for pseudostress-assisted diffusion problems

3.1 Introduction

In the Chapter 2, we employed a Banach spaces-based variational approach to derive a new mixed-primal finite element method for the nearly incompressible case of the pseudostress-assisted diffusion problem, which models the diffusion of a solute into an elastic material. More precisely, the aforementioned phenomenon refers to diffusion processes in deformable solids occupying originally a domain Ω of R^n, $n \in \{2,3\}$, and arises in diverse applications, including diffusion of boron and arsenic in silicon [69], voiding of aluminum conductor lines in integrated circuits [87], sorption in polymers [77], damage to electrodes in lithium-ion batteries [5], and anisotropy of cardiac dynamics [27], among others. The usual assumptions in most of them are, on one hand, that the solid satisfies an elastic regime, and on the other hand, that the diffusion obeys a Fickean law enriched with further contributions arising from local effects by exerted stresses. This second hypothesis means that the respective diffusion coefficient is a continuous function depending precisely on the stress, which acts then as a coupling variable. Mathematically, the underlying model is usually described by the following system of partial differential equations (cf. (2.12)):

$$
\begin{aligned}
\nabla \mathbf{u} &= \widehat{\mathcal{C}}^{-1}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \\
\widetilde{\boldsymbol{\sigma}} &= \vartheta(\boldsymbol{\sigma})\nabla\phi \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) = g(\mathbf{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,,
\end{aligned}
\tag{3.1}
$$

where

$$
\widehat{\mathcal{C}}^{-1}(\boldsymbol{\tau}) := \frac{1}{\mu}\boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{n\big(n\lambda + (n+1)\mu\big)}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I} \qquad \forall\,\boldsymbol{\tau} \in \mathrm{R}^{n\times n}\,. \tag{3.2}
$$

Here, $\boldsymbol{\sigma}$ is the non-symmetric pseudostress tensor, $\mathbf{u}$ is the displacement field, λ, $\mu > 0$ are the Lamé constants (dilation and shear moduli), which characterize the properties of the material, and $\mathbb{I}$ is the identity tensor of $\mathrm{R}^{n\times n}$. In turn, ϕ represents the local concentration of species, $\widetilde{\boldsymbol{\sigma}}$ is the diffusive flux, and $\vartheta : \mathrm{R}^{n\times n} \to \mathrm{R}^{n\times n}$ is a tensorial diffusivity function. In addition, $\mathbf{f} : \mathrm{R} \to \mathrm{R}^n$ is a vector field of body loads (which depends on the species concentration), $g : \mathrm{R}^n \to \mathrm{R}$ denotes an additional source term depending on the solid displacement $\mathbf{u}$, and $\mathbf{u}_D$ is the Dirichlet datum for $\mathbf{u}$, which belongs to a suitable trace space to be identified later on.

Now, throughout this chapter we suppose that ϑ is of class C^1 and uniformly positive definite, meaning the latter that there exists a positive constant ϑ_0 such that

$$\vartheta(\boldsymbol{\tau})\mathbf{w}\cdot\mathbf{w} \,\geq\, \vartheta_0\,|\mathbf{w}|^2 \quad \forall\,\mathbf{w}\in\mathbf{R}\,, \quad \forall\,\boldsymbol{\tau}\in\mathbb{R}\,. \tag{3.3}$$

We also require uniform boundedness and Lipschitz continuity of ϑ, that is that there exist positive constants ϑ_1, ϑ_2, and L_ϑ, such that

$$\vartheta_1 \,\leq\, |\vartheta(\boldsymbol{\tau})| \,\leq\, \vartheta_2 \quad \text{and} \quad |\vartheta(\boldsymbol{\tau})-\vartheta(\boldsymbol{\zeta})| \,\leq\, L_\vartheta\,|\boldsymbol{\tau}-\boldsymbol{\zeta}| \quad \forall\,\boldsymbol{\tau},\boldsymbol{\zeta}\in\mathbb{R}\,. \tag{3.4}$$

Moreover, thanks to (3.3), we have that the inverse of ϑ is uniformly positive definite as well, specifically, denoting from now on $\widetilde{\vartheta}(\boldsymbol{\tau}) := \vartheta(\boldsymbol{\tau})^{-1}$, there exists a positive constant $\widetilde{\vartheta}_0$ such that

$$\widetilde{\vartheta}(\boldsymbol{\tau})\mathbf{w}\cdot\mathbf{w} \,\geq\, \widetilde{\vartheta}_0\,|\mathbf{w}|^2 \quad \forall\,\mathbf{w}\in\mathbf{R}\,, \quad \forall\,\boldsymbol{\tau}\in\mathbb{R}\,. \tag{3.5}$$

We also require uniform boundedness and Lipschitz continuity of $\widetilde{\vartheta}$, that is that there exist positive constants $\widetilde{\vartheta}_1$, $\widetilde{\vartheta}_2$, and $L_{\widetilde{\vartheta}}$, such that

$$\widetilde{\vartheta}_1 \,\leq\, |\widetilde{\vartheta}(\boldsymbol{\tau})| \,\leq\, \widetilde{\vartheta}_2 \quad \text{and} \quad |\widetilde{\vartheta}(\boldsymbol{\tau})-\widetilde{\vartheta}(\boldsymbol{\zeta})| \,\leq\, L_{\widetilde{\vartheta}}\,|\boldsymbol{\tau}-\boldsymbol{\zeta}| \quad \forall\,\boldsymbol{\tau},\boldsymbol{\zeta}\in\mathbb{R}\,. \tag{3.6}$$

Similar hypotheses are assumed on the source functions $\mathbf{f}$ and g, which means that there exist positive constants f_1, f_2, L_f, g_1, g_2 and L_g, such that

$$f_1 \,\leq\, |\mathbf{f}(s)| \,\leq\, f_2\,, \quad |\mathbf{f}(s)-\mathbf{f}(t)| \,\leq\, L_f\,|s-t| \quad \forall\,s,t\in\mathrm{R}\,, \tag{3.7}$$

$$g_1 \,\leq\, |g(\mathbf{w})| \,\leq\, g_2\,, \quad \text{and} \quad |g(\mathbf{v})-g(\mathbf{w})| \,\leq\, L_g\,|\mathbf{v}-\mathbf{w}| \quad \forall\,\mathbf{v},\mathbf{w}\in\mathbf{R}\,. \tag{3.8}$$

The purpose of the present work is to continue contributing in the direction of Chapter 2 by introducing and analysing new fully-mixed finite element methods for the numerical solution of (3.1) - (3.2). In this way, the main novelty with respect to Chapter 2 is the utilization of a mixed variational formulation for the diffusion equation. As a consequence, and regarding the mixed approach for the elasticity equation, we certainly make use of the corresponding results from Chapter 2 either by stating or referring to them throughout the analysis. In some cases, and just for sake of completeness, the main aspects of the respective proofs are explicitly recalled. Needless to say, we remark that a fully-mixed approach for this model had basically been employed already in [47]. However, to be able to carry out the respective analysis within a Hilbertian framework, it was necessary to incorporate there augmented terms, thus increasing the complexity of the resulting discrete method. According to the above, and motivated by recent works using Banach spaces-based formulations (see, e.g. [11], [49], [52] and [54]), which do not need to resort to augmentation techniques, we proceed similarly to them and propose two mixed variational formulations for the diffusion equation in terms of suitable Lebesgue and Sobolev-type Banach spaces. For the first approach we perform integration by parts on the constitutive equation, while for the second one the diffusion gradient is introduced as an auxiliary unknown.

The chapter is organized as follows. The rest of this section collects first some preliminary notations, definitions, and results to be utilized throughout the chapter. In Section 3.2, we derive the two fully-mixed variational formulations of the problem. Suitable integration by parts formulae jointly with the Cauchy-Schwarz and Hölder inequalities are crucial for determining the right Lebesgue and related spaces to which the unknowns and corresponding test functions are required to belong. In Section 3.3, fixed-point strategies are adopted to analyse the solvability of the continuous formulations. The Babuška-Brezzi theory in Banach spaces is employed to study the corresponding uncoupled problems, and then the classical Banach theorem is applied to conclude the existence of a unique solution of the respective formulations. Analogue fixed-point approaches to those of Section 3.3 are utilized in Section 3.4 to study the well-posedness of the associated Galerkin scheme. In this way, and along with the corresponding versions of the theoretical tools employed in Section 3.3, a straightforward application of Brouwer's theorem allows us to conclude the existence of discrete solution. A priori error estimates in the form of Cea's estimate are also derived here. Next, in Section 3.5 we restrict ourselves to the 2D case and introduce specific finite element subspaces satisfying the theoretical hypotheses that were assumed in Section 3.4. The fact that a required boundedness property for a particular projector involved is still an open problem in 3D, stop us from extending the 2D analysis from Section 3.5 to that dimension. Finally, several numerical results illustrating the performance of the method and confirming the theoretical rates of convergence provided in Section 3.5, are reported in Section 3.6

3.2 The fully-mixed formulations

In this section we introduce two Banach spaces-based fully-mixed formulations of (3.1)-(3.2), which arise from a common formulation for elasticity (see Section 3.2.1 below) and two different approaches for the diffusion equation (see Sections 3.2.2 and 3.2.3 below). The integration by parts formulae provided by (9), along with the Cauchy-Schwarz and Hölder inequalities, play key roles in the derivation of the Banach spaces where the respective unknowns will be sought.

3.2.1 The elasticity equation

As explained in Chapter 2.3, given

$$r \in \left\{ \begin{array}{cl} (2,+\infty) & \text{if } n=2\,, \\ (2,6] & \text{if } n=3\,, \end{array}\right. \quad \text{and} \quad s \in \left\{ \begin{array}{cl} (1,2) & \text{if } n=2\,, \\ [6/5,2) & \text{if } n=3\,, \end{array}\right. \tag{3.9}$$

conjugate to each other, and given ϕ in a suitable space to be determined next, the Banach spaces-based mixed formulation for the elasticity equation reads: Find $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbf{X}_2 \times \mathbf{M}_1$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) + \mathbf{b}_1(\boldsymbol{\tau},\mathbf{u}) &= G(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in \mathbf{X}_1, \\ \mathbf{b}_2(\boldsymbol{\sigma},\mathbf{v}) &= F_\phi(\mathbf{v}) \qquad \forall\,\mathbf{v} \in \mathbf{M}_2, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} \mathbf{X}_2 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega) &:= \left\{ \boldsymbol{\tau} \in \mathbb{H}^r(\mathbf{div}_r;\Omega) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0 \right\}, \quad \mathbf{M}_1 := \mathbf{L}^r(\Omega)\,, \\ \mathbf{X}_1 := \mathbb{H}_0^s(\mathbf{div}_s;\Omega) &:= \left\{ \boldsymbol{\tau} \in \mathbb{H}^s(\mathbf{div}_s;\Omega) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0 \right\}, \quad \mathbf{M}_2 := \mathbf{L}^s(\Omega)\,, \end{aligned}$$

and the bilinear forms $\mathbf{a} : \mathbf{X}_2 \times \mathbf{X}_1 \to \mathrm{R}$ and $\mathbf{b}_i : \mathbf{X}_i \times \mathbf{M}_i \to \mathrm{R}$, $i \in \{1,2\}$, and the functionals $G \in \mathbf{X}_1'$ and $F_\phi \in \mathbf{M}_2'$, are defined, respectively, as

$$\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau}) := \frac{1}{\mu}\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \frac{1}{n(n\lambda+(n+1)\mu)}\int_\Omega \operatorname{tr}(\boldsymbol{\zeta})\operatorname{tr}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau}) \in \mathbf{X}_2 \times \mathbf{X}_1\,,$$

$$\mathbf{b}_i(\boldsymbol{\tau},\mathbf{v}) := \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau},\mathbf{v})\in\mathbf{X}_i\times\mathbf{M}_i\,,$$
$$G(\boldsymbol{\tau}) := \langle\boldsymbol{\tau\nu},\mathbf{u}_D\rangle_\Gamma, \qquad \forall\,\boldsymbol{\tau}\in\mathbf{X}_1\,, \tag{3.11}$$

and

$$F_\phi(\mathbf{v}) := -\int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v} \qquad \forall\,\mathbf{v}\in\mathbf{M}_2\,. \tag{3.12}$$

Furthermore, we have from (2.57) that $\mathbf{a}$, $\mathbf{b}_1$, $\mathbf{b}_2$, G and F_ϕ are all bounded with respective constants given by

$$\|\mathbf{a}\| = \frac{2}{\mu}\,, \quad \|\mathbf{b}_1\| = \|\mathbf{b}_2\| = 1\,, \quad \|G\| = C_r\,\|\mathbf{u}_D\|_{1/s,r;\Gamma}\,, \quad \text{and} \quad \|F_\phi\| = |\Omega|^{1/r} f_2\,, \tag{3.13}$$

where C_r is a positive constant such that (cf. (2.27))

$$\|\boldsymbol{\tau}\,\boldsymbol{\nu}\|_{-1/r,r;\Gamma} \le C_r\,\|\boldsymbol{\tau}\|_{r,\mathbf{div}_r;\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}^r(\mathbf{div}_r;\Omega)\,. \tag{3.14}$$

Having recalled the above from Chapter 2, we remark that in order to analyse the elasticity equation, we need to be able to control the expression

$$\int_\Omega (\mathbf{f}(\psi)-\mathbf{f}(\varphi))\cdot\mathbf{v}\,, \tag{3.15}$$

where $\mathbf{v}\in\mathbf{M}_2$, and ψ and φ are generic functions belonging to the same space in which we will seek the unknown ϕ. In this regard, employing the Lipschitz-continuity property of $\mathbf{f}$ (cf. (3.7)), a straightforward application of the Hölder inequality yields

$$\left|\int_\Omega (\mathbf{f}(\psi)-\mathbf{f}(\varphi))\cdot\mathbf{v}\right| \le L_f\|\psi-\varphi\|_{0,r;\Omega}\|\mathbf{v}\|_{0,s;\Omega}\,, \tag{3.16}$$

from which we deduce that we must look for the unknown ϕ in $\mathrm{L}^r(\Omega)$.

3.2.2 A first approach for the diffusion equation

In what follows we derive a first mixed variational formulation for the diffusion equation

$$\widetilde{\vartheta}(\boldsymbol{\sigma})\,\widetilde{\boldsymbol{\sigma}} = \nabla\phi \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) = g(\mathbf{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,, \tag{3.17}$$

where $\widetilde{\vartheta}(\boldsymbol{\sigma}) = \vartheta(\boldsymbol{\sigma})^{-1}$. To this end, we begin by considering $\phi\in\mathrm{H}^1(\Omega)$, which, thanks to the continuous embedding of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^r(\Omega)$, does not contradict what was discussed at the end of the previous section. Then, applying (9) with s specified in (3.9) and $\widetilde{\boldsymbol{\tau}}\in\mathbf{H}(\mathrm{div}_s;\Omega)$ (cf. (3)), and using the Dirichlet condition satisfied by ϕ, we get

$$\int_\Omega \widetilde{\boldsymbol{\tau}}\cdot\nabla\phi = -\int_\Omega \phi\,\mathrm{div}(\widetilde{\boldsymbol{\tau}})\,,$$

whence the corresponding testing of the first equation of (3.17) becomes

$$\int_\Omega \widetilde{\vartheta}(\boldsymbol{\sigma})\,\widetilde{\boldsymbol{\sigma}}\cdot\widetilde{\boldsymbol{\tau}} + \int_\Omega \phi\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}) = 0 \qquad \forall\,\widetilde{\boldsymbol{\tau}}\in\mathbf{H}(\mathrm{div}_s;\Omega)\,. \tag{3.18}$$

It is clear, thanks to (3.4) and Cauchy-Schwarz's inequality, that the first term of (3.18) makes sense for $\widetilde{\boldsymbol{\sigma}} \in \mathbf{L}^2(\Omega)$. In addition, formally testing the second equation of the second row of (3.1) against a function ψ, yields

$$\int_\Omega \psi \operatorname{div}(\widetilde{\boldsymbol{\sigma}}) \,=\, - \int_\Omega g(\mathbf{u})\psi\,, \tag{3.19}$$

whose right-hand side has a similar structure to (3.12). Hence, analogously to (3.15) and (3.16), and since $\mathbf{u} \in \mathbf{L}^r(\Omega)$ and $r > s$, Hölder's inequality allows us to conclude that it suffices to take ψ in $\mathrm{L}^r(\Omega)$. In fact, thanks to the Lipschitz continuity property of g (cf. (3.8)), we get

$$\left|\int_\Omega (g(\mathbf{u}) - g(\mathbf{v}))\psi\right| \,\le\, L_g\,\|\mathbf{u}-\mathbf{v}\|_{0,r;\Omega}\,\|\psi\|_{0,s;\Omega} \,\le\, |\Omega|^{\frac{r-s}{rs}}\,L_g\,\|\mathbf{u}-\mathbf{v}\|_{0,r;\Omega}\,\|\psi\|_{0,r;\Omega}\,, \tag{3.20}$$

from which we deduce that the left-hand side of (3.19) is finite if $\operatorname{div}(\widetilde{\boldsymbol{\sigma}}) \in \mathrm{L}^s(\Omega)$, and hence we will look for $\widetilde{\boldsymbol{\sigma}}$ in $\mathbf{H}(\operatorname{div}_s;\Omega)$ (cf. (3)). According to the foregoing discussion, we now set the following Banach spaces

$$\mathbf{Q} \,:=\, \mathbf{H}(\operatorname{div}_s;\Omega) \quad \text{and} \quad \mathrm{M} \,:=\, \mathrm{L}^r(\Omega)\,, \tag{3.21}$$

so that, given $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbf{X}_2 \times \mathbf{M}_1$, the mixed formulation for (3.17) reduces to: Find $(\widetilde{\boldsymbol{\sigma}},\phi) \in \mathbf{Q}\times \mathrm{M}$ such that

$$\begin{aligned} \widetilde{a}_{\boldsymbol{\sigma}}(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\tau}},\phi) \,&=\, 0 \qquad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,,\\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}},\psi) \,&=\, \widetilde{G}_{\mathbf{u}}(\psi) \quad \forall\,\psi \in \mathrm{M}\,, \end{aligned} \tag{3.22}$$

where, the bilinear forms $\widetilde{a}_{\boldsymbol{\sigma}} : \mathbf{Q}\times\mathbf{Q} \to \mathrm{R}$, $\widetilde{b} : \mathbf{Q}\times\mathrm{M} \to \mathrm{R}$, and the functional $G_{\mathbf{u}} \in \mathrm{M}$, are defined, respectively, as

$$\widetilde{a}_{\boldsymbol{\sigma}}(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\tau}}) \,:=\, \int_\Omega \widetilde{\vartheta}(\boldsymbol{\sigma})\,\widetilde{\boldsymbol{\sigma}}\cdot\widetilde{\boldsymbol{\tau}} \qquad \forall\,\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,, \tag{3.23}$$

$$\widetilde{b}(\widetilde{\boldsymbol{\tau}},\psi) \,:=\, \int_\Omega \psi \operatorname{div}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\,(\widetilde{\boldsymbol{\tau}},\psi) \in \mathbf{Q}\times\mathrm{M}\,, \tag{3.24}$$

and

$$\widetilde{G}_{\mathbf{u}}(\psi) \,:=\, -\int_\Omega g(\mathbf{u})\,\psi \qquad \forall\,\psi \in \mathrm{M}\,. \tag{3.25}$$

Next, a direct application of Hölder's inequality, and the bounds given by (3.6) and (3.8), allow to conclude that the bilinear forms $\widetilde{a}$ and $\widetilde{b}$, and the functional $\widetilde{G}_{\mathbf{u}}$, are all bounded with the corresponding norms given by

$$\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}} \,:=\, \|\widetilde{\boldsymbol{\tau}}\|_{\operatorname{div}_s;\Omega} \quad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{Q} \quad \text{and} \quad \|\psi\|_{\mathrm{M}} \,:=\, \|\psi\|_{0,r;\Omega} \quad \forall\,\psi \in \mathrm{M}\,.$$

In fact, there exist positive constants, given by

$$\|\widetilde{a}_{\boldsymbol{\sigma}}\| \,=\, \widetilde{\vartheta}_2\,, \quad \|\widetilde{b}\| \,=\, 1\,, \quad \text{and} \quad \|\widetilde{G}_{\mathbf{u}}\| \,=\, g_2\,|\Omega|^{1/s}\,, \tag{3.26}$$

such that

$$\begin{aligned} |\widetilde{a}_{\boldsymbol{\sigma}}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}})| \,&\le\, \|\widetilde{a}_{\boldsymbol{\sigma}}\|\,\|\widetilde{\boldsymbol{\zeta}}\|_{\mathbf{Q}}\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}} \qquad \forall\,\widetilde{\boldsymbol{\zeta}},\,\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,,\\ |\widetilde{b}(\widetilde{\boldsymbol{\tau}},\psi)| \,&\le\, \|\widetilde{b}\|\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}}\,\|\psi\|_{\mathrm{M}} \qquad \forall\,(\widetilde{\boldsymbol{\tau}},\psi) \in \mathbf{Q}\times\mathrm{M}\,, \end{aligned}$$

and

$$|\widetilde{G}_{\mathbf{u}}(\psi)| \,\le\, \|\widetilde{G}_{\mathbf{u}}\|\,\|\psi\|_{\mathrm{M}} \qquad \forall\,\psi \in \mathrm{M}\,.$$

3.2.3 A second approach for the diffusion equation

As an alternative to the previous formulation for the diffusion equation, and in order to obtain a more accurate approximation for the diffusion gradient, as well as to avoid inverting ϑ, we introduce the unknown $\mathbf{t} := \nabla\phi$ in Ω. Thus, the second row of (3.1) becomes

$$\begin{aligned} \mathbf{t} \,=\, \nabla\phi \quad \text{in} \quad \Omega\,, \qquad \widetilde{\boldsymbol{\sigma}} \,=\, \vartheta(\boldsymbol{\sigma})\,\mathbf{t} \quad \text{in} \quad \Omega\,, \\ \operatorname{div}(\widetilde{\boldsymbol{\sigma}}) \,=\, g(\mathbf{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi \,=\, 0 \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{3.27}$$

Then, bearing in mind that ϕ must be sought in $\mathrm{L}^r(\Omega)$, and thanks to the continuous embedding of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^r(\Omega)$, we initially look for ϕ in $\mathrm{H}^1(\Omega)$. In this way, testing the first equation of (3.27) against $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\operatorname{div}_s;\Omega)$, applying (9), with s specified in (3.9), and employing the Dirichlet boundary condition for ϕ, we obtain

$$\int_\Omega \mathbf{t}\cdot\widetilde{\boldsymbol{\tau}} \,+\, \int_\Omega \phi\,\operatorname{div}(\widetilde{\boldsymbol{\tau}}) \,=\, 0 \qquad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\operatorname{div}_s;\Omega)\,,$$

whence the first term makes sense for $\mathbf{t} \in \mathbf{L}^2(\Omega)$. In turn, testing the second equation of (3.27) against $\mathbf{s} \in \mathbf{L}^2(\Omega)$, we formally get

$$\int_\Omega \vartheta(\boldsymbol{\sigma})\,\mathbf{t}\cdot\mathbf{s} \,-\, \int_\Omega \widetilde{\boldsymbol{\sigma}}\cdot\mathbf{s} \,=\, 0 \qquad \forall\,\mathbf{s} \in \mathbf{L}^2(\Omega)\,, \tag{3.28}$$

from which we notice, thanks to Cauchy-Schwarz's inequality and (3.4), that the first term of (3.28) is finite, whereas its second term makes sense is $\widetilde{\boldsymbol{\sigma}}$ is sought in $\mathbf{L}^2(\Omega)$. Now, testing the third equation of (3.27) against a function φ, we have

$$\int_\Omega \varphi\,\operatorname{div}(\widetilde{\boldsymbol{\sigma}}) \,=\, \int_\Omega g(\mathbf{u})\,\varphi\,, \tag{3.29}$$

and, similarly to (3.20), we deduce from the right side of (3.29) that φ can be considered in $\mathrm{L}^r(\Omega)$. Hence, in order for the left-hand side of (3.29) to be well-defined we need that $\operatorname{div}(\widetilde{\boldsymbol{\sigma}}) \in \mathrm{L}^s(\Omega)$, which yields to look for $\widetilde{\boldsymbol{\sigma}}$ in $\mathbf{H}(\operatorname{div}_s;\Omega)$. Consequently, recalling from (3.21) the definition of M, we introduce the following notation

$$\vec{\phi} \,:=\, (\phi,\mathbf{t}), \quad \vec{\varphi} \,:=\, (\varphi,\mathbf{s}) \,\in\, \mathbf{H} \,:=\, \mathrm{M}\times\mathbf{L}^2(\Omega)\,.$$

Thus, given $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbf{X}_2\times\mathbf{M}_1$, we arrive at the following mixed formulation for (3.27): Find $(\vec{\phi},\widetilde{\boldsymbol{\sigma}}) \in \mathbf{H}\times\mathbf{Q}$ such that

$$\begin{aligned} a_{\boldsymbol{\sigma}}(\vec{\phi},\vec{\varphi}) \,+\, b(\vec{\varphi},\widetilde{\boldsymbol{\sigma}}) &\,=\, G_{\mathbf{u}}(\vec{\varphi}) \quad \forall\,\vec{\varphi}\in\mathbf{H}\,, \\ b(\vec{\phi},\widetilde{\boldsymbol{\tau}}) &\,=\, 0 \qquad\quad \forall\,\widetilde{\boldsymbol{\tau}}\in\mathbf{Q}\,, \end{aligned} \tag{3.30}$$

where the bilinear forms $a_{\boldsymbol{\sigma}} : \mathbf{H}\times\mathbf{H} \to \mathrm{R}$ and $b : \mathbf{H}\times\mathbf{Q} \to \mathrm{R}$ are defined as

$$a_{\boldsymbol{\sigma}}(\vec{\phi},\vec{\varphi}) \,:=\, \int_\Omega \vartheta(\boldsymbol{\sigma})\,\mathbf{t}\cdot\mathbf{s} \quad \forall\,\vec{\phi},\,\vec{\varphi}\in\mathbf{H}\,, \quad \text{and} \tag{3.31}$$

$$b(\vec{\varphi},\widetilde{\boldsymbol{\tau}}) \,:=\, -\int_\Omega \widetilde{\boldsymbol{\tau}}\cdot\mathbf{s} \,-\, \int_\Omega \varphi\,\operatorname{div}(\widetilde{\boldsymbol{\tau}}) \quad \forall\,(\vec{\varphi},\widetilde{\boldsymbol{\tau}})\in\mathbf{H}\times\mathbf{Q}\,, \tag{3.32}$$

whereas the linear functional $G_{\mathbf{u}} : \mathbf{H} \to \mathrm{R}$ is given by

$$G_{\mathbf{u}}(\vec{\varphi}) \,:=\, -\int_\Omega g(\mathbf{u})\,\varphi \quad \forall\,\vec{\varphi}\in\mathbf{H}\,. \tag{3.33}$$

Next, it is easily seen that $a_{\boldsymbol{\sigma}}$, b and $G_{\mathbf{u}}$ are bounded. In fact, endowing $\mathbf{H}$ with the product norm

$$\|\vec{\varphi}\|_{\mathbf{H}} := \|\varphi\|_{0,r;\Omega} + \|\mathbf{s}\|_{0,\Omega} \quad \forall\, \vec{\varphi} := (\varphi, \mathbf{s}) \in \mathbf{H}\,,$$

and applying (3.4), (3.8), and the Cauchy-Schwarz and Hölder inequalities, we find that there exist positive constants, denoted and given by

$$\|a_{\boldsymbol{\sigma}}\| = \vartheta_2\,, \quad \|b\| = 1\,, \quad \text{and} \quad \|G_{\mathbf{u}}\| = g_2\,|\Omega|^{1/s}\,, \tag{3.34}$$

such that

$$\begin{aligned} |a_{\boldsymbol{\sigma}}(\vec{\phi},\vec{\varphi})| &\le \|a_{\boldsymbol{\sigma}}\|\,\|\vec{\phi}\|_{\mathbf{H}}\,\|\vec{\varphi}\|_{\mathbf{H}} \qquad \forall\,\vec{\phi},\,\vec{\varphi}\in\mathbf{H}\,,\\ |b(\vec{\varphi},\widetilde{\boldsymbol{\tau}})| &\le \|b\|\,\|\vec{\varphi}\|_{\mathbf{H}}\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}} \qquad \forall\,(\vec{\varphi},\widetilde{\boldsymbol{\tau}})\in\mathbf{H}\times\mathbf{Q}\,, \end{aligned}$$

and

$$|G_{\mathbf{u}}(\vec{\varphi})| \le \|G_{\mathbf{u}}\|\,\|\vec{\varphi}\|_{\mathbf{H}} \qquad \forall\,\vec{\varphi}\in\mathbf{H}\,.$$

3.2.4 The coupled fully-mixed formulations

According to the analyses in Sections 3.2.1 and 3.2.2, our first fully-mixed formulation for (3.1)-(3.2) reduces to gathering (3.10) and (3.22), that is: Find $(\boldsymbol{\sigma},\mathbf{u})\in\mathbf{X}_2\times\mathbf{M}_1$ and $(\widetilde{\boldsymbol{\sigma}},\phi)\in\mathbf{Q}\times\mathrm{M}$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau})+\mathbf{b}_1(\boldsymbol{\tau},\mathbf{u}) &= G(\boldsymbol{\tau}) && \forall\,\boldsymbol{\tau}\in\mathbf{X}_1,\\ \mathbf{b}_2(\boldsymbol{\sigma},\mathbf{v}) &= F_\phi(\mathbf{v}) && \forall\,\mathbf{v}\in\mathbf{M}_2,\\ \widetilde{a}_{\boldsymbol{\sigma}}(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}},\phi) &= 0 && \forall\,\widetilde{\boldsymbol{\tau}}\in\mathbf{Q}\,,\\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}},\psi) &= \widetilde{G}_{\mathbf{u}}(\psi) && \forall\,\psi\in\mathrm{M}\,. \end{aligned} \tag{3.35}$$

In turn, as a consequence of the discussions in Sections 3.2.1 and 3.2.3, the second fully-mixed formulation for (3.1)-(3.2) is given by (3.10) jointly with (3.30), that is: Find $(\boldsymbol{\sigma},\mathbf{u})\in\mathbf{X}_2\times\mathbf{M}_1$ and $(\vec{\phi},\widetilde{\boldsymbol{\sigma}})\in\mathbf{H}\times\mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau})+\mathbf{b}_1(\boldsymbol{\tau},\mathbf{u}) &= G(\boldsymbol{\tau}) && \forall\,\boldsymbol{\tau}\in\mathbf{X}_1,\\ \mathbf{b}_2(\boldsymbol{\sigma},\mathbf{v}) &= F_\phi(\mathbf{v}) && \forall\,\mathbf{v}\in\mathbf{M}_2,\\ a_{\boldsymbol{\sigma}}(\vec{\phi},\vec{\varphi}) + b(\vec{\varphi},\widetilde{\boldsymbol{\sigma}}) &= G_{\mathbf{u}}(\vec{\varphi}) && \forall\,\vec{\varphi}\in\mathbf{H}\,,\\ b(\vec{\phi},\widetilde{\boldsymbol{\tau}}) &= 0 && \forall\,\widetilde{\boldsymbol{\tau}}\in\mathbf{Q}\,. \end{aligned} \tag{3.36}$$

3.3 The continuous solvability analysis

In this section we proceed similarly as in [29] and [52] (see also [21], [55], and some references therein), and adopt a fixed-point strategy to analyse the solvability of (3.35) and (3.36). To this end, we use the Babuška-Brezzi theory in Banach spaces (cf. [12, Theorem 2.1, Corollary 2.1, Section 2.1] for the general case, and [41, Theorem 2.34] for a particular one) to prove the well-posedness of the uncoupled problems (3.10), (3.22), and (3.30).

3.3.1 Well-posedness of the elasticity equation

We begin by letting $\mathbf{S} : \mathrm{M} \to \mathbf{X}_2 \times \mathbf{M}_1$ be the operator defined by

$$\mathbf{S}(\varphi) \,=\, (\mathbf{S}_1(\varphi), \mathbf{S}_2(\varphi)) \,:=\, (\boldsymbol{\sigma}, \mathbf{u}) \qquad \forall\, \varphi \in \mathrm{M}\,, \tag{3.37}$$

where $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X}_2 \times \mathbf{M}_1$ is the unique solution (to be confirmed below) of the mixed formulation for the elasticity equation (cf. (3.10)) with φ instead of ϕ, that is

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) \,+\, \mathbf{b}_1(\boldsymbol{\tau}, \widetilde{\mathbf{u}}) \quad &= \quad G(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{X}_1, \\ \mathbf{b}_2(\boldsymbol{\sigma}, \mathbf{v}) \quad &= \quad F_\varphi(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{M}_2\,. \end{aligned} \tag{3.38}$$

Then, assuming that the Lamé parameter λ is sufficiently large, namely $\lambda > M$, where M is specified in Lemma 2.4, we can establish that the operator $\mathbf{S}$ (cf. (3.37)) is well-defined. Indeed, letting $\boldsymbol{\alpha}$, $\boldsymbol{\beta}_1$, and $\boldsymbol{\beta}_2$ be the constants yielding the continuous inf-sup conditions for $\mathbf{a}$, $\mathbf{b}_1$, and $\mathbf{b}_2$ (cf. Lemmas 2.4 and 2.5), and bearing in mind the norms and the constant C_r defined in (3.13) and (3.14), respectively, a simple application of [12, Theorem 2.1, Corollary 2.1, Section 2.1] leads to the following result (cf. 2.6).

Lemma 3.1. *For each $\varphi \in \mathrm{M}$ there exists a unique $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X}_2 \times \mathbf{M}_1$ solution of (3.38), and hence one can define $\mathbf{S}(\varphi) = \big(\mathbf{S}_1(\varphi), \mathbf{S}_2(\varphi)\big) := (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X}_2 \times \mathbf{M}_1$. Moreover, there hold*

$$\begin{aligned} \|\mathbf{S}_1(\varphi)\|_{\mathbf{X}_2} \,&=\, \|\boldsymbol{\sigma}\|_{\mathbf{X}_2} \,\leq\, \frac{C_r}{\boldsymbol{\alpha}}\, \|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_2} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) f_2\,, \quad \textit{and} \\ \|\mathbf{S}_2(\varphi)\|_{\mathbf{M}_1} \,&=\, \|\mathbf{u}\|_{\mathbf{M}_1} \,\leq\, \frac{C_r}{\boldsymbol{\beta}_1} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) \|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu \boldsymbol{\beta}_1 \boldsymbol{\beta}_2} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) f_2\,. \end{aligned} \tag{3.39}$$

At this point we recall that the proof of the inf-sup condition for $\mathbf{a}$, given in Lemma 2.4, requires the result stating that for each $t \in (1, +\infty)$ there exists a positive constant $\widehat{C}_t$ such that

$$\|\boldsymbol{\tau}\|_{0,t;\Omega} \,\leq\, \widehat{C}_t \left\{ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,t;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \right\} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0^t(\mathbf{div}_t; \Omega)\,.$$

In this regard, we remark that the foregoing inequality, whose proof is provided in [48, Lemma 3.3], makes use of the surjectivity of the operator $\mathrm{div} : \mathbf{W}^{1,t}(\Omega) \to \mathrm{L}_0^t(\Omega) := \big\{ v \in \mathrm{L}^t(\Omega) : \quad \int_\Omega v = 0 \big\}$, which, in turn, requires that Ω be star-shaped with respect to a ball (cf. [41, Lemma B.69]). This is the reason why this hypothesis is assumed on Ω.

3.3.2 Well-posedness of the first approach for the diffusion equation

We now let $\widetilde{\mathrm{S}} : \mathbf{X}_2 \times \mathbf{M}_1 \to \mathbf{Q} \times \mathrm{M}$ be the operator defined by

$$\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \mathbf{w}) \,=\, (\widetilde{\mathrm{S}}_1(\boldsymbol{\zeta}, \mathbf{w}), \widetilde{\mathrm{S}}_2(\boldsymbol{\zeta}, \mathbf{w})) \,:=\, (\widetilde{\boldsymbol{\sigma}}, \phi) \qquad \forall\, (\boldsymbol{\zeta}, \mathbf{w}) \in \mathbf{X}_2 \times \mathbf{M}_1, \tag{3.40}$$

where $(\widetilde{\boldsymbol{\sigma}}, \phi) \in \mathbf{Q} \times \mathrm{M}$ is the unique solution (to be confirmed below) of (3.22) with $(\boldsymbol{\zeta}, \mathbf{w})$ instead of $(\boldsymbol{\sigma}, \mathbf{u})$, that is

$$\begin{aligned} \widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\tau}}, \phi) \quad &= \quad 0 \qquad\qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,, \\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}}, \psi) \quad &= \quad G_{\mathbf{w}}(\psi) \qquad \forall\, \psi \in \mathrm{M}\,. \end{aligned} \tag{3.41}$$

Next, we let $\widetilde{\mathcal{K}}$ be the kernel of the bilinear form $\widetilde{b}$ (cf. (3.24)), which reduces to

$$\widetilde{\mathcal{K}} := \left\{ \widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_s;\Omega) : \quad \mathrm{div}(\widetilde{\boldsymbol{\tau}}) = 0 \right\}.$$

Then, bearing in mind the uniform positiveness of $\widetilde{\vartheta}$ (cf. (3.5)), the definition of $\widetilde{a}_{\boldsymbol{\zeta}}$ (cf. (3.23)), and the norm of $\mathbf{H}(\mathrm{div}_s;\Omega)$ (cf. (6)), we readily deduce that

$$\widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\tau}},\widetilde{\boldsymbol{\tau}}) \geq \widetilde{\vartheta}_0 \|\widetilde{\boldsymbol{\tau}}\|^2_{\mathbf{Q}} \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \widetilde{\mathcal{K}}, \quad \forall\, \boldsymbol{\zeta} \in \mathbf{X}_2\,, \tag{3.42}$$

which yields the continuous inf-sup condition for $\widetilde{a}_{\boldsymbol{\zeta}}$ (cf. [41, eq. (2.28), Theorem 2.34]) with constant $\widetilde{\alpha} = \widetilde{\vartheta}_0$. In addition, we know from [52, Lemma 2.9] that there exists a positive constant $\widetilde{\beta}$ such that

$$\sup_{\substack{\widetilde{\boldsymbol{\tau}}\in\mathbf{Q}\\ \widetilde{\boldsymbol{\tau}}\neq 0}} \frac{\widetilde{b}(\widetilde{\boldsymbol{\tau}},\psi)}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}}} \geq \widetilde{\beta}\|\psi\|_{\mathrm{M}} \qquad \forall\, \psi \in \mathrm{M}\,, \tag{3.43}$$

which establishes the continuous inf-sup condition for $\widetilde{b}$.

Hence, we are in position to state that the operator $\widetilde{\mathrm{S}}$ is well-defined.

Lemma 3.2. *For each $(\boldsymbol{\zeta},\mathbf{w}) \in \mathbf{X}_2 \times \mathbf{M}_1$ there exists a unique $(\widetilde{\boldsymbol{\sigma}},\phi) \in \mathbf{Q}\times\mathrm{M}$ solution of (3.41), and hence one can define $\widetilde{\mathrm{S}}(\boldsymbol{\zeta},\mathbf{w}) := (\widetilde{\boldsymbol{\sigma}},\phi) \in \mathbf{Q}\times\mathrm{M}$. Moreover, there hold*

$$\|\widetilde{\mathrm{S}}_1(\boldsymbol{\zeta},\mathbf{w})\|_{\mathbf{Q}} = \|\widetilde{\boldsymbol{\sigma}}\|_{\mathbf{Q}} \leq \frac{1}{\widetilde{\beta}}\Big(1 + \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}}\Big)|\Omega|^{1/s} g_2\,, \quad \textit{and} \tag{3.44}$$

$$\|\widetilde{\mathrm{S}}_2(\boldsymbol{\zeta},\mathbf{w})\|_{\mathrm{M}} = \|\phi\|_{\mathrm{M}} \leq \frac{\widetilde{\vartheta}_2}{\widetilde{\beta}^2}\Big(1 + \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}}\Big)|\Omega|^{1/s} g_2\,. \tag{3.45}$$

Proof. Knowing from (3.42) and (3.43) that, given $(\boldsymbol{\zeta},\mathbf{w}) \in \mathbf{X}_2\times\mathbf{M}_1$, $\widetilde{a}_{\boldsymbol{\zeta}}$ and $\widetilde{b}$ satisfies the hypotheses of [41, Theorem 2.34], and noting that $\mathbf{Q} := \mathbf{H}(\mathrm{div}_s;\Omega)$ and $\mathrm{M} := \mathrm{L}^r(\Omega)$ are reflexive Banach spaces, the proof reduces to a straightforward application of the aforementioned theorem. In this way, the a priori estimates (3.44) and (3.45) follow from [41, eq. (2.30), Theorem 2.34] and (3.26). □

3.3.3 Well-posedness of the second approach for the diffusion equation

Similarly to the analysis of previous sections, we let $\mathrm{S} : \mathbf{X}_2 \times \mathbf{M}_1 \to \mathbf{H}$ be the operator given by

$$\mathrm{S}(\boldsymbol{\zeta},\mathbf{w}) = (\mathrm{S}_1(\boldsymbol{\zeta},\mathbf{w}), \mathrm{S}_2(\boldsymbol{\zeta},\mathbf{w})) := \vec{\phi} \quad \forall\,(\boldsymbol{\zeta},\mathbf{w}) \in \mathbf{X}_2\times\mathbf{M}_1\,, \tag{3.46}$$

where $(\vec{\phi},\widetilde{\boldsymbol{\sigma}}) := ((\phi,\mathbf{t}),\widetilde{\boldsymbol{\sigma}}) \in \mathbf{H}\times\mathbf{Q}$ is the unique solution (to be confirmed below) of problem (3.30) with $(\boldsymbol{\zeta},\mathbf{w})$ instead of $(\boldsymbol{\sigma},\mathbf{u})$, that is

$$\begin{aligned} a_{\boldsymbol{\zeta}}(\vec{\phi},\vec{\varphi}) + b(\vec{\varphi},\widetilde{\boldsymbol{\sigma}}) &= G_{\mathbf{w}}(\vec{\varphi}) \qquad \forall\,\vec{\varphi} \in \mathbf{H}\,, \\ b(\vec{\phi},\widetilde{\boldsymbol{\tau}}) &= 0 \qquad\qquad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,. \end{aligned} \tag{3.47}$$

Here we apply [41, Theorem 2.34] to prove that problem (3.47) is well-posed (equivalently, that S is well-defined). In this regard, it is important to stress that the structure of (3.47) is similar to the one of [29, eq. (3.23)], and hence, several results and techniques from there will be employed in what follows. Indeed, let V the kernel of the operator induced by b (cf. (3.32), which reduces to

$$V := \Big\{ \vec{\varphi} = (\varphi, \mathbf{s}) \in \mathbf{H} := \mathrm{M} \times \mathbf{L}^2(\Omega) : \quad \nabla\varphi = \mathbf{s} \Big\}. \tag{3.48}$$

Now, we let c_P be the positive constant yielding the Friedrichs-Poincaré inequality, which states that $|\varphi|_{1,\Omega}^2 \geq c_P \|\varphi\|_{1,\Omega}^2$ for all $\varphi \in \mathrm{H}_0^1(\Omega)$, and denote by i_r the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^r(\Omega)$. In addition, we consider an arbitrary $\boldsymbol{\zeta} \in \mathbf{X}_2$. Then, bearing in mind (3.3) and proceeding analogously to the proof of [29, eq. (3.41), Lemma 3.2], we find that

$$a_{\boldsymbol{\zeta}}(\vec{\varphi}, \vec{\varphi}) \geq \alpha \|\vec{\varphi}\|_{\mathbf{H}}^2 \quad \forall\, \vec{\varphi} \in V\,, \tag{3.49}$$

with

$$\alpha := \frac{\vartheta_0}{2} \min\{1, \frac{c_P}{\|i_r\|}\},$$

which proves the V-ellipticity of $a_{\boldsymbol{\zeta}}$. In turn, a slight modification of the proof of [29, Lemma 3.3] allows us to prove the existence of a positive constant β such that

$$\sup_{\substack{\vec{\varphi} \in \mathbf{H} \\ \vec{\varphi} \neq \mathbf{0}}} \frac{b(\vec{\varphi}, \widetilde{\boldsymbol{\tau}})}{\|\vec{\varphi}\|_{\mathbf{H}}} \geq \beta \|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}} \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,, \tag{3.50}$$

whence the bilinear form b satisfies the continuous inf-sup condition required by [41, Theorem 2.34].

We are now in position to confirm that the operator S is well-defined.

Lemma 3.3. *For each $(\boldsymbol{\zeta}, \mathbf{w}) \in \mathbf{X}_2 \times \mathbf{M}_1$ there exists a unique $(\vec{\phi}, \widetilde{\boldsymbol{\sigma}}) \in \mathbf{H} \times \mathbf{Q}$ solution of (3.47), and hence one can define $\mathrm{S}(\boldsymbol{\zeta}, \mathbf{w}) := \vec{\phi} \in \mathbf{H}$. Moreover, there holds*

$$\|\mathrm{S}(\boldsymbol{\zeta}, \mathbf{w})\|_{\mathbf{H}} = \|\vec{\phi}\|_{\mathbf{H}} = \|\phi\|_{0,r;\Omega} + \|\mathbf{t}\|_{0,\Omega} \leq \frac{|\Omega|^{1/s}}{\alpha}\, g_2\,. \tag{3.51}$$

Proof. Thanks to (3.34), (3.49) and (3.50), a straightforward application of [41, Theorem 2.34] yields the existence of a unique solution $(\vec{\phi}, \widetilde{\boldsymbol{\sigma}}) \in \mathbf{H} \times \mathbf{Q}$ to (3.30). Moreover, the corresponding a priori estimate given by the first inequality of [41, eq. (2.30)], along with the expression for $\|G_{\mathbf{w}}\|$ provided by (3.34), lead to (3.51). □

Regarding the a priori estimate for the component $\widetilde{\boldsymbol{\sigma}}$ of the unique solution of (3.30), which will be used later on, we recall that the second inequality in [41, eq. (2.30)] and (3.34) implies

$$\|\widetilde{\boldsymbol{\sigma}}\|_{\mathbf{Q}} \leq \frac{|\Omega|^{1/s}}{\beta} \left(1 + \frac{\vartheta_2}{\alpha}\right) g_2\,. \tag{3.52}$$

3.3.4 Solvability of the first fully-mixed formulation

We begin by defining the compose operator $\Xi : \mathrm{M} \to \mathrm{M}$ as

$$\Xi(\psi) := \widetilde{\mathrm{S}}_2\big(\mathbf{S}(\psi)\big) \qquad \forall\, \psi \in \mathrm{M}\,. \tag{3.53}$$

Then, knowing that the operators $\widetilde{\mathrm{S}}$ and $\mathbf{S}$, and hence Ξ as well, are well-defined, we notice that solving (3.35) is equivalent to seeking a fixed point of Ξ, that is: Find $\psi \in \mathrm{M}$ such that

$$\Xi(\psi) \,=\, \psi\,. \tag{3.54}$$

Next, in order to address the solvability of (3.54) (equivalently of (3.35)), we verify the hypotheses of the Banach fixed-point theorem. For this purpose, let us first introduce the ball

$$\widetilde{W} \,:=\, \Big\{\phi \in \mathrm{M}: \quad \|\phi\|_{\mathrm{M}} \,\le\, \widetilde{\delta}\Big\}\,, \tag{3.55}$$

with

$$\widetilde{\delta} \,:=\, \frac{\widetilde{\vartheta}_2}{\widetilde{\beta}^2}\Big(1 \,+\, \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}}\Big)\,|\Omega|^{1/s} g_2\,.$$

It follows from the definition of Ξ (cf. (3.53)) and the a priori estimate for $\widetilde{\mathrm{S}}_2$ (cf. (3.45)) that

$$\Xi(\widetilde{W}) \,\subseteq\, \widetilde{W}\,. \tag{3.56}$$

Now, in order to establish the continuity of Ξ, we previously establish those of $\mathbf{S}$ and $\widetilde{\mathrm{S}}$. Indeed, resorting to a slight modification of Lemma 2.9, we deduce the existence of a positive constant $C_{\mathbf{S}}$, depending only on μ, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}_1$, and $\boldsymbol{\beta}_2$, such that

$$\|\mathbf{S}(\phi) - \mathbf{S}(\varphi)\|_{\mathbf{X}_2\times\mathbf{M}_1} \,\le\, C_{\mathbf{S}}\, L_f\, \|\phi - \varphi\|_{\mathrm{M}} \qquad \forall\, \phi,\, \varphi \in \mathrm{M}\,, \tag{3.57}$$

which proves the Lipschitz-continuity of $\mathbf{S}$. Furthermore, for the same property of $\widetilde{\mathrm{S}}$, the approach from several previous works (see, e.g. [7], [30], [46], [47], and [52]) is adopted here, so that a regularity assumption on the solution of the problem defining this operator is introduced. More precisely, from now on we suppose that there exists $\varepsilon \ge \frac{n}{r}$ and a positive constant $\widetilde{C}_\varepsilon$, such that

($\mathbf{RA}_1$) for each $(\boldsymbol{\zeta}, \mathbf{w}) \in \mathbf{X}_2 \times \mathbf{M}_1$ there holds $\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \mathbf{w}) = (\widetilde{\boldsymbol{\sigma}}, \phi) \in (\mathbf{Q} \cap \mathbf{H}^\varepsilon(\Omega)) \times \mathbf{W}^{\varepsilon,r}(\Omega)$, and

$$\|\widetilde{\boldsymbol{\sigma}}\|_{\varepsilon,\Omega} \,+\, \|\phi\|_{\varepsilon,r;\Omega} \,\le\, \widetilde{C}_\varepsilon\, g_2\,. \tag{3.58}$$

The aforementioned lower bound of ε is explained within the proof of Lemma 3.4 below, which provides the Lipschitz-continuity of $\widetilde{\mathrm{S}}$. In this regard, we recall now from [74, Theorem 1.3.4, part a)] (see, also [56, Theorem 1.4.5.2, part e)]) that for each $\varepsilon < \frac{n}{2}$ there holds $\mathbf{H}^\varepsilon(\Omega) \subset \mathbf{L}^{\varepsilon^*}(\Omega)$, with continuous injection

$$i_\varepsilon : \mathbf{H}^\varepsilon(\Omega) \longrightarrow \mathbf{L}^{\varepsilon^*}(\Omega)\,, \quad \text{where} \quad \varepsilon^* = \frac{2n}{n - 2\varepsilon}\,. \tag{3.59}$$

Note that the indicated lower and upper bounds for the additional regularity ε, which turn out to require that $\varepsilon \in [\frac{n}{r}, \frac{n}{2})$, are compatible if and only if $r > 2$, which is coherent with the range stipulated in (3.9).

Now, regarding the feasibility of ($\mathbf{RA}_1$), we first stress that, given $(\boldsymbol{\zeta}, \mathbf{w}) \in \mathbf{X}_2 \times \mathbf{M}_1$, this hypothesis is actually determined by the regularity of the second order elliptic problem in divergence form arising from the second row of (3.1) (equivalently, (3.17)), that is

$$\widetilde{\boldsymbol{\sigma}} \,=\, \vartheta(\boldsymbol{\zeta})\nabla\phi \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \,=\, g(\mathbf{w}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,, \tag{3.60}$$

whose variational formulation has been set in the form given by (3.41). Then, assuming that $\vartheta(\boldsymbol{\zeta})$ and $g(\mathbf{w})$ are sufficiently smooth, and denoting by $\omega \in (0,\pi)\cup(\pi,2\pi)$ the largest interior angle of Ω, the respective elliptic regularity result (see, e.g. [35, Theorem 14.6] or [57]) establishes that in this case there holds $\phi \in \mathrm{H}^{1+\varepsilon}(\Omega)$ for all $\varepsilon \in (0,\frac{\pi}{\omega})$. It follows that $\nabla\phi \in \mathbf{H}^{\varepsilon}(\Omega)$, and hence, invoking again a suitable smoothness assumption on $\vartheta(\boldsymbol{\zeta})$, one would deduce that $\widetilde{\boldsymbol{\sigma}}$ belongs to $\mathbf{H}^{\varepsilon}(\Omega)$ as well. In addition, considering for simplicity the 2D case, we know from [56, Theorem 1.4.5.2, part e)] that the space $\mathrm{H}^t(\Omega) = \mathrm{W}^{t,2}(\Omega)$ is embedded in $\mathrm{W}^{\varepsilon,r}(\Omega)$ if $t = 1+\varepsilon-\frac{2}{r}$ and $\varepsilon \le t$. Thus, knowing that $r > 2$ (cf. (3.9)), it is easily seen that the inequality between ε and t is satisfied, and since $1+\varepsilon > t$, we get the continuous injections depicted as follows

$$\mathrm{H}^{1+\varepsilon}(\Omega) \hookrightarrow \mathrm{H}^{t}(\Omega) \hookrightarrow \mathrm{W}^{\varepsilon,r}(\Omega)\,,$$

from which we conclude that $\phi \in \mathrm{W}^{\varepsilon,r}(\Omega)$. In turn, (3.58) should follow from the a priori estimate for $\|\phi\|_{1+\varepsilon,\Omega}$ in terms of $g(\mathbf{w})$, and the fact that $|g|$ is bounded by g_2 (cf. (3.8)). Summarizing, the above discussion confirms $(\mathbf{RA}_1)$ as an achievable assumption, and hence, in order to be able to pick ε such that $\varepsilon \ge \frac{n}{r}$, which is needed below, it suffices to impose that $\frac{\pi}{\omega} > \frac{n}{r}$, that is $\omega < \frac{r}{n}\pi$, and hence $\omega < \min\left\{1,\frac{r}{n}\right\}2\pi$, which constitutes just a geometric condition on Ω.

We now use $(\mathbf{RA}_1)$ to prove the announced property of $\widetilde{\mathrm{S}}$.

Lemma 3.4. *There exists a positive constant $C_{\widetilde{\mathrm{S}}}$, depending only on $\widetilde{\alpha}$, $\widetilde{\beta}$, $|\Omega|$, r, ε, $\|i_\varepsilon\|$ M(cf. (3.59)M), and $\widetilde{C}_\varepsilon$ M(cf. (3.58)M), such that*

$$\|\widetilde{\mathrm{S}}(\boldsymbol{\zeta},\mathbf{w}) - \widetilde{\mathrm{S}}(\boldsymbol{\tau},\mathbf{v})\|_{\mathbf{Q}\times\mathrm{M}} \le C_{\widetilde{\mathrm{S}}}\left\{L_{\widetilde{\vartheta}}\,g_2 + L_g\right\}\|(\boldsymbol{\zeta},\mathbf{w}) - (\boldsymbol{\tau},\mathbf{v})\|_{\mathbf{X}_2\times\mathbf{M}_1} \tag{3.61}$$

for all $(\boldsymbol{\zeta},\mathbf{w}),\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbf{X}_2\times\mathbf{M}_1$.

Proof. We begin by noticing that the a priori estimates (3.44) and (3.45) of problem (3.41), with a given $(\boldsymbol{\zeta},\mathbf{w}) \in \mathbf{X}_2\times\mathbf{M}_1$, are equivalent to stating that

$$\|(\widetilde{\boldsymbol{\zeta}},\varphi)\|_{\mathbf{Q}\times\mathrm{M}} \le C \sup_{\substack{(\widetilde{\boldsymbol{\tau}},\psi)\in\mathbf{Q}\times\mathrm{M}\\(\widetilde{\boldsymbol{\tau}},\psi)\neq 0}} \frac{\widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}},\varphi) + \widetilde{b}(\widetilde{\boldsymbol{\zeta}},\psi)}{\|(\widetilde{\boldsymbol{\tau}},\psi)\|_{\mathbf{Q}\times\mathrm{M}}} \quad \forall\,(\widetilde{\boldsymbol{\zeta}},\varphi) \in \mathbf{Q}\times\mathrm{M}\,, \tag{3.62}$$

with a positive constant C that depends only on $\widetilde{\vartheta}_2$, $\widetilde{\alpha}$, and $\widetilde{\beta}$, and hence independent of $(\boldsymbol{\zeta},\mathbf{w})$. Next, given $(\boldsymbol{\zeta},\mathbf{w}),\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbf{X}_2\times\mathbf{M}_1$, we let

$$\widetilde{\mathrm{S}}(\boldsymbol{\zeta},\mathbf{w}) := (\widetilde{\boldsymbol{\sigma}},\phi) \quad \text{and} \quad \widetilde{\mathrm{S}}(\boldsymbol{\tau},\mathbf{v}) := (\widetilde{\boldsymbol{\zeta}},\varphi)\,,$$

which, according to (3.40) and (3.41), means, respectively, that

$$\begin{aligned} \widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}},\phi) &= 0 && \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,,\\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}},\psi) &= G_{\mathbf{w}}(\psi) && \forall\,\psi \in \mathrm{M}\,, \end{aligned} \tag{3.63}$$

and

$$\begin{aligned} \widetilde{a}_{\boldsymbol{\tau}}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}},\varphi) &= 0 && \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,,\\ \widetilde{b}(\widetilde{\boldsymbol{\zeta}},\psi) &= G_{\mathbf{v}}(\psi) && \forall\,\psi \in \mathrm{M}\,. \end{aligned} \tag{3.64}$$

Then, applying (3.62) to $\widetilde{\mathrm{S}}(\boldsymbol{\zeta},\mathbf{w})-\widetilde{\mathrm{S}}(\boldsymbol{\tau},\mathbf{v})=(\widetilde{\boldsymbol{\sigma}}-\widetilde{\boldsymbol{\zeta}},\phi-\varphi)$, and using (3.63) and (3.64), we get

$$\begin{aligned}\|\widetilde{\mathrm{S}}(\boldsymbol{\zeta},\mathbf{w})-\widetilde{\mathrm{S}}(\boldsymbol{\tau},\mathbf{v})\|_{\mathbf{Q}\times\mathrm{M}} &\le C \sup_{\substack{(\widetilde{\boldsymbol{\tau}},\psi)\in\mathbf{Q}\times\mathrm{M}\\ (\widetilde{\boldsymbol{\tau}},\psi)\neq 0}} \frac{\widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\sigma}}-\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}})+\widetilde{b}(\widetilde{\boldsymbol{\tau}},\phi-\varphi)+\widetilde{b}(\widetilde{\boldsymbol{\sigma}}-\widetilde{\boldsymbol{\zeta}},\psi)}{\|(\widetilde{\boldsymbol{\tau}},\psi)\|_{\mathbf{Q}\times\mathrm{M}}}\\ &\le C \sup_{\substack{(\widetilde{\boldsymbol{\tau}},\psi)\in\mathbf{Q}\times\mathrm{M}\\ (\widetilde{\boldsymbol{\tau}},\psi)\neq 0}} \frac{\widetilde{a}_{\boldsymbol{\tau}}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}})-\widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}})+(G_{\mathbf{w}}-G_{\mathbf{v}})(\psi)}{\|(\widetilde{\boldsymbol{\tau}},\psi)\|_{\mathbf{Q}\times\mathrm{M}}}\,.\end{aligned} \tag{3.65}$$

Thus, bearing in mind the definitions of $\widetilde{a}_{\boldsymbol{\tau}}$ and $\widetilde{a}_{\boldsymbol{\zeta}}$, and using the Lipschitz-continuity of $\widetilde{\vartheta}$ (cf. (3.6)) along with the Cauchy-Schwarz and Hölder inequalities, we find that

$$|\widetilde{a}_{\boldsymbol{\tau}}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}})-\widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}})| \le L_{\widetilde{\vartheta}}\,\|(\boldsymbol{\tau}-\boldsymbol{\zeta})\,\widetilde{\boldsymbol{\zeta}}\|_{0,\Omega}\,\|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega} \le L_{\widetilde{\vartheta}}\,\|\boldsymbol{\tau}-\boldsymbol{\zeta}\|_{0,2q;\Omega}\,\|\widetilde{\boldsymbol{\zeta}}\|_{0,2p,\Omega}\,\|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}\,, \tag{3.66}$$

where $p,\,q\in(1,+\infty)$ are conjugate to each other. Now, choosing p such that $2p=\varepsilon^*$ (cf. (3.59)), we get $2q=\frac{n}{\varepsilon}$, which, according to the range stipulated for ε, yields $2q\le r$, and thus the norm of the embedding of $\mathbf{L}^r(\Omega)$ into $\mathbf{L}^{2q}(\Omega)=\mathbf{L}^{\frac{n}{\varepsilon}}(\Omega)$ is given by $C_{r,\varepsilon}:=|\Omega|^{\frac{r\varepsilon-n}{rn}}$. In this way, using additionally the continuity of i_ε (cf. (3.59)) along with the regularity estimate (3.58), the inequality (3.66) becomes

$$\begin{aligned}|\widetilde{a}_{\boldsymbol{\tau}}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}})-\widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}})| &\le L_{\widetilde{\vartheta}}\,C_{r,\varepsilon}\,\|\boldsymbol{\tau}-\boldsymbol{\zeta}\|_{0,r;\Omega}\,\|i_\varepsilon\|\,\|\widetilde{\boldsymbol{\zeta}}\|_{\varepsilon,\Omega}\,\|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}\\ &\le L_{\widetilde{\vartheta}}\,C_{r,\varepsilon}\,\|i_\varepsilon\|\,\widetilde{C}_\varepsilon\,g_2\,\|\boldsymbol{\tau}-\boldsymbol{\zeta}\|_{\mathbf{X}_2}\,\|(\widetilde{\boldsymbol{\tau}},\psi)\|_{\mathbf{Q}\times\mathrm{M}}\,.\end{aligned} \tag{3.67}$$

In turn, the Lipschitz-continuity of g (cf. (3.8)), the fact that $s<r$ (cf. (3.9)), and Hölder's inequality, yield

$$\begin{aligned}|(G_{\mathbf{w}}-G_{\mathbf{v}})(\psi)| &\le L_g\,\|\mathbf{w}-\mathbf{v}\|_{0,r;\Omega}\,\|\psi\|_{0,s;\Omega} \le L_g\,|\Omega|^{\frac{r-s}{rs}}\,\|\mathbf{w}-\mathbf{v}\|_{0,r;\Omega}\,\|\psi\|_{0,r;\Omega}\\ &\le L_g\,|\Omega|^{\frac{r-s}{rs}}\,\|\mathbf{w}-\mathbf{v}\|_{\mathbf{M}_2}\,\|(\widetilde{\boldsymbol{\tau}},\psi)\|_{\mathbf{Q}\times\mathrm{M}}\,.\end{aligned} \tag{3.68}$$

Finally, replacing (3.67) and (3.68) back into (3.65), we arrive at (3.61), which ends the proof. □

We are able to prove now the Lipschitz-continuity of Ξ in the closed ball $\widetilde{W}$ of $\mathrm{M}:=\mathrm{L}^r(\Omega)$.

Lemma 3.5. *There exists a positive constant C_Ξ, depending only on $C_{\mathbf{S}}$ and $C_{\widetilde{\mathrm{S}}}$, such that*

$$\|\Xi(\phi)-\Xi(\varphi)\|_{\mathrm{M}} \le C_\Xi\,L_f\left\{L_g+L_{\widetilde{\vartheta}}\,g_2\right\}\|\phi-\varphi\|_{\mathrm{M}} \qquad \forall\,\phi,\,\varphi\in\mathrm{M}\,. \tag{3.69}$$

Proof. It readily follows from the definition of Ξ (cf. (3.53)), and the estimates (3.57) and (3.61), which yields $C_\Xi:=C_{\mathbf{S}}\,C_{\widetilde{\mathrm{S}}}$. □

Consequently, the main result of this subsection is stated as follows.

Theorem 3.6. *Assume the regularity assumption* $(\mathbf{RA_1})$ *(cf. (3.58)), and that the data L_f, L_g, $L_{\widetilde{\vartheta}}$, and g_2 are sufficiently small so that*

$$C_\Xi\,L_f\left\{L_g+L_{\widetilde{\vartheta}}\,g_2\right\} < 1\,. \tag{3.70}$$

Then, Ξ has a unique fixed point ϕ in $\widetilde{W}$. Equivalently, the coupled problem (3.35) has a unique solution $\big((\boldsymbol{\sigma},\mathbf{u}),(\widetilde{\boldsymbol{\sigma}},\phi)\big) \in \big(\mathbf{X}_2 \times \mathbf{M}_1\big) \times \big(\mathbf{Q} \times \mathrm{M}\big)$, with $\phi \in \widetilde{W}$ (cf. (3.55)). Moreover, there hold

$$\begin{aligned} \|\boldsymbol{\sigma}\|_{\mathbf{X}_2} \,&\le\, \frac{C_r}{\boldsymbol{\alpha}}\,\|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_2}\left(1+\frac{2}{\boldsymbol{\alpha}\mu}\right) f_2\,, \\ \|\mathbf{u}\|_{\mathbf{M}_1} \,&\le\, \frac{C_r}{\boldsymbol{\beta}_1}\left(1+\frac{2}{\boldsymbol{\alpha}\mu}\right) \|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu\boldsymbol{\beta}_1\boldsymbol{\beta}_2}\left(1+\frac{2}{\boldsymbol{\alpha}\mu}\right) f_2\,, \quad \textit{and} \\ \|\widetilde{\boldsymbol{\sigma}}\|_{\mathbf{Q}} \,&\le\, \frac{1}{\widetilde{\beta}_2}\Big(1\,+\,\frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}}\Big)|\Omega|^{1/s} g_2\,. \end{aligned} \tag{3.71}$$

Proof. Thanks to (3.56), Lemma 3.5, and the assumption (3.70), the existence of a unique $\phi \in \widetilde{W}$ solution to (3.54) (equivalently, the existence of a unique $\big((\boldsymbol{\sigma},\mathbf{u}),(\widetilde{\boldsymbol{\sigma}},\phi)\big) \in \big(\mathbf{X}_2 \times \mathbf{M}_1\big) \times \big(\mathbf{Q} \times \mathrm{M}\big)$ solution to (3.35)), follows from a straightforward application of the Banach fixed point Theorem. In addition, noting that $(\boldsymbol{\sigma},\mathbf{u}) = \mathbf{S}(\phi)$ and $(\widetilde{\boldsymbol{\sigma}},\phi) = \widetilde{\mathrm{S}}(\boldsymbol{\sigma},\mathbf{u})$, the a priori estimates (3.39) and (3.44) yield (3.71), which ends the proof. □

3.3.5 Solvability of the second fully-mixed formulation

Similarly to Section 3.3.4, for the solvability analysis of (3.36) we define the operator $\Lambda : \mathrm{M} \to \mathrm{M}$ as

$$\Lambda(\psi) \,:=\, \mathrm{S}_1\big(\mathbf{S}(\psi)\big) \qquad \forall\,\psi \in \mathrm{M}\,. \tag{3.72}$$

Then, noticing that S and $\mathbf{S}$, and hence Λ as well, are well-defined, we realize that solving (3.36) is equivalent to finding a fixed point of Λ, that is: Find $\psi \in \mathrm{M}$ such that

$$\Lambda(\psi) \,=\, \psi\,. \tag{3.73}$$

In what follows we show that Λ verifies the hypotheses of the respective Banach Theorem. We begin by defining the ball

$$W \,:=\, \Big\{\phi \in \mathrm{M}: \quad \|\phi\|_{\mathrm{M}} \,\le\, \delta\Big\}\,, \tag{3.74}$$

with

$$\delta \,:=\, \frac{|\Omega|^{1/s}}{\alpha}\, g_2\,,$$

so that from the definition of Λ (cf. (3.72)) and the a priori estimate for S_1 (cf. (3.51)), we get

$$\Lambda(W) \,\subseteq\, W\,. \tag{3.75}$$

Next, in order to prove that Λ is Lipschitz-continuous, and similarly to $(\mathbf{RA}_1)$, we need to introduce a regularity hypothesis on the solution of the problem defining the operator S. More precisely, we assume that there exists $\varepsilon \ge \frac{n}{r}$ and a positive constant C_ε such that

$(\mathbf{RA}_2)$ for each $(\boldsymbol{\zeta},\mathbf{w}) \in \mathbf{X}_2 \times \mathbf{M}_1$ there hold $\mathrm{S}(\boldsymbol{\zeta},\mathbf{w}) \,:=\, (\phi,\mathbf{t}) \in \mathbf{W}^{\varepsilon,r}(\Omega) \times \mathbf{H}^{\varepsilon}(\Omega)$, and

$$\|\phi\|_{\varepsilon,r;\Omega} \,+\, \|\mathbf{t}\|_{\varepsilon,\Omega} \,\le\, C_\varepsilon\, g_2\,. \tag{3.76}$$

Regarding the feasibility of $(\mathbf{RA}_2)$, we refer to the remark on $(\mathbf{RA}_1)$ provided in Section 3.3.4, which deals with the same boundary value problem defining the present operator S. In particular, it is also commented there that $\mathbf{t} := \nabla\phi$ would belong to $\mathbf{H}^\varepsilon(\Omega)$.

The Lipschitz-continuity of S is addressed next, whose corresponding proof requires again the lower bound of ε, and the embedding specified in (3.59). Therefore, following the same arguments from the previous section, we conclude that the feasible range for r and s are given by (3.9). The announced result is established as follows.

Lemma 3.7. *There exists a positive constant C_S, depending on α, $|\Omega|$, r, s, ε, $\|i_\varepsilon\|$ M(cf. (3.59)M), and C_ε (cf. (3.76)) such that*

$$\begin{aligned}\|\mathrm{S}(\boldsymbol{\zeta},\mathbf{w}) \,-\, \mathrm{S}(\boldsymbol{\tau},\mathbf{v})\|_{\mathbf{H}} \\ \le\, C_{\mathrm{S}}\Big\{L_g + L_\vartheta\, g_2\Big\}\,\|(\boldsymbol{\zeta},\mathbf{w}) - (\boldsymbol{\tau},\mathbf{v})\|_{\mathbf{X}_2\times\mathbf{M}_1} \quad \forall\,(\boldsymbol{\zeta},\mathbf{w}),\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbf{X}_2\times\mathbf{M}_1\,.\end{aligned} \tag{3.77}$$

Proof. Given $(\boldsymbol{\zeta},\mathbf{w})$, $(\boldsymbol{\tau},\mathbf{v}) \in \mathbf{X}_2\times\mathbf{M}_1$, we let

$$\mathrm{S}(\boldsymbol{\zeta},\mathbf{w}) \,:=\, \vec{\phi} \quad \text{and} \quad \mathrm{S}(\boldsymbol{\tau},\mathbf{v}) \,:=\, \vec{\psi}\,,$$

where $(\vec{\phi},\widetilde{\boldsymbol{\sigma}}) := ((\phi,\mathbf{t}),\widetilde{\boldsymbol{\sigma}}) \in \mathbf{H}\times\mathbf{Q}$ and $(\vec{\psi},\widetilde{\boldsymbol{\zeta}}) := ((\psi,\mathbf{r}),\widetilde{\boldsymbol{\zeta}}) \in \mathbf{H}\times\mathbf{Q}$ are the respective solutions of (3.47). It follows from the corresponding second equations of (3.47) that $\vec{\phi}-\vec{\psi} \in V$ (cf. (3.48)), and thus the V-ellipticity of $a_{\boldsymbol{\zeta}}$ (cf. (3.31)) gives

$$\alpha\,\|\vec{\phi}-\vec{\psi}\|^2_{\mathbf{H}} \,\le\, a_{\boldsymbol{\zeta}}(\vec{\phi}-\vec{\psi},\vec{\phi}-\vec{\psi})\,. \tag{3.78}$$

In turn, applying the corresponding first equations of (3.47) to $\vec{\varphi} = \vec{\phi}-\vec{\psi}$, we obtain

$$a_{\boldsymbol{\zeta}}(\vec{\phi},\vec{\phi}-\vec{\psi}) \,=\, G_{\mathbf{w}}(\vec{\phi}-\vec{\psi})\,, \quad \text{and} \tag{3.79}$$

$$a_{\boldsymbol{\tau}}(\vec{\psi},\vec{\phi}-\vec{\psi}) \,=\, G_{\mathbf{v}}(\vec{\phi}-\vec{\psi})\,, \tag{3.80}$$

so that employing (3.79), and then subtracting and adding $a_{\boldsymbol{\tau}}(\vec{\psi},\vec{\phi}-\vec{\psi})$ (cf. (3.80)), (3.78) becomes

$$\alpha\,\|\vec{\phi}-\vec{\psi}\|^2_{\mathbf{H}} \,\le\, (G_{\mathbf{w}}-G_{\mathbf{v}})(\vec{\phi}-\vec{\psi}) \,+\, (a_{\boldsymbol{\tau}}-a_{\boldsymbol{\zeta}})(\vec{\psi},\vec{\phi}-\vec{\psi})\,. \tag{3.81}$$

Next, proceeding as for (3.68), we easily get

$$(G_{\mathbf{w}}-G_{\mathbf{v}})(\vec{\phi}-\vec{\psi}) \,\le\, L_g\,|\Omega|^{\frac{r-s}{rs}}\,\|\mathbf{w}-\mathbf{v}\|_{\mathbf{M}_1}\,\|\phi-\psi\|_{\mathrm{M}}\,. \tag{3.82}$$

On the other hand, recalling that r and s are conjugate to each other with $s < r$ (cf. (3.9)), and employing the Lipschitz continuity of ϑ (cf. (3.4)) along with Hölder's inequality, we find that

$$(a_{\boldsymbol{\tau}}-a_{\boldsymbol{\zeta}})(\vec{\psi},\vec{\phi}-\vec{\psi}) \,\le L_\vartheta\,\|\boldsymbol{\tau}-\boldsymbol{\zeta}\|_{0,2q;\Omega}\,\|\mathbf{r}\|_{0,2p;\Omega}\,\|\mathbf{t}-\mathbf{r}\|_{0,\Omega}\,, \tag{3.83}$$

where p, $q \in (1,+\infty)$ are conjugate to each other as well. Then, similarly to the proof of Lemma 3.4, we choose p such that $2p = \varepsilon^*$ (cf. (3.59)), so that $2q = \frac{n}{\varepsilon} \le r$, and hence

$$(a_{\boldsymbol{\tau}}-a_{\boldsymbol{\zeta}})(\vec{\psi},\vec{\phi}-\vec{\psi}) \,\le\, L_\vartheta\, C_{r,\varepsilon}\,\|i_\varepsilon\|\, C_\varepsilon\, g_2\,\|\boldsymbol{\tau}-\boldsymbol{\zeta}\|_{0,r;\Omega}\,\|\mathbf{t}-\mathbf{r}\|_{0,\Omega}\,. \tag{3.84}$$

Thus, replacing the estimates (3.82) and (3.84) back into (3.81), we arrive at (3.77) with the constant $C_S \,:=\, \max\big\{|\Omega|^{\frac{r-s}{rs}}, C_{r,\varepsilon}\,\|i_\varepsilon\|\,C_\varepsilon\big\}$. □

We are now in position to conclude the Lipschitz-continuity of Λ.

Lemma 3.8. *There exists a positive constant C_Λ, depending only on $C_{\mathbf{S}}$ and C_S, such that*

$$\|\Lambda(\phi) - \Lambda(\varphi)\|_{\mathrm{M}} \le C_\Lambda L_f \left\{ L_g + L_\vartheta g_2 \right\} \|\phi - \varphi\|_{\mathrm{M}} \qquad \forall \phi, \varphi \in \mathrm{M}. \tag{3.85}$$

Proof. It is a direct consequence of the definition of Λ (cf. (3.72)) and the continuity properties given by (3.57) and Lemma 3.7. □

Finally, the well-posedness of (3.36) is established as follows.

Theorem 3.9. *Assume the regularity assumption* $(\mathbf{RA_2})$ *(cf. (3.76)), and that the data L_f, L_g, L_ϑ, and g_2 are sufficiently small so that*

$$C_\Lambda L_f \left\{ L_g + L_\vartheta g_2 \right\} < 1. \tag{3.86}$$

Then, Λ has a unique fixed point $\phi \in W$. Equivalently, the coupled problem (3.36) has a unique solution $\big((\boldsymbol{\sigma}, \mathbf{u}), (\vec{\phi}, \widetilde{\boldsymbol{\sigma}})\big) \in \big(\mathbf{X}_2 \times \mathbf{M}_1\big) \times \big(\mathbf{H} \times \mathbf{Q}\big)$, with $\vec{\phi} := (\phi, \mathbf{t}) \in \mathbf{H}$ and $\phi \in W$ (cf. (3.74)). Moreover, there hold

$$\begin{aligned}
\|\boldsymbol{\sigma}\|_{\mathbf{X}_2} &\le \frac{C_r}{\boldsymbol{\alpha}} \|\mathbf{u}_D\|_{1/s,r;\Gamma} + \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_2} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) f_2, \\
\|\mathbf{u}\|_{\mathbf{M}_1} &\le \frac{C_r}{\boldsymbol{\beta}_1} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) \|\mathbf{u}_D\|_{1/s,r;\Gamma} + \frac{2|\Omega|^{1/r}}{\mu \boldsymbol{\beta}_1 \boldsymbol{\beta}_2} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) f_2, \\
\|\vec{\phi}\|_{\mathbf{H}} &= \|\phi\|_{\mathrm{M}} + \|\mathbf{t}\|_{0,\Omega} \le \frac{|\Omega|^{1/s}}{\alpha} g_2, \quad \textit{and} \\
\|\widetilde{\boldsymbol{\sigma}}\|_{\mathbf{Q}} &\le \frac{|\Omega|^{1/s}}{\beta} \left(1 + \frac{\vartheta_2}{\alpha}\right) g_2.
\end{aligned} \tag{3.87}$$

Proof. Bearing in mind (3.75), Lemma 3.8, and the hypothesis (3.86), a direct application of the Banach fixed point Theorem implies the existence of a unique $\phi \in W$ solution to (3.73) (equivalently, the existence of a unique solution $\big((\boldsymbol{\sigma}, \mathbf{u}), (\vec{\phi}, \widetilde{\boldsymbol{\sigma}})\big) \in \big(\mathbf{X}_2 \times \mathbf{M}_1\big) \times \big(\mathbf{H} \times \mathbf{Q}\big)$ to (3.36)). In addition, recalling that $(\boldsymbol{\sigma}, \mathbf{u}) = \mathbf{S}(\phi)$ and $(\vec{\phi}, \widetilde{\boldsymbol{\sigma}}) = \mathrm{S}(\boldsymbol{\sigma}, \mathbf{u})$, the a priori estimates (3.39), (3.51), and (3.52) yield (3.87) and conclude the proof. □

We end this section by noticing that smallness data assumptions such as (3.70) and (3.86), which in this case yield unique solvability of the coupled problems (3.35) and (3.35), respectively, appear very often in the literature when combining primal and dual-mixed formulations with fixed-point strategies for addressing the solvability of diverse nonlinear problems in continuum mechanics. In particular, in our previous work on the present model, the analogue of each one of them is given by (2.98), whereas the same kind of hypotheses arise as well for similar and related problems, including, coupled flow-transport (cf. [11, Theorem 3.11, eq. (3.54)]), Navier-Stokes (cf. [18, Theorem 3.8, eq. (3.38)]), magneto-hydrodynamics (cf. [19, Theorem 3.7, eq. (3.39)]), Stokes/Poisson-Nernst-Planck (cf. [34, Theorem 4.11, eq. (105)]), chemotaxis/Navier-Stokes (cf. [23, Theorem 3.12, eq. (3.78)]), Boussinesq (cf. [24, Theorem 3.2, eq. (3.45)], [29, Theorem 3.11, eq. (3.78)], [33, Theorem 3.8, eq.

(3.34)]), and fluidizer bed (cf. [55, Theorem 3.12, eq. (3.93)]), among many others. While most of the aforementioned hypotheses have the same structure, in the sense that they reduce to constraints on linear combinations of data, which make them similar, though all different because of the constants involved, the fact that the latter are usually unknown, stops us from verifying them in practice, and hence of performing comparisons among them. Instead of it, what turns out to be very helpful and makes the difference in some models, is the fact that the respective fixed-point operator becomes compact, in which case the existence of solution simply follows from the Schauder theorem, without the need of imposing smallness data assumption, except the ones required to ensure that the operator maps a ball into itself. We refer to [7, Theorem 3.13] and [32, Theorem 3.10] as illustrative examples of the above for coupled flow-transport models.

3.4 The Galerkin schemes

In this section we introduce and analyse the Galerkin schemes of the fully-mixed formulations (3.35) and (3.36). In particular, for the solvability analyses of the discrete versions of the decoupled problems studied in Sections 3.3.1, 3.3.2, and 3.3.3, we employ the corresponding analogues of [12, Theorem 2.1, Corollary 2.1, Section 2.1] and [41, Theorem 2.34], which are given by [12, Corollary 2.2, eqs. (2.24), (2.25)] and [41, Proposition 2.42], respectively.

3.4.1 Preliminaries

We begin by letting $\mathbf{X}_{2,h}$, $\mathbf{M}_{1,h}$, $\mathbf{X}_{1,h}$, and $\mathbf{M}_{2,h}$ be the finite element subspaces of $\mathbf{X}_2$, $\mathbf{M}_1$, $\mathbf{X}_1$, and $\mathbf{M}_2$, respectively, that are described in (2.135). In addition, let $\mathbf{Q}_h$, M_h, and $\mathbf{H}_h^{\mathbf{t}}$ be arbitrary finite element subspaces of $\mathbf{Q}$, M, and $\mathbf{L}^2(\Omega)$, respectively. Hereafter, h stands for both the sub-index of each subspace and the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, that is, $h := \max\{h_K : \ K \in \mathcal{T}_h\}$. Then, the Galerkin scheme associated with (3.35) reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ and $(\widetilde{\boldsymbol{\sigma}}, \phi) \in \mathbf{Q}_h \times \mathrm{M}_h$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}_1(\boldsymbol{\tau}_h, \mathbf{u}_h) &= G(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbf{X}_{1,h}, \\ \mathbf{b}_2(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= F_{\phi_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{M}_{2,h}, \\ \widetilde{a}_{\boldsymbol{\sigma}_h}(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{\tau}}_h) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \phi_h) &= 0 && \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h\,, \\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}}_h, \psi_h) &= \widetilde{G}_{\mathbf{u}_h}(\psi_h) && \forall\, \psi_h \in \mathrm{M}_h\,. \end{aligned} \tag{3.88}$$

In turn, defining the product space $\mathbf{H}_h := \mathrm{M}_h \times \mathbf{H}_h^{\mathbf{t}}$ and setting the notation

$$\vec{\phi}_h := (\phi_h, \mathbf{t}_h)\,, \quad \vec{\varphi}_h := (\varphi_h, \mathbf{s}_h) \in \mathbf{H}_h\,,$$

the Galerkin scheme associated with (3.36) reduces to: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ and $(\vec{\phi}_h, \widetilde{\boldsymbol{\sigma}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)+\mathbf{b}_1(\boldsymbol{\tau}_h,\mathbf{u}_h) &= G(\boldsymbol{\tau}_h) && \forall\,\boldsymbol{\tau}_h \in \mathbf{X}_{1,h},\\ \mathbf{b}_2(\boldsymbol{\sigma}_h,\mathbf{v}_h) &= F_{\phi_h}(\mathbf{v}_h) && \forall\,\mathbf{v}_h \in \mathbf{M}_{2,h},\\ a_{\boldsymbol{\sigma}_h}(\vec{\phi}_h,\vec{\varphi}_h) + b(\vec{\varphi}_h,\widetilde{\boldsymbol{\sigma}}_h) &= G_{\mathbf{u}_h}(\vec{\varphi}_h) && \forall\,\vec{\varphi}_h \in \mathbf{H}_h\,,\\ b(\vec{\phi}_h,\widetilde{\boldsymbol{\tau}}_h) &= 0 && \forall\,\widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h\,. \end{aligned} \tag{3.89}$$

The aforementioned subspaces $\mathbf{X}_{2,h}$, $\mathbf{M}_{1,h}$, $\mathbf{X}_{1,h}$, and $\mathbf{M}_{2,h}$, along with specific examples of $\mathbf{Q}_h$, M_h, and $\mathbf{H}_h^{\mathbf{t}}$ satisfying the hypotheses to be assumed below in Sections 3.4.3 and 3.4.4, are described later on in Section 3.5.1.

3.4.2 Discrete well-posedness of the elasticity equation

We let $\mathbf{S}_h : \mathrm{M}_h \to \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ be the discrete version of the operator $\mathbf{S}$ (cf. (3.37)), that is

$$\mathbf{S}_h(\varphi_h) \,=\, (\mathbf{S}_{1,h}(\varphi_h),\mathbf{S}_{2,h}(\varphi_h)) \,:=\, (\boldsymbol{\sigma}_h,\mathbf{u}_h) \quad \forall\,\varphi_h \in \mathrm{M}_h\,, \tag{3.90}$$

where $(\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ is the unique solution (to be confirmed below) of the first two rows of (3.88) (or (3.89)) with φ_h instead of ϕ_h, namely

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)+\mathbf{b}_1(\boldsymbol{\tau}_h,\mathbf{u}_h) &= G(\boldsymbol{\tau}_h) && \forall\,\boldsymbol{\tau}_h \in \mathbf{X}_{1,h}\,,\\ \mathbf{b}_2(\boldsymbol{\sigma}_h,\mathbf{v}_h) &= F_{\varphi_h}(\mathbf{v}_h) && \forall\,\mathbf{v}_h \in \mathbf{M}_{2,h}\,. \end{aligned} \tag{3.91}$$

Then, under the same assumption on the Lamé parameter λ stipulated in Section 3.3.1, and letting $\boldsymbol{\alpha}_{\mathtt{d}}$, $\boldsymbol{\beta}_{1,\mathtt{d}}$, and $\boldsymbol{\beta}_{2,\mathtt{d}}$ be the constants yielding the discrete inf-sup conditions for $\mathbf{a}$, $\mathbf{b}_1$, and $\mathbf{b}_2$ (cf. Lemmas 2.20 and 2.21), a direct application of [12, Corollary 2.2, eqs. (2.24), (2.25)] yields the following result (cf. Lemma 2.16).

Lemma 3.10. *For each $\varphi_h \in \mathrm{M}_h$ there exists a unique $(\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ solution to (3.91), and hence one can define $\mathbf{S}_h(\varphi_h) \,=\, \big(\mathbf{S}_{1,h}(\varphi_h),\mathbf{S}_{2,h}(\varphi_h)\big) \,:=\, (\boldsymbol{\sigma}_h,\mathbf{u}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$. Moreover, there hold*

$$\begin{aligned} \|\mathbf{S}_{1,h}(\varphi_h)\|_{\mathbf{X}_2} = \|\boldsymbol{\sigma}_h\|_{\mathbf{X}_2} &\le \frac{C_r}{\boldsymbol{\alpha}_{\mathrm{d}}}\,\|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_{2,\mathrm{d}}}\left(1+\frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\,\mu}\right) f_2\,, \quad \textit{and}\\ \|\mathbf{S}_{2,h}(\varphi_h)\|_{\mathbf{M}_1} = \|\mathbf{u}_h\|_{\mathbf{M}_1} &\le \frac{C_r}{\boldsymbol{\beta}_{1,\mathrm{d}}}\left(1+\frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\,\mu}\right)\|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu\boldsymbol{\beta}_{1,\mathrm{d}}\,\boldsymbol{\beta}_{2,\mathrm{d}}}\left(1+\frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}}\,\mu\right) f_2\,. \end{aligned} \tag{3.92}$$

We stress here that the lack of a required boundedness property for a projector involved in the proof of the previous lemma, restricts the present discrete analysis to the 2D case. We refer to Chapter 2.5 for further details.

3.4.3 Discrete well-posedness of the first approach for the diffusion equation

We now let $\widetilde{\mathrm{S}}_h : \mathbf{X}_{2,h} \times \mathbf{M}_{1,h} \to \mathbf{Q}_h \times \mathrm{M}_h$ be the discrete version of $\widetilde{\mathrm{S}}$ (cf. (3.40)), that is

$$\widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h,\mathbf{w}_h) \,:=\, (\widetilde{\boldsymbol{\sigma}}_h,\phi_h) \quad \forall\,(\boldsymbol{\zeta}_h,\mathbf{w}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}\,, \tag{3.93}$$

where $(\widetilde{\boldsymbol{\sigma}}_h, \phi_h) \in \mathbf{Q}_h \times \mathrm{M}_h$ is the unique solution (to be confirmed below) of the third and fourth rows of (3.88) with $(\boldsymbol{\zeta}_h, \mathbf{w}_h)$ instead of $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$, namely

$$\begin{aligned} \widetilde{a}_{\boldsymbol{\zeta}_h}(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{\tau}}_h) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \phi_h) &= 0 \qquad && \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h\,, \\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}}_h, \psi_h) &= \widetilde{G}_{\mathbf{w}_h}(\psi_h) \qquad && \forall\, \psi_h \in \mathrm{M}_h\,. \end{aligned} \tag{3.94}$$

In order to establish the well-posedness of (3.94), we first consider the discrete kernel of $\widetilde{b}$, that is

$$\widetilde{\mathcal{K}}_h := \Big\{ \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h : \quad \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \phi_h) = 0 \quad \forall\, \phi_h \in \mathrm{M}_h \Big\}\,, \tag{3.95}$$

and suppose that

(H.1) $\mathrm{div}(\mathbf{Q}_h) \subseteq \mathrm{M}_h$.

Then, bearing mind the definition of $\widetilde{b}$ (cf. (3.24)), and employing **(H.1)**, we readily deduce from (3.95) that

$$\widetilde{\mathcal{K}}_h := \Big\{ \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h : \quad \mathrm{div}(\widetilde{\boldsymbol{\tau}}_h) = 0 \Big\}\,,$$

which yields the discrete analogue of (3.42), and hence the $\widetilde{\mathcal{K}}_h$-ellipticity of $\widetilde{a}_{\boldsymbol{\zeta}_h}$ with constant $\widetilde{\alpha}_{\mathtt{d}} = \widetilde{\vartheta}_0$.

Next, we also assume that

(H.2) there exists a positive constant $\widetilde{\beta}_{\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h \\ \widetilde{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{\widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \psi_h)}{\|\widetilde{\boldsymbol{\tau}}_h\|_{\mathbf{Q}}} \geq \widetilde{\beta}_{\mathrm{d}}\, \|\psi_h\|_{\mathrm{M}} \qquad \forall\, \psi_h \in \mathrm{M}_h\,.$$

Thus, straightforward applications of [41, Theorem 2.42] and the abstract estimates from [41, eq. (2.30)] imply the discrete analogue of Lemma 3.2, which is stated as follows.

Lemma 3.11. *For each $(\boldsymbol{\zeta}_h, \mathbf{w}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$, there exists a unique $(\widetilde{\boldsymbol{\sigma}}_h, \phi_h) \in \mathbf{Q}_h \times \mathrm{M}_h$ solution of (3.94), and hence one can define $\widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \mathbf{w}_h) = \big(\widetilde{\mathrm{S}}_{1,h}(\boldsymbol{\zeta}_h, \mathbf{w}_h), \widetilde{\mathrm{S}}_{2,h}(\boldsymbol{\zeta}_h, \mathbf{w}_h)\big) := (\widetilde{\boldsymbol{\sigma}}_h, \phi_h) \in \mathbf{Q}_h \times \mathrm{M}_h$. Moreover, there hold*

$$\begin{aligned} \|\widetilde{\mathrm{S}}_{1,h}(\boldsymbol{\zeta}_h, \mathbf{w}_h)\|_{\mathbf{Q}} &= \|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{Q}} \leq \frac{1}{\widetilde{\beta}_{\mathrm{d}}}\Big(1 + \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}_{\mathrm{d}}}\Big)|\Omega|^{1/s} g_2\,, \quad \textit{and} \\ \|\widetilde{\mathrm{S}}_{2,h}(\boldsymbol{\zeta}_h, \mathbf{w}_h)\|_{\mathrm{M}} &= \|\phi_h\|_{\mathrm{M}} \leq \frac{\widetilde{\vartheta}_2}{\widetilde{\beta}_{\mathrm{d}}^2}\Big(1 + \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}_{\mathrm{d}}}\Big)|\Omega|^{1/s} g_2\,. \end{aligned} \tag{3.96}$$

3.4.4 Discrete well-posedness of the second approach for the diffusion equation

Here we introduce the discrete operator $\mathrm{S}_h : \mathbf{X}_{2,h} \times \mathbf{M}_{1,h} \to \mathbf{H}_h$ given by

$$\mathrm{S}_h(\boldsymbol{\zeta}_h, \mathbf{w}_h) := \vec{\phi}_h \quad \forall\, (\boldsymbol{\zeta}_h, \mathbf{w}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}\,, \tag{3.97}$$

where $(\vec{\phi}_h, \widetilde{\boldsymbol{\sigma}}_h) := ((\phi_h, \mathbf{t}_h), \widetilde{\boldsymbol{\sigma}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the unique solution (to be confirmed below) of the third and fourth rows of (3.89) with $(\boldsymbol{\zeta}_h, \mathbf{w}_h)$ instead of $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$, that is

$$\begin{aligned} a_{\boldsymbol{\zeta}_h}(\vec{\phi}_h, \vec{\varphi}_h) + b(\vec{\varphi}_h, \widetilde{\boldsymbol{\sigma}}_h) &= G_{\mathbf{w}_h}(\vec{\varphi}_h) \qquad && \forall\, \vec{\varphi}_h \in \mathbf{H}_h\,, \\ b(\vec{\phi}_h, \widetilde{\boldsymbol{\tau}}_h) &= 0 \qquad && \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h\,. \end{aligned} \tag{3.98}$$

In order to prove that (3.98) is well-posed, we need to incorporate a couple of suitable hypotheses on the discrete spaces. Indeed, we first assume that

(**H.3**) there exists a positive constant $\beta_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\varphi}_h \in \mathbf{H}_h \\ \vec{\varphi}_h \neq \mathbf{0}}} \frac{b(\vec{\varphi}_h, \widetilde{\boldsymbol{\tau}}_h)}{\|\vec{\varphi}_h\|_{\mathbf{H}}} \geq \beta_{\mathtt{d}} \|\widetilde{\boldsymbol{\tau}}_h\|_{\mathbf{Q}} \qquad \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h \,.$$

Next, we let V_h be the discrete kernel of the bilinear form b, that is

$$V_h := \Big\{ \vec{\varphi}_h \in \mathbf{H}_h : \quad b(\vec{\varphi}_h, \widetilde{\boldsymbol{\tau}}_h) = 0 \quad \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h \Big\} ,$$

and suppose that

(**H.4**) there exists a positive constant $C_{\mathtt{d}}$, independent of h, such that

$$\|\mathbf{s}_h\|_{0,\Omega} \geq C_{\mathtt{d}} \|\varphi_h\|_{0,r,\Omega} \qquad \forall\, \vec{\varphi}_h := (\varphi_h, \mathbf{s}_h) \in V_h \,.$$

In this way, bearing in mind the definition of $a_{\boldsymbol{\zeta}_h}$ (cf. (3.31)), and employing the positive definiteness property of ϑ (cf. (3.3)) and (**H.4**), we deduce for each $\boldsymbol{\zeta}_h \in \mathbf{X}_{2,h}$ that

$$a_{\boldsymbol{\zeta}_h}(\vec{\varphi}_h, \vec{\varphi}_h) \geq \vartheta_0 \|\mathbf{s}_h\|^2_{0,\Omega} \geq \frac{\vartheta_0}{2} C_{\mathtt{d}}^2 \|\varphi_h\|^2_{0,r;\Omega} + \frac{\vartheta_0}{2} \|\mathbf{s}_h\|^2_{0,r;\Omega} \quad \forall\, \vec{\varphi}_h := (\varphi_h, \mathbf{s}_h) \in V_h \,, \tag{3.99}$$

from which it readily follows the V_h-ellipticity of $a_{\boldsymbol{\zeta}_h}$ with constant $\alpha_{\mathtt{d}} := \frac{\vartheta_0}{2} \min\{C_{\mathtt{d}}^2, 1\}$.

Consequently, applying [41, Proposition 2.42], and making use of the a priori estimate provided by [41, eq. (2.30)], we are lead to the discrete analogue of Lemma 3.3.

Lemma 3.12. *For each $(\boldsymbol{\zeta}_h, \mathbf{w}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ there exists a unique $(\vec{\phi}_h, \widetilde{\boldsymbol{\sigma}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ solution of (3.98), and hence one can define $\mathrm{S}_h(\boldsymbol{\zeta}_h, \mathbf{w}_h) := \vec{\phi}_h \in \mathbf{H}_h$. Moreover, there holds*

$$\|\mathrm{S}_h(\boldsymbol{\zeta}_h, \mathbf{w}_h)\|_{\mathbf{H}} = \|\vec{\phi}_h\|_{\mathbf{H}} = \|\phi_h\|_{0,r;\Omega} + \|\mathbf{t}_h\|_{0,\Omega} \leq \frac{|\Omega|^{1/s}}{\alpha_{\mathrm{d}}} g_2 \,. \tag{3.100}$$

We end this section by remarking that the discrete version of (3.52) becomes

$$\|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{Q}} = \|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathrm{div}_s;\Omega} \leq \frac{|\Omega|^{1/s}}{\beta_{\mathrm{d}}} \left(1 + \frac{\vartheta_2}{\alpha_{\mathrm{d}}}\right) g_2 \,. \tag{3.101}$$

3.4.5 Discrete solvability of the first fully-mixed formulation

In this section we adopt the discrete analogue of the fixed point strategy introduced in Section 3.3.4 to analyse the solvability of (3.88). According to it, we define the operator $\Xi_h : \mathrm{M}_h \to \mathrm{M}_h$ as

$$\Xi_h(\varphi_h) := \widetilde{\mathrm{S}}_{2,h}\big(\mathbf{S}_h(\varphi_h)\big) \qquad \forall\, \varphi_h \in \mathrm{M}_h \,, \tag{3.102}$$

and observe, being $\widetilde{\mathrm{S}}_h$ and $\mathbf{S}_h$, and hence Ξ_h as well, well-defined, that solving (3.88) is equivalent to seeking a fixed point of Ξ_h, that is: Find $\phi_h \in \mathrm{M}_h$ such that

$$\Xi_h(\phi_h) = \phi_h \,. \tag{3.103}$$

Thus, in what follows we show that Ξ_h verifies the hypotheses of the Brouwer theorem. In fact, introducing the ball

$$\widetilde{W}_h := \Big\{ \phi_h \in \mathrm{M}_{1,h} : \quad \|\phi_h\|_{0,r;\Omega} \le \widetilde{\delta}_{\mathrm{d}} \Big\}, \tag{3.104}$$

with

$$\widetilde{\delta}_{\mathrm{d}} := \frac{\widetilde{\vartheta}_2}{\widetilde{\beta}_{\mathrm{d}}^2}\Big(1 + \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}_{\mathrm{d}}}\Big)|\Omega|^{1/s} g_2 \,,$$

we realize, according to the definition of Ξ_h (cf. (3.102)) and the second a priori estimate in (3.96), that

$$\Xi_h\big(\widetilde{W}_h\big) \subseteq \widetilde{W}_h \,. \tag{3.105}$$

Next, in order to derive the continuity of Ξ_h, we first recall from (2.110) that there exists a positive constant $C_{\mathbf{S},\mathrm{d}}$, independent of h, such that

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\varphi_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} \le C_{\mathbf{S},\mathrm{d}}\, L_f\, \|\phi_h - \varphi_h\|_{0,r;\Omega} \qquad \forall\, \phi_h,\, \varphi_h \in \mathrm{M}_h \,. \tag{3.106}$$

On the other hand, for the continuity of $\widetilde{\mathrm{S}}_h$ the reasoning of the proof of Lemma 3.4 is slightly modified. Indeed, knowing that the regularity assumption $(\mathbf{RA}_1)$ is certainly not applicable in the present discrete context, we proceed to utilize a $\mathrm{L}^{2q} - \mathrm{L}^{2p} - \mathrm{L}^2$ argument to derive the discrete version of (3.61), where $p,\, q \in (1,+\infty)$, conjugate to each other, are chosen such that $2q = r$. The above is a feasible choice since, as stipulated in (3.9), there holds $r > 2$, which yields $r^* := 2p = \frac{2r}{r-2}$. In this way, given $(\boldsymbol{\zeta}_h, \mathbf{w}_h)$ $(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$, and denoting $(\widetilde{\boldsymbol{\sigma}}_h, \phi_h) = \widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \mathbf{w}_h) \in \mathbf{Q}_h \times \mathrm{M}_h$ and $(\widetilde{\boldsymbol{\zeta}}_h, \varphi_h) = \widetilde{\mathrm{S}}_h(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{Q}_h \times \mathrm{M}_h$, the discrete analogue of (3.66) becomes

$$\begin{aligned} |\widetilde{a}_{\boldsymbol{\tau}_h}(\widetilde{\boldsymbol{\zeta}}_h, \widetilde{\boldsymbol{\tau}}_h) - \widetilde{a}_{\boldsymbol{\zeta}_h}(\widetilde{\boldsymbol{\zeta}}_h, \widetilde{\boldsymbol{\tau}}_h)| &\le L_{\widetilde{\vartheta}}\, \|(\boldsymbol{\tau}_h - \boldsymbol{\zeta}_h)\widetilde{\boldsymbol{\zeta}}_h\|_{0,\Omega}\, \|\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega} \\ &\le L_{\widetilde{\vartheta}}\, \|\boldsymbol{\tau}_h - \boldsymbol{\zeta}_h\|_{0,2q;\Omega}\, \|\widetilde{\boldsymbol{\zeta}}_h\|_{0,2p,\Omega}\, \|\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega} \,. \end{aligned} \tag{3.107}$$

The foregoing inequality, along with the discrete versions of (3.65) and (3.68), whose details we omit here, imply the existence of a positive constant $C_{\widetilde{\mathrm{S}},\mathrm{d}}$, depending only on $\widetilde{\alpha}_{\mathrm{d}}$, $\widetilde{\beta}_{\mathrm{d}}$, and $|\Omega|$, and hence independent of h, such that

$$\begin{aligned} &\|\widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \mathbf{w}_h) - \widetilde{\mathrm{S}}_h(\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{\mathbf{Q}\times\mathrm{M}} \\ &\quad \le C_{\widetilde{\mathrm{S}},\mathrm{d}} \Big\{ L_g + L_{\widetilde{\vartheta}}\, \|\widetilde{\mathrm{S}}_{1,h}(\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{0,r^*;\Omega} \Big\}\, \|(\boldsymbol{\zeta}_h, \mathbf{w}_h) - (\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{\mathbf{Q}\times\mathrm{M}} \end{aligned} \tag{3.108}$$

for all $(\boldsymbol{\zeta}_h, \mathbf{w}_h)$, $(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$. In this way, recalling the definition of Ξ_h (cf. (3.102)), and employing the estimates (3.106) and (3.108), we conclude that

$$\begin{aligned} &\|\Xi_h(\phi_h) - \Xi_h(\varphi_h)\|_{0,r;\Omega} \\ &\quad \le C_{\Xi,\mathrm{d}}\, L_f \Big\{ L_g + L_{\widetilde{\vartheta}}\, \|\widetilde{\mathrm{S}}_{1,h}(\mathbf{S}_h(\varphi_h))\|_{0,r^*;\Omega} \Big\}\, \|\phi_h - \varphi_h\|_{0,r;\Omega} \qquad \forall\, \phi_h,\, \varphi_h \in \mathrm{M}_h \,, \end{aligned} \tag{3.109}$$

with the positive constant $C_{\Xi,\mathrm{d}} := C_{\mathbf{S},\mathrm{d}}\, C_{\widetilde{\mathrm{S}},\mathrm{d}}$. While the estimate (3.109) implies that Ξ_h is continuous, we emphasize that the lack of control of the term $\|\widetilde{\mathrm{S}}_{1,h}(\mathbf{S}_h(\varphi_h))\|_{0,r^*;\Omega}$ stop us from concluding Lipschitz-continuity and hence nor contractivity of this operator.

We are now in position to establish the following main result.

Theorem 3.13. *The operator Ξ_h has at least one fixed point $\phi_h \in \widetilde{W}_h$. Equivalently, the Galerkin scheme (3.88) has at least one solution $\big((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\widetilde{\boldsymbol{\sigma}}_h, \phi_h)\big) \in \big(\mathbf{X}_{2,h} \times \mathbf{M}_{1,h}\big) \times \big(\mathbf{Q}_h \times \mathrm{M}_h\big)$, with $\phi_h \in \widetilde{W}_h$ (cf. (3.104)). Moreover, there hold*

$$
\begin{aligned}
\|\boldsymbol{\sigma}_h\|_{\mathbf{X}_2} \,&\le\, \frac{C_r}{\boldsymbol{\alpha}_{\mathrm{d}}}\,\|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_{2,\mathrm{d}}}\left(1 + \frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\,\mu}\right) f_2\,,\\
\|\mathbf{u}_h\|_{\mathbf{M}_1} \,&\le\, \frac{C_r}{\boldsymbol{\beta}_{1,\mathrm{d}}}\left(1 + \frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\,\mu}\right)\|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu\boldsymbol{\beta}_{1,\mathrm{d}}\,\boldsymbol{\beta}_{2,\mathrm{d}}}\left(1 + \frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\,\mu}\right) f_2\,, \quad \textit{and}\\
\|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{Q}} \,&\le\, \frac{1}{\widetilde{\beta}_{\mathrm{d}}}\left(1 \,+\, \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}_{\mathrm{d}}}\right)|\Omega|^{1/s} g_2\,.
\end{aligned}
\tag{3.110}
$$

Proof. Thanks to (3.105), the continuity of Ξ_h (cf. (3.109)), and the equivalence between (3.88) and (3.103), a straightforward application of Brouwer's theorem (cf. [28, Theorem 9.9-2]) implies the first conclusion of this theorem. Next, noting that $(\boldsymbol{\sigma}_h, \mathbf{u}_h) = \mathbf{S}_h(\phi_h)$ and $(\widetilde{\boldsymbol{\sigma}}_h, \phi_h) = \widetilde{\mathrm{S}}_h(\boldsymbol{\sigma}_h, \mathbf{u}_h)$, the a priori estimate (3.110) follows from (3.92) and (3.96). □

3.4.6 A priori error analysis for the first fully-mixed formulation

In this section we establish the Céa estimate for the global error

$$
\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{X}_2 \times \mathbf{M}_1} \,+\, \|(\widetilde{\boldsymbol{\sigma}}, \phi) - (\widetilde{\boldsymbol{\sigma}}_h, \phi_h)\|_{\mathbf{Q} \times \mathrm{M}}\,,
$$

where $\big((\boldsymbol{\sigma}, \mathbf{u}), (\widetilde{\boldsymbol{\sigma}}, \phi)\big) \in \big(\mathbf{X}_2 \times \mathbf{M}_1\big) \times \big(\mathbf{Q} \times \mathrm{M}\big)$ and $\big((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\widetilde{\boldsymbol{\sigma}}_h, \phi_h)\big) \in \big(\mathbf{X}_{2,h} \times \mathbf{M}_{1,h}\big) \times \big(\mathbf{Q}_h \times \mathrm{M}_h\big)$ are the unique solutions of (3.35) and (3.88), respectively, with $\phi \in \widetilde{W}$ (cf. (3.55)) and $\phi_h \in \widetilde{W}_h$ (cf. (3.104)). In what follows, given a subspace Z_h of a generic Banach space $(Z, \|\cdot\|_Z)$, we set

$$
\mathrm{dist}\,(z, Z_h) \,:=\, \inf_{z_h \in Z_h} \|z - z_h\|_Z \qquad \forall\, z \in Z\,.
$$

Then, applying the Strang a priori error estimate provided by [12, Proposition 2.1, Corollary 2.3, and Theorem 2.3] to the pair of associated continuous and discrete formulations given by the first and second rows of (3.35) and (3.88), respectively, and proceeding as for the derivation of (2.119), but without using the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^r(\Omega)$ as done there, we deduce that there exists a positive constant $\widehat{C}_{ST}$, depending only on $\boldsymbol{\alpha}_{\mathrm{d}}$, $\boldsymbol{\beta}_{1,\mathrm{d}}$, $\boldsymbol{\beta}_{2,\mathrm{d}}$, $\|\mathbf{a}\|$, $\|\mathbf{b}_1\|$, and $\|\mathbf{b}_2\|$, and hence independent of h, such that

$$
\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{X}_2 \times \mathbf{M}_1} \,\le\, \widehat{C}_{ST}\left\{\mathrm{dist}\,(\boldsymbol{\sigma}, \mathbf{X}_{2,h}) + \mathrm{dist}\,(\mathbf{u}, \mathbf{M}_{1,h}) + L_f\,\|\phi - \phi_h\|_{\mathrm{M}}\right\}. \tag{3.111}
$$

Similarly, applying the Strang a priori error estimate from [12, Proposition 2.1, Corollary 2.3, and Theorem 2.3]to the pair of associated continuous and discrete formulations given by the third and fourth rows of (3.35) and (3.88), respectively, we find that there exists a positive constant $\widetilde{C}_{ST}$, depending only on $\widetilde{\alpha}_{\mathrm{d}}$, $\widetilde{\beta}_{\mathrm{d}}$, $\|\widetilde{a}_{\boldsymbol{\sigma}}\|$, and $\|\widetilde{b}\|$, and hence independent of h, as well, such that

$$
\begin{aligned}
&\|(\widetilde{\boldsymbol{\sigma}}, \phi) - (\widetilde{\boldsymbol{\sigma}}_h, \phi_h)\|_{\mathbf{Q} \times \mathrm{M}}\\
&\quad \le\, \widetilde{C}_{ST}\left\{\mathrm{dist}\,(\widetilde{\boldsymbol{\sigma}}, \mathbf{Q}_h) + \mathrm{dist}\,(\phi, \mathrm{M}_h) + \|(\widetilde{a}_{\boldsymbol{\sigma}} - \widetilde{a}_{\boldsymbol{\sigma}_h})(\widetilde{\boldsymbol{\sigma}}, \cdot)\|_{\mathbf{Q}_h'} + \|\widetilde{G}_{\mathbf{u}} - \widetilde{G}_{\mathbf{u}_h}\|_{\mathrm{M}_h'}\right\}.
\end{aligned}
\tag{3.112}
$$

Next, proceeding exactly as for the derivations of (3.67) and (3.68), we find that

$$\|(\widetilde{a}_{\boldsymbol{\sigma}} - \widetilde{a}_{\boldsymbol{\sigma}_h})(\widetilde{\boldsymbol{\sigma}}, \cdot)\|_{\mathbf{Q}'_h} \,\leq\, \widetilde{L}_{\widetilde{\mathbb{S}}}\, L_{\widetilde{\vartheta}}\, g_2 \, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{X}_2}\,, \tag{3.113}$$

where $\widetilde{L}_{\widetilde{\mathbb{S}}} := C_{r,\varepsilon}\, \|i_\varepsilon\|\, C_\varepsilon$, and

$$\|G_{\mathbf{u}} - G_{\mathbf{u}_h}\|_{\mathrm{M}'_h} \,\leq\, L_g\, |\Omega|^{\frac{r-s}{rs}}\, \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{M}_1}\,, \tag{3.114}$$

respectively. In this way, replacing (3.113) and (3.114) back into (3.112), we conclude that

$$\begin{aligned} &\|(\widetilde{\boldsymbol{\sigma}}, \phi) - (\widetilde{\boldsymbol{\sigma}}_h, \phi_h)\|_{\mathbf{Q}\times\mathrm{M}} \\ &\quad \leq\, \widetilde{C}_{ST} \Big\{ \mathrm{dist}\,(\widetilde{\boldsymbol{\sigma}}, \mathbf{Q}_h) + \mathrm{dist}\,(\phi, \mathrm{M}_h) + \widetilde{L}_{\widetilde{\mathbb{S}}}\, L_{\widetilde{\vartheta}}\, g_2\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{X}_2} + L_g\, |\Omega|^{\frac{r-s}{rs}}\, \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{M}_1} \Big\}. \end{aligned} \tag{3.115}$$

Thus, adding (3.111) and (3.115), we arrive at

$$\begin{aligned} &\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} + \|(\widetilde{\boldsymbol{\sigma}}, \phi) - (\widetilde{\boldsymbol{\sigma}}_h, \phi_h)\|_{\mathbf{Q}\times\mathrm{M}} \\ &\quad \leq\, C_{ST} \Big\{ \mathrm{dist}\,(\boldsymbol{\sigma}, \mathbf{X}_{2,h}) + \mathrm{dist}\,(\mathbf{u}, \mathbf{M}_{1,h}) + \mathrm{dist}\,(\widetilde{\boldsymbol{\sigma}}, \mathbf{Q}_h) + \mathrm{dist}\,(\phi, \mathrm{M}_h) \Big\} \\ &\quad + \mathcal{C}(\mathtt{data}) \Big\{ \|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} + \|\phi - \phi_h\|_{\mathrm{M}} \Big\}, \end{aligned} \tag{3.116}$$

where $C_{ST} \,:=\, \max\big\{\widehat{C}_{ST}, \widetilde{C}_{ST}\big\}$, and

$$\mathcal{C}(\mathtt{data}) \,:=\, \max\Big\{ \widehat{C}_{ST}\, L_f,\, \widetilde{C}_{ST}\, \widetilde{L}_{\widetilde{\mathbb{S}}}\, L_{\widetilde{\vartheta}}\, g_2,\, \widetilde{C}_{ST}\, L_g\, |\Omega|^{\frac{r-s}{rs}} \Big\}. \tag{3.117}$$

We are now in a position to state the announced Céa estimate for our first approach.

Theorem 3.14. *Assume that the data (cf. (3.117)) satisfy*

$$\mathcal{C}(\mathtt{data}) \,\leq\, \frac{1}{2}\,. \tag{3.118}$$

Then, there exists a positive constant C, independent of h, such that

$$\begin{aligned} &\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} + \|(\widetilde{\boldsymbol{\sigma}}, \phi) - (\widetilde{\boldsymbol{\sigma}}_h, \phi_h)\|_{\mathbf{Q}\times\mathrm{M}} \\ &\quad \leq\, C \Big\{ \mathrm{dist}\,(\boldsymbol{\sigma}, \mathbf{X}_{2,h}) + \mathrm{dist}\,(\mathbf{u}, \mathbf{M}_{1,h}) + \mathrm{dist}\,(\widetilde{\boldsymbol{\sigma}}, \mathbf{Q}_h) + \mathrm{dist}\,(\phi, \mathrm{M}_h) \Big\}. \end{aligned} \tag{3.119}$$

Proof. It follows directly from (3.116) and (3.118). □

3.4.7 Discrete solvability of the second fully-mixed formulation

The discrete analogue of the fixed point approach employed in Section 3.3.5 is adopted here to establish the solvability of (3.89). Thus, we now define the operator $\Lambda_h : \mathrm{M}_h \to \mathrm{M}_h$ as

$$\Lambda_h(\psi_h) \,:=\, \mathrm{S}_{1,h}\big(\mathbf{S}_h(\psi_h)\big) \qquad \forall\, \psi_h \in \mathrm{M}_h\,, \tag{3.120}$$

which is clearly well-defined since S_h and $\mathbf{S}_h$ are, and hence, solving (3.89) is equivalent to finding a fixed point of Λ_h, that is $\phi_h \in \mathrm{M}_h$ such that

$$\Lambda_h(\phi_h) \,=\, \phi_h\,. \tag{3.121}$$

Similarly to the analysis in Section 3.4.5, in what follows we prove that Λ_h verifies the hypotheses of the Brouwer theorem. Indeed, defining

$$W_h \,:=\, \Big\{\phi_h \in \mathrm{M}_h : \quad \|\phi_h\|_{0,r;\Omega} \,\le\, \delta_{\mathrm{d}}\Big\}\,, \tag{3.122}$$

with

$$\delta_{\mathrm{d}} \,:=\, \frac{|\Omega|^{1/s}}{\alpha_{\mathrm{d}}}\, g_2\,,$$

it is straightforward to see, from the definition of Λ_h (cf. (3.120)) and the a priori estimate for $\mathrm{S}_{1,h}$ (cf. (3.100)), that

$$\Lambda_h(W_h) \,\subseteq\, W_h\,. \tag{3.123}$$

Next, proceeding analogously to the proof of Lemma 3.7, but without using the regularity assumption $(\mathbf{RA}_2)$, which is not valid in the present discrete case, and letting $C_{\mathrm{S,d}} \,:=\, \max\big\{|\Omega|^{\frac{r-s}{rs}},\,1\big\}$, we are able to show that

$$\begin{aligned} &\|\mathrm{S}_h(\boldsymbol{\zeta}_h,\mathbf{w}_h) \,-\, \mathrm{S}_h(\boldsymbol{\tau}_h,\mathbf{v}_h)\|_{\mathbf{H}} \\ &\quad\le\, C_{\mathrm{S,d}}\Big\{L_g + L_\vartheta\,\|\mathrm{S}_{2,h}(\boldsymbol{\tau}_h,\mathbf{v}_h)\|_{0,r^*;\Omega}\Big\}\,\|(\boldsymbol{\zeta}_h,\mathbf{w}_h) - (\boldsymbol{\tau}_h,\mathbf{v}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} \end{aligned} \tag{3.124}$$

for all $(\boldsymbol{\zeta}_h,\mathbf{w}_h),\,(\boldsymbol{\tau}_h,\mathbf{v}_h) \in \mathbf{X}_{2,h}\times\mathbf{M}_{1,h}$. In this way, bearing in mind the definition of Λ_h (cf. (3.120)), and combining (3.124) with the Lipschitz-continuity of $\mathbf{S}_h$ (cf. (3.106)), we obtain

$$\begin{aligned} &\|\Lambda_h(\phi_h) - \Lambda_h(\varphi_h)\|_{0,r;\Omega} \\ &\quad\le\, L_{\Lambda,\mathrm{d}}\,L_f\Big\{L_g + L_\vartheta\,\|\mathrm{S}_{2,h}(\mathbf{S}_h(\varphi_h))\|_{0,r^*;\Omega}\Big\}\,\|\phi_h - \varphi_h\|_{0,r;\Omega} \quad \forall\,\phi_h,\,\varphi_h \in \mathrm{M}_h\,, \end{aligned} \tag{3.125}$$

with $L_{\Lambda,\mathrm{d}} \,:=\, C_{\mathbf{S},\mathrm{d}}\,C_{\mathrm{S,d}}$.

The main result of this section is then stated as follows.

Theorem 3.15. *The operator Λ_h has at least one fixed point $\phi_h \in \mathrm{M}_h$. Equivalently, the Galerkin scheme (3.89) has at least one solution $\big((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\big) \in \big(\mathbf{X}_{2,h}\times\mathbf{M}_{1,h}\big)\times\big(\mathbf{H}_h\times\mathbf{Q}_h\big)$, with $\phi_h \in W_h$ (cf. (3.122)). Moreover, there hold*

$$\begin{aligned} \|\boldsymbol{\sigma}_h\|_{\mathbf{X}_2} \,&\le\, \frac{C_r}{\boldsymbol{\alpha}_{\mathrm{d}}}\,\|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_{2,\mathrm{d}}}\left(1 + \frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\,\mu}\right) f_2\,, \\ \|\mathbf{u}_h\|_{\mathbf{M}_1} \,&\le\, \frac{C_r}{\boldsymbol{\beta}_{1,\mathrm{d}}}\left(1 + \frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\,\mu}\right)\|\mathbf{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu\boldsymbol{\beta}_{1,\mathrm{d}}\,\boldsymbol{\beta}_{2,\mathrm{d}}}\left(1 + \frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\,\mu}\right) f_2\,, \\ \|\vec{\phi}_h\|_{\mathbf{H}} \,&=\, \|\phi_h\|_{0,r;\Omega} + \|\mathbf{t}_h\|_{0,\Omega} \,\le\, \frac{|\Omega|^{1/s}}{\alpha_{\mathrm{d}}}\,g_2\,, \quad \textit{and} \\ \|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{Q}} \,&=\, \|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathrm{div}_r;\Omega} \,\le\, \frac{|\Omega|^{1/s}}{\beta_{\mathrm{d}}}\left(1 \,+\, \frac{\vartheta_2}{\alpha_{\mathrm{d}}}\right) g_2\,. \end{aligned} \tag{3.126}$$

Proof. Thanks to (3.123), the continuity of Λ_h (cf. (3.125)), and the fact that (3.89) and (3.121) are equivalent, the existence of solution follows from a direct application of the Brouwer theorem (cf. [28, Theorem 9.9-2]). In turn, the a priori estimates (3.92), (3.100), and (3.101) yield (3.126), which finishes the proof. □

3.4.8 A priori error analysis for the second fully-mixed formulation

In what follows we derive the Céa estimate for the global error

$$\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1}\;+\;\|(\vec{\phi},\widetilde{\boldsymbol{\sigma}})-(\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\,,$$

where $\big((\boldsymbol{\sigma},\mathbf{u}),(\vec{\phi},\widetilde{\boldsymbol{\sigma}})\big)\in\big(\mathbf{X}_2\times\mathbf{M}_1\big)\times\big(\mathbf{H}\times\mathbf{Q}\big)$ and $\big((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\big)\in\big(\mathbf{X}_{2,h}\times\mathbf{M}_{1,h}\big)\times\big(\mathbf{H}_h\times\mathbf{Q}_h\big)$ are the unique solutions of (3.36) and (3.89), respectively, with $\phi\in W$ (cf. (3.74)) and $\phi_h\in W_h$ (cf. (3.122)).

Since the first two rows of (3.35) and (3.88) coincide with those of (3.36) and (3.89), we realize that the a priori estimate for $\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1}$ is exactly the one given by (3.111). In turn, applying the Strang estimate provided by [29, Lemma 6.1] (whose proof is a simple modification of that of [44, Theorem 2.6]) to the pair of associated continuous and discrete formulations given by the last two rows of (3.36) and (3.89), we deduce the existence of a positive constant $\overline{C}_{ST}$, depending only on $\alpha_{\mathtt{d}}$, $\beta_{\mathtt{d}}$, $\|a_{\boldsymbol{\sigma}}\|$, and $\|b\|$, such that

$$\begin{aligned}&\|(\vec{\phi},\widetilde{\boldsymbol{\sigma}})\;-\;(\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\\&\quad\le\;\overline{C}_{ST}\left\{\operatorname{dist}(\vec{\phi},\mathbf{H}_h)\;+\;\operatorname{dist}(\widetilde{\boldsymbol{\sigma}},\mathbf{Q}_h)\;+\;\|(a_{\boldsymbol{\sigma}}-a_{\boldsymbol{\sigma}_h})(\vec{\phi},\cdot)\|_{\mathbf{H}_h'}\;+\;\|G_{\mathbf{u}}-G_{\mathbf{u}_h}\|_{\mathbf{H}_h'}\right\}.\end{aligned}\tag{3.127}$$

Then, proceeding exactly as for the derivations of (3.84) and (3.82), we readily obtain

$$\|(a_{\boldsymbol{\sigma}}-a_{\boldsymbol{\sigma}_h})(\vec{\phi},\cdot)\|_{\mathbf{H}_h'}\;\le\;L_{\mathrm{S}}\,L_{\vartheta}\,g_2\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,r;\Omega}\,,$$

where $L_{\mathrm{S}}\;:=\;C_{r,\varepsilon}\,\|i_{\varepsilon}\|\,C_{\varepsilon}$, and

$$\|G_{\mathbf{u}}-G_{\mathbf{u}_h}\|_{\mathbf{H}_h'}\;\le\;L_g\,|\Omega|^{\frac{r-s}{rs}}\,\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{M}_1}\,.$$

In this way, replacing the foregoing estimates back into (3.127), and adding the resulting inequality to (3.111), we arrive at

$$\begin{aligned}&\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1}+\|(\vec{\phi},\widetilde{\boldsymbol{\sigma}})\;-\;(\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\\&\quad\le\;C_{ST}\left\{\operatorname{dist}(\boldsymbol{\sigma},\mathbf{X}_{2,h})+\operatorname{dist}(\mathbf{u},\mathbf{M}_{1,h})+\operatorname{dist}(\vec{\phi},\mathbf{H}_h)+\operatorname{dist}(\widetilde{\boldsymbol{\sigma}},\mathbf{Q}_h)\right\}\\&\quad+\mathcal{D}(\mathtt{data})\left\{\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1}+\|\phi-\phi_h\|_{\mathrm{M}}\right\},\end{aligned}\tag{3.128}$$

where $C_{ST}\;:=\;\max\big\{\widehat{C}_{ST},\overline{C}_{ST}\big\}$, and

$$\mathcal{D}(\mathtt{data})\;:=\;\max\left\{\widehat{C}_{ST}\,L_f,\,\overline{C}_{ST}\,L_{\mathrm{S}}\,L_{\theta}\,g_2,\,\overline{C}_{ST}\,L_g\,|\Omega|^{\frac{r-s}{rs}}\right\}.\tag{3.129}$$

Thus, we conclude the Céa estimate for our second approach.

Theorem 3.16. *Assume that the data (cf. (3.129)) satisfy*

$$\mathcal{D}(\mathtt{data})\;\le\;\frac{1}{2}\,.\tag{3.130}$$

Then, there exists a positive constant C, independent of h, such that

$$\begin{aligned}&\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1}+\|(\vec{\phi},\widetilde{\boldsymbol{\sigma}})\;-\;(\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\\&\quad\le\;C\left\{\operatorname{dist}(\boldsymbol{\sigma},\mathbf{X}_{2,h})+\operatorname{dist}(\mathbf{u},\mathbf{M}_{1,h})+\operatorname{dist}(\vec{\phi},\mathbf{H}_h)+\operatorname{dist}(\widetilde{\boldsymbol{\sigma}},\mathbf{Q}_h)\right\}.\end{aligned}\tag{3.131}$$

Proof. It is a straightforward consequence of (3.128) and (3.130). □

We now refer to the eventual smallness data assumptions arising from the study of the Galerkin schemes. Indeed, similarly as for the continuous case, we first observe that, while sharing the same structure, they differ from each other between different methods, and are not verifiable in practice. The only coincidence is the amount of conditions that are required for each one of the specific goals of the analysis, say, for instance, existence or uniqueness of solution, and Céa's estimate. In particular, for the underlying discrete schemes, that is (2.100), (3.88), and (3.89), we are able to prove existence of solution by applying Brouwer's theorem, and without requiring any data assumption, but only continuity of the corresponding fixed-point operators. The respective results are given by Theorem 2.16, Theorem 3.13, and Theorem 3.15. In turn, the derivation of the Céa estimates requires of one smallness data condition each, which are included in the statements of the respective theorems, namely Theorem 2.17, Theorem 3.14, and Theorem 3.16.

3.5 Specific finite element subspaces

We now define specific finite element subspaces satisfying the stability conditions required by the respective discrete analyses developed in Section 3.4, and provide the rates of convergence of the resulting Galerkin schemes.

3.5.1 Preliminaries

Bearing in mind the mesh notations introduced at the beginning of Section 3.4.1, and given an integer $k \geq 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_k(K)$ be the space of polynomials defined on K of degree $\leq k$, and denote its vector version by $\mathbf{P}_k(K)$. In addition, we let $\widetilde{\mathrm{M}P}_k(K)$ be the space of polynomials defined on K of degree $= k$. Furthermore, we let $\mathbf{RT}_k(K) = \mathbf{P}_k(K) \oplus \widetilde{\mathrm{M}P}_k(K)\boldsymbol{x}$ be the local Raviart-Thomas space defined on K of order k, where $\boldsymbol{x}$ stands for a generic vector in R^2, and denote by $\mathbb{RT}_k(K)$ its corresponding tensor counterpart. In turn, we let $\mathrm{P}_k(\mathcal{T}_h)$, $\mathbf{P}_k(\mathcal{T}_h)$, $\mathbf{RT}_k(\mathcal{T}_h)$, and $\mathbb{RT}_k(\mathcal{T}_h)$ be the corresponding global versions of $\mathrm{P}_k(K)$, $\mathbf{P}_k(K)$, $\mathbf{RT}_k(K)$, and $\mathbb{RT}_k(K)$, respectively, that is

$$\mathrm{P}_k(\mathcal{T}_h) := \Big\{ \psi_h \in \mathrm{L}^2(\Omega) : \quad \psi_h|_K \in \mathrm{P}_k(K) \quad \forall K \in \mathcal{T}_h \Big\},$$

$$\mathbf{P}_k(\mathcal{T}_h) := \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h \Big\},$$

$$\mathbf{RT}_k(\mathcal{T}_h) := \Big\{ \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \widetilde{\boldsymbol{\tau}}_h|_K \in \mathbf{RT}_k(K) \quad \forall K \in \mathcal{T}_h \Big\},$$

and

$$\mathbb{RT}_k(\mathcal{T}_h) := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall K \in \mathcal{T}_h \Big\}.$$

We stress here that for each $t \in [1, +\infty]$ there hold $\mathrm{P}_k(\mathcal{T}_h) \subseteq \mathrm{L}^t(\Omega)$, $\mathbf{P}_k(\mathcal{T}_h) \subseteq \mathbf{L}^t(\Omega)$, $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}(\mathrm{div}_t;\Omega)$, and $\mathbb{RT}_k(\mathcal{T}_h) \subseteq \mathbb{H}^t(\mathbf{div}_t;\Omega)$, inclusions that are implicitly utilized in what follows.

As announced in Section 3.4.1, we first recall from (2.135) that the finite element subspaces of $\mathbf{X}_2$, $\mathbf{M}_1$, $\mathbf{X}_1$, and $\mathbf{M}_2$, are given, respectively, by

$$\begin{aligned} \mathbf{X}_{2,h} &:= \mathbb{H}_0^r(\mathbf{div}_r;\Omega) \cap \mathbb{RT}_k(\mathcal{T}_h)\,, \qquad \mathbf{M}_{1,h} := \mathbf{P}_k(\mathcal{T}_h)\,, \\ \mathbf{X}_{1,h} &:= \mathbb{H}_0^s(\mathbf{div}_s;\Omega) \cap \mathbb{RT}_k(\mathcal{T}_h)\,, \quad \text{and} \quad \mathbf{M}_{2,h} := \mathbf{P}_k(\mathcal{T}_h)\,, \end{aligned} \tag{3.132}$$

whereas those of $\mathbf{Q}$, M, and $\mathbf{L}^2(\Omega)$, are defined as

$$\mathbf{Q}_h := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \mathrm{M}_h := \mathrm{P}_k(\mathcal{T}_h)\,, \quad \text{and} \quad \mathbf{H}_h^{\mathbf{t}} := \mathbf{P}_k(\mathcal{T}_h)\,. \tag{3.133}$$

We stress here that $\mathbf{Q}_h$, M_h, and $\mathbf{H}_h^{\mathbf{t}}$ verify the assumptions (**H.1**) - (**H.4**). In fact, it is readily seen that $\mathrm{div}(\mathbf{Q}_h) \subseteq \mathrm{M}_h$, which confirms (**H.1**), whereas (**H.2**) is proved in [52, Lemma 4.5]. In turn, the assumptions **(H.3)** and **(H.4)** are shown in [11, Lemma 4.2].

3.5.2 The rates of convergence

The rates of convergence of the Galerkin schemes (3.88) and (3.89), with the specific finite element subspaces introduced in Section 3.5.1, are provided next. To this end, we require the approximation properties of $\mathbf{X}_{2,h}$, $\mathbf{M}_{1,h}$, $\mathbf{Q}_h$, M_h, and $\mathbf{H}_h^{\mathbf{t}}$, which are collected as follows (cf. [52, Section 4.5]):

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{W}^{l,r}(\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,r}(\Omega)$, there holds

$$\mathrm{dist}\,(\boldsymbol{\tau}, \mathbf{X}_{2,h}) := \inf_{\boldsymbol{\tau}_h \in \mathbf{X}_{2,h}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{r,\mathrm{div}_r;\Omega} \le C\, h^l \left\{ \|\boldsymbol{\tau}\|_{l,r;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{l,r;\Omega} \right\}.$$

($\mathbf{AP}_h^{\mathbf{u}}$) there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,r}(\Omega)$, there holds

$$\mathrm{dist}\,(\mathbf{v}, \mathbf{M}_{1,h}) := \inf_{\mathbf{v}_h \in \mathbf{M}_{1,h}} \|\mathbf{v} - \mathbf{v}_h\|_{0,r;\Omega} \le C\, h^l\, \|\mathbf{v}\|_{l,r;\Omega}\,.$$

($\mathbf{AP}_h^{\widetilde{\boldsymbol{\sigma}}}$) there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}^l(\Omega)$ with $\mathrm{div}(\widetilde{\boldsymbol{\tau}}) \in \mathbf{W}^{l,s}(\Omega)$, there holds

$$\mathrm{dist}\,(\widetilde{\boldsymbol{\tau}}, \mathbf{Q}_h) := \inf_{\widetilde{\boldsymbol{\tau}}_h \in X_{2,h}} \|\widetilde{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{div}_s;\Omega} \le C\, h^l \left\{ \|\widetilde{\boldsymbol{\tau}}\|_{l,\Omega} + \|\mathrm{div}(\widetilde{\boldsymbol{\tau}})\|_{l,s;\Omega} \right\}.$$

($\mathbf{AP}_h^{\phi}$) there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\psi \in \mathbf{W}^{l,r}(\Omega)$, there holds

$$\mathrm{dist}\,(\psi, \mathrm{M}_h) := \inf_{\psi_h \in \mathrm{M}_h} \|\psi - \psi_h\|_{0,r;\Omega} \le C\, h^l\, \|\psi\|_{l,r;\Omega}\,.$$

($\mathbf{AP}_h^{\mathbf{t}}$) there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{s} \in \mathbf{H}^l(\Omega)$, there holds

$$\mathrm{dist}\,(\mathbf{s}, \mathbf{H}_h^{\mathbf{t}}) := \inf_{\mathbf{s}_h \in \mathbf{H}_h^{\mathbf{t}}} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \le C\, h^l\, \|\mathbf{s}\|_{l,\Omega}\,.$$

Thus, the following two theorems establish the rates of convergence of (3.88) and (3.89).

Theorem 3.17. *Let $\big((\boldsymbol{\sigma},\mathbf{u}),(\widetilde{\boldsymbol{\sigma}},\phi)\big)\in\big(\mathbf{X}_2\times\mathbf{M}_1\big)\times\big(\mathbf{Q}\times\mathrm{M}\big)$ be the unique solution of (3.35), with $\phi\in\widetilde{W}$ (cf. (3.55)), and let $\big((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\widetilde{\boldsymbol{\sigma}}_h,\phi_h)\big)\in\big(\mathbf{X}_{2,h}\times\mathbf{M}_{1,h}\big)\times\big(\mathbf{Q}_h\times\mathrm{M}_h\big)$ be a solution of (3.88), with $\phi_h\in\widetilde{W}_h$ (cf. (3.104)), whose existences are guaranteed by Theorems 3.6 and 3.13, respectively. Assume that (3.118) (cf. Theorem 3.14) holds, and that there exists $l\in[1,k+1]$ such that $\boldsymbol{\sigma}\in\mathbb{W}^{l,r}(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma})\in\mathbf{W}^{l,r}(\Omega)$, $\mathbf{u}\in\mathbf{W}^{l,r}(\Omega)$, $\widetilde{\boldsymbol{\sigma}}\in\mathbf{H}^l(\Omega)$, $\mathrm{div}(\widetilde{\boldsymbol{\sigma}})\in\mathbf{W}^{l,s}(\Omega)$, and $\phi\in\mathbf{W}^{l,r}(\Omega)$. Then, there exists a positive constant C, independent of h, such that*

$$
\begin{aligned}
&\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1}+\|(\widetilde{\boldsymbol{\sigma}},\phi)-(\widetilde{\boldsymbol{\sigma}}_h,\phi_h)\|_{\mathbf{Q}\times\mathrm{M}}\\
&\quad\le C\,h^l\Big\{\|\boldsymbol{\sigma}\|_{l,r;\Omega}+\|\mathbf{div}(\boldsymbol{\sigma})\|_{l,r;\Omega}+\|\mathbf{u}\|_{l,r;\Omega}+\|\widetilde{\boldsymbol{\sigma}}\|_{l,\Omega}+\|\mathrm{div}(\widetilde{\boldsymbol{\sigma}})\|_{l,s;\Omega}+\|\phi\|_{l,r;\Omega}\Big\}\,.
\end{aligned}
$$

Proof. It follows from the Céa estimate (3.119) and the approximation properties $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ - $(\mathbf{AP}_h^{\phi})$. □

Theorem 3.18. *Let $\big((\boldsymbol{\sigma},\mathbf{u}),(\vec{\phi},\widetilde{\boldsymbol{\sigma}})\big)\in\big(\mathbf{X}_2\times\mathbf{M}_1\big)\times\big(\mathbf{H}\times\mathbf{Q}\big)$ be the unique solution of (3.36), with $\phi\in W$ (cf. (3.74)), and let $\big((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\big)\in\big(\mathbf{X}_{2,h}\times\mathbf{M}_{1,h}\big)\times\big(\mathbf{H}_h\times\mathbf{Q}_h\big)$ be a solution of (3.89), with $\phi_h\in W_h$ (cf. (3.122)), whose existences are guaranteed by Theorems 3.9 and 3.15, respectively. Assume that (3.130) (cf. Theorem 3.16) holds, and that there exists $l\in[1,k+1]$ such that $\boldsymbol{\sigma}\in\mathbb{W}^{l,r}(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma})\in\mathbf{W}^{l,r}(\Omega)$, $\mathbf{u}\in\mathbf{W}^{l,r}(\Omega)$, $\phi\in\mathbf{W}^{l,r}(\Omega)$, $\mathbf{t}\in\mathbf{H}^l(\Omega)$, $\widetilde{\boldsymbol{\sigma}}\in\mathbf{H}^l(\Omega)$, and $\mathrm{div}(\widetilde{\boldsymbol{\sigma}})\in\mathbf{W}^{l,s}(\Omega)$. Then there exists a positive constant C, independent of h, such that*

$$
\begin{aligned}
&\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1}+\|(\vec{\phi},\widetilde{\boldsymbol{\sigma}})-(\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\\
&\quad\le C\,h^l\Big\{\|\boldsymbol{\sigma}\|_{l,r;\Omega}+\|\mathbf{div}(\boldsymbol{\sigma})\|_{l,r;\Omega}+\|\mathbf{u}\|_{l,r;\Omega}+\|\phi\|_{l,r;\Omega}+\|\mathbf{t}\|_{l,\Omega}+\|\widetilde{\boldsymbol{\sigma}}\|_{l,\Omega}+\|\mathrm{div}(\widetilde{\boldsymbol{\sigma}})\|_{l,s;\Omega}\Big\}\,.
\end{aligned}
$$

Proof. It follows from the Céa estimate (3.131) and the approximation properties $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ - $(\mathbf{AP}_h^{\mathbf{t}})$. □

3.6 Numerical results

In this section we present three examples illustrating the performance of the fully-mixed finite schemes (3.88) and (3.89) with the finite element subspaces defined in Section 3.5.1 for $k\in\{0,1\}$, and confirming the rates of convergence provided by Theorems 3.17 and 3.18 on uniform refinements of the respective domains.

Letting N_e and N_t be the number of edges and triangles, respectively, of $\mathcal{T}_h$, and denoting by `DOF` the total number of degrees of freedom (or unknowns) of each approach, we deduce that in the 2D case the respective values for (3.88) and (3.89), are given by (cf. (3.132), (3.133), and [44, Chapter 3])

$$
\mathtt{DOF}\;=\;3\,(k+1)N_e\;+\;\frac{3}{2}\,(k+1)(3k+2)N_t\;+\;1\,,\tag{3.134}
$$

and

$$
\mathtt{DOF}\;=\;\text{previous expression}\;+\;(k+1)(k+2)N_t\,,\tag{3.135}
$$

where the extra degree of freedom at the end of (3.134) corresponds to the Lagrange multiplier taking care of the null mean value for the traces of the tensors in $\mathbf{X}_{2,h}$ and $\mathbf{X}_{1,h}$, whereas the ones at the end of (3.135) correspond to those required by the subspace $\mathbf{H}_h^{\mathbf{t}}$. The resulting nonlinear algebraic systems are solved employing the Picard iterative process suggested by the respective discrete fixed-point strategy (cf. Sections 3.4.5 and 3.4.6), whose computational implementation was done using a `FreeFem++` code [58]. We take as initial guess the trivial solution, and the iterations are stopped when the relative error between two consecutive vectors containing the full solutions of the aforementioned systems, namely $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|}{\|\mathbf{coeff}^{m+1}\|} \leq \mathsf{tol},$$

where $\|\cdot\|$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{DOF}}$, and tol is a given tolerance. In this regard, we remark in advance that for each one of the examples to be reported below, 3 iterations are required to achieve $\mathsf{tol} = 1\mathrm{e}-6$.

We now recall that the original Cauchy stress tensor $\boldsymbol{\rho}$ of our model can be computed in terms of $\boldsymbol{\sigma}$ according to the formula derived from (2.9) and (2.10) and (2.32), namely

$$\boldsymbol{\rho} := \boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{t}} - \left(\frac{\lambda + 2\mu}{n\lambda + (n+1)\mu}\,\mathrm{tr}(\boldsymbol{\sigma}) - \frac{n\lambda + 2\mu}{n|\Omega|}\int_\Gamma \mathbf{u}_D\cdot\boldsymbol{\nu}\right)\mathbb{I}, \tag{3.136}$$

which naturally suggests approximating this tensor by (2.175)

$$\boldsymbol{\rho}_h := \boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathrm{t}} - \left(\frac{\lambda + 2\mu}{n\lambda + (n+1)\mu}\,\mathrm{tr}(\boldsymbol{\sigma}_h) - \frac{n\lambda + 2\mu}{n|\Omega|}\int_\Gamma \mathbf{u}_D\cdot\boldsymbol{\nu}\right)\mathbb{I}. \tag{3.137}$$

It follows from (3.136) and (3.137) that there exists a constant $C > 0$, independent of h and λ, such that

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,r;\Omega} \leq C\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,r;\Omega},$$

whence the rate of convergence for $\boldsymbol{\rho}_h$ is at least the same of $\boldsymbol{\sigma}_h$.

Some additional notation is introduced next. We begin by defining the individual errors:

$$\begin{gathered}\mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{r,\mathbf{div}_r;\Omega}, \quad \mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,r;\Omega}, \quad \mathsf{e}(\boldsymbol{\rho}) := \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,r;\Omega},\\ \mathsf{e}(\widetilde{\boldsymbol{\sigma}}) := \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{\mathrm{div}_s;\Omega}, \quad \mathsf{e}(\phi) := \|\phi - \phi_h\|_{0,r;\Omega}, \quad \text{and} \quad \mathsf{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega},\end{gathered}$$

where r and s, taken from (3.9), will be specified in the examples below. In turn, for each $\star \in \{\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\rho}, \widetilde{\boldsymbol{\sigma}}, \phi, \mathbf{t}\}$ we let $\mathsf{r}(\star)$ be its experimental rate of convergence, which is defined as

$$\mathsf{r}(\star) := \log\left(\mathsf{e}(\star)/\widehat{\mathsf{e}}(\star)\right) / \log(h/\widehat{h}),$$

where e and $\widehat{\mathsf{e}}$ denote two consecutive errors with mesh sizes h and $\widehat{h}$, respectively.

The examples to be considered in this section are described next. In each case we let E and ν be the Young modulus and Poisson ratio, respectively, of the isotropic linear elastic solid occupying the region Ω, so that the corresponding Lamé parameters are given by

$$\mu := \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda := \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{3.138}$$

In addition, the mean value of $\mathrm{tr}(\boldsymbol{\sigma}_h)$ over Ω is fixed via a real Lagrange multiplier, which reduces to adding one row and one column to the matrix system that solves (3.91) for $\boldsymbol{\sigma}_h$ and $\mathbf{u}_h$.

3.6.1 Example 1: Convergence in a 2D domain

We begin by corroborating the rates of convergence against a smooth exact solution in the two-dimensional domain $\Omega = (0,1)^2$. To this end, we adequately manufacture the data so that the solution of (3.1)-(3.2) is given by

$$\mathbf{u}(\boldsymbol{x}) := \left(\begin{array}{c} 0.05\, \cos(\pi\, x_1)\, \sin(\pi\, x_2) + \dfrac{x_1^2(1-x_2)^2}{2\lambda} \\ -0.05\, \sin(\pi\, x_1)\, \cos(\pi\, x_2) + \dfrac{x_1^3(1-x_2)^3}{2\lambda} \end{array} \right) \quad \text{and} \quad \phi(\boldsymbol{x}) := (1-x_1)^2 x_1 (1-x_2) x_2^2 \,,$$

for all $\boldsymbol{x} := (x_1, x_2)^{\mathrm{t}} \in \Omega$, whereas the body load, the diffusive source, and the tensorial diffusivity, are given, respectively, by

$$\mathbf{f}(\phi) \,:=\, \frac{1}{10} \left(\begin{array}{c} \cos(\phi) \\ -\sin(\phi) \end{array} \right) , \quad g(\mathbf{u}) \,:=\, 2 + \frac{1}{1+|\mathbf{u}|^2} \,, \quad \vartheta(\boldsymbol{\sigma}) \,:=\, \mathbb{I} + \frac{1}{10}\, \boldsymbol{\sigma}^2 \,.$$

We note here that the second and fifth equation of (3.1), actually include additional explicit source terms that are added to $\mathbf{f}(\phi)$ and $g(\mathbf{u})$, respectively. However, yielding only slight modifications of the functionals G, F_ϕ, $\widetilde{G}_{\mathbf{u}}$ and $G_{\mathbf{u}}$ (cf. (3.11), (3.12), (3.25) and (3.33), respectively), this fact does not compromise the continuous and discrete analyses. Thus, in Tables 3.1 and 3.2 we summarize the convergence of (3.88) and (3.89), respectively, considering $r \in \{3, 4\}$, the Young's modulus $E = 1$, and the Poisson's ratio $\nu = 0.4999$, which, according to (3.138), yield $\lambda = 1666.44$ and $\mu = 0.3334$. The results confirm that the optimal rates of convergence $\mathcal{O}(h^{k+1})$ predicted by Theorems 3.17 and 3.18 are attained for $k \in \{0, 1\}$, and for both indexes r, as implicitly stated by those theorems as well. In addition, while not exactly the same values, the errors for $r = 3$ and $r = 4$ of each unknown are of the same order of magnitude in each given mesh. Some components and magnitudes of the discrete solutions of the first approach (3.88) are displayed in Figure 3.1.

On the other hand, we now compare the results of Example 1 with the mixed-primal method presented in Chapter 2 for the purpose of seeing the accuracy of each. Indeed, letting $\widetilde{\boldsymbol{\sigma}}_h$, $\widetilde{\mathbf{u}}_h$ and $\widetilde{\phi}_h$ be the solution of Galekin scheme introduced in (2.4) and (2.135). In addition, let $\widetilde{\boldsymbol{\rho}}_h$ be the corresponding postprocessing unknown computed in similar way of (3.137). Next, we add the following individual errors:

$$\mathsf{e}(\widetilde{\boldsymbol{\sigma}}) \,:=\, \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_h\|_{r,\mathbf{div}_r;\Omega} \,, \quad \mathsf{e}(\widetilde{\mathbf{u}}) \,:=\, \|\mathbf{u} - \widetilde{\mathbf{u}}_h\|_{0,r;\Omega} \,,$$
$$\mathsf{e}(\widetilde{\boldsymbol{\rho}}) \,:=\, \|\boldsymbol{\rho} - \widetilde{\boldsymbol{\rho}}_h\|_{0,r;\Omega} \,, \quad \text{and} \quad \mathsf{e}(\widetilde{\phi}) \,:=\, \|\phi - \widetilde{\phi}_h\|_{0,r;\Omega} \,,$$

and, for each $\star \in \{\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\rho}, \phi\}$ we let $\mathtt{d}(\star)$ be its relative error, which is defined as

$$\mathtt{d}(\star) \,:=\, \big|\mathsf{e}(\star) - \mathsf{e}(\widetilde{\star})\big| \,/\, \big|\mathsf{e}(\star)\big| \,,$$

where, for $\mathsf{e}(\star)$ we only employ the solution of the Galerkin scheme (3.88), since the results for (3.89) are the same. Finally, Table 3.3 shows the errors and convergence ratios for scheme (2.100) and (2.135), where the relative errors $\mathtt{d}(\cdot)$ suggest that the accuracy between the approximations for $\boldsymbol{\sigma}$, $\mathbf{u}$ and $\boldsymbol{\rho}$ are similar to that given by (3.88). Note that this is not the case for ϕ, which is expected since $\widetilde{\phi}_h \in \mathrm{H}^1(\Omega)$, whereas $\phi_h \in \mathrm{L}^r(\Omega)$. In this particular example, the results of Tables 3.1 and 3.3 establish, in the L^r-norm, that ϕ was calculated better for the new scheme (3.88).

k	h	DOF	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\rho}$)	r($\boldsymbol{\rho}$)	e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	e(ϕ)	r(ϕ)
	0.0471	13681	4.12e-2	−−	1.44e-3	−−	1.83e-2	−−	1.03e-2	−−	5.26e-4	−−
	0.0393	19657	3.43e-2	1.01	1.20e-3	1.03	1.57e-2	0.99	8.65e-3	1.00	4.38e-4	1.00
0	0.0337	26713	2.93e-2	1.01	1.02e-3	1.02	1.34e-2	0.99	7.40e-3	1.00	3.76e-4	1.00
	0.0295	34849	2.57e-2	1.01	8.93e-4	1.02	1.18e-2	0.99	6.48e-3	1.00	3.29e-4	1.00
	0.0262	44065	2.29e-2	1.01	7.92e-4	1.01	1.05e-2	1.00	5.76e-3	1.00	2.92e-4	1.00
	0.0471	43561	5.33e-4	−−	2.83e-5	−−	2.63e-4	−−	2.53e-4	−−	1.53e-5	−−
	0.0393	62641	3.70e-4	2.00	1.96e-5	2.00	1.83e-4	1.99	1.76e-4	2.00	1.06e-5	2.00
1	0.0337	85177	2.72e-4	2.00	1.44e-5	2.00	1.35e-4	1.99	1.29e-4	2.00	7.82e-6	2.00
	0.0295	111169	2.08e-4	2.00	1.10e-5	2.00	1.03e-4	1.99	9.89e-5	2.00	5.99e-6	2.00
	0.0262	140611	1.65e-4	2.00	8.71e-6	2.00	8.16e-5	1.99	7.82e-5	2.00	4.73e-6	2.00
	0.0471	13681	5.14e-2	−−	1.59e-3	−−	2.31e-2	−−	9.96e-3	−−	6.47e-4	−−
	0.0393	19657	4.27e-2	1.01	1.32e-3	1.03	1.93e-2	0.99	8.29e-3	1.00	5.40e-4	1.00
0	0.0337	26713	3.66e-2	1.01	1.12e-3	1.02	1.66e-2	0.99	7.11e-3	1.00	4.63e-4	1.00
	0.0295	34849	3.20e-2	1.01	9.82e-4	1.02	1.45e-2	0.99	6.22e-3	1.00	4.05e-4	1.00
	0.0262	44065	2.84e-2	1.01	8.72e-4	1.01	1.29e-2	1.00	5.53e-3	1.00	3.60e-4	1.00
	0.0471	43561	6.15e-4	−−	3.08e-5	−−	3.10e-4	−−	2.45e-4	−−	1.92e-5	−−
	0.0393	62641	4.27e-4	2.00	2.14e-5	2.00	2.15e-4	1.99	1.70e-4	2.00	1.33e-5	2.00
1	0.0337	85177	3.14e-4	2.00	1.57e-5	2.00	1.58e-4	2.00	1.25e-4	2.00	9.79e-6	2.00
	0.0295	111169	2.40e-4	2.00	1.20e-5	2.00	1.21e-4	2.00	9.68e-5	2.00	7.50e-6	2.00
	0.0262	140617	1.90e-4	2.00	9.50e-6	2.00	9.59e-5	2.00	7.57e-5	2.00	5.93e-6	2.00

Table 3.1: Example 1: History of convergence for the Galerkin scheme (3.88) with $r = 3$ (upper half), and $r = 4$ (lower half).

k	h	DOF	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\rho}$)	r($\boldsymbol{\rho}$)	e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	e(ϕ)	r(ϕ)	e($\mathbf{t}$)	r($\mathbf{t}$)
	0.0471	17281	4.12e-1	−−	1.44e-3	−−	1.88e-2	−−	1.04e-2	−−	5.25e-4	−−	3.24-03	−−
	0.0393	24841	3.43e-2	1.01	1.20e-3	1.03	1.57e-2	0.98	8.63e-3	1.00	4.38e-4	1.00	2.70e-3	1.03
0	0.0337	33769	2.93e-2	1.01	1.02e-3	1.02	1.34e-2	0.99	7.40e-3	1.00	3.76e-4	1.00	2.31e-3	1.02
	0.0295	44065	2.57e-2	1.01	8.93e-4	1.02	1.18e-2	0.99	6.48e-3	1.00	3.29e-4	1.00	2.02e-3	1.02
	0.0262	55729	2.28e-2	1.01	7.92e-4	1.01	1.05e-2	1.00	5.76e-3	1.00	2.92e-4	1.00	1.18e-3	1.01
	0.0471	54361	5.33e-4	−−	2.82e-5	−−	2.63e-4	−−	2.53e-4	−−	1.53e-5	−−	7.69e-5	2.00
	0.0393	78193	3.70e-4	2.00	1.96e-5	2.00	1.83e-4	1.99	1.76e-4	2.00	1.06e-5	2.00	5.34e-5	2.00
1	0.0337	106345	2.72e-4	2.00	1.44e-5	2.00	1.35e-4	1.99	1.29e-4	2.00	7.82e-6	2.00	3.93e-5	2.00
	0.0295	138817	2.08e-4	2.00	1.10e-5	2.00	1.03e-4	1.99	9.89e-5	2.00	5.99e-6	2.00	3.01e-5	2.00
	0.0262	175609	1.65e-4	2.00	8.71e-6	2.00	8.16e-5	1.99	7.82e-5	2.00	4.73e-6	2.00	2.48e-5	2.00
	0.0471	17281	5.14e-2	−−	1.59e-3	−−	2.31e-2	−−	9.95e-3	−−	6.47e-4	−−	3.24-03	1.03
	0.0393	24841	4.27e-2	1.01	1.32e-3	1.03	1.93e-2	0.99	8.29e-3	1.00	5.39e-4	1.00	2.70e-3	1.02
0	0.0337	33769	3.66e-2	1.01	1.12e-3	1.02	1.66e-2	0.99	7.11e-3	1.00	4.63e-4	1.00	2.31e-3	1.02
	0.0295	44065	3.20e-2	1.01	9.82e-4	1.01	1.45e-2	0.99	6.22e-3	1.00	4.05e-4	1.00	2.02e-3	1.01
	0.0262	55729	2.84e-2	1.01	8.72e-4	1.01	1.29e-2	1.00	5.53e-3	1.00	3.60e-4	1.00	1.18e-3	1.01
	0.0471	54361	6.15e-3	−−	3.08e-5	−−	3.10e-4	−−	2.45e-4	−−	1.92e-5	−−	7.69e-5	2.00
	0.0393	78193	4.27e-4	2.00	2.14e-5	2.00	2.15e-4	1.99	1.70e-4	2.00	1.33e-5	2.00	5.34e-5	2.00
1	0.0337	106345	3.14e-4	2.00	1.57e-5	2.00	1.58e-4	2.00	1.25e-4	2.00	9.79e-6	2.00	3.93e-5	2.00
	0.0295	138817	2.40e-4	2.00	1.20e-5	2.00	1.21e-4	2.00	9.58e-5	2.00	7.50e-6	2.00	3.01e-5	2.00
	0.0262	175609	1.90e-4	2.00	9.50e-6	2.00	9.59e-5	2.00	7.57e-5	2.00	5.93e-6	2.00	2.48e-5	2.00

Table 3.2: Example 1: History of convergence for the Galerkin scheme (3.89) with $r = 3$ (upper half), and $r = 4$ (lower half).

3.6.2 Example 2: Convergence in a non-convex 2D domain

We consider the L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1]^2$, and suitable perturbations of the definitions of the functionals G, F_ϕ, $\widetilde{G}_{\mathbf{u}}$, and $G_{\mathbf{u}}$, so that the exact solution of (3.1) - (3.2) reduces to the non-

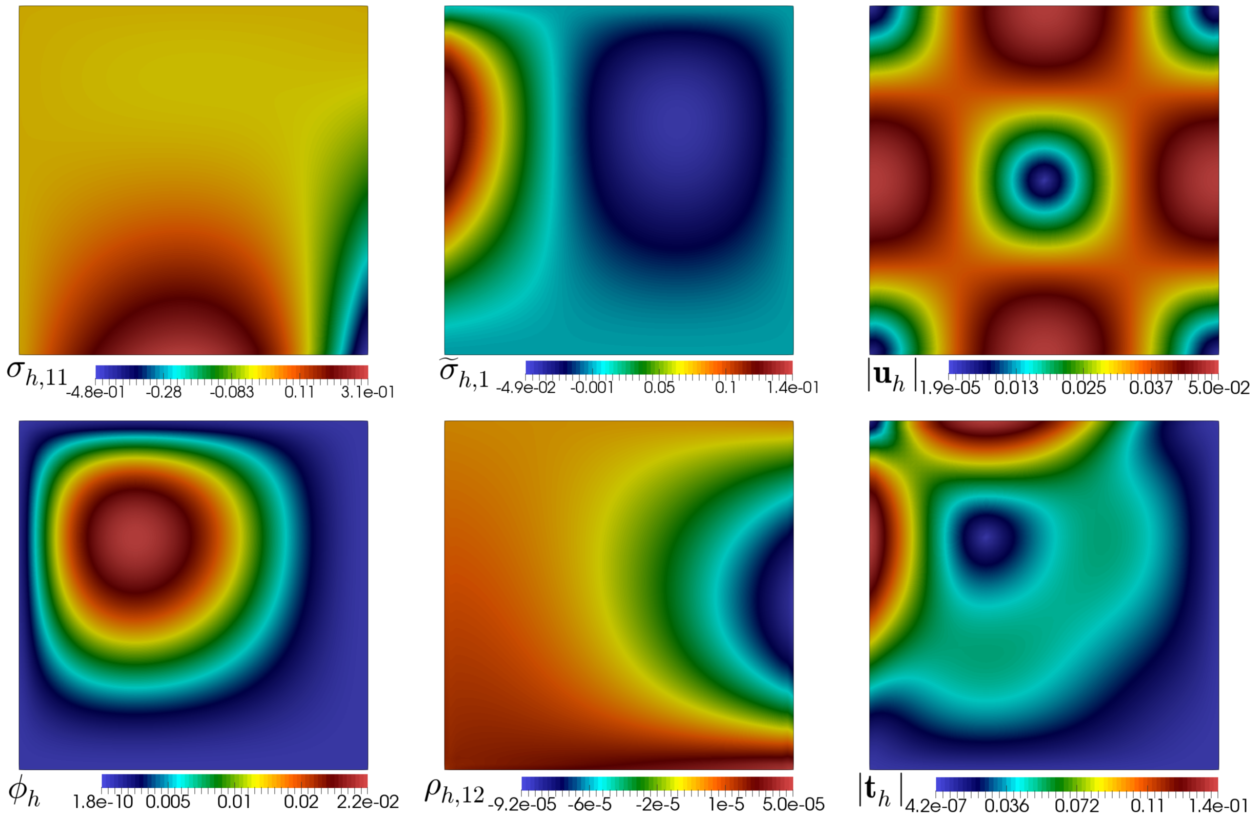

Figure 3.1: Example 1: Some components and magnitudes of the solution of the first approach (3.88) with $k = 1$, $\lambda = 1666.44$, and $\mu = 0.3334$.

smooth one defined as:

$$\mathbf{u}(\boldsymbol{x}) := \begin{pmatrix} |\boldsymbol{x}|^{2/3}\sin(\theta) \\ -|\boldsymbol{x}|^{2/3}\cos(\theta) \end{pmatrix} \quad \text{and} \quad \phi(\boldsymbol{x}) := \exp(x_1 + x_2)\sin(\pi x_1)\sin(\pi x_2)\,,$$

where $\theta = \arctan\left(\frac{x_2}{x_1}\right)$ for all $\boldsymbol{x} = (x_1, x_2)^{\mathtt{t}} \in \Omega$. In turn, the tensorial diffusivity is considered the same from the previous example, whereas the body load and the diffusive source are given, respectively, by

$$\mathbf{f}(\phi) := \begin{pmatrix} \dfrac{1}{40}\phi \\ \dfrac{1}{40}\phi(1-\phi) \end{pmatrix} \quad \text{and} \quad g(\mathbf{u}) := -|\mathbf{u}|\,.$$

In this case, we take $E = 100$ and $\nu = 0.4999$, which yields $\mu = 33.33$ and $\lambda = 166644.44$. Here we can see in Tables 3.4 and 3.5 that it was not possible to reach the convergence order $k+1$ indicated by Theorems 3.17 and 3.18. In particular, we notice that, for both formulations (cf. (3.88) and (3.89)), negative convergence orders are obtained for $\boldsymbol{\sigma}$, while for $\mathbf{u}$, $\boldsymbol{\rho}$, and $\widetilde{\boldsymbol{\sigma}}$, suboptimal ones are attained. Furthermore, as it was observed in Chapter 2.6, we remark that the convergence ratios depend not only on k but also on r and its conjugate s, which could be related to the $\mathbf{W}^{l,r}$- regularity of the solution, most likely with a non-integer l depending on r. We refer to [52, Lemma B.1] for a similar situation holding with a regularity result for the Poisson problem with homogeneous Neumann

k	h	DOF	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{d}(\boldsymbol{\sigma})$	$\mathtt{e}(\widetilde{\mathbf{u}})$	$\mathtt{r}(\widetilde{\mathbf{u}})$	$\mathtt{d}(\mathbf{u})$	$\mathtt{e}(\widetilde{\boldsymbol{\rho}})$	$\mathtt{r}(\widetilde{\boldsymbol{\rho}})$	$\mathtt{d}(\boldsymbol{\rho})$	$\mathtt{e}(\widetilde{\phi})$	$\mathtt{r}(\widetilde{\phi})$	$\mathtt{d}(\phi)$
	0.0471	10082	3.88e-2	−−	5.83e-2	1.35e-3	−−	6.23e-2	1.86e-2	−−	1.69e-2	9.88e-3	−−	1.78e+1
	0.0393	14474	3.23e-2	1.01	5.87e-2	1.12e-3	1.04	6.93e-2	1.55e-2	0.99	1.01e-2	8.33e-3	0.94	1.80e+1
0	0.0337	19658	2.76e-2	1.01	5.65e-2	9.53e-4	1.03	6.59e-2	1.33e-2	0.99	4.66e-3	7.18e-3	0.96	1.81e+1
	0.0295	25634	2.42e-2	1.01	5.96e-2	8.31e-4	1.02	6.93e-2	1.17e-2	0.99	1.02e-2	6.30e-3	0.98	1.82e+1
	0.0262	32402	2.15e-2	1.01	6.25e-2	7.37e-4	1.02	6.93e-2	1.04e-2	1.00	1.07e-2	5.61e-3	0.99	1.82e+1
	0.0471	32762	5.31e-4	−−	3.96e-3	2.87e-5	−−	1.56e-2	2.65e-4	−−	6.96e-3	1.78e-3	−−	1.15e+2
	0.0393	47090	3.69e-4	2.00	3.24e-3	2.00e-5	2.00	1.80e-2	1.84e-4	1.99	6.91e-3	1.25e-3	1.94	1.17e+2
1	0.0337	64010	2.71e-4	2.00	3.57e-3	1.47e-5	2.00	1.77e-2	1.36e-4	1.99	4.18e-3	9.07e-4	2.09	1.15e+2
	0.0295	83522	2.08e-4	2.00	2.15e-3	1.12e-5	2.00	1.98e-2	1.04e-4	1.99	8.72e-3	6.97e-4	1.97	1.15e+2
	0.0262	105626	1.64e-4	2.00	5.93e-3	8.86e-6	2.00	1.75e-2	8.22e-5	1.99	6.84e-3	5.51e-4	1.98	1.16e+2
	0.0471	10082	4.66e-2	−−	9.34e-2	1.42e-3	−−	1.04e-1	2.27e-2	−−	1.53e-2	1.31e-2	−−	1.92e+1
	0.0393	14474	3.87e-2	1.01	9.27e-2	1.18e-3	1.05	1.09e-1	1.90e-2	0.99	1.53e-2	1.10e-2	0.94	1.94e+1
0	0.0337	19658	3.31e-2	1.01	9.43e-2	1.00e-3	1.04	1.05e-1	1.63e-2	0.99	1.72e-2	9.50e-3	0.96	1.95e+1
	0.0295	25634	2.90e-2	1.01	9.49e-2	8.74e-4	1.03	1.10e-1	1.43e-2	0.99	1.45e-2	8.33e-3	0.98	1.96e+1
	0.0262	32402	2.57e-2	1.01	9.46e-2	7.75e-4	1.02	1.11e-1	1.27e-2	0.99	1.47e-2	7.42e-3	0.99	1.96e+1
	0.0471	32762	6.10e-4	−−	8.63e-3	3.18e-5	−−	3.18e-2	3.14e-4	−−	1.32e-2	2.18e-3	−−	1.12e+2
	0.0393	47090	4.23e-4	2.00	8.49e-3	2.21e-5	2.00	3.07e-2	2.18e-4	1.99	1.58e-2	1.53e-3	1.94	1.14e+2
1	0.0337	64010	3.11e-4	2.00	9.37e-3	1.62e-5	2.00	3.18e-2	1.61e-4	1.99	1.64e-2	1.13e-3	1.98	1.14e+2
	0.0295	83522	2.38e-4	2.00	7.65e-3	1.24e-5	2.00	3.32e-2	1.23e-4	2.00	1.68e-2	8.67e-4	1.96	1.15e+2
	0.0262	105626	1.88e-4	2.00	9.52e-3	9.79e-6	2.00	3.10e-2	9.73e-5	2.00	1.41e-2	6.87e-4	1.98	1.15e+2

Table 3.3: Example 1: History of convergence for the Galerkin scheme of Chapter 2 with $r = 3$ (upper half), and $r = 4$ (lower half).

boundary conditions and source term in a Lebesgue space. In order to recover the optimal rates of convergence, one could apply an adaptive strategy based on a posteriori error estimates, subject that we plan to address in a forthcoming work.

3.6.3 Example 3: Convergence in a 3D domain

In this example we confirm the rates of convergence in the three-dimensional domain $\Omega = (0,1)^3$ with the indexes $r = 3$ and $s = 3/2$ (cf. (3.9)). As in Example 1, we consider $\mu = 0.3334$ and $\lambda = 1666.44$, and suitably manufacture the data so that the exact solution is given by

$$\mathbf{u}(\boldsymbol{x}) := \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix} \quad \text{and} \quad \phi(\boldsymbol{x}) := x_1 x_2^2 x_3 (x_1-1)^2(x_2-1)(x_3-1)^2\,,$$

for all $\boldsymbol{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \Omega$, whereas the body load, the diffusive source, and the tensorial diffusivity, are given, respectively, by

$$\mathbf{f}(\phi) := \frac{1}{10}\begin{pmatrix} \cos(\phi) \\ -\sin(\phi) \\ \cos(\phi) \end{pmatrix}, \quad g(\mathbf{u}) := u_1 + u_2 + u_3\,, \quad \vartheta(\boldsymbol{\sigma}) := \frac{1}{2}\left(1 + \frac{1}{(1+|\boldsymbol{\sigma}|^2)^{1/2}}\right)\mathbb{I}\,.$$

The convergence histories for quasi-uniform refinements using $k = 0$ are reported in Tables 3.6 and 3.7. Again, the mixed finite element methods converge optimally, that is with order $\mathcal{O}(h)$ in this case, as it was proved by Theorems 3.17 and 3.18. This fact suggests that perhaps only technical difficulties

k	h	DOF	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$
	0.0471	40861	9.56e+2	−−	1.85e-2	−−	7.48e+0	−−	2.62e+1	−−	3.20e-2	−−
	0.0404	55546	1.06e+3	-0.66	1.59e-2	0.99	7.10e+0	0.34	2.31e+1	0.82	2.72e-2	1.05
0	0.0354	72481	1.16e+3	-0.66	1.39e-2	0.99	6.79e+0	0.33	2.07e+1	0.81	2.37e-2	1.04
	0.0314	91666	1.25e+3	-0.66	1.23e-2	0.99	6.53e+0	0.33	1.88e+1	0.80	2.10e-2	1.03
	0.0283	113101	1.34e+3	-0.66	1.11e-2	0.99	6.30e+0	0.33	1.73e+1	0.80	1.88e-2	1.03
	0.0471	130321	4.73e+2	−−	4.92e-4	−−	4.78e+0	−−	9.10e+0	−−	6.78e-4	−−
	0.0404	177241	5.24e+2	-0.66	4.00e-4	1.33	4.54e+0	0.33	8.15e+0	0.71	4.99e-4	1.99
1	0.0354	231361	5.73e+2	-0.66	3.35e-4	1.33	4.34e+0	0.33	7.42e+0	0.71	3.83e-4	1.99
	0.0314	292681	6.19e+2	-0.66	2.87e-4	1.33	4.18e+0	0.33	6.83e+0	0.70	3.03e-4	1.99
	0.0283	361201	6.64e+2	-0.66	2.49e-4	1.33	4.03e+0	0.33	6.34e+0	0.70	2.46e-4	1.98
	0.0471	40861	1.92e+3	−−	1.88e-2	−−	1.28e+1	−−	2.35e+1	−−	3.48e-2	−−
	0.0404	55546	2.18e+3	-0.83	1.61e-2	0.99	1.25e+1	0.17	2.04e+1	0.91	2.96e-2	1.03
0	0.0354	72481	2.44e+3	-0.83	1.41e-2	0.99	1.22e+1	0.17	1.81e+1	0.90	2.58e-2	1.03
	0.0314	91666	2.69e+3	-0.83	1.26e-2	0.99	1.20e+1	0.17	1.63e+1	0.90	2.29e-2	1.02
	0.0283	113101	2.91e+3	-0.83	1.13e-2	0.99	1.18e+1	0.17	1.48e+1	0.90	2.06e-2	1.02
	0.0471	130321	8.76e+2	−−	8.02e-4	−−	9.04e+0	−−	5.88e+0	−−	7.17e-4	−−
	0.0404	177241	9.95e+2	-0.83	6.70e-4	1.17	8.81e+0	0.17	5.12e+0	0.90	5.27e-4	2.00
1	0.0354	231361	1.11e+3	-0.83	5.74e-4	1.17	8.62e+0	0.17	4.55e+0	0.89	4.04e-4	2.00
	0.0314	292681	1.23e+3	-0.83	5.00e-4	1.17	8.45e+0	0.17	4.10e+0	0.89	3.19e-4	1.99
	0.0283	361201	1.34e+3	-0.83	4.42e-4	1.17	8.30e+0	0.17	3.74e+0	0.88	2.59e-4	1.99

Table 3.4: Example 2: History of convergence for the Galerkin scheme (3.88) with $r = 3$ (first half), and $r = 4$ (second half).

k	h	DOF	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$
	0.0471	51661	9.56e+2	−−	1.85e-2	−−	7.48e+0	−−	2.55e+1	−−	3.19e-2	−−	2.11e-1	−−
	0.0404	70246	1.06e+3	-0.66	1.59e-2	1.00	7.10e+0	0.34	2.25e+1	0.80	2.72e-2	1.04	1.80e-1	1.02
0	0.0354	91681	1.16e+3	-0.66	1.39e-2	1.00	6.79e+0	0.33	2.02e+1	0.80	2.37e-2	1.04	1.58e-1	1.02
	0.0314	115966	1.25e+3	-0.66	1.24e-2	1.00	6.53e+0	0.33	1.85e+1	0.79	2.10e-2	1.03	1.40e-1	1.01
	0.0283	143101	1.34e+3	-0.66	1.11e-2	1.00	6.30e+0	0.33	1.70e+1	0.79	1.88e-2	1.02	1.26e-1	1.01
	0.0471	162721	4.73e+2	−−	4.92e-4	−−	4.78e+0	−−	8.92e+0	−−	6.74e-04	−−	6.98e-3	−−
	0.0404	221341	5.24e+2	-0.66	4.01e-4	1.33	4.54e+0	0.33	8.01e+0	0.70	4.96e-04	1.99	5.24e-3	1.86
1	0.0354	288961	5.73e+2	-0.66	3.35e-4	1.33	4.34e+0	0.33	7.30e+0	0.70	3.80e-04	1.99	4.09e-3	1.86
	0.0314	365581	6.19e+2	-0.66	2.87e-4	1.33	4.18e+0	0.33	6.73e+0	0.69	3.01e-04	1.99	3.28e-3	1.87
	0.0283	451201	6.64e+2	-0.66	2.49e-4	1.33	4.03e+0	0.33	6.25e+0	0.69	2.44e-04	1.99	2.69e-3	1.87
	0.0471	51661	1.92e+3	−−	1.88e-2	−−	1.28e+1	−−	2.28e+1	−−	3.47e-2	−−	2.11e-1	−−
	0.0404	70246	2.18e+3	-0.83	1.61e-2	0.99	1.25e+1	0.17	1.99e+1	0.89	2.96e-2	1.04	1.80e-1	1.02
	0.0354	91681	2.44e+3	-0.83	1.41e-2	0.99	1.22e+1	0.17	1.77e+1	0.89	2.58e-2	1.04	1.58e-1	1.02
0	0.0314	115966	2.69e+3	-0.83	1.26e-2	0.99	1.20e+1	0.17	1.59e+1	0.89	2.29e-2	1.03	1.40e-1	1.01
	0.0283	143101	2.94e+3	-0.83	1.13e-2	0.99	1.18e+1	0.17	1.45e+1	0.89	2.06e-2	1.02	1.26e-1	1.01
	0.0471	162721	8.76e+2	−−	8.02e-4	−−	9.04e+0	−−	5.70e+0	−−	7.14e-04	−−	6.98e-3	−−
	0.0404	221341	9.95e+2	-0.83	6.70e-4	1.17	8.81e+0	0.17	4.98e+0	0.88	5.25e-04	2.00	5.24e-3	1.86
1	0.0354	288961	1.11e+3	-0.83	5.74e-4	1.17	8.62e+0	0.17	4.43e+0	0.88	4.02e-04	2.00	4.09e-3	1.86
	0.0314	365581	1.23e+3	-0.83	5.00e-4	1.17	8.45e+0	0.17	4.00e+0	0.87	3.18e-04	1.99	3.28e-3	1.87
	0.0283	451201	1.34e+3	-0.83	4.42e-4	1.17	8.30e+0	0.17	3.65e+0	0.87	2.58e-04	1.99	2.69e-3	1.87

Table 3.5: Example 2: History of convergence for the Galerkin scheme (3.89) with $r = 3$ (first half), and $r = 4$ (second half).

stop us from extending the analysis to the 3D framework. Finally, some components and magnitudes of the solution of the second approach (3.89) are displayed in Figure 3.2.

k	h	DOF	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$
	0.4330	4993	3.11e+2	−−	2.69e-1	−−	6.51e+1	−−	7.35e-3	−−	4.00e-4	−−
	0.3464	9601	2.51e+2	0.95	2.18e-1	0.95	5.26e+1	0.96	5.58e-3	1.23	3.30e-4	0.87
	0.2887	16417	2.11e+2	0.97	1.83e-1	0.97	4.40e+1	0.97	4.50e-3	1.18	2.79e-4	0.92
0	0.2474	25873	1.81e+2	0.98	1.57e-1	0.98	3.79e+1	0.98	3.78e-3	1.13	2.41e-4	0.95
	0.2165	38401	1.59e+2	0.98	1.38e-1	0.98	3.32e+1	0.99	3.25e-3	1.13	2.12e-4	0.96
	0.1925	54433	1.41e+2	0.99	1.23e-1	0.99	2.96e+1	0.99	2.87e-3	1.07	1.89e-4	0.97
	0.1732	74401	1.27e+2	0.99	1.11e-1	0.99	2.66e+1	0.99	2.55e-3	1.12	1.70e-4	0.98

Table 3.6: Example 3: History of convergence for the Galerkin scheme (3.88) with $r = 3$ and $s = 3/2$.

k	h	DOF	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\mathbf{t})$	$\mathtt{r}(\mathbf{t})$
	0.4330	6145	3.11e+2	−−	2.69e-1	−−	6.52e+1	−−	7.35e-3	−−	3.89e-4	−−	2.60e-3	−−
	0.3464	11851	2.51e+2	0.95	2.18e-1	0.95	5.26e+1	0.96	5.58e-3	1.23	3.23e-4	0.83	2.14e-3	0.89
	0.2887	20305	2.11e+2	0.97	1.83e-1	0.97	4.40e+1	0.97	4.50e-3	1.18	2.75e-4	0.89	1.81e-3	0.92
0	0.2474	32047	1.81e+2	0.98	1.57e-1	0.98	3.79e+1	0.98	3.78e-3	1.13	2.38e-4	0.93	1.56e-3	0.95
	0.2165	47617	1.59e+2	0.98	1.38e-1	0.98	3.32e+1	0.99	3.25e-3	1.13	2.10e-4	0.95	1.38e-3	0.96
	0.1925	67555	1.41e+2	0.99	1.23e-1	0.99	2.96e+1	0.99	2.87e-3	1.07	1.87e-4	0.96	1.23e-3	0.97
	0.1732	92401	1.27e+2	0.99	1.11e-1	0.99	2.66e+1	0.99	2.55e-3	1.12	1.69e-4	0.97	1.11e-3	0.98

Table 3.7: Example 3: History of convergence for the Galerkin scheme (3.89) with $r = 3$ and $s = 3/2$.

We find it important to remark at this point that, while the theoretical hypotheses on the coefficients and data of the model are usually not verified in practice, the numerical results reported in this section do not seem to be affected at all by the eventual lack of verification of them. Nevertheless, in order to illustrate the feasibility of these assumptions, we now prove, in particular, that, at least for Example 3, the Lipschitz-continuity of both $\vartheta(\boldsymbol{\tau})$ and $\widetilde{\vartheta}(\boldsymbol{\tau}) := \vartheta(\boldsymbol{\tau})^{-1}$ do hold. In fact, given $\boldsymbol{\zeta}$, $\boldsymbol{\tau} \in \mathbb{R} := \mathrm{R}^{n\times n}$, we obtain

$$\big|\vartheta(\boldsymbol{\zeta}) - \vartheta(\boldsymbol{\tau})\big| \,=\, \frac{1}{2}\left|\left\{\frac{1}{(1+|\boldsymbol{\zeta}|^2)^{1/2}} - \frac{1}{(1+|\boldsymbol{\tau}|^2)^{1/2}}\right\}\mathbb{I}\right| \,=\, \frac{1}{2}\left|\frac{(1+|\boldsymbol{\tau}|^2)^{1/2} - (1+|\boldsymbol{\zeta}|^2)^{1/2}}{(1+|\boldsymbol{\zeta}|^2)^{1/2}\,(1+|\boldsymbol{\tau}|^2)^{1/2}}\right|,$$

so that, defining $f(x) := (1+x)^{1/2}$ for all $x > 0$, and noting that $|f'(x)| \le \frac{1}{2}$ for all $x > 0$, which yields $|f(x) - f(y)| \,\le\, \frac{1}{2}\,|x - y|$ for all x, $y > 0$, it follows that

$$\big|\vartheta(\boldsymbol{\zeta}) - \vartheta(\boldsymbol{\tau})\big| \,\le\, \frac{1}{4}\left|\frac{|\boldsymbol{\zeta}|^2 - |\boldsymbol{\tau}|^2}{(1+|\boldsymbol{\zeta}|^2)^{1/2}\,(1+|\boldsymbol{\tau}|^2)^{1/2}}\right| \,=\, \frac{1}{4}\left|\frac{\big(|\boldsymbol{\zeta}| + |\boldsymbol{\tau}|\big)\big(|\boldsymbol{\zeta}| - |\boldsymbol{\tau}|\big)}{(1+|\boldsymbol{\zeta}|^2)^{1/2}\,(1+|\boldsymbol{\tau}|^2)^{1/2}}\right| \,\le\, \frac{1}{2}\,|\boldsymbol{\zeta} - \boldsymbol{\tau}|\,,$$

which proves the second inequality in (3.4) with $L_\vartheta \,=\, \frac{1}{2}$. In turn, performing some algebraic manipulations, we find that

$$\begin{aligned}\big|\widetilde{\vartheta}(\boldsymbol{\zeta}) - \widetilde{\vartheta}(\boldsymbol{\tau})\big| \,&=\, 2\left|\left\{\frac{(1+|\boldsymbol{\zeta}|^2)^{1/2}}{\big(1+(1+|\boldsymbol{\zeta}|^2)^{1/2}\big)} - \frac{(1+|\boldsymbol{\tau}|^2)^{1/2}}{\big(1+(1+|\boldsymbol{\tau}|^2)^{1/2}\big)}\right\}\mathbb{I}\right| \\ &=\, 2\left|\frac{(1+|\boldsymbol{\zeta}|^2)^{1/2} - (1+|\boldsymbol{\tau}|^2)^{1/2}}{\big(1+(1+|\boldsymbol{\zeta}|^2)^{1/2}\big)\big(1+(1+|\boldsymbol{\tau}|^2)^{1/2}\big)}\right|,\end{aligned}$$

from which, employing again the above bounds for f, we deduce the corresponding estimate in (3.6) with $L_{\widetilde{\vartheta}} = 4$. Furthermore, regarding ϑ in Example 1, it is easy to see that

$$\big|\vartheta(\boldsymbol{\zeta}) - \vartheta(\boldsymbol{\tau})\big| \,=\, \frac{1}{10}\,\big|\boldsymbol{\zeta}^2 - \boldsymbol{\tau}^2\big| \,=\, \frac{1}{10}\,\big|\boldsymbol{\zeta}(\boldsymbol{\zeta} - \boldsymbol{\tau}) + (\boldsymbol{\zeta} - \boldsymbol{\tau})\boldsymbol{\tau}\big| \,\le\, \frac{1}{10}\,\big(|\boldsymbol{\zeta}| + |\boldsymbol{\tau}|\big)\,|\boldsymbol{\zeta} - \boldsymbol{\tau}|\,,$$

which yields the Lipschitz-continuity of ϑ for $\boldsymbol{\zeta}$, $\boldsymbol{\tau}$ bounded, say $|\boldsymbol{\zeta}|$, $|\boldsymbol{\tau}| \le M$, whence $L_\vartheta = \frac{M}{5}$.

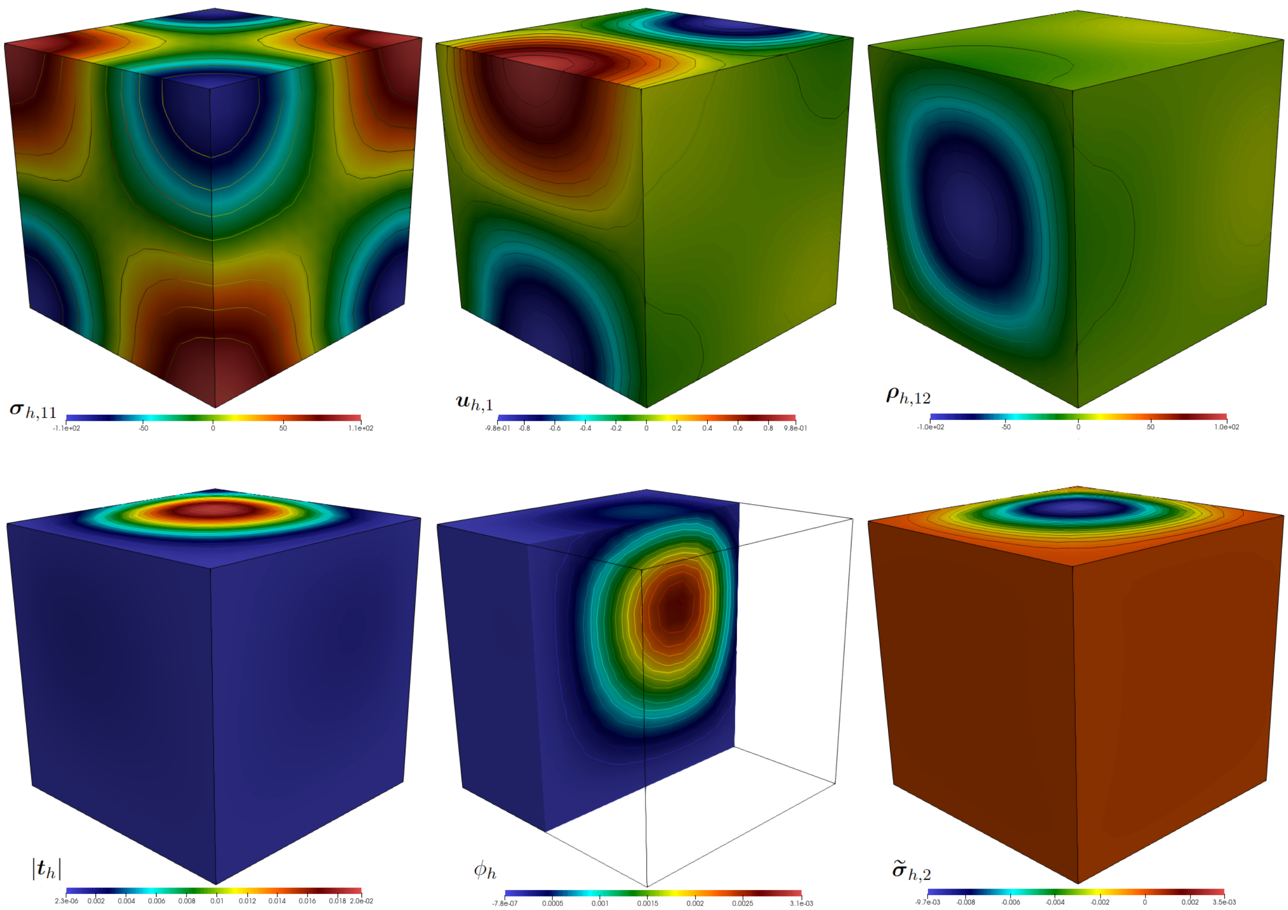

Figure 3.2: Example 3: Some components and magnitudes of the solution of the second approach (3.89) with $k = 0$, $\lambda = 1666.44$, and $\mu = 0.3334$.

3.6.4 Example 4: Convergence in a 2D domain with no manufactured solution

In the present example, we investigate the behavior of the model in a two-dimensional domain without a manufactured solution. The domain under consideration Ω is defined as $\Omega := [0,1] \times [0,1]$, with boundaries $\Gamma = \Gamma_{\text{top}} \cup \Gamma_{\text{left}} \cup \Gamma_{\text{right}} \cup \Gamma_{\text{bottom}}$. For this example, we set the Young's modulus and Poisson's ratio as $E = 1$ and $\nu = 0.4999$, respectively. This choice yields $\lambda = 1666.44$ and $\mu = 0.3334$ according to (3.138). The diffusive source follows Example 1, and the body load and tensorial diffusivity are given by

$$\mathbf{f}(\phi) := \begin{pmatrix} 0.1\,\phi \\ 0.1\,\phi(\phi-1) \end{pmatrix} \quad \text{and} \quad \vartheta(\boldsymbol{\sigma}) = \alpha\mathbf{I} + \alpha^2\boldsymbol{\sigma} + \alpha^3\boldsymbol{\sigma}^2,$$

respectively, with $\alpha = 0.005$. The boundary conditions are given by

$$\mathbf{u}_D := \begin{cases} (0,0) & \text{on} \quad \Gamma_{\text{top}} \cup \Gamma_{\text{bottom}}, \\ (-0.5y(y-1), 0) & \text{on} \quad \Gamma_{\text{left}}, \\ (0.5y(y-1), 0) & \text{on} \quad \Gamma_{\text{right}}, \end{cases}$$

k	h	DOF	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\rho}$)	r($\boldsymbol{\rho}$)	e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	e(ϕ)	r(ϕ)
	0.0471	13680	1.96e+0	−−	4.15e-2	−−	8.44e-1	−−	3.38e-2	−−	6.55e-1	−−
	0.0393	19652	1.63e+0	0.99	3.33e-2	1.20	7.01e-1	1.01	2.79e-2	1.05	5.49e-1	0.97
0	0.0337	26712	1.39e+0	1.05	2.83e-2	1.06	6.02e-1	0.99	2.38e-2	1.02	4.66e-1	1.07
	0.0294	34848	1.21e+0	1.01	2.44e-2	1.10	5.27e-1	1.00	2.08e-2	1.01	4.07e-1	1.01
	0.0262	44064	1.09e+0	0.92	2.16e-2	1.05	4.69e-1	9.78	1.86e-2	0.95	3.66e-1	0.90
	0.0236	54360	9.80e-1	1.00	1.94e-2	1.02	4.22e-1	1.01	1.68e-2	0.97	3.29e-1	1.01
	0.0471	13680	2.22e+0	−−	4.92e-2	−−	9.09e-1	−−	3.31e-2	−−	7.56e-1	−−
	0.0393	19652	1.86e+0	0.98	3.95e-2	1.20	7.55e-1	1.01	2.73e-2	1.06	6.36e-1	0.95
0	0.0337	26712	1.59e+0	1.07	3.40e-2	0.99	6.48e-1	0.98	2.33e-2	1.02	5.38e-1	1.08
	0.0294	34848	1.38e+0	1.01	2.96e-2	1.00	5.65e-1	1.00	2.04e-2	1.01	4.71e-1	1.00
	0.0262	44064	1.24e+0	0.91	2.61e-2	0.98	5.06e-1	0.98	1.83e-2	0.93	4.23e-1	0.90
	0.0236	54360	1.11e+0	1.04	2.34e-2	1.00	4.55e-1	1.00	1.65e-2	0.98	3.79e-1	1.06

Table 3.8: Example 4: History of convergence for the Galerkin scheme (3.88) with $r = 3$ (first half), and $r = 4$ (second half).

and $\phi_D := \cos(\pi x)$ on Γ. Table 3.8 provides a summary of the convergence history. We consider the finite element spaces introduced in Section 3.5.1 with $k = 0$ and solve the nonlinear problem, which typically requires approximately six iterations. As the analytical solution is unknown, we construct the convergence history by considering a solution obtained with 581,664 triangle elements as the exact solution on a sequence of uniform triangulations. It is observed that the method achieves optimal convergence with an order of $\mathcal{O}(h)$.

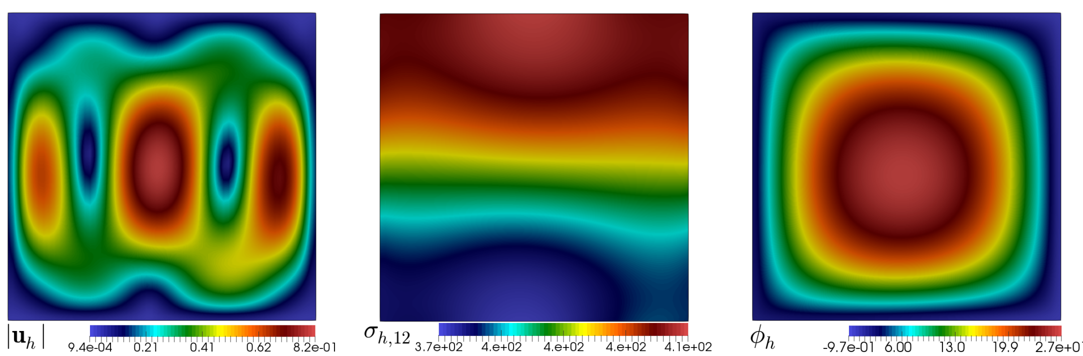

Figure 3.3: Example 4: Some components and magnitudes of the solution of the second approach (3.88) with $k = 0$, $\lambda = 1666.44$, and $\mu = 0.3334$.

Concluding remarks

In this chapter we have continued advancing in the direction of Chapter 2 by introducing and analysing two new Banach spaces-based fully-mixed finite element methods for the numerical solution of pseudostress-assisted diffusion problems. As compared with the mixed-primal method from Chapter 2, the main advantages of the schemes proposed here, which actually arise from the use of two different mixed approaches for the diffusion equation, are given by the fact that some additional variables of physical interest, such as the diffusive flux and the concentration gradient, are approximated directly.

In this way, and differently from what one would do to obtain approximations of those variables starting from the numerical solutions provided by the method from Chapter 2, no numerical differentiation, with the consequent loss of accuracy, is employed in the present case. In turn, regarding the number of degrees of freedom involved in each method, we now recall that for the one proposed in Chapter 2 there holds

$$\mathtt{DOF} \,=\, (3k+2)N_e \,+\, \Big\{(k+1)\,(3k+2) + \frac{1}{2}\,k\,(k-1)\Big\}N_t \,+\, N_v \,+\, 1\,, \tag{3.139}$$

where N_v denotes the number of vertices of $\mathcal{T}_h$. Thus, the difference between the degrees of freedom employed by (3.88) and those utilized by the mixed-primal scheme from Chapter 2 is obtained by subtracting (3.139) from (3.134), which gives

$$\mathtt{DIF}(k) \,:=\, N_e \,+\, (k^2+3k+1)\,N_t \,-\, N_v\,.$$

In Table 3.9 below we illustrate the behavior of $\mathtt{DIF}(k)$ through the sequence of quasi-uniform meshes from Example 1, with $k \in \{0,1\}$. Needless to say, the fact that (3.88), and hence (3.89) as well, requires more degrees of freedom than the method from Chapter 2, is largely compensated by the aforementioned advantages referring to the computation of further unknowns of interest. Furthermore,

h	N_e	N_t	N_v	$\mathtt{DIF}(0)$	$\mathtt{DIF}(1)$
0.0471	2760	1800	961	3599	10799
0.0393	3960	2592	1369	5183	15551
0.0337	5376	3528	1849	7055	21167
0.0295	7008	4608	2401	9215	27647
0.0262	8856	5832	3025	11663	34991

Table 3.9: $\mathtt{DIF}(k)$, $k \in \{0,1\}$, for the sequence of meshes of Example 1.

regarding a comparison between the two fully-mixed finite element methods developed here, we first notice from the respective theoretical results, which are confirmed by the reported numerical results, that, under assumed regularities of the exact solution, they provide the same rates of convergence, and hence this aspect does not yield a valid reason for choosing one or the other. However, we also observe from the tables that in order to attain a given accuracy, the second method requires a bit higher number of degrees of freedom, which is explained by the fact that the latter incorporates one more unknown than the first one, namely $\mathbf{t}_h \in \mathbf{H}_h^{\mathbf{t}}$. Indeed, as observed from (3.134) and (3.135), the difference between the number of unknowns of (3.89) and (3.88) is given by

$$(k+1)(k+2)N_t\,,$$

which, as already noticed at the beginning of this section, constitutes the number of degrees of freedom defining the subspace $\mathbf{H}_h^{\mathbf{t}}$. A minor aspect, though not that relevant, is that the tensorial diffusivity function does not need to be inverted in the second approach. Therefore, both methods are fully comparable, and deciding which one to employ for practical computations will depend on whether, besides the diffusive flux, the user is interested or not in obtaining also direct approximations of the concentration gradient.

CHAPTER 4

A Banach spaces-based fully-mixed finite element method for the stationary chemotaxis-Navier-Stokes problem

4.1 Introduction

Chemotaxis refers to the active and directed movement of cells triggered by a chemical stimulus in their surrounding microenvironment. From the development of multicellular organisms, to blood vessel formation, to immune system function, to cancer growth and metastasis, chemotaxis plays an essential role in many different biological processes [75]. The study of this phenomenon has particularly allowed valuable insights for basic research, drug discovery to decrease or inhibit certain infectious diseases and has ignited much hope for new prognostic tools and therapeutic interventions in oncology [59, 76]. From the mathematical point of view, the well-known Keller-Segel system and their variations [6, 64] are the simplest models for describing this phenomenon, which only relate the cell density and the concentration of the chemical signal, neglecting any interplay with further components. However, in many contexts, cell migration may influence the motion of a surrounding fluid through buoyant forces due to difference in densities, and vice versa the fluid-driven transport of cells and signal may substantially affect the overall behavior [37, 83]. In this regard, and for understanding the chemotaxis systems interaction with liquid environments, several models have been studied (see, e.g. [13, 62, 65, 80, 85, 86] and the references therein), which couple the Keller-Segel equations to a Navier-Stokes system. These works include models describing chemo-repulsion, chemo-attraction, the presence of either a signal production mechanism or a singular sensitivity, double-chemotaxis, among others. In particular, theoretical results on existence and uniqueness of solutions to the unsteady chemotaxis-Navier-Stokes system when the chemical signal is consumed by the organisms, case we focus on this work, are found in [61, 84, 85]. On the other hand, some results yielding existence of stationary classical solutions to a chemotaxis-consumption model with realistic boundary conditions have been established in [15].

Regarding the numerical solvability, a wide variety of techniques have been constructed so far to simulate the chemotaxis-fluid interaction [26, 36, 38, 66, 68]. These references include a combined finite volume-nonconforming finite element method [68], a high-resolution vorticity-based hybrid finite-volume finite-difference discretization [26], a splitting-type Navier-Stokes solver for a realistic three-dimensional setting [66] and an upwind finite element technique in two dimensions [36]. Other numerical techniques for close models can be found in the references of the aforementioned works. In turn,

[10], [38], [72], and [79] constitute, up to our knowledge, the few works available in the literature in which finite element methods for approximating the solutions of the full chemotaxis–Navier–Stokes system are proposed and analysed, including corresponding optimal errors estimates. In particular, an equivalent model in Hilbert spaces is proposed in [38] by using a splitting technique based on the introduction of the chemical concentration gradient as an extra unknown, allowing to control the strong regularity required by the model, which is one of the main difficulties appearing throughout the respective numerical analysis. Additionaly, a time-discrete scheme along with a finite element method with mass-lumping is introduced in [72] for solving a chemotaxis model describing tumor invasion. Furthermore, a new chemotaxis-Navier-Stokes system is derived in [79], and a linear, decoupled and unconditionally energy stable finite element scheme for its numerical solution is proposed there.

On the other hand, it is well-known that when dealing with problems involving couplings and nonlinearities, the introduction of additional variables, that is the use of mixed methods, yields the corresponding variational settings to be properly posed in terms of Banach spaces. This has become particularly frequent in recent years for a wide family of models (see, e.g. [11, 21, 25, 29, 33, 52, 55] and the references therein), whose resulting mixed formulations show mainly saddle-point, twofold saddle-point, or perturbed saddle-point structures. One of the advantages of keeping this functional framework, in addition to avoiding the incorporation of further redundant Galerkin-type penalty terms, as it has been usual, for instance, for diverse augmented schemes, lies on the fact that the sought variables belong to the natural Banach spaces that are originated after carrying out the respective testing and integration by parts procedures. Furthermore, the above not only allows to develop numerical schemes that are momentum conservative but also to compute additional physically relevant variables that might be introduced into the formulation or by employing postprocessing formulae in terms of the discrete solution. Nevertheless, no mixed methods with these features seem to be available in the literature so far to solve the chemotaxis-Navier-Stokes model, which certainly constitutes a gap in the field.

According to the previous discussion, and in order, on one hand, to fill the aforementioned gap, and on the other hand, to continue extending the applicability of Banach spaces-based approaches to study the continuous and discrete well-posedness of nonlinear coupled problems in fluid mechanics, our present purpose is to introduce and analyse a continuous Banach framework yielding a fully-mixed finite element method for the stationary Chemotaxis-Navier-Stokes model. Unfortunately, we must warn that the non-negativity of the cell density and chemical concentration, which, up to our knowledge, seems to be an open question for stationary chemotaxis problems, is not considered here. We believe, however, that a suitable time-discrete scheme ensuring that property for the non-stationary version of our model, should be obtained by combining the mass-lumping approach proposed in [72] with the mixed finite element method to be introduced and analysed in the present work. We plan to address this issue in a forthcoming study.

The chapter is organized as follows. The rest of this section first collects some preliminary notations, definitions, and results to be utilized throughout the chapter, and then describes the model of interest. In particular, the auxiliary unknowns are introduced here. In Section 4.2 we derive the fully-mixed variational formulation of the problem by splitting the analysis according to the three equations forming the coupled model. Suitable integration by parts formulae jointly with the Cauchy-Schwarz and Hölder inequalities are crucial for determining the right Lebesgue and related spaces to which the unknowns and corresponding test functions are required to belong. Some remarks on the boundary conditions are also provided here. In Section 4.3, a fixed-point strategy is adopted to analyse the solvability of the continuous formulation. The Babuška-Brezzi theory in Banach spaces is employed to study the corresponding uncoupled problems, and then the classical Banach theorem is applied to conclude the existence of a unique solution. An analogue fixed-point approach to that of Section 4.3 is utilized in Section 4.4 to study the well-posedness of the associated Galerkin scheme. Under suitable stability conditions on the finite element subspaces employed, existence and uniqueness of solution are proved by applying the Brouwer and Banach theorems along with the discrete Babuška-Brezzi theory. Specific finite element subspaces satisfying those assumptions are then introduced in Section 4.5, and the rates of convergence of the resulting discrete scheme are also established there. Several numerical examples confirming these theoretical findings and illustrating the good performance of the method as well as the local conservation of momentum in an approximate sense, are presented in Section 4.6. Finally, further properties of the Raviart-Thomas interpolator to be used in Section 4.5, are proved in Section 4.7.

4.1.1 The model problem

The stationary chemotaxis-Navier-Stokes problem consists of finding the velocity vector field $\mathbf{u}$ and the pressure scalar field p of an incompressible fluid occupying the region Ω, along with the additional scalar fields given by the cell density η, and the chemical signal concentration φ, satisfying the following system of coupled partial differential equations:

$$\begin{aligned} -\nu\,\Delta\mathbf{u} + \lambda\,(\nabla\mathbf{u})\,\mathbf{u} + \nabla p - \eta\,\nabla f &= \mathbf{f} && \text{in} \quad \Omega\,,\\ \mathbf{div}(\mathbf{u}) &= 0 && \text{in} \quad \Omega\,,\\ \textstyle\int_\Omega p &= 0\,, &&\\ -k_\eta\Delta\eta + \mu\,\mathbf{div}\big(\eta\nabla\varphi\big) + \mathbf{u}\cdot\nabla\eta &= f_\eta && \text{in} \quad \Omega\,,\\ -k_\varphi\Delta\varphi + \gamma\eta\varphi + \mathbf{u}\cdot\nabla\varphi &= f_\varphi && \text{in} \quad \Omega\,,\\ \mathbf{u} = \mathbf{u}_D\,, \quad \eta = \eta_D \quad \text{and} \quad \varphi &= \varphi_D && \text{on} \quad \Gamma\,, \end{aligned} \tag{4.1}$$

where f, $\mathbf{f}$, f_η, and f_φ are given functions belonging to suitable spaces to be indicated later on, whereas ν, λ, κ_η, κ_φ, μ, and γ are positive constants representing the fluid viscosity, the fluid density, the cell diffusion constant, the chemical diffusion constant, the chemotactic coefficient, and the consumption rate of the chemical signal, respectively. In turn, $\mathbf{u}_D$, η_D, and φ_D are corresponding Dirichlet data belonging to suitable spaces as well to be specified throughout the analysis. Meanwhile, we observe here that, due to the incompressibility of the fluid (cf. second equation of (4.1)), $\mathbf{u}_D$ must satisfy the compatibility condition

$$\int_\Gamma \mathbf{u}_D\cdot\mathbf{n} \,=\, 0\,. \tag{4.2}$$

In addition, we stress that while the external sources f_η and f_φ are usually null in most applications, we allow them here to be arbitrary in order to facilitate the construction of manufactured solutions when reporting numerical examples later on in Section 4.6.

Next, in order to derive a fully-mixed formulation of (4.1) in Section 4.2, we first adopt the approach from [33] (see also [29]) and introduce the velocity gradient and the Bernoulli-type stress tensor as further unknowns, that is

$$\mathbf{t} := \nabla \mathbf{u} \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{\sigma} := \nu\, \mathbf{t} - \frac{\lambda}{2}(\mathbf{u}\otimes\mathbf{u}) - p\,\mathbb{I} \quad \text{in} \quad \Omega\,, \tag{4.3}$$

so that the second equation of (4.3) is considered from now on as the constitutive law of the fluid. Then, noting that $\mathbf{div}(\mathbf{u}\otimes\mathbf{u}) = (\nabla\mathbf{u})\mathbf{u} = \mathbf{t}\mathbf{u}$, which follows from the fact that $\mathbf{div}(\mathbf{u}) = 0$, we find that the first equation of (4.1) can be rewritten as

$$-\mathbf{div}(\boldsymbol{\sigma}) + \frac{\lambda}{2}\mathbf{t}\mathbf{u} - \eta\,\nabla f = \mathbf{f} \quad \text{in} \quad \Omega\,.$$

In turn, it is straightforward to prove, taking matrix trace and the deviatoric part of the aforementioned constitutive equation, that the latter and the incompressibility condition, which can also be stated as the identity $\mathrm{tr}\,(\mathbf{t}) = 0$, are equivalent to

$$\boldsymbol{\sigma}^{\mathrm{d}} = \nu\,\mathbf{t} - \frac{\lambda}{2}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega \quad \text{and} \quad p = -\frac{1}{n}\mathrm{tr}\Big(\boldsymbol{\sigma} + \frac{\lambda}{2}(\mathbf{u}\otimes\mathbf{u})\Big) \quad \text{in} \quad \Omega\,, \tag{4.4}$$

and thus the pressure can be eliminated from the system and computed afterwards in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$ as indicated in the foregoing equation. As a consequence, the third equation of (4.1), which constitutes a uniqueness condition for p, is rewritten as

$$\int_\Omega \mathrm{tr}\Big(\boldsymbol{\sigma} + \frac{\lambda}{2}(\mathbf{u}\otimes\mathbf{u})\Big) = 0\,.$$

On the other hand, for the cell density and chemical signal concentration equations, we proceed similarly and define the auxiliary unknowns

$$\widetilde{\boldsymbol{\sigma}} := \nabla\eta - \kappa_\eta^{-1}\,\mu\,\eta\,\nabla\varphi - \kappa_\eta^{-1}\,\eta\,\mathbf{u} \quad \text{in} \quad \Omega \qquad \text{and} \qquad \mathbf{p} := \nabla\varphi \quad \text{in} \quad \Omega\,, \tag{4.5}$$

and observe that the fourth and fifth equations of (4.1) become, respectively,

$$\mathbf{div}(\widetilde{\boldsymbol{\sigma}}) = -\kappa_\eta^{-1}\, f_\eta \quad \text{in} \quad \Omega\,,$$

and

$$\mathbf{div}(\mathbf{p}) - \kappa_\varphi^{-1}\,\gamma\,\eta\varphi - \kappa_\varphi^{-1}\,\mathbf{u}\cdot\mathbf{p} = -\kappa_\varphi^{-1}\, f_\varphi \quad \text{in} \quad \Omega\,.$$

Note that $\widetilde{\boldsymbol{\sigma}}$ can be seen as the pseudostress associated with the cell density equation. Summarizing, (4.1) can be equivalently reformulated as: Find $\mathbf{u}, \mathbf{t}, \boldsymbol{\sigma}, \widetilde{\boldsymbol{\sigma}}, \eta, \mathbf{p}$ and φ in proper spaces to be introduced below, such that

$$
\begin{aligned}
\mathbf{t} \;&=\; \nabla \mathbf{u} && \text{in} \quad \Omega\,, \\
-\boldsymbol{\sigma}^{\mathrm{d}} + \nu\,\mathbf{t} - \frac{\lambda}{2}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \;&=\; 0 && \text{in} \quad \Omega\,, \\
-\mathbf{div}(\boldsymbol{\sigma}) + \frac{\lambda}{2}\,\mathbf{t}\,\mathbf{u} \;&=\; \eta\,\nabla f + \mathbf{f} && \text{in} \quad \Omega\,, \\
\int_\Omega \operatorname{tr}\Big(\boldsymbol{\sigma} + \frac{\lambda}{2}(\mathbf{u}\otimes\mathbf{u})\Big) \;&=\; 0\,, && \\
\widetilde{\boldsymbol{\sigma}} - \nabla\eta + \kappa_\eta^{-1}\,\mu\,\eta\,\mathbf{p} + \kappa_\eta^{-1}\,\eta\,\mathbf{u} \;&=\; 0 && \text{in} \quad \Omega\,, \\
\mathbf{div}(\widetilde{\boldsymbol{\sigma}}) \;&=\; -\kappa_\eta^{-1}\,f_\eta && \text{in} \quad \Omega\,, \\
\mathbf{p} \;&=\; \nabla\varphi && \text{in} \quad \Omega\,, \\
\mathbf{div}(\mathbf{p}) - k_\varphi^{-1}\,\gamma\,\eta\,\varphi - k_\varphi^{-1}\,\mathbf{u}\cdot\mathbf{p} \;&=\; -\kappa_\varphi^{-1}\,f_\varphi && \text{in} \quad \Omega\,, \\
\mathbf{u} = \mathbf{u}_D, \quad \eta = \eta_D \quad \text{and} \quad \varphi \;&=\; \varphi_D && \text{on} \quad \Gamma\,.
\end{aligned}
\tag{4.6}
$$

4.2 The fully-mixed formulation

In this section we derive a Banach spaces-based fully-mixed formulation of (4.6). The integration by parts formulae provided by (9) - (11), along with the Cauchy-Schwarz and Hölder inequalities, play a key role in this derivation. The corresponding analysis is split in the following three subsections, which correspond to the Navier-Stokes equations (first to fourth rows of (4.6)), the cell density equations (fifth and sixth rows of (4.6)), and the chemical signal concentration equations (seventh and eighth rows of (4.6)), respectively.

4.2.1 The Navier-Stokes equations

We begin by seeking originally $\mathbf{u} \in \mathbf{H}^1(\Omega)$, which requires to assume that $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$. Then, a straightforward application of (10) with $t \in \begin{cases} (1,+\infty) & \text{if } n = 2 \\ [6/5,+\infty) & \text{if } n = 3 \end{cases}$ and $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$, gives

$$
\int_\Omega \boldsymbol{\tau} : \nabla\mathbf{u} \;=\; -\int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) + \langle \boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_D\rangle\,,
$$

and hence the corresponding testing of the first equation of (4.6) becomes

$$
\int_\Omega \boldsymbol{\tau} : \mathbf{t} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \;=\; \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,. \tag{4.7}
$$

It is clear, thanks to Cauchy-Schwarz's inequality, that the first term of (4.7) makes sense for $\mathbf{t} \in \mathbb{L}^2(\Omega)$, so that according to its free trace property, we look for this unknown in the space

$$
\mathbb{L}^2_{\mathtt{tr}}(\Omega) \;:=\; \Big\{ \mathbf{s} \in \mathbb{L}^2(\Omega) : \quad \operatorname{tr}(\mathbf{s}) \;=\; 0 \Big\}\,. \tag{4.8}
$$

In addition, knowing that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)$, and using Hölder's inequality, we deduce from the second term of (4.7) that, instead of $\mathbf{H}^1(\Omega)$, it would suffice to look for $\mathbf{u}$ in $\mathbf{L}^{t'}(\Omega)$, where t' is the conjugate of t. Nevertheless, testing the second equation of (4.6) against tensors in $\mathbb{L}^2_{\mathtt{tr}}(\Omega)$, we formally get

$$-\int_\Omega \boldsymbol{\sigma} : \mathbf{s} + \nu \int_\Omega \mathbf{t} : \mathbf{s} - \frac{\lambda}{2} \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \mathbf{s} = 0 \qquad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathtt{tr}}(\Omega)\,, \tag{4.9}$$

from which, employing the Cauchy-Schwarz and Hölder inequalities, we deduce that its third term makes sense for $\mathbf{u} \in \mathbf{L}^4(\Omega)$, and hence from now we chose $t' = 4$, which yields $t = 4/3$. Needless to say, the first term in (4.9) is finite if $\boldsymbol{\sigma} \in \mathbb{L}^2(\Omega)$, and thus, aiming to use the same space for this unknown and its test functions $\boldsymbol{\tau}$, we seek $\boldsymbol{\sigma}$ in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ as well. In this way, knowing now that $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^{4/3}(\Omega)$, we test the third equation of (4.6) against the vector functions in $\mathbf{L}^4(\Omega)$, which yields

$$-\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) + \frac{\lambda}{2} \int_\Omega \mathbf{t}\mathbf{u} \cdot \mathbf{v} = \int_\Omega \eta \nabla f \cdot \mathbf{v} + \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,. \tag{4.10}$$

Note here, thanks again to the aforementioned inequalities and the already established spaces for $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$, that the first, second, and fourth terms of (4.10) are well-defined, the latter if the datum $\mathbf{f}$ belongs to $\mathbf{L}^{4/3}(\Omega)$, which is assumed from now on. Regarding the third one, which will depend on where to look for η, and where to assume the datum f, we will refer to it in Section 4.2.2. We now consider the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) = \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \oplus \mathrm{R}\,\mathbb{I}\,, \tag{4.11}$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}) = 0 \right\}, \tag{4.12}$$

and observe, in particular, that the unknown $\boldsymbol{\sigma}$ can be uniquely decomposed, according to (4.11) and the mean value condition given by the fourth equation of (4.6), as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c_0\mathbb{I}$, where

$$0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \qquad \text{and} \qquad c_0 := \frac{1}{n\,|\Omega|} \int_\Omega \mathrm{tr}\,(\boldsymbol{\sigma}) = -\frac{\lambda}{2\,n\,|\Omega|} \int_\Omega \mathrm{tr}(\mathbf{u} \otimes \mathbf{u})\,. \tag{4.13}$$

In this way, similarly as for the pressure, the constant c_0 can be computed once the velocity is known, and hence it only remains to obtain 0. In this regard, we notice that (4.9) and (4.10) remain unchanged if $\boldsymbol{\sigma}$ is replaced by 0. In addition, thanks to the fact that $\mathbf{t}$ is sought in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, and using the compatibility condition (4.2), we realize that testing (4.7) against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Consequently, bearing in mind the foregoing discussion, introducing the notations

$$\vec{\mathbf{u}} = (\mathbf{u}, \mathbf{t})\,, \quad \vec{\mathbf{v}} = (\mathbf{v}, \mathbf{s})\,, \quad \vec{\mathbf{w}} = (\mathbf{w}, \mathbf{o}) \in \mathbf{H} := \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \quad \text{and} \quad \mathbf{Q} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,$$

redenoting from now on 0 as simply $\boldsymbol{\sigma} \in \mathbf{Q}$, and suitably gathering (4.7), (4.9), and (4.10), we arrive at the following mixed formulation for the Navier-Stokes equations: Find $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) + \mathbf{c}(\mathbf{u}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) + \mathbf{b}(\vec{\mathbf{v}}, \boldsymbol{\sigma}) &= \mathbf{F}_\eta(\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,, \\ \mathbf{b}(\vec{\mathbf{u}}, \boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,, \end{aligned} \tag{4.14}$$

where, given $\mathbf{z} \in \mathbf{L}^4(\Omega)$, the bilinear forms $\mathbf{a} : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$, $\mathbf{b} : \mathbf{H} \times \mathbf{Q} \to \mathrm{R}$, and $\mathbf{c}(\mathbf{z}; \cdot, \cdot) : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$, are defined as

$$\mathbf{a}(\vec{\mathbf{w}}, \vec{\mathbf{v}}) := \nu \int_\Omega \mathbf{o} : \mathbf{s} \qquad \forall\, \vec{\mathbf{w}},\, \vec{\mathbf{v}} \in \mathbf{H}\,, \tag{4.15}$$

$$\mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\tau}) \,:=\, -\int_\Omega \boldsymbol{\tau} : \mathbf{s} \,-\, \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,(\vec{\mathbf{v}},\boldsymbol{\tau}) \in \mathbf{H}\times\mathbf{Q}\,, \tag{4.16}$$

and

$$\mathbf{c}(\mathbf{z};\vec{\mathbf{w}},\vec{\mathbf{v}}) \,:=\, \frac{\lambda}{2}\left\{\int_\Omega \mathbf{o}\,\mathbf{z}\cdot\mathbf{v} \,-\, \int_\Omega (\mathbf{w}\otimes\mathbf{z}) : \mathbf{s}\right\} \qquad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}} \in \mathbf{H}\,, \tag{4.17}$$

whereas, given χ in the same space where η will be sought, the linear functionals $\mathbf{F}_\chi : \mathbf{H} \to \mathrm{R}$ and $\mathbf{G} : \mathbf{Q} \to \mathrm{R}$ are given by

$$\mathbf{F}_\chi(\vec{\mathbf{v}}) \,:=\, \int_\Omega \chi\,\nabla f\cdot\mathbf{v} \,+\, \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\vec{\mathbf{v}} \in \mathbf{H}\,, \tag{4.18}$$

and

$$\mathbf{G}(\boldsymbol{\tau}) \,:=\, -\,\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma \qquad \forall\,\boldsymbol{\tau} \in \mathbf{Q}\,. \tag{4.19}$$

Next, it is easily seen that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}(\mathbf{z};\cdot,\cdot)$, and $\mathbf{G}$ are bounded. In fact, endowing $\mathbf{H}$ and $\mathbf{Q}$ with the norms

$$\|\vec{\mathbf{v}}\|_{\mathbf{H}} \,:=\, \|\mathbf{v}\|_{0,4;\Omega} \,+\, \|\mathbf{s}\|_{0,\Omega} \quad \forall\,\vec{\mathbf{v}} := (\mathbf{v},\mathbf{s}) \in \mathbf{H}\,, \quad \|\boldsymbol{\tau}\|_{\mathbf{Q}} \,:=\, \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \quad \forall\,\boldsymbol{\tau} \in \mathbf{Q}\,, \tag{4.20}$$

applying the Cauchy-Schwarz and Hölder inequalities, and invoking (10) along with the continuous injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, we find that there exists positive constants, denoted and given as

$$\|\mathbf{a}\| \,:=\, \nu\,, \qquad \|\mathbf{b}\| \,:=\, 1\,, \qquad \|\mathbf{c}\| \,:=\, \frac{\lambda}{2}\,, \qquad \text{and} \quad \|\mathbf{G}\| \,:=\, \big(1 + \|\mathbf{i}_4\|\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\,, \tag{4.21}$$

such that

$$|\mathbf{a}(\vec{\mathbf{w}},\vec{\mathbf{v}})| \,\le\, \|\mathbf{a}\|\,\|\vec{\mathbf{w}}\|_{\mathbf{H}}\,\|\vec{\mathbf{v}}\|_{\mathbf{H}} \qquad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}} \in \mathbf{H}\,, \tag{4.22}$$

$$|\mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\tau})| \,\le\, \|\mathbf{b}\|\,\|\vec{\mathbf{v}}\|_{\mathbf{H}}\,\|\boldsymbol{\tau}\|_{\mathbf{Q}} \qquad \forall\,(\vec{\mathbf{v}},\boldsymbol{\tau}) \in \mathbf{H}\times\mathbf{Q},\,, \tag{4.23}$$

$$|\mathbf{c}(\mathbf{z};\vec{\mathbf{w}},\vec{\mathbf{v}})| \,\le\, \|\mathbf{c}\|\,\|\mathbf{z}\|_{0,4;\Omega}\,\|\vec{\mathbf{w}}\|_{\mathbf{H}}\,\|\vec{\mathbf{v}}\|_{\mathbf{H}} \qquad \forall\,\mathbf{z} \in \mathbf{L}^4(\Omega)\,, \quad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}} \in \mathbf{H}\,, \tag{4.24}$$

and

$$|\mathbf{G}(\boldsymbol{\tau})| \,\le\, \|\mathbf{G}\|\,\|\boldsymbol{\tau}\|_{\mathbf{Q}} \qquad \forall\,\boldsymbol{\tau} \in \mathbf{Q}\,. \tag{4.25}$$

In addition, simple algebraic computations show that

$$\mathbf{c}(\mathbf{z};\vec{\mathbf{v}},\vec{\mathbf{v}}) \,=\, 0 \qquad \forall\,\mathbf{z} \in \mathbf{L}^4(\Omega)\,, \quad \forall\,\vec{\mathbf{v}} \in \mathbf{H}\,. \tag{4.26}$$

Regarding $\mathbf{F}_\chi$ (cf. (4.18)), and as already commented for its first term, we remark that its well-definedness will be concluded below at the end of Section 4.2.2.

4.2.2 The cell density equations

Testing the fifth equation of (4.6) against functions $\widetilde{\boldsymbol{\tau}} \in \mathbf{L}^2(\Omega)$, we formally obtain

$$\int_\Omega \widetilde{\boldsymbol{\sigma}}\cdot\widetilde{\boldsymbol{\tau}} \,-\, \int_\Omega \nabla\eta\cdot\widetilde{\boldsymbol{\tau}} \,+\, \kappa_\eta^{-1}\,\mu\int_\Omega \eta\,\mathbf{p}\cdot\widetilde{\boldsymbol{\tau}} \,+\, \kappa_\eta^{-1}\int_\Omega \eta\,\mathbf{u}\cdot\widetilde{\boldsymbol{\tau}} \,=\, 0\,, \tag{4.27}$$

from which we observe that the first and second terms of (4.27) are finite if $\widetilde{\boldsymbol{\sigma}} \in \mathbf{L}^2(\Omega)$ and $\eta \in \mathrm{H}^1_0(\Omega)$, respectively. In turn, using the Cauchy-Schwarz and Hölder inequalities, we find that for all $l,\, j \in (1,+\infty)$ conjugate to each other, there hold

$$\left| \int_\Omega \eta\, \mathbf{p} \cdot \widetilde{\boldsymbol{\tau}} \right| \,\le\, \|\eta\|_{0,2l;\Omega} \, \|\mathbf{p}\|_{0,2j;\Omega} \, \|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega} \tag{4.28}$$

and

$$\left| \int_\Omega \eta\, \mathbf{u} \cdot \widetilde{\boldsymbol{\tau}} \right| \,\le\, \|\eta\|_{0,2l;\Omega} \, \|\mathbf{u}\|_{0,2j;\Omega} \, \|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}\,, \tag{4.29}$$

from which we deduce that the third and fourth terms of (4.27) make sense for $\eta \in \mathrm{L}^{2l}(\Omega)$, $\mathbf{p} \in \mathbf{L}^{2j}(\Omega)$, and $\mathbf{u} \in \mathbf{L}^{2j}(\Omega)$. However, since we already know from Section 4.2.1 that $\mathbf{u}$ will be sought in $\mathbf{L}^4(\Omega)$, we have to impose here that $2j \le 4$. On the other hand, in order to be able to apply (9) to $\widetilde{\boldsymbol{\tau}}$ and η, so that we obtain

$$\int_\Omega \nabla\eta \cdot \widetilde{\boldsymbol{\tau}} \,=\, -\int_\Omega \eta\, \mathbf{div}(\widetilde{\boldsymbol{\tau}}) \,+\, \langle \widetilde{\boldsymbol{\tau}}\cdot\mathbf{n}, \eta_D\rangle_\Gamma\,, \tag{4.30}$$

with $\widetilde{\boldsymbol{\tau}}\cdot\mathbf{n} \in \mathrm{H}^{-1/2}(\Gamma)$ and $\langle\cdot,\cdot\rangle$ denoting the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, it suffices to assume that $\mathbf{div}(\widetilde{\boldsymbol{\tau}}) \in \mathrm{L}^{(2l)'}(\Omega)$, where $(2l)' := \frac{2l}{2l-1}$ is the conjugate of $2l$, the datum η_D belongs to $\mathrm{H}^{1/2}(\Gamma)$, and $\mathrm{H}^1(\Omega)$ is continuously embedded in $\mathrm{L}^{2l}(\Omega)$. The later is guaranteed for $2l \in [1,+\infty)$ when $n=2$, which is always satisfied, and for $2l \in [1,6]$ when $n=3$ (cf. [41, Corollary B.43]).

On the other hand, in order to utilize later on a result on the $\mathrm{W}^{1,2j}(\Omega)$-solvability of a Poisson equation, which will be required to establish a continuous inf-sup condition, and according to the result detailed in [48, Theorem 3.2] (see also [60, Theorems 1.1 and 1.3]), we need that $4/3 \le 2j \le 4$ when $n=2$, and $3/2 \le 2j \le 3$ when $n=3$. Note that these constraints are compatible with the previous requirement that $2j \le 4$. Now, since the respective lower bounds are already satisfied, we just look at the upper ones, and readily observe that for $n=2$, $2j = \dfrac{2l}{l-1} \le 4$ if and only if $2l \ge 4$, whereas for $n=3$, $2j = \dfrac{2l}{l-1} \le 3$ if and only if $2l \ge 6$. Thus, intersecting the above with the previous restrictions on $2l$, we find that when $n=2$ we require $4 \le 2l$, and when $n=3$ the only possible choice is $2l=6$. Therefore, denoting

$$(r,s) \,:=\, (2j,(2j)')\,, \quad \text{and} \quad (\rho,\varrho) \,:=\, (2l,(2l)')\,,$$

we conclude from the foregoing discussion that the feasible ranges for r, s, ρ, ϱ, j and l, are given by

$$\left\{ \begin{array}{lll} r \in (2,4] \quad \text{and} & s \in [4/3,2) & \text{if } n=2\,, \\[2ex] \;r \,=\, 3 \quad\;\; \text{and} & s \,=\, 3/2 & \text{if } n=3\,, \end{array} \right. \qquad \left\{ \begin{array}{lll} \rho \in [4,+\infty) \quad \text{and} & \varrho \in (1,4/3] & \text{if } n=2\,, \\[2ex] \;\rho \,=\, 6 \qquad\quad \text{and} & \varrho \,=\, 6/5 & \text{if } n=3\,, \end{array} \right. \tag{4.31}$$

and

$$\left\{ \begin{array}{lll} j \in (1,2] \quad \text{and} & l \in [2,+\infty) & \text{if } n=2\,, \\[2ex] j \,=\, 3/2 \quad \text{and} & l \,=\, 3 & \text{if } n=3\,. \end{array} \right. \tag{4.32}$$

Needless to say, once j (or its conjugate l) is chosen according to the indicated range, then r and ρ, and their respective conjugates s and ϱ, are fixed. For instance, taking for $n=2$, $j=l=2$ yields $r=\rho=4$ and $s=\varrho=4/3$.

Hence, in terms of these indexes, we look for $\eta \in \mathrm{L}^\rho(\Omega)$ and $\mathbf{p} \in \mathbf{L}^r(\Omega)$, whereas the test function $\widetilde{\boldsymbol{\tau}} \in \mathbf{L}^2(\Omega)$ is such that $\mathbf{div}(\widetilde{\boldsymbol{\tau}}) \in \mathrm{L}^\varrho(\Omega)$. In this way, replacing the resulting expression from (4.30) into (4.27), and taking into account the definition (3), we arrive at

$$\int_\Omega \widetilde{\boldsymbol{\sigma}} \cdot \widetilde{\boldsymbol{\tau}} + \int_\Omega \eta\, \mathbf{div}(\widetilde{\boldsymbol{\tau}}) \; + \widetilde{c}_{\mathbf{u},\mathbf{p}}(\widetilde{\boldsymbol{\tau}}, \eta) \;=\; \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathbf{div}_\varrho; \Omega)\,, \tag{4.33}$$

where, given $\mathbf{z} \in \mathbf{L}^4(\Omega)$ and $\mathbf{q} \in \mathbf{L}^r(\Omega)$, $\widetilde{c}_{\mathbf{z},\mathbf{q}} : \mathbf{H}(\mathbf{div}_\varrho; \Omega) \times \mathrm{L}^\rho(\Omega) \to \mathrm{R}$ is the bilinear form given by

$$\widetilde{c}_{\mathbf{z},\mathbf{q}}(\widetilde{\boldsymbol{\tau}}, \xi) \;:=\; \kappa_\eta^{-1}\, \mu \int_\Omega \xi\, \mathbf{q} \cdot \widetilde{\boldsymbol{\tau}} + \kappa_\eta^{-1} \int_\Omega \xi\, \mathbf{z} \cdot \widetilde{\boldsymbol{\tau}} \qquad \forall\, (\widetilde{\boldsymbol{\tau}}, \xi) \in \mathbf{H}(\mathbf{div}_\varrho; \Omega) \times \mathrm{L}^\rho(\Omega)\,, \tag{4.34}$$

and $\widetilde{\mathrm{F}} : \mathbf{H}(\mathbf{div}_\varrho; \Omega) \to \mathrm{R}$ is the linear functional defined as

$$\widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) \;:=\; \langle \widetilde{\boldsymbol{\tau}} \cdot \mathbf{n}, \eta_D \rangle_\Gamma \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathbf{div}_\varrho; \Omega)\,. \tag{4.35}$$

In turn, testing now the sixth equation of (4.6) against $\xi \in \mathrm{L}^\rho(\Omega)$, which implicitly impose the unknown $\widetilde{\boldsymbol{\sigma}}$ to live in $\mathbf{H}(\mathbf{div}_\varrho; \Omega)$, and assuming that the datum f_η belongs to $\mathrm{L}^\varrho(\Omega)$, we obtain

$$\int_\Omega \xi\, \mathbf{div}(\widetilde{\boldsymbol{\sigma}}) \;=\; \widetilde{\mathrm{G}}(\xi) \qquad \forall\, \xi \in \mathrm{L}^\rho(\Omega)\,, \tag{4.36}$$

where $\widetilde{\mathrm{G}} : \mathrm{L}^\rho(\Omega) \to \mathrm{R}$ is the functional given by

$$\widetilde{\mathrm{G}}(\xi) \;:=\; -\kappa_\eta^{-1} \int_\Omega f_\eta\, \xi \qquad \forall\, \xi \in \mathrm{L}^\rho(\Omega)\,. \tag{4.37}$$

In this way, given $\mathbf{p} \in \mathbf{L}^r(\Omega)$ and $\mathbf{u} \in \mathbf{L}^4(\Omega)$, and denoting the spaces

$$\mathrm{H} \;:=\; \mathbf{H}(\mathbf{div}_\varrho; \Omega) \quad \text{and} \quad \mathrm{Q} \;:=\; \mathrm{L}^\rho(\Omega)\,, \tag{4.38}$$

the mixed formulation for the cell density equation reduces to: Find $(\widetilde{\boldsymbol{\sigma}}, \eta) \in \mathrm{H} \times \mathrm{Q}$ such that

$$\begin{aligned} \widetilde{a}(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}, \eta) + \widetilde{c}_{\mathbf{u},\mathbf{p}}(\widetilde{\boldsymbol{\tau}}, \eta) &= \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathrm{H}\,, \\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}}, \xi) &= \widetilde{\mathrm{G}}(\xi) \qquad \forall\, \xi \in \mathrm{Q}\,, \end{aligned} \tag{4.39}$$

where $\widetilde{a} : \mathrm{H} \times \mathrm{H} \to \mathrm{R}$ and $\widetilde{b} : \mathrm{H} \times \mathrm{Q} \to \mathrm{R}$ are the bilinear forms defined as

$$\widetilde{a}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}}) \;:=\; \int_\Omega \widetilde{\boldsymbol{\zeta}} \cdot \widetilde{\boldsymbol{\tau}} \qquad \forall\, \widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}} \in \mathrm{H}\,, \tag{4.40}$$

and

$$\widetilde{b}(\widetilde{\boldsymbol{\tau}}, \xi) \;:=\; \int_\Omega \xi\, \mathbf{div}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\, (\widetilde{\boldsymbol{\tau}}, \xi) \in \mathrm{H} \times \mathrm{Q}\,. \tag{4.41}$$

It is easily seen that $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}_{\mathbf{z},\mathbf{q}}$, $\widetilde{\mathrm{F}}$, and $\widetilde{\mathrm{G}}$ are bounded with the corresponding norms given by $\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}} := \|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{div}_\varrho;\Omega}$ for all $\widetilde{\boldsymbol{\tau}} \in \mathrm{H}$, and $\|\xi\|_{\mathrm{Q}} := \|\xi\|_{0,\rho;\Omega}$ for all $\xi \in \mathrm{Q}$. Indeed, applying the Hölder and Cauchy-Schwarz inequalities, invoking the bounds provided by (4.28) and (4.29), along with the fact that $\|\cdot\|_{0,r;\Omega} \,\leq\, |\Omega|^{(4-r)/4r}\, \|\cdot\|_{0,4;\Omega}$ for $\widetilde{c}_{\mathbf{z},\mathbf{q}}$, and proceeding similarly to $\mathbf{G}$ (cf. (4.21), (4.25)) for $\widetilde{\mathrm{F}}$, besides letting $\mathrm{i}_\rho : \mathrm{H}^1(\Omega) \to \mathrm{L}^\rho(\Omega)$ be the respective continuous injection, we deduce the existence of positive constants, denoted and given as

$$\begin{gathered} \|\widetilde{a}\| := 1\,, \qquad \|\widetilde{b}\| := 1\,, \qquad \|\widetilde{c}\| := \kappa_\eta^{-1} \max\left\{\mu, |\Omega|^{(4-r)/4r}\right\}, \\ \|\widetilde{\mathrm{F}}\| := \big(1 + \|\mathrm{i}_\rho\|\big)\, \|\eta_D\|_{1/2,\Gamma}\,, \qquad \text{and} \qquad \|\widetilde{\mathrm{G}}\| := \kappa_\eta^{-1}\, \|f_\eta\|_{0,\varrho;\Omega}\,, \end{gathered} \tag{4.42}$$

such that

$$|\widetilde{a}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}})| \,\le\, \|\widetilde{a}\|\, \|\widetilde{\boldsymbol{\zeta}}\|_{\mathrm{H}}\, \|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}} \qquad \forall\, \widetilde{\boldsymbol{\zeta}},\, \widetilde{\boldsymbol{\tau}} \in \mathrm{H}\,, \tag{4.43}$$

$$|\widetilde{b}(\widetilde{\boldsymbol{\tau}}, \xi)| \,\le\, \|\widetilde{b}\|\, \|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}}\, \|\xi\|_{\mathrm{Q}} \qquad \forall\, (\widetilde{\boldsymbol{\tau}}, \xi) \in \mathrm{H}\times\mathrm{Q}\,, \tag{4.44}$$

$$|\widetilde{c}_{\mathbf{z},\mathbf{q}}(\widetilde{\boldsymbol{\tau}}, \xi)| \,\le\, \|\widetilde{c}\|\, \big(\|\mathbf{z}\|_{0,4;O} + \|\mathbf{q}\|_{0,r;\Omega}\big)\, \|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}}\, \|\xi\|_{\mathrm{Q}} \qquad \forall\, (\widetilde{\boldsymbol{\tau}}, \xi) \in \mathrm{H}\times\mathrm{Q}\,, \tag{4.45}$$

$$|\widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}})| \,\le\, \|\widetilde{\mathrm{F}}\|\, \|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}} \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathrm{H}\,, \tag{4.46}$$

and

$$|\widetilde{\mathrm{G}}(\xi)| \,\le\, \|\widetilde{\mathrm{G}}\|\, \|\xi\|_{\mathrm{Q}} \qquad \forall\, \xi \in \mathrm{Q}\,. \tag{4.47}$$

Finally, knowing that η will be sought in $\mathrm{L}^\rho(\Omega)$, we consider $\chi \in \mathrm{L}^\rho(\Omega)$, proceed similarly to the derivation of (4.28) and (4.29), and use that $\|\cdot\|_{0,\Omega} \,\le\, |\Omega|^{1/4}\, \|\cdot\|_{0,4;\Omega}$, to bound the first term defining $\mathbf{F}_\chi$ (cf. (4.18)) as

$$\left|\int_\Omega \chi\, \nabla f \cdot \mathbf{v}\right| \,\le\, |\Omega|^{1/4}\, \|\chi\|_{0,\rho;\Omega}\, \|\nabla f\|_{0,r;\Omega}\, \|\mathbf{v}\|_{0,4;\Omega} \qquad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,, \tag{4.48}$$

which requires to assume from now on that $\nabla f \in \mathbf{L}^r(\Omega)$. Then, bearing in mind the definition of $\mathbf{F}_\chi$ (cf. (4.18)) and the foregoing estimate, and setting the constant

$$\|\mathbf{F}\| \,:=\, \max\left\{1, |\Omega|^{1/4}\right\}, \tag{4.49}$$

we readily conclude that

$$|\mathbf{F}_\chi(\vec{\mathbf{v}})| \,\le\, \|\mathbf{F}\|\, \big(\|\chi\|_{0,\rho;\Omega}\, \|\nabla f\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega}\big)\, \|\vec{\mathbf{v}}\|_{\mathbf{H}} \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,, \tag{4.50}$$

thus confirming the announced well-definedness and boundedness of $\mathbf{F}_\chi$.

4.2.3 The chemical signal concentration equations

Knowing already that $\mathbf{p} \in \mathbf{L}^r(\Omega)$, the seventh equation of (4.6) suggests to look originally for φ in $\mathrm{W}^{1,r}(\Omega)$. In this way, testing that equation against $\mathbf{q} \in \mathbf{H}^s(\mathrm{div}_s;\Omega)$ (cf. (5)), and then employing (11) and the Dirichlet boundary condition for φ, we obtain

$$\int_\Omega \mathbf{p}\cdot\mathbf{q} + \int_\Omega \varphi\, \mathbf{div}(\mathbf{q}) \,=\, \langle \mathbf{q}\cdot\mathbf{n}, \varphi_D\rangle_\Gamma\,, \tag{4.51}$$

which requires to assume that $\varphi_D \in \mathrm{W}^{1/s,r}(\Gamma)$. It follows from (4.51) that it suffices to seek the concentration φ of the chemical signal in the space $\mathrm{L}^r(\Omega)$. In turn, testing the eighth equation of (4.6) against an arbitrary function ϕ belonging to a space to be determined, we formally get

$$\int_\Omega \phi\, \mathbf{div}(\mathbf{p}) \,-\, \kappa_\varphi^{-1}\, \gamma \int_\Omega \eta\, \varphi\, \phi \,-\, \kappa_\varphi^{-1} \int_\Omega \mathbf{u}\cdot\mathbf{p}\, \phi \,=\, -\kappa_\varphi^{-1} \int_\Omega f_\varphi\, \phi\,. \tag{4.52}$$

Next, given the same l, $j \in (1,+\infty)$ conjugate to each other as before, and proceeding similarly to the derivation of (4.28) and (4.29), we find that

$$\left|\int_\Omega \eta\,\varphi\,\phi\right| \,\le\, \|\eta\|_{0,2j;\Omega}\,\|\varphi\|_{0,2j;\Omega}\,\|\phi\|_{0,l;\Omega} \,=\, \|\eta\|_{0,r;\Omega}\,\|\varphi\|_{0,r;\Omega}\,\|\phi\|_{0,l;\Omega} \tag{4.53}$$

and

$$\left|\int_\Omega \mathbf{u}\cdot\mathbf{p}\,\phi\right| \,\le\, \|\mathbf{u}\|_{0,2j;\Omega}\,\|\mathbf{p}\|_{0,2j;\Omega}\,\|\phi\|_{0,l;\Omega} \,=\, \|\mathbf{u}\|_{0,r;\Omega}\,\|\mathbf{p}\|_{0,r;\Omega}\,\|\phi\|_{0,l;\Omega}\,, \tag{4.54}$$

whence, recalling from (4.31) that $r \,\le\, 4 \,\le\, \rho$, we deduce that the second and third terms of (4.52) make sense for $\eta \in \mathrm{L}^\rho(\Omega)$, $\varphi \in \mathrm{L}^r(\Omega)$, $\phi \in \mathrm{L}^l(\Omega)$, $\mathbf{u} \in \mathbf{L}^4(\Omega)$, and $\mathbf{p} \in \mathbf{L}^r(\Omega)$. In addition, in order for the first and fourth terms to be well-defined, we need that both $\mathbf{div}(\mathbf{p})$ and the datum f_φ belong to $\mathrm{L}^j(\Omega)$, which yields, in particular, to look for $\mathbf{p}$ in $\mathbf{H}^r(\mathrm{div}_j;\Omega)$ (cf. (5)).

According to the foregoing discussion, we now set the Banach spaces

$$X_2 \,:=\, \mathbf{H}^r(\mathrm{div}_j;\Omega)\,,\quad X_1 \,:=\, \mathbf{H}^s(\mathrm{div}_s;\Omega)\,,\quad M_1 \,:=\, \mathrm{L}^r(\Omega)\,,\quad \text{and}\quad M_2 \,:=\, \mathrm{L}^l(\Omega)\,, \tag{4.55}$$

so that, given $\mathbf{u} \in \mathbf{L}^4(\Omega)$ and $\eta \in \mathrm{L}^\rho(\Omega)$, the mixed formulation for the chemical signal concentration equation reduces to: Find $(\mathbf{p},\varphi) \in X_2 \times M_1$ such that

$$\begin{aligned} a(\mathbf{p},\mathbf{q}) \,+\, b_1(\mathbf{q},\varphi) \quad &= \quad \mathrm{F}(\mathbf{q}) \quad \forall\,\mathbf{q} \in X_1\,,\\ b_2(\mathbf{p},\phi) \,-\, c_{\mathbf{u},\eta}\big((\mathbf{p},\varphi),\phi\big) \quad &= \quad \mathrm{G}(\phi) \quad \forall\,\phi \in M_2\,, \end{aligned} \tag{4.56}$$

where, given $\mathbf{z} \in \mathbf{L}^4(\Omega)$ and $\chi \in \mathrm{L}^\rho(\Omega)$, the bilinear forms $a : X_2 \times X_1 \to \mathrm{R}$, $b_i : X_i \times M_i \to \mathrm{R}$, $i \in \{1,2\}$, and $c_{\mathbf{z},\chi} : \big(X_2 \times M_1\big) \times M_2 \to \mathrm{R}$, are defined as

$$a(\mathbf{r},\mathbf{q}) \,:=\, \int_\Omega \mathbf{r}\cdot\mathbf{q} \qquad \forall\,(\mathbf{r},\mathbf{q}) \in X_2 \times X_1\,, \tag{4.57}$$

$$b_i(\mathbf{q},\phi) \,:=\, \int_\Omega \phi\,\mathbf{div}(\mathbf{q}) \qquad \forall\,(\mathbf{q},\phi) \in X_i \times M_i\,, \tag{4.58}$$

and

$$c_{\mathbf{z},\chi}\big((\mathbf{r},\psi),\phi\big) \,:=\, \kappa_\varphi^{-1}\int_\Omega \mathbf{z}\cdot\mathbf{r}\,\phi \,+\, \kappa_\varphi^{-1}\,\gamma\int_\Omega \chi\,\psi\,\phi \qquad \forall\,\big((\mathbf{r},\psi),\phi\big) \in \big(X_2 \times M_1\big) \times M_2\,, \tag{4.59}$$

whereas the linear functionals $\mathrm{F} : X_1 \to \mathrm{R}$ and $\mathrm{G} : M_2 \to \mathrm{R}$ are given by

$$\mathrm{F}(\mathbf{q}) \,:=\, \langle \mathbf{q}\cdot\mathbf{n}, \varphi_D\rangle_\Gamma \qquad \forall\,\mathbf{q} \in X_1\,, \tag{4.60}$$

and

$$\mathrm{G}(\phi) \,:=\, -\kappa_\varphi^{-1}\int_\Omega f_\varphi\,\phi \qquad \forall\,\phi \in M_2\,. \tag{4.61}$$

Next, it is straightforward to see that the bilinear forms a, b_i, $i \in \{1,2\}$, and $c_{\mathbf{z},\chi}$, as well as the functionals F and G, are all bounded. In fact, applying Hölder's inequality, appealing to the bounds given by (4.53) and (4.54), and making use of the fact that $\|\cdot\|_{0,r;\Omega} \,\le\, |\Omega|^{(4-r)/4r}\,\|\cdot\|_{0,4;\Omega}$ and $\|\cdot\|_{0,r;\Omega} \,\le\, |\Omega|^{(\rho-r)/\rho r}\,\|\cdot\|_{0,\rho;\Omega}$ for $c_{\mathbf{z},\chi}$, we find that there exist positive constants, given by

$$\begin{gathered} \|a\| \,:=\, 1\,,\qquad \|b_i\| \,:=\, 1\ \big(i \in \{1,2\}\big)\,,\qquad \|c\| \,:=\, \kappa_\varphi^{-1}\,\max\big\{|\Omega|^{(4-r)/4r}, \gamma\,|\Omega|^{(\rho-r)/\rho r}\big\}\,,\\ \text{and}\qquad \|\mathrm{G}\| \,=\, \kappa_\varphi^{-1}\,\|f_\varphi\|_{0,j;\Omega}\,, \end{gathered} \tag{4.62}$$

such that

$$|a(\mathbf{r},\mathbf{q})| \,\le\, \|a\|\,\|\mathbf{r}\|_{X_2}\,\|\mathbf{q}\|_{M_1} \qquad \forall\,(\mathbf{r},\mathbf{q})\in X_2\times M_1\,, \tag{4.63}$$

$$|b_i(\mathbf{q},\phi)| \,\le\, \|b_i\|\,\|\mathbf{q}\|_{X_i}\,\|\phi\|_{M_i} \qquad \forall\,(\mathbf{q},\phi)\in X_i\times M_i\,, \tag{4.64}$$

$$\begin{gathered}|c_{\mathbf{z},\chi}\big((\mathbf{r},\psi),\phi\big)| \,\le\, \|c\|\,\big(\|\mathbf{z}\|_{0,4;\Omega} \,+\, \|\chi\|_{0,\rho;\Omega}\big)\,\|(\mathbf{r},\psi)\|_{X_2\times M_1}\,\|\phi\|_{M_2}\\ \forall\,\big((\mathbf{r},\psi),\phi\big)\in\big(X_2\times M_1\big)\times M_2\,,\end{gathered} \tag{4.65}$$

and

$$|\mathrm{G}(\phi)| \,\le\, \|\mathrm{G}\|\,\|\phi\|_{M_2} \qquad \forall\,\phi\in M_2\,. \tag{4.66}$$

In turn, for the boundedness F we first observe, thanks to [41, Lemma A.36] and the surjectivity of the trace operator mapping $\mathbf{W}^{1,r}(\Omega)$ onto $\mathbf{W}^{1/s,r}(\Gamma)$, that there exists a fixed constant $C_r>0$ such that for each $\varphi\in\mathbf{W}^{1/s,r}(\Gamma)$ there exists $v\in\mathbf{W}^{1,r}(\Omega)$ satisfying $v|_\Gamma=\varphi$ and

$$\|v\|_{1,r;\Omega} := \|v\|_{0,r;\Omega} + \|\nabla v\|_{0,r;\Omega} \,\le\, C_r\,\|\varphi\|_{1/s,r;\Gamma}\,.$$

In particular, denoting by $v_D\in\mathbf{W}^{1,r}(\Omega)$ a corresponding function for $\varphi_D\in\mathbf{W}^{1/s,r}(\Gamma)$, applying (11) to $(t,t')=(s,r)$ and $(\boldsymbol{\tau},v)=(\mathbf{q},v_D)$, and then using Hölder's inequality, we deduce that

$$|\mathrm{F}(\mathbf{q})| \,\le\, \|\mathrm{F}\|\,\|\mathbf{q}\|_{X_1} \qquad \forall\,\mathbf{q}\in X_1\,, \tag{4.67}$$

with the constant

$$\|\mathrm{F}\| \,:=\, C_r\,\|\varphi_D\|_{1/s,r;\Gamma}\,. \tag{4.68}$$

As a consequence of the analysis developed in Sections 4.2.1 and 4.2.2, and the present Section 4.2.3, and under the assumption that the data belong to the indicated spaces, namely $\nabla f\in\mathbf{L}^r(\Omega)$, $\mathbf{f}\in\mathbf{L}^{4/3}(\Omega)$, $f_\eta\in\mathrm{L}^\varrho(\Omega)$, $f_\varphi\in\mathrm{L}^j(\Omega)$, $\mathbf{u}_D\in\mathbf{H}^{1/2}(\Gamma)$, $\eta_D\in\mathrm{H}^{1/2}(\Gamma)$, and $\varphi_D\in\mathrm{W}^{1/s,r}(\Gamma)$, we conclude that the fully-mixed formulation of the chemotaxis-Navier-Stokes problem (4.6) can be summarized by gathering (4.14), (4.39) and (4.56), that is: Find $(\vec{\mathbf{u}},\boldsymbol{\sigma})\in\mathbf{H}\times\mathbf{Q}$, $(\widetilde{\boldsymbol{\sigma}},\eta)\in\mathrm{H}\times\mathrm{Q}$, and $(\mathbf{p},\varphi)\in X_2\times M_1$, such that

$$\begin{array}{rcll}
\mathbf{a}(\vec{\mathbf{u}},\vec{\mathbf{v}}) \,+\, \mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\vec{\mathbf{v}}) \,+\, \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\sigma}) & = & \mathbf{F}_\eta(\vec{\mathbf{v}}) & \forall\,\vec{\mathbf{v}}\in\mathbf{H}\,,\\
\mathbf{b}(\vec{\mathbf{u}},\boldsymbol{\tau}) & = & \mathbf{G}(\boldsymbol{\tau}) & \forall\,\boldsymbol{\tau}\in\mathbf{Q}\,,\\
\widetilde{a}(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\tau}},\eta) \,+\, \widetilde{c}_{\mathbf{u},\mathbf{p}}(\widetilde{\boldsymbol{\tau}},\eta) & = & \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) & \forall\,\widetilde{\boldsymbol{\tau}}\in\mathrm{H}\,,\\
\widetilde{b}(\widetilde{\boldsymbol{\sigma}},\xi) & = & \widetilde{\mathrm{G}}(\xi) & \forall\,\xi\in\mathrm{Q}\,,\\
a(\mathbf{p},\mathbf{q}) \,+\, b_1(\mathbf{q},\varphi) & = & \mathrm{F}(\mathbf{q}) & \forall\,\mathbf{q}\in X_1\,,\\
b_2(\mathbf{p},\phi) \,-\, c_{\mathbf{u},\eta}\big((\mathbf{p},\varphi),\phi\big) & = & \mathrm{G}(\phi) & \forall\,\phi\in M_2\,.
\end{array} \tag{4.69}$$

We notice here that the first, fourth, and sixth rows of (4.69), the first one with $\vec{\mathbf{v}}:=(\mathbf{v},\mathbf{0})\in\mathbf{H}$, constitute the conservation of momentum properties for the three equations involved. We will refer to the discrete versions of them later on in Sections 4.4.1 and 4.5.1.

4.2.4 Remarks on the boundary conditions

We find it important to highlight here that, while Dirichlet boundary conditions for the cell density and chemical signal concentration are not the usual ones in applications, the variational formulations resulting from more interesting boundary conditions are just minor modifications of (4.69). Indeed, let us assume for instance the no-slip boundary condition for $\mathbf{u}$ and the no-flux boundary conditions for η and φ considered in [38, eq. (1.2)] (see, also [72, eq. (1.2)], [10, eq. (1.1)], and [79, eq. (2.8)]), that is

$$\mathbf{u} \,=\, \mathbf{0}\,, \quad \nabla\eta\cdot\mathbf{n} \,=\, 0\,, \quad \text{and} \quad \nabla\varphi\cdot\mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma\,. \tag{4.70}$$

In this case, we first observe from (4.2.1) that the functional $\mathbf{G}$ (cf. (4.19)) becomes the null one, which constitutes the only change in the first two rows of (4.69). Moreover, it follows from the definitions of $\widetilde{\boldsymbol{\sigma}}$ and $\mathbf{p}$ (cf. (4.5)) that

$$\widetilde{\boldsymbol{\sigma}}\cdot\mathbf{n} \,=\, 0 \quad \text{and} \quad \mathbf{p}\cdot\mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma\,, \tag{4.71}$$

whence the unknowns $\widetilde{\boldsymbol{\sigma}}$ and $\mathbf{p}$ are sought now in the spaces arising from (4.38) and (4.55), respectively, after incorporating into them the above homogeneous boundary conditions, that is

$$\begin{aligned} \mathrm{H} \quad &:= \quad \Big\{\widetilde{\boldsymbol{\tau}}\in\mathbf{H}(\mathbf{div}_\varrho;\Omega): \quad \widetilde{\boldsymbol{\tau}}\cdot\mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma\Big\}, \quad \text{and} \\ X_2 \quad &:= \quad \Big\{\mathbf{q}\in\mathbf{H}^r(\mathrm{div}_j;\Omega): \quad \mathbf{q}\cdot\mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma\Big\}. \end{aligned} \tag{4.72}$$

In this way, bearing in mind the new space H (cf. (4.72)), we notice that the last term of (4.30) now vanishes, whence, besides H, the only change in the third row of (4.69) (cf. (4.33)) is given by the fact that $\widetilde{\mathrm{F}}$ (cf. (4.35)) becomes the null functional as well. In addition, taking $\xi\equiv 1$ in the fourth equation of (4.69) (cf. (4.36)) and using the first identity of (4.71), we realize that the datum f_η must belong to $\mathrm{L}^\varrho_0(\Omega) := \Big\{g\in\mathrm{L}^\varrho(\Omega): \quad \int_\Omega g = 0\Big\}$. It follows that testing this fourth equation against $\xi\in\mathrm{Q}:=\mathrm{L}^\rho(\Omega)$ (cf. (4.38)) is equivalent to doing it against $\xi\in\mathrm{Q}:=\mathrm{L}^\rho_0(\Omega)$, thus suggesting to look for the corresponding unknown η in this new space Q as well, which actually would be required for the corresponding solvability analysis. We refer in a more precise way to this fact in Section 4.3.2.

Furthermore, we proceed analogously for the chemical signal concentration, so that, according to the definition of X_2 in (4.72), we now incorporate the same homogeneous boundary condition into the new version of the test space X_1, that is

$$X_1 \,:=\, \Big\{\mathbf{q}\in\mathbf{H}^s(\mathrm{div}_s;\Omega): \quad \mathbf{q}\cdot\mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma\Big\}. \tag{4.73}$$

As a consequence, the last term of (4.51) vanishes, that is the functional F (cf. (4.60)) becomes the null one, which, besides X_2 and X_1, constitutes the only change in the fifth row of (4.69). Similarly as for the cell density, and regarding the spaces M_1 and M_2, one would just need to redefine them as $M_1 := \mathrm{L}^r_0(\Omega)$ and $M_2 := \mathrm{L}^l_0(\Omega)$ in order to be able to perform the respective solvability analysis. We further explain the above in Section 4.3.2.

On the other hand, let us next consider the more realistic boundary conditions from [15], that is

$$\mathbf{u} \,=\, \mathbf{0}\,, \quad \nabla\eta\cdot\mathbf{n} \,=\, 0\,, \quad \text{and} \quad \nabla\varphi\cdot\mathbf{n} \,=\, \big(c-\varphi\big)\,g \quad \text{on} \quad \Gamma\,, \tag{4.74}$$

where c is a positive constant and g is a sufficiently smooth function defined on $\bar{\Omega}$, such that $g\neq 0$ and $g\geq 0$ on Γ. In this case, it is easy to see from the definitions of $\widetilde{\boldsymbol{\sigma}}$ and $\mathbf{p}$ (cf. (4.5)) that

$$\widetilde{\boldsymbol{\sigma}}\cdot\mathbf{n} \,=\, \widetilde{\mathbf{g}}(\eta,\varphi) \,:=\, -\kappa_\eta^{-1}\,\mu\,\eta\big(c-\varphi\big)\,g \quad \text{and} \quad \mathbf{p}\cdot\mathbf{n} \,=\, \mathbf{g}(\varphi) \,:=\, \big(c-\varphi\big)\,g \quad \text{on} \quad \Gamma\,, \tag{4.75}$$

whence the spaces H and X_2 are kept as defined originally by (4.38) and (4.55). However, the derivations of the weak formulations for the cell density and chemical signal concentration equations require to define the negative traces of η and φ as the Lagrange multipliers associated to the non-homogeneous boundary conditions in (4.75). In particular, letting $\eta_D := -\eta|_\Gamma \in \mathrm{H}^{1/2}(\Gamma)$, which is now an unknown, and bearing in mind the integration by parts in (4.30), the identity (4.33) becomes

$$\int_\Omega \widetilde{\boldsymbol{\sigma}} \cdot \widetilde{\boldsymbol{\tau}} \,+\, \int_\Omega \eta\, \mathbf{div}(\widetilde{\boldsymbol{\tau}}) \,+\, \langle \widetilde{\boldsymbol{\tau}} \cdot \mathbf{n}, \eta_D \rangle_\Gamma \,+\, \widetilde{c}_{\mathbf{u},\mathbf{p}}(\widetilde{\boldsymbol{\tau}}, \eta) \,=\, 0 \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathbf{div}_\varrho;\Omega)\,, \tag{4.76}$$

whereas the testing of the first boundary condition in (4.75) yields

$$\langle \widetilde{\boldsymbol{\sigma}} \cdot \mathbf{n}, \xi_D \rangle_\Gamma \,=\, \langle \widetilde{\mathbf{g}}(\eta,\varphi), \xi_D \rangle_\Gamma \qquad \forall\, \xi_D \in \mathrm{H}^{1/2}(\Gamma)\,, \tag{4.77}$$

which is then added to (4.36). Consequently, H becomes the product space $\mathbf{H}(\mathbf{div}_\varrho;\Omega) \times \mathrm{H}^{1/2}(\Gamma)$, the bilinear form $\widetilde{b}$ incorporates the additional boundary expression given by both the third and first terms of (4.76) and (4.77), respectively, the functional $\widetilde{\mathrm{F}}$ is again the null one, and the functional $\widetilde{\mathrm{G}}$ results from adding the right-hand sides of (4.37) and (4.77). In turn, introducing $\varphi_D := -\varphi|_\Gamma \in \mathbf{W}^{1/s,r}(\Gamma)$ as a further unknown of the chemical signal equations, the fifth and sixth rows of (4.69) suffer analogous modifications to those aforedescribed for the third and fourth ones. We omit further details here.

Summarizing, we believe that the discussion in this section has made clear that there is no loss of generality when utilizing Dirichet boundary conditions for the chemotaxis-Navier-Stokes problem since, at least for the case described by (4.70), the continuous and discrete analyses to be developed throughout the rest of the chapter will need to be slighted modified only. In order to emphasize this fact from a numerical point of view, in Section 4.6 we also include examples reporting the applicability of our method to the aforementioned situation, and even to the particular case of (4.75) in which $\widetilde{\mathbf{g}}$ and $\mathbf{g}$ are explicitly known data. A detailed discussion of the full case (4.75) certainly merits to be part of a separate work. In particular, the way in which the right-hand side of (4.77) - which meantime forms part of $\widetilde{\mathrm{G}}$ - is incorporated into the fixed-point strategy, is one of the key aspects to be elucidated.

4.3 The continuous solvability analysis

In this section we proceed similarly as in [29] and [52] (see also [11], [21], [55], and some of the references therein) and adopt a fixed-point strategy to analyse the solvability of (4.69).

4.3.1 The fixed-point approach

We begin by rewriting (4.69) as an equivalent fixed point equation. To this end, we first let $\mathbf{S} : \mathbf{L}^4(\Omega) \times \mathrm{Q} \to \mathbf{L}^4(\Omega)$ be the operator defined by

$$\mathbf{S}(\mathbf{z},\chi) \,:=\, \mathbf{u} \qquad \forall\, (\mathbf{z},\chi) \in \mathbf{L}^4(\Omega) \times \mathrm{Q}\,, \tag{4.78}$$

where $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) = \big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution (to be confirmed below) of problem (4.14) (equivalently, the first and second rows of (4.69)) when $\mathbf{c}(\mathbf{u};\cdot,\cdot)$ and $\mathbf{F}_\eta$ are replaced by $\mathbf{c}(\mathbf{z};\cdot,\cdot)$ and $\mathbf{F}_\chi$, respectively, that is

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{u}},\vec{\mathbf{v}}) \,+\, \mathbf{c}(\mathbf{z};\vec{\mathbf{u}},\vec{\mathbf{v}}) \,+\, \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\sigma}) \quad &= \quad \mathbf{F}_\chi(\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,, \\ \mathbf{b}(\vec{\mathbf{u}},\boldsymbol{\tau}) \quad &= \quad \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,. \end{aligned} \tag{4.79}$$

Similarly, we let $\widetilde{\mathrm{S}} : \mathbf{L}^4(\Omega) \times X_2 \to \mathrm{Q}$ be the operator given by

$$\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r}) := \eta \qquad \forall\,(\mathbf{z},\mathbf{r}) \in \mathbf{L}^4(\Omega) \times X_2\,, \tag{4.80}$$

where $(\widetilde{\boldsymbol{\sigma}},\eta) \in \mathrm{H} \times \mathrm{Q}$ is the unique solution (to be confirmed below) of problem (4.39) (equivalently, the third and fourth rows of (4.69)) when $\widetilde{c}_{\mathbf{u},\mathbf{p}}$ is replaced by $\widetilde{c}_{\mathbf{z},\mathbf{r}}$, that is

$$\begin{aligned} \widetilde{a}(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}},\eta) + \widetilde{c}_{\mathbf{z},\mathbf{r}}(\widetilde{\boldsymbol{\tau}},\eta) &= \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathrm{H}\,,\\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}},\xi) &= \widetilde{\mathrm{G}}(\xi) \qquad \forall\,\xi \in \mathrm{Q}\,. \end{aligned} \tag{4.81}$$

In turn, we let $\mathrm{S} : \mathbf{L}^4(\Omega) \times \mathrm{Q} \to X_2$ be the operator given by

$$\mathrm{S}(\mathbf{z},\chi) := \mathbf{p} \quad \forall\,(\mathbf{z},\chi) \in \mathbf{L}^4(\Omega) \times \mathrm{Q}\,, \tag{4.82}$$

where $(\mathbf{p},\varphi) \in X_2 \times M_1$ is the unique solution (to be confirmed below) of problem (4.56) (equivalently, the fifth and sixth rows of (4.69)) when $c_{\mathbf{u},\eta}$ is replaced by $c_{\mathbf{z},\chi}$, that is

$$\begin{aligned} a(\mathbf{p},\mathbf{q}) + b_1(\mathbf{q},\varphi) &= \mathrm{F}(\mathbf{q}) \qquad \forall\,\mathbf{q} \in X_1\,,\\ b_2(\mathbf{p},\phi) - c_{\mathbf{z},\chi}\big((\mathbf{p},\varphi),\phi\big) &= \mathrm{G}(\phi) \qquad \forall\,\phi \in M_2\,. \end{aligned} \tag{4.83}$$

Thus, defining the operator $\mathrm{T} : \mathbf{L}^4(\Omega) \times X_2 \to \mathbf{L}^4(\Omega) \times X_2$ as

$$\mathrm{T}(\mathbf{z},\mathbf{r}) := \Big(\mathbf{S}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big), \mathrm{S}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big)\Big) \qquad \forall\,(\mathbf{z},\mathbf{r}) \in \mathbf{L}^4(\Omega) \times X_2\,, \tag{4.84}$$

we realize that solving (4.69) is equivalent to seeking a fixed point of T, that is: Find $(\mathbf{u},\mathbf{p}) \in \mathbf{L}^4(\Omega)\times X_2$ such that

$$\mathrm{T}(\mathbf{u},\mathbf{p}) = (\mathbf{u},\mathbf{p})\,. \tag{4.85}$$

4.3.2 Well-posedness of the uncoupled problems

We now employ the Babuska-Brezzi theory in Banach spaces (cf. [12, Theorem 2.1, Corollary 2.1, Section 2.1] for the general case, and [41, Theorem 2.34] for a particular one), and the Banach-Nečas-Babuška Theorem (also known as the generalized Lax-Milgram Lemma) (cf. [41, Theorem 2.6]), to establish the well-posedness of the problems (4.79), (4.81), and (4.83), defining the operators $\mathbf{S}$, $\widetilde{\mathrm{S}}$, and S, respectively.

Well-definedness of operator S

Here we apply [41, Theorem 2.34] to prove that problem (4.79) is well-posed (equivalently, that $\mathbf{S}$ is well-defined). In this regard, it is important to stress that the structure of (4.79) is the same of the problem stated in [29, eq. (3.23)], and hence, several results and techniques from there will be employed in what follows. Indeed, given $(\mathbf{z},\chi) \in \mathbf{L}^4(\Omega) \times \mathrm{Q}$, we proceed as in [29, Section 3.3], and introduce first the bilinear form $\mathcal{A}_{\mathbf{z}} : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$ defined by

$$\mathcal{A}_{\mathbf{z}}(\vec{\mathbf{w}},\vec{\mathbf{v}}) := \mathbf{a}(\vec{\mathbf{w}},\vec{\mathbf{v}}) + \mathbf{c}(\mathbf{z};\vec{\mathbf{w}},\vec{\mathbf{v}}) \qquad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}} \in \mathbf{H}\,, \tag{4.86}$$

so that problem (4.79) can be rewritten as: Find $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathcal{A}_{\mathbf{z}}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) + \mathbf{b}(\vec{\mathbf{v}}, \boldsymbol{\sigma}) &= \mathbf{F}_{\chi}(\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,, \\ \mathbf{b}(\vec{\mathbf{u}}, \boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,. \end{aligned} \tag{4.87}$$

Now, we let $\mathbf{V}$ be the kernel of the operator induced by the bilinear form $\mathbf{b}$ (cf. (4.16)), that is

$$\mathbf{V} := \Big\{ \vec{\mathbf{v}} := (\mathbf{v}, \mathbf{s}) \in \mathbf{H} : \quad \mathbf{b}(\vec{\mathbf{v}}, \boldsymbol{\tau}) = 0 \quad \forall\, \boldsymbol{\tau} \in \mathbf{Q} \Big\}\,,$$

which, exactly as [29, eq. (3.34)], reduces to

$$\mathbf{V} := \Big\{ \vec{\mathbf{v}} := (\mathbf{v}, \mathbf{s}) \in \mathbf{H} : \quad \nabla \mathbf{v} = \mathbf{s} \quad \text{and} \quad \mathbf{v} \in \mathbf{H}^1_0(\Omega) \Big\}\,. \tag{4.88}$$

Then, letting c_P be the positive constant yielding the Friedrichs-Poincaré inequality, which states that $|\mathbf{v}|^2_{1,\Omega} \ge c_P\,\|\mathbf{v}\|^2_{1,\Omega}$ for all $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$, denoting by $\mathbf{i}_4$ the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, bearing in mind (4.86) and (4.26), and proceeding analogously to the proof of [29, eq. (3.41), Lemma 3.2], we find that

$$\mathcal{A}_{\mathbf{z}}(\vec{\mathbf{v}}, \vec{\mathbf{v}}) = \mathbf{a}(\vec{\mathbf{v}}, \vec{\mathbf{v}}) \ge \boldsymbol{\alpha}\,\|\vec{\mathbf{v}}\|^2_{\mathbf{H}} \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{V}\,, \tag{4.89}$$

with $\boldsymbol{\alpha} := \frac{\nu}{2}\min\big\{1, \frac{c_P}{\|\mathbf{i}_4\|^2}\big\}$, which gives the $\mathbf{V}$-ellipticity of $\mathcal{A}_{\mathbf{z}}$. Thus, it is easily seen, thanks to (4.89), that $\mathcal{A}_{\mathbf{z}}$ satisfies the hypothesis specified in [41, Theorem 2.34, eq. (2.28)] with the constant $\boldsymbol{\alpha}$ defined above. In addition, it follows from (4.86), along with (4.21), (4.22), and (4.24), that there holds

$$|\mathcal{A}_{\mathbf{z}}(\vec{\mathbf{w}}, \vec{\mathbf{v}})| \le \|\mathcal{A}_{\mathbf{z}}\|\,\|\vec{\mathbf{w}}\|_{\mathbf{H}}\,\|\vec{\mathbf{v}}\|_{\mathbf{H}} \qquad \forall\, \vec{\mathbf{w}},\, \vec{\mathbf{v}} \in \mathbf{H}\,, \tag{4.90}$$

with the constant

$$\|\mathcal{A}_{\mathbf{z}}\| := \|\mathbf{a}\| + \|\mathbf{c}\|\,\|\mathbf{z}\|_{0,4;\Omega} = \nu + \frac{\lambda}{2}\,\|\mathbf{z}\|\,, \tag{4.91}$$

which says that $\mathcal{A}$ is bounded.

In turn, using that for each $t \in \begin{cases} (1, +\infty) & \text{if } n = 2 \\ [6/5, +\infty) & \text{if } n = 3 \end{cases}$ there exists a constant $C_t > 0$, depending only on Ω, such that

$$C_t\,\|\boldsymbol{\tau}\|^2_{0,\Omega} \le \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_t; \Omega)\,, \tag{4.92}$$

which follows from a slight modification of the proof of [44, Lemma 2.3], one can prove the continuous inf-sup condition for the bilinear form $\mathbf{b}$. More precisely, employing (4.92) with $t = 4/3$, it is shown in [29, Lemma 3.3, eq. (3.44)] that there exists a positive constant $\boldsymbol{\beta}$, depending only on $C_{4/3}$, such that

$$\sup_{\substack{\vec{\mathbf{v}} \in \mathbf{H} \\ \vec{\mathbf{v}} \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{v}}, \vec{\boldsymbol{\tau}})}{\|\vec{\mathbf{v}}\|_{\mathbf{H}}} \ge \boldsymbol{\beta}\,\|\boldsymbol{\tau}\|_{\mathbf{Q}} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,, \tag{4.93}$$

whence the bilinear form $\mathbf{b}$ satisfies the hypothesis indicated in [41, Theorem 2.34, eq. (2.29)].

For sake of completeness we remark here that, exactly as for the proof of [44, Lemma 2.3], the derivation of (4.92) is based on the fact that the divergence operator $\mathbf{div}$ is an isomorphism from the closed subspace of $\mathbf{H}^1_0(\Omega)$ given by $W^{\perp}$, where $W := \Big\{\mathbf{z} \in \mathbf{H}^1_0(\Omega) : \quad \mathbf{div}(\mathbf{z}) = 0\Big\}$, onto $\mathrm{L}^2_0(\Omega)$. In this way, given $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_t;\Omega)$, that is $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$ and $\mathrm{tr}\,(\boldsymbol{\tau}) \in \mathrm{L}_0^2(\Omega)$, we let $\mathbf{z}$ be the unique element in $W^\perp$ such that $\mathbf{div}(\mathbf{z}) = \mathrm{tr}\,(\boldsymbol{\tau})$ and $\|\mathbf{z}\|_{1,\Omega} \le C_0\,\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,\Omega}$, with a positive constant C_0 independent of $\mathbf{z}$ and $\boldsymbol{\tau}$. Then, a key identity stated a few lines after [44, eq. (2.53)] establishes that

$$\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,\Omega}^2 \,=\, -n\int_\Omega \mathbf{z}\cdot\mathbf{div}(\boldsymbol{\tau}) \,-\, n\int_\Omega \boldsymbol{\tau}^{\mathrm{d}} : \nabla\mathbf{z}\,,$$

from which, applying the Hölder and Cauchy-Schwarz inequalities, letting $t' \in (1,+\infty)$ be the conjugate of t, and denoting by $\mathbf{i}_{t'}$ the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^{t'}(\Omega)$, which holds for $t' \in [1,+\infty)$ in 2D and $t' \in [1,6]$ in 3D, we get

$$\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,\Omega}^2 \le\, n\,C_0\Big\{\|\mathbf{i}_{t'}\|\,\|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} + \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}\Big\}\,\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,\Omega}\,,$$

and hence

$$\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,\Omega} \,\le\, n\,C_0\Big\{\|\mathbf{i}_{t'}\|\,\|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} + \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}\Big\}\,.$$

Finally, it is readily seen that (4.92) follows straightforwardly from the foregoing inequality and the fact that $\|\boldsymbol{\tau}\|_{0,\Omega}^2 \,=\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 \,+\, \dfrac{1}{n}\,\|\mathrm{tr}\,(\boldsymbol{\tau})\|_{0,\Omega}^2$.

We are now in position to confirm that the operator $\mathbf{S}$ is well-defined.

Lemma 4.1. *For each $(\mathbf{z},\chi) \in \mathbf{L}^4(\Omega)\times\mathrm{Q}$ there exists a unique $(\vec{\mathbf{u}},\boldsymbol{\sigma}) := \big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big) \in \mathbf{H}\times\mathbf{Q}$ solution of (4.87) (equivalently (4.79)), and hence one can define $\mathbf{S}(\mathbf{z},\chi) := \mathbf{u} \in \mathbf{L}^4(\Omega)$. Moreover, there exists a positive constant $C_{\mathbf{S}}$, depending only on $|\Omega|$, $\|\mathbf{i}_4\|$, ν, λ, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$, such that*

$$\begin{aligned}\|\mathbf{S}(\mathbf{z},\chi)\|_{0,4;\Omega} \,&=\, \|\mathbf{u}\|_{0,4;\Omega} \,\le\, \|\vec{\mathbf{u}}\|_{\mathbf{H}}\\ &\le\, C_{\mathbf{S}}\,\Big\{\|\chi\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \big(1+\|\mathbf{z}\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,.\end{aligned} \tag{4.94}$$

Proof. Having previously established that $\mathcal{A}_{\mathbf{z}}$ and $\mathbf{b}$ satisfy [41, eqs. (2.28) and (2.29)], and knowing that $\mathcal{A}_{\mathbf{z}}$, $\mathbf{b}$, $\mathbf{F}_\chi$, and $\mathbf{G}$ are all bounded, a straightforward application of [41, Theorem 2.34] confirms the existence of a unique $(\vec{\mathbf{u}},\boldsymbol{\sigma}) := \big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big) \in \mathbf{H}\times\mathbf{Q}$ solution of (4.87). In addition, the corresponding a priori estimate in [41, Theorem 2.34, eq. (2.30)] yields

$$\|\vec{\mathbf{u}}\|_{\mathbf{H}} \,\le\, \frac{1}{\boldsymbol{\alpha}}\,\|\mathbf{F}_\chi\| \,+\, \frac{1}{\boldsymbol{\beta}}\left(1+\frac{\|\mathcal{A}_{\mathbf{z}}\|}{\boldsymbol{\alpha}}\right)\|\mathbf{G}\|\,. \tag{4.95}$$

Then, noting from (4.49) and (4.50) that

$$\|\mathbf{F}_\chi\| \,\le\, \max\big\{1,|\Omega|^{1/4}\big\}\,\big(\|\chi\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega}\big)\,, \tag{4.96}$$

invoking the expressions for $\|\mathbf{G}\|$ and $\|\mathcal{A}_{\mathbf{z}}\|$ provided in (4.21) and (4.91), respectively, and performing some minor algebraic manipulations, we readily derive from (4.95) the required inequality (4.94). □

Regarding the a priori estimate for the component $\boldsymbol{\sigma}$ of the unique solution of (4.87), which will be used later on, we recall that the second inequality in [41, Theorem 2.34, eq. (2.30)] gives

$$\|\boldsymbol{\sigma}\|_{\mathbf{Q}} \,\le\, \frac{1}{\boldsymbol{\beta}}\left(1+\frac{\|\mathcal{A}_{\mathbf{z}}\|}{\boldsymbol{\alpha}}\right)\|\mathbf{F}_\chi\| \,+\, \frac{\|\mathcal{A}_{\mathbf{z}}\|}{\boldsymbol{\beta}^2}\left(1+\frac{\|\mathcal{A}_{\mathbf{z}}\|}{\boldsymbol{\alpha}}\right)\|\mathbf{G}\|\,,$$

which, proceeding similarly to the derivation of (4.94), yields

$$\|\boldsymbol{\sigma}\|_{\mathbf{Q}} \,=\, \|\boldsymbol{\sigma}\|_{\mathbf{div}_{4/3};\Omega} \,\leq\, \bar{C}_{\mathbf{S}}\,\left(1 \,+\, \|\mathbf{z}\|_{0,4;\Omega}\right) \Big\{ \|\chi\|_{0,\rho;\Omega}\, \|\nabla f\|_{0,r;\Omega} \\ +\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \left(1 + \|\mathbf{z}\|_{0,4;\Omega}\right) \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\}\,, \tag{4.97}$$

where $\bar{C}_{\mathbf{S}}$ is a positive constant depending as well on $|\Omega|$, $\|\mathbf{i}_4\|$, ν, λ, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$.

Well-definedness of operator $\widetilde{\mathrm{S}}$

In this section we make use of [41, Theorems 2.34 and 2.6] to show that (4.81) is well-posed (equivalently, that $\widetilde{\mathrm{S}}$ is well-defined). To this end, and similarly to Section 4.3.2, we notice that, given $(\mathbf{z},\mathbf{r}) \in \mathbf{L}^4(\Omega) \times X_2$, the structure of (4.81) is analogous to that of the problem specified in [52, eq. (2.33), Section 2.3], so that some results and techniques from its corresponding analysis are employed below. In particular, following the approach from [52, Section 2.4.3], we first apply [41, Theorem 2.34] to a perturbation of (4.81), and then employ [41, Theorem 2.6] to conclude that the whole problem (4.81) is well-posed. More precisely, we let $\widetilde{A} : (\mathrm{H} \times \mathrm{Q}) \times (\mathrm{H} \times \mathrm{Q}) \to \mathrm{R}$ be the bounded bilinear form arising from (4.81) after adding the left-hand sides of its equations, but without including $\widetilde{c}_{\mathbf{z},\mathbf{r}}$, that is

$$\widetilde{A}((\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi)) \,:=\, \widetilde{a}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\tau}},\chi) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\zeta}},\xi) \tag{4.98}$$

for all $(\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi) \in \mathrm{H} \times \mathrm{Q}$, and show next that $\widetilde{A}$ satisfies a global continuous inf-sup condition. Note that, being $\widetilde{A}$ symmetric, the latter will be valid with respect to any of its components. We also remark that the boundedness of $\widetilde{A}$ follows from those of $\widetilde{a}$ and $\widetilde{b}$ (cf. (4.42), (4.43), and (4.44)).

Since establishing the aforementioned property for $\widetilde{A}$ is equivalent to proving that the bilinear forms $\widetilde{a}$ and $\widetilde{b}$ satisfy the hypotheses of [41, Theorem 2.34], we proceed with the latter in what follows. We begin by letting $\widetilde{\mathrm{V}}$ be the null space of the operator induced by the bilinear form $\widetilde{b}$, that is

$$\widetilde{\mathrm{V}} \,:=\, \Big\{ \widetilde{\boldsymbol{\tau}} \in \mathrm{H} : \quad b(\widetilde{\boldsymbol{\tau}},\xi) \,=\, 0 \quad \forall\, \xi \in \mathrm{Q} \Big\}\,,$$

which, according to the definitions of $\widetilde{b}$ (cf. (4.41)) and the spaces H and Q (cf. (4.38)), yields

$$\widetilde{\mathrm{V}} \,:=\, \Big\{ \widetilde{\boldsymbol{\tau}} \in \mathrm{H} : \quad \mathbf{div}(\widetilde{\boldsymbol{\tau}}) \,=\, 0 \Big\}\,. \tag{4.99}$$

Then, it is straightforward to see from the definitions of $\widetilde{a}$ (cf. (4.40)) and the norm of $\mathrm{H} := \mathbf{H}(\mathbf{div}_{\varrho};\Omega)$ (cf. (6)) that there holds

$$\widetilde{a}(\widetilde{\boldsymbol{\tau}},\widetilde{\boldsymbol{\tau}}) \,=\, \|\widetilde{\boldsymbol{\tau}}^2\|_{\mathrm{H}} \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \widetilde{\mathrm{V}}\,, \tag{4.100}$$

from which one easily deduces that $\widetilde{a}$ satisfies the hypotheses given by [41, Theorem 2.34, eq. (2.28)] with the constant $\widetilde{\alpha} = 1$.

Furthermore, since the continuous inf-sup condition for $\widetilde{b}$ has already been established (see, e.g. [21, Lemma 2.1], [52, Lemma 2.9], and also [55, Lemma 3.5] for a closely related result), we provide next only the main details of its corresponding proof. In fact, given $\xi \in \mathrm{Q} := \mathrm{L}^\rho(\Omega)$, we note from (4.31) that $\rho > 2$, introduce $\xi_\varrho \,:=\, |\xi|^{\rho-2}\,\xi$, and observe that

$$\xi_\varrho \in \mathrm{L}^\varrho(\Omega) \qquad \text{and} \qquad \int_\Omega \xi\, \xi_\varrho \,=\, \|\xi\|_{0,\rho;\Omega}\, \|\xi_\varrho\|_{0,\varrho;\Omega}\,. \tag{4.101}$$

Then, letting $w \in \mathrm{H}_0^1(\Omega)$ be the unique weak solution of $\Delta w = -\xi_\varrho$ in Ω, $w = 0$ on Γ, for which there holds $\|w\|_{1,\Omega} \le \frac{\|i_\rho\|}{c_P} \|\xi_\varrho\|_{0,\varrho;\Omega}$, where c_P is the constant yielding the Friedrichs-Poincaré inequality, and i_ρ is the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^\rho(\Omega)$, we define $\widetilde{\boldsymbol{\zeta}} := -\nabla w \in \mathbf{L}^2(\Omega)$ and notice that $\mathbf{div}(\widetilde{\boldsymbol{\zeta}}) = \xi_\varrho$, so that $\widetilde{\boldsymbol{\zeta}} \in \mathrm{H} := \mathbf{H}(\mathbf{div}_\varrho;\Omega)$. In this way, bounding by below with $\widetilde{\boldsymbol{\tau}} = \widetilde{\boldsymbol{\zeta}}$, and using the above identities and estimates, we arrive at

$$\sup_{\substack{\widetilde{\boldsymbol{\tau}}\in \mathrm{H}\\ \widetilde{\boldsymbol{\tau}}\neq \mathbf{0}}} \frac{\widetilde{b}(\widetilde{\boldsymbol{\tau}},\xi)}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}}} \ge \widetilde{\beta}\, \|\xi\|_{\mathrm{Q}}\,, \tag{4.102}$$

with $\widetilde{\beta} := \big(1 + \frac{\|i_\rho\|}{c_P}\big)^{-1}$.

At this point we recall from Section 4.2.4 that in the case of the boundary conditions given by (4.70) (which yields (4.71)), the elements of H have a null normal trace, and hence the proof of the inf-sup condition for $\widetilde{b}$ needs to be slightly modified. In fact, the auxiliary boundary value problem with solution w must consider now a homogeneous Neumann boundary condition instead of a Dirichlet one, so that, in order for the compatibility condition between the data be satisfied, the mean value of the source term must be 0. For achieving the latter as well as the identity in (4.101), which is a key aspect of the proof of (4.102), it suffices that Q becomes $\mathrm{L}_0^\rho(\Omega)$, thus confirming what was announced in Section 4.2.4. Alternatively, and coherently with [15, Theorem 1.1], we may assume that $\int_\Omega \eta$ is given, which allows to uniquely decompose η as $\eta = \eta_0 + c_0$, with the new unknown $\eta_0 \in \mathrm{L}_0^\rho(\Omega)$, and a real constant c_0 that is explicitly known in terms of the aforementioned given value.

Consequently, thanks to (4.100) and (4.102), the hypotheses of [41, Theorem 2.34] are satisfied, and hence the a priori estimates given by [41, Theorem 2.34, eq. (2.30)] imply the existence of a positive constant $\alpha_{\widetilde{\mathrm{S}}}$, depending only on $\widetilde{\alpha}$, $\widetilde{\beta}$, and $\|\widetilde{a}\|$, such that

$$\sup_{\substack{(\widetilde{\boldsymbol{\tau}},\xi)\in \mathrm{H}\times\mathrm{Q}\\ (\widetilde{\boldsymbol{\tau}},\xi)\neq 0}} \frac{\widetilde{A}((\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi))}{\|(\widetilde{\boldsymbol{\tau}},\xi)\|_{\mathrm{H}\times\mathrm{Q}}} \ge \alpha_{\widetilde{\mathrm{S}}}\, \|(\widetilde{\boldsymbol{\zeta}},\chi)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\, (\widetilde{\boldsymbol{\zeta}},\chi) \in \mathrm{H}\times\mathrm{Q}\,. \tag{4.103}$$

Next, we let $\widetilde{A}_{\mathbf{z},\mathbf{r}} : \big(\mathrm{H}\times\mathrm{Q}\big)\times\big(\mathrm{H}\times\mathrm{Q}\big) \to \mathrm{R}$ be the bounded bilinear form that results after adding the full left-hand sides of the equations of (4.81), that is

$$\widetilde{A}_{\mathbf{z},\mathbf{r}}((\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi)) := \widetilde{A}((\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi)) + \widetilde{c}_{\mathbf{z},\mathbf{r}}(\widetilde{\boldsymbol{\tau}},\chi) \qquad \forall\, (\widetilde{\boldsymbol{\zeta}},\chi),\, (\widetilde{\boldsymbol{\tau}},\xi) \in \mathrm{H}\times\mathrm{Q}\,, \tag{4.104}$$

whence problem (4.81) can be rewritten, equivalently, as: Find $(\widetilde{\boldsymbol{\sigma}},\eta) \in \mathrm{H}\times\mathrm{Q}$ such that

$$\widetilde{A}_{\mathbf{z},\mathbf{r}}((\widetilde{\boldsymbol{\sigma}},\eta),(\widetilde{\boldsymbol{\tau}},\xi)) = \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) + \widetilde{\mathrm{G}}(\xi) \qquad \forall\, (\widetilde{\boldsymbol{\tau}},\xi) \in \mathrm{H}\times\mathrm{Q}\,. \tag{4.105}$$

We remark that the boundedness of $\widetilde{A}$ and $\widetilde{c}_{\mathbf{z},\mathbf{r}}$ (cf. (4.45)) implies the same property for $\widetilde{A}_{\mathbf{z},\mathbf{r}}$. In turn, it follows from (4.103), (4.104), and the boundedness of $\widetilde{c}_{\mathbf{z},\mathbf{r}}$ (cf. (4.42) and (4.45)), that for each $(\widetilde{\boldsymbol{\zeta}},\chi) \in \mathrm{H}\times\mathrm{Q}$ there holds

$$\begin{aligned}\sup_{\substack{(\widetilde{\boldsymbol{\tau}},\xi)\in \mathrm{H}\times\mathrm{Q}\\ (\widetilde{\boldsymbol{\tau}},\xi)\neq \mathbf{0}}} \frac{\widetilde{A}_{\mathbf{z},\mathbf{r}}((\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi))}{\|(\widetilde{\boldsymbol{\tau}},\xi)\|_{\mathrm{H}\times\mathrm{Q}}} &\ge \alpha_{\widetilde{\mathrm{S}}}\, \|(\widetilde{\boldsymbol{\zeta}},\chi)\|_{\mathrm{H}\times\mathrm{Q}} - \|\widetilde{c}\|\, \big(\|\mathbf{z}\|_{0,4;O} + \|\mathbf{r}\|_{0,r;\Omega}\big)\, \|\chi\|_{\mathrm{Q}} \\ &\ge \Big\{\alpha_{\widetilde{\mathrm{S}}} - \|\widetilde{c}\|\, \big(\|\mathbf{z}\|_{0,4;\Omega} + \|\mathbf{r}\|_{0,r;\Omega}\big)\Big\}\, \|(\widetilde{\boldsymbol{\zeta}},\chi)\|_{\mathrm{H}\times\mathrm{Q}}\,,\end{aligned}$$

and thus, under the assumption that

$$\|\mathbf{z}\|_{0,4;\Omega} + \|\mathbf{r}\|_{0,r;\Omega} \,\le\, \frac{\alpha_{\widetilde{\mathrm{S}}}}{2\,\|\widetilde{c}\|}\,, \tag{4.106}$$

we arrive at

$$\sup_{\substack{(\widetilde{\boldsymbol{\tau}},\xi)\in \mathrm{H}\times\mathrm{Q}\\ (\widetilde{\boldsymbol{\tau}},\xi)\neq\mathbf{0}}} \frac{\widetilde{A}_{\mathbf{z},\mathbf{r}}((\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi))}{\|(\widetilde{\boldsymbol{\tau}},\xi)\|_{\mathrm{H}\times\mathrm{Q}}} \,\ge\, \frac{\alpha_{\widetilde{\mathrm{S}}}}{2}\,\|(\widetilde{\boldsymbol{\zeta}},\chi)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\widetilde{\boldsymbol{\zeta}},\chi)\in \mathrm{H}\times\mathrm{Q}\,. \tag{4.107}$$

Analogously, noting that $\widetilde{A}$ is symmetric, proceeding as before, and under the same assumption (4.106), we obtain

$$\sup_{\substack{(\widetilde{\boldsymbol{\zeta}},\chi)\in \mathrm{H}\times\mathrm{Q}\\ (\widetilde{\boldsymbol{\zeta}},\chi)\neq\mathbf{0}}} \frac{\widetilde{A}_{\mathbf{z},\mathbf{r}}((\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi))}{\|(\widetilde{\boldsymbol{\zeta}},\chi)\|_{\mathrm{H}\times\mathrm{Q}}} \,\ge\, \frac{\alpha_{\widetilde{\mathrm{S}}}}{2}\,\|(\widetilde{\boldsymbol{\tau}},\xi)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\widetilde{\boldsymbol{\tau}},\xi)\in \mathrm{H}\times\mathrm{Q}\,. \tag{4.108}$$

According to the foregoing analysis, the well-definedness of $\widetilde{\mathrm{S}}$ is established as follows.

Lemma 4.2. *For each $(\mathbf{z},\mathbf{r})\in \mathbf{L}^4(\Omega)\times X_2$ satisfying (4.106) there exists a unique $(\widetilde{\boldsymbol{\sigma}},\eta)\in \mathrm{H}\times\mathrm{Q}$ solution of (4.105) (equivalently (4.81)), and hence one can define $\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r}) := \eta\in\mathrm{Q}$. Moreover, there exists a positive constant $C_{\widetilde{\mathrm{S}}}$, depending only on $\alpha_{\widetilde{\mathrm{S}}}$, $\|\mathrm{i}_\rho\|$, and κ_η, such that*

$$\|\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\|_{\mathrm{Q}} \,=\, \|\eta\|_{0,\rho;\Omega} \,\le\, \|(\widetilde{\boldsymbol{\sigma}},\eta)\|_{\mathrm{H}\times\mathrm{Q}} \,\le\, C_{\widetilde{\mathrm{S}}}\left\{\|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\right\}. \tag{4.109}$$

Proof. Bearing in mind the boundedness of $\widetilde{A}_{\mathbf{z},\mathbf{r}}$, (4.107), and (4.108), a straightforward application of [41, Theorem 2.6] yields the existence of a unique solution $(\widetilde{\boldsymbol{\sigma}},\eta)\in\mathrm{H}\times\mathrm{Q}$ to (4.105). In addition, the corresponding a priori estimate (cf. [41, Theorem 2.6, eq. (2.5)]) gives

$$\|(\widetilde{\boldsymbol{\sigma}},\eta)\|_{\mathrm{H}\times\mathrm{Q}} \,\le\, \frac{2}{\alpha_{\widetilde{\mathrm{S}}}}\left\{\|\widetilde{\mathrm{F}}\| \,+\, \|\widetilde{\mathrm{G}}\|\right\},$$

which, along with the expressions for $\|\widetilde{\mathrm{F}}\|$ and $\|\widetilde{\mathrm{G}}\|$ provided in (4.42), lead to (4.109) with the constant $C_{\widetilde{\mathrm{S}}} := \frac{2}{\alpha_{\widetilde{\mathrm{S}}}}\max\left\{1+\|\mathrm{i}_\rho\|, \kappa_\eta^{-1}\right\}$. □

Well-definedness of operator S

Our goal now is to show that (4.83) is well-posed (equivalently, that S is well-defined), for which we will make use of the most general Babuška-Brezzi theory in Banach spaces (cf. [12, Theorem 2.1, Corollary 2.1, Section 2.1]) and the Banach-Nečas-Babuška Theorem (cf. [41, Theorem 2.6]). To this end, and as observed for Sections 4.3.2 and 4.3.2, we notice here that, given $(\mathbf{z},\chi)\in\mathbf{L}^4(\Omega)\times\mathrm{Q}$, the structure of (4.83) is similar to a perturbation of the problem described by [52, eq. (2.32)], so that some techniques employed there will be adapted for our analysis below. In particular, proceeding as in [52, Section 2.4.2], we first employ [12, Theorem 2.1, Corollary 2.1, Section 2.1] to analyse part of (4.83), and then we apply [41, Theorem 2.6] to conclude the well-posedness of the whole problem. According to this, we now let $\mathbf{A} : \big(X_2\times M_1\big)\times\big(X_1\times M_2\big)\to \mathrm{R}$ be the bounded bilinear form arising from (4.83) after adding the left-hand sides of its equations, but without including $c_{\mathbf{z},\chi}$, that is

$$\begin{gathered}\mathbf{A}((\mathbf{r},\psi),(\mathbf{q},\phi)) \,:=\, a(\mathbf{r},\mathbf{q}) \,+\, b_1(\mathbf{q},\psi) \,+\, b_2(\mathbf{r},\phi)\\ \forall\,(\mathbf{r},\psi)\in\big(X_2\times M_1\big),\quad \forall\,(\mathbf{q},\phi)\in\big(X_1\times M_2\big)\,,\end{gathered} \tag{4.110}$$

and aim to prove next that $\mathbf{A}$ satisfies global continuous inf-sup conditions with respect to both its first and second component. Note that the boundedness of $\mathbf{A}$ follows from those of a, b_1 and b_2 (cf. (4.63), (4.64)). The verification of the aforementioned properties of $\mathbf{A}$ is equivalent to establishing that the bilinear forms a, b_1, and b_2 verify the hypotheses of [12, Theorem 2.1, Section 2.1], which we address in what follows. Firstly, for each $i \in \{1,2\}$ we let K_i be the kernel of the bilinear form b_i (cf. (4.58)), that is

$$K_i := \Big\{ \mathbf{q} \in X_i : \quad b_i(\mathbf{q}, \phi) = 0 \quad \forall \phi \in M_i \Big\},$$

which, according to the definitions of X_1, X_2, M_1, and M_2 (cf. (4.55)), and b_i (cf. (4.58)), gives

$$K_1 = \Big\{ \mathbf{q} \in \mathbf{H}^s(\mathrm{div}_s;\Omega) : \quad \mathbf{div}(\mathbf{q}) = 0 \quad \text{in} \quad \Omega \Big\}, \quad \text{and} \tag{4.111}$$

$$K_2 = \Big\{ \mathbf{q} \in \mathbf{H}^r(\mathrm{div}_j;\Omega) : \quad \mathbf{div}(\mathbf{q}) = 0 \quad \text{in} \quad \Omega \Big\}. \tag{4.112}$$

The following lemma introduces a suitable linear operator mapping $\mathbf{L}^t(\Omega)$ into itself for a range of t containing the one specified for s in (4.31). This result will be utilized next to establish the inf-sup conditions required by [12, Theorem 2.1] (equivalently, [12, eqs. (2.8) and (2.9)]) for our bilinear form a (cf. (4.57)).

Lemma 4.3. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let $t, t' \in (1,+\infty)$ conjugate to each other with t (and hence t') lying in* $\begin{cases} [4/3,4] & \text{if } n=2 \\ [3/2,3] & \text{if } n=3 \end{cases}$. *Then, there exists a linear and bounded operator $D_t : \mathbf{L}^t(\Omega) \to \mathbf{L}^t(\Omega)$ such that*

$$\mathbf{div}(D_t(\mathbf{w})) = 0 \quad \text{in} \quad \Omega \qquad \forall \, \mathbf{w} \in \mathbf{L}^t(\Omega). \tag{4.113}$$

In addition, for each $\mathbf{z} \in \mathbf{L}^{t'}(\Omega)$ such that $\mathbf{div}(\mathbf{z}) = 0$ in Ω, there holds

$$\int_\Omega \mathbf{z} \cdot D_t(\mathbf{w}) = \int_\Omega \mathbf{z} \cdot \mathbf{w} \qquad \forall \, \mathbf{w} \in \mathbf{L}^t(\Omega). \tag{4.114}$$

Proof. It is a slight modification of the proof of [52, Lemma 2.3]. Indeed, given $\mathbf{w} \in \mathbf{L}^t(\Omega)$, with t in the range indicated, we know from the scalar version of [48, Theorem 3.2] (see also [60, Theorems 1.1 and 1.3]) that there exists a unique $u \in \mathrm{W}^{1,t}(\Omega)$ such that

$$\mathrm{div}(\nabla u + \mathbf{w}) = 0 \quad \text{in} \quad \Omega\,, \quad u = 0 \quad \text{on} \quad \partial\Omega\,,$$

and there exists a constant $C_t > 0$ such that $\|u\|_{1,t;\Omega} \le C_t \, \|\mathbf{w}\|_{0,t;\Omega}$. Then, defining $D_t(\mathbf{w}) := \nabla u + \mathbf{w}$, it is readily seen that D_t is linear and bounded, and satisfies (4.113). In turn, given $\mathbf{z} \in \mathbf{L}^{t'}(\Omega)$ such that $\mathbf{div}(\mathbf{z}) = 0$ in Ω, it is clear that $\mathbf{z} \in \mathbf{H}^{t'}(\mathbf{div}_{t'};\Omega)$, so that applying (11) to $\mathbf{z}$ and u, we obtain

$$\int_\Omega \mathbf{z} \cdot \nabla u = -\int_\Omega u \, \mathbf{div}(\mathbf{z}) + \langle \mathbf{z} \cdot \mathbf{n}, u \rangle = 0\,,$$

which yields (4.114) and finishes the proof. □

The following result, which makes use of Lemma 4.3, resembles [52, Lemma 2.6], which, in turn, employs [52, Lemma 2.3]. Note that the difference between Lemma 4.3 and [52, Lemma 2.3] lies on the boundary conditions involved.

Lemma 4.4. *There exists a positive constant α such that*

$$\sup_{\substack{\mathbf{q}\in K_1\\ \mathbf{q}\neq\mathbf{0}}} \frac{a(\mathbf{r},\mathbf{q})}{\|\mathbf{q}\|_{X_1}} \,\geq\, \alpha\,\|\mathbf{r}\|_{X_2} \qquad \forall\,\mathbf{r}\in K_2\,, \quad \textit{and} \tag{4.115}$$

$$\sup_{\mathbf{r}\in K_2} a(\mathbf{r},\mathbf{q}) \,>\, 0 \qquad \forall\,\mathbf{q}\in K_1,\, \mathbf{q}\neq\mathbf{0}\,. \tag{4.116}$$

Proof. Given $\mathbf{r}\in K_2$ (cf. (4.112)), that is $\mathbf{r}\in\mathbf{H}^r(\mathrm{div}_j;\Omega)$ such that $\mathbf{div}(\mathbf{r}) = 0$ in Ω, and recalling from (4.31) that $r>2$, we set $\mathbf{r}_s := |\mathbf{r}|^{r-2}\,\mathbf{r}$, and observe, similarly to (4.101), that

$$\mathbf{r}_s \,\in\, \mathbf{L}^s(\Omega) \qquad \text{and} \qquad \int_\Omega \mathbf{r}\cdot\mathbf{r}_s \,=\, \|\mathbf{r}\|_{0,r;\Omega}\,\|\mathbf{r}_s\|_{0,s;\Omega}\,. \tag{4.117}$$

Then, noting from (4.31) that s does belong to the range required by Lemma 4.3, an application of this result to $t=s$ yields $D_s(\mathbf{r}_s)\in K_1$, and hence, using (4.114), the identity given in (4.117), and the boundedness of D_s, we find that

$$\sup_{\substack{\mathbf{q}\in K_1\\ \mathbf{q}\neq\mathbf{0}}} \frac{a(\mathbf{r},\mathbf{q})}{\|\mathbf{q}\|_{X_1}} \,\geq\, \frac{a(\mathbf{r},D_s(\mathbf{r}_s))}{\|D_s(\mathbf{r}_s)\|_{X_1}} \,=\, \frac{\displaystyle\int_\Omega \mathbf{r}\cdot D_s(\mathbf{r}_s)}{\|D_s(\mathbf{r}_s)\|_{0,s;\Omega}} \,=\, \frac{\displaystyle\int_\Omega \mathbf{r}\cdot\mathbf{r}_s}{\|D_s(\mathbf{r}_s)\|_{0,s;\Omega}} \,\geq\, \frac{1}{\|D_s\|}\,\|\mathbf{r}\|_{0,r;\Omega}\,,$$

which proves (4.115) with $\alpha = \dfrac{1}{\|D_s\|}$. In turn, we now take $\mathbf{q}\in K_1$ (cf. (4.111)), $\mathbf{q}\neq\mathbf{0}$, define $\mathbf{q}_r := \begin{cases} |\mathbf{q}|^{s-2}\,\mathbf{q} & \text{if } \mathbf{q}\neq\mathbf{0}\\ \mathbf{0} & \text{if } \mathbf{q}=\mathbf{0}\end{cases}$, and observe, similarly to (4.101) and (4.117), that

$$\mathbf{q}_r \,\in\, \mathbf{L}^r(\Omega) \qquad \text{and} \qquad \int_\Omega \mathbf{q}\cdot\mathbf{q}_r \,=\, \|\mathbf{q}\|^s_{0,s;\Omega}\,. \tag{4.118}$$

In this way, noting from Lemma 4.3 that $D_r(\mathbf{q}_r)\in K_2$ (cf. (4.112)), and using (4.114) and the identity in (4.118), we obtain

$$\sup_{\mathbf{r}\in K_2} a(\mathbf{r},\mathbf{q}) \,\geq\, \int_\Omega D_r(\mathbf{q}_r)\cdot\mathbf{q} \,=\, \int_\Omega \mathbf{q}_r\cdot\mathbf{q} \,=\, \|\mathbf{q}\|^s_{0,s;\Omega} \,>\, 0\,,$$

which shows (4.116) and finishes the proof of the lemma. □

The continuous inf-sup conditions for the bilinear forms b_i, $i\in\{1,2\}$, which resemble, though with some differences, the results given by [52, Lemma 2.7], are established in the following lemma.

Lemma 4.5. *For each $i\in\{1,2\}$ there exists a positive constant β_i such that*

$$\sup_{\substack{\mathbf{q}\in X_i\\ \mathbf{q}\neq\mathbf{0}}} \frac{b_i(\mathbf{q},\phi)}{\|\mathbf{q}\|_{X_i}} \,\geq\, \beta_i\,\|\phi\|_{M_i} \qquad \forall\,\phi\in M_i\,. \tag{4.119}$$

Proof. For the case $i=1$, in which $X_i = \mathbf{H}^s(\mathrm{div}_s;\Omega)$ and $M_i = \mathrm{L}^r(\Omega)$, with r and s conjugate to each other (cf. (4.31)), the present proof proceeds similarly to that of [52, Lemma 2.7], except for the fact that the boundary conditions of the auxiliary problems utilized are homogeneous Dirichlet and Neumann, respectively. We omit further details and refer to [52, Lemma 2.7]. On the other hand, for the case $i = 2$, in which $X_i = \mathbf{H}^r(\mathrm{div}_j;\Omega)$ and $M_i = \mathrm{L}^l(\Omega)$, with j and l conjugate to each other (cf. (4.32)), we first let $\mathcal{O}$ be a bounded convex polygonal domain containing $\bar{\Omega}$. Then, given $\phi \in M_2 = \mathrm{L}^l(\Omega)$, we recall from (4.32) that $l \geq 2$, set $\phi_j := |\phi|^{l-2}\,\phi$, and observe, as before, that

$$\phi_j \in \mathrm{L}^j(\Omega) \qquad \text{and} \qquad \int_\Omega \phi\,\phi_j = \|\phi\|_{0,l;\Omega}\,\|\phi_j\|_{0,j;\Omega}\,. \tag{4.120}$$

Next, we define $g := \begin{cases} \phi_j & \text{in} \quad \Omega\,, \\ 0 & \text{in} \quad \mathcal{O}\setminus\bar{\Omega}\,. \end{cases}$, which clearly belongs to $\mathrm{L}^j(\mathcal{O})$, and deduce, applying [43, Corollary 1] to $j \in (1,2]$ (cf. (4.32)), that there exists a unique $z \in \mathrm{W}^{1,j}_0(\mathcal{O}) \cap \mathrm{W}^{2,j}(\mathcal{O})$ such that

$$\Delta z = g \quad \text{in} \quad \mathcal{O}\,, \quad z = 0 \quad \text{on} \quad \partial\mathcal{O}\,,$$

and

$$\|z\|_{2,j;\mathcal{O}} \leq C_j\,\|g\|_{0,j;\mathcal{O}} = C_j\,\|\phi_j\|_{0,j;\Omega}\,,$$

with a positive constant C_j depending only on j and $\mathcal{O}$. Thus, letting $\bar{\mathbf{q}} := \nabla z|_\Omega \in \mathrm{W}^{1,j}(\Omega)$, it follows that $\mathbf{div}(\bar{\mathbf{q}}) = \phi_j$ in Ω, whereas using the continuous embedding $i_{j,r}$ from $\mathrm{W}^{1,j}(\Omega)$ into $\mathrm{L}^r(\Omega)$, which is valid (cf. [41, Corollary B.43]) for the ranges of r and j specified in (4.31) and (4.32), respectively, we get

$$\|\bar{\mathbf{q}}\|_{0,r;\Omega} \leq \|i_{j,r}\|\,\|\bar{\mathbf{q}}\|_{1,j;\Omega} \leq \|i_{j,r}\|\,\|z\|_{2,j;\mathcal{O}} \leq \|i_{j,r}\|\,C_j\,\|\phi_j\|_{0,j;\Omega}\,.$$

In this way, we conclude that $\bar{\mathbf{q}} \in X_2 := \mathbf{H}^r(\mathrm{div}_j;\Omega)$, and that

$$\|\bar{\mathbf{q}}\|_{X_2} = \|\bar{\mathbf{q}}\|_{0,r;\Omega} + \|\mathbf{div}(\bar{\mathbf{q}})\|_{0,j;\Omega} \leq \big(1 + \|i_{j,r}\|\,C_j\big)\,\|\phi_j\|_{0,j;\Omega}\,,$$

whence, using the identity in (4.120) as well, we find that

$$\sup_{\substack{\mathbf{q}\in X_2\\ \mathbf{q}\neq\mathbf{0}}} \frac{b_2(\mathbf{q},\phi)}{\|\mathbf{q}\|_{X_2}} \geq \frac{b_2(\bar{\mathbf{q}},\phi)}{\|\bar{\mathbf{q}}\|_{X_2}} \geq \frac{1}{\big(1 + \|i_{j,r}\|\,C_j\big)}\,\frac{\displaystyle\int_\Omega \phi\,\phi_j}{\|\phi_j\|_{0,j;\Omega}} = \frac{1}{\big(1 + \|i_{j,r}\|\,C_j\big)}\,\|\phi\|_{0,l;\Omega}\,,$$

which proves (4.119) with $\beta_2 = \big(1 + \|i_{j,r}\|\,C_j\big)^{-1}$. □

We now stress, in virtue of the discussion in Section 4.2.4 and the analysis developed in [52], that when the boundary conditions (4.70) (which yields (4.71)) are considered, the spaces and bilinear forms resulting from the weak formulation of the chemical signal equations, namely X_2, M_1, X_1, M_2, a, b_1, and b_2, are actually the same ones that arise for the Darcy part of the coupled model studied in [52], except that the corresponding spaces X_2 and M_2 differ in some Lebesgue indexes involved. Consequently, being the proofs for Dirichlet boundary conditions basically the same from [52] for all the forms, except the one for b_2, which needs some additional technical aspects, as emphasized in the present section, we conclude that those for the case (4.70) do not differ much either from them.

According to Lemmas 4.4 and 4.5, the required hypotheses of [12, Theorem 2.1, Section 2.1] are satisfied, and hence the a priori estimates provided by [12, Corollary 2.1, Section 2.1] imply the existence of a positive constant α_{S}, depending only on α, β_1, β_2, and $\|a\|$, such that

$$\sup_{\substack{(\mathbf{q},\phi)\in X_1\times M_2\\ (\mathbf{q},\phi)\neq\mathbf{0}}} \frac{\mathbf{A}((\mathbf{r},\psi),(\mathbf{q},\phi))}{\|(\mathbf{q},\phi)\|_{X_1\times M_2}} \geq \alpha_{\mathrm{S}}\,\|(\mathbf{r},\psi)\|_{X_2\times M_1} \qquad \forall\,(\mathbf{r},\psi) \in X_2\times M_1\,, \tag{4.121}$$

and

$$\sup_{\substack{(\mathbf{r},\psi)\in X_2\times M_1\\ (\mathbf{r},\psi)\neq\mathbf{0}}} \frac{\mathbf{A}((\mathbf{r},\psi),(\mathbf{q},\phi))}{\|(\mathbf{r},\psi)\|_{X_2\times M_1}} \,\geq\, \alpha_{\mathrm{S}}\,\|(\mathbf{q},\phi)\|_{X_1\times M_2} \qquad \forall\,(\mathbf{q},\phi)\in X_1\times M_2\,. \tag{4.122}$$

Now, we let $\mathbf{A}_{\mathbf{z},\chi}:(X_2\times M_1)\times(X_1\times M_2)\to \mathrm{R}$ be the bounded bilinear form arising from (4.83) after adding the full left-hand sides of its equations, that is

$$\begin{gathered}\mathbf{A}_{\mathbf{z},\chi}((\mathbf{r},\psi),(\mathbf{q},\phi)) \,:=\, \mathbf{A}((\mathbf{r},\psi),(\mathbf{q},\phi)) \,-\, c_{\mathbf{z},\chi}\big((\mathbf{r},\psi),\phi\big)\\ \forall\,(\mathbf{r},\psi)\in\big(X_2\times M_1\big),\quad \forall\,(\mathbf{q},\phi)\in\big(X_1\times M_2\big)\,,\end{gathered} \tag{4.123}$$

and realize that (4.83) can be rewritten, equivalently, as: Find $(\mathbf{p},\varphi)\,\in\,X_2\times M_1$ such that

$$\mathbf{A}_{\mathbf{z},\chi}((\mathbf{p},\varphi),(\mathbf{q},\phi)) \,=\, \mathrm{F}(\mathbf{q}) \,+\, \mathrm{G}(\phi) \qquad \forall\,(\mathbf{q},\phi)\,\in\,X_1\times M_2\,. \tag{4.124}$$

Note that the boundedness of $\mathbf{A}$ and $c_{\mathbf{z},\chi}$ (cf. (4.65)) guarantees that $\mathbf{A}_{\mathbf{z},\chi}$ is bounded as well. Thus, bearing in mind (4.123), and employing (4.121) and (4.65), we find that for each $(\mathbf{r},\psi)\,\in\,X_2\times M_1$ there holds

$$\sup_{\substack{(\mathbf{q},\phi)\in X_1\times M_2\\ (\mathbf{q},\phi)\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{z},\chi}((\mathbf{r},\psi),(\mathbf{q},\phi))}{\|(\mathbf{q},\phi)\|_{X_1\times M_2}} \,\geq\, \Big\{\alpha_{\mathrm{S}} \,-\, \|c\|\,\big(\|\mathbf{z}\|_{0,4;\Omega} \,+\, \|\chi\|_{0,\rho;\Omega}\big)\Big\}\,\|(\mathbf{r},\psi)\|_{X_2\times M_1}\,, \tag{4.125}$$

and then, under the assumption that

$$\|\mathbf{z}\|_{0,4;\Omega}+\|\chi\|_{0,\rho;\Omega} \,\leq\, \frac{\alpha_{\mathrm{S}}}{2\,\|c\|}\,, \tag{4.126}$$

we arrive at

$$\sup_{\substack{(\mathbf{q},\phi)\in X_1\times M_2\\ (\mathbf{q},\phi)\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{z},\chi}((\mathbf{r},\psi),(\mathbf{q},\phi))}{\|(\mathbf{q},\phi)\|_{X_1\times M_2}} \,\geq\, \frac{\alpha_{\mathrm{S}}}{2}\,\|(\mathbf{r},\psi)\|_{X_2\times M_1} \qquad \forall\,(\mathbf{r},\psi)\,\in\,X_2\times M_1\,. \tag{4.127}$$

Similarly, but employing now (4.122) instead of (4.121), and under the same assumption (4.126), we obtain

$$\sup_{\substack{(\mathbf{r},\psi)\in X_2\times M_1\\ (\mathbf{r},\psi)\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{z},\chi}((\mathbf{r},\psi),(\mathbf{q},\phi))}{\|(\mathbf{r},\psi)\|_{X_2\times M_1}} \,\geq\, \frac{\alpha_{\mathrm{S}}}{2}\,\|(\mathbf{q},\phi)\|_{X_1\times M_2} \qquad \forall\,(\mathbf{q},\phi)\,\in\,X_1\times M_2\,. \tag{4.128}$$

We are now in position to establish the well-definedness of S.

Lemma 4.6. *For each $(\mathbf{z},\chi)\in\mathbf{L}^4(\Omega)\times\mathrm{Q}$ satisfying (4.126), there exists a unique $(\mathbf{p},\varphi)\,\in\,X_2\times M_1$ solution of (4.124) (equivalently (4.83)), and hence one can define $\mathrm{S}(\mathbf{z},\chi)\,:=\,\mathbf{p}\in X_2$. Moreover, there exists a positive constant C_{S}, depending only on α_{S}, C_r, and κ_φ, such that*

$$\|\mathrm{S}(\mathbf{z},\chi)\|_{X_2} \,=\, \|\mathbf{p}\|_{X_2} \,\leq\, \|(\mathbf{p},\varphi)\|_{X_2\times M_1} \,\leq\, C_{\mathrm{S}}\,\Big\{\|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega}\Big\}\,. \tag{4.129}$$

Proof. Thanks to the boundedness of $\mathbf{A}_{\mathbf{z},\chi}$, and the global inf-sup conditions (4.127) and (4.128), a direct application of [41, Theorem 2.6] provides the existence of a unique solution $(\mathbf{p},\varphi)\,\in\,X_2\times M_1$ to (4.124). Moreover, the corresponding a priori estimate (cf. [41, Theorem 2.6, eq. (2.5)]) yields

$$\|(\mathbf{p},\varphi)\|_{X_2\times M_1} \,\leq\, \frac{2}{\alpha_{\mathrm{S}}}\,\Big\{\|\mathrm{F}\| \,+\, \|\mathrm{G}\|\Big\}\,,$$

which, together with the expressions for $\|\mathrm{F}\|$ and $\|\mathrm{G}\|$ given in (4.68) and (4.62), imply (4.129) with $C_{\mathrm{S}}\,:=\,\frac{2}{\alpha_{\mathrm{S}}}\,\max\big\{C_r,\kappa_\varphi^{-1}\big\}$. □

4.3.3 Solvability analysis of the fixed-point equation

Knowing that the operators $\mathbf{S}$, $\widetilde{\mathrm{S}}$, S and hence T as well, are well-defined, in this section we address the solvability of the fixed point equation (4.84). To this end, in what follows we aim to verify the hypotheses of the respective Banach Theorem. We begin the analysis by establishing sufficient conditions under which T maps a closed ball of $\mathbf{L}^4(\Omega) \times X_2$ into itself. Indeed, given a radius δ to be explicitly defined later on, we first set

$$W_\delta := \Big\{ (\mathbf{z}, \mathbf{r}) \in \mathbf{L}^4(\Omega) \times X_2 : \quad \|(\mathbf{z}, \mathbf{r})\| := \|\mathbf{z}\|_{0,4;\Omega} + \|\mathbf{r}\|_{X_2} \le \delta \Big\}. \tag{4.130}$$

Then, given $(\mathbf{z}, \mathbf{r}) \in W_\delta$, we have from the a priori estimate for $\mathbf{S}$ (cf. (4.94) in Lemma 4.1) that

$$\begin{aligned} &\|\mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big)\|_{0,4;\Omega} \\ &\quad \le C_{\mathbf{S}} \Big\{ \|\widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\|_{0,\rho;\Omega} \, \|\nabla f\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega} + \big(1 + \|\mathbf{z}\|_{0,4;\Omega}\big) \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\}, \end{aligned} \tag{4.131}$$

from which, using the corresponding estimate for $\widetilde{\mathrm{S}}$ (cf. (4.109), Lemma 4.2), and assuming (cf. (4.106))

$$\|\mathbf{z}\|_{0,4;O} + \|\mathbf{r}\|_{0,r;\Omega} \le \frac{\alpha_{\widetilde{\mathrm{S}}}}{2\,\|\widetilde{c}\|}, \tag{4.132}$$

we get

$$\begin{aligned} \|\mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big)\|_{0,4;\Omega} \le C_{\mathbf{S}} \Big\{ & C_{\widetilde{\mathrm{S}}} \left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega} \right) \|\nabla f\|_{0,r;\Omega} \\ & + \|\mathbf{f}\|_{0,4/3;\Omega} + \big(1 + \|\mathbf{z}\|_{0,4;\Omega}\big) \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\}. \end{aligned} \tag{4.133}$$

Furthermore, supposing now that (cf. (4.126))

$$\|\mathbf{z}\|_{0,4;O} + \|\widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\|_{\mathrm{Q}} \le \frac{\alpha_{\mathrm{S}}}{2\,\|c\|}, \tag{4.134}$$

the a priori estimate for S (cf. (4.129) in Lemma 4.6) gives

$$\|\mathrm{S}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big)\|_{X_2} \le C_{\mathrm{S}} \Big\{ \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega} \Big\}. \tag{4.135}$$

Regarding (4.132), we observe that it is satisfied if we consider δ such that $\delta \le \frac{\alpha_{\widetilde{\mathrm{S}}}}{2\|\widetilde{c}\|}$. In turn, noting that certainly $\|\mathbf{z}\|_{0,4;\Omega} \le \delta$, and according to the estimate for $\|\widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\|_{\mathrm{Q}}$ (cf. (4.109)), we deduce that a sufficient condition for (4.134) is given by the assumptions

$$\delta \le \frac{\alpha_{\mathrm{S}}}{4\,\|c\|} \quad \text{and} \quad C_{\widetilde{\mathrm{S}}} \left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega} \right) \le \frac{\alpha_{\mathrm{S}}}{4\,\|c\|}. \tag{4.136}$$

In this way, defining

$$\delta := \min \left\{ \frac{\alpha_{\widetilde{\mathrm{S}}}}{2\,\|\widetilde{c}\|}, \frac{\alpha_{\mathrm{S}}}{4\,\|c\|} \right\}, \tag{4.137}$$

(4.132) and (4.134) are satisfied, whence (4.133) and (4.135) are valid, and thus, assuming the second inequality in (4.136), and recalling that $\|\mathrm{T}(\mathbf{z}, \mathbf{r})\| := \|\mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big)\|_{0,4;\Omega} + \|\mathrm{S}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big)\|_{X_2}$, we obtain

$$\begin{aligned} \|\mathrm{T}(\mathbf{z}, \mathbf{r})\| \le C(\delta) \Big\{ & \left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega} \right) \|\nabla f\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega} \\ & + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega} \Big\}, \end{aligned} \tag{4.138}$$

where $C(\delta)$ is a positive constant depending explicitly on $C_{\mathbf{S}}$, $C_{\widetilde{\mathrm{S}}}$, $(1+\delta)$, and C_{S}.

We have then proved the following result.

Lemma 4.7. *Assume that the data are sufficiently small so that*

$$C_{\widetilde{\mathrm{S}}}\left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\right) \,\leq\, \frac{\alpha_{\mathrm{S}}}{4\,\|c\|}\,, \tag{4.139}$$

and

$$\begin{aligned} C(\delta)\Big\{&\left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\right)\|\nabla f\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \\ +\;\; &\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega}\Big\} \,\leq\, \delta\,. \end{aligned} \tag{4.140}$$

Then, $\mathrm{T}(W_\delta) \,\subseteq\, W_\delta$.

We now aim to prove that the operator T is Lipschitz-continuous, for which, according to its definition (cf. (4.84)), it suffices to show that $\mathbf{S}$, $\widetilde{\mathrm{S}}$ and S satisfy suitable continuity properties. We begin with the corresponding result for $\mathbf{S}$.

Lemma 4.8. *There exists a positive constant* $L_{\mathbf{S}}$*, depending on* $\boldsymbol{\alpha}$*,* $|\Omega|$*, and* $\|\mathbf{c}\|$*, such that*

$$\begin{aligned} &\|\mathbf{S}(\mathbf{z},\chi) \,-\, \mathbf{S}(\mathbf{z}_0,\chi_0)\|_{\mathbf{H}} \\ &\qquad \leq\, L_{\mathbf{S}}\left\{\|\nabla f\|_{0,r;\Omega}\,\|\chi \,-\, \chi_0\|_{0,\rho;\Omega} \,+\, \boldsymbol{\mathcal{F}}(\mathbf{z}_0,\chi_0)\,\|\mathbf{z} \,-\, \mathbf{z}_0\|_{0,4;\Omega}\right\} \end{aligned} \tag{4.141}$$

for all $(\mathbf{z},\chi),\ (\mathbf{z}_0,\chi_0) \in \mathbf{L}^4(\Omega)\times \mathrm{Q}$*, where*

$$\boldsymbol{\mathcal{F}}(\mathbf{z}_0,\chi_0) \,:=\, C_{\mathbf{S}}\left\{\|\chi_0\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \left(1 + \|\mathbf{z}_0\|_{0,4;\Omega}\right)\|\mathbf{u}_D\|_{1/2,\Gamma}\right\}. \tag{4.142}$$

Proof. Given $(\mathbf{z},\chi),\ (\mathbf{z}_0,\chi_0) \in \mathbf{L}^4(\Omega)\times \mathrm{Q}$, we let $\mathbf{S}(\mathbf{z},\chi) := \mathbf{u} \in \mathrm{L}^4(\Omega)$ and $\mathbf{S}(\mathbf{z}_0,\chi_0) := \mathbf{u}_0 \in \mathrm{L}^4(\Omega)$, where $(\vec{\mathbf{u}},\boldsymbol{\sigma}) = \big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big) \in \mathbf{H}\times\mathbf{Q}$ and $(\vec{\mathbf{u}}_0, 0) = \big((\mathbf{u}_0,\mathbf{t}_0),0\big) \in \mathbf{H}\times\mathbf{Q}$ are the respective solutions of (4.79). It follows from the corresponding second equations of (4.79) that $\vec{\mathbf{u}} - \vec{\mathbf{u}}_0 \in \mathbf{V}$ (cf. (4.88)), and then the $\mathbf{V}-$ellipticity of $\mathbf{a}$ (cf. (4.89)) gives

$$\boldsymbol{\alpha}\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|^2_{\mathbf{H}} \,\leq\, \mathbf{a}(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0)\,. \tag{4.143}$$

In turn, applying the corresponding first equations of (4.79) to $\vec{\mathbf{v}} = \vec{\mathbf{u}} - \vec{\mathbf{u}}_0$, we obtain

$$\mathbf{a}(\vec{\mathbf{u}}, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,+\, \mathbf{c}(\mathbf{z};\vec{\mathbf{u}}, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,=\, \mathbf{F}_\chi(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0)\,, \tag{4.144}$$

and

$$\mathbf{a}(\vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,+\, \mathbf{c}(\mathbf{z}_0;\vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,=\, \mathbf{F}_{\chi_0}(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0)\,, \tag{4.145}$$

so that, subtracting (4.145) from (4.144), and using, thanks to the bilinearity of $\mathbf{c}(\mathbf{z};\cdot,\cdot)$ and (4.26), that

$$\mathbf{c}(\mathbf{z};\vec{\mathbf{u}}, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,=\, \mathbf{c}(\mathbf{z};\vec{\mathbf{u}} - \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,+\, \mathbf{c}(\mathbf{z};\vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,=\, \mathbf{c}(\mathbf{z};\vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0)\,,$$

we find

$$\mathbf{a}(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,=\, \big(\mathbf{F}_\chi - \mathbf{F}_{\chi_0}\big)(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,+\, \mathbf{c}(\mathbf{z}_0 - \mathbf{z};\vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0)\,. \tag{4.146}$$

Next, utilizing (4.48), we get

$$\begin{aligned} \big(\mathbf{F}_\chi - \mathbf{F}_{\chi_0}\big)(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,&=\, \int_\Omega (\chi - \chi_0)\,\nabla f\cdot(\mathbf{u} - \mathbf{u}_0) \\ &\leq\, |\Omega|^{1/4}\,\|\chi - \chi_0\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega}\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_{\mathbf{H}}\,, \end{aligned} \tag{4.147}$$

whereas the boundedness property of $\mathbf{c}$ (cf. (4.24)) yields

$$\mathbf{c}(\mathbf{z}_0 - \mathbf{z}; \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,\le\, \|\mathbf{c}\| \, \|\mathbf{z} - \mathbf{z}_0\|_{0,4;\Omega} \, \|\vec{\mathbf{u}}_0\|_{\mathbf{H}} \, \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_{\mathbf{H}} \,. \tag{4.148}$$

Finally, employing (4.147) and (4.148) in (4.146), replacing the resulting estimate back into (4.143), simplifying by $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_{\mathbf{H}}$, and bounding $\|\vec{\mathbf{u}}_0\|_{\mathbf{H}}$ by the corresponding upper bound in (4.94), we arrive at the required inequality (4.141) with $L_{\mathbf{S}} := \boldsymbol{\alpha}^{-1} \max\big\{|\Omega|^{1/4}, \|\mathbf{c}\|\big\}$. □

The continuity of $\widetilde{\mathrm{S}}$ is addressed next. More precisely, we have the following result.

Lemma 4.9. *There exists a positive constant $L_{\widetilde{\mathrm{S}}}$, depending only on $\|\widetilde{c}\|$, $\alpha_{\widetilde{\mathrm{S}}}$, and $C_{\widetilde{\mathrm{S}}}$, such that*

$$\begin{aligned} &\|\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r}) \,-\, \widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\|_{\mathrm{Q}} \\ &\qquad \le\, L_{\widetilde{\mathrm{S}}} \Big\{ \|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega} \Big\} \, \|(\mathbf{z},\mathbf{r}) \,-\, (\mathbf{z}_0,\mathbf{r}_0)\| \end{aligned} \tag{4.149}$$

for all $(\mathbf{z},\mathbf{r})$, $(\mathbf{z}_0,\mathbf{r}_0) \in \mathbf{L}^4(\Omega) \times X_2$ satisfying (4.106).

Proof. Given $(\mathbf{z},\mathbf{r})$, $(\mathbf{z}_0,\mathbf{r}_0) \in \mathbf{L}^4(\Omega) \times X_2$, we let $\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r}) := \eta \in \mathrm{Q}$ and $\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0) := \eta_0 \in \mathrm{Q}$, where $(\widetilde{\boldsymbol{\sigma}},\eta) \in \mathrm{H} \times \mathrm{Q}$ and $(\widetilde{\boldsymbol{\sigma}}_0,\eta_0) \in \mathrm{H} \times \mathrm{Q}$ are the respective solutions of (4.81), equivalently (4.105), that is

$$\widetilde{A}_{\mathbf{z},\mathbf{r}}\big((\widetilde{\boldsymbol{\sigma}},\eta),(\widetilde{\boldsymbol{\tau}},\xi)\big) \,=\, \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{\mathrm{G}}(\xi) \qquad \forall\, (\widetilde{\boldsymbol{\tau}},\xi) \in \mathrm{H} \times \mathrm{Q}\,, \tag{4.150}$$

and

$$\widetilde{A}_{\mathbf{z}_0,\mathbf{r}_0}\big((\widetilde{\boldsymbol{\sigma}}_0,\eta_0),(\widetilde{\boldsymbol{\tau}},\xi)\big) \,=\, \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{\mathrm{G}}(\xi) \qquad \forall\, (\widetilde{\boldsymbol{\tau}},\xi) \in \mathrm{H} \times \mathrm{Q}\,. \tag{4.151}$$

It follows from the foregoing identities and the definitions of $\widetilde{A}_{\mathbf{z},\mathbf{r}}$ (cf. (4.104)) and $\widetilde{c}_{\mathbf{z},\mathbf{q}}$ (cf. (4.34)) that

$$\begin{aligned} \widetilde{A}_{\mathbf{z},\mathbf{r}}\big((\widetilde{\boldsymbol{\sigma}},\eta) - (\widetilde{\boldsymbol{\sigma}}_0,\eta_0),(\widetilde{\boldsymbol{\tau}},\xi)\big) \,&=\, \widetilde{A}_{\mathbf{z},\mathbf{r}}\big((\widetilde{\boldsymbol{\sigma}},\eta),(\widetilde{\boldsymbol{\tau}},\xi)\big) - \widetilde{A}_{\mathbf{z},\mathbf{r}}\big((\widetilde{\boldsymbol{\sigma}}_0,\eta_0),(\widetilde{\boldsymbol{\tau}},\xi)\big) \\ =\, \widetilde{A}_{\mathbf{z}_0,\mathbf{r}_0}\big((\widetilde{\boldsymbol{\sigma}}_0,\eta_0),(\widetilde{\boldsymbol{\tau}},\xi)\big) &- \widetilde{A}_{\mathbf{z},\mathbf{r}}\big((\widetilde{\boldsymbol{\sigma}}_0,\eta_0),(\widetilde{\boldsymbol{\tau}},\xi)\big) \,=\, \widetilde{c}_{\mathbf{z}_0-\mathbf{z},\mathbf{r}_0-\mathbf{r}}(\widetilde{\boldsymbol{\tau}},\eta_0)\,, \end{aligned} \tag{4.152}$$

and hence, applying the global inf-sup condition (4.107) to $(\widetilde{\boldsymbol{\sigma}},\eta) - (\widetilde{\boldsymbol{\sigma}}_0,\eta_0)$, and employing (4.152) and the boundedness of $\widetilde{c}_{\mathbf{z},\mathbf{r}}$ (cf. (4.45)), we find that

$$\begin{aligned} \|(\widetilde{\boldsymbol{\sigma}},\eta) - (\widetilde{\boldsymbol{\sigma}}_0,\eta_0)\|_{\mathrm{H}\times\mathrm{Q}} \,&\le\, \frac{2}{\alpha_{\widetilde{\mathrm{S}}}} \sup_{\substack{(\widetilde{\boldsymbol{\tau}},\xi)\in\mathrm{H}\times\mathrm{Q} \\ (\widetilde{\boldsymbol{\tau}},\xi)\neq\mathbf{0}}} \frac{\widetilde{c}_{\mathbf{z}_0-\mathbf{z},\mathbf{r}_0-\mathbf{r}}(\widetilde{\boldsymbol{\tau}},\eta_0)}{\|(\widetilde{\boldsymbol{\tau}},\xi)\|_{\mathrm{H}\times\mathrm{Q}}} \\ &\le\, \frac{2\,\|\widetilde{c}\|}{\alpha_{\widetilde{\mathrm{S}}}} \, \|\eta_0\|_{\mathrm{Q}} \Big\{ \|\mathbf{z} - \mathbf{z}\|_{0,4;\Omega} + \|\mathbf{r} - \mathbf{r}_0\|_{0,r;\Omega} \Big\}\,, \end{aligned}$$

which, together with the a priori estimate (4.109) for $\|\eta_0\|_{\mathrm{Q}}$, yields (4.149) with $L_{\widetilde{\mathrm{S}}} := 2\,\|\widetilde{c}\|\,\alpha_{\widetilde{\mathrm{S}}}^{-1}\, C_{\widetilde{\mathrm{S}}}$. □

It remains to establish the continuity of S, which is the purpose of the following lemma.

Lemma 4.10. *There exists a positive constant L_{S}, depending only on $\|c\|$, α_{S}, and C_{S}, such that*

$$\begin{aligned} &\|\mathrm{S}(\mathbf{z},\chi) \,-\, \mathrm{S}(\mathbf{z}_0,\chi_0)\|_{X_2} \\ &\qquad \le\, L_{\mathrm{S}} \Big\{ \|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega} \Big\} \, \|(\mathbf{z},\chi) \,-\, (\mathbf{z}_0,\chi_0)\| \end{aligned} \tag{4.153}$$

for all $(\mathbf{z},\chi)$, $(\mathbf{z}_0,\chi_0) \in \mathbf{L}^4(\Omega) \times \mathrm{Q}$ satisfying (4.126).

Proof. Given $(\mathbf{z},\chi)$, $(\mathbf{z}_0,\chi_0) \in \mathbf{L}^4(\Omega)\times \mathrm{Q}$ as indicated, we proceed similarly to the proof of Lemma 4.9 and let $\mathrm{S}(\mathbf{z},\chi) := \mathbf{p} \in X_2$ and $\mathrm{S}(\mathbf{z}_0,\chi_0) := \mathbf{p}_0 \in X_2$, where $(\mathbf{p},\varphi) \in X_2 \times M_1$ and $(\mathbf{p}_0,\varphi_0) \in X_2 \times M_1$ are the respective solutions of (4.83), equivalently (4.124), that is

$$\mathbf{A}_{\mathbf{z},\chi}\big((\mathbf{p},\varphi),(\mathbf{q},\phi)\big) \,=\, \mathrm{F}(\mathbf{q}) \,+\, \mathrm{G}(\phi) \qquad \forall\,(\mathbf{q},\phi)\,\in\, X_1\times M_2\,, \tag{4.154}$$

and

$$\mathbf{A}_{\mathbf{z}_0,\chi_0}\big((\mathbf{p}_0,\varphi_0),(\mathbf{q},\phi)\big) \,=\, \mathrm{F}(\mathbf{q}) \,+\, \mathrm{G}(\phi) \qquad \forall\,(\mathbf{q},\phi)\,\in\, X_1\times M_2\,. \tag{4.155}$$

Next, proceeding analogously to the derivation of (4.152), we deduce from the identities (4.154) and (4.155), along with the definitions of $\mathbf{A}_{\mathbf{z},\chi}$ (cf. (4.123)) and $c_{\mathbf{z},\chi}$ (cf. (4.59)) that

$$\mathbf{A}_{\mathbf{z},\chi}\big((\mathbf{p},\varphi)-(\mathbf{p}_0,\varphi_0),(\mathbf{q},\phi)\big) \,=\, c_{\mathbf{z}-\mathbf{z}_0,\chi-\chi_0}\big((\mathbf{p}_0,\varphi_0),\phi\big)\,, \tag{4.156}$$

and thus, applying the global inf-sup condition (4.127) to $(\mathbf{p},\varphi)-(\mathbf{p}_0,\varphi_0)$, and making use of (4.156) and the boundedness of $c_{\mathbf{z},\chi}$ (cf. (4.65)), we get

$$\begin{aligned}\|(\mathbf{p},\varphi)-(\mathbf{p}_0,\varphi_0)\|_{X_2\times M_1} \,&\leq\, \frac{2}{\alpha_{\mathrm{S}}} \sup_{\substack{(\mathbf{q},\phi)\in X_1\times M_2\\ (\mathbf{q},\phi)\neq\mathbf{0}}} \frac{c_{\mathbf{z}-\mathbf{z}_0,\chi-\chi_0}\big((\mathbf{p}_0,\varphi_0),\phi\big)}{\|(\mathbf{q},\phi)\|_{X_1\times M_2}}\\ &\leq\, \frac{2\,\|c\|}{\alpha_{\mathrm{S}}}\,\|(\mathbf{p}_0,\varphi_0)\|_{X_2\times M_1}\Big\{\|\mathbf{z}-\mathbf{z}_0\|_{0,4;\Omega} \,+\, \|\chi-\chi_0\|_{0,\rho;\Omega}\Big\}\,,\end{aligned}$$

which, together with the a priori estimate (4.129) for $\|(\mathbf{p}_0,\varphi_0)\|_{X_2\times M_1}$, yields (4.153) with $L_{\mathrm{S}} := 2\,\|c\|\,\alpha_{\mathrm{S}}^{-1}\,C_{\mathrm{S}}$. □

Having proved Lemmas 4.8, 4.9 and 4.10, we now aim to establish the continuity property of the fixed point operator T in the closed ball W_δ (cf. (4.130)). Indeed, given $(\mathbf{z},\mathbf{r})$, $(\mathbf{z}_0,\mathbf{r}_0) \in W_\delta$, we first observe from the definition of T (cf. (4.84)) that

$$\begin{aligned}\|\mathrm{T}(\mathbf{z},\mathbf{r})-\mathrm{T}(\mathbf{z}_0,\mathbf{r}_0)\| \,&=\, \|\mathbf{S}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big)-\mathbf{S}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\|_{0,4;\Omega}\\ &\quad+\,\|\mathrm{S}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big)-\mathrm{S}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\|_{X_2}\,.\end{aligned} \tag{4.157}$$

Then, employing the continuity properties of $\mathbf{S}$ (cf. Lemma 4.8, (4.141)) and $\widetilde{\mathrm{S}}$ (cf. Lemma 4.9, (4.149)), we find that

$$\begin{aligned}&\|\mathbf{S}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big)-\mathbf{S}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\|_{0,4;\Omega}\\ &\quad\leq\, L_{\mathbf{S}}\Big\{\|\nabla f\|_{0,r;\Omega}\,\|\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\,-\,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\|_{0,\rho;\Omega}\,+\,\boldsymbol{\mathcal{F}}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\,\|\mathbf{z}\,-\,\mathbf{z}_0\|_{0,4;\Omega}\Big\}\\ &\quad\leq\, L_{\mathbf{S}}\Big\{L_{\widetilde{\mathrm{S}}}\left(\|\eta_D\|_{1/2,\Gamma}\,+\,\|f_\eta\|_{0,\varrho;\Omega}\right)\|\nabla f\|_{0,r;\Omega}+\boldsymbol{\mathcal{F}}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\Big\}\,\|(\mathbf{z},\mathbf{r})-(\mathbf{z}_0,\mathbf{r}_0)\|\end{aligned} \tag{4.158}$$

whereas (4.142) and the a priori estimate of $\widetilde{\mathrm{S}}$ (cf. (4.109)) gives

$$\begin{aligned}&\boldsymbol{\mathcal{F}}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\\ &\quad\leq\, C_{\mathbf{S}}\Big\{\|\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega}\,+\,\|\mathbf{f}\|_{0,4/3;\Omega}\,+\,\big(1+\|\mathbf{z}_0\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\\ &\quad\leq\, C_{\mathbf{S}}\Big\{C_{\widetilde{\mathrm{S}}}\left(\|\eta_D\|_{1/2,\Gamma}\,+\,\|f_\eta\|_{0,\varrho;\Omega}\right)\|\nabla f\|_{0,r;\Omega}\,+\,\|\mathbf{f}\|_{0,4/3;\Omega}\,+\,\big(1+\|\mathbf{z}_0\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,.\end{aligned} \tag{4.159}$$

In this way, replacing the bound from (4.159) into (4.158), and using that $\|\mathbf{z}_0\|_{0,4;\Omega} \leq \delta$, we deduce the existence of a positive constant $L_{\mathrm{T},\mathbf{S}}$, depending only on $L_{\mathbf{S}}$, $L_{\widetilde{\mathrm{S}}}$, $C_{\mathbf{S}}$, $C_{\widetilde{\mathrm{S}}}$, and δ, such that

$$\begin{aligned} \|\mathbf{S}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big) - \mathbf{S}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\|_{0,4;\Omega} \,\leq\, L_{\mathrm{T},\mathbf{S}} \Big\{ \big(\|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\big) \,\|\nabla f\|_{0,r;\Omega} \\ +\quad \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\} \,\|(\mathbf{z},\mathbf{r}) - (\mathbf{z}_0,\mathbf{r}_0)\|\,. \end{aligned} \tag{4.160}$$

In turn, proceeding similarly as before, but applying now the continuity properties of S (cf. Lemma 4.10, (4.153)) and $\widetilde{\mathrm{S}}$ (cf. Lemma 4.9, (4.149)), we arrive at

$$\begin{aligned} &\|\mathrm{S}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big) - \mathrm{S}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\|_{X_2} \\ &\leq\, L_{\mathrm{T,S}}\,\big(1 \,+\, \|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\big) \Big\{ \|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega} \Big\} \,\|(\mathbf{z},\mathbf{r}) - (\mathbf{z}_0,\mathbf{r}_0)\|\,, \end{aligned} \tag{4.161}$$

where $L_{\mathrm{T,S}}$ is a positive constant depending only on L_{S} and $L_{\widetilde{\mathrm{S}}}$.

Defining $L_{\mathrm{T}} \,:=\, \max\big\{L_{\mathrm{T},\mathbf{S}}, L_{\mathrm{T,S}}\big\}$, we summarize the above discussion in the following result.

Lemma 4.11. *Assume* (4.139)*, that is*

$$C_{\widetilde{\mathrm{S}}}\,\big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big) \,\leq\, \frac{\alpha_{\mathrm{S}}}{4\,\|c\|}\,.$$

Then, there holds

$$\begin{aligned} \|\mathrm{T}(\mathbf{z},\mathbf{r}) - \mathrm{T}(\mathbf{z}_0,\mathbf{r}_0)\| \leq L_{\mathrm{T}}\Big\{ \big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big)\big(\|\nabla f\|_{0,r;\Omega} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\big) \\ +\quad \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\}\|(\mathbf{z},\mathbf{r}) - (\mathbf{z}_0,\mathbf{r}_0)\|\,, \end{aligned} \tag{4.162}$$

for all $(\mathbf{z},\mathbf{r})$*,* $(\mathbf{z}_0,\mathbf{r}_0) \in W_\delta$*.*

Proof. We first stress that (4.139) is assumed here to ensure that both $\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big)$ and $\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)$ verify the hypothesis (4.126), which is required by the definition of S and its continuity property. Then, it is readily seen that (4.162) follows directly from (4.157), (4.160), and (4.161) □

The main result of this section is hence stated as follows.

Theorem 4.12. *Assume that the data are sufficiently small so that* (4.139) *and* (4.140) *hold. In addition, suppose that*

$$\begin{aligned} L_{\mathrm{T}} \Big\{ \big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big)\big(\|\nabla f\|_{0,r;\Omega} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\big) \\ +\quad \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\} \,<\, 1\,. \end{aligned} \tag{4.163}$$

Then, the operator T *has a unique fixed point* $(\mathbf{u},\mathbf{p}) \in W_\delta$*. Equivalently, the coupled problem* (4.69) *has a unique solution* $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{H}\times\mathbf{Q}$*,* $(\widetilde{\boldsymbol{\sigma}},\eta) \in \mathrm{H}\times\mathrm{Q}$*, and* $(\mathbf{p},\varphi) \in X_2 \times M_1$*, with* $(\mathbf{u},\mathbf{p}) \in W_\delta$*. Moreover, there hold the following a priori estimates*

$$\|(\vec{\mathbf{u}},\boldsymbol{\sigma})\|_{\mathbf{H}\times\mathbf{Q}} \,\leq\, C_{\vec{\mathbf{u}},\boldsymbol{\sigma}} \Big\{ \|\nabla f\|_{0,r;\Omega}\,\big(\|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\big) \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\}\,,$$

$$\|(\widetilde{\boldsymbol{\sigma}},\eta)\|_{\mathrm{H}\times\mathrm{Q}} \,\leq\, C_{\widetilde{\mathrm{S}}} \Big\{ \|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega} \Big\}\,,$$

$$\|(\mathbf{p},\varphi)\|_{X_2\times M_1} \,\leq\, C_{\mathrm{S}} \Big\{ \|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega} \Big\}\,,$$

where $C_{\vec{\mathbf{u}},\boldsymbol{\sigma}}$ *is a positive constant depending only on* $C_{\mathbf{S}}$*,* $\bar{C}_{\mathbf{S}}$*,* $C_{\widetilde{\mathrm{S}}}$*, and* δ*.*

Proof. We begin by recalling from Lemma 4.7 that (4.139) and (4.140) guarantee that T maps W_δ into itself. Hence, in virtue of the equivalence between (4.69) and (4.85), and bearing in mind the Lipschitz-continuity of T (cf. Lemma 4.11) and the hypothesis (4.163), a straightforward application of the Banach fixed point Theorem implies the existence of a unique solution $(\mathbf{u},\mathbf{p}) \in W_\delta$ of (4.69). In addition, the a priori estimates follow straightforwardly from (4.94), (4.97), (4.109) and (4.129), and bounding $\|\mathbf{u}\|_{0,4;\Omega}$ by δ. □

4.4 The Galerkin scheme

In this section we introduce the Galerkin scheme of the fully-mixed formulation (4.69), analyse its solvability by employing a discrete version of the fixed point strategy introduced in Section 4.3.1, and develop the corresponding a priori error analysis.

4.4.1 Preliminaries

We begin by considering arbitrary finite element subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, H_h, Q_h, $X_{2,h}$, $M_{1,h}$, $X_{1,h}$ and $M_{2,h}$ of the spaces $\mathbf{L}^4(\Omega)$, $\mathbb{L}^2_{\mathbf{tr}}(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, H, Q, X_2, M_1, X_1, and M_2, respectively. Hereafter, h stands for both the sub-index of each foregoing subspace and the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $n=2$) or tetrahedra K (when $n=3$) of diameter h_K, that is $h := \max\left\{h_K : \quad K \in \mathcal{T}_h\right\}$. Specific finite element subspaces satisfying the stability conditions to be introduced along the analysis will be provided later on in Section 4.5. Then, defining the spaces

$$\mathbf{H}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}}, \quad \mathbf{Q}_h := \mathbb{H}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega),$$

and setting the notations

$$\vec{\mathbf{u}}_h := (\mathbf{u}_h, \mathbf{t}_h), \quad \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{H}_h,$$

the Galerkin scheme associated with (4.69) reads: Find $(\vec{\mathbf{u}}_h, h) \in \mathbf{H}_h \times \mathbf{Q}_h$, $(\widetilde{\boldsymbol{\sigma}}_h, \eta_h) \in \mathrm{H}_h \times \mathrm{Q}_h$, and $(\mathbf{p}_h, \varphi_h) \in X_{2,h} \times M_{1,h}$, such that

$$\begin{aligned}
\mathbf{a}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{c}(\mathbf{u}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{b}(\vec{\mathbf{v}}_h, h) &= \mathbf{F}_{\eta_h}(\vec{\mathbf{v}}_h) & \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h\,,\\
\mathbf{b}(\vec{\mathbf{u}}_h, \widetilde{a}_h) &= \mathbf{G}(\widetilde{a}_h) & \forall\, \widetilde{a}_h \in \mathbf{Q}_h\,,\\
\widetilde{a}(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{\tau}}_h) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \eta_h) + \widetilde{c}_{\mathbf{u}_h,\mathbf{p}_h}(\widetilde{\boldsymbol{\tau}}_h, \eta_h) &= \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}_h) & \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathrm{H}_h\,,\\
\widetilde{b}(\widetilde{\boldsymbol{\sigma}}_h, \xi_h) &= \widetilde{\mathrm{G}}(\xi_h) & \forall\, \xi_h \in \mathrm{Q}_h\,,\\
a(\mathbf{p}_h, \mathbf{q}_h) + b_1(\mathbf{q}_h, \varphi_h) &= \mathrm{F}(\mathbf{q}_h) & \forall\, \mathbf{q}_h \in X_{1,h}\,,\\
b_2(\mathbf{p}_h, \phi_h) - c_{\mathbf{u}_h,\eta_h}\big((\mathbf{p}_h, \varphi_h), \phi_h\big) &= \mathrm{G}(\phi_h) & \forall\, \phi_h \in M_{2,h}\,.
\end{aligned} \tag{4.164}$$

Following the remark at the end of Section 4.2, we now stress that the first, fourth, and sixth rows of (4.164), the first one with $\vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{0}) \in \mathbf{H}_h$, constitute the discrete conservation of momentum properties. The fact that they are satisfied in an approximate sense will become clear in Section 4.5.1.

Throughout the rest of this section, we adopt the discrete analogue of the fixed point strategy introduced in Section 4.3.1 to analyse the solvability of (4.164). According to it, we now let $\mathbf{S}_h$:

$\mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h \to \mathbf{H}_h^{\mathbf{u}}$ be the operator defined by

$$\mathbf{S}_h(\mathbf{z}_h, \chi_h) := \mathbf{u}_h \qquad \forall\, (\mathbf{z}_h, \chi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h\,, \tag{4.165}$$

where $(\vec{\mathbf{u}}_h, h) = \big((\mathbf{u}_h, \mathbf{t}_h), h\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the unique solution (to be confirmed) of the first and second rows of (4.164) when $\mathbf{c}(\mathbf{u}_h; \cdot, \cdot)$ and $\mathbf{F}_{\eta_h}$ are replaced by $\mathbf{c}(\mathbf{z}_h; \cdot, \cdot)$ and $\mathbf{F}_{\chi_h}$, respectively, that is

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \,+\, \mathbf{c}(\mathbf{z}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \,+\, \mathbf{b}(\vec{\mathbf{v}}_h, h) \;&=\; \mathbf{F}_{\chi_h}(\vec{\mathbf{v}}_h) \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h\,, \\ \mathbf{b}(\vec{\mathbf{u}}_h, \widetilde{a}_h) \;&=\; \mathbf{G}(\widetilde{a}_h) \qquad \forall\, \widetilde{a}_h \in \mathbf{Q}_h\,. \end{aligned} \tag{4.166}$$

In turn, we let $\widetilde{\mathrm{S}}_h : \mathbf{H}_h^{\mathbf{u}} \times X_{2,h} \to \mathrm{Q}_h$ be the operator given by

$$\widetilde{\mathrm{S}}_h(\mathbf{z}_h, \mathbf{r}_h) := \eta_h \qquad \forall\, (\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{H}_h^{\mathbf{u}} \times X_{2,h}\,, \tag{4.167}$$

where $(\widetilde{\boldsymbol{\sigma}}_h, \eta_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ is the unique solution (to be confirmed) of the third and fourth rows of (4.164) when $\widetilde{c}_{\mathbf{u}_h, \mathbf{p}_h}$ is replaced by $\widetilde{c}_{\mathbf{z}_h, \mathbf{r}_h}$, that is

$$\begin{aligned} \widetilde{a}(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{\tau}}_h) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \eta_h) \,+\, \widetilde{c}_{\mathbf{z}_h, \mathbf{r}_h}(\widetilde{\boldsymbol{\tau}}_h, \eta_h) \;&=\; \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}_h) \qquad \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathrm{H}_h\,, \\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}}_h, \xi_h) \;&=\; \widetilde{\mathrm{G}}(\xi_h) \qquad \forall\, \xi_h \in \mathrm{Q}_h\,. \end{aligned} \tag{4.168}$$

Similarly, we let $\mathrm{S}_h : \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h \to X_{2,h}$ be the operator given by

$$\mathrm{S}_h(\mathbf{z}_h, \chi_h) := \mathbf{p}_h \quad \forall\, (\mathbf{z}_h, \chi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h\,, \tag{4.169}$$

where $(\mathbf{p}_h, \varphi_h) \in X_{2,h} \times M_{1,h}$ is the unique solution (to be confirmed) of the fifth and sixth rows of (4.164) when $c_{\mathbf{u}_h, \eta_h}$ is replaced by $c_{\mathbf{z}_h, \chi_h}$, that is

$$\begin{aligned} a(\mathbf{p}_h, \mathbf{q}_h) \,+\, b_1(\mathbf{q}_h, \varphi_h) \;&=\; \mathrm{F}(\mathbf{q}_h) \qquad \forall\, \mathbf{q}_h \in X_{1,h}\,, \\ b_2(\mathbf{p}_h, \phi_h) \,-\, c_{\mathbf{z}_h, \chi_h}\big((\mathbf{p}_h, \varphi_h), \phi_h\big) \;&=\; \mathrm{G}(\phi_h) \qquad \forall\, \phi_h \in M_{2,h}\,. \end{aligned} \tag{4.170}$$

Finally, we define $\mathrm{T}_h : \mathbf{H}_h^{\mathbf{u}} \times X_{2,h} \to \mathbf{H}_h^{\mathbf{u}} \times X_{2,h}$ as

$$\mathrm{T}_h(\mathbf{z}_h, \mathbf{r}_h) := \Big(\mathbf{S}_h\big(\mathbf{z}_h, \widetilde{\mathrm{S}}_h(\mathbf{z}_h, \mathbf{r}_h)\big), \mathrm{S}_h\big(\mathbf{z}_h, \widetilde{\mathrm{S}}_h(\mathbf{z}_h, \mathbf{r}_h)\big)\Big) \qquad \forall\, (\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{H}_h^{\mathbf{u}} \times X_{2,h}\,, \tag{4.171}$$

and notice that solving (4.164) is equivalent to seeking a fixed point of T_h, that is: Find $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{H}_h^{\mathbf{u}} \times X_{2,h}$ such that

$$\mathrm{T}_h(\mathbf{u}_h, \mathbf{p}_h) \,=\, (\mathbf{u}_h, \mathbf{p}_h)\,. \tag{4.172}$$

4.4.2 Discrete solvability analysis

Similarly to the approach from Section 4.3, here we establish the well-posedness of the discrete system (4.164) by studying the equivalent fixed-point equation (4.172). More precisely, being the respective analyses fully analogous to those developed in Sections 4.3.2 and 4.3.3, in what follows we basically collect the corresponding results and, eventually, discuss some details of the respective proofs.

We begin by stating next that the discrete operators $\mathbf{S}_h$, $\widetilde{\mathrm{S}}_h$, and S_h are well-defined, equivalently, that the problems (4.166), (4.168), and (4.170) are well-posed. Certainly, instead of [12, Theorem 2.1, Corollary 2.1, Section 2.1], [41, Theorem 2.34], and [41, Theorem 2.6], we now resort to the respective discrete versions given by [12, Corollary 2.2, Section 2.2], [41, Proposition 2.42], and [41, Theorem 2.22]. To this end, we need to introduce general hypotheses on the finite element subspaces to be utilized in (4.164), and later on in Section 4.5 we will introduce specific examples of the latter satisfying them. According to the above, and in order to address first the well-definedness of $\mathbf{S}$, we assume that

(H.1) there exists a positive constant $\boldsymbol{\beta}_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\mathbf{v}}_h \in \mathbf{H}_h \\ \vec{\mathbf{v}}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{v}}_h, \widetilde{a}_h)}{\|\vec{\mathbf{v}}_h\|_{\mathbf{H}}} \geq \boldsymbol{\beta}_{\mathtt{d}} \, \|\widetilde{a}_h\|_{\mathbf{Q}} \quad \forall \, \widetilde{a}_h \in \mathbf{Q}_h \, .$$

In addition, we let $\mathbf{V}_h$ be the discrete kernel of the bilinear form $\mathbf{b}$, that is

$$\mathbf{V}_h \, := \, \Big\{ \vec{\mathbf{v}}_h \in \mathbf{H}_h : \quad \mathbf{b}(\vec{\mathbf{v}}_h, \widetilde{a}_h) \, = \, 0 \quad \forall \, \widetilde{a}_h \in \mathbf{Q}_h \Big\} \, , \tag{4.173}$$

and suppose that

(H.2) there exists a positive constant $C_{\mathtt{d}}$, independent of h, such that

$$\|\mathbf{s}_h\|_{0,\Omega} \, \geq \, C_{\mathtt{d}} \, \|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall \, \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \, \in \, \mathbf{V}_h \, .$$

Then, given $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$, it readily follows from the definitions of $\mathcal{A}_{\mathbf{z}_h}$ (cf. (4.86)) and $\mathbf{a}$ (cf. (4.15)), the identity (4.26), and the assumption **(H.2)**, that

$$\mathcal{A}_{\mathbf{z}_h}(\vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) \, = \, \mathbf{a}(\vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) \, = \, \nu \, \|\mathbf{s}_h\|^2_{0,\Omega} \, \geq \, \frac{\nu}{2} \, C_{\mathtt{d}}^2 \, \|\mathbf{v}_h\|^2_{0,4;\Omega} + \frac{\nu}{2} \, \|\mathbf{s}_h\|^2_{0,\Omega} \quad \forall \, \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \, \in \, \mathbf{V}_h \, , \tag{4.174}$$

which proves the $\mathbf{V}_h$-ellipticity of $\mathcal{A}_{\mathbf{z}_h}$ with constant $\boldsymbol{\alpha}_{\mathtt{d}} \, := \, \frac{\nu}{2} \min \big\{ C_{\mathtt{d}}^2, 1 \big\}$. Thus, the discrete analogue of Lemma 4.1 reads as follows.

Lemma 4.13. *For each* $(\mathbf{z}_h, \chi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h$ *there exists a unique* $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \, := \, \big((\mathbf{u}_h, \mathbf{t}_h), h \big) \in \mathbf{H}_h \times \mathbf{Q}_h$ *solution of* (4.166)*, and hence one can define* $\mathbf{S}_h(\mathbf{z}_h, \chi_h) \, := \, \mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$*. Moreover, there exists a positive constant* $C_{\mathbf{S},\mathtt{d}}$*, depending only on* $|\Omega|$*,* $\|\mathbf{i}_4\|$*,* ν*,* λ*,* $\boldsymbol{\alpha}_{\mathtt{d}}$*, and* $\boldsymbol{\beta}_{\mathtt{d}}$*, such that*

$$\begin{aligned} \|\mathbf{S}_h(\mathbf{z}_h, \chi_h)\|_{0,4;\Omega} \, &= \, \|\mathbf{u}_h\|_{0,4;\Omega} \, \leq \, \|\vec{\mathbf{u}}_h\|_{\mathbf{H}} \\ &\leq \, C_{\mathbf{S},\mathtt{d}} \Big\{ \|\chi_h\|_{0,\rho;\Omega} \, \|\nabla f\|_{0,r;\Omega} \, + \, \|\mathbf{f}\|_{0,4/3;\Omega} \, + \, \big(1 + \|\mathbf{z}_h\|_{0,4;\Omega}\big) \, \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\} \, . \end{aligned} \tag{4.175}$$

Proof. Having the discrete inf-sup condition for $\mathbf{b}$ (cf. **(H.1)**) and the $\mathbf{V}_h$-ellipticity of $\mathcal{A}_{\mathbf{z}_h}$ for each $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$ (cf. (4.174)), the existence of a unique solution to (4.166) is a straightforward application of [41, Proposition 2.42], whereas the a priori estimate (4.175) follows from [41, eq. (2.30)]. □

We remark here that the discrete analogue of (4.97) reads

$$\begin{aligned} \|h\|_{\mathbf{Q}} \, = \, \|h\|_{\mathbf{div}_{4/3};\Omega} \, \leq \, \bar{C}_{\mathbf{S},\mathtt{d}} \, (1 \, + \, \|\mathbf{z}_h\|_{0,4;\Omega}) \, \Big\{ \|\chi_h\|_{0,\rho;\Omega} \, \|\nabla f\|_{0,r;\Omega} \\ + \, \|\mathbf{f}\|_{0,4/3;\Omega} \, + \, (1 + \|\mathbf{z}_h\|_{0,4;\Omega}) \, \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\} \, , \end{aligned} \tag{4.176}$$

where $\bar{C}_{\mathbf{S},\mathtt{d}}$ is a positive constant depending as well on $|\Omega|$, $\|\mathbf{i}_4\|$, ν, λ, $\boldsymbol{\alpha}_{\mathtt{d}}$, and $\boldsymbol{\beta}_{\mathtt{d}}$.

In turn, for the well-definedness of $\widetilde{\mathrm{S}}_h$, we now look at the discrete kernel of $\widetilde{b}$, that is

$$\widetilde{\mathrm{V}}_h \,:=\, \Big\{ \widetilde{\boldsymbol{\tau}}_h \in \mathrm{H}_h : \quad \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \xi_h) \,=\, 0 \quad \forall\, \xi_h \in \mathrm{Q}_h \Big\}\,, \tag{4.177}$$

and suppose that

(H.3) there holds $\mathbf{div}(\mathrm{H}_h) \,\subseteq\, \mathrm{Q}_h$,

(H.4) there exists a positive constant $\widetilde{\beta}_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\widetilde{\boldsymbol{\tau}}_h \in \mathrm{H}_h \\ \widetilde{\boldsymbol{\tau}}_h \neq 0}} \frac{\widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \xi_h)}{\|\widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{H}}} \,\geq\, \widetilde{\beta}_{\mathtt{d}}\, \|\xi_h\|_{\mathrm{Q}} \qquad \forall\, \xi_h \in \mathrm{Q}_h\,.$$

Bearing in mind the definition of $\widetilde{b}$ (cf. (4.41)), and employing **(H.3)**, we deduce from (4.177) that $\widetilde{\mathrm{V}}_h \,=\, \Big\{ \widetilde{\boldsymbol{\tau}}_h \in \mathrm{H}_h : \quad \mathbf{div}(\widetilde{\boldsymbol{\tau}}_h) \,=\, 0 \Big\}$, which yields the discrete analogue of (4.99), and hence the $\widetilde{\mathrm{V}}_h$-ellipticity of $\widetilde{a}$ (cf. (4.40)) with constant $\widetilde{\alpha}_{\mathtt{d}} = 1$. This fact together with **(H.4)** guarantee, thanks to [41, Proposition 2.42], the discrete global inf-sup condition for $\widetilde{A}$ (cf. (4.98)) with a positive constant $\alpha_{\widetilde{\mathrm{S}},\mathtt{d}}$ depending only on $\widetilde{\alpha}_{\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, and $\|\widetilde{a}\|$, and thus the same property is transferred to $\widetilde{A}_{\mathbf{z}_h,\mathbf{r}_h}$ (cf. (4.104)) for each $(\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{H}_h^{\mathbf{u}} \times X_{2,h}$ satisfying the discrete version of (4.106), that is

$$\|\mathbf{z}_h\|_{0,4;\Omega} + \|\mathbf{r}_h\|_{0,r;\Omega} \,\leq\, \frac{\alpha_{\widetilde{\mathrm{S}},\mathtt{d}}}{2\,\|\widetilde{c}\|}\,. \tag{4.178}$$

In this way, the well-definedness of $\widetilde{\mathrm{S}}_h$ is established by the following lemma.

Lemma 4.14. *For each $(\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{H}_h^{\mathbf{u}} \times X_{2,h}$ verifying (4.178), there exists a unique $(\widetilde{\boldsymbol{\sigma}}_h, \eta_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ solution of (4.168), and hence one can define $\widetilde{\mathrm{S}}_h(\mathbf{z}_h, \mathbf{r}_h) \,:=\, \eta_h \in \mathrm{Q}_h$. Moreover, there exists a positive constant $C_{\widetilde{\mathrm{S}},\mathtt{d}}$, depending only on $\alpha_{\widetilde{\mathrm{S}},\mathtt{d}}$, $\|\mathrm{i}_\rho\|$, and κ_η, such that*

$$\|\widetilde{\mathrm{S}}_h(\mathbf{z}_h, \mathbf{r}_h)\|_{\mathrm{Q}} \,=\, \|\eta_h\|_{0,\rho;\Omega} \,\leq\, \|(\widetilde{\boldsymbol{\sigma}}_h, \eta_h)\|_{\mathrm{H}\times\mathrm{Q}} \,\leq\, C_{\widetilde{\mathrm{S}},\mathtt{d}} \Big\{ \|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega} \Big\}\,. \tag{4.179}$$

Proof. It is a direct application of [41, Theorem 2.22]. □

Furthermore, the well-definedness of S_h requires the introduction of the discrete kernels of b_1 and b_2, namely

$$K_{1,h} \,:=\, \Big\{ \mathbf{q}_h \in X_{1,h} : \quad b_1(\mathbf{q}_h, \phi_h) \,=\, 0 \quad \forall\, \phi_h \in M_{1,h} \Big\}\,,$$

and

$$K_{2,h} \,:=\, \Big\{ \mathbf{q}_h \in X_{2,h} : \quad b_2(\mathbf{q}_h, \phi_h) \,=\, 0 \quad \forall\, \phi_h \in M_{2,h} \Big\}\,,$$

and the following hypotheses:

(H.5) there exists a positive constant $\alpha_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\mathbf{q}_h \in K_{1,h} \\ \mathbf{q}_h \neq 0}} \frac{a(\mathbf{r}_h, \mathbf{q}_h)}{\|\mathbf{q}_h\|_{X_1}} \,\geq\, \alpha_{\mathtt{d}}\, \|\mathbf{r}_h\|_{X_2} \qquad \forall\, \mathbf{r}_h \in K_{2,h}\,, \quad \text{and}$$

$$\sup_{\mathbf{r}_h \in \mathcal{K}_{2,h}} a(\mathbf{r}_h, \mathbf{q}_h) > 0 \qquad \forall\, \mathbf{q}_h \in K_{1,h}, \ \mathbf{q}_h \neq \mathbf{0},$$

(H.6) for each $i \in \{1,2\}$ there exists a positive constant $\beta_{i,\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\mathbf{q}_h \in X_{i,h}\\ \mathbf{q}_h \neq 0}} \frac{b_i(\mathbf{q}_h, \phi_h)}{\|\mathbf{q}_h\|_{X_i}} \geq \beta_{i,\mathtt{d}}\, \|\phi_h\|_{M_i} \qquad \forall\, \phi_h \in M_{i,h}\,.$$

Thanks to **(H.5)** and **(H.6)**, a straightforward application of [12, Corollary 2.2, Section 2.2] implies the discrete global inf-sup condition for $\mathbf{A}$ (cf. (4.110)) with a positive constant $\alpha_{\mathrm{S},\mathtt{d}}$ depending only on $\alpha_{\mathtt{d}}$, $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$ and $\|a\|$, and hence the same property is shared by $\mathbf{A}_{\mathbf{z}_h,\chi_h}$ (cf. (4.123)) for each $(\mathbf{z}_h, \chi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h$ satisfying the discrete version of (4.126), that is

$$\|\mathbf{z}_h\|_{0,4;\Omega} + \|\chi_h\|_{0,\rho;\Omega} \leq \frac{\alpha_{\mathrm{S},\mathtt{d}}}{2\,\|c\|}\,. \tag{4.180}$$

In this way, the well-definedness of S_h is stated as follows.

Lemma 4.15. *For each $(\mathbf{z}_h, \chi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h$ verifying (4.180), there exists a unique $(\mathbf{p}_h, \varphi_h) \in X_{2,h} \times M_{1,h}$ solution of (4.170), and hence one can define $\mathrm{S}_h(\mathbf{z}_h, \chi_h) := \mathbf{p}_h \in X_{2,h}$. Moreover, there exists a positive constant $C_{\mathrm{S},\mathtt{d}}$, depending only on $\alpha_{\mathrm{S},\mathtt{d}}$, C_r, and κ_φ, such that*

$$\|\mathrm{S}_h(\mathbf{z}_h, \chi_h)\|_{X_2} = \|\mathbf{p}_h\|_{X_2} \leq \|(\mathbf{p}_h, \varphi_h)\|_{X_2 \times M_1} \leq C_{\mathrm{S},\mathtt{d}} \Big\{ \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega} \Big\}\,. \tag{4.181}$$

Proof. Similarly to the proof of Lemma 4.14, it reduces to a simple application of [41, Theorem 2.22]. □

Having established that the discrete operators $\mathbf{S}_h$, $\widetilde{\mathrm{S}}_h$, S_h, and hence T_h (under the constraint imposed by (4.180)), are all well-defined, we now proceed as in Section 4.3.3 to address the solvability of the corresponding fixed point equation (4.172). Indeed, letting $\delta_{\mathtt{d}}$ be the discrete version of (4.137), that is

$$\delta_{\mathtt{d}} := \min\left\{ \frac{\alpha_{\widetilde{\mathrm{S}},\mathtt{d}}}{2\,\|\widetilde{c}\|}, \frac{\alpha_{\mathrm{S},\mathtt{d}}}{4\,\|c\|} \right\}, \tag{4.182}$$

we first introduce the ball

$$W_{\delta_{\mathtt{d}}} := \Big\{ (\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{H}_h^{\mathbf{u}} \times X_{2,h} : \quad \|(\mathbf{z}_h, \mathbf{r}_h)\| := \|\mathbf{z}_h\|_{0,4;\Omega} + \|\mathbf{r}_h\|_{X_2} \leq \delta_{\mathtt{d}} \Big\}\,. \tag{4.183}$$

Then, analogously to the derivation of Lemma 4.7 (cf. beginning of Section 4.3.3), we deduce that T_h maps $W_{\delta_{\mathtt{d}}}$ into itself under the discrete versions of (4.139) and (4.140), which read exactly as those, except that the constants $C_{\widetilde{\mathrm{S}}}$, α_{S}, and $C(\delta)$, and the radius δ utilized there are replaced by $C_{\widetilde{\mathrm{S}},\mathtt{d}}$, $\alpha_{\mathrm{S},\mathtt{d}}$, $C_{\mathtt{d}}(\delta)$, and $\delta_{\mathtt{d}}$, respectively, where, similarly to $C(\delta)$, $C_{\mathtt{d}}(\delta)$ depends explicitly on $C_{\mathbf{S},\mathtt{d}}$, $C_{\widetilde{\mathrm{S}},\mathtt{d}}$, $(1+\delta)$, and $C_{\mathrm{S},\mathtt{d}}$. Moreover, following analogue arguments to those employed in the proofs of Lemmas 4.8, 4.9, and 4.10, we are able to prove the continuity properties of $\mathbf{S}_h$, $\widetilde{\mathrm{S}}_h$, and S_h, that is the discrete versions of (4.141), (4.149), and (4.153), which are the same as the latter, but instead of $L_{\mathbf{S}}$, $L_{\widetilde{\mathrm{S}}}$, and L_{S}, the resulting constants are given by

$$L_{\mathbf{S},\mathtt{d}} := \boldsymbol{\alpha}_{\mathtt{d}}^{-1} \max\left\{ |\Omega|^{1/4}, \|\mathbf{c}\| \right\}, \quad L_{\widetilde{\mathrm{S}},\mathtt{d}} := 2\,\|\widetilde{c}\|\, \alpha_{\widetilde{\mathrm{S}},\mathtt{d}}^{-1}\, C_{\widetilde{\mathrm{S}},\mathtt{d}}, \quad \text{and} \quad L_{\mathrm{S},\mathtt{d}} := 2\,\|c\|\, \alpha_{\mathrm{S},\mathtt{d}}^{-1}\, C_{\mathrm{S},\mathtt{d}}\,,$$

respectively. Hence, proceeding analogously to the derivation of (4.160), (4.161), and the consequent Lemma 4.11, we are able to show that, under the discrete version of (4.139), there holds

$$
\begin{aligned}
&\|\mathrm{T}_h(\mathbf{z}_h,\mathbf{r}_h) - \mathrm{T}_h(\mathbf{z}_{0,h},\mathbf{r}_{0,h})\| \\
&\quad \leq L_{\mathrm{T},\mathtt{d}} \Big\{ \big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big)\big(\|\nabla f\|_{0,r;\Omega} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\big) \\
&\qquad + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\} \|(\mathbf{z}_h,\mathbf{r}_h) - (\mathbf{z}_{0,h},\mathbf{r}_{0,h})\|\,,
\end{aligned}
\tag{4.184}
$$

for all $(\mathbf{z}_h,\mathbf{r}_h)$, $(\mathbf{z}_{0,h},\mathbf{r}_{0,h}) \in W_{\delta_\mathtt{d}}$, where $L_{\mathrm{T},\mathtt{d}}$ is a positive constant depending only on $L_{\mathbf{S},\mathtt{d}}$, $L_{\widetilde{\mathrm{S}},\mathtt{d}}$, $L_{\mathrm{S},\mathtt{d}}$, $C_{\mathbf{S},\mathtt{d}}$, $C_{\widetilde{\mathrm{S}},\mathtt{d}}$, and δ.

According to the above, the main result of this section is established as follows.

Theorem 4.16. *Assume that the data are sufficiently small so that the discrete versions of* (4.139) *and* (4.140) *hold, that is*

$$
C_{\widetilde{\mathrm{S}},\mathtt{d}}\,\big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big) \,\leq\, \frac{\alpha_{\mathrm{S},\mathtt{d}}}{4\,\|c\|}\,,
\tag{4.185}
$$

and

$$
\begin{aligned}
&C_\mathtt{d}(\delta)\Big\{\big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big)\,\|\nabla f\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \\
&\qquad + \;\; \|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega}\Big\} \,\leq\, \delta_\mathtt{d}\,.
\end{aligned}
\tag{4.186}
$$

Then, the operator T_h *has a fixed point* $(\mathbf{u}_h,\mathbf{p}_h) \in W_{\delta_\mathtt{d}}$. *Equivalently, the coupled problem* (4.164) *has a solution* $(\vec{\mathbf{u}}_h, h) \in \mathbf{H}_h \times \mathbf{Q}_h$, $(\widetilde{\boldsymbol{\sigma}}_h,\eta_h) \in \mathrm{H}_h \times \mathrm{Q}_h$, *and* $(\mathbf{p}_h,\varphi_h) \in X_{2,h} \times M_{1,h}$, *with* $(\mathbf{u}_h,\mathbf{p}_h) \in W_{\delta_\mathtt{d}}$. *Moreover, there hold the following a priori estimates*

$$
\begin{aligned}
&\|(\vec{\mathbf{u}}_h,h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leq\, C_{\vec{\mathbf{u}},\boldsymbol{\sigma},\mathtt{d}}\Big\{\|\nabla f\|_{0,r;\Omega}\,\big(\|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\big) \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,, \\
&\|(\widetilde{\boldsymbol{\sigma}}_h,\eta_h)\|_{\mathrm{H}\times\mathrm{Q}} \,\leq\, C_{\widetilde{\mathrm{S}},\mathtt{d}}\Big\{\|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\Big\}\,, \\
&\|(\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1} \,\leq\, C_{\mathrm{S},\mathtt{d}}\Big\{\|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega}\Big\}\,,
\end{aligned}
$$

where $C_{\vec{\mathbf{u}},\boldsymbol{\sigma},\mathtt{d}}$ *is a positive constant depending only on* $C_{\mathbf{S},\mathtt{d}}$, $\bar{C}_{\mathbf{S},\mathtt{d}}$, $C_{\widetilde{\mathrm{S}},\mathtt{d}}$, *and* $\delta_\mathtt{d}$. *Furthermore, under the additional assumption*

$$
\begin{aligned}
&L_{\mathrm{T},\mathtt{d}}\Big\{\big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big)\big(\|\nabla f\|_{0,r;\Omega} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\big) \\
&\qquad + \;\; \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\} \,<\, 1\,,
\end{aligned}
\tag{4.187}
$$

the aforementioned solutions of (4.172) *and* (4.164) *are unique.*

Proof. As previously mentioned, (4.185) and (4.186) guarantee that T_h maps $W_{\delta_\mathtt{d}}$ into itself. Then, knowing from (4.184) that $\mathrm{T}_h : W_{\delta_\mathtt{d}} \to W_{\delta_\mathtt{d}}$ is continuous, a straightforward application of Brouwer's theorem (cf. [28, Theorem 9.9-2]) implies the existence of solution of (4.172), and hence of (4.164). In turn, under the further hypotheses (4.187), the Banach fixed-point theorem yields the respective uniqueness of solution. Finally, in any case, the a priori estimates are consequences of (4.175), (4.176), (4.179) and (4.181), and the fact that $\|\mathbf{u}_h\|_{0,4;\Omega} \,\leq\, \delta_\mathtt{d}$. □

Needless to say, analogue remarks to those stated in Sections 4.3.2 and 4.3.2 for the case of the boundary conditions (4.70), hold in the present discrete case as well.

4.4.3 A priori error analysis

In this section we derive an a priori error estimate for the Galerkin scheme (4.164) with arbitrary finite element subspaces satisfying the hypotheses introduced in Section 4.4.2. More precisely, we are interested in establishing the Céa estimate for the error

$$\mathtt{E} := \|(\vec{\mathbf{u}},\boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\widetilde{\boldsymbol{\sigma}},\eta) - (\widetilde{\boldsymbol{\sigma}}_h,\eta_h)\|_{\mathrm{H}\times\mathrm{Q}} + \|(\mathbf{p},\varphi) - (\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1}\,, \tag{4.188}$$

where $\big((\vec{\mathbf{u}},\boldsymbol{\sigma}),(\widetilde{\boldsymbol{\sigma}},\eta),(\mathbf{p},\varphi)\big) \in (\mathbf{H}\times\mathbf{Q})\times(\mathrm{H}\times\mathrm{Q})\times(X_2\times M_1)$ is the unique solution of (4.69) with $(\mathbf{u},\mathbf{p}) \in W_\delta$ (cf. (4.130)), and $\big((\vec{\mathbf{u}}_h, h),(\widetilde{\boldsymbol{\sigma}}_h,\eta_h),(\mathbf{p}_h,\varphi_h)\big) \in (\mathbf{H}_h\times\mathbf{Q}_h)\times(\mathrm{H}_h\times\mathrm{Q}_h)\times(X_{2,h}\times M_{1,h})$ is a solution of (4.164) with $(\mathbf{u}_h,\mathbf{p}_h) \in W_{\delta_\mathtt{d}}$ (cf. (4.183)). To this end, we consider the pairs of associated continuous and discrete formulations arising from (4.69) and (4.164) once the latter are split according to the three equations forming the full model. In what follows, given a subspace Z_h of a generic Banach space $(Z,\|\cdot\|_Z)$, we set for each $z \in Z$

$$\mathrm{dist}(z, Z_h) := \inf_{z_h\in Z_h} \|z - z_h\|_Z\,. \tag{4.189}$$

We begin by applying the Strang estimate provided by [29, Lemma 6.1], whose proof is a simple modification of that of [44, Theorem 2.6], to the context given by the first two rows of (4.69) and (4.164). As a consequence, we deduce the existence of a positive constant $\widehat{C}_\mathbf{S}$, depending only on $\boldsymbol{\alpha}_\mathtt{d}$ $\boldsymbol{\beta}_\mathtt{d}$, $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, $\|\mathbf{c}\|$, δ, and $\delta_\mathtt{d}$, such that

$$\begin{aligned}\|(\vec{\mathbf{u}},\boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, h)\|_{\mathbf{H}\times\mathbf{Q}} \le \widehat{C}_\mathbf{S}\Big\{&\mathrm{dist}\,(\vec{\mathbf{u}},\mathbf{H}_h) + \mathrm{dist}\,(\boldsymbol{\sigma},\mathbf{Q}_h)\\ &+ \|\mathbf{F}_\eta - \mathbf{F}_{\eta_h}\|_{\mathbf{H}_h'} + \|\mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\cdot) - \mathbf{c}(\mathbf{u}_h;\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'}\Big\}\,.\end{aligned} \tag{4.190}$$

In fact, we first observe that the first two rows of (4.69) and (4.164) can be rewritten, respectively, as

$$\begin{aligned}\widetilde{\mathbf{a}}(\vec{\mathbf{u}},\vec{\mathbf{v}}) + \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\sigma}) &= \mathbf{F}_\eta(\vec{\mathbf{v}}) \qquad \forall\,\vec{\mathbf{v}}\in\mathbf{H}\,,\\ \mathbf{b}(\vec{\mathbf{u}},\boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbf{Q}\,,\end{aligned} \tag{4.191}$$

and

$$\begin{aligned}\widetilde{\mathbf{a}}_h(\vec{\mathbf{u}}_h,\vec{\mathbf{v}}_h) + \mathbf{b}(\vec{\mathbf{v}}_h, h) &= \mathbf{F}_{\eta_h}(\vec{\mathbf{v}}_h) \qquad \forall\,\vec{\mathbf{v}}_h\in\mathbf{H}_h\,,\\ \mathbf{b}(\vec{\mathbf{u}}_h,\widetilde{a}_h) &= \mathbf{G}(\widetilde{a}_h) \qquad \forall\,\widetilde{a}_h\in\mathbf{Q}_h\,,\end{aligned} \tag{4.192}$$

where $\widetilde{\mathbf{a}} : \mathbf{H}\times\mathbf{H}\to\mathrm{R}$ and $\widetilde{\mathbf{a}}_h : \mathbf{H}_h\times\mathbf{H}_h\to\mathrm{R}$, depending on $\mathbf{u}$ and $\mathbf{u}_h$, respectively, are given by

$$\widetilde{\mathbf{a}}(\vec{\mathbf{w}},\vec{\mathbf{v}}) := \mathbf{a}(\vec{\mathbf{w}},\vec{\mathbf{v}}) + \mathbf{c}(\mathbf{u};\vec{\mathbf{w}},\vec{\mathbf{v}}) \qquad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}}\in\mathbf{H}\,, \tag{4.193}$$

and

$$\widetilde{\mathbf{a}}_h(\vec{\mathbf{w}}_h,\vec{\mathbf{v}}_h) := \mathbf{a}(\vec{\mathbf{w}}_h,\vec{\mathbf{v}}_h) + \mathbf{c}(\mathbf{u}_h;\vec{\mathbf{w}}_h,\vec{\mathbf{v}}_h) \qquad \forall\,\vec{\mathbf{w}}_h,\,\vec{\mathbf{v}}_h\in\mathbf{H}_h\,. \tag{4.194}$$

It is clear from (4.193) and (4.194) that

$$\|\widetilde{\mathbf{a}}(\vec{\mathbf{u}},\cdot) - \widetilde{\mathbf{a}}_h(\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'} = \|\mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\cdot) - \mathbf{c}(\mathbf{u}_h;\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'}\,,$$

so that (4.190) follows from a straightforward application of [29, Lemma 6.1] to the pair (4.191) - (4.192).

Then, using the boundedness properties of $\mathbf{F}_\eta$ (cf. (4.48) and (4.147)) and $\mathbf{c}$ (cf. (4.24) and (4.148)), we readily obtain

$$\|\mathbf{F}_\eta - \mathbf{F}_{\eta_h}\|_{\mathbf{H}_h'} \,\le\, |\Omega|^{1/4}\,\|\eta-\eta_h\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega}\,,$$

and

$$\|\mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\cdot) - \mathbf{c}(\mathbf{u}_h;\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'} \,\le\, \|\mathbf{c}\|\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}\|_{\mathbf{H}}\,,$$

which, replaced back in (4.190), give

$$\begin{aligned}\|(\vec{\mathbf{u}},\boldsymbol{\sigma}) \,-\, (\vec{\mathbf{u}}_h,h)\|_{\mathbf{H}\times\mathbf{Q}} \,&\le\, \widehat{C}_{\mathbf{S}}\Big\{\mathrm{dist}\,(\vec{\mathbf{u}},\mathbf{H}_h) \,+\, \mathrm{dist}\,(\boldsymbol{\sigma},\mathbf{Q}_h)\Big\}\\ &+\,\overline{C}_{\mathbf{S}}\Big\{\|\nabla f\|_{0,r;\Omega}\,\|\eta-\eta_h\|_{0,\rho;\Omega} \,+\, \|\vec{\mathbf{u}}\|_{\mathbf{H}}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big\}\,,\end{aligned} \tag{4.195}$$

where $\overline{C}_{\mathbf{S}} \,:=\, \widehat{C}_{\mathbf{S}}\,\max\big\{|\Omega|^{1/4},\|\mathbf{c}\|\big\}$.

Next, we apply the Strang a priori error estimate from [12, Proposition 2.1, Corollary 2.3, and Theorem 2.3] to the context given by the third and fourth rows of (4.69) and (4.164), in which each term involving $\widetilde{c}$ is considered as part of the respective functional on the right-hand side. In this way, we deduce the existence of a positive constant $\widehat{C}_{\widetilde{\mathrm{S}}}$, depending only on $\widetilde{\alpha}_{\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, $\|\widetilde{a}\|$ and $\|\widetilde{b}\|$, such that

$$\|(\widetilde{\boldsymbol{\sigma}},\eta) \,-\, (\widetilde{\boldsymbol{\sigma}}_h,\eta_h)\|_{\mathrm{H}\times\mathrm{Q}} \le \widehat{C}_{\widetilde{\mathrm{S}}}\Big\{\mathrm{dist}\,(\widetilde{\boldsymbol{\sigma}},\mathrm{H}_h) \,+\, \mathrm{dist}\,(\eta,\mathrm{Q}_h) \,+\, \|\widetilde{c}_{\mathbf{u},\mathbf{p}}(\cdot,\eta) - \widetilde{c}_{\mathbf{u}_h,\mathbf{p}_h}(\cdot,\eta_h)\|_{\mathrm{H}_h'}\Big\}. \tag{4.196}$$

In turn, subtracting and adding η_h to the second component of $\widetilde{c}_{\mathbf{u},\mathbf{p}}(\cdot,\eta)$, making use of the triangle inequality, bearing in mind the definition of $\widetilde{c}_{\mathbf{z},\mathbf{q}}$ (cf. (4.34)), and employing its boundedness property (cf. (4.45)), we find that

$$\begin{aligned}\|\widetilde{c}_{\mathbf{u},\mathbf{p}}(\cdot,\eta) - \widetilde{c}_{\mathbf{u}_h,\mathbf{p}_h}(\cdot,\eta_h)\|_{\mathrm{H}_h'} \,&\le\, \|\widetilde{c}_{\mathbf{u},\mathbf{p}}(\cdot,\eta-\eta_h)\|_{\mathrm{H}_h'} \,+\, \|\widetilde{c}_{\mathbf{u},\mathbf{p}}(\cdot,\eta_h) - \widetilde{c}_{\mathbf{u}_h,\mathbf{p}_h}(\cdot,\eta_h)\|_{\mathrm{H}_h'}\\ &\le\, \|\widetilde{c}\|\Big\{\big(\|\mathbf{u}\|_{0,4;\Omega} + \|\mathbf{p}\|_{0,r;\Omega}\big)\,\|\eta-\eta_h\|_{\mathrm{Q}} + \|\eta_h\|_{\mathrm{Q}}\,\big(\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega} + \|\mathbf{p}-\mathbf{p}_h\|_{0,r;\Omega}\big)\Big\}\,,\end{aligned}$$

which, along with (4.196), yield

$$\begin{aligned}\|(\widetilde{\boldsymbol{\sigma}},\eta) \,-\, (\widetilde{\boldsymbol{\sigma}}_h,\eta_h)\|_{\mathrm{H}\times\mathrm{Q}} &\le \widehat{C}_{\widetilde{\mathrm{S}}}\Big\{\mathrm{dist}\,(\widetilde{\boldsymbol{\sigma}},\mathrm{H}_h) \,+\, \mathrm{dist}\,(\eta,\mathrm{Q}_h)\Big\}\\ &+\overline{C}_{\widetilde{\mathrm{S}}}\Big\{\big(\|\mathbf{u}\|_{0,4;\Omega} + \|\mathbf{p}\|_{0,r;\Omega}\big)\,\|\eta-\eta_h\|_{\mathrm{Q}} + \|\eta_h\|_{\mathrm{Q}}\,\big(\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega} + \|\mathbf{p}-\mathbf{p}_h\|_{0,r;\Omega}\big)\Big\}\,,\end{aligned} \tag{4.197}$$

where $\overline{C}_{\widetilde{\mathrm{S}}} \,:=\, \widehat{C}_{\widetilde{\mathrm{S}}}\,\|\widetilde{c}\|$.

Furthermore, we proceed analogously to the previous case for the context given by the fifth and sixth rows of (4.69) and (4.164), that is, we consider each term involving c as part of the respective functional on the right-hand side, and then apply the Strang a priori error estimate from [12, Proposition 2.1, Corollary 2.3, and Theorem 2.3]. As a result of it we obtain

$$\begin{aligned}&\|(\mathbf{p},\varphi) \,-\, (\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1}\\ &\quad\le\, \widehat{C}_{\mathrm{S}}\Big\{\mathrm{dist}\,(\mathbf{p},X_{2,h}) \,+\, \mathrm{dist}\,(\varphi,M_{1,h}) \,+\, \|c_{\mathbf{u},\eta}\big((\mathbf{p},\varphi),\cdot\big) - c_{\mathbf{u}_h,\eta_h}\big((\mathbf{p}_h,\varphi_h),\cdot\big)\|_{M_{2,h}'}\Big\}\,,\end{aligned} \tag{4.198}$$

where $\widehat{C}_{\mathrm{S}}$ is a positive constant depending only on $\alpha_{\mathtt{d}}$, $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$, $\|a\|$, $\|b_1\|$, and $\|b_2\|$. Now, in order to estimate the consistency error term of (4.198), we subtract and add $(\mathbf{p}_h,\varphi_h)$ in the first component of $c_{\mathbf{u},\eta}\big((\mathbf{p},\varphi),\cdot\big)$, employ triangle inequality, and invoke the definition of $c_{\mathbf{z},\chi}$ (cf. (4.59)) and its boundedness property (cf. (4.65)), to arrive at

$$\begin{aligned}
&\|c_{\mathbf{u},\eta}\big((\mathbf{p},\varphi),\cdot\big) - c_{\mathbf{u}_h,\eta_h}\big((\mathbf{p}_h,\varphi_h),\cdot\big)\|_{M'_{2,h}}\\
&\quad\le \|c_{\mathbf{u},\eta}\big((\mathbf{p},\varphi)-(\mathbf{p}_h,\varphi_h),\cdot\big)\|_{M'_{2,h}} + \|c_{\mathbf{u},\eta}\big((\mathbf{p}_h,\varphi_h),\cdot\big) - c_{\mathbf{u}_h,\eta_h}\big((\mathbf{p}_h,\varphi_h),\cdot\big)\|_{M'_{2,h}}\\
&\quad\le \|c\|\Big\{\big(\|\mathbf{u}\|_{0,4;\Omega} + \|\eta\|_{\mathrm{Q}}\big)\,\|(\mathbf{p},\varphi)-(\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1}\\
&\quad + \big(\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega} + \|\eta-\eta_h\|_{\mathrm{Q}}\big)\,\|(\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1}\Big\},
\end{aligned}$$

which, jointly with (4.198), imply

$$\begin{aligned}
\|(\mathbf{p},\varphi) - (\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1} &\le \widehat{C}_{\mathrm{S}}\Big\{\mathrm{dist}\,(\mathbf{p},X_{2,h}) + \mathrm{dist}\,(\varphi,M_{1,h})\Big\}\\
&+\overline{C}_{\mathrm{S}}\Big\{\big(\|\mathbf{u}\|_{0,4;\Omega} + \|\eta\|_{\mathrm{Q}}\big)\,\|(\mathbf{p},\varphi)-(\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1}\\
&+\|(\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1}\,\big(\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega} + \|\eta-\eta_h\|_{\mathrm{Q}}\big)\Big\},
\end{aligned}\tag{4.199}$$

with $\overline{C}_{\mathrm{S}} := \widehat{C}_{\mathrm{S}}\,\|c\|$.

Consequently, adding the inequalities (4.195), (4.197), and (4.199), denoting $\widehat{C} := \max\big\{\widehat{C}_{\mathbf{S}},\widehat{C}_{\widetilde{\mathrm{S}}},\widehat{C}_{\mathrm{S}}\big\}$, employing the bounds for $\|\vec{\mathbf{u}}\|_{\mathbf{H}}$, $\|\mathbf{p}\|_{X_2}$, $\|\eta\|_{\mathrm{Q}}$, $\|\eta_h\|_{\mathrm{Q}}$, and $\|(\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1}$ provided by Theorems 4.12 and 4.16, and performing some algebraic manipulations, we find, in terms of the notations introduced in (4.188) and (4.189), that

$$\begin{aligned}
\mathtt{E} \le \widehat{C}&\Big\{\mathrm{dist}\,\big((\vec{\mathbf{u}},\boldsymbol{\sigma}),\mathbf{H}_h\times\mathbf{Q}_h\big) + \mathrm{dist}\,\big((\widetilde{\boldsymbol{\sigma}},\eta),\mathrm{H}_h\times\mathrm{Q}_h\big) + \mathrm{dist}\,\big((\mathbf{p},\varphi),X_{2,h}\times M_{1,h}\big)\Big\}\\
&+ \widehat{C}_0\Big\{\big(1+\|\nabla f\|_{0,r;\Omega}\big)\big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big) + \|\nabla f\|_{0,r;\Omega}\\
&+ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\}\,\mathtt{E}\,,
\end{aligned}\tag{4.200}$$

where $\widehat{C}_0$ is a positive constant depending on $\overline{C}_{\mathbf{S}}$, $\overline{C}_{\widetilde{\mathrm{S}}}$, $\overline{C}_{\mathrm{S}}$, $C_{\vec{\mathbf{u}},\boldsymbol{\sigma}}$, $C_{\widetilde{\mathrm{S}}}$, C_{S}, $C_{\widetilde{\mathrm{S}},\mathtt{d}}$, and $C_{\mathrm{S},\mathtt{d}}$.

We are now in a position to establish the announced Céa estimate.

Theorem 4.17. *In addition to the hypotheses of Theorems 4.12 and 4.16, assume that*

$$\begin{aligned}
&\widehat{C}_0\Big\{\big(1+\|\nabla f\|_{0,r;\Omega}\big)\big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big) + \|\nabla f\|_{0,r;\Omega}\\
&+ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\} \le \frac{1}{2}\,.
\end{aligned}\tag{4.201}$$

Then, denoting $\overline{C} := 2\,\widehat{C}$, there holds

$$\begin{aligned}
&\|(\vec{\mathbf{u}},\boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h,h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\widetilde{\boldsymbol{\sigma}},\eta) - (\widetilde{\boldsymbol{\sigma}}_h,\eta_h)\|_{\mathrm{H}\times\mathrm{Q}} + \|(\mathbf{p},\varphi)-(\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1}\\
&\quad\le \overline{C}\Big\{\mathrm{dist}\,(\vec{\mathbf{u}},\mathbf{H}_h) + \mathrm{dist}\,(\boldsymbol{\sigma},\mathbf{Q}_h) + \mathrm{dist}\,(\widetilde{\boldsymbol{\sigma}},\mathrm{H}_h) + \mathrm{dist}\,(\eta,\mathrm{Q}_h) + \mathrm{dist}\,(\mathbf{p},X_{2,h}) + \mathrm{dist}\,(\varphi,M_{1,h})\Big\}.
\end{aligned}$$

Proof. It follows straightforwardly from (4.200). □

We end the section with the a priori estimate for $\|p - p_h\|_{0,\Omega}$, where p_h is the discrete pressure suggested by the postprocessing formula given by the second identity in (4.4), which, according to (4.13), becomes

$$p_h \,=\, -\,\frac{1}{n}\mathrm{tr}\Big(h + c_h\,\mathbb{I} \,+\, \frac{\lambda}{2}(\mathbf{u}_h\otimes\mathbf{u}_h)\Big)\,, \tag{4.202}$$

with

$$c_h \,:=\, -\frac{\lambda}{2\,n\,|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}_h\otimes\mathbf{u}_h)\,. \tag{4.203}$$

Then, applying Cauchy-Schwarz's inequality, performing some algebraic manipulations, and employing the a priori bounds for $\|\mathbf{u}\|_{0,4;\Omega}$ and $\|\mathbf{u}_h\|_{0,4;\Omega}$, we deduce the existence of a positive constant C, depending on data, but independent of h, such that

$$\|p - p_h\|_{0,\Omega} \,\leq\, C\,\Big\{\|\boldsymbol{\sigma} - h\|_{0,\Omega} \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\Big\}\,. \tag{4.204}$$

4.5 Specific finite element subspaces

We now define specific finite element subspaces satisfying the conditions (**H.1**) - (**H.6**) that were introduced in Section 4.4.2, and provide the rates of convergence of the resulting discrete method.

4.5.1 Preliminaries

Bearing in mind the notations introduced at the beginning of Section 4.4.1, and given an integer $k \geq 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_k(K)$ be the space of polynomials of degree $\leq k$ defined on K, and denote its vector and tensor versions by $\mathbf{P}_k(K)$ and $\mathbb{P}_k(K)$, respectively. In addition, we let $\mathbf{RT}_k(K) = \mathbf{P}_k(K) + \mathrm{P}_k(K)\boldsymbol{x}$ be the local Raviart-Thomas space of order k defined on K, where $\boldsymbol{x}$ stands for a generic vector in R^n, and denote by $\mathbb{RT}_k(K)$ its corresponding tensor counterpart. In turn, we let $\mathrm{P}_k(\mathcal{T}_h)$, $\mathbf{P}_k(\mathcal{T}_h)$, $\mathbb{P}_k(\mathcal{T}_h)$, $\mathbf{RT}_k(\mathcal{T}_h)$ and $\mathbb{RT}_k(\mathcal{T}_h)$ be the corresponding global versions of $\mathrm{P}_k(K)$, $\mathbf{P}_k(K)$, $\mathbb{P}_k(K)$, $\mathbf{RT}_k(K)$ and $\mathbb{RT}_k(K)$, respectively, that is

$$\mathrm{P}_k(\mathcal{T}_h) \,:=\, \Big\{\phi_h \in \mathrm{L}^2(\Omega): \quad \phi_h|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,,$$

$$\mathbf{P}_k(\mathcal{T}_h) \,:=\, \Big\{\mathbf{v}_h \in \mathbf{L}^2(\Omega): \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,,$$

$$\mathbb{P}_k(\mathcal{T}_h) \,:=\, \Big\{\mathbf{s}_h \in \mathbb{L}^2(\Omega): \quad \mathbf{s}_h|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,,$$

$$\mathbf{RT}_k(\mathcal{T}_h) \,:=\, \Big\{\mathbf{q}_h \in \mathbf{H}(\mathbf{div};\Omega): \quad \mathbf{q}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,,$$

and

$$\mathbb{RT}_k(\mathcal{T}_h) \,:=\, \Big\{\widetilde{a}_h \in \mathbb{H}(\mathbf{div};\Omega): \quad \widetilde{a}_h|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,.$$

We stress here that for each $t,\, s \in (1,+\infty)$ such that $t \geq s$, there hold $\mathrm{P}_k(\mathcal{T}_h) \subseteq \mathrm{L}^t(\Omega)$, $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}(\mathbf{div}_t;\Omega)$, $\mathbb{RT}_k(\mathcal{T}_h) \subseteq \mathbb{H}(\mathbf{div}_t;\Omega)$, and $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}^t(\mathrm{div}_s;\Omega)$, inclusions that are implicitly utilized below to introduce the announced specific finite element subspaces. Indeed, we now define

$$\begin{gathered}\mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_k(\mathcal{T}_h)\,, \quad \mathbb{H}_h^{\mathbf{t}} := \mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{P}_k(\mathcal{T}_h)\,, \quad \mathbf{H}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}}\,, \quad \mathbb{H}_h^{\boldsymbol{\sigma}} := \mathbb{RT}_k(\mathcal{T}_h)\,, \\ \mathbf{Q}_h := \mathbb{H}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \quad \mathrm{H}_h := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \mathrm{Q}_h := \mathrm{P}_k(\mathcal{T}_h)\,, \\ X_{2,h} := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad M_{1,h} := \mathrm{P}_k(\mathcal{T}_h)\,, \quad X_{1,h} := \mathbf{RT}_k(\mathcal{T}_h)\,, \quad M_{2,h} := \mathrm{P}_k(\mathcal{T}_h)\,. \end{gathered} \tag{4.205}$$

Next, as a complement of the remark provided in Section 4.4.1, we now observe that for the finite element subspaces introduced in (4.205), the discrete conservation of momentum properties become

$$\mathbf{div}(h) - \boldsymbol{\mathcal{P}}_h^k(\mathbf{f}_{\mathtt{NS}}) = \mathbf{0} \quad \text{in} \quad \Omega\,, \quad \text{with} \quad \mathbf{f}_{\mathtt{NS}} := \frac{\lambda}{2}\,\mathbf{t}_h\mathbf{u}_h - \eta_h\,\nabla f - \mathbf{f}\,, \tag{4.206}$$

$$\mathrm{div}(\widetilde{\boldsymbol{\sigma}}_h) + \mathcal{P}_h^k(f_{\mathtt{CD}}) = 0 \quad \text{in} \quad \Omega\,, \quad \text{with} \quad f_{\mathtt{CD}} := k_\eta^{-1}\,f_\eta\,, \quad \text{and} \tag{4.207}$$

$$\mathrm{div}(\mathbf{p}_h) - \mathcal{P}_h^k(f_{\mathtt{CS}}) = 0 \quad \text{in} \quad \Omega\,, \quad \text{with} \quad f_{\mathtt{CS}} := k_\varphi^{-1}\big(\gamma\,\eta_h\,\varphi_h + \mathbf{u}_h\cdot\mathbf{p}_h - f_\varphi\big)\,, \tag{4.208}$$

where $\boldsymbol{\mathcal{P}}_h^k : \mathbf{L}^1(\Omega) \to \mathbf{P}_k(\mathcal{T}_h)$ is the projector defined, for each $\mathbf{v} \in \mathbf{L}^1(\Omega)$, as the unique element $\boldsymbol{\mathcal{P}}_h^k(\mathbf{v}) \in \mathbf{P}_k(\mathcal{T}_h)$ such that

$$\int_\Omega \boldsymbol{\mathcal{P}}_h^k(v)\cdot\mathbf{q}_h = \int_\Omega \mathbf{v}\cdot\mathbf{q}_h \qquad \forall\,\mathbf{q}_h \in \mathbf{P}_k(\mathcal{T}_h)\,, \tag{4.209}$$

and $\mathcal{P}_h^k : \mathrm{L}^1(\Omega) \to \mathrm{P}_k(\mathcal{T}_h)$ is the corresponding scalar version, that is for each $v \in \mathrm{L}^1(\Omega)$, $\mathcal{P}_h^k(v)$ is the unique element in $\mathrm{P}_k(\mathcal{T}_h)$ such that

$$\int_\Omega \mathcal{P}_h^k(v)\,q_h = \int_\Omega v\,q_h \qquad \forall\,q_h \in \mathrm{P}_k(\mathcal{T}_h)\,. \tag{4.210}$$

4.5.2 Verification of the stability conditions

In this section we prove that the specific finite element subspaces given by (4.205) verify the assumptions **(H.1) - (H.6)**. We begin with the following lemma establishing **(H.1)** and **(H.2)**, for which we recall that the definition of the discrete kernel $\mathbf{V}_h$ of the bilinear form $\mathbf{b}$ is given in (4.173).

Lemma 4.18. *There exist positive constants $\boldsymbol{\beta}_{\mathtt{d}}$ and $C_{\mathtt{d}}$, independent of h, such that*

$$\sup_{\substack{\vec{\mathbf{v}}_h\in\mathbf{H}_h\\ \vec{\mathbf{v}}_h\neq\mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{v}}_h,\widetilde{a}_h)}{\|\vec{\mathbf{v}}_h\|_{\mathbf{H}}} \ge \boldsymbol{\beta}_{\mathtt{d}}\,\|\widetilde{a}_h\|_{\mathbf{Q}} \quad \forall\,\widetilde{a}_h \in \mathbf{Q}_h\,, \tag{4.211}$$

and

$$\|\mathbf{s}_h\|_{0,\Omega} \ge C_{\mathtt{d}}\,\|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\,\vec{\mathbf{v}}_h := (\mathbf{v}_h,\mathbf{s}_h) \in \mathbf{V}_h\,. \tag{4.212}$$

Proof. We first introduce the subspace

$$\mathbf{Q}_{0,h} := \Big\{\widetilde{a}_h \in \mathbf{Q}_h : \quad \mathbf{b}((\mathbf{v}_h,\mathbf{0}),\widetilde{a}_h) := \int_\Omega \mathbf{v}_h\cdot\mathbf{div}(\widetilde{a}_h) = 0 \quad \forall\,\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\Big\}\,,$$

which, using from (4.205) that $\mathbf{div}\big(\mathbf{Q}_h\big) \subseteq \mathbf{H}_h^{\mathbf{u}}$, reduces to

$$\mathbf{Q}_{0,h} = \Big\{\widetilde{a}_h \in \mathbf{Q}_h : \quad \mathbf{div}\big(\widetilde{a}_h\big) = 0 \quad \text{in} \quad \Omega\Big\}\,.$$

Next, we proceed as in [11, Lemma 4.2] and apply the abstract equivalence result provided by [29, Lemma 5.1] to the setting $X = \mathbf{H}_h^{\mathbf{u}}$, $Y = Y_1 = \mathbb{H}_h^{\mathbf{t}}$, $Y_2 = \{\mathbf{0}\}$, $V = \mathbf{V}_h$, $Z = \mathbf{Q}_h$, and $Z_0 = \mathbf{Q}_{0,h}$, where X, Y, Y_1, Y_2, V, Z, and Z_0 correspond to the notations employed in [29, Lemma 5.1]. As a consequence of it, we deduce that (4.211) and (4.212) are jointly equivalent to the existence of positive constants $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, independent of h, such that there hold

$$\sup_{\substack{\widetilde{a}_h\in\mathbf{Q}_h\\ \widetilde{a}_h\neq\mathbf{0}}} \frac{\mathbf{b}((\mathbf{v}_h,0),\widetilde{a}_h)}{\|\widetilde{a}_h\|_{\mathbf{Q}}} = \sup_{\substack{\widetilde{a}_h\in\mathbf{Q}_h\\ \widetilde{a}_h\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{v}_h\cdot\mathbf{div}(\widetilde{a}_h)}{\|\widetilde{a}_h\|_{\mathbf{Q}}} \geq \boldsymbol{\beta}_1\,\|\mathbf{v}_h\|_{0,4;\Omega} \quad \forall\,\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{u}}\,, \tag{4.213}$$

and

$$\sup_{\substack{\mathbf{s}_h\in\mathbb{H}_h^{\mathbf{t}}\\ \mathbf{s}_h\neq\mathbf{0}}} \frac{\mathbf{b}((\mathbf{0},\mathbf{s}_h),\widetilde{a}_h)}{\|\mathbf{s}_h\|_{0,\Omega}} = \sup_{\substack{\mathbf{s}_h\in\mathbb{H}_h^{\mathbf{t}}\\ \mathbf{s}_h\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{s}_h:\widetilde{a}_h}{\|\mathbf{s}_h\|_{0,\Omega}} \geq \boldsymbol{\beta}_2\,\|\widetilde{a}_h\|_{\mathbf{Q}} \quad \forall\,\widetilde{a}_h\in\mathbf{Q}_{0,h}\,. \tag{4.214}$$

Regarding (4.213), we stress that this result was already established in [29, Lemma 5.5]. In turn, for the proof of (4.214), we first recall from [44, proof of Theorem 3.3] that, being $\mathbf{Q}_h \subseteq \mathbb{RT}_k(\mathcal{T}_h)$, there holds $\mathbf{Q}_{0,h} \subseteq \mathbb{P}_k(\mathcal{T}_h)$. In this way, given $\widetilde{a}_h \in \mathbf{Q}_{0,h}$, it is clear that $\boldsymbol{\tau}_h^{\mathbf{d}} \in \mathbb{H}_h^{\mathbf{t}}$, and hence bounding below the supremum in (4.214) with $\mathbf{s}_h := \boldsymbol{\tau}_h^{\mathbf{d}}$, and employing (4.92) for $t = 4/3$, gives the required inequality with $\boldsymbol{\beta}_2 := C_{4/3}^{1/2}$. □

Now, as far as **(H.3)** and **(H.4)** are concerned, we observe from (4.205) that $\mathbf{div}(\mathrm{H}_h) \subseteq \mathrm{Q}_h$, which confirms the former hypothesis, whereas the latter is proved in [52, Lemma 4.8].

On the other hand, in order to address the verification of **(H.5)** and **(H.6)**, we first notice from (4.205) that $\mathbf{div}\big(X_{i,h}\big) \subseteq M_{i,h}$ for all $i \in \{1,2\}$. Thus, being the pairs $\big(X_{2,h}, M_{2,h}\big)$ and $\big(X_{1,h}, M_{1,h}\big)$ algebraically equal, the corresponding discrete kernels of the bilinear forms b_1 and b_2 (cf. (4.58)) coincide as well, and it is easily seen that they become the space

$$K_h^k := \Big\{\mathbf{q}_h \in \mathbf{RT}_k\big(\mathcal{T}_h\big) : \quad \mathbf{div}(\mathbf{q}_h) = 0 \quad \text{in} \quad \Omega\Big\}. \tag{4.215}$$

In turn, analogously to (4.209), we let $\Theta_h^k : \mathbf{L}^1(\Omega) \to K_h^k$ be the projector defined for each $\mathbf{r} \in \mathbf{L}^1(\Omega)$ as the unique $\Theta_h^k(\mathbf{r}) \in K_h^k$ satisfying

$$\int_\Omega \Theta_h^k(\mathbf{r})\cdot\mathbf{q}_h = \int_\Omega \mathbf{r}\cdot\mathbf{q}_h \quad \forall\,\mathbf{q}_h \in K_h^k\,. \tag{4.216}$$

Then, we recall from [39, Theorem 3.1] (see also [52, Lemma 4.2] for a slight variant of it), that in the 2D case, given $t \in (1,+\infty)$ and an integer $k \geq 0$, there exist positive constants C_t^k and $\bar{C}_t^k$, independent of h, such that, defining

$$c_t^k := \begin{cases} C_t^k & \text{if } \Omega \text{ is convex,}\\ \bar{C}_t^k\{-\log(h)\}^{|1-2/t|} & \text{if } \Omega \text{ is non-convex and } k=0,\\ \bar{C}_t^k & \text{if } \Omega \text{ is non-convex and } k\geq 1 \end{cases}$$

there holds

$$\|\Theta_h^k(\mathbf{r})\|_{0,t;\Omega} \leq c_t^k\,\|\mathbf{r}\|_{0,t;\Omega} \qquad \forall\,\mathbf{r}\in\widetilde{\mathbf{H}}^t(\mathbf{div}_j;\Omega)\,, \tag{4.217}$$

where

$$\widetilde{\mathbf{H}}^t(\mathbf{div}_j;\Omega) := \Big\{ \mathbf{r}\in \mathbf{H}^t(\mathbf{div}_j;\Omega) : \quad \mathbf{div}(\mathbf{r}) = 0 \quad \text{in} \quad \Omega \Big\}.$$

We stress here that only when Ω is non-convex and $k=0$, c_t^k depends on h, though in a very harmless manner. In fact, the term $\{-\log(h)\}^{|1-2/t|}$ grows very slowly when h approaches 0, and thus it remains reasonably bounded for very small values of the mesh size. In particular, taking $t = 3/2$, which lies in the range for s (cf. (4.31)), index with which (4.217) will be applied below, we observe that for $h \geq 10^{-10}$ there holds $\{-\log(h)\}^{|1-2/t|} = \{-\log(h)\}^{1/3} < 3$. Additionally, we remark that whether the boundedness property (4.217) is satisfied or not in 3D is still an open problem, and hence the hypothesis **(H.5)**, to be established next by using (4.217), constitutes the only aspect of the analysis of the present section that is not valid in 3D. All the other stability conditions hold in both 2D and 3D.

Lemma 4.19. *There exists a positive constant $\alpha_{\mathtt{d}}$, independent of h, such that*

$$\sup_{\substack{\mathbf{q}_h\in K_h^k\\ \mathbf{q}_h\neq 0}} \frac{a(\mathbf{r}_h,\mathbf{q}_h)}{\|\mathbf{q}_h\|_{X_1}} \geq \alpha_{\mathtt{d}}\,\|\mathbf{r}_h\|_{X_2} \qquad \forall\,\mathbf{r}_h\in K_h^k\,, \tag{4.218}$$

and

$$\sup_{\mathbf{r}_h\in K_h^k} a(\mathbf{r}_h,\mathbf{q}_h) > 0 \qquad \forall\,\mathbf{q}_h\in K_h^k,\ \mathbf{q}_h\neq \mathbf{0}\,. \tag{4.219}$$

Proof. Indeed, given $\mathbf{r}_h \in K_h^k$ (cf. (4.215)), $\mathbf{r}_h\neq\mathbf{0}$, one first defines $\mathbf{r}_{h,s} := |\mathbf{r}_h|^{r-2}\,\mathbf{r}_h$, which belongs to $\mathbf{L}^s(\Omega)$. Note from (4.31) that $r>2$. Next, bounding below the supremum in (4.218) with $\mathbf{q}_h := \Theta_h^k\big(D_s(\mathbf{r}_{h,s})\big) \in K_h^k$, and then employing (4.216), (4.114) (cf. Lemma 4.3), and the boundedness of Θ_h^k (cf. (4.217)) and D_s (cf. Lemma 4.3), we arrive at (4.218) with $\alpha_{\mathtt{d}} := \big(c_s^k\,\|D_s\|\big)^{-1}$. A similar procedure is applied to derive (4.219). We omit further details and refer to the proof of [52, Lemma 4.3]. □

We now employ the notations and results from the Appendix (cf. Section 4.7) to prove **(H.6)**, that is the discrete inf-sup conditions for the bilinear forms b_i, $i\in\{1,2\}$. Actually, being the proof for $i=1$ a slight modification of that for [52, Lemma 4.5], we omit its details and just focus on the case $i=2$.

Lemma 4.20. *There exists a positive constant $\beta_{2,\mathtt{d}}$, independent of h, such that*

$$\sup_{\substack{\mathbf{q}_h\in X_{2,h}\\ \mathbf{q}_h\neq \mathbf{0}}} \frac{b_2(\mathbf{q}_h,\phi_h)}{\|\mathbf{q}_h\|_{X_2}} \geq \beta_{2,\mathtt{d}}\,\|\phi_h\|_{M_2} \qquad \forall\,\phi_h\in M_{2,h}\,. \tag{4.220}$$

Proof. Given $\phi_h\in M_{2,h}$, we set $\phi_{h,j} := |\phi_h|^{l-2}\,\phi_h$, which belongs to $\mathrm{L}^j(\Omega)$, and notice that

$$\int_\Omega \phi_{h,j}\,\phi_h = \|\phi_{h,j}\|_{0,j;\Omega}\,\|\phi_h\|_{0,l;\Omega}\,. \tag{4.221}$$

Note from (4.32) that $l\geq 2$. Also, we let $\mathcal{O}$ be a bounded convex polygonal domain containing $\bar{\Omega}$, and set

$$g := \begin{cases} \phi_{h,j} & \text{in} \quad \Omega\,,\\ 0 & \text{in} \quad \mathcal{O}\setminus\overline{\Omega}\,. \end{cases}$$

It is clear that $g \in \mathrm{L}^j(\mathcal{O})$ and $\|g\|_{0,j;\mathcal{O}} = \|\phi_{h,j}\|_{0,j;\Omega}$. Then, applying the elliptic regularity result provided in [43, Corollary 1], we deduce that there exists a unique $z \in \mathrm{W}^{2,j}(\mathcal{O}) \cap \mathrm{W}^{1,j}_0(\mathcal{O})$ such that: $\Delta z = g$ in $\mathcal{O}$, $z = 0$ on $\partial\mathcal{O}$, and there exists a positive constant $C_{\mathtt{reg}}$, depending only on $\mathcal{O}$, such that

$$\|z\|_{2,j;\mathcal{O}} \le C_{\mathtt{reg}} \|g\|_{0,j;\mathcal{O}} = C_{\mathtt{reg}} \|\phi_{h,j}\|_{0,j;\Omega} . \tag{4.222}$$

Thus, defining $\mathbf{r} := \nabla z|_\Omega \in \mathbf{W}^{1,j}(\Omega)$, we observe that $\mathbf{div}(\mathbf{r}) = \phi_{h,j}$ in Ω, and, using (4.222), there holds

$$\|\mathbf{r}\|_{1,j;\Omega} \le \|z\|_{2,j;\mathcal{O}} \le C_{\mathtt{reg}} \|\phi_{h,j}\|_{0,j;\Omega} . \tag{4.223}$$

In addition, letting $\mathbf{r}_h$ be the global Raviart-Thomas interpolant of $\mathbf{r}$, that is $\mathbf{r}_h := \Pi^k_h(\mathbf{r})$, and employing (4.227), we find that

$$\mathbf{div}(\mathbf{r}_h) = \mathbf{div}\big(\Pi^k_h(\mathbf{r})\big) = \mathcal{P}^k_h\big(\mathbf{div}(\mathbf{r})\big) = \mathcal{P}^k_h(\phi_{h,j}) , \tag{4.224}$$

so that, thanks to the stability estimate (4.230), it follows that

$$\|\mathbf{div}(\mathbf{r}_h)\|_{0,j;\Omega} \le C_{\mathcal{P}} \|\phi_{h,j}\|_{0,j;\Omega} . \tag{4.225}$$

In turn, noting from (4.31) and (4.32) that $j < r \le \dfrac{nj}{n-j}$, Lemma 4.24 and (4.223) yield

$$\|\mathbf{r}_h\|_{0,r;\Omega} = \|\Pi^k_h(\mathbf{r})\|_{0,r;\Omega} \le C_\Pi \|\mathbf{r}\|_{1,j;\Omega} \le C_\Pi C_{\mathtt{reg}} \|\phi_{h,j}\|_{0,j;\Omega} ,$$

which, jointly with (4.225), imply

$$\|\mathbf{r}_h\|_{X_2} = \|\mathbf{r}_h\|_{0,r;\Omega} + \|\mathbf{div}(\mathbf{r}_h)\|_{0,j;\Omega} \le \big(C_{\mathcal{P}} + C_\Pi C_{\mathtt{reg}}\big) \|\phi_{h,j}\|_{0,j;\Omega} . \tag{4.226}$$

Finally, bounding below the supremum in (4.220) with $\mathbf{r}_h \in X_{2,h}$, and using (4.224), (4.210), (4.221), and (4.226), we conclude the required discrete inf-sup condition for b_2 with $\beta_{2,\mathtt{d}} := \big(C_{\mathcal{P}} + C_\Pi C_{\mathtt{reg}}\big)^{-1}$. □

4.5.3 The rates of convergence

In this section we provide the rates of convergence of the Galerkin scheme (4.164) with the specific finite element subspaces introduced in Section 4.5.1. To this end, we first collect the approximation properties of the latter. Indeed, it is easily seen from (4.228) and its corresponding vector and tensorial versions, along with interpolation estimates of Sobolev spaces, that those of $\mathbf{H}^{\mathbf{u}}_h$, $\mathbb{H}^{\mathbf{t}}_h$, Q_h, and $M_{1,h}$, are given as follows

$(\mathbf{AP}^{\mathbf{u}}_h)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,4}(\Omega)$, there holds

$$\mathrm{dist}\,\big(\mathbf{v}, \mathbf{H}^{\mathbf{u}}_h\big) := \inf_{\mathbf{v}_h \in \mathbf{H}^{\mathbf{u}}_h} \|\mathbf{v} - \mathbf{v}_h\|_{0,4;\Omega} \le C\, h^l \, \|\mathbf{v}\|_{l,4;\Omega} ,$$

$\big(\mathbf{AP}^{\mathbf{t}}_h\big)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{s} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathtt{tr}}(\Omega)$, there holds

$$\mathrm{dist}\,\big(\mathbf{s}, \mathbb{H}^{\mathbf{t}}_h\big) := \inf_{\mathbf{s}_h \in \mathbb{H}^{\mathbf{t}}_h} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \le C\, h^l \, \|\mathbf{s}\|_{l,\Omega} ,$$

$\left(\mathbf{AP}_h^{\eta}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\xi \in \mathrm{W}^{l,\rho}(\Omega)$, there holds

$$\operatorname{dist}\big(\xi, \mathrm{Q}_h\big) \,:=\, \inf_{\xi_h \in \mathrm{Q}_h} \|\xi \,-\, \xi_h\|_{0,\rho;\Omega} \,\leq\, C\,h^l\,\|\xi\|_{l,\rho;\Omega}\,,$$

$(\mathbf{AP}_{\mathbf{h}}^{\phi})$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\psi \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}\big(\psi, M_{1,h}\big) \,:=\, \inf_{\psi_h \in M_{1,h}} \|\psi \,-\, \psi_h\|_{0,r;\Omega} \,\leq\, C\,h^l\,\|\mathbf{r}\|_{l,r;\Omega}\,.$$

In turn, from [52, eq. (4.6), Section 4.1] and its tensorial version, along with interpolation estimates of Sobolev spaces as well, we obtain the approximation properties of $\mathbf{Q}_h$ and H_h, which reduce to

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,4/3}(\Omega)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\tau}, \mathbf{Q}_h\big) \,:=\, \inf_{\widetilde{a}_h \in \mathbf{Q}_h} \|\boldsymbol{\tau} \,-\, \widetilde{a}_h\|_{\mathbf{div}_{4/3};\Omega} \,\leq\, C\,h^l\,\Big\{\|\boldsymbol{\tau}\|_{l,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{l,4/3;\Omega}\Big\}\,,$$

$\left(\mathbf{AP}_h^{\widetilde{\boldsymbol{\sigma}}}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}^l(\Omega)$ with $\mathbf{div}(\widetilde{\boldsymbol{\tau}}) \in \mathrm{W}^{l,\varrho}(\Omega)$, there holds

$$\operatorname{dist}\big(\widetilde{\boldsymbol{\tau}}, \mathrm{H}_h\big) \,:=\, \inf_{\widetilde{\boldsymbol{\tau}}_h \in \mathrm{H}_h} \|\widetilde{\boldsymbol{\tau}} \,-\, \widetilde{\boldsymbol{\tau}}_h\|_{\mathbf{div}_{\varrho};\Omega} \,\leq\, C\,h^l\,\Big\{\|\widetilde{\boldsymbol{\tau}}\|_{l,\Omega} \,+\, \|\mathbf{div}(\widetilde{\boldsymbol{\tau}})\|_{l,\varrho;\Omega}\Big\}\,.$$

Finally, that of $X_{2,h}$, which follows from Lemma 4.23 and (4.229) (with $m = 0$), and applying again interpolation estimates of Sobolev spaces, becomes

$(\mathbf{AP}_{\mathbf{h}}^{\mathbf{p}})$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\mathbf{q} \in \mathbf{W}^{l,r}(\Omega)$ with $\mathbf{div}(\mathbf{q}) \in \mathrm{W}^{l,j}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{q}, X_{2,h}\big) \,:=\, \inf_{\mathbf{q}_h \in X_{2,h}} \|\mathbf{q} - \mathbf{q}_h\|_{r,\mathrm{div}_j;\Omega} \,\leq\, C\,h^l\,\Big\{\|\mathbf{q}\|_{l,r;\Omega} \,+\, \|\mathbf{div}(\mathbf{q})\|_{l,j;\Omega}\Big\}\,.$$

Hence, we can state the following main theorem.

Theorem 4.21. *Let* $\big((\vec{\mathbf{u}}, \boldsymbol{\sigma}), (\widetilde{\boldsymbol{\sigma}}, \eta), (\mathbf{p}, \varphi)\big) \in (\mathbf{H} \times \mathbf{Q}) \times (\mathrm{H} \times \mathrm{Q}) \times (X_2 \times M_1)$ *be the unique solution of* (4.69) *with* $(\mathbf{u}, \mathbf{p}) \in W_\delta$ *(cf.* (4.130)*), and let* $\big((\vec{\mathbf{u}}_h, h), (\widetilde{\boldsymbol{\sigma}}_h, \eta_h), (\mathbf{p}_h, \varphi_h)\big) \in (\mathbf{H}_h \times \mathbf{Q}_h) \times (\mathrm{H}_h \times \mathrm{Q}_h) \times (X_{2,h} \times M_{1,h})$ *be a solution of* (4.164) *with* $(\mathbf{u}_h, \mathbf{p}_h) \in W_{\delta_{\mathtt{d}}}$ *(cf.* (4.183)*), which is guaranteed by Theorems* 4.12 *and* 4.16*, respectively. In turn, let* p *and* p_h *given by* (4.4) *and* (4.202)*, respectively. Assume the hypotheses of Theorem* 4.17*, and that there exists* $l \in [1, k+1]$ *such that* $\mathbf{u} \in \mathbf{W}^{l,4}(\Omega)$, $\mathbf{t} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathbf{tr}}(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,4/3}(\Omega)$, $\widetilde{\boldsymbol{\sigma}} \in \mathbf{H}^l(\Omega)$, $\mathbf{div}(\widetilde{\boldsymbol{\sigma}}) \in \mathrm{W}^{l,\varrho}(\Omega)$, $\eta \in \mathrm{W}^{l,\rho}(\Omega)$, $\mathbf{p} \in \mathbf{W}^{l,r}(\Omega)$, $\mathbf{div}(\mathbf{p}) \in \mathrm{W}^{l,j}(\Omega)$, *and* $\varphi \in \mathrm{W}^{l,r}(\Omega)$*. Then, there exists a positive constant* C*, independent of* h*, such that*

$$\begin{aligned}
&\|(\vec{\mathbf{u}}, \boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \|(\widetilde{\boldsymbol{\sigma}}, \eta) - (\widetilde{\boldsymbol{\sigma}}_h, \eta_h)\|_{\mathrm{H}\times\mathrm{Q}} \,+\, \|(\mathbf{p}, \varphi) - (\mathbf{p}_h, \varphi_h)\|_{X_2\times M_1} \,+\, \|p - p_h\|_{0,\Omega} \\
&\qquad \leq\, C\,h^l\,\Big\{\|\mathbf{u}\|_{l,4;\Omega} \,+\, \|\mathbf{t}\|_{l,\Omega} \,+\, \|\boldsymbol{\sigma}\|_{l,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,4/3;\Omega} \,+\, \|\widetilde{\boldsymbol{\sigma}}\|_{l,\Omega} \\
&\qquad +\|\mathbf{div}(\widetilde{\boldsymbol{\sigma}})\|_{l,\varrho;\Omega} \,+\, \|\eta\|_{l,\rho;\Omega} \,+\, \|\mathbf{p}\|_{l,r;\Omega} \,+\, \|\mathbf{div}(\mathbf{p})\|_{l,j;\Omega} \,+\, \|\varphi\|_{l,r;\Omega}\Big\}\,.
\end{aligned}$$

Proof. It follows straightforwardly from Theorem 4.17, (4.204), and the above approximation properties. □

4.6 Numerical results

In this section we present three examples illustrating the performance of the fully-mixed finite element method (4.164) on a set of quasi-uniform triangulations of the respective domains, and considering the finite element subspaces defined by (4.205) (cf. Section 4.5.1). In what follows, we refer to the corresponding sets of finite element subspaces generated by $k = 0$ and $k = 1$, as simply $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ and $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{RT}_1 - \mathrm{P}_1 - \mathbf{RT}_1 - \mathrm{P}_1$, respectively. The implementation of the numerical method is based on a `FreeFem++` code [58]. A Newton–Raphson algorithm with a fixed tolerance $\mathsf{tol} = 1\mathrm{E}-6$ is used for the resolution of the nonlinear problem (4.164). As usual, the iterative method is finished when the relative error between two consecutive iterations of the complete coefficient vector, namely $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|}{\|\mathbf{coeff}^{m+1}\|} \leq \mathsf{tol}\,,$$

where $\|\cdot\|$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{DOF}}$ with `DOF` denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h^{\mathbf{u}}, \mathbb{H}_h^{\mathbf{t}}, \mathbb{H}_h^{\boldsymbol{\sigma}}, \mathrm{H}_h, \mathrm{Q}_h, X_{2,h}$, and $M_{1,h}$ (cf. (4.205)).

We now introduce some additional notation. The individual errors are denoted by:

$$\mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}, \quad \mathsf{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}, \quad \mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - h\|_{\mathbf{div}_{4/3};\Omega}, \quad \mathsf{e}(p) := \|p - p_h\|_{0,\Omega},$$

$$\mathsf{e}(\widetilde{\boldsymbol{\sigma}}) := \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{div}_\varrho;\Omega}, \quad \mathsf{e}(\eta) := \|\eta - \eta_h\|_{0,\rho;\Omega}, \quad \mathsf{e}(\mathbf{p}) := \|\mathbf{p} - \mathbf{p}_h\|_{r,\mathbf{div}_j;\Omega}, \quad \mathsf{e}(\varphi) := \|\varphi - \varphi_h\|_{0,r;\Omega},$$

where ϱ, ρ, r and j are described in (4.31)–(4.32), and will be specified in the examples below. Next, as usual, for each $\star \in \{\mathbf{u}, \mathbf{t}, \boldsymbol{\sigma}, p, \widetilde{\boldsymbol{\sigma}}, \eta, \mathbf{p}, \varphi\}$ we let $\mathsf{r}(\star)$ be the experimental rate of convergence given by $\mathsf{r}(\star) := \log\big(\mathsf{e}(\star)/\widehat{\mathsf{e}}(\star)\big)/\log(h/\widehat{h})$, where h and $\widehat{h}$ denote two consecutive meshsizes with errors e and $\widehat{\mathsf{e}}$, respectively.

The examples to be considered in this section are described next. In the first two examples, for the sake of simplicity, we take $\nu = 1$, $\lambda = 1$, $\kappa_\eta = 1$, $\mu = 1$, $\kappa_\varphi = 1$, and $\gamma = 1$. In addition, the mean value of $\operatorname{tr}(h)$ over Ω is fixed via a Lagrange multiplier strategy (adding one row and one column to the matrix system that solves (4.166) for $\mathbf{u}_h, \mathbf{t}_h$, and h).

Example 1: Convergence against smooth exact solutions in a 2D domain

In this test we corroborate the rates of convergence in a non-convex two-dimensional domain. We consider an L-shaped domain $\Omega = (-1,1)^2 \setminus (0,1)^2$. We choose $j = l = 2$, whence the remaining parameters become $r = \rho = 4$ and $\varrho = 4/3$ (cf. (4.31)–(4.32)). In turn, we consider the given function $f(x_1,x_2) = \sin(x_1+x_2)$, and choose the data $\mathbf{f}, f_\eta, f_\varphi$ (cf. (4.6)) such that the exact solution is given by

$$\mathbf{u}(x_1,x_2) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \quad p(x_1,x_2) = \cos(\pi x_1)\exp(x_2)\,,$$

$$\eta(x_1,x_2) = 0.5 + 0.5\cos(x_1 x_2)\,, \quad \text{and} \quad \varphi(x_1,x_2) = 0.1 + 0.3\exp(x_1 x_2)\,.$$

The model problem is then complemented with the appropriate Dirichlet boundary conditions. Tables 4.1 and 4.2 show the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations. Notice that we are able not only to approximate the original unknowns but also the pressure field through the formula (4.202). The results confirm that the optimal rates of convergence $\mathcal{O}(h^{k+1})$ predicted by Theorem 4.21 are attained for $k=0,1$. The Newton method exhibits a behavior independent of the meshsize, converging in five iterations in all cases. In addition, in the case $k=0$, and since $\mathbf{f}_{\mathtt{NS}}$, $f_{\mathtt{CD}}$, and $f_{\mathtt{CS}}$ (cf. (4.206) - (4.208)) are not necessarily piecewise constant, we observe that our Galerkin scheme provides conservation of momentum in an approximate sense. We illustrate this fact in Table 4.3, where the computed ℓ^∞-norm for $\mathbf{div}(h) - \boldsymbol{\mathcal{P}}_h^0(\mathbf{f}_{\mathtt{NS}})$, $\mathbf{div}(\widetilde{\boldsymbol{\sigma}}_h) + \mathcal{P}_h^0(f_{\mathtt{CD}})$, and $\mathbf{div}(\mathbf{p}_h) - \mathcal{P}_h^0(f_{\mathtt{CS}})$, are displayed. As expected, these values are certainly close to zero.

Example 2: Convergence against smooth exact solutions in a 3D domain

In the second example we consider the cube domain $\Omega = (0,1)^3$ and the only possible choice of parameters in 3D, that is $j=3/2$, $r=3$, $\rho=6$, and $\varrho = 6/5$ (cf. (4.31)–(4.32)). The solution is given by

$$\mathbf{u}(x_1,x_2,x_3) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix}, \quad p(x_1,x_2,x_3) = \cos(\pi x_1)\exp(x_2+x_3)\,,$$
$$\eta(x_1,x_2,x_3) = 0.5 + 0.5\cos(x_1x_2x_3)\,, \quad \text{and} \quad \varphi(x_1,x_2,x_3) = 0.1 + 0.3\exp(x_1x_2x_3)\,.$$

Similarly to the first example, we consider $f(x_1,x_2,x_3) = \sin(x_1+x_2+x_3)$, whereas the data $\mathbf{f}, f_\eta, f_\varphi$ are computed from (4.6) using the above solution. The convergence history for a set of quasi-uniform mesh refinements using $k=0$ is shown in Table 4.4. Again, the mixed finite element method converges optimally with order $\mathcal{O}(h)$, as it was proved by Theorem 4.21. In addition, some components of the numerical solution are displayed in Figure 4.1, which were built using the fully-mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation with meshsize $h = 0.0643$ and $63,888$ tetrahedral elements (actually representing $1,483,944$ DOF). The numerical results suggest that perhaps only technical difficulties stop us from proving (4.217) for the 3D framework.

Example 3: Movement of cells guided by the concentration of a chemical signal

In the last example, inspired by [38, Test1, Section 7], we consider the rectangle domain $\Omega = (0,2)\times(0,1)$, and the unsteady version of the problem (4.6) with physical parameters $\nu = 10, \lambda = 1, \kappa_\eta = 4, \mu = 8, \kappa_\varphi = 1, \gamma = 6$, data $f(x_1,x_2) = -1000\,x_2$, $\mathbf{f} = 0, f_\eta = 0, f_\varphi = 0$, boundary conditions $\mathbf{u} = 0$ on Γ, $\widetilde{\boldsymbol{\sigma}}\cdot\mathbf{n} = \widetilde{\mathbf{g}}$ on Γ, $\mathbf{p}\cdot\mathbf{n} = \mathbf{g}$ on Γ, where, we distinct two cases. Case 1: $\widetilde{\mathbf{g}} = \mathbf{g} = 0$ (cf. (4.71)). Case 2: $\widetilde{\mathbf{g}}$ and $\mathbf{g}$ are not homogeneous (cf. (4.75)), and are defined as follows

$$\widetilde{\mathbf{g}} := \begin{cases} 200\,x_2\,(1-x_2) & \text{on } \Gamma_\ell\,, \\ 0 & \text{on } \partial\Omega\setminus\Gamma_\ell\,, \end{cases} \quad \text{and} \quad \mathbf{g} := \begin{cases} 200\,\exp(1-x_2) & \text{on } \Gamma_\ell\,, \\ 200\,\exp(2-x_1) & \text{on } \Gamma_{\mathrm{b}}\,, \\ 0 & \text{on } \partial\Omega\setminus\big(\Gamma_\ell\cup\Gamma_{\mathrm{b}}\big)\,, \end{cases}$$

where Γ_ℓ and Γ_b represent the left and lower part of the boundary Γ. In both cases, we consider the initial conditions

$$\mathbf{u}_0 = 0, \quad \eta_0 = \sum_{i=1}^{3} 70 \exp(-8(x_1 - s_i)^2 - 10(x_2 - 1)^2), \quad \varphi_0 = 30 \exp(-5(x_1 - 1)^2 - 5(x_2 - 0.5)^2),$$

where $s_1 = 0.2, s_2 = 0.5$ and $s_3 = 1.2$. We employ a suitable backward Euler time discretization, with time step $\Delta t = 10^{-5}$ and final time $T = 5 \times 10^{-3}$. We observe that at each time step we are solving a slight adaptation of the stationary problem (4.164) for Case 1, whereas a discrete version of the Lagrange multiplier approach detailed in (4.76)–(4.77) is required for Case 2 (see [44, Section 4.4] for details of a similar approach). Next, we discuss the numerical results obtained in each case. We begin by detailing the Case 1, which is in agreement with [38, Test1, Section 7]. In Figure 4.2, we display the computed magnitude of the velocity, and the cell density and chemical signal concentration fields, which were built using the fully-mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation on a mesh with meshsize $h = 0.0298$ and $18,566$ triangle elements (actually representing $242,126$ `DOF`). Similarly to [38], the cells are in two clusters in the upper part of the domain at time $T = 10^{-5}$, and then they begin to orient their movement in the direction of greater concentration of the chemical signal (the center of the domain) as we can see at time $T = 10^{-3}$, where the organisms tend to agglomerate in the center of the rectangle. This interesting behavior occurs because the chemotaxis/cross-diffusion term is the dominant one in the initial times. However, as time progresses, the chemical signal is consumed, which causes that the cross-diffusion loses strength, and the self-diffusion of the cells begins to dominate, and therefore they begin to distribute themselves homogeneously over the domain. At final time $T = 5 \times 10^{-3}$ the cells move towards the bottom of the domain, which is due to the external force $\nabla f = (0, -1000)$. In addition, some changes in the velocity field are evidenced, influenced by the movement of the cells. Finally, we detail the results for Case 2, which, besides being for academic purposes, illustrates the capability of our method to handle non-homogeneous Neumann boundary conditions, as it is explained in Section 4.2.4. We observe from Figure 4.3 and the middle plots of Figure 4.2 that in this case the magnitudes of the velocity are similar to the ones obtained in Case 1, but the cell density and chemical signal concentration fields are affected on the left and bottom of the domain, respectively, because of the non-homogeneous data $\widetilde{\mathbf{g}}$ and $\mathbf{g}$, as we expected.

4.7 Further properties of the Raviart-Thomas interpolator

We begin by introducing for all t, $s \in (1, +\infty)$ such that $t \geq s$, the space

$$\mathbf{H}_s^t := \Big\{ \boldsymbol{\tau} \in \mathbf{H}^t(\mathbf{div}_s; \Omega) : \quad \boldsymbol{\tau}|_K \in \mathbf{W}^{1,s}(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

and let $\Pi_h^k : \mathbf{H}_s^t \to \mathbf{RT}_k(\mathcal{T}_h)$ be the global Raviart-Thomas interpolation operator (cf. [14, Section 2.5]). Then, we recall from [14, Proposition 2.5.2 and eq. (2.5.27)] that the commuting diagram property states that

$$\mathbf{div}\big(\Pi_h^k(\mathbf{q})\big) \;=\; \mathcal{P}_h^k\big(\mathbf{div}(\mathbf{q})\big) \qquad \forall\, \mathbf{q} \in \mathbf{H}_s^t\,, \tag{4.227}$$

where $\mathcal{P}_h^k : \mathrm{L}^1(\Omega) \to \mathrm{P}_k(\mathcal{T}_h)$ is the projector defined by (4.210). In turn, employing the $\mathrm{W}^{m,t}$ version of the Deny-Lions Lemma (cf. [41, Lemma B.67]) with integer $m \geq 0$ and $t \in (1, +\infty)$, along with the associated scaling estimates (cf. [41, Lemma 1.101]) and the regularity of $\big\{\mathcal{T}_h\big\}_{h>0}$, we deduce

DOF	h	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(p)	r(p)
1416	0.4002	2.98E-01	–	1.59E-00	–	8.62E-00	–	5.70E-01	–
3398	0.2652	1.87E-01	1.133	1.03E-00	1.058	5.44E-00	1.119	3.43E-01	1.235
8922	0.1551	1.15E-01	0.914	6.26E-01	0.928	3.32E-00	0.920	2.11E-01	0.904
28498	0.0892	6.43E-02	1.046	3.51E-01	1.047	1.86E-00	1.046	1.18E-01	1.059
102828	0.0502	3.36E-02	1.127	1.85E-01	1.115	9.72E-01	1.131	6.03E-02	1.160
382112	0.0258	1.74E-02	0.988	9.54E-02	0.992	5.03E-01	0.988	3.13E-02	0.984

e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	e(η)	r(η)	e($\mathbf{p}$)	r($\mathbf{p}$)	e(φ)	r(φ)	iter
5.21E-01	–	2.02E-02	–	1.51E-01	–	2.84E-02	–	5
3.29E-01	1.122	1.32E-02	1.042	9.85E-02	1.034	1.90E-02	0.979	5
2.07E-01	0.866	7.86E-03	0.963	5.98E-02	0.930	1.16E-02	0.925	5
1.16E-01	1.049	4.67E-03	0.942	3.32E-02	1.062	6.61E-03	1.011	5
6.10E-02	1.110	2.50E-03	1.088	1.84E-02	1.031	3.82E-03	0.951	5
3.15E-02	0.994	1.32E-03	0.952	9.54E-03	0.982	2.03E-03	0.951	5

Table 4.1: Example 1, Number of degrees of freedom, meshsizes, errors, rates of convergence, and number of Newton iterations for the fully-mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation of the chemotaxis–Navier–Stokes model.

DOF	h	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(p)	r(p)
4392	0.4002	4.06E-02	–	2.20E-01	–	1.17E-00	–	6.70E-02	–
10606	0.2652	1.79E-02	1.985	8.61E-02	2.280	4.94E-01	2.093	2.66E-02	2.246
27954	0.1551	6.24E-03	1.969	3.22E-02	1.837	1.82E-01	1.857	9.85E-03	1.851
89546	0.0892	1.95E-03	2.106	1.01E-02	2.093	5.67E-02	2.111	2.98E-03	2.160
323676	0.0502	5.38E-04	2.234	2.76E-03	2.253	1.56E-02	2.239	8.20E-04	2.244
1203904	0.0258	1.42E-04	1.995	7.39E-04	1.979	4.17E-03	1.983	2.19E-04	1.983

e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	e(η)	r(η)	e($\mathbf{p}$)	r($\mathbf{p}$)	e(φ)	r(φ)	iter
6.52E-02	–	1.59E-03	–	8.68E-03	–	1.67E-03	–	5
2.85E-02	2.006	7.31E-04	1.888	3.75E-03	2.037	7.23E-04	2.025	5
1.05E-02	1.872	2.44E-04	2.049	1.40E-03	1.843	2.68E-04	1.857	5
3.28E-03	2.097	7.84E-05	2.049	4.37E-04	2.099	8.52E-05	2.074	5
9.16E-04	2.216	2.16E-05	2.239	1.40E-04	1.984	2.88E-05	1.888	5
2.42E-04	1.999	5.88E-06	1.955	3.84E-05	1.937	8.04E-06	1.913	5

Table 4.2: Example 1, Number of degrees of freedom, meshsizes, errors, rates of convergence, and number of Newton iterations for the fully-mixed $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{RT}_1 - \mathrm{P}_1 - \mathbf{RT}_1 - \mathrm{P}_1$ approximation of the chemotaxis-Navier–Stokes model.

the existence of positive constants C_1, C_2, independent of h, such that for integers l and m verifying $0 \le l \le k+1$ and $0 \le m \le l$, there hold

$$|\phi \,-\, \mathcal{P}_k^h(\phi)|_{m,s;\Omega} \,\le\, C_1\, h^{l-m}\, |\phi|_{l,s;\Omega} \tag{4.228}$$

h	0.4002	0.2652	0.1551	0.0892	0.0502	0.0258
$\Vert\mathbf{div}(h) - \boldsymbol{\mathcal{P}}_h^0(\mathbf{f}_{\mathtt{NS}})\Vert_{\ell^\infty}$	1.07E-14	2.13E-14	2.66E-14	5.42E-14	1.51E-13	3.73E-13
$\Vert\mathbf{div}(\widetilde{\boldsymbol{\sigma}}_h) + \mathcal{P}_h^0(f_{\mathtt{CD}})\Vert_{\ell^\infty}$	2.00E-15	5.22E-15	9.10E-15	2.80E-14	4.49E-14	8.42E-14
$\Vert\mathbf{div}(\mathbf{p}_h) - \mathcal{P}_h^0(f_{\mathtt{CS}})\Vert_{\ell^\infty}$	4.28E-09	3.52E-09	3.89E-09	4.07E-09	4.08E-09	4.06E-09

Table 4.3: Example 1, Conservation of momentum for the fully-mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation of the chemotaxis-Navier–Stokes model.

DOF	h	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(p)	r(p)
1224	0.7071	5.74E-01	–	2.63E-00	–	1.50E+01	–	1.18E-00	–
9312	0.3536	3.02E-01	0.927	1.44E-00	0.872	8.00E-00	0.911	6.46E-01	0.874
72576	0.1768	1.55E-01	0.961	7.41E-01	0.955	4.03E-00	0.989	3.00E-01	1.110
384552	0.1010	8.90E-02	0.990	4.27E-01	0.982	2.29E-00	1.007	1.54E-01	1.185
1483944	0.0643	5.68E-02	0.997	2.73E-01	0.992	1.46E-00	1.007	9.17E-02	1.152

e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	e(η)	r(η)	e($\mathbf{p}$)	r($\mathbf{p}$)	e(φ)	r(φ)	iter
6.11E-01	–	3.90E-02	–	2.13E-01	–	4.52E-02	–	5
3.48E-01	0.811	2.34E-02	0.734	1.12E-01	0.929	2.37E-02	0.930	5
1.83E-01	0.927	1.22E-02	0.945	5.66E-02	0.985	1.20E-02	0.982	5
1.06E-01	0.974	6.98E-03	0.995	3.24E-02	0.997	6.87E-03	0.995	5
6.79E-02	0.989	4.44E-03	1.001	2.06E-02	0.999	4.38E-03	0.998	5

Table 4.4: Example 2, Number of degrees of freedom, meshsizes, errors, rates of convergence, and number of Newton iterations for the fully-mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation of the chemotaxis-Navier–Stokes model.

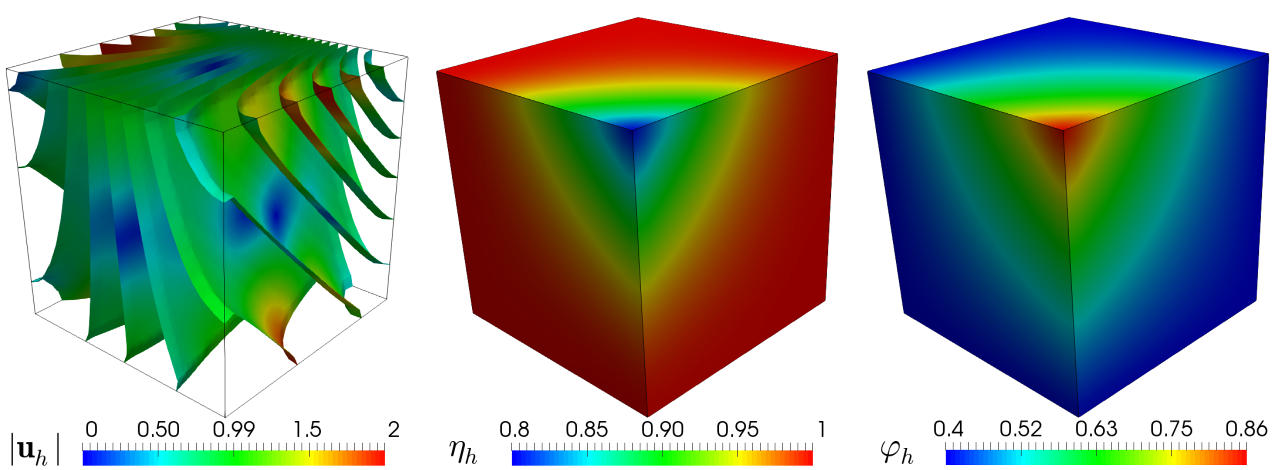

Figure 4.1: Example 2, Computed magnitude of the velocity, cell density field and chemical signal concentration field.

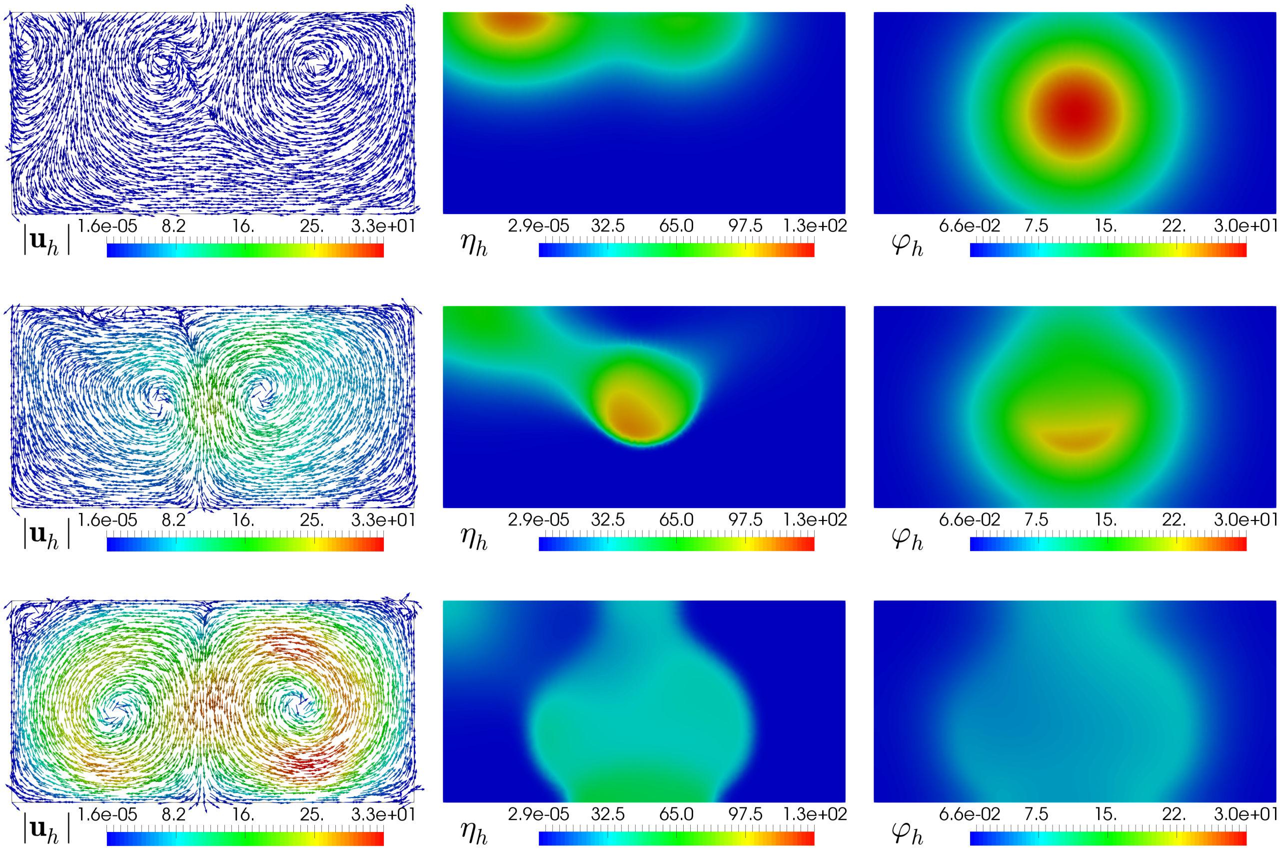

Figure 4.2: Example 3 - Case 1, Computed magnitude of the velocity, cell density field and chemical signal concentration field at time $T = 10^{-5}$ (top plots), at time $T = 10^{-3}$ (middle plots), and at time $T = 5 \times 10^{-3}$ (bottom plots).

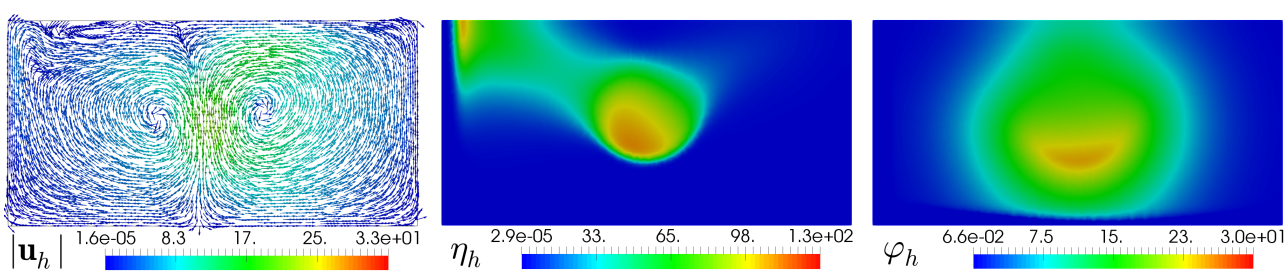

Figure 4.3: Example 3 - Case 2, Computed magnitude of the velocity, cell density field and chemical signal concentration field at time $T = 10^{-3}$.

for all $\phi \in \mathbf{W}^{l,s}(\Omega)$, and

$$|\mathbf{div}(\mathbf{q}) - \mathbf{div}(\Pi_h^k(\mathbf{q}))|_{m,s;\Omega} \le C_2\, h^{l-m}\, |\mathbf{div}(\mathbf{q})|_{l,s;\Omega} \tag{4.229}$$

$\forall\, \mathbf{q} \in \mathbf{W}^{1,s}(\Omega)$ with $\mathbf{div}(\mathbf{q}) \in \mathbf{W}^{l,s}(\Omega)$. Note that (4.229) follows from (4.227) and a direct application of (4.228) to $\phi = \mathbf{div}(\mathbf{q})$. In turn, taking in particular $m = l = 0$ in (4.228), we deduce the stability of $\mathcal{P}_h^k$ with respect to $\|\cdot\|_{0,s;\Omega}$, that is the existence of a positive constant $C_{\mathcal{P}}$, independent of h, such that

$$\|\mathcal{P}_h^k(\phi)\|_{0,s;\Omega} \,\le\, C_{\mathcal{P}}\,\|\phi\|_{0,s;\Omega} \quad \forall\,\phi\in \mathrm{L}^s(\Omega)\,. \tag{4.230}$$

In what follows we prove additional approximation properties of Π_h^k. To this end, we now denote the reference element of $\mathcal{T}_h$ by $\widehat{K}$, so that, given $K\in\mathcal{T}_h$, we let $F_K:\widehat{K}\to K$ be the bijective affine mapping defined by $F_K(\widehat{\mathbf{x}}) := B_K\,\widehat{\mathbf{x}}+b_K\ \forall\,\widehat{\mathbf{x}}\in\widehat{K}$, with $B_K\in\mathrm{R}^{n\times n}$ invertible and $b_K\in\mathrm{R}^n$. Then, the scaling properties via Piola's transformation between $\mathbf{W}^{m,t}(K)$ and $\mathbf{W}^{m,t}(\widehat{K})$, with m a non-negative integer and $t\in(1,+\infty)$, establish the existence of positive constants $\widehat{C}_{\mathtt{P}}$ and $C_{\mathtt{P}}$, such that for each $K\in\mathcal{T}_h$ there hold

$$|\widehat{\mathbf{q}}|_{m,t;\widehat{K}} \,\le\, \widehat{C}_{\mathtt{P}}\,\|B_K\|^m\,\|B_K^{-1}\|\,|\det(B_K)|^{1-1/t}\,|\mathbf{q}|_{m,t;K} \qquad \forall\,\mathbf{q}\in\mathbf{W}^{m,t}(K)\,, \tag{4.231}$$

and

$$|\mathbf{q}|_{m,t;K} \,\le\, C_{\mathtt{P}}\,\|B_K^{-1}\|^m\,\|B_K\|\,|\det(B_K)|^{1/t-1}\,|\widehat{\mathbf{q}}|_{m,t;\widehat{K}} \qquad \forall\,\widehat{\mathbf{q}}\in\mathbf{W}^{m,t}(\widehat{K})\,. \tag{4.232}$$

Then, letting $\Pi_K^k:\mathbf{W}^{1,s}(K)\to\mathbf{RT}_k(K)$ be the local Raviart-Thomas interpolator for each $K\in\mathcal{T}_h$, and letting $\Pi_{\widehat{K}}^k$ be the corresponding operator for $\widehat{K}$, we have the following approximation property.

Lemma 4.22. *Let k and l be integers such that $1\le l\le k+1$, and let t and s such that $1\le t\le \frac{ns}{n-s}$ if $s<n$, or $s\le t<+\infty$ if $s=n$. Then, there exists a positive constant C, depending only on $\widehat{K}$, $\Pi_{\widehat{K}}^k$, k, n, t, and s, such that*

$$\|\mathbf{q}\,-\,\Pi_K^k(\mathbf{q})\|_{0,t;K} \,\le\, C\,h_K^{l+\frac{n}{t}-\frac{n}{s}}\,|\mathbf{q}|_{l,s;K} \qquad \forall\,\mathbf{q}\in\mathbf{W}^{l,s}(K)\,. \tag{4.233}$$

Proof. Given $\mathbf{q}\in\mathbf{W}^{l,s}(K)$, we use (4.232) with $m=0$ to obtain

$$\|\mathbf{q}-\Pi_K^k(\mathbf{q})\|_{0,t;K} \,\le\, C_{\mathtt{P}}\,\|B_K\|\,|\mathrm{det}B_K|^{1/t-1}\,|\widehat{\mathbf{q}}-\Pi_{\widehat{K}}^k(\widehat{\mathbf{q}})|_{0,t;\widehat{K}}\,,$$

which, thanks to the continuous embedding of $\mathbf{W}^{1,s}(\widehat{K})$ in $\mathbf{L}^t(\widehat{K})$ for the indicated ranges of s and t, yields

$$\|\mathbf{q}-\Pi_K^k(\mathbf{q})\|_{0,t;K} \,\le\, C\,\|B_K\|\,|\det(B_K)|^{1/t-1}\,\|\widehat{\mathbf{q}}-\Pi_{\widehat{K}}^k(\widehat{\mathbf{q}})\|_{1,s;\widehat{K}}\,. \tag{4.234}$$

Next, since $\Pi_{\widehat{K}}^k(\widehat{\mathbf{q}})=\widehat{\mathbf{q}}\ \forall\,\widehat{\mathbf{q}}\in\mathbf{RT}_k(\widehat{K})$, and there holds $\mathbf{P}_{l-1}(\widehat{K})\subseteq\mathbf{P}_k(\widehat{K})\subseteq\mathbf{RT}_k(\widehat{K})$, the Bramble-Hilbert Lemma implies that

$$\|\widehat{\mathbf{q}}-\Pi_{\widehat{K}}^k(\widehat{\mathbf{q}})\|_{m,s;\widehat{K}} \,\le\, C\,|\widehat{\mathbf{q}}|_{l,s;\widehat{K}} \quad \text{for } 0\le m\le l,$$

and hence, using in particular the above with $m=1$ we deduce

$$\|\widehat{\mathbf{q}}\,-\,\Pi_{\widehat{K}}^k(\widehat{\mathbf{q}})\|_{1,s;\widehat{K}} \,\le\, C\,|\widehat{\mathbf{q}}|_{l,s;\widehat{K}}\,. \tag{4.235}$$

In this way, replacing (4.235) into (4.234), and then employing (4.231), it follows that

$$\begin{aligned}\|\mathbf{q}\,-\,\Pi_K^k(\mathbf{q})\|_{0,t;K} \,&\le\, C\,\|B_K\|\,|\det(B_K)|^{1/t-1}\,|\widehat{\mathbf{q}}|_{l,s;\widehat{K}}\\ &\le\, C\,\widehat{C}_{\mathtt{P}}\,\|B_K\|^{l+1}\,\|B_K^{-1}\|\,|\det(B_K)|^{1/t-1/s}\,|\mathbf{q}|_{l,s;K}\,,\end{aligned}$$

from which, using that $\|B_K\|\,\le\,C\,h_K$, $\|B_K^{-1}\|\,\le\,C\,h_K^{-1}$, and $|\det(B_K)|\,\cong\,h_K^n$, we arrive at (4.233) and end the proof. □

The extension of Lemma 4.22 to the global Raviart-Thomas interpolator Π_h^k is stated next.

Lemma 4.23. *Let k and l be integers such that $1 \le l \le k+1$, and let t and s such that $1 \le t \le \frac{ns}{n-s}$ if $s < n$, or $s \le t < +\infty$ if $s = n$. Then, with the same constant C from (4.233), there holds*

$$\|\mathbf{q} - \Pi_h^k(\mathbf{q})\|_{0,t;\Omega} \le C\, h^{l+\frac{n}{t}-\frac{n}{s}}\, |\mathbf{q}|_{l,s;\Omega} \qquad \forall\, \mathbf{q} \in \mathbf{W}^{l,s}(\Omega)\,.$$

Proof. Given $\mathbf{q} \in \mathbf{W}^{l,s}(\Omega)$, it suffices to see that

$$\|\mathbf{q} - \Pi_h^k(\mathbf{q})\|_{0,t;\Omega} = \Big\{ \sum_{K\in\mathcal{T}_h} \|\mathbf{q} - \Pi_K^k(\mathbf{q})\|_{0,t;K}^t \Big\}^{1/t} = \Big\{ \Big(\sum_{K\in\mathcal{T}_h} \|\mathbf{q} - \Pi_K^k(\mathbf{q})\|_{0,t;K}^t \Big)^{s/t} \Big\}^{1/s},$$

and then apply the sub-additivity property with exponent $\frac{s}{t} \in (0,1]$, and Lemma 4.22. □

Finally, a simple corollary of Lemma 4.23 reads as follows.

Lemma 4.24. *Let k be integer such that $1 \le k+1$, and let t and s such that $1 \le t \le \frac{ns}{n-s}$ if $s < n$, or $s \le t < +\infty$ if $s = n$. Then, there exists $C_\Pi > 0$, depending only on C, $|\Omega|$, n, t, and s, such that*

$$\|\Pi_h^k(\mathbf{q})\|_{0,t;\Omega} \le C_\Pi\, \|\mathbf{q}\|_{1,s;\Omega} \qquad \forall\, \mathbf{q} \in \mathbf{W}^{1,s}(\Omega)\,. \tag{4.236}$$

Proof. Given $\mathbf{q} \in \mathbf{W}^{1,s}(\Omega)$, the embedding $\mathbf{i}_{s,t} : \mathbf{W}^{1,s}(\Omega) \to \mathbf{L}^t(\Omega)$ and Lemma 4.23 (with $l = 1$) imply

$$\|\Pi_h^k(\mathbf{q})\|_{0,t;\Omega} \le \|\mathbf{q}\|_{0,t;\Omega} + \|\mathbf{q} - \Pi_h^k(\mathbf{q})\|_{0,t;\Omega} \le \|\mathbf{i}_{s,t}\|\, \|\mathbf{q}\|_{1,s;\Omega} + C\,|\Omega|^{1+\frac{n}{t}-\frac{n}{s}}\, \|\mathbf{q}\|_{1,s;\Omega}\,,$$

which yields (4.236) with $C_\Pi := \|\mathbf{i}_{s,t}\| + C\,|\Omega|^{1+\frac{n}{t}-\frac{n}{s}}$. □

Conclusions and future works

Conclusions

In this thesis, we have developed mixed finite element methods based on Banach spaces to numerically solve partial differential equation systems relevant in solid and fluid mechanics. We have specifically focused on the following models:

Stress-Assisted Diffusion: Aiming to develop the necessary tools to tackle this problem in Banach spaces, due to the similarities in the resulting continuous formulations, we initially focused on the continuous analysis of the nearly incompressible linear elasticity problem and the Stokes problem. We reformulated both models with respect to the non-symmetric pseudostress tensor, which allowed us to avoid the weak symmetry impositions. It should be noted that the original Cauchy stress tensor can be obtained from the pseudostress tensor through simple post-processing. Then, using the integration by parts formula suitable for the Sobolev spaces in which we worked, we obtained the corresponding mixed variational formulations. These continuous schemes have the property that the search and test spaces do not coincide, as is usually the case in formulations in a Hilbertian framework, therefore, to establish the existence and uniqueness of the solution, it was necessary to apply the generalized Babuška-Brezzi theorem on Banach spaces. To this end, verifying the inf-sup conditions constituted one of the first challenges of this thesis. For the study of these, we developed and employed results such as the well-posedness of primal formulations based in Banach spaces for the Stokes and Poisson equations, an operator that maps a space L^t onto itself, and a generalization to Lebesgue spaces of a key inequality in the analysis for linear elasticity.

Subsequently, having established these tools, we focused on the stress-assisted diffusion problem, which we reformulated in terms of the pseudostress tensor, allowing us to avoid the imposition of symmetry on the Cauchy tensor. We carried out a mixed formulation for the linear elasticity problem and first coupled it with a primal formulation for the diffusion equation, and then with two mixed formulation alternatives for the latter. We detailed their search spaces over Banach spaces. We highlight that the choice of these more general spaces allowed us to avoid the restrictions of two-dimensional, polygonal, and convex domains that had arisen in previous works. We applied the previously obtained results and, with them, the Babuška-Brezzi theory, along with the Banach fixed-point theorem, allowed us to establish the existence and uniqueness at the continuous level of the coupled problem.

At the discrete level, we initially considered arbitrary inf-sup stable finite element spaces. Through Brouwer's Theorem, we established the existence of a solution and, in addition, obtained its corresponding Céa estimate. Since it is not possible to control a certain term with respect to the data, establishing the uniqueness of the solution at the discrete level is not possible. Subsequently, we chose finite element spaces and demonstrated that they indeed satisfy the assumed hypotheses. Thanks to the choice of these subspaces, which, although different at the topological level, coincide algebraically, and, therefore, the stiffness matrices associated with the bilinear forms b_1 and b_2 are the same, which constitutes an advantage at the computational level. In addition, we defined an L^t-stable projector that acts on the space of deviators of kernel elements. Finally, regarding the achievement of results concerning the orders of convergence, which were empirically verified through numerical trials.

Chemotaxis–Navier–Stokes: Chemotaxis–Navier–Stokes: In the context of the stationary chemotaxis-Navier-Stokes problem, we have developed a fully mixed finite element method. Aiming to improve approximations for the gradients of $\mathbf{u}$, the cellular density η, and the chemical signal concentration φ, we introduced these variables into the system. Due to the presence of trilinear terms, it was necessary to address the resulting variational formulations in Banach spaces, where we note that although we previously worked in spaces where the variables and their divergences were considered in L^t, in this case, we had to address the divergence in $\mathrm{L}^{t/2}$. It is important to highlight that, although we have opted to consider Dirichlet boundary conditions, the resulting formulations require minimal adjustments to handle more complex boundary conditions. The Babuška-Brezzi theory, along with the Banach fixed-point theorem, allowed us to establish the existence and uniqueness, at the continuous level, of our fully mixed scheme.

Concerning the Galerkin scheme, the introduction of this new space $\mathbf{H}$-div led us to develop new techniques and tools, such as additional properties of the Raviart-Thomas interpolant. In this way, we were able to verify the necessary inf-sup conditions at the discrete level to establish that each decoupled problem is well-posed. Then, thanks to Brouwer's theorem and the Banach fixed-point theorem, we were able to establish the uniqueness of the solution for the discrete scheme. We also demonstrated the Céa estimate and, thus, along with the approximation properties of the chosen finite element spaces, we theoretically concluded the orders of convergence. Numerical experiments empirically confirm these orders. Additionally, we conducted experiments that illustrate our method's ability to address more realistic boundary conditions.

Future Work

The methods developed and the results obtained this thesis have motivated several and future projects. Descriptions of some of these projects are provided below:

1. **A Posteriori Analysis for Stress-Assisted Diffusion and chemotaxis-Navier-Stokes Problems.**
   As a natural continuation, we are interested in conducting a posteriori error analysis for the problems studied in this thesis to enhance their robustness in cases involving complex geometries or solutions with high gradients.

2. **Development of a Mixed Finite Element Method for the Coupled Poroelasticity-Heat Problem.**
   We are interested in applying a mixed finite element method in Banach spaces to the coupled poroelasticity problem with heat. This problem involves a homogeneous porous medium com-

posed of a mixture of incompressible grains and an interstitial fluid. Our focus will be on the following coupled Biot equations with a convection-diffusion equation:

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{e}(\mathbf{u}) + \lambda\mathrm{div}(\mathbf{u})\mathbb{I} - (\alpha p + \beta\theta)\mathbb{I} \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{f} \quad \text{in}\ \Omega\,, \tag{4.237a}$$

$$c_0 p + \alpha\mathrm{div}(\mathbf{u}) - \mathrm{div}(\mathbf{w}) = l \quad \text{in} \quad \Omega\,, \quad \mathbf{w} = \frac{\kappa}{\eta}\nabla p \quad \text{in}\ \Omega\,, \tag{4.237b}$$

$$\theta + \mathbf{w}\cdot\nabla\theta - \mathrm{div}(\delta(\boldsymbol{\sigma})\nabla\theta) = g \quad \text{in}\ \Omega\,, \tag{4.237c}$$

where $\boldsymbol{e}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathbf{t}})$ is the strain tensor, κ is the permeability of the porous solid, λ and μ are the Lamé constants of the solid (dilation and shear moduli, respectively), $c_0 > 0$ is the restricted specific storage coefficient, $0 < \alpha \leq 1$ is the Biot-Willis parameter, β is an active stress scale indicating bidirectional coupling between diffusion and movement, η is the fluid viscosity in the pores, and $\delta : \mathbb{R} \to \mathbb{R}$ is a stress-dependent diffusivity accounting for altered diffusion in the poroelastic domain.

This problem exhibits a structure similar to what has been studied in this thesis. Furthermore, the second and third terms on the left-hand side of (4.237c) will require that $\mathbf{w}$ and $\boldsymbol{\sigma}$ be sought in the Banach spaces $\mathbf{H}^t(\mathrm{div}_t;\Omega)$ and $\mathbb{H}^t(\mathbf{div}_t;\Omega)$, respectively. Therefore, many of the tools developed in this thesis will be valuable for addressing this problem.

3. **A More Generalized and Robust Analysis**

While in this thesis we extended the analysis to Banach spaces, this led to the necessity of restricting our analysis to certain cases. Specifically, for the stress-assisted diffusion problem, continuous analysis required imposing constraints on the dilation modulus, assuming it to be sufficiently large. On the other hand, proving inf-sup conditions at the discrete level involved the use of a projector, whose stability in terms of the L^t-norm is only valid for the two-dimensional case. Therefore, a possible future direction would be to find new ways to conduct the analysis that address these constraints on data and problem dimensionality.

Conclusiones y trabajos futuros

Conclusiones

En esta tesis, hemos desarrollamos métodos de elementos finitos mixtos basados en espacios de Banach para resolver numéricamente sistemas de ecuaciones diferenciales parciales relevantes en la mecánica de sólidos y fluidos. Nos hemos enfocado específicamente en los siguientes modelos:

Difusión asistida por esfuerzo:

Con el objetivo de desarrollar las herramientas necesarias para abordar este problema sobre espacios de Banach, debido a las similitudes de las formulaciones continuas resultantes, nos enfocamos inicialmente en el análisis continuo del problema de elasticidad lineal casi incompresible y el problema de Stokes. Reformulamos ambos problemas con respecto al tensor no simétrico de pseudoesfuerzos, lo cual nos permitió evadir imposiciones de simetría débil. Cabe destacar que los tensores de esfuerzos de Cauchy originales son recuperables a partir del tensor de pseudoesfuerzos mediante una simple post-proceso. Luego, utilizando formulas de integraciones por partes adecuadas a los espacios de Sobolev en los cuales trabajamos obtuvimos las correspondientes formulaciones variacionales mixtas. Dichos esquemas continuos tienen la propiedad de que los espacios de búsqueda y de testeo no coinciden como usualmente sucede en las formulaciones en un marco Hilbertiano, por lo cual para establecer existencia y unicidad de solución, fue necesario la aplicación del teorema de Babŭzka-Brezzi sobre espacios de Banach generalizado. Con este fin, la verificación de las condiciones inf-sup constituyo uno de los primeros desafíos de esta tesis. Para el estudio de estas, desarrollamos y empleamos resultados tales como el buen-planteamiento de las formulaciones primales basadas en espacios de Banach para las ecuaciones de Stokes y Poisson, un operador que mapea un espacio L^t dentro de sí mismo, y una generalización a espacios de Lebesgue de una desigualdad clave en el análisis para elasticidad lineal.

Posteriormente, habiendo establecido dichas herramientas, nos concentramos en el problema de difusión asistida por esfuerzo, el cual reformulamos en términos del tensor de pseudoesfuerzo, que nos permitió evitar la imposición de simetría sobre el tensor de Cauchy. Realizamos una formulación mixta para el problema de elasticidad lineal y lo acoplamos primero con una formulación primal para la ecuación de difusión, y luego con dos alternativas de formulaciones mixtas para este último. Establecimos en detalle sus espacios de búsqueda sobre espacios de Banach. Resaltamos que la elección de estos espacios más generales permitió evitar las restricciones de dominio bidimensional, poligonal y convexo que habían surgido en trabajos anteriores. Aplicamos los resultados obtenidos previamente y, con ello, la teoría de Babuška-Brezzi, junto con el teorema del punto fijo de Banach, nos permitió establecer la existencia y unicidad a nivel continuo del problema acoplado.

A nivel discreto, consideramos inicialmente espacios de elementos finitos arbitrarios inf-sup estables. Mediante el Teorema de Brouwer, establecimos la existencia de solución y, además, obtuvimos su estimación de Céa correspondiente. Dado que no es posible controlar cierto término con respecto a los datos, establecer la unicidad de solución a nivel discreto no es posible. Posteriormente, elegimos espacios de elementos finitos y demostramos que efectivamente satisfacen las hipótesis asumidas. Gracias a la elección de estos subespacios, que si bien son distintos a nivel topológico, coinciden algebraicamente, y, por lo tanto, las matrices de rigidez asociadas a las formas bilineales b_1 y b_2 son las mismas, lo cual constituye una ventaja a nivel computacional. Además, definimos un proyector L^t-estable que actúa sobre el espacio de desviadores de los elementos del kernel. Finalmente, respecto a la obtención de resultados concernientes a los órdenes de convergencia, los cuales fueron comprobados empíricamente mediante ensayos numéricos.

Chemotaxis–Navier–Stokes: En el contexto del problema estacionario de chemotaxis-Navier-Stokes, hemos realizado un método de elementos finitos completamente mixto. Con el objetivo de mejorar las aproximaciones para los gradientes de $\mathbf{u}$, la densidad celular η y la concentración de la señal química φ, hemos introducido estas variables al sistema. Debido a la presencia de términos trilineales, fue necesario a abordar las formulaciones variacionales resultantes en espacios de Banach, en donde destacamos que si bien previamente trabajábamos en espacios donde las variables y sus divergencias eran consideradas en L^t, en este caso, hemos tenido que abordar la divergencia en $\mathrm{L}^{t/2}$. Es importante destacar que, aunque hemos optado por considerar condiciones de Dirichlet para la frontera, las formulaciones resultantes requieren ajustes mínimos para manejar condiciones de frontera más complejas. La teoría de Babuška-Brezzi, junto con el teorema del punto fijo de Banach, nos permitió establecer la existencia y unicidad, a nivel continuo, de nuestro esquema completamente mixto.

Concerniente al esquema de Galerkin, la introducción de este nuevo espacio $\mathbf{H}$-div nos indujo a desarrollar nuevas técnicas y herramientas, tales como propiedades adicionales del interpolante de Raviart-Thomas. De esta manera, pudimos verificar las condiciones inf-sup a nivel discreto necesarias para establecer que cada problema desacoplado está bien planteado. Luego, gracias al teorema de Brouwer y al teorema del punto fijo de Banach, pudimos establecer la unicidad de la solución para el esquema discreto. También demostramos la estimación de Céa y, así, junto con las propiedades de aproximación de los espacios de elementos finitos elegidos, concluimos teóricamente los órdenes de convergencia. Los experimentos numéricos confirman empíricamente el cumplimiento de estos órdenes. Además, realizamos experimentos que ilustran la capacidad de nuestro método para abordar condiciones de frontera más realistas.

Trabajos futuros

Los métodos desarrollados y los resultados obtenidos en esta tesis han motivado varios proyectos en proceso y a futuro. Algunos de ellos son descritos a continuación:

1. **Análisis a posteriori para los problemas de difusión asistida por esfuerzo y chemotaxis–Navier–Stokes.**
   Como una continuación natural, estamos interesados en llevar a cabo un análisis de error a posteriori para los problemas estudiados en esta tesis, para mejorar su robustez ante problemas en los cuales se involucran geometrías complejas o soluciones con altos gradientes.

2. **Desarrollo de un método de elementos finitos mixtos para el problema de poroelasticidad acoplado con calor.**
Estamos interesados en aplicar un método de elementos finitos mixtos en espacios de Banach para el problema de poroelasticidad acoplado con calor, el cual consiste en un medio poroso homogéneo constituido por una mezcla de granos incompresibles y un fluido intersticial. Y dado una fuerza volumétrica $\mathbf{f}$ y términos fuente dados l y g, centraremos la discusión en las siguientes ecuaciones de Biot acopladas con una ecuación de convección-difusión:

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{e}(\mathbf{u}) + \lambda\mathrm{div}(\mathbf{u})\mathbb{I} - (\alpha p + \beta\theta)\mathbb{I} \quad \text{en} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{f} \quad \text{en} \quad \Omega\,, \tag{4.238a}$$

$$c_0 p + \alpha\mathrm{div}(\mathbf{u}) - \mathrm{div}(\mathbf{w}) = l \quad \text{en} \quad \Omega\,, \quad \mathbf{w} = \frac{\kappa}{\eta}\nabla p \quad \text{en} \quad \Omega\,, \tag{4.238b}$$

$$\theta + \mathbf{w}\cdot\nabla\theta - \mathrm{div}(\delta(\boldsymbol{\sigma})\nabla\theta) = g \quad \text{en} \quad \Omega\,, \tag{4.238c}$$

donde $\boldsymbol{e}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathbf{t}})$ es el tensor de deformaciones infinitesimales, κ es la permeabilidad del sólido poroso, λ y μ son las constantes de Lamé del sólido (módulos de dilatación y corte, respectivamente), $c_0 > 0$ es el coeficiente de almacenamiento específico restringido, $0 < \alpha \leq 1$ es el parámetro de Biot-Willis, β es una escala de tensión activa que indica un acoplamiento bidireccional entre difusión y movimiento, η es la viscosidad del fluido en los poros y $\delta : \mathbb{R} \to \mathbb{R}$ es una difusividad dependiente del estrés que considera una difusión alterada en el dominio poroelástico.

Acá notamos, que el segundo y tercer término en el lado izquierdo de (4.238c), implicarán que $\mathbf{w}$ y $\boldsymbol{\sigma}$ deberán ser buscados en los espacios de Banach $\mathbf{H}^t(\mathrm{div}_t;\Omega)$ y $\mathbb{H}^t(\mathbf{div}_t;\Omega)$, respectivamente. Y en consecuencia, las herramientas desarrolladas en esta tesis serán de gran utilidad para el análisis de este problema.

3. **Un Análisis más Generalizado y Robusto**

Si bien en esta tesis generalizamos el análisis a espacios de Banach, esto conllevó la necesidad de restringir nuestro análisis a ciertos casos. El primero, para el problema de difusión asistido por esfuerzo, requería imponer restricciones en el módulo de dilatación, suponiendo que este fuera suficientemente grande. Por otra parte, el principal desafió de demostrar condiciones inf-sup a nivel discreto implicó la utilización de un proyector, cuya estabilidad con respecto a la norma L^t solo es válida para el caso bidimensional. Por lo tanto, un posible trabajo a futuro consistiría en encontrar nuevas formas de realizar el análisis de manera que se evadan estas restricciones sobre los datos y la dimensionalidad del problema.

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