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# Geometric Pure and Matter Coupled Supergravity Theories

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*To my parents, sister and Pat*

# Table of contents

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# Publications

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# Abstract

This thesis deals with the formulation of pure and matter coupled supergravity theories in three and four dimensions. Different supergravity Lagrangians are constructed in geometrical terms, by using the useful properties of the abelian semigroup expansion method. Furthermore, a supergravity model with partial breaking of $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry which, in the low energy limit, gives rise to a rigid supersymmetric theory, is presented.

In Chapter 1, we briefly review General Relativity in both, Einstein and Cartan formalism. It is also revised a natural extension of Einstein theory to $D$-dimensions, namely the Lanczos-Lovelock theory. Then, we study the Maxwell type algebras, and we show that standard General Relativity can be obtained in a certain limit of Chern-Simons and Born-Infeld theories, invariant under these algebras. Chapter 2 deals with the supersymmetric extension of gravity. We mainly study the MacDowell-Mansouri supergravity and the AdS Chern-Simons supergravity.

In Chapters 3, 4, 5, 6 and 7, we present our main results, which are based on five articles written during the doctoral research. First, we present supersymmetric extensions of the Maxwell type algebras. We show that considering different choices of semigroups, inequivalent Maxwell superalgebras are obtained, when using the $S$-expansion procedure. Then, we construct the $\mathcal{N} = 1$ supergravity action à la MacDowell-Mansouri from the minimal Maxwell superalgebra. Interestingly, the action describes pure supergravity. Based on the AdS-Lorentz superalgebra, we also build the minimal $D = 4$ supergravity action which includes a generalized supersymmetric cosmological constant term. The construction of the Chern-Simons supergravity action from a generalized minimal Maxwell superalgebra is also presented.

Eventually, in Chapter 7 we present the multi-vector generalization of a rigid, partially broken $\mathcal{N} = 2$ supersymmetric theory as a rigid limit of a gauged $\mathcal{N} = 2$ supergravity with electric and magnetic charges.

# Introduction

*"Learn from yesterday, live for today, hope for tomorrow.*
*The important thing is not to stop questioning"*
Albert Einstein

This year marks an outstanding milestone in the history of physics with the one-hundredth anniversary of Einstein's Theory of General Relativity. The theory of General Relativity explains all gravitational phenomena we know, and it has been survived to various tests supporting its validity. However, this theory requires extensions since it has certain shortcomings; for instance, the failure to unify gravity with the other three fundamentals interactions of nature, which are described consistently by the Standard Model (SM) through Yang-Mills (YM) quantum theories. The SM is based on the gauge group $SU(3) \times SU(2) \times U(1)$ and defines a consistent quantum theory free of anomalies and widely verified experimentally.

Gravity is described by General Relativity as a dynamic manifestation of the geometric properties of space-time. Thus, the possibility of unifying gravity with the other interactions in a same geometric framework would require to incorporate internal and space-time symmetries in a same group. A possible way to achieve this task is supersymmetry (SUSY), a kind of symmetry against which we shall demand the laws of nature be invariant, at least at a certain level. The idea that SUSY is actually an underlying symmetry of Nature is supported by various phenomenological arguments. For instance, the presence of this symmetry makes field theories better behaved in the ultraviolet (UV) by virtue of the cancellation of fermionic and bosonic contributions to divergent loop integrals. This is a very interesting property from the point of view of a quantum gravity theory.

Supersymmetry is a symmetry that relates bosonic and fermionic particles. From a theoretical point of view, SUSY has a most interesting aspect since it unifies bosonic space-time symmetries with other internal bosonic symmetries (like the $SU(3) \times SU(2) \times U(1)$ invariance of the standard model), giving the possibility of unify gravity with the other interactions in a same geometric framework. Indeed, the group containing supersymmetry transformations generalizes the Poincaré group, and in addition to the Lorentz generators $J_{ab}$ and the space-time translations $P_a$, we have also supersymmetry generators $Q$ and internal generators $B_i$. Then, the corresponding algebra is called the super-Poincaré algebra.

The supersymmetric extension of General Relativity is known as Supergravity (SUGRA) and it is a theory of local supersymmetry. In SUGRA, the gravitational field is coupled to its super-partners and possibly to other supermultiplets containing matter multiplets. In its simplest version Supergravity can be viewed as the "gauge" theory of the super-Poincaré group whose action is given by the Einstein-Hilbert term representing the graviton, plus a Rarita-Schwinger kinetic term describing the gravitino $\psi$, a spin-3/2 particle.

There are several different supersymmetric theories, which differ in the space-time dimension $D$ and in the number $\mathcal{N}$ of supersymmetry charges. $\mathcal{N}$ supersymmetry generators define an $\mathcal{N}$-extended supersymmetry. SUGRA theories of particular relevance are defined in $D = 10$ and $D = 11$ since they describe the low-energy dynamics of superstring theory and M-theory, on at space-time, respectively. In supergravity, the limit on the amount $\mathcal{N}$ of supersymmetry comes from the possibility of a consistent coupling to gravity, which restricts the maximum spin of the fields to be two, thus implying $\mathcal{N} \leq 8$.

On the other hand, the successful AdS/CFT (Anti-de-Sitter/Conformal Field Theory) correspondence, that is the conjectured equivalence between superstring theory realized on an anti-de Sitter space-time and the conformal field theory on its boundary at infinity, made supergravity a useful tool for studying non-perturbative properties of gauge theories.

Global and local supersymmetric theories exhibit deep geometrical structures inherent to the non-linear interactions of matter multiplets. In the $D = 4$, $\mathcal{N} = 2$ case, the geometrical structure is described by the Special Kähler geometry and the Hypergeometry, when vector multiplets and hypermultiplets are present. When matter is added, the underlying geometrical structure is much richer since $\mathcal{N} = 2$ matter hypermultiplets are associated with quaternionic geometry.

The purpose of this thesis is to study different pure and matter coupled supergravity theories in three and four dimensions. First, we will present a supersymmetric extension of the Maxwell type algebras. Using the properties of the Abelian semigroup expansion method ($S$-expansion), we will show that inequivalent Maxwell superalgebras can be obtained when different semigroups are chosen. Thus, we will obtain a family of Maxwell superalgebras having the Maxwell type algebras as subalgebras. The $\mathcal{N}$-extended cases will be also studied.

Then, we will construct different supergravity Lagrangians in three and four dimensions following a geometrical approach and using the useful properties of the $S$-expansion. In four dimensions, we will construct the $\mathcal{N}=1$ supergravity action à la MacDowell-Mansouri from the minimal Maxwell superalgebra. Based on the AdS-Lorentz superalgebra, we will also build the minimal supergravity action which includes a generalized supersymmetric cosmological constant term. In three dimensions, the construction of a Chern-Simons supergravity action from a minimal Maxwell superalgebra will be a further result of this work.

Finally, we will construct an appropriate dyonic gauging of an $\mathcal{N}=2$ supergravity coupled to $n$ vector multiplets and to one hypermultiplet allowing for a well-defined rigid limit to a multi-vector APT model. This will clarify the supergravity origin of the multifield Born Infeld (BI) and, in particular, to understand the origin of the dyonic Fayet Iliopoulus (FI) terms as deriving from electric and magnetic charges in the supergravity gauged model. Furthermore, we will give a general proof of the Ward identity for generic dyonic gaugings.

# Part I

# Gravity, Maxwell symmetries and Supergravity

# Chapter 1

# General Relativity and Maxwell type algebras

## 1.1 Introduction

One hundred years ago Albert Einstein wrote down the field equations of General Relativity, his masterwork describing gravity as the curvature of the space-time. The theory of General Relativity explains all gravitational phenomena we know, such as falling apples and orbiting planets, and it has been survived to various tests supporting its validity. For instance, some experimental evidences are the gravitational lensing, the changes in the orbit of Mercury, gravitational redshift of light, the deflection of light by the sun and frame-dragging of space-time around rotating bodies.

General Relativity describes gravity as a dynamic manifestation of the space-time geometry, and its main underlying assumptions are the requirements of general covariance and second order field equations for the metric. On the other hand, the possibility that space-time may have more than four dimensions is a standard assumption in high-energy physics. Although many different approaches have been followed, most of them assume the simplest generalization of General Relativity to higher dimensions, namely the Einstein-Hilbert action. Based on the same principles of General Relativity, the most general metric theory of gravity is the Lanczos-Lovelock gravity theory (LL) [1], [2]. This theory refers to a family parametrized by a set of real coefficients $\alpha_p$, which are not fixed from first principles. In [5], these parameters were fixed in terms of the gravitational and the cosmological constants. As a consequence, the action in odd dimensions can be formulated as a Chern-Simons (CS) theory of the AdS group, while in even dimensions the action has a Born-Infeld (BI) form invariant only under local Lorentz rotations.

If CS and BI theories are the appropriate gauge theories to describe the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to General Relativity.

In this chapter, we shall show that standard odd-and even-dimensional General Relativity can be obtained from Chern-Simons and Born-Infeld like theories, invariant under the Maxwell type algebras, when certain conditions are imposed [12], [17], [19]. These Maxwell type algebras are obtained from the AdS algebra and a particular choice of the semigroup by means of the $S$-expansion procedure. Furthermore, we present the Einstein-Lovelock-Cartan Lagrangian leading to General Relativity in a certain limit of the coupling constant, both in odd and even dimensions [20].

Before introducing the Maxwell type algebras and its applications to gravity, let us briefly review the first order formalism of gravity and the Lanczos-Lovelock theory.

## 1.2 First order formulation of gravity

General Relativity describes gravity as a dynamic manifestation of the space-time geometry. This idea is encoded in the *metric tensor* $g_{\mu\nu}(x)$, which provides the notion of distance between the two nearby spacetime points $x^\mu$ and $x^\mu + dx^\mu$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu . \tag{1.1}$$

Another important concept in the understanding of the space-time geometry is *parallelism*, which is encoded in the affine connection $\Gamma^\alpha_{\beta\gamma}(x)$: a vector $\xi^\alpha_\parallel$ is said to be parallel to the vector $\xi^\alpha$, if their components are related by "parallel transport"

$$\begin{aligned} \xi^\alpha_\parallel (x + dx; x) &= \xi^\alpha (x + dx) + dx^\mu \Gamma^\alpha_{\ \mu\beta} \xi^\beta (x) \\ &= \xi^\alpha (x) + dx \left[ \partial_\mu \xi^\alpha + \Gamma^\alpha_{\ \mu\beta} \xi^\beta (x) \right] . \end{aligned} \tag{1.2}$$

The expression $\partial_\mu \xi^\alpha + \Gamma^\alpha_{\ \mu\beta} \xi^\beta (x)$ corresponds to the *covariant derivative* of $\xi^\alpha$, and we will denote it by

$$D_\mu \xi^\alpha \equiv \partial_\mu \xi^\alpha + \Gamma^\alpha_{\ \mu\beta} \xi^\beta . \tag{1.3}$$

In General Relativity, the affine connection is required to be symmetric in the lower index, i.e,

$$\Gamma^\alpha_{\ \mu\beta} = \Gamma^\alpha_{\ \beta\mu} . \tag{1.4}$$

This equation expresses the vanishing of the *torsion tensor,*

$$T^{\alpha}_{\ \mu\beta} \equiv \Gamma^{\alpha}_{\ \mu\beta} - \Gamma^{\alpha}_{\ \beta\mu}. \tag{1.5}$$

The affine connection $\Gamma^{\alpha}_{\beta\gamma}$ satisfying (1.4) is known as the connection or the *Christoffel symbol*, and becomes a function of the metric

$$\Gamma^{\alpha}_{\mu\beta} = \frac{1}{2} g^{\alpha\lambda} \left( \partial_{\mu} g_{\lambda\beta} + \partial_{\beta} g_{\lambda\mu} - \partial_{\lambda} g_{\mu\beta} \right). \tag{1.6}$$

Using the definition (1.3), we can compute the commutator of two covariant derivatives acting on a vector $\xi^{\alpha}$,

$$[D_{\mu}, D_{\nu}] \xi^{\alpha} = R^{\alpha}_{\ \beta\mu\nu} \xi^{\beta} - T^{\lambda}_{\ \mu\nu} D_{\lambda} \xi^{\alpha} \tag{1.7}$$

where $T^{\lambda}_{\ \mu\nu}$ is the torsion, and $R^{\alpha}_{\ \beta\mu\nu}$ is the Riemann tensor defined by

$$R^{\alpha}_{\ \beta\mu\nu} \equiv \partial_{\mu} \Gamma^{\alpha}_{\ \nu\beta} - \partial_{\nu} \Gamma^{\alpha}_{\ \mu\beta} + \Gamma^{\alpha}_{\ \mu\lambda} \Gamma^{\lambda}_{\ \nu\beta} - \Gamma^{\alpha}_{\ \nu\lambda} \Gamma^{\lambda}_{\ \mu\beta} \tag{1.8}$$

Besides, we define the Ricci tensor $R_{\mu\nu} \equiv R^{\alpha}_{\ \mu\alpha\nu}$ and the curvature scalar $R \equiv g^{\mu\nu} R_{\mu\nu}$. We use these ingredients to construct the Einstein-Hilbert (EH) action

$$S^{(4)}_{EH} = \int d^4 x \sqrt{-g} R, \tag{1.9}$$

where $g = \det(g_{\mu\nu}) < 0$. The variation of the action leads to the Einstein field equations (in the vacumm),

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{1.10}$$

So far, we have reviewed the formulation of General Relativity considering that the metric and affine properties are not independent. For this, it was necessary to introduce a constraint: the torsion tensor was assumed to be zero. However, these properties can be considered as independent notions. The formulation of General Relativity considering the metricity and parallelism as independent properties is known as Cartan gravity (in the differential forms formulation) or Palatini formalism (in the tensorial formulation).

Let us consider the mapping between the space-time $D$-dimensional manifold $M$ and a flat Minkowski tangent space $T_x$, which is a good approximation of the manifold on an open set in the neighborhood of $x$. The relation between $M$ and the collection $\{T_x\}$ is given by an isomorphism represented by means of a linear map $e$,

$$e^{a}_{\ i} = \frac{\partial z^a}{\partial x^i}, \tag{1.11}$$

where the matrices $e^a_{\ i} = e^a_\mu(x)$ $(a = 1, ..., D = \dim M)$ are called the vielbein, and define a local orthonormal frame on $M$. Thus the infinitesimal $dx^\mu(x) = dx^i$ on $M$ is mapped to corresponding separation $dz^a$ in $T_x$,

$$dz^a = e^a_{\ i} dx^i. \tag{1.12}$$

Furthermore, the Lorentzian metric defined in Minkowski space can be used to induce a metric on $M$ through the isomorphism $e^a_{\ i}$. In fact, from a given tetrad

$$e^a_{\ i} = \eta^{ab} e^j_{\ b} g_{ij}, \tag{1.13}$$

one can find the metric on $M$,

$$g_{ij} = e^a_{\ i} e^b_{\ j} \eta_{ab}. \tag{1.14}$$

The definition (1.11) implies that $e^a_{\ i}$ transforms as a covariant vector under diffeomorphisms on $M$, and as a contravariant vector under local Lorentz rotations of $T_x$, $SO(D-1,1)$. In fact, under Lorentz transformation the vielbein are transformed as follows

$$e^a_{\ i} \to e'^a_{\ i} = \Lambda^a_b(x)\, e^b_{\ i}, \tag{1.15}$$

where $\Lambda(x) \in SO(D-1,1)$. By definition of the Lorentz group, the matrices $\Lambda(x)$ leave the metric in the tangent space unchanged

$$\Lambda^a_c(x)\, \Lambda^b_d(x)\, \eta_{ab} = \eta_{cd}. \tag{1.16}$$

We define now the vielbein 1-form

$$e^a \equiv e^a_{\ \mu} dx^\mu, \tag{1.17}$$

and its covariant derivative

$$De^a \equiv de^a + \omega^a_{\ b} e^a, \tag{1.18}$$

where $\omega^{ab} = \omega^{ab}_\mu dx^\mu$ is called the spin connection 1-form and it is transformed as

$$\omega^a_{\ b} \to \omega'^a_{\ b} = \Lambda^a_{\ c}(x)\, \Lambda_b^{\ d}(x)\, \omega^c_{\ d} + \Lambda^a_{\ c}(x)\, d\Lambda_b^{\ c}(x). \tag{1.19}$$

This object plays the role of the gauge potential and defines the curvature two-form

$$R^a_{\ b} \equiv d\omega^a_{\ b} + \omega^a_{\ c} \omega^c_{\ b}. \tag{1.20}$$

In addition, we introduce the torsion two-form

$$T^a \equiv De^a = de^a + \omega^a_{\ b} e^b, \tag{1.21}$$

which involves both the vielbein and the connection.

The equations (1.20) and (1.21) are called structure equations because they describe the geometrical structure of the manifold $M$. These 2-forms satisfy the following Bianchi identities

$$\begin{aligned} DT^a &= R^a_{\ b} e^b, & (1.22) \\ DR^a_{\ b} &= 0. & (1.23) \end{aligned}$$

In the Cartan formalism the Einstein-Hilbert action is given by

$$S^{(4)}_{EH} = \int_M \epsilon_{abcd} R^{ab} e^c e^d, \tag{1.24}$$

where $R^{ab} = d\omega^{ab} + \omega^a_{\ c}\omega^{cb}$ is the two-form curvature and $e^a$ is the vierbein. Moreover, we have used that $\kappa = 1$, with $\kappa$ the gravitational coupling constant. The action (1.11) is equivalent to the EH action in the tensorial formalism (1.9).

Considering the variation of the action (1.11), we have that $\delta S^{(4)}_{EH} = 0$ leads us to

$$-2\int \epsilon_{abcd}\delta\omega^{ab} T^c e^d + 2\int \epsilon_{abcd} R^{ab} \left(\delta e^c\right) e^d = 0. \tag{1.25}$$

Because the variations $\delta\omega^{ab}$ and $\delta e^c$ are arbitrary, we have

$$\begin{aligned} \epsilon_{abcd} R^{ab} e^c &= 0, & (1.26) \\ \epsilon_{abcd} T^c e^d &= 0. & (1.27) \end{aligned}$$

The first equation is equivalent to the Einstein field equations (1.10), while the second one expresses the vanishing of the torsion

$$T^a = de^a + \omega^a_{\ b} e^b = 0. \tag{1.28}$$

This equation can be solved for the spin connection $\omega^a_{\ b}$, allowing to express it in terms of the vielbein and its derivatives.

By construction, the action (1.24) is invariant under general coordinate transformations and under (local) Lorentz transformations, but is not invariant under Poincaré local translations[1]. In fact, a gauge theory for the Poincaré group should be based on the one-form

[1] The invariance of the action requires to impose de condition $T^a = 0$. However, this constraint is not invariant under Poincaré local translations, because $\delta T^a \neq 0$, for $\delta e^a = D\rho^a$, $\delta\omega^{ab} = 0$.

connection $A = e^a P_a + \omega^{ab} J_{ab}$, with $\{J_{ab}, P_c\}$ the generators of the group. Since there is no Poincaré-invariant 4-form that can be constructed with this field, then no Poincaré-invariant gravity action can be constructed in $D = 4$ dimensions. An alternative approach could be consider another group $G$ containing the Lorentz transformations as a subgroup. The smallest nontrivial choices for $G$ are the de Sitter (dS) and the anti-de Sitter (AdS) groups. These are semisimple groups, and the Poincaré group can be obtained as a contraction of them. This property could mean that these groups are better candidates in order to become physically relevant for gravity.

Nevertheless, so far it is not possible to describe gravity as a gauge theory for the dS or AdS groups. The Einstein-Hilbert action (1.24) is basically the only action for gravity $D = 4$, but many more options exist in higher dimensions. As we will see next, in $D = 2n-1$ dimensions, gravity can be expressed as a gauge theory of the groups $SO(D, 1)$, $SO(D-1, 2)$, or $ISO(D-1, 1)$. This will not be the same for even dimensions, $D = 2n$.

## 1.3 Lanczos-Lovelock theory

So far, the possibility that space-time may have more than four dimensions is a standard assumption in High-energy physics. Nevertheless, if we want to extend the space-time dimension to dimensions greater than four, the reformulation of the structure of the equations for the gravitational field is required, and we have to critically examine the minimal requirements for a consistent theory of gravity in any dimension, including both general covariance and second order field equations for the metric.

Although many different approaches have been followed, most of them assume the simplest generalization of General Relativity to higher dimensions, namely the Einstein-Hilbert action. Based on the same principles of General Relativity, the most general metric theory of gravity satisfying the criteria of general covariance and leading to second-order field equations is a polynomial of degree $[D/2]$ in the curvature known as the Lanczos-Lovelock gravity theory (LL) [1], [2]. The Lovelock action can be written as the most general $D$-form invariant under local Lorentz transformations, constructed with the spin connection, the vielbein and their exterior derivatives, without the Hodge dual [3, 4]

$$S_G = \int \sum_{p=0}^{[D/2]} \alpha_p L^{(p)}, \tag{1.29}$$

where

$$L^{(p)} = \varepsilon_{a_1 \cdots a_D} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_D}. \tag{1.30}$$

Here $R^{ab}$ is the curvature two-form defined in (1.20), $e^a$ corresponds to the one-form vielbein and the coefficients $\alpha_p$, $p = 0, 1, ..., [D/2]$, are arbitrary constants. The LL theories allow to construct the most general gravity theory in $D$-dimensions as a natural extension of the Einstein theory, and thus they have the same degrees of freedom $(D(D-3)/2)$ as the Einstein-Hilbert Lagrangian in each dimension.

The Lanczos-Lovelock theory refers to a family parametrized by a set of real coefficients $\alpha_p$, which are not fixed from first principles. In [5], R. Troncoso and J. Zanelli showed that these parameters are fixed in terms of the gravitational and the cosmological constants, through the requirement that the theory possess the largest possible number of degrees of freedom. As a consequence, the action in odd dimensions can be formulated as a Chern-Simons (CS) theory of the AdS group, while in even dimensions the action has a Born-Infeld (BI) form invariant only under local Lorentz rotations, in the same way as the Einstein–Hilbert action [5], [6], [7], [8]. Let us briefly review the approach developed in [5].

Consider the LL action (1.29), as a functional of the spin connection and the vielbein, $S_G = S_G \left[\omega^{ab}, e^a\right]$. Varying with respect to these fields, the following field equations are obtained

$$\delta e^a \rightarrow \varepsilon_a = 0, \qquad \delta\omega^{ab} \rightarrow \varepsilon_{ab} = 0, \tag{1.31}$$

where we have defined

$$\varepsilon_a = \sum_{p=0}^{[(D-1)/2]} \alpha_p \left(D - 2p\right) \varepsilon_a^p, \tag{1.32}$$

$$\varepsilon_{ab} = \sum_{p=1}^{[(D-1)/2]} \alpha_p p \left(D - 2p\right) \varepsilon_{ab}^p, \tag{1.33}$$

and

$$\varepsilon_a^p \equiv \varepsilon_{ab_1 \cdots b_{d-1}} R^{b_1 b_2} \cdots R^{b_{2p-1} b_2} e^{b_{2p+1}} \cdots e^{b_{D-1}}, \tag{1.34}$$

$$\varepsilon_{ab}^p \equiv \varepsilon_{aba_3 \cdots a_d} R^{a_3 a_4} \cdots R^{a_{2p-1} a_p} T^{a_{2p+1}} e^{a_{2p+2}} \cdots e^{a_D}. \tag{1.35}$$

Since the $(D-1)$-forms $\varepsilon_a$ and $\varepsilon_{ab}$ are independent Lorentz tensors they vanish independently, which means that the metric and the affine properties are independent. If there were algebraic relations among these tensors, then the fields $\omega^{ab}$ and $e^a$ would be relate and as a con sequence, some components of the torsion tensor must vanish freezing out some degrees of freedom in the theory. On the other hand, considering the Bianchi identities (1.22) and (1.23), we have the following equations

$$D\varepsilon_a = \sum_{p=1}^{[(D+1)/2]} \alpha_{p-1}\,(D-2p+2)\,(D-2p+1)\,e^b\varepsilon_{ba}^p, \tag{1.36}$$

which by consistency with $\varepsilon_a = 0$ must also vanish. Furthermore, the exterior product of $\varepsilon_{ab}$with $e^b$ gives us

$$e^b\varepsilon_{ba} = \sum_{p=1}^{[(D-1)/2]} \alpha_p p\,(D-2p)\,e^b\varepsilon_{ba}^p, \tag{1.37}$$

which also vanish by virtue of $\varepsilon_{ab} = 0$. If the coefficients $\alpha_p$ were generic, then the equations (1.31) would imply in general additional restrictions of the form $e^b\varepsilon_{ba}^p = 0$ for some $p's$. Thus, different choices of $\alpha_p$ correspond, in general, to theories with different numbers of physical degrees of freedom depending on how many additional *off-shell* constraints are imposed on the geometry. As shown in [5] among all the possible choices for the $\alpha_p$, there is a special one which occurs only in odd dimensions, and where non additional constraints are imposed. In fact, equations (1.36) and (1.37) are proportional to each other term by term for $D = 2n-1$ but for $D = 2n$, both equations have different number of terms.

### 1.3.1 $D = 2n-1$: Local (A)dS Chern-Simons Gravity

As we said before, equations (1.36) and (1.37) have the same number of terms for odd dimensions. Thus, the two series must be proportional term by term, leading to the following recursion relation

$$\gamma\frac{\alpha_{p-1}}{\alpha_p} = \frac{p\,(D-2p)}{(D-2p+2)\,(D-2p+1)} \tag{1.38}$$

where $0 \leq p \leq n$, and $\gamma$ is an arbitrary constant of dimension [length$^2$]. The solution to this equation reads

$$\alpha_p = \alpha_0\frac{(2n-1)\,(2\gamma)^p}{(2n-2p-1)}\binom{n-1}{p}, \tag{1.39}$$

where

$$\alpha_0 = \frac{\kappa}{l^{D-1}D}, \qquad \gamma = -sign\,(\Lambda)\,\frac{l^2}{2}. \tag{1.40}$$

Here $\kappa$ is the gravitational constant, $\Lambda$ is the cosmological constant and $l$ is a length parameter.

With this choice of the $\alpha_p$ parameters, the LL action is not only invariant under local Lorentz rotations, but also under AdS boost

$$\begin{aligned}\delta e^a &= -D\lambda^a, \\ \delta\omega^{ab} &= \frac{1}{l^2}\left(\lambda^a e^b - \lambda^b e^a\right).\end{aligned}$$

Thus the LL Lagrangian in (1.29) with the coefficients (1.39), corresponds to the Euler-Chern-Simons form for the AdS group [9],[10], [11],

$$L_{AdS}^{(2n-1)} = \kappa\varepsilon_{a_1\cdots a_{2n-1}} \sum_{k=0}^{n} \frac{c_k}{l^{2(n-k)-1}} R^{a_1a_2}\cdots R^{a_{2k-1}a_{2k}} e^{a_{2k+1}}\cdots e^{a_{2n-1}}, \tag{1.41}$$

where

$$c_k = \frac{1}{2\left(n-k\right)-1}\binom{n-1}{k}. \tag{1.42}$$

In this case, the vielbein and the spin connection can be seen as the different components of an (A)dS connection, so that the local symmetry is extended from Lorentz to (A)dS, or Poincaré when $\Lambda \to 0$. In fact, in the limit $l \to \infty$ we obtain Chern-Simons gravity for the Poincaré group,

$$L^{(2n-1)} = \kappa\varepsilon_{a_1\cdots a_{2n-1}} R^{a_1a_2}\cdots R^{a_{2n-3}a_{2n-2}} e^{a_{2n-1}}.$$

### 1.3.2 $D = 2n$: Born-Infeld-Like Gravity

For even dimensions, equations (1.36) and (1.37) are not proportional term by term, and the procedure is a little bit longer. In this case it was shown that the solution which allows the maximum number of degrees of freedom leads to the following recursion relation for the $\alpha_p$'s:

$$2\gamma\left(n-p+1\right)\alpha_{p-1} = p\alpha_p, \tag{1.43}$$

for some fixed $\gamma$. The solution to this equation is

$$\alpha_p = \alpha_0\left(2\gamma\right)^p \binom{n}{p}, \tag{1.44}$$

with $0 \leq p \leq n-1$. This formula can be extended to $p = n$, adding the Euler density to the Lagrangian with the weight $\alpha_n = \alpha_0\left(2\gamma\right)^n$.

As in the odd dimensional case, the action depends only on the gravitational and the cosmological constants. The choice of coefficients (1.44) implies that the LL Lagrangian takes the form

$$L = \frac{\kappa}{2n}\epsilon_{a_1\cdots a_{2n}} \bar{R}^{a_1a_2}\cdots\bar{R}^{a_{2n-1}a_{2n}}, \tag{1.45}$$

where $\bar{R}^{ab} = R^{ab} + \frac{1}{l^2}e^a e^b$, and can be written as the Born-Infeld like form [6],

$$L = 2^{n-1}(n-1)!\kappa\sqrt{\det\left(R^{ab} + \frac{1}{l^2}e^a e^b\right)}. \tag{1.46}$$

In four dimensions (1.45) reduces to a particular linear combination of the Einstein-Hilbert action, the cosmological constant and the Euler density:

$$L_{BI}^{(4)} = \frac{\kappa}{4}\epsilon_{abcd}\left(R^{ab}R^{cd} + \frac{2}{l^2}R^{ab}e^c e^d + \frac{1}{l^4}e^a e^b e^c e^d\right). \tag{1.47}$$

Although the first term does not contribute to the field equations (it is a boundary term), it plays a fundamental role in the definition od conserved charges for gravity theories in $2n \geq 4$ dimensions [13], [14], [15].

It is important to note that in even dimensions, the Lagrangian (1.45) is invariant only under local Lorentz transformations and not under the AdS group. In contrast, as shown above, in odd dimensions it is possible to construct gauge invariant theories of gravity under the full (A)dS group (or Poincaré).

## 1.4 Chern-Simons gravity and Maxwell type algebras

As seen before, the Chern-Simons forms can be used to construct gauge invariant actions. In odd dimensions the LL action corresponds to a Chern-Simons form, when the coefficients are chosen in a particular way. In this case the action is invariant not only under local Lorentz rotations, but also under AdS boost. If CS theories are the appropriate gauge theories to describe the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to General Relativity.

In ref. [12] it was shown that the standard, odd-dimensional General Relativity can be obtained from a Chern–Simons gravity theory for a certain $\mathfrak{B}_m$ Lie algebra, which was called generalized Poincaré algebra (where the particular case $\mathfrak{B}_4$ corresponds to the so-called Maxwell algebra [16]). The generalized Poincaré algebras can be obtained by a resonant reduced $S$-expansion [2] of the AdS Lie algebra using $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$ as a semigroup. Subsequently, in Ref.[17] it was found that standard odd-dimensional General Relativity emerges as a weak coupling constant limit of a $(2p+1)$-dimensional Chern-Simons Lagrangian

[2]See appendix A for a review of the $S$-expansion method

invariant under $\mathfrak{B}_{2m+1}$, if and only if $m \geq p$. Let us briefly review here the results obtained in [12] and [17].

Let us consider the $S$-expansion of the AdS Lie algebra, $\mathfrak{so}(2n,2)$, using as a semigroup $S_E^{(2n-1)} = \{\lambda_0, \lambda_1, \cdots, \lambda_{2n}\}$, endowed with the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leq 2n, \\ \lambda_{2n}, & \text{when } \alpha+\beta > 2n. \end{cases} \tag{1.48}$$

The AdS generators $\left\{\tilde{J}_{ab}, \tilde{P}_c\right\}$ satisfy the following commutation relations

$$\begin{aligned} \left[\tilde{P}_a, \tilde{P}_b\right] &= \tilde{J}_{ab}, \\ \left[\tilde{J}_{ab}, \tilde{P}_c\right] &= \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b, \\ \left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] &= \eta_{cb}\tilde{J}_{ad} - \eta_{ca}\tilde{J}_{bd} + \eta_{db}\tilde{J}_{ca} - \eta_{da}\tilde{J}_{cb}. \end{aligned} \tag{1.49}$$

Let us consider the following subset decomposition $S_E^{(2n-1)} = S_0 \cup S_1$, with

$$S_0 = \{\lambda_{2m}, \text{ with } m = 0, 1 \ldots, n-1\} \cup \{\lambda_{2n}\}, \tag{1.50}$$

$$S_1 = \{\lambda_{2m+1}, \text{ with } m = 0, 1 \ldots, n-1\} \cup \{\lambda_{2n}\}, \tag{1.51}$$

where $\lambda_{2n}$ corresponds to the zero element of the semigroup ($\lambda_{2n} = 0_S$). After extracting a resonant subalgebra and performing its $0_S$-reduction, one finds the generalized Poincaré algebra $\mathfrak{B}_{2n+1}$, whose generators defined by

$$J_{(ab,2k)} = \lambda_{2k} \otimes \tilde{J}_{ab}, \tag{1.52}$$

$$P_{(a,2k+1)} = \lambda_{2k+1} \otimes \tilde{P}_a, \tag{1.53}$$

with $k = 0, \cdots, n-1$,satisfy the following commutation relations

$$[P_a, P_b] = Z_{ab}^{(1)}, \qquad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b \tag{1.54}$$

$$[J_{ab}, J_{cd}] = \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb} \tag{1.55}$$

$$\left[J_{ab}, Z_c^{(i)}\right] = \eta_{bc} Z_a^{(i)} - \eta_{ac} Z_b^{(i)}, \tag{1.56}$$

$$\left[Z_{ab}^{(i)}, P_c\right] = \eta_{bc} Z_a^{(i)} - \eta_{ac} Z_b^{(i)}, \tag{1.57}$$

$$\left[Z_{ab}^{(i)}, Z_c^{(j)}\right] = \eta_{bc} Z_a^{(i+j)} - \eta_{ac} Z_b^{(i+j)} \tag{1.58}$$

$$\left[J_{ab}, Z_{cd}^{(i)}\right] = \eta_{cb} Z_{ad}^{(i)} - \eta_{ca} Z_{bd}^{(i)} + \eta_{db} Z_{ca}^{(i)} - \eta_{da} Z_{cb}^{(i)} \tag{1.59}$$

$$\left[Z_{ab}^{(i)}, Z_{cd}^{(j)}\right] = \eta_{cb} Z_{ad}^{(i+j)} - \eta_{ca} Z_{bd}^{(i+j)} + \eta_{db} Z_{ca}^{(i+j)} - \eta_{da} Z_{cb}^{(i+j)} \tag{1.60}$$

$$\left[P_a, Z_c^{(i)}\right] = Z_{ab}^{(i+1)}, \qquad \left[Z_a^{(i)}, Z_c^{(j)}\right] = Z_{ab}^{(i+j+1)}. \tag{1.61}$$

where $J_{ab} = \lambda_0 \otimes \tilde{J}_{ab}$, $Z_{ab}^{(i)} = \lambda_{2i} \otimes \tilde{J}_{ab}$, $P_a = \lambda_1 \otimes \tilde{P}_a$ and $Z_a^{(i)} = \lambda_{2i+1} \otimes \tilde{P}_a$ with $i, j = 1, ..., n-1$. The generalized Poincaré algebra is also known as the Maxwell type algebra $\mathcal{M}_{2n+1}$ [17]. These algebras are particularly interesting in the context of gravity since, as we shall see now, standard odd-dimensional General Relativity may emerge as the weak coupling constant limit ($l \to 0$) of a $(2n+1)$-dimensional Chern-Simons Lagrangian invariant under the $\mathcal{M}_{2n+1}$ algebra.

### 1.4.1 General Relativity from Chern-Simons gravity

In this subsection, it is shown that the odd-dimensional Einstein-Hilbert Lagrangians can be obtained from Chern-Simons Lagrangians invariant under the Maxwell type algebras. According to Theorem VII.2 from [15], the only non-vanishing components of a symmetric invariant tensor of order $n+1$ for the $\mathcal{M}_{2n+1}$algebra, are given by

$$\left\langle J_{(a_1 a_2, i_1)} \cdots J_{(a_{2n-1} a_{2n}, i_n)} P_{(a_{2n+1}, i_{n+1})} \right\rangle = \frac{2^n l^{2n-1}}{n+1} \alpha_j \delta^j_{i_1 + \cdots i_{n+1}} \epsilon_{a_1 \cdots a_{2n+1}}, \tag{1.62}$$

where $i_p, j = 0, \ldots, 2n-1$, and the $\alpha_i$'s are arbitrary constants of dimension $[length]^{-2n+1}$.

The $\mathcal{M}_{2n+1}$-valued, one-form gauge connection $A$ takes the form

$$A = \sum_{k=0}^{n-1} \left[ \frac{1}{2} \omega^{(ab,2k)} J_{(ab,2k)} + \frac{1}{l} e^{(a,2k+1)} P_{(a,2k+1)} \right], \tag{1.63}$$

while the associated curvature two-form $F = dA + AA$,is given by

$$F = \sum_{k=0}^{n-1} \left[ \frac{1}{2} F^{(ab,2k)} J_{(ab,2k)} + \frac{1}{l} F^{(a,2k+1)} P_{(a,2k+1)} \right], \tag{1.64}$$

where

$$\begin{aligned} F^{(ab,2k)} &= d\omega^{(ab,2k)} + \eta_{cd}\omega^{(ac,2i)}\omega^{(db,2j)}\delta_{i+j}^{k} + \frac{1}{l^2} e^{(a,2i+1)} e^{(b,2j+1)} \delta_{i+j+1}^{k}, & (1.65) \\ F^{(a,2k+1)} &= de^{(a,2k+1)} + \eta_{bc}\omega^{(ab,2i)} e^{(c,2j)} \delta_{i+j}^{k}. & (1.66) \end{aligned}$$

From de definition of the one-form gauge connection $A$, we see that it depends on a scale parameter $l$, which can be interpreted as a coupling constant that characterizes different regimes within the theory. The $(2n+1)$-dimensional Chern-Simons Lagrangian invariant under the $\mathcal{M}_{2n+1}$ algebra can be written as [12]

$$\begin{aligned} L_{CS\ (2n+1)}^{\mathcal{M}_{2n+1}} = \sum_{k=1}^{n} l^{2k-2} c_k \alpha_j \delta_{i_1+\cdots+i_{n+1}}^{j} \delta_{p_1+q_1}^{i_{k+1}} \cdots \delta_{p_{n-k}+q_{n-k}}^{i_n} \\ \varepsilon_{a_1\cdots a_{2n+1}} R^{(a_1 a_2,i_1)} \cdots R^{(a_{2k-1}a_{2k},i_k)} e^{(a_{2k+1},p_1)} e^{(a_{2k+2},q_1)} \cdots \\ \cdots e^{(a_{2n-1},p_{n-k})} e^{(a_{2n},q_{n-k})} e^{(a_{2n+1},i_{n+1})}, \end{aligned} \tag{1.67}$$

where

$$\begin{aligned} c_k &= \frac{1}{2(n-k)+1} \binom{n}{k}, \\ R^{(ab,2k)} &= d\omega^{(ab,2k)} + \eta_{cd}\omega^{(ac,2i)}\omega^{(db,2j)}\delta_{i+j}^{k}, \end{aligned}$$

and $\alpha_j$ are arbitrary constants which appear as a consequence of the $S$-expansion process. Let us note that the $S$-expanded fields are related to the original AdS fields $\left\{\tilde{\omega}^{ab}, \tilde{e}^{a}\right\}$: $\omega^{(ab,2k)} = \lambda_{2k} \oplus \tilde{\omega}^{ab}$, $e^{(a,2k+1)} = \lambda_{2k+1} \oplus \tilde{e}^{a}$. The Lagrangian (1.67) is called the Einstein-Chern-Simons (ECS) Lagrangian.

In the limit $l \to 0$, the only non vanishing term in (1.67) corresponds to the case $k = 1$, whose only non-vanishing component occurs for $p = q_1 = \cdots = q_{2n-1} = 0$ and is proportional to the Einstein-Hilbert Lagrangian in odd-dimensions [12]

$$L_{CS}^{(2n+1)}\Big|_{l=0} = \frac{n}{2n-1} \alpha_{2n-1} \varepsilon_{a_1\cdots a_{2n+1}} R^{a_1 a_2} e^{a_3} \cdots e^{a_{2n+1}} \tag{1.68}$$

**Example for** $D = 5$

Let us consider as an example the case of $D = 5$ dimensions. In this case, the CS AdS Lagrangian for gravity is given by (see eq.(1.41))

$$L_{AdS}^{(5)} = \kappa\epsilon_{abcde}\left(\frac{1}{l}R^{ab}R^{cd}e^{e} + \frac{2}{3l^{3}}R^{ab}e^{c}e^{d}e^{e} + \frac{1}{5l^{5}}e^{a}e^{b}e^{c}e^{d}e^{e}\right). \tag{1.69}$$

From this Lagrangian, we see that neither the $l \to \infty$ nor the $l \to 0$ limits yield the EH term $\epsilon_{abcde}R^{ab}e^{c}e^{d}e^{e}$. Rescaling $\kappa$ properly, those limits will lead to either the Gauss-Bonnet term or the cosmological constant term by itself, respectively.

Following the above definitions, let us consider the $S$-expansion of the AdS Lie algebra $\mathfrak{so}(4,2)$, using $S_{E}^{(3)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ as a semigroup. After extracting a resonant subalgebra and performing its $0_S$-reduction, one finds the new Lie algebra $\mathcal{M}_5$, whose generators $\{J_{ab}, P_a, Z_{ab}, Z_a\}$, satisfy the following commutation relations

$$\begin{aligned}
[P_a, P_b] &= Z_{ab}, \qquad [J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b \\
[J_{ab}, J_{cd}] &= \eta_{cb}J_{ad} - \eta_{ca}J_{bd} + \eta_{db}J_{ca} - \eta_{da}J_{cb} \\
[J_{ab}, Z_c] &= \eta_{bc}Z_a - \eta_{ac}Z_b, \\
[Z_{ab}, P_c] &= \eta_{bc}Z_a - \eta_{ac}Z_b, \\
[J_{ab}, Z_{cd}] &= \eta_{cb}Z_{ad} - \eta_{ca}Z_{bd} + \eta_{db}Z_{ca} - \eta_{da}Z_{cb} \\
[Z_{ab}, Z_c] &= [P_a, Z_c] = [Z_a, Z_c] = [Z_{ab}, Z_{cd}] = 0,
\end{aligned} \tag{1.70}$$

which are given in terms of the original AdS generators $\left\{\tilde{J}_{ab}, \tilde{P}_a\right\}$ as follows

$$J_{ab} = \lambda_o \otimes \tilde{J}_{ab}, \qquad P_a = \lambda_1 \otimes \tilde{P}_a, \tag{1.71}$$

$$Z_{ab} = \lambda_2 \otimes \tilde{J}_{ab}, \qquad Z_a = \lambda_3 \otimes \tilde{P}_a. \tag{1.72}$$

From the expression (1.62), we have that the only non-vanishing components of a symmetric invariant tensor for the $\mathcal{M}_5$ algebra are

$$\begin{aligned}
\langle J_{ab}J_{cd}P_e\rangle &= \frac{4}{3}l^{3}\alpha_1\epsilon_{abcde}, \\
\langle J_{ab}J_{cd}Z_e\rangle &= \frac{4}{3}l^{3}\alpha_3\epsilon_{abcde}, \\
\langle J_{ab}Z_{cd}P_e\rangle &= \frac{4}{3}l^{3}\alpha_3\epsilon_{abcde}.
\end{aligned} \tag{1.73}$$

Then the $\mathcal{M}_5$-valued, one-form gauge connection $A$ takes the form

$$A = \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{l}e^{a}P_a + \frac{1}{2}k^{ab}Z_{ab} + \frac{1}{l}h^{a}Z_a, \tag{1.74}$$

and the corresponding curvature 2-form is given by

$$F = \frac{1}{2}R^{ab}J_{ab} + \frac{1}{l}T^{a}P_{a} + \frac{1}{2}\left(D_{\omega}k^{ab} + \frac{1}{l^{2}}e^{a}e^{b}\right)Z_{ab} + \frac{1}{l}\left(D_{\omega}h^{a} + k^{a}{}_{b}e^{b}\right)Z_{a}. \tag{1.75}$$

Using the dual formulation of the $S$-expansion in terms of the Maurer-Cartan forms [22], we can write down the CS Lagrangian in $D=5$ dimensions for the $\mathcal{M}_5$ algebra as

$$L^{\mathcal{M}_5}_{CS\ (5)} = \alpha_1 l^2 \epsilon_{abcde}R^{ab}R^{cd}e^{e} + \alpha_3\epsilon_{abcde}\left(\frac{2}{3}R^{ab}e^{c}e^{d}e^{e} + 2l^{2}k^{ab}R^{cd}T^{e} + l^{2}R^{ab}R^{cd}h^{e}\right). \tag{1.76}$$

From this Lagrangian, we see that it is split in two independent pieces, one proportional to $\alpha_1$ and the other proportional to $\alpha_3$. The former corresponds to the Iïonü–Wigner contraction [21] of the Lagrangian (1.69), and therefore it is the CS Lagrangian for the Poincaré Lie group $ISO(4,1)$. The latter contains the EH term $\epsilon_{abcde}R^{ab}e^{c}e^{d}e^{e}$ plus non-linear couplings between the curvature and the new bosonic fields $k_{ab}$ and $h_a$. Let us note that these couplings are all proportional to $l^2$.

Remarkably, considering the strict limit $l=0$ in the Lagrangian, we obtain solely the EH term

$$L^{\mathcal{M}_5}_{CS\ (5)}\Big|_{l=0} = \frac{2}{3}\alpha_3\epsilon_{abcde}R^{ab}e^{c}e^{d}e^{e}. \tag{1.77}$$

These results have been generalized in [17], where we have shown that the $(2n+1)$-dimensional Lagrangians $L^{\mathcal{M}_{2m+1}}_{CS\ (2n+1)}$ invariant under the $\mathcal{M}_{2m+1}$ algebra, lead to the Einstein-Hilbert Lagrangian in a weak coupling constant limit, if and only if $m \geq n$. In fact, the following theorem was announced:

**Theorem 1** *Let $\mathcal{M}_{2m+1}$ be the Maxwell type algebra, which is obtained from the AdS algebra by a resonant reduced $S^{(2m-1)}_{E}$-expansion. If $L^{\mathcal{M}_{2m+1}}_{CS\ (2p+1)}$ is a Chern-Simons Lagrangian $(2p+1)$-dimensional invariant under the $\mathcal{M}_{2m+1}$ algebra, then the $(2p+1)$-dimensional Chern-Simons Lagrangian leads to the Einstein-Hilbert Lagrangian in a certain limit of the coupling constant $l$, if and only if $m \geq p$.*

## 1.5 Born-Infeld gravity and Lorentz type Maxwell algebras

In even dimensions, the closest one can get to a Chern-Simons theory is with the so called Born-Infeld theories [5], [6], [7], [8]. As seen before, the Born-Infeld Lagrangian is obtained by a particular choice of the parameters in the LL action. In this case, the action is invariant only under local Lorentz rotations, in the same way as the Einstein–Hilbert action. If BI theories are the appropriate even-dimensional theories to provide a framework for the gravitational interaction, then these theories must satisfy the correspondence principle, namely they must be related to General Relativity.

In [19] it was shown that the standard, even-dimensional General Relativity can be obtained from a Born-Infeld theory invariant under a certain Lorentz type algebra, $\mathfrak{L}^{\mathfrak{B}}(=\mathfrak{L}^{\mathcal{M}})$. This algebra can be obtained from the Lorentz algebra and a particular semigroup by means of the $S$-expansion procedure, and corresponds to a subalgebra of the Maxwell type algebra. Then, in [17] it was found that standard even-dimensional General Relativity emerges as a weak coupling constant limit of a $2p$-dimensional Born-Infeld Lagrangian invariant under $\mathfrak{L}^{\mathcal{M}_{2m+1}}$, if and only if $m \geq p$. Let us briefly review here the results obtained in [17] and [19].

Let us consider the $S$-expansion of the Lie algebra $\mathfrak{so}(2n-1,2)$ using as a semigroup the sub-semigroup $S_0^{(2n-1)} = \{\lambda_0, \lambda_2, \lambda_4, \ldots, \lambda_{2n}\}$ of the semigroup $S_E^{(2n-1)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{2n}\}$. The semigroup $S_0^{(2n-1)}$ is endowed with the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leq 2n, \\ \lambda_{2n}, & \text{when } \alpha+\beta > 2n. \end{cases} \tag{1.78}$$

The Lorentz generators $\left\{\tilde{J}_{ab}\right\}$ satisfy the following commutation relations

$$\left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] = \eta_{cb}\tilde{J}_{ad} - \eta_{ca}\tilde{J}_{bd} + \eta_{db}\tilde{J}_{ca} - \eta_{da}\tilde{J}_{cb}. \tag{1.79}$$

After performing a $0_S\,(=\lambda_{2n})$-reduction, one finds a new Lie algebra $\mathfrak{L}^{\mathcal{M}_{2n+1}}$ whose generators $J_{ab} = \lambda_0 \otimes \tilde{J}_{ab}$, $Z_{ab}^{(i)} = \lambda_{2i} \otimes \tilde{J}_{ab}$ with $i, j = 1, ..., n-1$, satisfy the following commutation relations

$$[J_{ab}, J_{cd}] = \eta_{cb}J_{ad} - \eta_{ca}J_{bd} + \eta_{db}J_{ca} - \eta_{da}J_{cb}, \tag{1.80}$$

$$\left[J_{ab}, Z_{cd}^{(i)}\right] = \eta_{cb}Z_{ad}^{(i)} - \eta_{ca}Z_{bd}^{(i)} + \eta_{db}Z_{ca}^{(i)} - \eta_{da}Z_{cb}^{(i)}, \tag{1.81}$$

$$\left[Z_{ab}^{(i)}, Z_{cd}^{(j)}\right] = \eta_{cb}Z_{ad}^{(i+j)} - \eta_{ca}Z_{bd}^{(i+j)} + \eta_{db}Z_{ca}^{(i+j)} - \eta_{da}Z_{cb}^{(i+j)}. \tag{1.82}$$

Comparing these commutators with eqs. (1.55), (1.59) and (1.55), we can see that the Lorentz type algebra $\mathfrak{L}^{\mathcal{M}_{2n+1}}$ is a subalgebra of the Maxwell type algebra $\mathcal{M}_{2n+1}$. As we shall see now, standard even-dimensional General Relativity may emerge as the weak coupling constant limit $(l \to 0)$ of a Born-Infeld Lagrangian invariant under the $\mathfrak{L}^{\mathcal{M}_{2n+1}}$ algebra.

### 1.5.1 General Relativity from Born-Infeld gravity

In this subsection, it is shown that the even-dimensional Einstein-Hilbert Lagrangians can be obtained from Born-Infeld Lagrangians invariant under the Lorentz type algebras. Using Theorem VII.2 of [15], it is possible to show that the only non-vanishing components of a symmetric invariant tensor for the $\mathfrak{L}^{\mathcal{M}_{2n+1}}$ algebra are given by

$$\left\langle J_{(a_1a_2,i_1)}\cdots J_{(a_{2n-1}a_{2n},i_n)}\right\rangle = \frac{2^{n-1}l^{2n-2}}{n}\alpha_j\delta^j_{i_1+\cdots i_n}\epsilon_{a_1\cdots a_{2n}}, \tag{1.83}$$

where $j=0,\ldots,2n-2$, and the $\alpha_j$'s are arbitrary constants of dimension $[length]^{-2n+2}$.

In this case the curvature 2-form is given by

$$F=\sum_{k=0}^{n-1}\frac{1}{2}F^{(ab,2k)}J_{(ab,2k)} \tag{1.84}$$

where

$$F^{(ab,2k)}=d\omega^{(ab,2k)}+\eta_{cd}\omega^{(ac,2i)}\omega^{(db,2j)}\delta^k_{i+j}+\frac{1}{l^2}e^{(a,2i+1)}e^{(b,2j+1)}\delta^k_{i+j+1}, \tag{1.85}$$

which depends on a scale parameter $l$ which can be interpreted as a coupling constant that characterizes different regimes within the theory. Then, using the dual formulation of the $S$-expansion in terms of the Maurer-Cartan forms, we find that the $2n$-dimensional Born-Infeld Lagrangian invariant under the $\mathfrak{L}^{\mathcal{M}_{2n+1}}$ algebra can be written as [19]

$$\begin{aligned}L^{\mathfrak{L}^{\mathcal{M}_{2n+1}}}_{BI\ (2n)} = \sum_{k=1}^{n} l^{2k-2}\binom{n}{k}\alpha_j\delta^j_{i_1+\cdots+i_n}\delta^{i_{k+1}}_{p_1+q_1}\cdots\delta^{i_n}_{p_{n-k}+q_{n-k}} \\ \varepsilon_{a_1\cdots a_{2n}}R^{(a_1a_2,i_1)}\cdots R^{(a_{2k-1}a_{2k},i_k)}e^{(a_{2k+1},p_1)}e^{(a_{2k+2},q_1)}\cdots \\ \cdots e^{(a_{2n-1},p_{n-k})}e^{(a_{2n},q_{n-k})},\end{aligned} \tag{1.86}$$

where

$$R^{(ab,2k)}=d\omega^{(ab,2k)}+\eta_{cd}\omega^{(ac,2i)}\omega^{(db,2j)}\delta^k_{i+j},$$

and $\alpha_j$ are arbitrary constants which appear as a consequence of the $S$-expansion method. Let us note that the $S$-expanded fields are related to the AdS $\left\{\tilde{\omega}^{ab},\tilde{e}^a\right\}$ fields: $\omega^{(ab,2k)}=\lambda_{2k}\oplus\tilde{\omega}^{ab}$, $e^{(a,2k+1)}=\lambda_{2k+1}\oplus\tilde{e}^a$. The Lagrangian (1.86) was called the Einstein-Born-Infeld (EBI) Lagrangian.

In the limit $l\to 0$, the only non zero term in (1.86) corresponds to the case $k=1$, whose only non-vanishing component occurs for $p=q_1=\cdots=q_{2n-1}=0$ and is proportional to the Einstein-Hilbert Lagrangian in even-dimensions [19]

$$L^{(2n)}_{BI}\Big|_{l=0}=\frac{1}{2}\alpha_{2n-2}\varepsilon_{a_1\cdots a_{2n}}R^{a_1a_2}e^{a_3}\cdots e^{a_{2n}} \tag{1.87}$$

**Example for $D = 4$**

Let us consider the case of $D = 4$ dimensions. In this case, the Born-Infeld Lagrangian for gravity is written as (see eq.(1.45))

$$L_{BI}^{(4)} = \frac{\kappa}{4}\varepsilon_{abcd}\left(R^{ab}R^{cd} + \frac{2}{l^2}R^{ab}e^c e^d + \frac{1}{l^4}e^a e^b e^c e^d\right). \tag{1.88}$$

From this Lagrangian, it is apparent that neither the $l \to \infty$ nor $l \to 0$ limit yields the Einstein-Hilbert term alone. Rescaling $\kappa$ properly, those limits will lead either to the Euler density or to the cosmological constant term by itself, respectively. Since the Euler density is a topological invariant, it does not contribute to the equations of motion. Thus, in $D = 4$ dimensions and considering $l \to \infty$, the dominant term would be the EH term $\varepsilon_{abcd}R^{ab}e^c e^d$. Nevertheless, for $D > 4$ this statement is not valid anymore and we have that no limit allows us to obtain the desired term.

Following the above definitions, let us consider the $S$-expansion of the Lorentz Lie algebra $\mathfrak{so}(3,1)$ using the semigroup $S_0^{(3)} = \{\lambda_0, \lambda_2, \lambda_4\}$. After performing its $0_S$-reduction, we find the new Lie algebra $\mathfrak{L}^{\mathcal{M}_5}$ (or $\mathfrak{L}^{\mathfrak{B}_5}$ as was introduced in [19]), which corresponds to a subalgebra of the Maxwell type algebra $\mathcal{M}_5$. The generators defined by $J_{ab} = \lambda_0 \tilde{J}_{ab}$, $Z_{ab} = \lambda_2 \tilde{J}_{ab}$ (where $\tilde{J}_{ab}$ are the $\mathfrak{so}(3,1)$ generators), satisfy

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{cb}J_{ad} - \eta_{ca}J_{bd} + \eta_{db}J_{ca} - \eta_{da}J_{cb}, \\
[J_{ab}, Z_{cd}] &= \eta_{cb}Z_{ad} - \eta_{ca}Z_{bd} + \eta_{db}Z_{ca} - \eta_{da}Z_{cb}, \\
[Z_{ab}, Z_{cd}] &= 0.
\end{aligned} \tag{1.89}$$

From (1.83) we find the $\mathfrak{L}^{\mathcal{M}_5}$ invariant tensors, which are given by

$$\langle J_{ab}J_{cd}\rangle_{\mathfrak{L}^{\mathcal{M}_5}} = \alpha_0 l^2 \varepsilon_{abcd}, \tag{1.90}$$

$$\langle J_{ab}Z_{cd}\rangle_{\mathfrak{L}^{\mathcal{M}_5}} = \alpha_2 l^2 \varepsilon_{abcd} \tag{1.91}$$

and the curvature two-form is

$$F = \frac{1}{2}R^{ab}J_{ab} + \frac{1}{2}\left(D_\omega k^{ab} + \frac{1}{l^2}e^a e^b\right)Z_{ab}$$

Using the dual formulation of the $S$-expansion in terms of the Maurer-Cartan forms [22], we can write down the Born-Infeld Lagrangian invariant under $\mathfrak{L}^{\mathcal{M}_5}$ algebra, as follows

$$L_{BI\ (4)}^{\mathfrak{L}^{\mathcal{M}_5}} = \frac{\alpha_0}{4}\epsilon_{abcd}l^2 R^{ab}R^{cd} + \frac{\alpha_2}{2}\epsilon_{abcd}\left(R^{ab}e^c e^d + l^2 D_\omega k^{ab}R^{cd}\right). \tag{1.92}$$

This Lagrangian is split in two independent pieces, one proportional to $\alpha_0$ and the other proportional to $\alpha_2$. The term proportional to $\alpha_0$ corresponds to the Euler invariant, while the piece proportional to $\alpha_2$ contains the EH term $\epsilon_{abcd}R^{ab}e^c e^d$ plus a boundary term which contains, besides the usual curvature $R^{ab}$, a bosonic matter field $k^{ab}$.

Then, considering the strict limit $l = 0$ in the Lagrangian, we recover the four-dimensional EH Lagrangian

$$L_{BI\ (4)}^{\mathfrak{L}^{\mathcal{M}_5}}\Big|_{l=0} = \frac{\alpha_2}{2}\epsilon_{abcd}R^{ab}e^c e^d. \tag{1.93}$$

These results have been generalized in [17], where we have shown that the $2n$-dimensional Lagrangians $L_{BI\ (2n)}^{\mathfrak{L}^{\mathcal{M}_{2m+1}}}$ invariant under the $\mathfrak{L}^{\mathcal{M}_{2m+1}}$ algebra, lead to the Einstein–Hilbert Lagrangian in a weak coupling constant limit, if and only if $m \geq n$. In fact, the following theorem was announced:

**Theorem 2** *Let $\mathfrak{L}^{\mathcal{M}_{2m+1}}$ be the algebra obtained from the Lorentz algebra by a reduced $S_0^{(2m-2)}$-expansion, which corresponds to a subalgebra of the $\mathcal{M}_{2m+1}$ algebra.. If $L_{BI\ (2p)}^{\mathfrak{L}^{\mathcal{M}_{2m+1}}}$ is a Born-Infeld type $2p$-dimensional Lagrangian invariant under the $\mathfrak{L}^{\mathcal{M}_{2m+1}}$ algebra, then the $2p$-dimensional Born-Infeld type Lagrangian leads to the Einstein-Hilbert Lagrangian in a certain limit of the coupling constant $l$, if and only if $m \geq p$.*

## 1.6 Einstein-Lovelock-Cartan gravity theory

In this section, we shall briefly discuss the main results of [20], where we have shown that it is possible to construct an Einstein-Lovelock-Cartan (ELC) Lagrangian leading to the Einstein-Chern-Simons Lagrangian in $D = 2n-1$ invariant under the $\mathcal{M}_{2n-1}$ algebra , and to the Einstein-Born-Infeld Lagrangian in $D = 2n$ invariant under the $\mathfrak{L}^{\mathcal{M}_{2m}}$ algebra. The ECS and EBI theories are particularly interesting since as was shown in the previous sections, General Relativity can be obtained as a certain limit of these gravity theories. For our purpose, we shall use the useful properties of the $S$-expansion procedure using $S_E^{(D-2)}$ as the relevant semigroup.

The expanded action is given by [20]

$$S_{\mathcal{ELC}} = \int \sum_{p=0}^{[D/2]} \mu_i \alpha_p L_{\mathcal{ELC}}^{(p,i)} \tag{1.94}$$

where $\alpha_p$ and $\mu_i$, with $i = 0, \ldots, D-2$ are arbitrary constants and $L^{(p,i)}_{\mathcal{ELC}}$ is given by

$$L^{(p,i)}_{\mathcal{ELC}} = l^{d-2}\delta^{i}_{i_1+\cdots+i_{d-p}}\varepsilon_{a_1a_2\cdots a_D}R^{(a_1a_2,i_1)}\cdots R^{(a_{2p-1}a_{2p},i_p)}e^{(a_{2p+1},i_{p+1})}\cdots e^{\left(a_D,i_{D-p}\right)}, \tag{1.95}$$

where

$$R^{(ab,2i)} = d\omega^{(ab,2i)} + \eta_{cd}\omega^{(ac,2j)}\omega^{(db,2k)}\delta^{i}_{j+k}. \tag{1.96}$$

The expanded fields $\left\{e^{(a,2i+1)}, \omega^{(ab,2i)}\right\}$ are related to the AdS fields $\left\{\tilde{e}^a, \tilde{\omega}^{ab}\right\}$ as follows

$$\omega^{(ab,2i)} = \lambda_{2i} \otimes \tilde{\omega}^{ab} \tag{1.97}$$

$$e^{(a,2i+1)} = \lambda_{2i+1} \otimes \tilde{e}^{a} \tag{1.98}$$

with $\lambda_\alpha \in S^{(D-2)}_E$, and where $S^{(D-2)}_E$ is the semigroup whose elements obey the following multiplication law

$$\lambda_\alpha\lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leq D-1, \\ \lambda_{2n}, & \text{when } \alpha+\beta > D-1. \end{cases} \tag{1.99}$$

The Lagrangian in (1.94) corresponds to the Einstein-Lovelock-Cartan Lagrangian and can be used in both odd and even dimensions. Following the same procedure of [5], and considering the action as a functional of the expanded fields $S_{\mathcal{ELC}} = S_{\mathcal{ELC}}\left[e^{(a,j)}, \omega^{(ab,j)}\right]$, we have that the variation of the action with respect to $e^{(a,i)}$and $\omega^{(ab,i)}$ lead to the following equations:

$$\varepsilon^{(i)}_a = \sum_{p=0}^{[(D-1)/2]} \mu_i\alpha_p\left(d-2p\right)\varepsilon^{(p,i)}_a = 0\ , \tag{1.100}$$

$$\varepsilon^{(i)}_{ab} = \sum_{p=1}^{[(D-1)/2]} \mu_i\alpha_p p\left(d-2p\right)\varepsilon^{(p,i)}_{ab} = 0, \tag{1.101}$$

where

$$\begin{aligned}\varepsilon^{(p,i)}_a :\ &= l^{d-2}\delta^{i}_{i_1+\cdots+i_{d-p-1}}\varepsilon_{ab_1\cdots b_{d-1}}R^{(b_1b_2,i_1)}\cdots R^{(b_{2p-1}b_{2p},i_p)} \\ &\times e^{(b_{2p+1},i_{p+1})}\cdots e^{\left(b_{D-1},i_{D-p-1}\right)},\end{aligned} \tag{1.102}$$

$$\begin{aligned}\varepsilon^{(p,i)}_{ab} :\ &= l^{d-2}\delta^{i}_{i_1+\cdots+i_{d-p-1}}\varepsilon_{aba_3\cdots a_d}R^{(a_3a_4,i_1)}\cdots R^{(a_{2p-1}a_{2p},i_{p-1})} \\ &T^{(a_{2p+1},i_p)}e^{(a_{2p+2},i_{p+1})}\cdots e^{\left(a_D,i_{D-p-1}\right)},\end{aligned} \tag{1.103}$$

and where $T^{(a,i)} = de^{(a,i)} + \eta_{dc}\omega^{(ad,j)}e^{(c,k)}\delta^{i}_{j+k}$ is the expanded torsion 2-form . In general, there are different ways of choosing the coefficients $\alpha_p$ which in general correspond to different theories with different numbers of degrees of freedom. It is possible to choose the $\alpha_p$ such that $\varepsilon_a^{(i)}$ and $\varepsilon_{ab}^{(i)}$ are independent. This last condition corresponds to the maximum number of independent components.

As in [5], we showed that in odd-dimensions the $\alpha_p$ coefficients are given by

$$\alpha_p = \alpha_0 \frac{(2n-1)(2\gamma)^p}{(2n-2p-1)} \binom{n-1}{p}, \tag{1.104}$$

where

$$\alpha_0 = \frac{\kappa}{(l^{D-1}D)}; \qquad \gamma = -sgn(\Lambda)\frac{l^2}{2}, \tag{1.105}$$

and, for any dimension $D$, $l$ is a length parameter related to the cosmological constant by

$$\Lambda = \pm \frac{(D-1)(D-2)}{2l^2}. \tag{1.106}$$

With these coefficients the Lagrangian (1.94) can be written as the Chern–Simons form

$$\begin{aligned} L_{CS\;(2n-1)}^{\mathcal{M}_{2n-1}} &= \sum_{p=0}^{n-1} l^{2p-2} \frac{\kappa}{2(n-p)-1} \binom{n-1}{p} \mu_i \delta_{i_1+\cdots+i_{2n-1-p}}^{i} \\ &\quad \varepsilon_{a_1 a_2 \cdots a_{2n-1}} R^{(a_1 a_2, i_1)} \cdots R^{(a_{2p-1} a_{2p}, i_p)} e^{(a_{2p+1}, i_{p+1})} \cdots e^{(a_{2n-1}, i_{2n-1-p})}. \end{aligned} \tag{1.107}$$

Thus, we conclude that in odd-dimensions the choice of the coefficients (1.104), allows us to write the Lagrangian (1.94) as a Chern-Simons form for the Maxwell type algebra $\mathcal{M}_{2n-1}$, called the Einstein-Chern-Simons Lagrangian in [12]. Furthermore, let us note that the $S$-expansion process did not modify the $\alpha_p$ coefficients of [18] for the odd-dimensional case.

In the even dimensional case the $\alpha_p$ coefficients are given by

$$\alpha_p = \alpha_0 (2\gamma)^p \binom{n}{p}. \tag{1.108}$$

With these coefficients the ELC Lagrangian (1.94) is written as

$$\begin{aligned} L_{BI\;(2n)}^{\mathfrak{E}^{\mathcal{M}_{2n}}} &= \sum_{p=0}^{n} \frac{\kappa}{2n} l^{2p-2} \binom{n}{p} \mu_i \delta_{i_1+\cdots+i_{2n-p}}^{i} \\ &\quad \varepsilon_{a_1 a_2 \cdots a_{2n}} R^{(a_1 a_2, i_1)} \cdots R^{(a_{2p-1} a_{2p}, i_p)} e^{(a_{2p+1}, i_{p+1})} \cdots e^{(a_{2n}, i_{2n-p})}, \end{aligned} \tag{1.109}$$

which corresponds to the Einstein-Born-Infeld Lagrangian found in [19].

In this way, we have shown that the $S$-expansion procedure does not modify the $\alpha_p$'s coefficients defined in [5]. Unlike the Lanczos-Lovelock action, the ELC action (1.94) has the property of leading to General Relativity in a certain limit of the coupling constant $l$, both even and odd dimensions. The Einstein-Lovelock Lagrangian ((1.94) can be interpreted as the most general $D$-form invariant under a Lorentz type subalgebra $\mathfrak{L}^{\mathcal{M}_{2n}}$ of the Maxwell type algebra. This Lagrangian is constructed from the expanded vielbein and the expanded spin connection $e^{(a,2k+1)}$, $\omega^{(ab,2k)}$ $(k = 0, \ldots, n-1)$[3] and their exterior derivatives.

Furthermore, in [20] we have shown that the Einstein-Lovelock-Cartan Lagrangian can be generalized adding torsional terms, following a similar procedure to that of [5].

[3]When $k = 0$, $e^{(a,1)}$ and $\omega^{(ab,0)}$ are identified with the usual vielbein $e^a$ and the spin connection $\omega^{ab}$, respectively.

# Chapter 2

# Supersymmetric extension of Gravity

## 2.1 Introduction

It is well known that the Standard Model describes consistently three of the four fundamental interactions of Nature, through Yang-Mills quantum theories. Despite numerous attempts, the fourth fundamental interaction, gravity, has resisted quantization. Gravity is described by General Relativity as a manifestation of the geometric properties of space-time. The possibility of unifying gravity with the other interactions in a same geometric framework would require to incorporate internal and space-time symmetries in a same group. A possible way to achieve this task is supersymmetry, a kind of symmetry against which we shall demand the laws of nature be invariant, at least at a certain level (for an introduction of supersymmetry see for instance [23]). The idea that SUSY is actually an underlying symmetry of Nature is supported by various phenomenological arguments. For example, the presence of this symmetry makes field theories better behaved in the UV by virtue of the cancellation of fermionic and bosonic contributions to divergent loop integrals (see for instance [24]). This solves an important problem with the Standard Model, namely the hierarchy problem.

The supersymmetric extension of General Relativity is known as Supergravity (see refs. [25], [26] and [27]) and it is a theory of local supersymmetry. In SUGRA, the gravitational field is coupled to its super-partners and possibly to other supermultiplets containing matter multiplets. Furthermore, it can be viewed as the gauge theory of the superPoincaré group, which unifies space-time and internal symmetries. Mathematically, it is about a graded Lie algebra, also called super Lie algebra (or superalgebra) having bosonic ($B$) and fermionic

($F$) generators satisfying (anti)commutation relations; $[B, B] = B$, $[B, F] = F$, $\{F, F\} = B$ (see for instance [23]).

Initially, Supergravity was conceived as a theory described by an action including the Einstein-Hilbert term representing the graviton plus a Rarita-Schwinger kinetic term describing the gravitino. The $\mathcal{N} = 1$ pure Supergravity was constructed first by D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara in [29], and was derived in the second order formalism, i.e. writing $\omega^{ab}$ in terms of the other fields by imposing the vanishing of the supertorsion. Then, the same results were found by S. Deser and B. Zumino in the first order formalism [29]. Subsequently, the model was extended to incorporate other features like enlarged symmetries, matter couplings, higher dimensions with their corresponding reductions to four dimensions and cosmological constant.

In the next section, we briefly describe some general aspects of supersymmetry and supergravity, which are essential concepts in this thesis. Then, we shall introduce a supergravity theory with $\mathcal{N} = 1$ in four dimensions with cosmological constant. In particular, we consider the geometrical approach presented by S.W. MacDowell and F. Mansouri in [28]. Eventually, in the last section we consider the construction of the most general three-dimensional CS Supergravity action for the AdS superalgebra.

## 2.2 Supersymmetry and Supergravity: General aspects

Supersymmetry is a symmetry that mixes bosonic and fermionic particles. As we said before, the idea that this curious symmetry is actually an underlying symmetry of Nature is supported by many phenomenological arguments. For instance, the presence of this symmetry makes many field theories better behaved in the UV by virtue of the cancellation of divergences of the bosons by divergences coming from the fermionic sector. This is a very interesting property from the point of view of a quantum gravity theory. In fact, it was shown in [24] that in a supersymmetric extension of General Relativity, the ultraviolet divergences at the one-loop level are exactly cancelled.

From a theoretical point of view, SUSY has a most interesting aspect since it unifies bosonic space-time symmetries with other internal bosonic symmetries (like the $SU(3) \times SU(2) \times U(1)$ invariance of the standard model), giving the possibility of unify gravity with the other interactions in a same geometric framework. In fact, the group containing supersymmetry transformations generalizes the Poincaré group, and in addition to the Lorentz generators $J_{ab}$ and the space-time translations $P_a$, we have also supersymmetry generators

$Q$ (the fermionic generators) and internal generators $B_i$. Then, the corresponding algebra is called the super-Poincaré algebra. Furthermore, we have that the action of the $Q$ generators on the states $|\text{fermion}\rangle$ or $|\text{boson}\rangle$ is given by

$$Q\,|\text{fermion}\rangle = |\text{boson}\rangle\,; \qquad Q\,|\text{boson}\rangle = \quad |\text{fermion}\rangle \tag{2.1}$$

The super-Poincaré algebra must be defined in terms of both commutators $[,]$ and anticommutators $\{,\}$ as follows,

$$[B,B] \sim B, \qquad [B,F] \sim F, \qquad \{F,F\} \sim B, \tag{2.2}$$

where the generators of the Poincaré group are included in the bosonic sector, and the $F$'s are the supersymmetry generators. A Lie algebra containing fermionic generators obeying anti-commutation relations as above is called a graded Lie algebra, or simply superalgebra. For an arbitrary bosonic group is not always possible to find a set of fermionic generators in order to close the superalgebra. In this way, a consistency condition is required and is given by the super-Jacobi identity

$$\left[G_I\left[G_J,G_K\right]_{\pm}\right]_{\pm} + (-)^{\sigma(JKI)}\left[G_J,\left[G_K,G_I\right]_{\pm}\right]_{\pm} + (-)^{\sigma(KIJ)}\left[G_K,\left[G_I,G_J\right]_{\pm}\right]_{\pm} = 0, \tag{2.3}$$

where $G_I$ represents any generator in the algebra, $[A,B]_{\pm} = AB \pm BA$, and $\sigma$ corresponds to the number of permutations of fermionic generators required for $(IJK) \to (JKI)$.

As said before, the supersymmetric extension of General Relativity is known as Supergravity and it is a theory of local supersymmetry. In its simplest version Supergravity can be viewed as the "gauge" theory of the super-Poincaré group whose action is given by the Einstein-Hilbert term representing the graviton, plus a Rarita-Schwinger kinetic term describing the gravitino $\psi$, a spin-3/2 particle,

$$S = \int \epsilon_{abcd} R^{ab} e^c e^d + 4\bar{\psi} e^a \gamma_a \gamma_5 D\psi.$$

Standard SUGRAs are not gauge theories for a group or a supergroup, and the local (super)symmetry algebra closes naturally on-shell only. When we said that Supergravity can be viewed as the "gauge" theory of the super-Poincaré group, we mean that the "gauge group" describes external, i.e. space-time symmetries. On the other hand, in the case of the Standard Model, the gauge group is an internal symmetry, namely acts on internal degrees of freedom.

There are several different supersymmetric theories, which differ in the space-time dimension $D$ and in the number $\mathcal{N}$ of supersymmetry charges. SUGRA theories of particular relevance are defined in $D = 10$ and $D = 11$, since they describe the low-energy dynamics of superstring theory and M-theory, on at space-time, respectively. Regarding the number of supersymmetry, $\mathcal{N}$ supersymmetry generators define an $\mathcal{N}$-extended supersymmetry. Theories which are only invariant under global superPoincaré transformations (rigid supersymmetry), do not contain gravity and are thus defined on flat space-time. Renormalizability requires their fields not to have spin greater than 1, and thus $\mathcal{N} \leq 4$. The $\mathcal{N} = 4$ case describes a supersymmetric extension of the Yang-Mills theory (super-YM theory). In Supergravity, the limit on the amount $\mathcal{N}$ of supersymmetry comes from the possibility of a consistent coupling to gravity, which restricts the maximum spin of the fields to be 2, thus implying $\mathcal{N} \leq 8$.

In particular, global and local supersymmetric theories display deep geometrical structures inherent to the non-linear interactions of matter multiplets. In the $D = 4$, $\mathcal{N} = 2$ case the geometrical structure is described by the Special Kähler geometry and the Hypergeometry, when vector multiplets and hypermultiplets are present. There are two kinds of special Kähler geometry: the local and the rigid one. In the local case, the special Kähler geometry describes the scalar field sector of vector multiplets in $\mathcal{N} = 2$ SUGRA, while in the latter case the rigid special Kähler geometry describes the same sector in a $\mathcal{N} = 2$ Yang–Mills theory.

When matter is added, the underlying geometrical structure is much richer, since $\mathcal{N} = 2$ matter hypermultiplets are associated with the quaternionic geometry. There are four real scalar fields for each hypermultiplet, which can be viewed (locally) as the four components of a quaternion. As in the vector multiplet case, there are two kinds of hypergeometry, the local and the rigid one. The former is called Quaternionic geometry, while the latter is called the HyperKähler geometry.

A complete study of the $\mathcal{N} = 2$ Supergravity and $\mathcal{N} = 2$ Super Yang-Mills theory coupled to vector multiplets and hypermultiplets can be found in [31]. As we will see in the Part III of this thesis, interesting results have been found in the study of rigid and local supersymmetric $\mathcal{N} = 2$ field theories in $D = 4$. In particular, in the study of spontaneous breaking of $\mathcal{N} = 2$ to $\mathcal{N} = 1$ in local supersymmetric theories, and the corresponding low energy limit to a rigid supersymmetric theory, we have shown that a well-defined limit exists where the low energy, $\mathcal{N} = 1$ residual theory appears as a supersymmetric Born-Infeld theory.

## 2.3 Mac Dowell-Mansouri Supergravity

Several supergravity theories are known for all $D \leq 11$. For $D = 4$ dimensions a supergravity action with a cosmological constant was first presented by P. K Towsand in [34] and then by S.W. MacDowell and F. Mansouri [28]. Nevertheless, to find a supergravity action with cosmological constant in an arbitrary dimension is a nontrivial task. For instance, in the case of the standard supergravity in $D = 11$ dimensions [32] it has been shown that it is not possible to accommodate a cosmological constant [35], [36].

In [28] S.W. MacDowell and F. Mansouri presented a geometric formulation of $\mathcal{N} = 1$ supergravity in four dimensions, where the relevant gauge fields of the theory are those corresponding to the $Osp(4|1)$ supergroup. The resulting action, constructed exclusively in terms of the components of the curvature, led to the $\mathcal{N} = 1$ supergravity plus cosmological and topological terms, and corresponds to a generalization of [29] with the addition of cosmological terms. In this section, we consider a brief review of this construction, whose results will be essential in the formulation of new supergravity models which will be presented throughout this thesis.

The (anti)-commutation relations for the $\mathfrak{osp}(4|1)$ superalgebra are given by

$$\left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] = \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} - \eta_{bd}\tilde{J}_{ac} + \eta_{ad}\tilde{J}_{bc}, \tag{2.4}$$

$$\left[\tilde{J}_{ab}, \tilde{P}_{c}\right] = \eta_{bc}\tilde{P}_{a} - \eta_{ac}\tilde{P}_{b}, \tag{2.5}$$

$$\left[\tilde{P}_{a}, \tilde{P}_{b}\right] = \tilde{J}_{ab}, \tag{2.6}$$

$$\left[\tilde{J}_{ab}, \tilde{Q}_{\alpha}\right] = -\frac{1}{2}\left(\gamma_{ab}\tilde{Q}\right)_{\alpha}, \qquad \left[\tilde{P}_{a}, \tilde{Q}_{\alpha}\right] = -\frac{1}{2}\left(\gamma_{a}\tilde{Q}\right)_{\alpha}, \tag{2.7}$$

$$\left\{\tilde{Q}_{\alpha}, \tilde{Q}_{\beta}\right\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{J}_{ab} - 2\left(\gamma^{a}C\right)_{\alpha\beta}\tilde{P}_{a}\right]. \tag{2.8}$$

where $\tilde{J}_{ab}$, $\tilde{P}_{a}$ and $\tilde{Q}_{\alpha}$ correspond to the Lorentz generators, the AdS boost generators and the fermionic generators, respectively. Here, $C$ stands for the charge conjugation matrix and $\gamma_{a}$ are Dirac matrices.

In order to write down a Lagrangian for this superalgebra, we start from the one-form gauge connection

$$A = \frac{1}{2}\omega^{ab}\tilde{J}_{ab} + \frac{1}{l}e^{a}\tilde{P}_{a} + \frac{1}{\sqrt{l}}\psi^{\alpha}\tilde{Q}_{\alpha}, \tag{2.9}$$

and the associated curvature two-form $F = dA + A \wedge A$,

$$F = F^{A}T_{A} = \frac{1}{2}\mathcal{R}^{ab}\tilde{J}_{ab} + \frac{1}{l}R^{a}\tilde{P}_{a} + \frac{1}{\sqrt{l}}\rho^{\alpha}\tilde{Q}_{\alpha}, \tag{2.10}$$

where

$$\mathcal{R}^{ab} = d\omega^{ab} + \omega^{a}_{\ c}\omega^{cb} + \frac{1}{l^2}e^a e^b + \frac{1}{2l}\bar{\psi}\gamma^{ab}\psi, \tag{2.11}$$

$$R^a = de^a + \omega^{a}_{\ b}e^b - \frac{1}{2}\bar{\psi}\gamma^a\psi, \tag{2.12}$$

$$\rho = d\psi + \frac{1}{4}\omega_{ab}\gamma^{ab}\psi + \frac{1}{2l}e^a\gamma_a\psi = D\psi + \frac{1}{2l}e^a\gamma_a\psi. \tag{2.13}$$

From the Bianchi identity $\nabla F = 0$, where $\nabla = d + [A, \cdot]$, it is possible to show that the Lorentz covariant exterior derivatives of the curvatures are given by,

$$D\mathcal{R}^{ab} = \frac{1}{l^2}R^a e^b - \frac{1}{l^2}e^a R^b - \frac{1}{l}\bar{\psi}\gamma^{ab}\Psi, \tag{2.14}$$

$$DR^a = R^{a}_{\ b}e^b + \bar{\psi}\gamma^a\Psi, \tag{2.15}$$

$$D\rho = \frac{1}{4}R_{ab}\gamma^{ab}\psi + \frac{1}{2l}R^a\gamma_a\psi - \frac{1}{2l}e^a\gamma_a\Psi. \tag{2.16}$$

The one-forms $e^a, \omega^{ab}$ and $\psi$ are respectively the vierbein, the spin connection and the gravitino field (a Majorana spinor, i.e, $\bar{\psi} = \psi^T C$, where $C$ is the charge conjugation matrix).

Unlike the original approach in [28], here we have introduced a length scale $l$. This is done because we have chosen the Lie algebra generators $T_A = \left\{\tilde{J}_{ab}, \tilde{P}_a, \tilde{Q}_\alpha\right\}$ as dimensionless and thus the one form connection $A = A^{A}_{\ \mu}T_A dx^\mu$ must also be dimensionless. Nevertheless, the vierbein $e^a = e^{a}_{\ \mu}dx^\mu$ must have dimensions of length if it is related to the spacetime metric $g_{\mu\nu}$ through the usual equation $g_{\mu\nu} = e^{a}_{\ \mu}e^{b}_{\ \nu}\eta_{ab}$. This means that the "true" gauge field must be considered as $e^a/l$, with $l$ a length parameter. In the same way, as the gravitino $\psi = \psi_\mu dx^\mu$ has dimensions of (length)$^{1/2}$, we must consider that $\psi/\sqrt{l}$ is the gauge field of supersymmetry.

The general form of an action constructed with the curvature 2-form (2.10) is given by

$$S = 2\int \langle F \wedge F \rangle = 2\int F^A \wedge F^B \langle T_A T_B \rangle . \tag{2.17}$$

Let us note that if we choose $\langle T_A T_B \rangle$ as an invariant tensor (which satisfies the Bianchi identity) for the $Osp\,(4|1)$ supergroup, then the action (2.17) is a topological invariant and thus, gives no equations of motion. However, with the following choice of the invariant tensor

$$\langle T_A T_B \rangle = \begin{cases} \left\langle \tilde{J}_{ab}\tilde{J}_{cd} \right\rangle = \epsilon_{abcd} \\ \left\langle \tilde{Q}_\alpha \tilde{Q}_\beta \right\rangle = 2\left(\gamma_5\right)_{\alpha\beta} \end{cases} \tag{2.18}$$

the action (2.17) becomes

$$S = 2\int \frac{1}{4}\mathcal{R}^{ab}\mathcal{R}^{ab}\epsilon_{abcd} + \frac{2}{l}\bar{\rho}\gamma_5\rho \tag{2.19}$$

which corresponds to the Mac Dowell-Mansouri action [28]. This choice of the invariant tensor, which is required in order to reproduce a dynamical action, breaks the $Osp\,(4|1)$ supergroup to their Lorentz subgroup.

The explicit form of the action is given by,

$$\begin{aligned} S = \int \frac{1}{2}\epsilon_{abcd}\left(R^{ab}R^{cd} + \frac{2}{l^2}R^{ab}e^c e^d + \frac{1}{l^4}e^a e^b e^c e^d + \frac{2}{l^3}\bar{\psi}\gamma^{ab}\psi e^c e^d\right) \\ + \frac{4}{l^2}\bar{\psi}e^a\gamma_a\gamma_5 D\psi + \frac{4}{l}d\left(\bar{\psi}\gamma_5 D\psi\right). \end{aligned} \tag{2.20}$$

Here, we have used the gravitino Bianchi identity

$$DD\psi = \frac{1}{4}R_{ab}\gamma^{ab}\psi, \tag{2.21}$$

and the gamma matrix identity

$$2\gamma_{ab}\gamma_5 = -\epsilon_{abcd}\gamma^{cd}, \tag{2.22}$$

to recognize that

$$\frac{1}{2}\epsilon_{abcd}R^{ab}\bar{\psi}\gamma^{ab}\psi + 4D\bar{\psi}\gamma_5 D\psi = 4d\left(\bar{\psi}\gamma_5 D\psi\right). \tag{2.23}$$

Thus the action can be written, modulo boundary terms, as follows

$$S = \int \frac{1}{l^2}\left(\epsilon_{abcd}R^{ab}e^c e^d + 4\bar{\psi}e^a\gamma_a\gamma_5 D\psi\right) + \frac{1}{2}\epsilon_{abcd}\left(\frac{1}{l^4}e^a e^b e^c e^d + \frac{2}{l^3}\bar{\psi}\gamma^{ab}\psi e^c e^d\right), \tag{2.24}$$

where $R^{ab} = d\omega^{ab} + \omega^a_{\ c}\omega^{cb}$. The action (2.24) corresponds to the Mac Dowell-Mansouri action for the $\mathfrak{osp}\,(4|1)$ superalgebra. This action describes $\mathcal{N}=1$, $D=4$ AdS Supergravity, and the last term is the supersymmetric cosmological term. We can see that in the limit $l \to \infty$ the usual $\mathcal{N}=1$, $D=4$ supergravity is recovered, namely

$$S = \int \left(\epsilon_{abcd}R^{ab}e^c e^d + 4\bar{\psi}e^a\gamma_a\gamma_5 D\psi\right) \tag{2.25}$$

which is the simplest version of supergravity for the super-Poincaré group. In fact, this limit corresponds to the Inönü-Wigner contraction of $OSp(4|1)$ to the superPoincaré group.

The action (2.24) is not invariant under the $\mathfrak{osp}\,(4|1)$ gauge transformations. Nevertheless, the invariance of the action under supersymmetry transformations can be obtained modifying the spin connection supersymmetry transformation [33].

### 2.3.1 $Osp(4|1)$ gauge transformations and supersymmetry

The gauge transformation of the one-form gauge connection $A$ is

$$\delta_{\rho}A = D\rho = d\rho + [A, \rho] \tag{2.26}$$

where $\rho$ is the $Osp(4|1)$ gauge parameter,

$$\rho = \frac{1}{2}\rho^{ab}J_{ab} + \frac{1}{l}\rho^{a}P_{a} + \frac{1}{\sqrt{l}}\epsilon^{\alpha}Q_{\alpha}. \tag{2.27}$$

Then, using

$$\delta\left(A^{A}T_{A}\right) = d\rho + \left[A^{B}T_{B}, \rho^{C}T_{C}\right], \tag{2.28}$$

the $Osp(4|1)$ gauge transformations are given by

$$\delta\omega^{ab} = D\rho^{ab} + \frac{2}{l^{2}}e^{a}\rho^{b} - \frac{1}{l}\bar{\epsilon}\gamma^{ab}\psi, \tag{2.29}$$

$$\delta e^{a} = D\rho^{a} + e^{b}\rho_{b}^{\ a} + \bar{\epsilon}\gamma^{a}\psi, \tag{2.30}$$

$$\delta\psi = d\epsilon + \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon + \frac{1}{2l}e^{a}\gamma_{a}\epsilon - \frac{1}{4}\rho^{ab}\gamma_{ab}\psi - \frac{1}{2l}\rho^{a}\gamma_{a}\psi. \tag{2.31}$$

Although the MacDowell-Mansouri action (2.24) is built from the $Osp(4|1)$ curvature, it is not invariant under the $Osp(4|1)$ gauge transformations. Furthermore, the action does not correspond to a Yang-Mills action, nor a topological invariant.

Moreover, the action is not invariant under gauge supersymmetry. In fact, if we consider the variation of the action (2.24) under gauge supersymmetry, we find that

$$\delta_{susy}S = -\frac{4}{l^{2}}\int R^{a}\bar{\Psi}\gamma_{a}\gamma_{5}\epsilon. \tag{2.32}$$

As in the super-Poincaré case, the action is invariant under gauge supersymmetry imposing the super torsion constraint $R^{a} = 0$. This yields to express the spin connection $\omega^{ab}$ in terms of the vielbein and the gravitino fields, leading to the supersymmetric action for the $\mathfrak{osp}\left(4|1\right)$ superalgebra in second order formalism.

On the other hand, it is possible to have supersymmetry in first order formalism if we modify the supersymmetry transformation for the spin connection $\omega^{ab}$. In fact, if we consider the variation of the action under an arbitrary $\delta\omega^{ab}$ we have

$$\delta_{\omega}S = \frac{2}{l^{2}}\int \epsilon_{abcd}R^{a}e^{b}\delta\omega^{cd}, \tag{2.33}$$

thus the variation vanishes for arbitrary $\delta\omega^{ab}$ if $R^a = 0$. It is possible to modify $\delta\omega^{ab}$ adding an extra piece to the gauge transformation such that the variation of the action can be written as

$$\delta S = -\frac{4}{l^2}\int R^a\left(\bar{\Psi}\gamma_a\gamma_5\epsilon - \frac{1}{2}\epsilon_{abcd}e^b\delta_{extra}\omega^{cd}\right). \tag{2.34}$$

In order to have an invariant action, $\delta_{extra}\omega^{ab}$ must be given by

$$\delta_{extra}\omega^{ab} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_d\gamma_5\epsilon + \bar{\Psi}_{de}\gamma_c\gamma_5\epsilon - \bar{\Psi}_{cd}\gamma_e\gamma_5\epsilon\right)e^e, \tag{2.35}$$

with $\bar{\Psi} = \bar{\Psi}_{ab}e^a e^b$.

Then, in the first order formalism the action is invariant under the following supersymmetry transformations

$$\delta\omega^{ab} = -\frac{1}{l}\bar{\epsilon}\gamma^{ab}\psi + 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_d\gamma_5\epsilon + \bar{\Psi}_{de}\gamma_c\gamma_5\epsilon - \bar{\Psi}_{cd}\gamma_e\gamma_5\epsilon\right)e^e, \tag{2.36}$$

$$\delta e^a = \bar{\epsilon}\gamma^a\psi, \tag{2.37}$$

$$\delta\psi = d\epsilon + \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon + \frac{1}{2l}e^a\gamma_a\epsilon \equiv \nabla\epsilon. \tag{2.38}$$

## 2.4 AdS Chern-Simons Supergravity.

As we have seen before, in odd dimensions the Lanczos-Lovelock Lagrangian is a Chern-Simons form for the (Anti)-de Sitter or Poincaré groups. In particular, in three dimensions a CS theory for these gauge groups is equivalent to General Relativity, but with different cosmological constants [38], [39]. CS models for gravity are interesting because they provide with a truly gauge-invariant action principle in the fiber-bundle sense.

In general, in all odd-dimensions $D = 2n-1$ a CS form is defined by the condition that its exterior derivative be an invariant polynomial of degree $n$ in the curvature $F$. Thus, a generic CS Lagrangian, $L_{CS}^{(2n-1)}$ for a Lie algebra $g$ can be written as $dL_{CS}^{(2n-1)} = \langle F^n\rangle$, where $\langle\cdots\rangle$ corresponds to a symmetric invariant tensor for $g$, and

$$L_{CS}^{(2n-1)} = n\int_0^1 dt\left\langle A\left(tdA + t^2A^2\right)^{n-1}\right\rangle. \tag{2.39}$$

In $D = 3$ dimensions, the locally supersymmetric extension of General Relativity was done in [40], and it has been shown that it can be written as a CS theory for the Poincaré or the (anti)-De Sitter supergroups in [38], [39].

As mentioned before, a good candidate to describe a three-dimensional CS supergravity theory with cosmological constant is the AdS supergroup. The most generalized supersymmetric extension of the three-dimensional AdS algebra is given by the direct product [39]

$$\mathfrak{osp}\left(2|p\right)\otimes\mathfrak{osp}\left(2|q\right), \tag{2.40}$$

describing a $(p,q)$-type AdS-Chern-Simons supergravity in presence of a cosmological constant. Interestingly, the $\mathfrak{osp}\left(2|p\right)\otimes\mathfrak{osp}\left(2|q\right)$ superalgebra allows to construct a non minimal three-dimensional *AdS* CS supergravity theory. In particular, the minimal AdS CS supergravity is obtained when $p=1$ and $q=0$ ($\mathfrak{osp}\left(2|1\right)\otimes\mathfrak{sp}\left(2\right)$) [37]. As was pointed out in ref. [39], the presence of $\mathcal{N}=p+q$ supersymmetries allows to introduce CS terms related to the $O\left(p\right)\otimes O\left(q\right)$ gauge symmetry.

In this section, we consider the construction of the most general three-dimensional CS Supergravity action for the AdS superalgebra, $\mathfrak{osp}\left(2|1\right)\otimes\mathfrak{sp}\left(2\right)$, containing a cosmological constant. This corresponds to the supersymmetric extension of the most general action for gravity in $D=3$ dimensions, which apart from the Einstein-Hilbert term with cosmological constant, contains the Lorentz-Chern-Simons form (or "exotic" Lagrangian [8]) and a term involving the torsion [41], [42].

The (anti)-commutation relations for the $D=3$ AdS superalgebra are given by

$$\left[\tilde{J}_{ab},\tilde{J}_{cd}\right]=\eta_{bc}\tilde{J}_{ad}-\eta_{ac}\tilde{J}_{bd}-\eta_{bd}\tilde{J}_{ac}+\eta_{ad}\tilde{J}_{bc}, \tag{2.41}$$

$$\left[\tilde{J}_{ab},\tilde{P}_{c}\right]=\eta_{bc}\tilde{P}_{a}-\eta_{ac}\tilde{P}_{b},\qquad\left[\tilde{P}_{a},\tilde{P}_{b}\right]=\tilde{J}_{ab}, \tag{2.42}$$

$$\left[\tilde{P}_{a},\tilde{Q}_{\alpha}\right]=-\frac{1}{2}\left(\Gamma_{a}\tilde{Q}\right)_{\alpha}, \tag{2.43}$$

$$\left[\tilde{J}_{ab},\tilde{Q}_{\alpha}\right]=-\frac{1}{2}\left(\Gamma_{ab}\tilde{Q}\right)_{\alpha}, \tag{2.44}$$

$$\left\{\tilde{Q}_{\alpha},\tilde{Q}_{\beta}\right\}=-\frac{1}{2}\left[\left(\Gamma^{ab}C\right)_{\alpha\beta}\tilde{J}_{ab}-2\left(\Gamma^{a}C\right)_{\alpha\beta}\tilde{P}_{a}\right], \tag{2.45}$$

where $\tilde{J}_{ab}$, $\tilde{P}_{a}$ and $\tilde{Q}_{\alpha}$ are the generators of Lorentz transformations, the *AdS* boost and supersymmetry, respectively. Here $C$ stands for the charge conjugation matrix, $\Gamma_{a}$ are Dirac matrices and $\Gamma_{ab}=\frac{1}{2}\left[\Gamma_{a},\Gamma_{b}\right]$.

The Chern-Simons action in $(2+1)$ dimensions [9], [11] is given by

$$S_{CS}^{(2+1)}=k\int\left\langle A\left(dA+\frac{2}{3}A^{2}\right)\right\rangle, \tag{2.46}$$

In our case $A$ is the one-form gauge connection for the $\mathfrak{osp}\left(2|1\right) \oplus \mathfrak{sp}\left(2\right)$ superalgebra

$$A = \frac{1}{2}\omega^{ab}\tilde{J}_{ab} + \frac{1}{l}e^{a}\tilde{P}_{a} + \frac{1}{\sqrt{l}}\psi^{\alpha}\tilde{Q}_{\alpha}, \tag{2.47}$$

whose associated curvature two-form $F = dA + A \wedge A$ is

$$F = F^{A}T_{A} = \frac{1}{2}\mathcal{R}^{ab}\tilde{J}_{ab} + \frac{1}{l}R^{a}\tilde{P}_{a} + \frac{1}{\sqrt{l}}\Psi^{\alpha}\tilde{Q}_{\alpha}, \tag{2.48}$$

with

$$\begin{aligned}
\mathcal{R}^{ab} &= d\omega^{ab} + \omega^{a}_{\ c}\omega^{cb} + \frac{1}{l^{2}}e^{a}e^{b} + \frac{1}{2l}\bar{\psi}\Gamma^{ab}\psi, \\
R^{a} &= de^{a} + \omega^{a}_{\ b}e^{b} - \frac{1}{2}\bar{\psi}\Gamma^{a}\psi, \\
\Psi &= \nabla\psi = d\psi + \frac{1}{4}\omega_{ab}\Gamma^{ab}\psi + \frac{1}{2l}e^{a}\Gamma_{a}\psi.
\end{aligned}$$

In (2.46) the bracket $\langle\cdots\rangle$ stands for the non-vanishing components of an invariant tensor for the $\mathfrak{osp}\left(2|1\right) \otimes \mathfrak{sp}\left(2\right)$ superalgebra in $(2+1)$-dimensions:

$$\left\langle \tilde{J}_{ab}\tilde{J}_{cd} \right\rangle = \mu_{0}\left(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}\right), \tag{2.49}$$

$$\left\langle \tilde{J}_{ab}\tilde{P}_{c} \right\rangle = \mu_{1}\epsilon_{abc}, \tag{2.50}$$

$$\left\langle \tilde{P}_{a}\tilde{P}_{b} \right\rangle = \mu_{0}\eta_{ab}, \tag{2.51}$$

$$\left\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta} \right\rangle = \left(\mu_{0} - \mu_{1}\right)C_{\alpha\beta}, \tag{2.52}$$

where $\mu_0$ and $\mu_1$ are arbitrary constants.

Considering (2.49)-(2.52) and the one-form connection (2.47), the CS action (2.46) for the $\mathfrak{osp}\left(2|1\right) \otimes \mathfrak{sp}\left(2\right)$ superalgebra can be written as

$$\begin{aligned}
S_{CS}^{(2+1)} = k\int_{M} \frac{\mu_{0}}{2}&\left(\omega^{a}_{\ b}d\omega^{b}_{\ a} + \frac{2}{3}\omega^{a}_{\ c}\omega^{c}_{\ b}\omega^{b}_{\ a} + \frac{2}{l^{2}}e^{a}T_{a} + \frac{2}{l}\bar{\psi}\Psi\right) \\
&+ \frac{\mu_{1}}{l}\left(\epsilon_{abc}\left(R^{ab}e^{c} + \frac{1}{3l^{2}}e^{a}e^{b}e^{c}\right) - \bar{\psi}\Psi\right) - d\left(\frac{\mu_{1}}{2l}\epsilon_{abc}\omega^{ab}e^{c}\right)
\end{aligned} \tag{2.53}$$

where $T^{a} = de^{a} + \omega^{a}_{\ b}e^{b}$ is the torsion 2-form and $R^{ab} = d\omega^{ab} + \omega^{a}_{\ c}\omega^{cb}$ is the Lorentz curvature. This action describes the most general $\mathcal{N} = 1$, $D = 3$ CS supergravity action with cosmological constant for the AdS supergroup [37]. There are two independent terms, the one proportional to $\mu_0$ contains the "exotic" Lagrangian and a term involving the torsion, while the second one proportional to $\mu_1$ contains the EH Lagrangian with a cosmological constant.

It is straightforward to show that the action (2.47) is invariant (up to boundary terms) under supersymmetry,

$$\delta_\epsilon \psi = \nabla \epsilon, \qquad \delta_\epsilon e^a = \bar{\epsilon}\Gamma^a \psi, \quad \delta_\epsilon \omega^{ab} = -\frac{1}{l}\bar{\epsilon}\Gamma^{ab}\psi. \tag{2.54}$$

As no field equations are required in order to prove this invariance, we said that it is an off-shell local SUSY.

Furthermore, the Inönü Wigner contraction of the the $Osp\,(2|1) \otimes sp\,(2)$ group leads us to the superPoincaré in three dimensions, in a similar way as the Poincaré group is obtained as an Inönü Wigner contraction of the AdS group.

# Part II

# $\mathcal{N} = 1$ Supergravity theories, Maxwell and AdS-Lorentz superalgebras

# Chapter 3

# Maxwell superalgebras and Abelian semigroup expansion

## 3.1 Introduction

The derivation of new Lie algebras from a given one is particularly interesting in Physics since it allows us to find new physical theories from an already known. In fact, an important example consists in obtaining the Poincaré algebra from the Galileo algebra using a deformation procedure which can be seen as an algebraic prediction of Relativity. At present, there are at least four different ways to relate new Lie algebras; deformation, contraction, extension and expansion. In particular, the expansion method leads to higher dimensional new Lie algebras from a given one. The expansion procedure was first introduced by Hadsuda and Sakaguchi in [43] in the context of AdS superstring. An interesting expansion method was proposed by Azcarraga, Izquierdo, Picón and Varela in [44] and subsequently developed in [45], [46]. This expansion method known as Maurer-Cartan (MC) forms power-series expansion consists in rescaling some group parameters by a factor $\lambda$, and then apply an expansion as a power series in $\lambda$. This series is truncated in a way that the Maurer-Cartan equations of the new algebra are satisfied.

Another expansion method was proposed by F. Izaurieta, E. Rodríguez and P. Salgado in [18] which is based on operations performed directly on the algebra generators. This method consists in combining the inner multiplication law of a semigroup $S$ with the structure constants of a Lie (super)algebra $\mathfrak{g}$ in order to define the Lie bracket of a new (super)algebra $\mathfrak{G} = S \times \mathfrak{g}$. This abelian semigroup expansion procedure, or simply $S$-expansion, can reproduce all Maurer-Cartan forms power series expansion for a particular choice of a semigroup $S$. Interestingly, different choices of the semigroup lead to new expanded Lie algebras that cannot be obtained by the MC expansion.

Some examples of (super)algebras obtained as an $S$-expansion can be found in [18], [47] where the D'auria-Fré superalgebra introduced originally in [48] and the M algebra are derived alternatively as an $S$-expansion of $\mathfrak{osp}\left(32|1\right)$. As we have seen in previous sections, the $S$-expansion method allows to obtain the Maxwell type algebras $\mathcal{M}_m$ from the AdS algebra using $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$ as the relevant semigroup.

The Maxwell algebra (and its supersymmetric extensions) has been extensively studied in [50]-[59]. This algebra describes the symmetries of a particle moving in a background in the presence of a constant electromagnetic field [50]. In [53] the minimal $D = 4$ Maxwell superalgebra $s\mathcal{M}$ which contains the Maxwell algebra as its bosonic subalgebra was presented. In [57] the Maurer-Cartan expansion allowed to obtain the minimal Maxwell superalgebra and its $\mathcal{N}$-extended generalization from the $\mathfrak{osp}\left(4|N\right)$ superalgebra. This Maxwell superalgebra can be used to obtain the minimal $D = 4$ pure supergravity from the curvature 2-form associated to $s\mathcal{M}$ [58].

In this chapter, we present the results of [60], where we have shown that the abelian semigroup expansion is an alternative expansion method to obtain the Maxwell superalgebra and the $\mathcal{N}$-extended cases. In this way, we showed that the results of [57] can be derived alternatively as an $S$-expansion of the $\mathfrak{osp}\left(4|\mathcal{N}\right)$ superalgebra choosing appropriate semigroups. In particular, the minimal Maxwell superalgebra $s\mathcal{M}$ is obtained as an $S$-expansion setting a generator equals to zero. We finally generalize these results proposing new Maxwell superalgebras namely, the minimal Maxwell type superalgebras $s\mathcal{M}_{m+2}$ and the $\mathcal{N}$-extended superalgebras $s\mathcal{M}_{m+2}^{(N)}$, which can be derived from the $\mathfrak{osp}\left(4|\mathcal{N}\right)$ superalgebra.

As we will see in the next chapter, these superalgebras can be used to construct dynamical actions in $D = 4$, leading to standard pure supergravity in a very similar way to the bosonic case considered in [17], [19].

## 3.2 Maxwell algebra as an $S$-expansion

Before considering the supersymmetric case, let us review here how to obtain the Maxwell algebra $\mathcal{M}$ as an $S$-expansion of AdS. This algebra describes the symmetries of a particle moving in a background in presence of a constant electromagnetic field, and is provided by $\{J_{ab}, P_a, Z_{ab}\}$, where $\{P_a, J_{ab}\}$ do not generate the Poincaré algebra. In fact, a particular feature of the Maxwell algebra (which is also a feature shared by all the family of Maxwell type algebras) is given by the relation

$$[P_a, P_b] = Z_{ab} \tag{3.1}$$

where $Z_{ab}$ commutes with all generators of the algebra except the Lorentz generators $J_{ab}$,

$$[J_{ab}, Z_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \tag{3.2}$$

$$[Z_{ab}, P_a] = [Z_{ab}, Z_{cd}] = 0. \tag{3.3}$$

The other commutators of the algebra are

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \tag{3.4}$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b. \tag{3.5}$$

Following [16] and [18], it is possible to obtain the Maxwell algebra $\mathcal{M}$ as an $S$-expansion of the AdS Lie algebra $\mathfrak{g}$ using $S_E^{(2)}$ as the appropriate abelian semigroup. Before applying the $S$-expansion procedure it is necessary to consider a decomposition of the original algebra $\mathfrak{g}$ in subspaces $V_p$,

$$\mathfrak{g} = \mathfrak{so}\left(3,2\right) = \mathfrak{so}\left(3,1\right) \oplus \frac{\mathfrak{so}\left(3,2\right)}{\mathfrak{so}\left(3,1\right)} = V_0 \oplus V_1, \tag{3.6}$$

where $V_0$ is generated by the Lorentz generator $\tilde{J}_{ab}$ and $V_1$ is generated by the $AdS$ boost generator $\tilde{P}_a$. The $\tilde{J}_{ab}$, $\tilde{P}_a$ generators satisfy the following relations

$$\left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] = \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} - \eta_{bd}\tilde{J}_{ac} + \eta_{ad}\tilde{J}_{bc}, \tag{3.7}$$

$$\left[\tilde{J}_{ab}, \tilde{P}_c\right] = \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b, \tag{3.8}$$

$$\left[\tilde{P}_a, \tilde{P}_b\right] = \tilde{J}_{ab}. \tag{3.9}$$

The subspace structure may be written as

$$[V_0, V_0] \subset V_0, \qquad [V_0, V_1] \subset V_1, \qquad [V_1, V_1] \subset V_0. \tag{3.10}$$

Let $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ be an abelian semigroup with the following subset decomposition $S_E^{(2)} = S_0 \cup S_1$, where the subsets $S_0, S_1$ are given by

$$S_0 = \{\lambda_0, \lambda_2, \lambda_3\}, \tag{3.11}$$

$$S_1 = \{\lambda_1, \lambda_3\}, \tag{3.12}$$

where $\lambda_3$ corresponds to the zero element of the semigroup ($0_s = \lambda_3$). This subset decomposition is said to be "resonant" because it satisfies [compare with eqs.(3.10).]

$$S_0 \cdot S_0 \subset S_0, \qquad S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_1 \subset S_{0.} \tag{3.13}$$

In this case, the elements of the semigroup $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ satisfy the following multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 3, \\ \lambda_3, & \text{when } \alpha + \beta > 3. \end{cases} \tag{3.14}$$

Following the definitions of [18], after extracting a resonant subalgebra and performing its $0_S$-reduction, one finds the Maxwell algebra $\mathcal{M} = \{J_{ab}, P_a, Z_{ab}\}$, whose generators can be written in terms of the original ones,

$$J_{ab} = \lambda_0 \otimes \tilde{J}_{ab}, \tag{3.15}$$

$$P_a = \lambda_1 \otimes \tilde{P}_a, \tag{3.16}$$

$$Z_{ab} = \lambda_2 \otimes \tilde{J}_{ab.} \tag{3.17}$$

Furthermore, as we have seen in previous sections, it is possible to extend this procedure and obtain all the Maxwell type algebras using the appropriate semigroup [17].

## 3.3 $S$-expansion of the $\mathfrak{osp}(4|1)$ superalgebra

In this section, we shall consider the AdS superalgebra $\mathfrak{osp}(4|1)$ as a starting point. We will see that different choices of abelian semigroup $S$ lead to new $D = 4$ superalgebras. In every case, before applying the $S$-expansion procedure it is necessary to decompose the original algebra $\mathfrak{g}$ as a direct sum of subspaces $V_p$,

$$\begin{aligned} \mathfrak{g} = \mathfrak{osp}(4|1) &= \mathfrak{so}(3,1) \oplus \frac{\mathfrak{osp}(4|1)}{\mathfrak{sp}(4)} \oplus \frac{\mathfrak{sp}(4)}{\mathfrak{so}(3,1)} \\ &= V_0 \oplus V_1 \oplus V_2, \end{aligned} \tag{3.18}$$

where $V_0$ corresponds to the Lorentz subspace generated by $\tilde{J}_{ab}$, $V_1$ corresponds to the fermionic subspace generated by a 4-component Majorana spinor charge $\tilde{Q}_\alpha$ and $V_2$ corresponds to the $AdS$ boost generated by $\tilde{P}_a$. The $\mathfrak{osp}\,(4|1)$ generators satisfy the (anti)commutation relations given by $(2.4) - (2.8)$.

The subspace structure can be written as

$$[V_0, V_0] \subset V_0, \qquad [V_1, V_1] \subset V_0 \oplus V_2, \tag{3.19}$$
$$[V_0, V_1] \subset V_1, \qquad [V_1, V_2] \subset V_1, \tag{3.20}$$
$$[V_0, V_2] \subset V_2, \qquad [V_2, V_2] \subset V_0. \tag{3.21}$$

The next step consists in finding a subset decomposition of a semigroup $S$ which is "resonant" with respect to $(3.19) - (3.21)$.

### 3.3.1 Minimal $D = 4$ Maxwell superalgebra

Let us consider $S_E^{(4)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ as the relevant abelian semigroup whose elements obey the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 5, \\ \lambda_5, & \text{when } \alpha + \beta > 5. \end{cases} \tag{3.22}$$

In this case, $\lambda_5$ plays the role of the zero element of the semigroup $S_E^{(4)}$, so we have for each $\lambda_\alpha \in S_E^{(4)}$, $\lambda_5 \lambda_\alpha = \lambda_5 = 0_s$. Let us consider the decomposition $S = S_0 \cup S_1 \cup S_2$, with

$$S_0 = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\}, \tag{3.23}$$
$$S_1 = \{\lambda_1, \lambda_3, \lambda_5\}, \tag{3.24}$$
$$S_2 = \{\lambda_2, \lambda_4, \lambda_5\}. \tag{3.25}$$

One sees that this decomposition is resonant since it satisfies the same structure as the subspaces $V_p$ [compare with eqs. $(3.19) - (3.21)$]

$$S_0 \cdot S_0 \subset S_0, \qquad S_1 \cdot S_1 \subset S_0 \cap S_2, \tag{3.26}$$
$$S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_2 \subset S_1, \tag{3.27}$$
$$S_0 \cdot S_2 \subset S_2, \qquad S_2 \cdot S_2 \subset S_0. \tag{3.28}$$

Following theorem IV.2 of [18], we can say that the superalgebra

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{3.29}$$

is a resonant super subalgebra of $S_E^{(4)} \times \mathfrak{g}$, where

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\} \times \left\{\tilde{J}_{ab}\right\} = \left\{\lambda_0 \tilde{J}_{ab}, \lambda_2 \tilde{J}_{ab}, \lambda_4 \tilde{J}_{ab}, \lambda_5 \tilde{J}_{ab}\right\}, \tag{3.30}$$

$$W_1 = (S_1 \times V_1) = \{\lambda_1, \lambda_3, \lambda_5\} \times \left\{\tilde{Q}_\alpha\right\} = \left\{\lambda_1 \tilde{Q}_\alpha, \lambda_3 \tilde{Q}_\alpha, \lambda_5 \tilde{Q}_\alpha\right\}, \tag{3.31}$$

$$W_2 = (S_2 \times V_2) = \{\lambda_2, \lambda_4, \lambda_5\} \times \left\{\tilde{P}_a\right\} = \left\{\lambda_2 \tilde{P}_a, \lambda_4 \tilde{P}_a, \lambda_5 \tilde{P}_a\right\}. \tag{3.32}$$

In order to extract a smaller superalgebra from the resonant super subalgebra $\mathfrak{G}_R$ it is necessary to apply the reduction procedure.

Let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$ where

$$\check{S}_0 = \{\lambda_0, \lambda_2, \lambda_4\}, \qquad \hat{S}_0 = \{\lambda_5\}, \tag{3.33}$$

$$\check{S}_1 = \{\lambda_1, \lambda_3\}, \qquad \hat{S}_1 = \{\lambda_5\}, \tag{3.34}$$

$$\check{S}_2 = \{\lambda_2\}, \qquad \hat{S}_2 = \{\lambda_4, \lambda_5\}. \tag{3.35}$$

For each $p$, $\hat{S}_p \cap \check{S}_p = \varnothing$, and using the product (3.22) one sees that the partition satisfies [compare with ecs. (3.19) – (3.21)]

$$\begin{array}{ll} \check{S}_0 \cdot \hat{S}_0 \subset \hat{S}_0, & \check{S}_1 \cdot \hat{S}_1 \subset \hat{S}_0 \cap \hat{S}_2, \\ \check{S}_0 \cdot \hat{S}_1 \subset \hat{S}_1, & \check{S}_1 \cdot \hat{S}_2 \subset \hat{S}_1, \\ \check{S}_0 \cdot \hat{S}_2 \subset \hat{S}_2, & \check{S}_2 \cdot \hat{S}_2 \subset \hat{S}_0. \end{array} \tag{3.36}$$

Then, following definitions of [18], we have

$$\check{\mathfrak{G}}_R = \left(\check{S}_0 \times V_0\right) \oplus \left(\check{S}_1 \times V_1\right) \oplus \left(\check{S}_2 \times V_2\right), \tag{3.37}$$

$$\hat{\mathfrak{G}}_R = \left(\hat{S}_0 \times V_0\right) \oplus \left(\hat{S}_1 \times V_1\right) \oplus \left(\hat{S}_2 \times V_2\right), \tag{3.38}$$

where

$$\left[\check{\mathfrak{G}}_R, \hat{\mathfrak{G}}_R\right] \subset \hat{\mathfrak{G}}_R, \tag{3.39}$$

and therefore $\left|\check{\mathfrak{G}}_R\right|$ corresponds to a reduced algebra of $\mathfrak{G}_R$. This $S$-expansion process can be seen explicitly in the following diagrams:

$$\begin{array}{c|c|c|c|} \lambda_5 & J_{ab,5} & Q_{\alpha,5} & P_{a,5} \\ \hline \lambda_4 & J_{ab,4} & & P_{a,4} \\ \hline \lambda_3 & & Q_{\alpha,3} & \\ \hline \lambda_2 & J_{ab,2} & & P_{a,2} \\ \hline \lambda_1 & & Q_{\alpha,1} & \\ \hline \lambda_0 & J_{ab,0} & & \\ \hline & V_0 & V_1 & V_2 \end{array} \qquad \begin{array}{c|c|c|c|} \lambda_5 & & & \\ \hline \lambda_4 & J_{ab,4} & & \\ \hline \lambda_3 & & Q_{\alpha,3} & \\ \hline \lambda_2 & J_{ab,2} & & P_{a,2} \\ \hline \lambda_1 & & Q_{\alpha,1} & \\ \hline \lambda_0 & J_{ab,0} & & \\ \hline & V_0 & V_1 & V_2 \end{array}, \tag{3.40}$$

where we have defined $J_{ab,i} = \lambda_i \tilde{J}_{ab}$, $P_{a,i} = \lambda_i \tilde{P}_a$ and $Q_{\alpha,i} = \lambda_i \tilde{Q}_\alpha$. We can observe that the first diagram corresponds to the resonant subalgebra of the $S$-expanded superalgebra $S_E^{(4)} \times \mathfrak{osp}(4|1)$. The second one consists in a particular reduction of the resonant subalgebra.

Thus, the new superalgebra is generated by $\left\{ J_{ab}, P_a, \tilde{Z}_{ab}, Z_{ab}, Q_\alpha, \Sigma_\alpha \right\}$ where these new generators can be written in terms of the original AdS generators as

$$\begin{aligned} J_{ab} &= J_{ab,0} = \lambda_0 \tilde{J}_{ab}, \quad & P_a &= P_{a,2} = \lambda_2 \tilde{P}_a, \\ \tilde{Z}_{ab} &= J_{ab,2} = \lambda_2 \tilde{J}_{ab}, \quad & Z_{ab} &= J_{ab,4} = \lambda_4 \tilde{J}_{ab}, \\ Q_\alpha &= Q_{\alpha,1} = \lambda_1 \tilde{Q}_\alpha, \quad & \Sigma_\alpha &= Q_{\alpha,3} = \lambda_3 \tilde{Q}_\alpha. \end{aligned} \tag{3.41}$$

These new generators satisfy the commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{3.42}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{3.43}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{3.44}$$

$$[P_a, Q_\alpha] = -\frac{1}{2} (\gamma_a \Sigma)_\alpha, \tag{3.45}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2} (\gamma_{ab} Q)_\alpha, \tag{3.46}$$

$$[J_{ab}, \Sigma_\alpha] = -\frac{1}{2} (\gamma_{ab} \Sigma)_\alpha, \tag{3.47}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2} \left[ \left(\gamma^{ab} C\right)_{\alpha\beta} \tilde{Z}_{ab} - 2 (\gamma^a C)_{\alpha\beta} P_a \right], \tag{3.48}$$

$$\{Q_\alpha, \Sigma_\beta\} = -\frac{1}{2} \left(\gamma^{ab} C\right)_{\alpha\beta} Z_{ab}, \tag{3.49}$$

$$\left[J_{ab}, \tilde{Z}_{ab}\right] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \tag{3.50}$$

$$\left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{3.51}$$

$$\left[\tilde{Z}_{ab}, Q_\alpha\right] = -\frac{1}{2} (\gamma_{ab} \Sigma)_\alpha, \tag{3.52}$$

$$\text{others} = 0, \tag{3.53}$$

where we have used the multiplication law of the semigroup (3.22) and the commutation relations of the original superalgebra. The new superalgebra obtained after a reduced resonant $S$-expansion of the $\mathfrak{osp}(4|1)$ superalgebra corresponds to the minimal Maxwell type superalgebra $s\mathcal{M}_4$ in $D = 4$ . One can see that imposing $\tilde{Z}_{ab} = 0$ leads us to the minimal Maxwell superalgebra $s\mathcal{M}$ [55, 57]. This can be done since the Jacobi identities for spinors generators are satisfied due to the gamma matrix identity $(C\gamma^a)_{(\alpha\beta}\,(C\gamma_a)_{\gamma\delta)} = 0$ (cyclic permutations of $\alpha, \beta, \gamma$).

In this case, the $S$-expansion procedure produces a new Majorana spinor charge $\Sigma$. The introduction of a second abelian spinorial generator has been initially proposed in [48] in the context of $D = 11$ supergravity and subsequently in [59] in the context of superstring theory.

The $s\mathcal{M}$ superalgebra seems particularly interesting in the context of $D = 4$ supergravity. In fact, in [58], it was shown that the $D = 4$, $\mathcal{N} = 1$ pure supergravity Lagrangian can be written as a quadratic expression in the curvatures of the gauge fields associated with the minimal Maxwell superalgebra. As we will see in the next chapter, the same result can be found for the minimal Maxwell type superalgebra $s\mathcal{M}_4$ (and its generalization $s\mathcal{M}_{m+2}$), using the $S$-expansion method.

### 3.3.2 Minimal $D = 4$ Maxwell type superalgebra $s\mathcal{M}_5$

In [17] it was shown that the Maxwell type algebra $\mathcal{M}_m$ can be obtained from an $S$-expansion of AdS algebra. These bigger algebras require semigroups with more elements but with the same type of multiplication law. Since our main motivation is to obtain a $D = 4$ Maxwell type superalgebra $s\mathcal{M}_m$ it seems natural to consider a semigroup bigger than $S_E^{(4)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$. As in the previous case, we shall consider $\mathfrak{g} = \mathfrak{osp}\,(4|1)$ as a starting point with the subspace structure given by eqs. (3.19) – (3.21).

Let us consider $S_E^{(6)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\}$ as the relevant finite abelian semigroup whose elements are dimensionless and obey the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leq 7, \\ \lambda_7, & \text{when } \alpha+\beta > 7, \end{cases} \tag{3.54}$$

where $\lambda_7$ plays the role of the zero element of the semigroup $S_E^{(6)}$. Let us consider the decomposition $S = S_0 \cup S_1 \cup S_2$, with

$$S_0 = \{\lambda_0, \lambda_2, \lambda_4, \lambda_6, \lambda_7\}\,, \tag{3.55}$$

$$S_1 = \{\lambda_1, \lambda_3, \lambda_5, \lambda_7\}\,, \tag{3.56}$$

$$S_2 = \{\lambda_2, \lambda_4, \lambda_6, \lambda_7\}\,. \tag{3.57}$$

This subset decomposition of $S_E^{(6)}$ satisfies the resonance condition since it satisfies the same structure that the subspaces $V_p$ [compare with eqs. (3.19) – (3.21)]

$$S_0 \cdot S_0 \subset S_0, \qquad S_1 \cdot S_1 \subset S_0 \cap S_2, \tag{3.58}$$
$$S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_2 \subset S_1, \tag{3.59}$$
$$S_0 \cdot S_2 \subset S_2, \qquad S_2 \cdot S_2 \subset S_0. \tag{3.60}$$

Therefore, according to Theorem IV.2 of [18], we have that

$$\mathfrak{G}_R = W_0 + W_1 + W_2, \tag{3.61}$$

with

$$W_p = S_p \times V_p, \tag{3.62}$$

is a resonant super subalgebra of $\mathfrak{G} = S \times \mathfrak{g}$.

As in the previous case, it is possible to extract a smaller superalgebra from the resonant subalgebra $\mathfrak{G}_R$ using the reduction procedure. Let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$ where

$$\check{S}_0 = \{\lambda_0, \lambda_2, \lambda_4\}, \qquad \hat{S}_0 = \{\lambda_6, \lambda_7\}, \tag{3.63}$$
$$\check{S}_1 = \{\lambda_1, \lambda_3, \lambda_5\}, \qquad \hat{S}_1 = \{\lambda_7\}, \tag{3.64}$$
$$\check{S}_2 = \{\lambda_2, \lambda_4, \lambda_6\}, \qquad \hat{S}_2 = \{\lambda_7\}. \tag{3.65}$$

For each $p$, $\hat{S}_p \cap \check{S}_p = \varnothing$, and using the product (3.54) one can see that the partition satisfies [compare with ecs. (3.19) – (3.21)]

$$\begin{array}{ll} \check{S}_0 \cdot \hat{S}_0 \subset \hat{S}_0, & \check{S}_1 \cdot \hat{S}_1 \subset \hat{S}_0 \cap \hat{S}_2, \\ \check{S}_0 \cdot \hat{S}_1 \subset \hat{S}_1, & \check{S}_1 \cdot \hat{S}_2 \subset \hat{S}_1, \\ \check{S}_0 \cdot \hat{S}_2 \subset \hat{S}_2, & \check{S}_2 \cdot \hat{S}_2 \subset \hat{S}_0. \end{array} \tag{3.66}$$

Then, we have

$$\check{\mathfrak{G}}_R = \left(\check{S}_0 \times V_0\right) \oplus \left(\check{S}_1 \times V_1\right) \oplus \left(\check{S}_2 \times V_2\right), \tag{3.67}$$
$$\hat{\mathfrak{G}}_R = \left(\hat{S}_0 \times V_0\right) \oplus \left(\hat{S}_1 \times V_1\right) \oplus \left(\hat{S}_2 \times V_2\right), \tag{3.68}$$

where

$$\left[\check{\mathfrak{G}}_R, \hat{\mathfrak{G}}_R\right] \subset \hat{\mathfrak{G}}_R, \tag{3.69}$$

and therefore $\left|\check{\mathfrak{G}}_R\right|$ corresponds to a reduced algebra of $\mathfrak{G}_R$.

The new superalgebra is generated by $\left\{ J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, Z_a, \tilde{Z}_a, Q_\alpha, \Sigma_\alpha, \Phi_\alpha \right\}$ where these new generators can be written as

$$\begin{aligned} J_{ab} &= J_{ab,0} = \lambda_0 \tilde{J}_{ab}, & \tilde{Z}_a &= P_{a,4} = \lambda_4 \tilde{P}_a, \\ P_a &= P_{a,2} = \lambda_2 \tilde{P}_a, & Q_\alpha &= Q_{\alpha,1} = \lambda_1 \tilde{Q}_\alpha, \\ Z_{ab} &= J_{ab,4} = \lambda_4 \tilde{J}_{ab}, & \Sigma_\alpha &= Q_{\alpha,3} = \lambda_3 \tilde{Q}_\alpha, \\ \tilde{Z}_{ab} &= J_{ab,2} = \lambda_2 \tilde{J}_{ab}, & \Phi_\alpha &= Q_{\alpha,5} = \lambda_5 \tilde{Q}_\alpha, \\ Z_a &= P_{a,6} = \lambda_6 \tilde{P}_a. \end{aligned} \tag{3.70}$$

These new generators satisfy the commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{3.71}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{3.72}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{3.73}$$

$$[Z_{ab}, P_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \qquad [J_{ab}, Z_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \tag{3.74}$$

$$\left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{3.75}$$

$$\left[J_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \tag{3.76}$$

$$\left[\tilde{Z}_{ab}, P_c\right] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \qquad \left[\tilde{Z}_{ab}, \tilde{Z}_c\right] = \eta_{bc} Z_a - \eta_{ac} Z_b \tag{3.77}$$

$$\left[J_{ab}, \tilde{Z}_c\right] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \tag{3.78}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2} \left(\gamma_{ab} Q\right)_\alpha, \qquad [J_{ab}, \Sigma_\alpha] = -\frac{1}{2} \left(\gamma_{ab} \Sigma\right)_\alpha, \tag{3.79}$$

$$[J_{ab}, \Phi_\alpha] = -\frac{1}{2} \left(\gamma_{ab} \Phi\right)_\alpha, \qquad \left[\tilde{Z}_{ab}, Q_\alpha\right] = -\frac{1}{2} \left(\gamma_{ab} \Sigma\right)_\alpha, \tag{3.80}$$

$$\left[\tilde{Z}_{ab}, \Sigma_\alpha\right] = -\frac{1}{2} \left(\gamma_{ab} \Phi\right)_\alpha, \qquad [Z_{ab}, Q_\alpha] = -\frac{1}{2} \left(\gamma_{ab} \Phi\right)_\alpha, \tag{3.81}$$

$$[P_a, Q_\alpha] = -\frac{1}{2} \left(\gamma_a \Sigma\right)_\alpha, \qquad [P_a, \Sigma_\alpha] = -\frac{1}{2} \left(\gamma_a \Phi\right)_\alpha, \tag{3.82}$$

$$\left[\tilde{Z}_a, Q_\alpha\right] = -\frac{1}{2} \left(\gamma_a \Phi\right)_\alpha, \tag{3.83}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2} \left[ \left(\gamma^{ab} C\right)_{\alpha\beta} \tilde{Z}_{ab} - 2 \left(\gamma^a C\right)_{\alpha\beta} P_a \right], \tag{3.84}$$

$$\{Q_\alpha, \Sigma_\beta\} = -\frac{1}{2} \left[ \left(\gamma^{ab} C\right)_{\alpha\beta} Z_{ab} - 2 \left(\gamma^a C\right)_{\alpha\beta} \tilde{Z}_a \right], \tag{3.85}$$

$$\{Q_\alpha, \Phi_\beta\} = \left(\gamma^a C\right)_{\alpha\beta} Z_a = \{\Sigma_\alpha, \Sigma_\beta\}, \tag{3.86}$$

$$\text{others} = 0, \tag{3.87}$$

where we have used the multiplication law of the semigroup (3.54) and the commutation relations of the original superalgebra (2.4) – (2.8). The new superalgebra obtained after a reduced resonant $S$-expansion of $\mathfrak{osp}(4|1)$ superalgebra corresponds to a minimal Maxwell type superalgebra $s\mathcal{M}_5$ in $D = 4$. Interestingly, this new superalgebra contains the Maxwell type algebra $\mathcal{M}_5 = \{J_{ab}, P_a, Z_{ab}, Z_a\}$ as a subalgebra [12], [17].

In this case, two new Majorana spinor charges $\Sigma$ and $\Phi$ appear as a consequence of the $S$-expansion. These fermionic generators transform as spinors under Lorentz transformations. One sees that the minimal Maxwell type superalgebra $s\mathcal{M}_5$ requires new bosonic generators $\left(\tilde{Z}_{ab}, \tilde{Z}_a, Z_a\right)$ and $\Sigma$ is not abelian anymore. It is important to note that setting $\tilde{Z}_{ab}$ and $\tilde{Z}_a$ equal to zero does not lead to a subalgebra. In fact, these generators are required in Jacobi identity for $(Q_\alpha, Q_\beta, \Sigma_\gamma)$ due to the gamma matrix identity $(C\gamma^a)_{(\alpha\beta}(C\gamma_a)_{\gamma\delta)} = \left(C\gamma^{ab}\right)_{(\alpha\beta}(C\gamma_{ab})_{\gamma\delta)} = 0$ (cyclic permutations of $\alpha, \beta, \gamma$).

### 3.3.3 Minimal $D = 4$ Maxwell type superalgebra $s\mathcal{M}_{m+2}$

Let us generalize the previous setting. In order to obtain the minimal $D = 4$ Mawell type superalgebra $s\mathcal{M}_{m+2}$, it is necessary to consider a bigger semigroup. Let us consider $S_E^{(2m)} = \{\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{2m+1}\}$ as the relevant finite abelian semigroup whose elements are dimensionless and obey the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leq \lambda_{2m+1}, \\ \lambda_{2m+1}, & \text{when } \alpha+\beta > \lambda_{2m+1}. \end{cases} \tag{3.88}$$

where $\lambda_{2m+1}$ plays the role of the zero element of the semigroup. Let us consider the decomposition $S_E^{(2m)} = S_0 \cup S_1 \cup S_2$, where the subsets $S_0, S_1, S_2$ are given by the general expression

$$S_p = \left\{\lambda_{2n+p}, \text{ with } n = 0, \cdots, \left[\frac{2m-p}{2}\right]\right\} \cup \{\lambda_{2m+1}\}, \qquad p = 0, 1, 2. \tag{3.89}$$

This decomposition is said to be resonant since it satisfies [compare with eqs. (3.19) – (3.21)]

$$S_0 \cdot S_0 \subset S_0, \qquad S_1 \cdot S_1 \subset S_0 \cap S_2, \tag{3.90}$$

$$S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_2 \subset S_1, \tag{3.91}$$

$$S_0 \cdot S_2 \subset S_2, \qquad S_2 \cdot S_2 \subset S_0. \tag{3.92}$$

Thus, we have that

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{3.93}$$

with

$$W_p = S_p \times V_p, \tag{3.94}$$

is a resonant subalgebra of $\mathfrak{G} = S \times \mathfrak{g}$.

As in previous cases, it is possible to extract a smaller algebra from the resonant subalgebra $\mathfrak{G}_R$ using the reduction procedure. Let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$, with

$$\check{S}_0 = \{\lambda_{2n}, \text{ with } n = 0, \cdots, 2[m/2]\}, \quad \hat{S}_0 = \{(\lambda_{2m}), \lambda_{2m+1}\}, \tag{3.95}$$

$$\check{S}_1 = \{\lambda_{2n+1}, \text{ with } n = 0, \cdots, m-1\}, \quad \hat{S}_1 = \{\lambda_{2m+1}\}, \tag{3.96}$$

$$\check{S}_2 = \{\lambda_{2n+2}, \text{ with } n = 0, \cdots, 2[(m-1)/2]\}, \quad \hat{S}_2 = \{(\lambda_{2m}), \lambda_{2m+1}\}, \tag{3.97}$$

where $(\lambda_{2m})$ means that $\lambda_{2m} \in \hat{S}_0$ if $m$ is odd and $\lambda_{2m} \in \hat{S}_2$ if $m$ is even. For each $p$, $\hat{S}_p \cap \check{S}_p = \varnothing$, and using the product (3.88) one sees that the partition satisfies [compare with ecs. (3.19) – (3.21)]

$$\begin{aligned} &\check{S}_0 \cdot \hat{S}_0 \subset \hat{S}_0, \quad \check{S}_1 \cdot \hat{S}_1 \subset \hat{S}_0 \cap \hat{S}_2, \\ &\check{S}_0 \cdot \hat{S}_1 \subset \hat{S}_1, \quad \check{S}_1 \cdot \hat{S}_2 \subset \hat{S}_1, \\ &\check{S}_0 \cdot \hat{S}_2 \subset \hat{S}_2, \quad \check{S}_2 \cdot \hat{S}_2 \subset \hat{S}_0. \end{aligned} \tag{3.98}$$

Thus,

$$\check{\mathfrak{G}}_R = \check{W}_0 \oplus \check{W}_1 \oplus \check{W}_2, \tag{3.99}$$

corresponds to a reduced algebra of $\mathfrak{G}_R$, where

$$\check{W}_0 = \left(\check{S}_0 \times V_0\right) = \{\lambda_{2n}, \text{ with } n = 0, \cdots, 2[m/2]\} \times \left\{\tilde{J}_{ab}\right\}, \tag{3.100}$$

$$\check{W}_1 = \left(\check{S}_1 \times V_1\right) = \{\lambda_{2n+1}, \text{ with } n = 0, \cdots, m-1\} \times \left\{\tilde{Q}_\alpha\right\}, \tag{3.101}$$

$$\check{W}_2 = \left(\check{S}_2 \times V_2\right) = \{\lambda_{2n+2}, \text{ with } n = 0, \cdots, 2[(m-1)/2]\} \times \left\{\tilde{P}_a\right\}. \tag{3.102}$$

Here, $\tilde{J}_{ab}, \tilde{P}_a$ and $\tilde{Q}_\alpha$ correspond to the generators of $\mathfrak{osp}(4|1)$ superalgebra. The new superalgebra obtained by the $S$-expansion procedure is generated by

$$\left\{J_{ab}, P_a, Z_{ab}^{(k)}, \tilde{Z}_{ab}^{(k)}, Z_a^{(l)}, \tilde{Z}_a^{(l)}, Q_\alpha, \Sigma_\alpha^{(k)}, \Phi_\alpha^{(l)}\right\}, \tag{3.103}$$

where these new generators can be written as

$$\begin{aligned} J_{ab} &= J_{ab,0} = \lambda_0 \tilde{J}_{ab}, & \tilde{Z}_a^{(l)} &= P_{a,4l} = \lambda_{4l}\tilde{P}_a, \\ P_a &= P_{a,2} = \lambda_2 \tilde{P}_a, & Q_\alpha &= Q_{\alpha,1} = \lambda_1 \tilde{Q}_\alpha, \\ Z_{ab}^{(k)} &= J_{ab,4k} = \lambda_{4k}\tilde{J}_{ab}, & \Sigma_\alpha^{(k)} &= Q_{\alpha,4k-1} = \lambda_{4k-1}\tilde{Q}_\alpha, \\ \tilde{Z}_{ab}^{(k)} &= J_{ab,4k-2} = \lambda_{4k-2}\tilde{J}_{ab}, & \Phi_\alpha^{(l)} &= Q_{\alpha,4l+1} = \lambda_{4l+1}\tilde{Q}_\alpha, \\ Z_a^{(l)} &= P_{a,4l+2} = \lambda_{4l+2}\tilde{P}_a. \end{aligned} \tag{3.104}$$

with $k=1,\ldots,\left[\frac{m}{2}\right]$, $l=1,\ldots,\left[\frac{m-1}{2}\right]$. It is important to note that the super indices $k$ and $l$ of spinor generators correspond to the expansion labels and they do not define an $\mathcal{N}$-extended superalgebra. The $\mathcal{N}$-extended case will be considered in the next section.

These new generators satisfy the commutation relations

$$[J_{ab},J_{cd}]=\eta_{bc}J_{ad}-\eta_{ac}J_{bd}-\eta_{bd}J_{ac}+\eta_{ad}J_{bc}, \tag{3.105}$$
$$[J_{ab},P_{c}]=\eta_{bc}P_{a}-\eta_{ac}P_{b}, \qquad [P_{a},P_{b}]=Z_{ab}^{(1)}, \tag{3.106}$$
$$\left[J_{ab},Z_{cd}^{(k)}\right]=\eta_{bc}Z_{ad}^{(k)}-\eta_{ac}Z_{bd}^{(k)}-\eta_{bd}Z_{ac}^{(k)}+\eta_{ad}Z_{bc}^{(k)}, \tag{3.107}$$
$$\left[Z_{ab}^{(k)},P_{c}\right]=\eta_{bc}Z_{a}^{(k)}-\eta_{ac}Z_{b}^{(k)}, \qquad \left[J_{ab},Z_{c}^{(l)}\right]=\eta_{bc}Z_{a}^{(l)}-\eta_{ac}Z_{b}^{(l)}, \tag{3.108}$$
$$\left[Z_{ab}^{(k)},Z_{c}^{(l)}\right]=\eta_{bc}Z_{a}^{(k+l)}-\eta_{ac}Z_{b}^{(k+l)}, \tag{3.109}$$
$$\left[Z_{ab}^{(k)},Z_{cd}^{(j)}\right]=\eta_{bc}Z_{ad}^{(k+j)}-\eta_{ac}Z_{bd}^{(k+j)}-\eta_{bd}Z_{ac}^{(k+j)}+\eta_{ad}Z_{bc}^{(k+j)}, \tag{3.110}$$
$$\left[P_{a},Z_{c}^{(k)}\right]=Z_{ab}^{(k+1)}, \qquad \left[Z_{a}^{(l)},Z_{c}^{(n)}\right]=Z_{ab}^{(l+n+1)} \tag{3.111}$$

$$\left[\tilde{Z}_{ab}^{(k)},\tilde{Z}_{cd}^{(j)}\right]=\eta_{bc}Z_{ad}^{(k+j-1)}-\eta_{ac}Z_{bd}^{(k+j-1)}-\eta_{bd}Z_{ac}^{(k+j-1)}+\eta_{ad}Z_{bc}^{(k+j-1)}, \tag{3.112}$$
$$\left[J_{ab},\tilde{Z}_{cd}^{(k)}\right]=\eta_{bc}\tilde{Z}_{ad}^{(k)}-\eta_{ac}\tilde{Z}_{bd}^{(k)}-\eta_{bd}\tilde{Z}_{ac}^{(k)}+\eta_{ad}\tilde{Z}_{bc}^{(k)}, \tag{3.113}$$
$$\left[\tilde{Z}_{ab}^{(k)},P_{c}\right]=\eta_{bc}\tilde{Z}_{a}^{(k)}-\eta_{ac}\tilde{Z}_{b}^{(k)}, \qquad \left[J_{ab},\tilde{Z}_{c}^{(l)}\right]=\eta_{bc}\tilde{Z}_{a}^{(l)}-\eta_{ac}\tilde{Z}_{b}^{(l)}, \tag{3.114}$$
$$\left[Z_{ab}^{(k)},\tilde{Z}_{c}^{(l)}\right]=\eta_{bc}\tilde{Z}_{a}^{(k+l)}-\eta_{ac}\tilde{Z}_{b}^{(k+l)}, \qquad \left[\tilde{Z}_{ab}^{(k)},Z_{c}^{(l)}\right]=\eta_{bc}\tilde{Z}_{a}^{(k+l)}-\eta_{ac}\tilde{Z}_{b}^{(k+l)}, \tag{3.115}$$
$$\left[\tilde{Z}_{ab}^{(k)},\tilde{Z}_{c}^{(l)}\right]=\eta_{bc}Z_{a}^{(k+l-1)}-\eta_{ac}Z_{b}^{(k+l-1)}, \qquad \left[P_{a},\tilde{Z}_{b}^{(l)}\right]=\tilde{Z}_{ab}^{(l+1)} \tag{3.116}$$
$$\left[\tilde{Z}_{a}^{(l)},\tilde{Z}_{b}^{(n)}\right]=Z_{ab}^{(l+n)}, \qquad \left[Z_{a}^{(l)},\tilde{Z}_{b}^{(n)}\right]=\tilde{Z}_{ab}^{(l+n+1)}, \tag{3.117}$$
$$\left[Z_{ab}^{(k)},\tilde{Z}_{cd}^{(j)}\right]=\eta_{bc}\tilde{Z}_{ad}^{(k+j)}-\eta_{ac}\tilde{Z}_{bd}^{(k+j)}-\eta_{bd}\tilde{Z}_{ac}^{(k+j)}+\eta_{ad}\tilde{Z}_{bc}^{(k+j)}, \tag{3.118}$$

$$\left[J_{ab},Q_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha, \qquad \left[J_{ab},\Sigma_\alpha^{(k)}\right] = -\frac{1}{2}\left(\gamma_{ab}\Sigma^{(k)}\right)_\alpha, \tag{3.119}$$

$$\left[J_{ab},\Phi_\alpha^{(l)}\right] = -\frac{1}{2}\left(\gamma_{ab}\Phi^{(l)}\right)_\alpha, \qquad \left[\tilde{Z}_{ab}^{(k)},Q_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab}\Sigma^{(k)}\right)_\alpha, \tag{3.120}$$

$$\left[\tilde{Z}_{ab}^{(k)},\Sigma_\alpha^{(j)}\right] = -\frac{1}{2}\left(\gamma_{ab}\Phi^{(k+j-1)}\right)_\alpha, \qquad \left[\tilde{Z}_{ab}^{(k)},\Phi_\alpha^{(l)}\right] = -\frac{1}{2}\left(\gamma_{ab}\Sigma^{(k+l)}\right)_\alpha, \tag{3.121}$$

$$\left[Z_{ab}^{(k)},Q_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab}\Phi^{(k)}\right)_\alpha, \qquad \left[Z_{ab}^{(k)},\Sigma_\alpha^{(j)}\right] = -\frac{1}{2}\left(\gamma_{ab}\Sigma^{(k+j)}\right)_\alpha, \tag{3.122}$$

$$\left[Z_{ab}^{(k)},\Phi_\alpha^{(l)}\right] = -\frac{1}{2}\left(\gamma_{ab}\Phi^{(k+l)}\right)_\alpha, \qquad \left[P_a,Q_\alpha\right] = -\frac{1}{2}\left(\gamma_a\Sigma^{(1)}\right)_\alpha \tag{3.123}$$

$$\left[P_a,\Sigma_\alpha^{(k)}\right] = -\frac{1}{2}\left(\gamma_a\Phi^{(k)}\right)_\alpha, \qquad \left[P_a,\Phi_\alpha^{(l)}\right] = -\frac{1}{2}\left(\gamma_a\Sigma^{(l+1)}\right)_\alpha, \tag{3.124}$$

$$\left[\tilde{Z}_a^{(l)},Q_\alpha\right] = -\frac{1}{2}\left(\gamma_a\Phi^{(l)}\right)_\alpha, \qquad \left[\tilde{Z}_a^{(l)},\Sigma_\alpha^{(k)}\right] = -\frac{1}{2}\left(\gamma_a\Sigma^{(l+k)}\right)_\alpha, \tag{3.125}$$

$$\left[\tilde{Z}_a^{(l)},\Phi_\alpha^{(n)}\right] = -\frac{1}{2}\left(\gamma_a\Phi^{(l+n)}\right)_\alpha, \qquad \left[Z_a^{(l)},Q_\alpha\right] = -\frac{1}{2}\left(\gamma_a\Sigma^{(l+1)}\right)_\alpha, \tag{3.126}$$

$$\left[Z_a^{(l)},\Sigma_\alpha^{(n)}\right] = -\frac{1}{2}\left(\gamma_a\Phi^{(l+n)}\right)_\alpha, \qquad \left[Z_a^{(l)},\Phi_\alpha^{(n)}\right] = -\frac{1}{2}\left(\gamma_a\Sigma^{(l+n+1)}\right)_\alpha, \tag{3.127}$$

$$\left\{Q_\alpha,Q_\beta\right\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{Z}_{ab}^{(1)} - 2\left(\gamma^aC\right)_{\alpha\beta}P_a\right], \tag{3.128}$$

$$\left\{Q_\alpha,\Sigma_\beta^{(k)}\right\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}Z_{ab}^{(k)} - 2\left(\gamma^aC\right)_{\alpha\beta}\tilde{Z}_a^{(k)}\right], \tag{3.129}$$

$$\left\{Q_\alpha,\Phi_\beta^{(l)}\right\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{Z}_{ab}^{(l+1)} - 2\left(\gamma^aC\right)_{\alpha\beta}Z_a^{(l)}\right], \tag{3.130}$$

$$\left\{\Sigma_\alpha^{(k)},\Sigma_\beta^{(j)}\right\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{Z}_{ab}^{(k+j)} - 2\left(\gamma^aC\right)_{\alpha\beta}Z_a^{(k+j-1)}\right], \tag{3.131}$$

$$\left\{\Sigma_\alpha^{(k)},\Phi_\beta^{(l)}\right\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}Z_{ab}^{(k+l)} - 2\left(\gamma^aC\right)_{\alpha\beta}\tilde{Z}_a^{(k+l)}\right], \tag{3.132}$$

$$\left\{\Phi_\alpha^{(l)},\Phi_\beta^{(n)}\right\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{Z}_{ab}^{(l+n+1)} - 2\left(\gamma^aC\right)_{\alpha\beta}Z_a^{(l+n)}\right], \tag{3.133}$$

with $k,j=1,\ldots,\left[\frac{m}{2}\right]$, $l,n=1,\ldots,\left[\frac{m-1}{2}\right]$. These (anti)commutation relations are obtained using the multiplication law of the semigroup (3.88) and the (anti)commutation relations of the original superalgebra (2.4) – (2.8). One sees that when $k+l>\left[\frac{m}{2}\right]$ the generatos $T_A^{(k)}$ and $T_B^{(l)}$ are abelian.

The new superalgebra obtained after a reduced resonant $S$-expansion of $\mathfrak{osp}\,(4|1)$ superalgebra corresponds to the $D=4$ minimal Maxwell type superalgebra $s\mathcal{M}_{m+2}$. This superalgebra contains the Maxwell type algebra $\mathcal{M}_{m+2}=\left\{J_{ab},P_a,Z_{ab}^{(k)},Z_a^{(l)}\right\}$ as a subalgebra (eqs. (3.105) – (3.111)) [12], [17]. Interestingly, when $m=2$ and imposing $\tilde{Z}_{ab}^{(1)}=0$ we recover the minimal Maxwell superalgebra $s\mathcal{M}$. The case $m=1$ corresponds to $D=4$

Poincaré superalgebra $s\mathcal{P} = \{J_{ab}, P_a, Q_\alpha\}$. This is not a surprise since the reduced resonant $S_E^{(2)}$-expansion of $\mathfrak{osp}(4|1)$ coincides with an Inönü-Wigner contraction.

In this case, the $S$-expansion method produces new Majorana spinors charge $\Sigma^{(k)}$ and $\Phi^{(l)}$. These fermionic generators transform as spinors under Lorentz transformations. One can see that the Jacobi identities for spinors generators are satisfied due to the gamma matrix identity $(C\gamma^a)_{(\alpha\beta}(C\gamma_a)_{\gamma\delta)} = (C\gamma^{a\beta})_{(\alpha\beta}(C\gamma_{a\beta})_{\gamma\delta)} = 0$ (cyclic permutations of $\alpha, \beta, \gamma$).

## 3.4 $S$-expansion of the $\mathfrak{osp}(4|\mathcal{N})$ superalgebra

### 3.4.1 $\mathcal{N}$-extended Maxwell superalgebras

We have shown that the minimal $D = 4$ Maxwell type superalgebras $s\mathcal{M}_{m+2}$ can be obtained from a reduced resonant $S_E^{(2m)}$-expansion of $\mathfrak{osp}(4|1)$ superalgebra. It seems natural to expect to obtain the $D = 4$ $\mathcal{N}$-extended Maxwell superalgebras from an $S$-expansion of the $\mathfrak{osp}(4|\mathcal{N})$ superalgebra.

Let us consider the following decomposition of the original superalgebra $\mathfrak{g}$ as a direct sum of subspaces $V_p$,

$$\begin{aligned} \mathfrak{g} = \mathfrak{osp}(4|\mathcal{N}) &= (\mathfrak{so}(3,1) \oplus \mathfrak{so}(\mathcal{N})) \oplus \frac{\mathfrak{osp}(4|\mathcal{N})}{\mathfrak{sp}(4) \oplus \mathfrak{so}(\mathcal{N})} \oplus \frac{\mathfrak{sp}(4)}{\mathfrak{so}(3,1)} \\ &= V_0 \oplus V_1 \oplus V_2, \end{aligned} \tag{3.134}$$

where $V_0$ corresponds to the subspace generated by Lorentz generators $\tilde{J}_{ab}$ and by $\frac{\mathcal{N}(\mathcal{N}-1)}{2}$ internal symmetry generators $T^{ij}$, $V_1$ corresponds to the fermionic subspace generated by $\mathcal{N}$ Majorana spinor charges $\tilde{Q}^i_\alpha$ ($i = 1, \cdots, \mathcal{N}$ ; $\alpha = 1, \cdots, 4$) and $V_2$ corresponds to the $AdS$ boost generated by $\tilde{P}_a$. The $\mathfrak{osp}(4|\mathcal{N})$ (anti)commutation relations read

$$\left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] = \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} - \eta_{bd}\tilde{J}_{ac} + \eta_{ad}\tilde{J}_{bc}, \tag{3.135}$$

$$\left[T^{ij}, T^{kl}\right] = \delta^{jk}T^{il} - \delta^{ik}T^{jl} - \delta^{jl}T^{ik} + \delta^{il}T^{jk}, \tag{3.136}$$

$$\left[\tilde{J}_{ab}, \tilde{P}_c\right] = \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b, \tag{3.137}$$

$$\left[\tilde{P}_a, \tilde{P}_b\right] = \tilde{J}_{ab}, \tag{3.138}$$

$$\left[\tilde{J}_{ab},\tilde{Q}^i_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab}\tilde{Q}^i\right)_\alpha, \qquad \left[\tilde{P}_a,\tilde{Q}^i_\alpha\right] = -\frac{1}{2}\left(\gamma_a\tilde{Q}^i\right)_\alpha, \tag{3.139}$$

$$\left[T^{ij},\tilde{Q}^k_\alpha\right] = \left(\delta^{jk}\tilde{Q}^i_\alpha - \delta^{ik}\tilde{Q}^i_\alpha\right), \tag{3.140}$$

$$\left\{\tilde{Q}^i_\alpha,\tilde{Q}^j_\beta\right\} = -\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{J}_{ab} - 2\left(\gamma^a C\right)_{\alpha\beta}\tilde{P}_a\right] + C_{\alpha\beta}T^{ij}, \tag{3.141}$$

where $i,j,k,l = 1,\ldots,\mathcal{N}$.

The subspace structure may be written as

$$[V_0,V_0] \subset V_0, \qquad [V_1,V_1] \subset V_0 \oplus V_2, \tag{3.142}$$

$$[V_0,V_1] \subset V_1, \qquad [V_1,V_2] \subset V_1, \tag{3.143}$$

$$[V_0,V_2] \subset V_2, \qquad [V_2,V_2] \subset V_0. \tag{3.144}$$

Let us consider $S_E^{(4)} = \{\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\}$ as the relevant finite abelian semigroup whose elements obey the multiplication law

$$\lambda_\alpha\lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leq 5, \\ \lambda_5, & \text{when } \alpha+\beta > 5. \end{cases} \tag{3.145}$$

In this case, $\lambda_5$ plays the role of the zero element of the semigroup $S_E^{(4)}$.

Let $S_E^{(4)} = S_0 \cup S_1 \cup S_2$ be a subset decomposition of $S_E^{(4)}$ with

$$S_0 = \{\lambda_0,\lambda_2,\lambda_4,\lambda_5\}, \tag{3.146}$$

$$S_1 = \{\lambda_1,\lambda_3,\lambda_5\}, \tag{3.147}$$

$$S_2 = \{\lambda_2,\lambda_4,\lambda_5\}, \tag{3.148}$$

This subset decomposition satisfies the resonance condition since we have [compare with eqs. (3.142) – (3.144)]

$$\begin{array}{ll} S_0 \cdot S_0 \subset S_0, & S_1 \cdot S_1 \subset S_0 \cap S_2, \\ S_0 \cdot S_1 \subset S_1, & S_1 \cdot S_2 \subset S_1, \\ S_0 \cdot S_2 \subset S_2, & S_2 \cdot S_2 \subset S_0. \end{array} \tag{3.149}$$

Thus, according to Theorem IV.2 of [18], we have that

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{3.150}$$

is a resonant subalgebra of $S_E^{(4)} \times \mathfrak{g}$, where

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\} \times \left\{ \tilde{J}_{ab}, T^{ij} \right\} \tag{3.151}$$
$$= \left\{ \lambda_0 \tilde{J}_{ab}, \lambda_2 \tilde{J}_{ab}, \lambda_4 \tilde{J}_{ab}, \lambda_5 \tilde{J}_{ab}, \lambda_0 T^{ij}, \lambda_2 T^{ij}, \lambda_4 T^{ij}, \lambda_5 T^{ij} \right\},$$
$$W_1 = (S_1 \times V_1) = \{\lambda_1, \lambda_3, \lambda_5\} \times \left\{ \tilde{Q}_\alpha \right\} = \left\{ \lambda_1 \tilde{Q}_\alpha, \lambda_3 \tilde{Q}_\alpha, \lambda_5 \tilde{Q}_\alpha \right\}, \tag{3.152}$$
$$W_2 = (S_2 \times V_2) = \{\lambda_2, \lambda_4, \lambda_5\} \times \left\{ \tilde{P}_a \right\} = \left\{ \lambda_2 \tilde{P}_a, \lambda_4 \tilde{P}_a, \lambda_5 \tilde{P}_a \right\}. \tag{3.153}$$

Imposing $\lambda_5 T_A = 0$, the $0_S$-reduced resonant superalgebra is obtained. Thus, the new superalgebra is generated by $\left\{ J_{ab}, P_a, Z_{ab}, \tilde{Z}_{ab}, \tilde{Z}_a, Q_\alpha^i, \Sigma_\alpha^i, T^{ij}, Y^{ij}, \tilde{Y}^{ij} \right\}$ where

$$\begin{array}{ll} J_{ab} = J_{ab,0} = \lambda_0 \tilde{J}_{ab}, & Q_\alpha^i = Q_{\alpha,1}^i = \lambda_1 \tilde{Q}_\alpha^i, \\ P_a = P_{a,2} = \lambda_2 \tilde{P}_a, & \Sigma_\alpha^i = \Sigma_{\alpha,3}^i = \lambda_3 \tilde{Q}_\alpha^i, \\ Z_{ab} = J_{ab,4} = \lambda_4 \tilde{J}_{ab}, & T^{ij} = T^{ij}_{,0} = \lambda_0 T^{ij}, \\ \tilde{Z}_{ab} = J_{ab,2} = \lambda_2 \tilde{J}_{ab}, & Y^{ij} = T^{ij}_{,4} = \lambda_4 T^{ij}, \\ \tilde{Z}_a = P_{a,4} = \lambda_4 \tilde{P}_a, & \tilde{Y}^{ij} = T^{ij}_{,2} = \lambda_2 T^{ij}. \end{array} \tag{3.154}$$

Then using the multiplication law of the semigroup (3.145) and the (anti)commutations relations of the original superalgebra (3.135) – (3.141) we find the new superalgebra

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{3.155}$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{3.156}$$
$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{3.157}$$
$$\left[J_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \tag{3.158}$$
$$\left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{3.159}$$
$$\left[J_{ab}, \tilde{Z}_c\right] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \tag{3.160}$$
$$\left[\tilde{Z}_{ab}, P_c\right] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \tag{3.161}$$

$$\left[T^{ij}, T^{kl}\right] = \delta^{jk} T^{il} - \delta^{ik} T^{jl} - \delta^{jl} T^{ik} + \delta^{il} T^{jk}, \tag{3.162}$$
$$\left[T^{ij}, Y^{kl}\right] = \delta^{jk} Y^{il} - \delta^{ik} Y^{jl} - \delta^{jl} Y^{ik} + \delta^{il} Y^{jk}, \tag{3.163}$$
$$\left[T^{ij}, \tilde{Y}^{kl}\right] = \delta^{jk} \tilde{Y}^{il} - \delta^{ik} \tilde{Y}^{jl} - \delta^{jl} \tilde{Y}^{ik} + \delta^{il} \tilde{Y}^{jk}, \tag{3.164}$$
$$\left[\tilde{Y}^{ij}, \tilde{Y}^{kl}\right] = \delta^{jk} Y^{il} - \delta^{ik} Y^{jl} - \delta^{jl} Y^{ik} + \delta^{il} Y^{jk}, \tag{3.165}$$

$$\left[J_{ab},Q_{\alpha}^{i}\right]=-\frac{1}{2}\left(\gamma_{ab}Q^{i}\right)_{\alpha}, \qquad \left[\tilde{Z}_{ab},Q_{\alpha}^{i}\right]=-\frac{1}{2}\left(\gamma_{ab}\Sigma^{i}\right)_{\alpha}, \tag{3.166}$$

$$\left[J_{ab},\Sigma_{\alpha}^{i}\right]=-\frac{1}{2}\left(\gamma_{ab}\Sigma^{i}\right)_{\alpha}, \qquad \left[T^{ij},Q_{\alpha}^{i}\right]=\left(\delta^{jk}Q_{\alpha}^{i}-\delta^{ik}Q_{\alpha}^{i}\right), \tag{3.167}$$

$$\left[T^{ij},\Sigma_{\alpha}^{k}\right]=\left(\delta^{jk}\Sigma_{\alpha}^{i}-\delta^{ik}\Sigma_{\alpha}^{i}\right), \tag{3.168}$$

$$\left[\tilde{Y}^{ij},Q_{\alpha}^{k}\right]=\left(\delta^{jk}\Sigma_{\alpha}^{i}-\delta^{ik}\Sigma_{\alpha}^{i}\right), \tag{3.169}$$

$$\left[P_{a},Q_{\alpha}^{i}\right]=-\frac{1}{2}\left(\gamma_{a}\Sigma^{i}\right)_{\alpha}, \tag{3.170}$$

$$\left\{Q_{\alpha}^{i},Q_{\beta}^{j}\right\}=-\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{Z}_{ab}-2\left(\gamma^{a}C\right)_{\alpha\beta}P_{a}\right]+C_{\alpha\beta}\tilde{Y}^{ij}, \tag{3.171}$$

$$\left\{Q_{\alpha}^{i},\Sigma_{\beta}^{j}\right\}=-\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}Z_{ab}-2\left(\gamma^{a}C\right)_{\alpha\beta}\tilde{Z}_{a}\right]+C_{\alpha\beta}Y^{ij}, \tag{3.172}$$

$$\text{others }=0. \tag{3.173}$$

The new superalgebra obtained after a reduced resonant $S_E^{(4)}$-expansion of $\mathfrak{osp}\left(4|\mathcal{N}\right)$ superalgebra corresponds to the $D=4$ $\mathcal{N}$-extended Maxwell superalgebra $s\mathcal{M}_4^{(\mathcal{N})}$. An alternative expansion procedure to obtain the $\mathcal{N}$-extended Maxwell superalgebra has been proposed in [57]. Interestingly, this superalgebra contains the generalized Maxwell algebra $g\mathcal{M}=\left\{J_{ab},P_a,Z_{ab},\tilde{Z}_{ab},\tilde{Z}_a\right\}$ as a subalgebra (see Appendix B). One sees that the $S$-expansion procedure introduces additional bosonic generators which modify the minimal Maxwell superalgebra [see eqs. (3.171), (3.172)]. Naturally when $\tilde{Z}_a=\tilde{Z}_{ab}=Y^{ij}=\tilde{Y}^{ij}=0$, we obtain the simplest $D=4$ $\mathcal{N}$-extended Maxwell superalgebra $s\mathcal{M}^{(\mathcal{N})}$ generated by $\left\{J_{ab},P_a,Z_{ab},Q_{\alpha}^{i},\Sigma_{\alpha}^{i},T_{ab}\right\}$ . Eventually for $\mathcal{N}=1$, with $T_{ab}=0$, the $D=4$ minimal Maxwell superalgebra $s\mathcal{M}$ is recovered.

It is important to note that setting some generators equals to zero does not always lead to a Lie superalgebra. However, the properties of the gamma matrices in 4 dimensions allow us to impose some generators equals to zero without breaking the Jacobi identity.

We can generalize this procedure and obtain the $\mathcal{N}$-extended Maxwell type superalgebra $s\mathcal{M}_{m+2}^{(\mathcal{N})}$ as a reduced resonant $S$-expansion of $\mathfrak{osp}\left(4|\mathcal{N}\right)$, when $S_E^{(2m)}=\left\{\lambda_0,\lambda_1,\lambda_2,\cdots,\lambda_{2m+1}\right\}$ is the relevant abelian semigroup. In fact, if we consider a resonant subset decomposition $S_E^{(2m)}=S_0\cup S_1\cup S_2$, where

$$S_p=\left\{\lambda_{2n+p},\text{ with }n=0,\cdots,\left[\frac{2m-p}{2}\right]\right\}\cup\left\{\lambda_{2m+1}\right\}, \qquad p=0,1,2, \tag{3.174}$$

and let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$ where

$$\check{S}_0 = \{\lambda_{2n}, \text{ with } n = 0, \cdots, 2[m/2]\}, \quad \hat{S}_0 = \{(\lambda_{2m}), \lambda_{2m+1}\}, \tag{3.175}$$

$$\check{S}_1 = \{\lambda_{2n+1}, \text{ with } n = 0, \cdots, m-1\}, \quad \hat{S}_1 = \{\lambda_{2m+1}\}, \tag{3.176}$$

$$\check{S}_2 = \{\lambda_{2n+2}, \text{ with } n = 0, \cdots, 2[(m-1)/2]\}, \quad \hat{S}_2 = \{(\lambda_{2m}), \lambda_{2m+1}\}, \tag{3.177}$$

where $(\lambda_{2m})$ means that $\lambda_{2m} \in \hat{S}_0$ if $m$ is odd and $\lambda_{2m} \in \hat{S}_2$ if $m$ is even. This decomposition satisfies the resonant condition for any value of $m$ and we find that

$$\check{\mathfrak{G}}_R = \left(\check{S}_0 \times V_0\right) \oplus \left(\check{S}_1 \times V_1\right) \oplus \left(\check{S}_2 \times V_2\right), \tag{3.178}$$

corresponds to a reduced resonant algebra. This new superalgebra correspond to the $\mathcal{N}$-extended Maxwell superalgebra type $s\mathcal{M}_{m+2}^{(\mathcal{N})}$ which is generated by

$$\left\{J_{ab}, P_a, Z_{ab}^{(k)}, \tilde{Z}_{ab}^{(k)}, Z_a^{(k)}, \tilde{Z}_a^{(k)}, Q_\alpha^i, \Sigma_\alpha^{i(k)}, \Phi_\alpha^{i(k)}, T^{ij}, Y^{ij(k)}, \tilde{Y}^{ij(k)}\right\}. \tag{3.179}$$

These generators can be written as

$$\begin{array}{ll}
J_{ab} = J_{ab,0} = \lambda_0 \tilde{J}_{ab}, & P_a = P_{a,2} = \lambda_2 \tilde{P}_a, \\
Z_{ab}^{(k)} = J_{ab,4k} = \lambda_{4k} \tilde{J}_{ab}, & \tilde{Z}_{ab}^{(k)} = J_{ab,4k-2} = \lambda_{4k-2} \tilde{J}_{ab}, \\
Z_a^{(l)} = P_{a,4l+2} = \lambda_{4l+2} \tilde{P}_a, & \tilde{Z}_a^{(l)} = P_{a,4l} = \lambda_{4l} \tilde{P}_a, \\
Q_\alpha^i = Q_{\alpha,1}^i = \lambda_1 \tilde{Q}_\alpha^i, & \Sigma_\alpha^{i(k)} = Q_{\alpha,4k-1}^i = \lambda_{4k-1} \tilde{Q}_\alpha^i, \\
\Phi_\alpha^{i(l)} = Q_{\alpha,4l+1}^i = \lambda_{4l+1} \tilde{Q}_\alpha^i, & T^{ij} = T_{,0}^{ij} = \lambda_0 T^{ij}, \\
Y^{ij(k)} = T_{,4k}^{ij} = \lambda_{4k} T^{ij}, & \tilde{Y}^{ij(k)} = T_{,4k-2}^{ij} = \lambda_{4k-2} T^{ij},
\end{array} \tag{3.180}$$

with $k = 1, \ldots, \left[\frac{m}{2}\right]$, $l = 1, \ldots, \left[\frac{m-1}{2}\right]$, $i, j = 1, \ldots, \mathcal{N}$. The new bosonic generators $\left\{Z_{ab}, \tilde{Z}_{ab}, Z_a, \tilde{Z}_a, Y^{ij}, \tilde{Y}^{ij}\right\}$ modify some anticommutators of the minimal Maxwell type superalgebra ((3.128) – (3.133)). Now we have

$$\left\{Q_\alpha^i, Q_\beta^j\right\} = -\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab}C\right)_{\alpha\beta} \tilde{Z}_{ab}^{(1)} - 2\left(\gamma^a C\right)_{\alpha\beta} P_a\right] + C_{\alpha\beta} \tilde{Y}^{ij(1)}, \tag{3.181}$$

$$\left\{Q_\alpha^i, \Sigma_\beta^{j(k)}\right\} = -\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab}C\right)_{\alpha\beta} Z_{ab}^{(k)} - 2\left(\gamma^a C\right)_{\alpha\beta} \tilde{Z}_a^{(k)}\right] + C_{\alpha\beta} Y^{ij(k)}, \tag{3.182}$$

$$\left\{Q_\alpha^i, \Phi_\beta^{j(l)}\right\} = -\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab}C\right)_{\alpha\beta} \tilde{Z}_{ab}^{(l+1)} - 2\left(\gamma^a C\right)_{\alpha\beta} Z_a^{(l)}\right] + C_{\alpha\beta} \tilde{Y}^{ij(l+1)}, \tag{3.183}$$

$$\left\{\Sigma_\alpha^{i(k)}, \Sigma_\beta^{j(q)}\right\} = -\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab}C\right)_{\alpha\beta} \tilde{Z}_{ab}^{(k+q)} - 2\left(\gamma^a C\right)_{\alpha\beta} Z_a^{(k+q-1)}\right] + C_{\alpha\beta} \tilde{Y}^{ij(k+q)}, \tag{3.184}$$

$$\left\{\Sigma_\alpha^{i(k)}, \Phi_\beta^{j(l)}\right\} = -\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab}C\right)_{\alpha\beta} Z_{ab}^{(k+l)} - 2\left(\gamma^a C\right)_{\alpha\beta} \tilde{Z}_a^{(k+l)}\right] + C_{\alpha\beta} Y^{ij(k+l)}, \tag{3.185}$$

$$\left\{\Phi_\alpha^{i(l)}, \Phi_\beta^{j(n)}\right\} = -\frac{1}{2}\delta^{ij}\left[\left(\gamma^{ab}C\right)_{\alpha\beta} \tilde{Z}_{ab}^{(l+n+1)} - 2\left(\gamma^a C\right)_{\alpha\beta} Z_a^{(l+n)}\right] + C_{\alpha\beta} \tilde{Y}^{ij(l+n+1)}, \tag{3.186}$$

with $k,q=1,\ldots,\left[\frac{m}{2}\right]$, $l,n=1,\ldots,\left[\frac{m-1}{2}\right]$, $i,j=1,\ldots,\mathcal{N}$. The internal symmetries generators also brings some new commutation relations besides the commutators (3.105) – (3.127),

$$\left[T^{ij},T^{gh}\right]=\delta^{jg}T^{ih}-\delta^{ig}T^{jh}-\delta^{jh}T^{ig}+\delta^{ih}T^{jg}, \tag{3.187}$$

$$\left[T^{ij},Y^{gh(k)}\right]=\delta^{jg}Y^{ih(k)}-\delta^{ig}Y^{jh(k)}-\delta^{jh}Y^{ig(k)}+\delta^{ih}Y^{jg(k)}, \tag{3.188}$$

$$\left[T^{ij},\tilde{Y}^{gh(k)}\right]=\delta^{jg}\tilde{Y}^{ih(k)}-\delta^{ig}\tilde{Y}^{jh(k)}-\delta^{jh}\tilde{Y}^{ig(k)}+\delta^{ih}\tilde{Y}^{jg(k)}, \tag{3.189}$$

$$\left[\tilde{Y}^{ij(k)},\tilde{Y}^{gh(q)}\right]=\delta^{jg}Y^{ih(k+q-1)}-\delta^{ig}Y^{jh(k+q-1)}-\delta^{jh}Y^{ig(k+q-1)}+\delta^{ih}Y^{jg(k+q-1)}, \tag{3.190}$$

$$\left[\tilde{Y}^{ij(k)},Y^{gh(q)}\right]=\delta^{jg}\tilde{Y}^{ih(k+q)}-\delta^{ig}\tilde{Y}^{jh(k+q)}-\delta^{jh}\tilde{Y}^{ig(k+q)}+\delta^{ih}\tilde{Y}^{jg(k+q)}, \tag{3.191}$$

$$\left[Y^{ij(k)},Y^{gh(q)}\right]=\delta^{jg}Y^{ih(k+q)}-\delta^{ig}Y^{jh(k+q)}-\delta^{jh}Y^{ig(k+q)}+\delta^{ih}Y^{jg(k+q)}, \tag{3.192}$$

$$\left[T^{ij},Q_{\alpha}^{i}\right]=\left(\delta^{jk}Q_{\alpha}^{i}-\delta^{ik}Q_{\alpha}^{i}\right), \tag{3.193}$$

$$\left[T^{ij},\Sigma_{\alpha}^{g(k)}\right]=\left[\tilde{Y}^{ij(k)},Q_{\alpha}^{g}\right]=\left(\delta^{jg}\Sigma_{\alpha}^{i(k)}-\delta^{ig}\Sigma_{\alpha}^{i(k)}\right), \tag{3.194}$$

$$\left[T^{ij},\Phi_{\alpha}^{g(k)}\right]=\left[Y^{ij(k)},Q_{\alpha}^{g}\right]=\left(\delta^{jg}\Phi_{\alpha}^{i(k)}-\delta^{ig}\Phi_{\alpha}^{i(k)}\right), \tag{3.195}$$

$$\left[\tilde{Y}^{ij(k)},\Phi_{\alpha}^{g(q)}\right]=\left[Y^{ij(k)},\Sigma_{\alpha}^{g(q)}\right]=\left(\delta^{jg}\Sigma_{\alpha}^{i(k+q)}-\delta^{ig}\Sigma_{\alpha}^{i(k+q)}\right), \tag{3.196}$$

$$\left[\tilde{Y}^{ij(k)},\Sigma_{\alpha}^{g(q)}\right]=\left(\delta^{jg}\Phi_{\alpha}^{i(k+q-1)}-\delta^{ig}\Phi_{\alpha}^{i(k+q-1)}\right), \tag{3.197}$$

$$\left[Y^{ij(k)},\Phi_{\alpha}^{g(q)}\right]=\left(\delta^{jg}\Phi_{\alpha}^{i(k+q)}-\delta^{ig}\Phi_{\alpha}^{i(k+q)}\right). \tag{3.198}$$

As in the case of the minimal Maxwell type superalgebra, one sees that when $k+q>\left[\frac{m}{2}\right]$ then the generators $T_{A}^{(k)}$ and $T_{B}^{(q)}$ are abelian. As in the previous case, the $S$-expansion method produces new Majorana spinors charge $\Sigma^{i(k)}$ and $\Phi^{i(l)}$ which transform as spinors under Lorentz transformations.

The $\mathcal{N}$-extended Maxwell type superalgebra $s\mathcal{M}_{m+2}^{(\mathcal{N})}$ contains the Maxwell type algebra $\mathcal{M}_{m+2}=\left\{J_{ab},P_{a},Z_{ab}^{(k)},Z_{a}^{(l)}\right\}$ as a subalgebra (eqs. (3.105) – (3.111)) [17]. We can see that for $m=2$, we recover the $D=4$ $\mathcal{N}$-extended Maxwell superalgebra $s\mathcal{M}_{4}^{(\mathcal{N})}$. It is interesting to observe that for $m=1$, we obtain the $D=4$ $\mathcal{N}$-extended Poincaré superalgebra $s\mathcal{P}^{(\mathcal{N})}=\{J_{ab},P_{a},Q_{\alpha},T^{ij}\}$. This is not a surprise because the reduced resonant $S_{E}^{(2)}$-expansion of $\mathfrak{osp}(4|\mathcal{N})$ coincides with an Inönü-Wigner contraction.

In summary, we have shown that the Maxwell superalgebras found by the MC expansion method in [57], can be alternatively derived by the $S$-expansion procedure. In particular, the $S$-expansion of $\mathfrak{osp}(4|1)$ allowed us to obtain the minimal Maxwell superalgebra $s\mathcal{M}$.

Then, choosing different semigroups we have shown that it is possible to define new minimal $D = 4$ Maxwell type superalgebras $s\mathcal{M}_{m+2}$, which can be seen as a generalization of the D'Auria-Fré superalgebra and the Green algebras introduced in [48], [59] respectively.

We also have shown that the $D = 4$, $\mathcal{N}$-extended Maxwell superalgebra $s\mathcal{M}^{(\mathcal{N})}$ derived initially as a MC expansion in [57], can be alternatively obtained as an $S$-expansion of $\mathfrak{osp}\,(4|\mathcal{N})$. Choosing bigger semigroups we have defined new $D = 4$ $\mathcal{N}$-extended Maxwell type superalgebras $s\mathcal{M}^{(\mathcal{N})}_{m+2}$. Clearly, when $m = 2$ we recover the $s\mathcal{M}^{(\mathcal{N})}$ superalgebra and for $\mathcal{N} = 1$ we recover the Maxwell type algebra $s\mathcal{M}_{m+2}$.

As we shall see in the next chapter, the minimal Maxwell type superalgebra $s\mathcal{M}_4$ (and its generalization $s\mathcal{M}_{m+2}$) can be used to construct dynamical actions in $D = 4$.

# Chapter 4

# $\mathcal{N} = 1$, $D = 4$ Supergravity and Maxwell Superalgebras

## 4.1 Introduction

The so-called Maxwell algebra $\mathcal{M}$ corresponds to a modification of the Poincaré symmetries, where a constant electromagnetic field background is added to the Minkowski space [49], [50], [51], [52], [54], [61]. In $D = 4$ dimensions this algebra is obtained by adding to the Poincaré generators $\{J_{ab}, P_a\}$ the tensorial Abelian charges $Z_{ab}$, modifying the commutator of the translation generators $P_a$ as follows

$$[P_a, P_b] = Z_{ab}. \tag{4.1}$$

In this way, the Maxwell algebra is an enlargement of Poincaré algebra, i.e., if we consider $Z_{ab} = 0$ we recover the Poincaré algebra. As we discussed above this Maxwell algebra can also be obtained through an expansion procedure from the AdS Lie algebra $\mathfrak{so}(3,2)$ [16], [57] using $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$. Moreover, this result was extended to all Maxwell type algebras $\mathcal{M}_m$ which can be obtained as an $S$-expansion of the AdS algebra using bigger semigroups [17].

In the context of supersymmetry, the minimal $D = 4$ Maxwell superalgebra $s\mathcal{M}$ is obtained as an enlargement of the Poincaré superalgebra [53]. This is particularly interesting since it describes the supersymmetries of generalized $\mathcal{N} = 1$, $D = 4$ superspace in the presence of a constant abelian supersymmetric field strength background. This superalgebra can also be obtained using the Maurer Cartan expansion method [57], and can be used to obtain the minimal $D = 4$ pure supergravity from the curvature 2-form associated to $s\mathcal{M}$ [58].

Furthermore, in the previous chapter we saw that this superalgebra and its generalization $s\mathcal{M}_{m+2}$ can be found as an $S$-expansion of the $\mathfrak{osp}(4|1)$ superalgebra [60].

In this chapter, we present one of the main results of this thesis. Following Ref.[62], we shall construct the minimal $D=4$ supergravity action from the minimal Maxwell type superalgebra $s\mathcal{M}_4$. To this aim, we will apply the $S$-expansion procedure and we will build a geometric action à la Mac Dowell-Mansouri with the expanded curvature 2-form. We show that $\mathcal{N}=1$, $D=4$ pure supergravity can be derived alternatively as the MacDowell-Mansouri like action, which is constructed exclusively in terms of the curvatures of the Maxwell type superalgebra $s\mathcal{M}_4$. Eventually, we extend this result to all minimal Maxwell type superalgebras $s\mathcal{M}_{m+2}$ in $D=4$.

## 4.2 $D=4$ pure Supergravity from $s\mathcal{M}_4$

In the previous chapter, we introduced the minimal Maxwell type superalgebra $s\mathcal{M}_4$ in $D=4$. This superalgebra was obtained after a reduced resonant $S_E^{(4)}$-expansion of the $\mathfrak{osp}(4|1)$ superalgebra, and its generators $\left\{J_{ab}, \tilde{Z}_{ab}, Z_{ab}, P_a, Q_\alpha, \Sigma_\alpha\right\}$ satisfy the (anti)-commutation relations $(3.42)-(3.53)$.

In this section, we present a geometric formulation of $\mathcal{N}=1$ supergravity in four dimensions, where the relevant gauge fields of the theory are those corresponding to the minimal Maxwell superalgebra $s\mathcal{M}_4$. In order to write down an action for $s\mathcal{M}_4$, we start from the one-form gauge connection

$$A=\frac{1}{2}\omega^{ab}J_{ab}+\frac{1}{2}\tilde{k}^{ab}\tilde{Z}_{ab}+\frac{1}{2}k^{ab}Z_{ab}+\frac{1}{l}e^aP_a+\frac{1}{\sqrt{l}}\psi^\alpha Q_\alpha+\frac{1}{\sqrt{l}}\xi^\alpha\Sigma_\alpha, \tag{4.2}$$

where the 1-form gauge fields are given by

$$\begin{aligned}
\omega^{ab} &= \omega^{(ab,0)}=\lambda_0\tilde{\omega}^{ab}, & e^a &= e^{(a,2)}=\lambda_2\tilde{e}^a,\\
\tilde{k}^{ab} &= \omega^{(ab,2)}=\lambda_2\tilde{\omega}^{ab}, & \psi^\alpha &= \psi^{(\alpha,1)}=\lambda_1\tilde{\psi}^\alpha,\\
k^{ab} &= \omega^{(ab,4)}=\lambda_4\tilde{\omega}^{ab}, & \xi^\alpha &= \psi^{(\alpha,3)}=\lambda_3\tilde{\psi}^\alpha,
\end{aligned}$$

where $\tilde{e}^a, \tilde{\omega}^{ab}$ and $\tilde{\psi}$ are the components of the $\mathfrak{osp}(4|1)$ connection (see eq. (2.9)) and the $\lambda_\alpha$ are the elements of the $S_E^{(4)}=\{\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\}$ semigroup.

The associated curvature two-form $F=dA+A\wedge A$ is

$$F=F^AT_A=\frac{1}{2}R^{ab}J_{ab}+\frac{1}{l}R^aP_a+\frac{1}{2}\tilde{F}^{ab}\tilde{Z}_{ab}+\frac{1}{2}F^{ab}Z_{ab}+\frac{1}{\sqrt{l}}\Psi^\alpha Q_\alpha+\frac{1}{\sqrt{l}}\Xi^\alpha\Sigma_\alpha, \tag{4.3}$$

where

$$R^{ab} = d\omega^{ab} + \omega^a_{\ c}\omega^{cb}, \tag{4.4}$$

$$R^a = de^a + \omega^a_{\ b}e^b - \frac{1}{2}\bar{\psi}\gamma^a\psi, \tag{4.5}$$

$$\tilde{F}^{ab} = d\tilde{k}^{ab} + \omega^a_{\ c}\tilde{k}^{cb} - \omega^b_{\ c}\tilde{k}^{ca} + \frac{1}{2l}\bar{\psi}\gamma^{ab}\psi, \tag{4.6}$$

$$F^{ab} = dk^{ab} + \omega^a_{\ c}k^{cb} - \omega^b_{\ c}k^{ca} + \tilde{k}^a_{\ c}\tilde{k}^{cb} + \frac{1}{l^2}e^a e^b + \frac{1}{l}\bar{\xi}\gamma^{ab}\psi, \tag{4.7}$$

$$\Psi = d\psi + \frac{1}{4}\omega_{ab}\gamma^{ab}\psi = D\psi, \tag{4.8}$$

$$\begin{aligned}\Xi &= d\xi + \frac{1}{4}\omega_{ab}\gamma^{ab}\xi + \frac{1}{4}\tilde{k}_{ab}\gamma^{ab}\psi + \frac{1}{2l}e^a\gamma_a\psi \\ &= D\xi + \frac{1}{4}\tilde{k}_{ab}\gamma^{ab}\psi + \frac{1}{2l}e^a\gamma_a\psi.\end{aligned} \tag{4.9}$$

From the Bianchi identity $\nabla F = 0$, with $\nabla = d + [A, \cdot]$, one finds the Lorentz covariant exterior derivatives of the curvatures,

$$DR^{ab} = 0, \tag{4.10}$$

$$DR^a = R^a_{\ b}e^b + \bar{\psi}\gamma^a\Psi, \tag{4.11}$$

$$D\tilde{F}^{ab} = R^a_{\ c}\tilde{k}^{cb} - R^b_{\ c}\tilde{k}^{ca} - \frac{1}{l}\bar{\psi}\gamma^{ab}\Psi \tag{4.12}$$

$$DF^{ab} = R^a_{\ c}k^{cb} - R^b_{\ c}k^{ca} + \tilde{F}^a_{\ c}\tilde{k}^{cb} - \tilde{F}^b_{\ c}\tilde{k}^{ca} + \frac{1}{l^2}R^a e^b - \frac{1}{l^2}e^a R^b \tag{4.13}$$

$$+ \frac{1}{l}\bar{\Xi}\gamma^{ab}\psi - \frac{1}{l}\bar{\xi}\gamma^{ab}\Psi, \tag{4.14}$$

$$D\Psi = \frac{1}{4}R_{ab}\gamma^{ab}\psi, \tag{4.15}$$

$$D\Xi = \frac{1}{4}R_{ab}\gamma^{ab}\xi - \frac{1}{4}\tilde{k}_{ab}\gamma^{ab}\Psi + \frac{1}{4}\tilde{F}_{ab}\gamma^{ab}\psi + \frac{1}{2l}R^a\gamma_a\psi - \frac{1}{2l}e^a\gamma_a\Psi. \tag{4.16}$$

Then, the action can be written as [28]

$$S = 2\int \langle F \wedge F \rangle = 2\int F^A \wedge F^B \langle T_A T_B \rangle_{s\mathcal{M}_4}, \tag{4.17}$$

where $\langle T_A T_B \rangle$ corresponds to an $S$-expanded invariant tensor which is obtained from (2.18). In fact, using Theorem VII.1 of [18] it is possible to show that the non-vanishing components of the $S$-expanded invariant tensor are given by

$$\langle J_{ab}J_{cd}\rangle_{s\mathcal{M}_4} = \alpha_0 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \tag{4.18}$$

$$\left\langle J_{ab}\tilde{Z}_{cd}\right\rangle_{s\mathcal{M}_4} = \alpha_2 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \tag{4.19}$$

$$\left\langle \tilde{Z}_{ab}\tilde{Z}_{cd}\right\rangle_{s\mathcal{M}_4} = \alpha_4 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \tag{4.20}$$

$$\langle J_{ab}Z_{cd}\rangle_{s\mathcal{M}_4} = \alpha_4 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \tag{4.21}$$

$$\langle Q_\alpha Q_\beta\rangle_{s\mathcal{M}_4} = \alpha_2 \left\langle \tilde{Q}_\alpha\tilde{Q}_\beta\right\rangle, \tag{4.22}$$

$$\langle Q_\alpha \Sigma_\beta\rangle_{s\mathcal{M}_4} = \alpha_4 \left\langle \tilde{Q}_\alpha\tilde{Q}_\beta\right\rangle, \tag{4.23}$$

where

$$\left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle = \epsilon_{abcd}, \tag{4.24}$$

$$\left\langle \tilde{Q}_\alpha\tilde{Q}_\beta\right\rangle = 2\left(\gamma_5\right)_{\alpha\beta}, \tag{4.25}$$

are the invariant tensors required to reproduce the MacDowell-Mansouri action for the $\mathfrak{osp}\left(4|1\right)$ superalgebra [28] (see Chapter 2, Section 2.3), and the $\alpha$'s are dimensionless arbitrary constants.

Considering the different non-vanishing components of the invariant tensor $(4.18)-(4.23)$ and the curvature two-form (4.3), we found that the action can be written as

$$\begin{aligned} S = 2\int \Big( & \frac{1}{4}\alpha_0\epsilon_{abcd}R^{ab}R^{cd} + \frac{1}{2}\alpha_2\epsilon_{abcd}R^{ab}\tilde{F}^{cd} + \frac{1}{2}\alpha_4\epsilon_{abcd}R^{ab}F^{cd} \\ & + \frac{1}{4}\alpha_4\epsilon_{abcd}\tilde{F}^{ab}\tilde{F}^{cd} + \frac{2}{l}\alpha_2\bar{\Psi}\gamma_5\Psi + \frac{4}{l}\alpha_4\bar{\Psi}\gamma_5\Xi\Big) \end{aligned} \tag{4.26}$$

or explicitly,

$$\begin{aligned} S = \int & \frac{\alpha_0}{2}\epsilon_{abcd}R^{ab}R^{cd} + \alpha_2\epsilon_{abcd}\left(R^{ab}D\tilde{k}^{cd} + \frac{1}{2l}R^{ab}\bar{\psi}\gamma^{cd}\psi\right) \\ & + \frac{4}{l}\alpha_2 D\bar{\psi}\gamma_5 D\psi + \alpha_4\epsilon_{abcd}\Big(R^{ab}Dk^{cd} + \frac{1}{2}D\tilde{k}^{ab}D\tilde{k}^{cd} + \frac{1}{l^2}R^{ab}e^c e^d \\ & + \frac{1}{2l}D\tilde{k}^{ab}\bar{\psi}\gamma^{cd}\psi + R^{ab}\tilde{k}^{c}{}_{f}\tilde{k}^{fd} + \frac{1}{l}R^{ab}\bar{\xi}\gamma^{cd}\psi\Big) \\ & + \frac{8}{l}\alpha_4 D\bar{\psi}\gamma_5 D\xi + \frac{2}{l}\alpha_4 D\bar{\psi}\gamma_5\tilde{k}_{ab}\gamma^{ab}\psi + \frac{4}{l^2}\alpha_4\bar{\psi}e^a\gamma_a\gamma_5 D\psi \end{aligned} \tag{4.27}$$

where $D = d + [\omega, \cdot]$ . Using the gravitino Bianchi identity

$$D\Psi = \frac{1}{4} R^{ab} \gamma_{ab} \psi, \tag{4.28}$$

and the gamma matrix identity $(C.1)$

$$2\gamma_{ab}\gamma_5 = -\epsilon_{abcd}\gamma^{cd}, \tag{4.29}$$

it is straightforward to show that,

$$\begin{aligned}
\frac{1}{2}\epsilon_{abcd}R^{ab}\bar{\psi}\gamma^{ab}\psi + 4D\bar{\psi}\gamma_5 D\psi &= d\left(4D\bar{\psi}\gamma_5\psi\right), \\
\epsilon_{abcd}R^{ab}\bar{\xi}\gamma^{cd}\psi + 8D\bar{\xi}\gamma_5 D\psi &= d\left(8D\bar{\xi}\gamma_5\psi\right), \\
\frac{1}{2}\epsilon_{abcd}D\tilde{k}^{ab}\bar{\psi}\gamma^{cd}\psi + 2\bar{\psi}\tilde{k}^{ab}\gamma_{ab}\gamma_5 D\psi &= d\left(\bar{\psi}\tilde{k}^{ab}\gamma_{ab}\gamma_5\psi\right).
\end{aligned}$$

Thus the geometric Mac Dowell-Mansouri like action for the $s\mathcal{M}_4$ superalgebra is given by

$$\begin{aligned}
S = \int & \frac{\alpha_0}{2}\epsilon_{abcd}R^{ab}R^{cd} + \alpha_2 d\left(\epsilon_{abcd}R^{ab}\tilde{k}^{cd} + \frac{4}{l}D\bar{\psi}\gamma_5\psi\right) \\
& + \alpha_4\left[\frac{1}{l^2}\epsilon_{abcd}R^{ab}e^c e^d + \frac{4}{l^2}\bar{\psi}e^a\gamma_a\gamma_5 D\psi\right. \\
& \left. + d\left(\epsilon_{abcd}\left(R^{ab}k^{cd} + \frac{1}{2}D\tilde{k}^{ab}\tilde{k}^{cd}\right) + \frac{8}{l}\bar{\xi}\gamma_5 D\psi + \frac{1}{l}\bar{\psi}\tilde{k}^{ab}\gamma_{ab}\gamma_5\psi\right)\right]
\end{aligned} \tag{4.30}$$

From this action, we see that it is split into three independent pieces proportional to $\alpha_0$, $\alpha_2$ and $\alpha_4$. The first term corresponds to the Euler invariant and can be written as a boundary term. The piece proportional to $\alpha_2$ is also a boundary term. The term proportional to $\alpha_4$ contains the Einstein-Hilbert term $\epsilon_{abcd}R^{ab}e^c e^d$ plus the Rarita-Schwinger Lagrangian $4\bar{\psi}e^a\gamma_a\gamma_5 D\psi$, and a boundary term.

From (4.30) we can see that the minimal Maxwell superalgebra $s\mathcal{M}_4$ leads us to the pure Supergravity action. In this way the new Maxwell gauge fields do not contribute to the dynamics and enlarge only the boundary terms. Moreover, as a consequence of the $S$-expansion procedure the supersymmetric cosmological term disappears completely from the action for $s\mathcal{M}_4$ [compare (2.24) and (4.30)]. Although the boundary terms does not contribute to the dynamics of the theory, they play an important role in the context of AdS/CFT correspondence [75].

The result found here can be seen as the supersymmetric case of [17], [19] where the Einstein-Hilbert action was obtained from the Maxwell algebra as a Born-Infeld like action.

Note that if we consider $\tilde{k}^{ab}=0$, the term proportional to $\alpha_4$ corresponds to the action found in [58], namely

$$S|_{\tilde{k}^{ab}=0}=\alpha_4\int\frac{1}{l^2}\left(\epsilon_{abcd}R^{ab}e^ce^d+4\bar{\psi}e^a\gamma_a\gamma_5D_\omega\psi\right)+d\left(\epsilon_{abcd}R^{ab}k^{cd}+\frac{8}{l}\bar{\xi}\gamma_5D_\omega\psi\right) \tag{4.31}$$

which corresponds to four-dimensional pure supergravity plus a boundary term. This is not a surprise but something expected because as we said before, setting $\tilde{Z}_{ab}=0$ in $s\mathcal{M}_4$ leads us to the simplest minimal Maxwell superalgebra [57], whose curvature two-form allows the construction of (4.31) as was shown in [58].

### 4.2.1 $s\mathcal{M}_4$ gauge transformations and supersymmetry

The gauge transformation of the one-form gauge connection $A$ is

$$\delta_\rho A=D\rho=d\rho+[A,\rho] \tag{4.32}$$

where $\rho$ is the $s\mathcal{M}_4$ gauge parameter,

$$\rho=\frac{1}{2}\rho^{ab}J_{ab}+\frac{1}{2}\tilde{\kappa}^{ab}\tilde{Z}_{ab}+\frac{1}{2}\kappa^{ab}Z_{ab}+\frac{1}{l}\rho^aP_a+\frac{1}{\sqrt{l}}\epsilon^\alpha Q_\alpha+\frac{1}{\sqrt{l}}\varrho^\alpha\Sigma_\alpha. \tag{4.33}$$

Then, using eq. (2.28) we have that the $s\mathcal{M}_4$ gauge transformation are given by

$$\delta\omega^{ab}=D\rho^{ab}, \tag{4.34}$$
$$\delta\tilde{k}^{ab}=D\tilde{\kappa}^{ab}-\left(\tilde{k}^a_{\ c}\,\rho^b_{\ c}-\tilde{k}^{bc}\rho^a_{\ c}\right)-\frac{1}{l}\bar{\epsilon}\gamma^{ab}\psi, \tag{4.35}$$
$$\begin{aligned}\delta k^{ab}=&\,D\kappa^{ab}-\left(k^{ac}\rho^b_{\ c}-k^{bc}\rho^a_{\ c}\right)-\left(\tilde{k}^{ac}\tilde{\kappa}^b_{\ c}-\tilde{k}^{bc}\tilde{\kappa}^a_{\ c}\right)\\&+\frac{2}{l^2}e^a\rho^b-\frac{1}{l}\bar{\varrho}\gamma^{ab}\psi-\frac{1}{l}\bar{\epsilon}\gamma^{ab}\xi,\end{aligned} \tag{4.36}$$
$$\delta e^a=D\rho^a+e^b\rho_b^{\ a}+\bar{\epsilon}\gamma^a\psi, \tag{4.37}$$
$$\delta\psi=d\epsilon+\frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon-\frac{1}{4}\rho^{ab}\gamma_{ab}\psi, \tag{4.38}$$
$$\begin{aligned}\delta\xi=&\,d\varrho+\frac{1}{4}\omega^{ab}\gamma_{ab}\varrho+\frac{1}{2l}e^a\gamma_a\epsilon-\frac{1}{2l}\rho^a\gamma_a\psi-\frac{1}{4}\rho^{ab}\gamma_{ab}\xi\\&+\frac{1}{4}\tilde{k}^{ab}\gamma_{ab}\epsilon-\frac{1}{4}\tilde{\kappa}^{ab}\gamma_{ab}\psi.\end{aligned} \tag{4.39}$$

In the same way, from the gauge variation of the curvature

$$\delta_\rho F=[F,\rho] \tag{4.40}$$

it is possible to show that the gauge transformations of the curvature $F$ are given by

$$\delta R^{ab} = R^{ac}\rho_c^{\ b} - R^{cb}\rho^a_{\ c}, \tag{4.41}$$

$$\delta \tilde{F}^{ab} = \left(R^{ac}\tilde{\kappa}_c^{\ b} - R^{bc}\tilde{\kappa}^a_{\ c}\right) - \left(\tilde{F}^{ac}\rho^b_{\ c} - \tilde{F}^{bc}\rho^a_{\ c}\right) - \frac{1}{l}\bar{\epsilon}\gamma^{ab}\Psi, \tag{4.42}$$

$$\begin{aligned}\delta F^{ab} = & \left(R^{ac}\kappa_c^{\ b} - R^{bc}\kappa^a_{\ c}\right) - \left(F^{ac}\rho^b_{\ c} - F^{bc}\rho^a_{\ c}\right) - \left(\tilde{F}^{ac}\tilde{\kappa}^b_{\ c} - \tilde{F}^{ac}\tilde{\kappa}^a_{\ c}\right) \\ & + \frac{2}{l^2}R^a\rho^b - \frac{1}{l}\bar{\varrho}\gamma^{ab}\Psi - \frac{1}{l}\bar{\epsilon}\gamma^{ab}\Xi,\end{aligned} \tag{4.43}$$

$$\delta R^a = R^a_{\ b}\rho^b + R^b\rho_b^{\ a} + \bar{\epsilon}\gamma^a\Psi, \tag{4.44}$$

$$\delta\Psi = \frac{1}{4}R^{ab}\gamma_{ab}\epsilon - \frac{1}{4}\rho^{ab}\gamma_{ab}\Psi, \tag{4.45}$$

$$\delta\Xi = \frac{1}{4}R^{ab}\gamma_{ab}\varrho + \frac{1}{2l}R^a\gamma_a\epsilon - \frac{1}{2l}\rho^a\gamma_a\Psi - \frac{1}{4}\rho^{ab}\gamma_{ab}\Xi + \frac{1}{4}\tilde{F}^{ab}\gamma_{ab}\epsilon - \frac{1}{4}\tilde{\kappa}^{ab}\gamma_{ab}\Psi, \tag{4.46}$$

Although the MacDowell-Mansouri like action (4.30) is built from the $s\mathcal{M}_4$ curvature, it is not invariant under the $s\mathcal{M}_4$ gauge transformations. As we can see the action does not correspond to a Yang-Mills action, nor a topological invariant.

Furthermore, the action is not invariant under gauge supersymmetry. In fact, if we consider the variation of the action (4.30) under gauge supersymmetry, we find

$$\delta_{susy}S = -\frac{4}{l^2}\alpha_4\int R^a\bar{\Psi}\gamma_a\gamma_5\epsilon. \tag{4.47}$$

As in $\mathfrak{osp}\,(4|1)$ and super-Poincaré cases, the action is invariant under gauge supersymmetry imposing that the super torsion vanishes $R^a = 0$, leading to the supersymmetric action for the $s\mathcal{M}_4$ superalgebra in second order formalism.

Alternatively, it is possible to have supersymmetry in first order formalism if we modify the supersymmetry transformation for the spin connection $\omega^{ab}$. In fact, if we consider the variation of the action under an arbitrary $\delta\omega^{ab}$ we have that

$$\delta_\omega S = \frac{2}{l^2}\alpha_4\int \epsilon_{abcd}R^a e^b\delta\omega^{cd}, \tag{4.48}$$

and thus the variation vanishes for arbitrary $\delta\omega^{ab}$ if $R^a = 0$. It is possible to modify $\delta\omega^{ab}$ adding an extra piece to the gauge transformation such that

$$\delta S = -\frac{4}{l^2}\alpha_4\int R^a\left(\bar{\Psi}\gamma_a\gamma_5\epsilon - \frac{1}{2}\epsilon_{abcd}e^b\delta_{extra}\omega^{cd}\right). \tag{4.49}$$

In order to have an invariant action, $\delta_{extra}\omega^{ab}$ is given by

$$\delta_{extra}\omega^{ab} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_d\gamma_5\epsilon + \bar{\Psi}_{de}\gamma_c\gamma_5\epsilon - \bar{\Psi}_{cd}\gamma_e\gamma_5\epsilon\right)e^e, \tag{4.50}$$

with $\bar{\Psi} = \bar{\Psi}_{ab} e^a e^b$.

Then, the action in the first order formalism is invariant under the following supersymmetry transformations

$$\delta\omega^{ab} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_d\gamma_5\epsilon + \bar{\Psi}_{de}\gamma_c\gamma_5\epsilon - \bar{\Psi}_{cd}\gamma_e\gamma_5\epsilon\right)e^e, \tag{4.51}$$

$$\delta\tilde{k}^{ab} = -\frac{1}{l}\bar{\epsilon}\gamma^{ab}\psi, \tag{4.52}$$

$$\delta k^{ab} = -\frac{1}{l}\bar{\epsilon}\gamma^{ab}\xi, \tag{4.53}$$

$$\delta e^a = \bar{\epsilon}\gamma^a\psi, \tag{4.54}$$

$$\delta\psi = d\epsilon + \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon = D\epsilon, \tag{4.55}$$

$$\delta\xi = \frac{1}{2l}e^a\gamma_a\epsilon + \frac{1}{4}\tilde{k}^{ab}\gamma_{ab}\epsilon. \tag{4.56}$$

Note that there is a new supersymmetry related to the spinor charge $\Sigma$. The new supersymmetry transformations are given by

$$\delta\omega^{ab} = 0, \qquad \delta\tilde{k}^{ab} = 0, \tag{4.57}$$

$$\delta k^{ab} = -\frac{1}{l}\bar{\varrho}\gamma^{ab}\psi, \qquad \delta e^a = 0, \tag{4.58}$$

$$\delta\psi = 0, \qquad \delta\xi = d\varrho + \frac{1}{4}\omega^{ab}\gamma_{ab}\varrho. \tag{4.59}$$

Considering the variation of the action (4.30) under the new gauge supersymmetry transformations, we find that the action is truly invariant

$$\delta_\varrho S = 0. \tag{4.60}$$

Then one can see that the action is *off-shell* invariant under a subalgebra of $s\mathcal{M}_4$ given by $s\mathcal{L}_{\mathcal{M}_4} = \left\{J_{ab}, \tilde{Z}_{ab}, Z_{ab}, \Sigma_\alpha\right\}$ which corresponds to a Lorentz type superalgebra.

## 4.3 $D = 4$ Supergravity from $s\mathcal{M}_{m+2}$

In the previous chapter, we introduce the minimal Maxwell type superalgebra $s\mathcal{M}_{m+2}$ in $D = 4$. This superalgebra was obtained after a reduced resonant $S_E^{(2m)}$-expansion of the $\mathfrak{osp}(4|1)$ superalgebra, and its generators $\left\{J_{ab}, P_a, Z_{ab}^{(k)}, \tilde{Z}_{ab}^{(k)}, Z_a^{(l)}, \tilde{Z}_a^{(l)}, Q_\alpha, \Sigma_\alpha^{(p)}\right\}$ satisfy the (anti)-commutation relations (3.105) $-$ (3.133). In order to write down an action for this superalgebra, we will consider a more compact notation for the (anti)-commutation relations, namely

$$\left\{J_{ab,(k)}, P_{a,(l)}, Q_{\alpha,(p)}\right\}, \tag{4.61}$$

where these new generators can be written as

$$J_{ab,(k)} = \lambda_{2k}\tilde{J}_{ab}, \tag{4.62}$$
$$P_{a,(l)} = \lambda_{2l}\tilde{P}_{a}, \tag{4.63}$$
$$Q_{\alpha,(p)} = \lambda_{2p-1}\tilde{Q}_{\alpha}, \tag{4.64}$$

with $k = 0, \ldots, m$; $l = p = 1, \ldots, m$ and where $\tilde{J}_{ab}, \tilde{P}_a$ and $\tilde{Q}_\alpha$ are the $\mathfrak{osp}(4|1)$ generators. The new generators satisfy the commutation relations

$$\left[J_{ab,(k)}, J_{cd,(j)}\right] = \eta_{bc}J_{ad,(k+j)} - \eta_{ac}J_{bd,(k+j)} - \eta_{bd}J_{ac,(k+j)} + \eta_{ad}J_{bc,(k+j)}, \tag{4.65}$$
$$\left[J_{ab,(k)}, P_{a,(l)}\right] = \eta_{bc}P_{a,(k+l)} - \eta_{ac}P_{b,(k+l)}, \tag{4.66}$$
$$\left[P_{a,(l)}, P_{b,(n)}\right] = J_{ab,(l+n)}, \tag{4.67}$$
$$\left[J_{ab,(k)}, Q_{\alpha,(p)}\right] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_{\alpha,(k+p)}, \tag{4.68}$$
$$\left[P_{a,(l)}, Q_{\alpha,(p)}\right] = -\frac{1}{2}\left(\gamma_{a}Q\right)_{\alpha,(l+p)}, \tag{4.69}$$
$$\left\{Q_{\alpha,(p)}, Q_{\beta,(q)}\right\} = -\frac{1}{2}\left[\left(\gamma^{ab}C\right)_{\alpha\beta}J_{ab,(p+q)} - 2\left(\gamma^{a}C\right)_{\alpha\beta}P_{a,(p+q)}\right]. \tag{4.70}$$

Naturally, when $k+j > m$ the generators $T_A^{(k)}$ and $T_B^{(j)}$ are abelian. If we redefine the generators as

$$\begin{array}{ll}
J_{ab} = J_{ab,0} = \lambda_0\tilde{J}_{ab}, & P_a = P_{a,2} = \lambda_2\tilde{P}_a, \\
Z_{ab}^{(k)} = J_{ab,4k} = \lambda_{4k}\tilde{J}_{ab}, & Z_a^{(l)} = P_{a,4l+2} = \lambda_{4l+2}\tilde{P}_a, \\
\tilde{Z}_{ab}^{(k)} = J_{ab,4k-2} = \lambda_{4k-2}\tilde{J}_{ab}, & \tilde{Z}_a^{(l)} = P_{a,4l} = \lambda_{4l}\tilde{P}_a, \\
Q_\alpha = Q_{\alpha,1} = \lambda_1\tilde{Q}_\alpha, & \Sigma_\alpha^{(k)} = Q_{\alpha,4k-1} = \lambda_{4k-1}\tilde{Q}_\alpha, \\
\Phi_\alpha^{(l)} = Q_{\alpha,4k+1} = \lambda_{4k+1}\tilde{Q}_\alpha, &
\end{array}$$

we obtain the (anti)commutation relations (3.105) – (3.133).

In order to write down a Lagrangian for $s\mathcal{M}_{m+2}$, we start from the one-form gauge connection

$$A = \frac{1}{2}\sum_k \omega^{ab,(k)}J_{ab,(k)} + \frac{1}{l}\sum_l e^{a,(l)}P_{a,(l)} + \frac{1}{\sqrt{l}}\sum_p \psi^{\alpha,(p)}Q_{\alpha,(p)}, \tag{4.71}$$

where the different components are given by

$$\omega^{ab,(k)} = \lambda_{2k}\tilde{\omega}^{ab}, \tag{4.72}$$
$$e^{a,(l)} = \lambda_{2l}\tilde{e}^{a}, \tag{4.73}$$
$$\psi^{\alpha,(p)} = \lambda_{2p-1}\tilde{\psi}^{\alpha}, \tag{4.74}$$

in terms of $\tilde{e}^a, \tilde{\omega}^{ab}$ and $\tilde{\psi}$ which are the components of the $\mathfrak{osp}\,(4|1)$ connection.

The associated curvature two-form $F = dA + A \wedge A$ is

$$F = F^A T_A = \frac{1}{2}\sum_k \mathcal{R}^{ab,(k)} J_{ab,(k)} + \frac{1}{l}\sum_l R^{a,(l)} P_{a,(l)} + \frac{1}{\sqrt{l}}\sum_p \Psi^{\alpha,(p)} Q_{\alpha,(p)}, \tag{4.75}$$

where

$$\begin{aligned} \mathcal{R}^{ab,(k)} &= d\omega^{ab,(k)} + \omega_{\ c}^{a\ (i)} \wedge \omega^{cb,(j)}\delta_{i+j}^{k} + \frac{1}{l^2} e^{a,(l)} e^{b,(n)}\delta_{l+n}^{k} \\ &\quad + \frac{1}{2l}\bar{\psi}^{(p)}\gamma^{ab} \wedge \psi^{(q)}\delta_{p+q}^{2k}, \end{aligned} \tag{4.76}$$
$$R^{a,(l)} = de^{a,(l)} + \omega_{\ b}^{a\ (k)} \wedge e^{b,(n)}\delta_{k+n}^{l} - \frac{1}{2}\bar{\psi}^{(p)}\gamma^{a} \wedge \psi^{(q)}\delta_{p+q}^{2l}, \tag{4.77}$$
$$\Psi^{(p)} = d\psi^{(p)} + \frac{1}{4}\omega_{ab}^{\ \ (k)}\gamma^{ab} \wedge \psi^{(q)}\delta_{k+q}^{p} + \frac{1}{2l} e^{a,(l)}\gamma_{a} \wedge \psi^{(q)}\delta_{l+q}^{p}, \tag{4.78}$$

with $k = 0, \ldots, m$; $l = p = 1, \ldots, m$. Considering the Bianchi identity $\nabla F = 0$, where $\nabla = d + [A, \cdot]$, it is possible to show that

$$\begin{aligned} D\mathcal{R}^{ab,(k)} &= \left(\mathcal{R}^{ac,(i)}\omega_{c}^{\ b,(j+1)} - \mathcal{R}^{bc,(i)}\omega_{c}^{\ a,(j+1)}\right)\delta_{i+j+1}^{k} \\ &\quad + \frac{1}{l}\left(R^{a,(l)}e^{b,(n)} - e^{a,(n)}R^{b,(l)}\right)\delta_{l+n}^{k} - \frac{1}{l}\bar{\psi}^{(p)}\gamma^{ab}\Psi^{(q)}\delta_{p+q}^{2k}, \end{aligned} \tag{4.79}$$
$$DR^{a,(l)} = \mathcal{R}^{ab,(i)}e_{b}^{\ ,(j)}\delta_{i+j}^{l} + R^{c,(n)}\omega_{c}^{\ a,(j+1)}\delta_{n+j+1}^{l} + \bar{\psi}^{(p)}\gamma^{a}\Psi^{(q)}\delta_{p+q}^{2l}, \tag{4.80}$$
$$\begin{aligned} D\Psi^{(p)} &= \frac{1}{4}\left(\mathcal{R}^{ab,(i)}\gamma_{ab}\psi^{(q)}\right)\delta_{i+q}^{p} - \frac{1}{4}\left(\omega^{ab,(i+1)}\gamma_{ab}\Psi^{(q)}\right)\delta_{i+1+q}^{p} \\ &\quad + \frac{1}{2l}\left(T^{a,(l)}\gamma_{a}\psi^{(q)}\right)\delta_{l+q}^{p} - \frac{1}{2l}\left(e^{a,(l)}\gamma_{a}\Psi^{(q)}\right)\delta_{l+q}^{p}, \end{aligned} \tag{4.81}$$

where $D$ corresponds to the Lorentz covariant exterior derivative $D = d + [\omega, \cdot]$.

Then, the action can be written as

$$S = 2\int \langle F \wedge F \rangle = 2\int F^A \wedge F^B \langle T_A T_B \rangle_{s\mathcal{M}_{m+2}}, \tag{4.82}$$

where $\langle T_A T_B \rangle$ corresponds to the non-vanishing components of an $S$-expanded invariant tensor which is obtained from (2.18). Using Theorem VII.1 of [18] it is possible to show that these components are given by

$$\left\langle J_{ab,(k)} J_{cd,(j)} \right\rangle_{s\mathcal{M}_{m+2}} = \alpha_{2(k+j)} \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \right\rangle, \tag{4.83}$$

$$\left\langle Q_{\alpha,(p)} Q_{\beta,(q)} \right\rangle_{s\mathcal{M}_{m+2}} = \alpha_{2(p+q-1)} \left\langle \tilde{Q}_{\alpha} \tilde{Q}_{\beta} \right\rangle, \tag{4.84}$$

which can be written as

$$\left\langle J_{ab,(k)} J_{cd,(j)} \right\rangle_{s\mathcal{M}_{m+2}} = \alpha_{2(k+j)} \epsilon_{abcd}, \tag{4.85}$$

$$\left\langle Q_{\alpha,(p)} Q_{\beta,(q)} \right\rangle_{s\mathcal{M}_{m+2}} = 2\alpha_{2(p+q-1)} \left(\gamma_5\right)_{\alpha\beta}, \tag{4.86}$$

where the $\alpha$'s are arbitrary independent constants and $J_{ab,(k)}$, $Q_{\alpha,(p)}$ are given by (4.62) , (4.64), respectively. Using the different components of the invariant tensor (4.85) − (4.86) and the curvature two-form (4.75), we found that the action is given by

$$S = 2 \int \sum_{k,j} \frac{\alpha_{2(k+j)}}{2} \epsilon_{abcd} \mathcal{R}^{ab,(k)} \mathcal{R}^{cd,(j)} + \sum_{p,q} \alpha_{2(p+q-1)} \frac{4}{l} \bar{\Psi}^{(p)} \wedge \gamma_5 \Psi^{(q)}, \tag{4.87}$$

with $k, j = 0, \ldots, m$; $p, q = 1, \ldots, m$.

### 4.3.1 $s\mathcal{M}_{m+2}$ gauge transformations and supersymmetry

Using the multiplication law of the semigroup (3.88) and eq. (2.28) it is possible to show that the gauge transformations are given by

$$\begin{aligned}
\delta\omega^{ab,(k)} &= D\rho^{ab,(k)} - \left(\omega^{ac,(i+1)} \rho^{b\ ,(j)}_{\ c} - \omega^{bc,(i+1)} \rho^{a\ ,(j)}_{\ c}\right) \delta^{k}_{i+j+1} \\
&\quad + \frac{2}{l^2} e^{a,(l)} \rho^{b,(n)} \delta^{k}_{l+n} - \frac{1}{l} \bar{\epsilon}^{(p)} \gamma^{ab} \psi^{(q)} \delta^{2k}_{p+q},
\end{aligned} \tag{4.88}$$

$$\delta e^{a,(l)} = D\rho^{a,(l)} + \omega^{a\ ,(k+1)}_{\ b} \rho^{b,(n)} \delta^{l}_{k+n+1} + e^{b,(n)} \rho_{b}^{\ a,(k)} \delta^{l}_{n+k} + \bar{\epsilon}^{(p)} \gamma^{a} \psi^{(q)} \delta^{2l}_{p+q}, \tag{4.89}$$

$$\begin{aligned}
\delta\psi^{(p)} &= d\epsilon^{(p)} + \frac{1}{4} \omega^{ab,(k)} \gamma_{ab} \epsilon^{(q)} \delta^{p}_{k+q} + \frac{1}{2l} e^{a,(l)} \gamma_{a} \epsilon^{(q)} \delta^{p}_{l+q} \\
&\quad - \frac{1}{4} \rho^{ab,(k)} \gamma_{ab} \psi^{(q)} \delta^{p}_{k+q} - \frac{1}{2l} \rho^{a,(l)} \gamma_{a} \psi^{(q)} \delta^{p}_{l+q}.
\end{aligned} \tag{4.90}$$

where the $s\mathcal{M}_{m+2}$ gauge parameter is

$$\rho = \frac{1}{2} \sum_{k} \rho^{ab,(k)} J_{ab,(k)} + \frac{1}{l} \sum_{l} \rho^{a,(l)} P_{a,(l)} + \frac{1}{\sqrt{l}} \sum_{p} \epsilon^{\alpha,(p)} Q_{\alpha,(p)}, \tag{4.91}$$

and where we have written the components of the gauge parameter as an $S$-expansion of the component of the $\mathfrak{osp}(4|1)$ gauge parameter,

$$\rho^{ab,(k)} = \lambda_{2k} \tilde{\rho}^{ab}, \qquad \rho^{a,(l)} = \lambda_{2l} \tilde{\rho}^{a}, \qquad \epsilon^{\alpha,(p)} = \lambda_{2p-1} \tilde{\epsilon}^{\alpha},$$

with $k = 0, \ldots, m$; $l = p = 1, \ldots, m$ and $\lambda_\alpha \in S_E^{(2m)} = \{\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{2m+1}\}$.

In the same way, from the gauge variation of the curvature $\delta_\lambda F = [F, \lambda]$ , it is possible to show that the gauge transformations of the curvature $F$ are given by

$$\begin{aligned}
\delta \mathcal{R}^{ab,(k)} &= \left( \mathcal{R}^{ac,(i)} \rho_c^{\ b,(j)} - \mathcal{R}^{cb,(i)} \rho^{a\ ,(j)}_{\ c} \right) \delta^k_{i+j} + \frac{2}{l^2} R^{a,(l)} \rho^{b,(n)} \delta^k_{l+n} \\
&\quad - \frac{1}{l} \bar{\epsilon}^{(p)} \gamma^{ab} \Psi^{(q)} \delta^{2k}_{p+q}, && (4.92) \\
\delta R^{a,(l)} &= \mathcal{R}^{a\ ,(k)}_{\ b} \rho^{b,(n)} \delta^l_{k+n} + R^{b,(n)} \rho_b^{\ a,(k)} \delta^l_{k+n} + \bar{\epsilon}^{(p)} \gamma^a \Psi^{(q)} \delta^{2l}_{p+q}, && (4.93) \\
\delta \Psi^{(p)} &= \frac{1}{4} \mathcal{R}^{ab,(k)} \gamma_{ab} \epsilon^{(q)} \delta^p_{k+q} + \frac{1}{2l} R^{a,(l)} \gamma_a \epsilon^{(q)} \delta^p_{l+q} - \frac{1}{4} \rho^{ab,(k)} \gamma_{ab} \Psi^{(q)} \delta^p_{k+q} \\
&\quad - \frac{1}{2l} \rho^{a,(l)} \gamma_a \Psi^{(q)} \delta^p_{l+q}, && (4.94)
\end{aligned}$$

with $k = i = j = 0, \ldots, m$; $l = n = p = q = 1, \ldots, m$.

Although the Mac Dowell-Mansouri like action (4.87) is built from the $s\mathcal{M}_{m+2}$ curvature, it is not invariant under $s\mathcal{M}_{m+2}$ gauge transformations.

Moreover, the action is not invariant under gauge supersymmetry. In fact, if we consider the variation of the action (4.87) under gauge supersymmetry related to $Q_{(1)}$, we find

$$\delta_{susy} S = -\frac{4}{l^2} \int \sum_k \alpha_{2k} R^{a,(l)} \bar{\Psi}^{(p)} \gamma_a \gamma_5 \epsilon \delta^k_{l+p}, \tag{4.95}$$

with $k = 2, \ldots, m$; $l, p \geq 1$ and where $\epsilon$ is the gauge parameter associated to the spinor charge $Q_{(1)}$.

As in the previous case the action is invariant for every value of $k$ under gauge supersymmetry imposing the expanded super torsion constraint $R^{a,(l)} = 0$. This yields to express the expanded spin connection $\omega^{ab,(k)}$ in terms of the expanded fields as we can see in (4.77), leading to the supersymmetric action for the $s\mathcal{M}_{m+2}$ superalgebra in the second order formalism.

Alternatively, since the $\alpha$'s are arbitrary and independent we can study the supersymmetry in each term separately. Then if we consider the variation of the action proportional to $\alpha_{2k}$ under gauge supersymmetry transformations asociated to $Q_{(k-1)}$, we find

$$\delta_{susy} S = -\frac{4}{l^2} \alpha_{2k} \int R^a \bar{\Psi} \gamma_a \gamma_5 \epsilon^{(k-1)},$$

with $k = 2, \ldots, m$ and where $\epsilon^{(k-1)}$ is the gauge parameter associated to the spinor charge $Q_{(k-1)}$. Here $R^a$ and $\Psi$ correspond to $R^{a,(1)}$ and $\Psi^{(1)}$ respectively.

It is possible to have invariance under supersymmetry in first order formalism in every term if we modify the supersymmetry transformation for every expanded spin connection. In fact, if we consider the variation of the action under an arbitrary $\delta\omega^{ab,(k-2)}$ we find

$$\delta_\omega S = \frac{2}{l^2}\alpha_{2k}\int \epsilon_{abcd}R^a e^b \delta\omega^{cd,(k-2)}, \tag{4.96}$$

with $k = 2, \ldots, m$; $R^a = R^{a,(1)}$ and $e^a = e^{a,(1)}$. One can see that the variation vanishes for arbitrary $\delta\omega^{ab,(k-2)}$ if $R^a = 0$.

Nevertheless, it is possible to modify $\delta\omega^{ab,(k-2)}$ by adding an extra piece such that the variation of the action ($\sim \alpha_{2k}$) can be written as

$$\delta S = -\frac{4}{l^2}\alpha_{2k}\int R^a\left(\bar{\Psi}\gamma_a\gamma_5\epsilon^{(k-1)} - \frac{1}{2}\epsilon_{abcd}e^b\delta_{extra}\omega^{cd,(k-2)}\right). \tag{4.97}$$

Thus the transformation of the $\omega^{ab,(k-2)}$ field leaving the term proportional to $\alpha_{2k}$ invariant is

$$\delta_{extra}\omega^{ab,(k-2)} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_d\gamma_5\epsilon^{(k-1)} + \bar{\Psi}_{de}\gamma_c\gamma_5\epsilon^{(k-1)} - \bar{\Psi}_{cd}\gamma_e\gamma_5\epsilon^{(k-1)}\right)e^e,$$

with $\bar{\Psi} = \bar{\Psi}_{ab}e^a e^b$.

Note that the term proportional to $\alpha_{2k}$ is truly invariant under gauge supersymmetry transformations associated to $Q_{(q)}$, with $q \geq k$. Moreover, when $m = 2$ in $s\mathcal{M}_{m+2}$ we obtain the results presented in the previous section.

### 4.3.2 Pure supergravity from $s\mathcal{M}_{m+2}$

Since we are interested in obtaining the Einstein-Hilbert and the Rarita-Schwinger Lagrangians, we will consider only the terms proportional to $\alpha_4$. Then, the following choice for the non-vanishing components of an invariant tensor is requiered

$$\left\langle J_{ab,(0)}J_{cd,(4)}\right\rangle_{s\mathcal{M}_{m+2}} = \alpha_4\left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \tag{4.98}$$

$$\left\langle J_{ab,(2)}J_{cd,(2)}\right\rangle_{s\mathcal{M}_{m+2}} = \alpha_4\left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \tag{4.99}$$

$$\left\langle Q_{\alpha,(1)}Q_{\beta,(3)}\right\rangle_{s\mathcal{M}_{m+2}} = \alpha_4\left\langle \tilde{Q}_\alpha\tilde{Q}_\beta\right\rangle, \tag{4.100}$$

which can be expressed as

$$\left\langle J_{ab,(0)}J_{cd,(4)}\right\rangle_{s\mathcal{M}_{m+2}} = \alpha_4\epsilon_{abcd}, \tag{4.101}$$

$$\left\langle J_{ab,(2)}J_{cd,(2)}\right\rangle_{s\mathcal{M}_{m+2}} = \alpha_4\epsilon_{abcd}, \tag{4.102}$$

$$\left\langle Q_{\alpha,(1)}Q_{\beta,(3)}\right\rangle_{s\mathcal{M}_{m+2}} = 2\alpha_4\left(\gamma_5\right)_{\alpha\beta}. \tag{4.103}$$

Thus, we only have to consider curvatures the two-form associated to the $J_{ab,(0)}$, $J_{ab,(2)}$, $J_{(ab,4)}$, $Q_{\alpha,(1)}$ and $Q_{\alpha,(3)}$ generators, which can be derived from (4.76) – (4.78)

Considering the non-vanishing components of the invariant tensor and the respective curvatures two-form we obtain the following action for the $S$-expanded superalgebra

$$S = 2\alpha_4 \int \left( \frac{1}{2}\epsilon_{abcd}\mathcal{R}^{ab,(0)}\mathcal{R}^{cd,(4)} + \frac{1}{4}\epsilon_{abcd}\mathcal{R}^{ab,(2)}\mathcal{R}^{cd,(2)} + \frac{4}{l}\bar{\Psi}^{(3)} \wedge \gamma_5 \Psi^{(1)} \right), \tag{4.104}$$

which can be written explicitly as follows

$$\begin{aligned} S = \alpha_4 \int & \epsilon_{abcd}\frac{1}{l^2}\left(\mathcal{R}^{ab,(0)}e^{c,(2)}e^{d,(2)} + 4\bar{\psi}^{(1)}e^{a,(2)}\gamma_a\gamma_5 D\psi^{(1)}\right) \\ & + d\left(\epsilon_{abcd}\left(\mathcal{R}^{ab,(0)}\omega^{ab,(4)} + \frac{1}{2}D\omega^{ab,(2)}\omega^{cd,(2)}\right)\right. \\ & \left. + \frac{8}{l}D\bar{\psi}^{(1)}\gamma_5\psi^{(3)} + \frac{1}{l}\bar{\psi}^{(1)}\omega^{ab,(2)}\gamma_{ab}\gamma_5\psi^{(1)}\right) \end{aligned} \tag{4.105}$$

Here we have used the gravitino Bianchi identity $D\Psi^{(1)} = \frac{1}{4}R^{ab}\gamma_{ab}\Psi^{(1)}$ and the matrix gamma identity (4.29) to show that

$$\begin{aligned} \epsilon_{abcd}\mathcal{R}^{ab,(0)}\bar{\psi}^{(3)}\gamma^{cd}\psi^{(1)} + 8D\bar{\psi}^{(1)}\gamma_5 D\psi^{(3)} &= D\left(8D\bar{\psi}^{(1)}\gamma_5\psi^{(3)}\right) \\ \frac{1}{2}\epsilon_{abcd}D\omega^{ab,(2)}\bar{\psi}^{(1)}\gamma^{cd}\psi^{(1)} + 2\bar{\psi}^{(1)}\omega^{ab,(2)}\gamma_{ab}\gamma_5 D\psi^{(1)} &= D\left(\bar{\psi}^{(1)}\omega^{ab,(2)}\gamma_{ab}\gamma_5\psi^{(1)}\right). \end{aligned}$$

Then, using the following identification

$$\begin{aligned} \omega^{ab,(0)} &= \omega^{ab}, \quad \omega^{ab,(2)} = \tilde{k}^{ab}, \\ \omega^{ab,(4)} &= k^{ab}, \quad e^{a,(2)} = e^a, \\ \mathcal{R}^{ab,(0)} &= R^{ab}, \quad \psi^{(1)} = \psi, \\ \psi^{(3)} &= \xi, \end{aligned}$$

the action is given by

$$\begin{aligned} S = \alpha_4 \int & \epsilon_{abcd}\frac{1}{l^2}\left(R^{ab}e^c e^d + 4\bar{\psi}e^a\gamma_a\gamma_5 D\psi\right) \\ & + d\left(\epsilon_{abcd}\left(R^{ab}k^{cd} + \frac{1}{2}D_\omega\tilde{k}^{ab}\tilde{k}^{cd}\right) + \frac{8}{l}\bar{\xi}\gamma_5 D\psi + \frac{1}{l}\bar{\psi}\tilde{k}^{ab}\gamma_{ab}\gamma_5\psi\right). \end{aligned} \tag{4.106}$$

Here, we can see that the action proportional to $\alpha_4$ contains the Einstein-Hilbert term $\epsilon_{abcd}R^{ab}e^c e^d$, the Rarita-Schwinger Lagrangian $4\bar{\psi}e^a\gamma_a\gamma_5 D\psi$ and a boundary term involving the new fields $k_{ab}$, $\tilde{k}_{ab}$, $\xi$ and the original ones.

Unlike the Mac Dowell-Mansouri Lagrangian for the $\mathfrak{osp}\,(4|1)$ superalgebra the supersymmetric cosmological constant does not appear explicitly in this action. This is due to the $S$-expansion procedure since if we want to obtain the supersymmetric cosmological constant

$$\frac{1}{2l^4}e^a e^b e^c e^d + \frac{1}{l^3}\bar{\psi}\gamma^{ab}\psi e^c e^d$$

in the action, it should be necessary to consider the components $\left\langle J_{ab,(4)} J_{cd,(4)} \right\rangle$ and $\left\langle J_{ab,(2)} J_{cd,(4)} \right\rangle$ which are proportional to $\alpha_8$ and $\alpha_6$, respectively.

Regardless of the number of new generators of the Maxwell type superalgebra, the new Maxwell fields do not contribute to the dynamics of the term proportional to $\alpha_4$. In this way, we have shown that $\mathcal{N}=1$, $D=4$ pure supergravity can be obtained as a Mac Dowell-Mansouri like action for the minimal Maxwell superalgebras $s\mathcal{M}_{m+2}$ (with $m>1$).

$$S=\alpha_4 \int \frac{1}{l^2}\left[\epsilon_{abcd}R^{ab}e^c e^d + 4\bar{\psi}e^a\gamma_a\gamma_5 D\psi\right] + \text{ boundary terms.} \tag{4.107}$$

It is important to note that the $m=1$ case corresponds to the Poincaré superalgebra $s\mathcal{P}=\{J_{ab}, P_a, Q_\alpha\}$. Nevertheless, in this case we cannot derive the pure supergravity action as a Mac Dowell-Mansouri like action since it is not possible to obtain the Eintein-Hilbert term from $\langle J_{ab}J_{cd}\rangle$ for $s\mathcal{P}$.

In sumary, we have derived the minimal $D=4$ supergravity action from the minimal Maxwell type superalgebra $s\mathcal{M}_4$. The action was constructed in geometrical terms as the Mac Dowell-Mansouri like action and interestingly describes pure supergravity. Then we have obtained the minimal supergravity action in four dimensions from the $s\mathcal{M}_{m+2}$ superalgebra. The invariance under supersymmetry was also discussed.

# Chapter 5

# $\mathcal{N}=1$, $D=4$ Supergravity with supersymmetric cosmological term

## 5.1 Introduction

It is well-known that a cosmological constant can be introduced in gravity when we consider the (A)dS algebra instead of Poincaré. As was pointed out in [63], [64] the presence of a cosmological constant seems to be an interesting alternative in order to describe the dark energy. Furthermore, the supersymmetric extension of gravity including a cosmological constant can be derived in geometrical terms from the AdS superalgebra. As we have seen in a previous chapter, in this approach the theory is built in terms of the $\mathfrak{osp}\,(4|1)$ curvature and the action is known as the Mac Dowell-Mansouri action [28].

Recently, an alternative way of introducing a generalized cosmological constant term in gravity was proposed using the Maxwell symmetries [55]. Moreover, the deformations of these symmetries lead to the $\mathfrak{so}(D-1,2)\oplus\mathfrak{so}(D-1,1)$ algebra [65], [66]. In [66] this algebra was found as a semi-simple extension of the Poincaré algebra. From now on we will refer to this algebra as the AdS-Lorentz ($AdS-\mathcal{L}_4$) algebra.

The $AdS-\mathcal{L}_4$ algebra (and its generalizations) has been extensively studied in [16]. In particular, it was shown that a generalized cosmological constant can be included in a four-dimensional Born-Infeld like action constructed out from the curvature 2-form of the AdS-Lorentz algebra. Interestingly, this algebra can also be obtained as an abelian semigroup expansion ($S$-expansion) of the AdS algebra [67].

In this chapter we analyze the physical consequences of considering the supersymmetric extension of the AdS-Lorentz algebra in the construction of a minimal supergravity theory. Following [68] we present an alternative way of introducing the supersymmetric cosmological constant to supergravity. Based on the AdS-Lorentz superalgebra we build the minimal $D = 4$ supergravity action which includes a generalized supersymmetric cosmological constant term.

## 5.2 AdS-Lorentz superalgebra

Following [18], [69] in this section we will show the procedure to obtain the AdS-Lorentz superalgebra as an $S$-expansion of the $\mathfrak{osp}\,(4|1)$ superalgebra using $S_{\mathcal{M}}^{(2)}$ as the abelian semigroup.

Before applying the $S$-expansion method it is necessary to consider a decomposition of the original algebra in subspaces $\mathfrak{g} = \mathfrak{osp}\,(4|1) = V_0 \oplus V_1 \oplus V_2$ , where $V_0$ is generated by the Lorentz generator $\tilde{J}_{ab}$, $V_1$ corresponds to the fermionic subspace generated by a 4-component Majorana spinor charge $\tilde{Q}_\alpha$ and $V_2$ corresponds to the $AdS$ boost generated by $\tilde{P}_a$. These generators satisfy the (anti)commutation relations given by $(2.4) - (2.8)$.

The subspace structure can be written as

$$\begin{aligned} &[V_0, V_0] \subset V_0, \quad [V_1, V_1] \subset V_0 \oplus V_2, \\ &[V_0, V_1] \subset V_1, \quad [V_1, V_2] \subset V_1, \\ &[V_0, V_2] \subset V_2, \quad [V_2, V_2] \subset V_0. \end{aligned} \tag{5.1}$$

Let $S_{\mathcal{M}}^{(2)} = \{\lambda_0, \lambda_1, \lambda_2\}$ be an abelian semigroup whose elements satisfy the multiplication law,

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leq 2 \\ \lambda_{\alpha+\beta-2}, & \text{when } \alpha+\beta > 2 \end{cases} \tag{5.2}$$

Let us consider the subset decomposition $S_{\mathcal{M}}^{(2)} = S_0 \cup S_1 \cup S_2$, with

$$S_0 = \{\lambda_0, \lambda_2\}\,, \tag{5.3}$$

$$S_1 = \{\lambda_1\}\,, \tag{5.4}$$

$$S_2 = \{\lambda_2\}\,. \tag{5.5}$$

One sees that this decomposition is said to be resonant since it satisfies the same structure as the subspaces $V_p$ [compare with eqs (5.1)]

$$\begin{aligned} &S_0 \cdot S_0 \subset S_0, \quad S_1 \cdot S_1 \subset S_0 \cap S_2, \\ &S_0 \cdot S_1 \subset S_1, \quad S_1 \cdot S_2 \subset S_1, \\ &S_0 \cdot S_2 \subset S_2, \quad S_2 \cdot S_2 \subset S_0. \end{aligned} \tag{5.6}$$

Following theorem IV.2 of [18], we can say that the superalgebra

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{5.7}$$

is a resonant subalgebra of $S_{\mathcal{M}}^{(2)} \times \mathfrak{g}$, where

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_2\} \times \left\{ \tilde{J}_{ab} \right\} = \left\{ \lambda_0 \tilde{J}_{ab}, \lambda_2 \tilde{J}_{ab} \right\}, \tag{5.8}$$

$$W_1 = (S_1 \times V_1) = \{\lambda_1\} \times \left\{ \tilde{Q}_\alpha \right\} = \left\{ \lambda_1 \tilde{Q}_\alpha \right\}, \tag{5.9}$$

$$W_2 = (S_2 \times V_2) = \{\lambda_2\} \times \left\{ \tilde{P}_a \right\} = \left\{ \lambda_2 \tilde{P}_a \right\}. \tag{5.10}$$

Thus, we obtain a new superalgebra generated by $\{J_{ab}, P_a, Z_{ab}, Q_\alpha\}$. These new generators can be written as

$$\begin{aligned} J_{ab} &= \lambda_0 \tilde{J}_{ab}, \qquad P_a = \lambda_2 \tilde{P}_a \\ Z_{ab} &= \lambda_2 \tilde{J}_{ab}, \qquad Q_\alpha = \lambda_1 \tilde{Q}_\alpha, \end{aligned}$$

and satisfy the following (anti)commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{5.11}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{5.12}$$

$$[Z_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{5.13}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{5.14}$$

$$[Z_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \tag{5.15}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2} \left( \gamma_{ab} Q \right)_\alpha, \qquad [P_a, Q_\alpha] = -\frac{1}{2} \left( \gamma_a Q \right)_\alpha, \tag{5.16}$$

$$[Z_{ab}, Q_\alpha] = -\frac{1}{2} \left( \gamma_{ab} Q \right)_\alpha, \tag{5.17}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2} \left[ \left( \gamma^{ab} C \right)_{\alpha\beta} Z_{ab} - 2 \left( \gamma^a C \right)_{\alpha\beta} P_a \right], \tag{5.18}$$

where we have used the multiplication law of the semigroup (5.2) and the commutation relations of the $\mathfrak{osp}(4|1)$ superalgebra. The new superalgebra obtained after a resonant

$S_{\mathcal{M}}^{(2)}$-expansion of $\mathfrak{osp}\,(4|1)$ corresponds to the AdS-Lorentz superalgebra in four dimensions, which will be denote as $sAdS-\mathcal{L}_4$ .

From the above relations we see that the AdS-Lorentz superalgebra contains the $AdS-\mathcal{L}_4$ algebra $=\{J_{ab}, P_a, Z_{ab}\}$[1] as a subalgebra. Unlike the Maxwell superalgebra the $Z_{ab}$ generators are not abelian and behave as a Lorentz generator.

On the other hand, it is well known that an Inönü-Wigner contraction of the AdS-Lorentz superalgebra leads to the non-standard Maxwell superalgebra [70]. In fact, the rescaling

$$Z_{ab} \rightarrow \mu^2 Z_{ab}, \quad P_a \rightarrow \mu P_a \text{ and } Q_\alpha \rightarrow \mu Q_\alpha \tag{5.19}$$

provides the Maxwell superalgebra in the limit $\mu \rightarrow \infty$,

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, \tag{5.20}$$
$$[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, \tag{5.21}$$
$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{5.22}$$
$$[Z_{ab}, P_c] = 0, \qquad [Z_{ab}, Z_{cd}] = 0 \tag{5.23}$$
$$[J_{ab}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha, \tag{5.24}$$
$$[Z_{ab}, Q_\alpha] = 0, \qquad [P_a, Q_\alpha] = 0 \tag{5.25}$$
$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left(\gamma^{ab}C\right)_{\alpha\beta} Z_{ab}, \tag{5.26}$$

## 5.3 Supergravity action for $sAdS-\mathcal{L}_4$

In this section, we present a geometric formulation of $\mathcal{N}=1$ supergravity in four dimensions, where the relevant gauge fields of the theory are those corresponding to the AdS-Lorentz superalgebra $sAdS-\mathcal{L}_4$. The action will be constructed exclusively in terms of the curvature 2-form following the same approach of [28], and using the useful properties of the $S$-expansion procedure.

The one-form connection is given by

$$A = A^A T_A = \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{l}e^a P_a + \frac{1}{2}k^{ab}Z_{ab} + \frac{1}{\sqrt{l}}\psi^\alpha Q_\alpha, \tag{5.27}$$

[1] Also known as Poincaré semi-simple extended algebra.

where the one-form gauge fields are given in terms of the AdS fields $\left\{\tilde{\omega}^{ab}, \tilde{e}^{a}, \tilde{\psi}^{\alpha}\right\}$,

$$\begin{aligned} \omega^{ab} &= \lambda_0 \tilde{\omega}^{ab}, \qquad & k^{ab} &= \lambda_2 \tilde{\omega}^{ab}, \\ e^a &= \lambda_2 \tilde{e}^a, \qquad & \psi^\alpha &= \lambda_1 \tilde{\psi}^\alpha. \end{aligned}$$

The associated curvature two-form $F = dA + A \wedge A$ is given by

$$F = F^A T_A = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} R^a P_a + \frac{1}{2} F^{ab} Z_{ab} + \frac{1}{\sqrt{l}} \Psi^\alpha Q_\alpha, \tag{5.28}$$

where

$$R^{ab} = d\omega^{ab} + \omega^a_{\ c} \omega^{cb}, \tag{5.29}$$

$$R^a = de^a + \omega^a_{\ b} e^b + k^a_{\ b} e^b - \frac{1}{2} \bar{\psi} \gamma^a \psi, \tag{5.30}$$

$$F^{ab} = dk^{ab} + \omega^a_{\ c} k^{cb} - \omega^b_{\ c} k^{ca} + k^a_{\ c} k^{cb} + \frac{1}{l^2} e^a e^b + \frac{1}{2l} \bar{\psi} \gamma^{ab} \psi, \tag{5.31}$$

$$\Psi = d\psi + \frac{1}{4} \omega_{ab} \gamma^{ab} \psi + \frac{1}{2l} e^a \gamma_a \psi + \frac{1}{4} k_{ab} \gamma^{ab} \psi. \tag{5.32}$$

Let us note that the presence of the generator $Z_{ab}$ implies the introduction of a bosonic "matter" field $k^{ab}$, which modifies the definition of the curvatures.

From the Bianchi identity $\nabla F = 0$, with $\nabla = d + [A, \cdot]$, we can write the Lorentz covariant exterior derivatives of the curvatures as

$$DR^{ab} = 0, \tag{5.33}$$

$$DR^a = R^a_{\ b} e^b + F^a_{\ b} e^b + R^c k_c^{\ a} + \bar{\psi} \gamma^a \Psi, \tag{5.34}$$

$$\begin{aligned} DF^{ab} &= R^a_{\ c} k^{cb} - R^b_{\ c} k^{ca} + F^a_{\ c} k^{cb} - F^b_{\ c} k^{ca} + \frac{1}{l^2} \left( R^a e^b - e^a R^b \right) \\ &\quad + \frac{1}{l} \bar{\Psi} \gamma^{ab} \psi, \end{aligned} \tag{5.35}$$

$$\begin{aligned} D\Psi &= \frac{1}{4} R_{ab} \gamma^{ab} \psi + \frac{1}{4} F_{ab} \gamma^{ab} \psi - \frac{1}{4} k_{ab} \gamma^{ab} \Psi + \frac{1}{2l} R^a \gamma_a \psi \\ &\quad - \frac{1}{2l} e^a \gamma_a \Psi. \end{aligned} \tag{5.36}$$

The MacDowell-Mansouri like action for the AdS-Lorentz superalgebra can be written as

$$S = 2 \int F^A \wedge F^B \left\langle T_A T_B \right\rangle_{sAdS-\mathcal{L}_4}, \tag{5.37}$$

where $\left\langle T_A T_B \right\rangle_{sAdS-\mathcal{L}_4}$ are non-vanishing components of an invariant tensor which can be derived from the components of the invariant tensor (2.18). In fact, using Theorem VII.1

of [18], it is possible to show that the non-vanishing components of $\langle T_A T_B \rangle_{sAdS-\mathcal{L}_4}$ are given by

$$\langle J_{ab} J_{cd} \rangle_{sAdS-\mathcal{L}_4} = \alpha_0 \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \right\rangle , \tag{5.38}$$

$$\langle J_{ab} Z_{cd} \rangle_{sAdS-\mathcal{L}_4} = \alpha_2 \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \right\rangle , \tag{5.39}$$

$$\langle Z_{ab} Z_{cd} \rangle_{sAdS-\mathcal{L}_4} = \alpha_2 \left\langle \tilde{J}_{ab} \tilde{J}_{cd} \right\rangle , \tag{5.40}$$

$$\langle Q_{\alpha} Q_{\beta} \rangle_{sAdS-\mathcal{L}_4} = \alpha_2 \left\langle \tilde{Q}_{\alpha} \tilde{Q}_{\beta} \right\rangle , \tag{5.41}$$

where $\alpha_0$ and $\alpha_2$ are dimensionless arbitrary constants and

$$\left\langle \tilde{J}_{ab} \tilde{J}_{cd} \right\rangle = \epsilon_{abcd}, \tag{5.42}$$

$$\left\langle \tilde{Q}_{\alpha} \tilde{Q}_{\beta} \right\rangle = 2 \left( \gamma_5 \right)_{\alpha\beta} , \tag{5.43}$$

are the invariant tensors requiered to reproduce the MacDowell-Mansouri action for the $\mathfrak{osp}\,(4|1)$ superalgebra (see Chapter 2, Section 2.3). This choice of the invariant tensor breaks the AdS-Lorentz supergroup to their Lorentz like subgroup.

Then considering the invariant tensors (5.38) − (5.41) and the curvature 2-form (5.28), it is possible to write down the action à la Mac Dowell-Mansouri as follows

$$S = 2 \int \left( \frac{1}{4} \alpha_0 \epsilon_{abcd} R^{ab} R^{cd} + \frac{1}{2} \alpha_2 \epsilon_{abcd} R^{ab} F^{cd} + \frac{1}{4} \alpha_2 \epsilon_{abcd} F^{ab} F^{cd} + \frac{2}{l} \alpha_2 \bar{\Psi} \gamma_5 \Psi \right) . \tag{5.44}$$

or explicitly,

$$\begin{aligned} S = \int & \frac{\alpha_0}{2} \epsilon_{abcd} R^{ab} R^{cd} + \alpha_2 \epsilon_{abcd} \bigg( R^{ab} D k^{cd} + R^{ab} k^{c}_{\ e} k^{ed} + \frac{1}{l^2} R^{ab} e^c e^d \\ & + \frac{1}{2l} R^{ab} \bar{\psi} \gamma^{cd} \psi + \frac{1}{2} D k^{ab} D k^{cd} + D k^{ab} k^{c}_{\ e} k^{ed} + \frac{1}{l^2} D k^{ab} e^c e^d \\ & + \frac{1}{2l} D k^{ab} \bar{\psi} \gamma^{cd} \psi + \frac{1}{2} k^{a}_{\ f} k^{fb} k^{c}_{\ g} k^{gd} + \frac{1}{l^2} k^{a}_{\ f} k^{fb} e^c e^d + \frac{1}{2l} k^{a}_{\ f} k^{fb} \bar{\psi} \gamma^{cd} \psi \bigg) \\ & + \frac{1}{2l^3} e^a e^b \bar{\psi} \gamma^{cd} \psi + \frac{1}{2l^4} e^a e^b e^c e^d \bigg) + \alpha_2 \bigg( \frac{4}{l} D \bar{\psi} \gamma_5 D \psi + \frac{4}{l^2} \bar{\psi} e^a \gamma_a \gamma_5 D \psi \\ & + \frac{2}{l} D \bar{\psi} \gamma_5 k_{ab} \gamma^{ab} \psi + \frac{1}{l^3} \bar{\psi} e^a \gamma_a \gamma_5 e^b \gamma_b \psi + \frac{1}{l^2} \bar{\psi} e^a \gamma_a \gamma_5 k^{bc} \gamma_{bc} \psi \\ & + \frac{1}{4l} \bar{\psi} k_{ab} \gamma^{ab} \gamma_5 k_{cd} \gamma^{cd} \psi \bigg) . \end{aligned} \tag{5.45}$$

The action can be written in a more compact way using the gamma matrix identity $(C.1)$

$$\gamma_{ab}\gamma_5 = -\frac{1}{2}\epsilon_{abcd}\gamma^{cd}, \tag{5.46}$$

and the gravitino Bianchi identity

$$DD\psi = \frac{1}{4}R^{ab}\gamma_{ab}\psi, \tag{5.47}$$

to show the following relations

$$\frac{1}{2}\epsilon_{abcd}R^{ab}\bar{\psi}\gamma^{cd}\psi + 4D\bar{\psi}\gamma_5 D\psi = d\left(4D\bar{\psi}\gamma_5\psi\right), \tag{5.48}$$

$$\frac{1}{2}\epsilon_{abcd}Dk^{ab}\bar{\psi}\gamma^{cd}\psi + 2D\bar{\psi}\gamma_5 k^{ab}\gamma_{ab}\psi = d\left(\bar{\psi}k^{ab}\gamma_{ab}\gamma_5\psi\right). \tag{5.49}$$

Furthermore,

$$\bar{\psi}e^a\gamma_a\gamma_5 e^b\gamma_b\psi = \frac{1}{2}e^a e^b\bar{\psi}\gamma^{cd}\psi\epsilon_{abcd}, \tag{5.50}$$

$$\frac{1}{4}\bar{\psi}k_{ab}\gamma^{ab}\gamma_5 k_{cd}\gamma^{cd}\psi = -\frac{1}{2}k^a{}_f k^{fb}\bar{\psi}\gamma^{cd}\psi\epsilon_{abcd}, \tag{5.51}$$

$$\bar{\psi}e^a\gamma_a\gamma_5 k^{bc}\gamma_{bc}\psi = \epsilon_{abcd}k^{ab}e^c\bar{\psi}\gamma^d\psi, \tag{5.52}$$

where we have used the identities $(C.2) - (C.5)$ and that $\gamma_5\gamma_a$ is an antisymmetric matrix.

Thus the MacDowell-Mansouri like action for the $sAdS - \mathcal{L}_4$ superalgebra is

$$\begin{aligned} S = \int & \frac{\alpha_0}{2}\epsilon_{abcd}R^{ab}R^{cd} + \frac{\alpha_2}{l^2}\left(\epsilon_{abcd}R^{ab}e^c e^d + 4\bar{\psi}e^a\gamma_a\gamma_5 D\psi\right) \\ & + \alpha_2\epsilon_{abcd}\left(R^{ab}Dk^{cd} + R^{ab}k^c{}_e k^{ed} + \frac{1}{2}Dk^{ab}Dk^{cd} + Dk^{ab}k^c{}_e k^{ed} + \frac{1}{2}k^a{}_f k^{fb}k^c{}_g k^{gd}\right) \\ & + \alpha_2\epsilon_{abcd}\left(\frac{1}{l^2}Dk^{ab}e^c e^d + \frac{1}{l^2}k^a{}_f k^{fb}e^c e^d + \frac{1}{l^3}e^a e^b\bar{\psi}\gamma^{cd}\psi\right. \\ & \left. + \frac{1}{l^2}k^{ab}e^c\bar{\psi}\gamma^d\psi + \frac{1}{2l^4}e^a e^b e^c e^d\right) + \alpha_2 d\left(4D\bar{\psi}\gamma_5\psi + \bar{\psi}k^{ab}\gamma_{ab}\gamma_5\psi\right). \end{aligned} \tag{5.53}$$

We have separated the action in five pieces in order to analyze each one of them. The first term is proportional to $\alpha_0$ and corresponds to the Gauss Bonnet term. The second term contains the Einstein-Hilbert term plus the Rarita-Schwinger Lagrangian. The third piece corresponds to a Gauss Bonnet like term and does not contribute to the dynamics because it can be written as a boundary term. The fourth term corresponds to a generalized supersymmetric cosmological term which contains the usual supersymmetric cosmological constant plus three additional terms depending on the field $k^{ab}$. The last piece is a boundary term.

Then, the action written à la MacDowell-Mansouri for the AdS-Lorentz superalgebra describes a supergravity theory with a generalized supersymmetric cosmological term. From (5.53) we can see that the bosonic part of the action corresponds to the one found in [16] for AdS-Lorentz algebra. Moreover, the action contains the generalized cosmological term introduced in [55] for the Maxwell algebra.

Neglecting boundary terms, the action can be written as

$$
\begin{aligned}
S = \int \frac{\alpha_2}{l^2}\left(\epsilon_{abcd}R^{ab}e^c e^d + 4\bar{\psi}e^a\gamma_a\gamma_5 D\psi\right) + \alpha_2\epsilon_{abcd}\Bigg(\frac{1}{l^2}Dk^{ab}e^c e^d + \frac{1}{l^2}k^a{}_f k^{fb}e^c e^d \\
+\frac{1}{l^3}e^a e^b\bar{\psi}\gamma^{cd}\psi + \frac{1}{l^2}k^{ab}e^c\bar{\psi}\gamma^d\psi + \frac{1}{2l^4}e^a e^b e^c e^d\Bigg),
\end{aligned} \tag{5.54}
$$

and using that

$$
\begin{aligned}
\epsilon_{abcd}Dk^{ab}e^c e^d &= 2\epsilon_{abcd}k^{ab}T^c e^d + d\left(\frac{1}{l^2}\epsilon_{abcd}k^{ab}e^c e^d\right), \\
\hat{T}^a &\equiv De^a - \frac{1}{2}\bar{\psi}\gamma^a\psi = T^a - \frac{1}{2}\bar{\psi}\gamma^a\psi,
\end{aligned} \tag{5.55}
$$

it can be rewritten as follows

$$
\begin{aligned}
S = \int \frac{\alpha_2}{l^2}&\left(\epsilon_{abcd}R^{ab}e^c e^d + 4\bar{\psi}e^a\gamma_a\gamma_5 D\psi\right) \\
&+ \alpha_2\epsilon_{abcd}\left(\frac{2}{l^2}k^{ab}\hat{T}^c e^d + \frac{1}{l^2}k^a{}_f k^{fb}e^c e^d + \frac{1}{l^3}e^a e^b\bar{\psi}\gamma^{cd}\psi + \frac{1}{2l^4}e^a e^b e^c e^d\right).
\end{aligned} \tag{5.56}
$$

## 5.4 The equations of motion of $D = 4$, $\mathcal{N} = 1$ AdS-Lorentz supergravity

Let us find the equations of motion associated to the four independent space-time fields $\omega^{ab}$, $k^{ab}$, $e^a$ and $\psi$. The variation of the Lagrangian with respect to the spin connection $\omega^{ab}$ yields (modulo boundary terms)

$$
\begin{aligned}
\delta_\omega\mathcal{L} &= \frac{\alpha_2}{l^2}\epsilon_{abcd}\left(2\delta\omega^{ab}De^c e^d + 2\delta\omega^a{}_f k^{fb}e^c e^d\right) + \frac{\alpha_2}{l^2}\bar{\psi}e^a\gamma_a\gamma_5\delta\omega^{cd}\gamma_{cd}\psi \\
&= \frac{2\alpha_2}{l^2}\epsilon_{abcd}\delta\omega^{ab}\left(T^c + k^c{}_f e^f - \frac{1}{2}\bar{\psi}\gamma^c\psi\right)e^d \\
&= \frac{2\alpha_2}{l^2}\epsilon_{abcd}\delta\omega^{ab}R^c e^d.
\end{aligned} \tag{5.57}
$$

Thus, for arbitrary $\delta\omega^{ab}$ we have that $\delta_\omega\mathcal{L}=0$ leads to the following field equation

$$2\epsilon_{abcd}R^c e^d = 0. \tag{5.58}$$

Considering now the variation of the Lagrangian with respect to the vielbein $e^a$, we found

$$\begin{aligned}\delta_e\mathcal{L} &= \frac{\alpha_2}{l^2}\epsilon_{abcd}\left(2R^{ab}e^c + 2Dk^{ab}e^c + 2k^a{}_f k^{fb}e^c + \frac{2}{l}\bar{\psi}\gamma^{ab}\psi e^c + \frac{2}{l^2}e^a e^b e^c\right)\delta e^d \\ &+ \frac{\alpha_2}{l^2}\left(4\bar{\psi}\gamma_d\gamma_5 D\psi + \bar{\psi}\gamma_d\gamma_5 k^{ab}\gamma_{ab}\psi\right)\delta e^d. \\ &= \frac{2\alpha_2}{l^2}\epsilon_{abcd}\left(R^{ab}e^c + F^{ab}e^c\right)\delta e^d + \frac{\alpha_2}{l^2}\left(4\bar{\psi}\gamma_d\gamma_5\Psi\right)\delta e^d,\end{aligned} \tag{5.59}$$

where we have used the AdS-Lorentz curvatures 2-form (5.28) and eqs.(5.50) $-$ (5.51). Then the field equation is obtained imposing $\delta_e\mathcal{L}=0$

$$2\epsilon_{abcd}\left(R^{ab}+F^{ab}\right)e^c + 4\bar{\psi}\gamma_d\gamma_5\Psi = 0. \tag{5.60}$$

The variation of the Lagrangian with respect to the new AdS-Lorentz field $k^{ab}$ gives

$$\begin{aligned}\delta_k\mathcal{L} &= \frac{\alpha_2}{l^2}\epsilon_{abcd}\left(2\delta k^{ab}De^c e^d + 2\delta k^a{}_f k^{fb}e^c e^d + \frac{1}{l^2}\delta k^{ab}\bar{\psi}\gamma^d\psi e^c\right) \\ &= \frac{2\alpha_2}{l^2}\epsilon_{abcd}\delta k^{ab}\left(T^c + k^c{}_f e^f - \frac{1}{2}\bar{\psi}\gamma^c\psi\right)e^d \\ &= \frac{2\alpha_2}{l^2}\epsilon_{abcd}\delta k^{ab}R^c e^d,\end{aligned} \tag{5.61}$$

where we have used the gamma matrix identities ($C$.1) and ($C$.5). Thus, $\delta_k\mathcal{L}=0$ leads to the same field equation that $\delta_\omega\mathcal{L}=0$,

$$2\epsilon_{abcd}R^c e^d = 0. \tag{5.62}$$

Let us consider the variation of the Lagrangian with respect to the gravitino field $\psi$,

$$\begin{aligned}\delta_\psi\mathcal{L} &= \frac{\alpha_2}{l^2}\left(4\delta\bar{\psi}e^a\gamma_a\gamma_5 D\psi - 4D\bar{\psi}e^a\gamma_a\gamma_5\delta\psi + 4\bar{\psi}De^a\gamma_a\gamma_5\delta\psi\right) \\ &+ \frac{\alpha_2}{l^2}\epsilon_{abcd}\left(2k^{ab}e^c\delta\bar{\psi}\gamma^d\psi + \frac{2}{l}e^a e^b\delta\bar{\psi}\gamma^{cd}\psi\right) \\ &= \frac{\alpha_2}{l^2}\delta\bar{\psi}\left(8e^a\gamma_a\gamma_5 D\psi - 4\gamma_a\gamma_5\psi De^a + 2e^a\gamma_a\gamma_5 k^{bc}\gamma_{bc}\psi - 4\gamma_a\gamma_5 k^a_b e^b\psi + \frac{4}{l}e^a\gamma_a\gamma_5 e^b\gamma_b\psi\right) \\ &= \frac{\alpha_2}{l^2}\delta\bar{\psi}\left(8e^a\gamma_a\gamma_5\Psi - 4\gamma_a\gamma_5\psi R^a\right).\end{aligned} \tag{5.63}$$

Then, we find the following field equation,

$$8e^a\gamma_a\gamma_5\Psi - 4\gamma_a\gamma_5\psi R^a = 0. \tag{5.64}$$

We can see that the presence of a generalized supersymmetric cosmological constant leads to field equations very similar to those of standard supergravity. The differences appear in the definition of the curvatures two-form due to the presence of the new matter field $k^{ab}$.

As we said before, from eqs. (5.58) and (5.62), we see that the equation of motion coming from the variation of the Lagrangian with respect to the bosonic field $k^{ab}$ reduces to that of the spin connection $\omega^{ab}$. From this equation we have that

$$R^a \equiv T^a + k^a_{\ b}e^b - \frac{1}{2}\bar{\psi}\gamma^a\psi = 0. \tag{5.65}$$

Let us define a new bosonic field as

$$\varpi^{ab} = \omega^{ab} + k^{ab}, \tag{5.66}$$

and its respective covariant derivative,

$$\mathcal{D} = d + \varpi. \tag{5.67}$$

Then, eq.(5.65) can be written as

$$\mathcal{D}e^a - \frac{1}{2}\bar{\psi}\gamma^a\psi = 0. \tag{5.68}$$

This allows to express the bosonic field $\varpi^{ab}$ in terms of the vielbein $e^a$ and gravitino field $\psi^\alpha$. The equation can be solved considering the following decomposition,

$$\varpi^{ab} = \mathring{\varpi}^{ab} + \tilde{\varpi}^{ab}, \tag{5.69}$$

where $\mathring{\varpi}^{ab}$ corresponds to the solution of $\mathcal{D}e^c = 0$ and is given by

$$\mathring{\varpi}^{ab}_\mu = \left(e^c_\lambda\partial_{[\mu}e^d_{\nu]}\eta_{cd} + e^c_\nu\partial_{[\lambda}e^d_{\mu]}\eta_{cd} - e^c_\mu\partial_{[\nu}e^d_{\lambda]}\eta_{cd}\right)e^{\lambda|a}e^{\nu|b}. \tag{5.70}$$

Now we have that

$$\mathcal{D}e^a = de^a + \mathring{\varpi}^{ab}e_b + \tilde{\varpi}^{ab}e_b = \frac{1}{2}\bar{\psi}\gamma^a\psi, \tag{5.71}$$

implies

$$\tilde{\varpi}^{ab}_{[\mu}e_{\nu]b} = \frac{1}{2}\bar{\psi}_\mu\gamma^a\psi_\nu. \tag{5.72}$$

Then we may solve $\tilde{\varpi}^{ab}$ in terms of the two other fields,

$$\tilde{\varpi}^{ab}_{\mu} = \frac{1}{4} e^{a|\lambda} e^{b|\nu} \left( \bar{\psi}_{\mu}\gamma_{\lambda}\psi_{\nu} + \bar{\psi}_{\lambda}\gamma_{\nu}\psi_{\mu} - \bar{\psi}_{\nu}\gamma_{\mu}\psi_{\lambda} - \bar{\psi}_{\mu}\gamma_{\nu}\psi_{\lambda} - \bar{\psi}_{\nu}\gamma_{\lambda}\psi_{\mu} + \bar{\psi}_{\lambda}\gamma_{\mu}\psi_{\nu} \right). \tag{5.73}$$

Thus, the bosonic field $\varpi^{ab}$ is completely determined in terms of $e^{a}_{\mu}$ and $\psi^{\alpha}_{\mu}$ and does not carry additional physical degrees of freedom. In fact, when the supertorsion $R^{a} = \mathcal{D}e^{c} - \frac{1}{2}\bar{\psi}\gamma^{c}\psi$ is set equals to zero, the number of bosonic degrees of freedom is 2 as the Einstein-Hilbert gravity theory.

## 5.5 Supersymmetry transformations and action invariance

Although the action is built from the AdS-Lorentz superalgebra, it is not invariant under gauge transformations. The variation of the action (5.53) under gauge supersymmetry can be derived using $\delta F = [F, \epsilon]$, with $\epsilon$ the supersymmetry parameter,

$$\delta_{susy} S = -\frac{4\alpha_2}{l^2} \int R^{a} \bar{\Psi} \gamma_{a} \gamma_{5} \epsilon. \tag{5.74}$$

Thus in order to have gauge supersymmetry invariance it is necessary to impose the AdS-Lorentz supertorsion constraint

$$R^{a} = 0. \tag{5.75}$$

However this leads to express the spin connection $\omega^{ab}$ in terms of the others fields $\left\{ e^{a}, k^{ab}, \psi \right\}$.

Nevertheless, it is possible to have supersymmetry invariance in the first formalism adding an extra piece to the gauge transformation $\delta\omega^{ab}$ such that the variation of the action can be written as

$$\delta S = -\frac{4\alpha_2}{l^2} \int R^{a} \left[ \bar{\Psi}\gamma_{a}\gamma_{5}\epsilon - \frac{1}{2}\epsilon_{abcd} e^{b} \delta_{extra} \omega^{cd} \right], \tag{5.76}$$

where the supersymmetry invariance is fullfilled when

$$\delta_{extra}\omega^{ab} = 2\epsilon^{abcd} \left( \bar{\Psi}_{ec}\gamma_{d}\gamma_{5}\epsilon + \bar{\Psi}_{de}\gamma_{c}\gamma_{5}\epsilon - \bar{\Psi}_{cd}\gamma_{e}\gamma_{5}\epsilon \right) e^{e}, \tag{5.77}$$

with $\bar{\Psi} = \bar{\Psi}_{ab} e^{a} e^{b}$.

Thus, the action (5.53) in the first order formalism is invariant under the following supersymmetry transformations

$$\delta\omega^{ab} = 2\epsilon^{abcd}\left(\bar{\Psi}_{ec}\gamma_d\gamma_5\epsilon + \bar{\Psi}_{de}\gamma_c\gamma_5\epsilon - \bar{\Psi}_{cd}\gamma_e\gamma_5\epsilon\right)e^e, \tag{5.78}$$

$$\delta k^{ab} = -\frac{1}{l}\bar{\epsilon}\gamma^{ab}\psi, \tag{5.79}$$

$$\delta e^a = \bar{\epsilon}\gamma^a\psi, \tag{5.80}$$

$$\delta\psi = d\epsilon + \frac{1}{4}\omega^{ab}\gamma_{ab}\epsilon + \frac{1}{4}k^{ab}\gamma_{ab}\epsilon + \frac{1}{2l}e^a\gamma_a\epsilon. \tag{5.81}$$

Let us note that supersymmetry is not a gauge symmetry of the action, since it is broken to a Lorentz like symmetry.

# Chapter 6

# Maxwell Chern-Simons Supergravity

Analogously to the four dimensional case seen above, the AdS-Lorentz superalgebra in $D=3$ can be derived as an $S$-expansion of the $\mathfrak{osp}\left(2|1\right)\oplus\mathfrak{sp}\left(2\right)$ superalgebra [69]. Furthermore, as we said before, the non-standard Maxwell superalgebra $s\mathcal{M}$ can be obtained as a Inönü-Wigner contraction of the AdS-Lorentz superalgebra [16], [70]. Then it seems natural to derive the non-standard Maxwell superalgebra combining the $S$-expansion procedure with the Inönü-Wigner contraction. In particular, as we will see later the non-vanishing components of an invariant tensor for this superalgebra can be found in this way.

Following [71], in this chapter we construct a $D=3$ supergravity action from a minimal Maxwell superalgebra $s\mathcal{M}^{g}$. The $s\mathcal{M}^{g}$ superalgebra is obtained as an $S$-expansion of the $\mathfrak{osp}\left(2|1\right)\otimes\mathfrak{sp}\left(2\right)$ superalgebra by considering an appropriate semigroup, and corresponds to a supersymmetric extension of the generalized Maxwell algebra $\mathcal{M}^{g}$ (see Appendix B).

Let us first consider an algebraic construction of a three-dimensional supersymmetric action invariant under the usual Maxwell supergroup. To this aim, we shall combine the $S$-expansion procedure and the Inönü-Wigner contraction in order to derive the non-standard $s\mathcal{M}$ superalgebra and the non-vanishing components of an invariant tensor for this superalgebra.

## 6.1 CS supersymmetric action from $s\mathcal{M}$

In this section, we present a $D=3$ Chern-Simons supersymmetric action for the non-standard Maxwell superalgebra. As we will see next, the Maxwell superalgebra $s\mathcal{M}$ can be obtained alternatively combining the $S$-expansion method and the Inönü-Wigner contraction.

### 6.1.1 $D = 3$ Maxwell superalgebra $s\mathcal{M}$

Following [69] and [18], it is possible to obtain the AdS-Lorentz superalgebra as an $S$-expansion of the $\mathfrak{osp}(2|1) \oplus \mathfrak{sp}(2)$ superalgebra using $S_\wedge = \{\lambda_0, \lambda_1\}$ as the relevant semigroup. As in the previous cases, we have to consider a decomposition of the original algebra in subspaces $\mathfrak{g} = \mathfrak{osp}(2|1) \oplus \mathfrak{sp}(2) = V_0 \oplus V_1 \oplus V_2$, where $V_0$ corresponds to a Lorentz subalgebra and it is generated by the Lorentz generator $\tilde{J}_{ab}$, $V_1$ corresponds to the fermionic subspace generated by a 3-component Majorana spinor charge $\tilde{Q}_\alpha$ and $V_2$ corresponds to the $AdS$ boost generated by $\tilde{P}_a$. These generators satisfy the (anti)commutation relations (2.41) – (2.45). The subspace structure may be written as

$$\begin{aligned} &[V_0, V_0] \subset V_0, \quad [V_1, V_1] \subset V_0 \oplus V_2, \\ &[V_0, V_1] \subset V_1, \quad [V_1, V_2] \subset V_1, \\ &[V_0, V_2] \subset V_2, \quad [V_2, V_2] \subset V_0. \end{aligned} \tag{6.1}$$

Consider the abelian semigroup $S_\wedge = \{\lambda_0, \lambda_1\}$ whose elements are dimensionless and satisfy the multiplication law,

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_1, & \text{if } \alpha = \beta = 1 \\ \lambda_0, & \text{all others} \end{cases} \tag{6.2}$$

Let us consider the subset decomposition $S_\wedge = S_0 \cup S_1 \cup S_2$, with

$$S_0 = \{\lambda_0, \lambda_1\}, \quad S_1 = \{\lambda_0\}, \quad S_2 = \{\lambda_0\}. \tag{6.3}$$

One sees that this decomposition is said to be resonant since it satisfies the same structure as the subspaces $V_p$ [compare with eqs. (6.1)]

$$\begin{aligned} &S_0 \cdot S_0 \subset S_0, \quad S_1 \cdot S_1 \subset S_0 \cap S_2, \\ &S_0 \cdot S_1 \subset S_1, \quad S_1 \cdot S_2 \subset S_1, \\ &S_0 \cdot S_2 \subset S_2, \quad S_2 \cdot S_2 \subset S_0. \end{aligned} \tag{6.4}$$

Following theorem IV.1 of [18], we can say that the superalgebra

$$\mathfrak{G}_R = W_0 \oplus W_1 \oplus W_2, \tag{6.5}$$

is a resonant subalgebra of $S_\wedge \times \mathfrak{g}$, where

$$W_0 = (S_0 \times V_0) = \{\lambda_0, \lambda_1\} \times \left\{\tilde{J}_{ab}\right\} = \left\{\lambda_0 \tilde{J}_{ab}, \lambda_1 \tilde{J}_{ab}\right\}, \tag{6.6}$$

$$W_1 = (S_1 \times V_1) = \{\lambda_0\} \times \left\{\tilde{Q}_\alpha\right\} = \left\{\lambda_0 \tilde{Q}_\alpha\right\}, \tag{6.7}$$

$$W_2 = (S_2 \times V_2) = \{\lambda_0\} \times \left\{\tilde{P}_a\right\} = \left\{\lambda_0 \tilde{P}_a\right\}. \tag{6.8}$$

The new superalgebra is generated by $\{J_{ab}, P_a, Z_{ab}, Q_\alpha\}$, where these generators are defined by

$$J_{ab} = \lambda_1 \tilde{J}_{ab}, \qquad P_a = \lambda_0 \tilde{P}_a, \tag{6.9}$$

$$Z_{ab} = \lambda_0 \tilde{J}_{ab}, \qquad Q_\alpha = \lambda_0 \tilde{Q}_\alpha. \tag{6.10}$$

and satisfy the (anti)commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{6.11}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{6.12}$$

$$[Z_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{6.13}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{6.14}$$

$$[Z_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \tag{6.15}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2} \left(\Gamma_{ab} Q\right)_\alpha, \qquad [P_a, Q_\alpha] = -\frac{1}{2} \left(\Gamma_a Q\right)_\alpha, \tag{6.16}$$

$$[Z_{ab}, Q_\alpha] = -\frac{1}{2} \left(\Gamma_{ab} Q\right)_\alpha, \tag{6.17}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2} \left[ \left(\Gamma^{ab} C\right)_{\alpha\beta} Z_{ab} - 2 \left(\Gamma^a C\right)_{\alpha\beta} P_a \right], \tag{6.18}$$

where we have used the multiplication law of the semigroup (6.2) and the commutation relations of the original superalgebra (2.41) – (2.45). The new superalgebra obtained after a resonant $S_\wedge$-expansion of $\mathfrak{osp}\,(2|1) \oplus \mathfrak{sp}\,(2)$ corresponds to the AdS-Lorentz superalgebra in three dimensions. As we have seen in the previous chapter this superalgebra has an interesting application in $D = 4$ supergravity since it allows to include a generalized cosmological constant in a MacDowell-Mansouri like action [68]. The generalization of the AdS-Lorentz superalgebra (6.11) – (6.17) and its extension to $\mathcal{N}$ supersymmetries can be found in [68] and [74], respectively.

Let us now consider the Inönü-Wigner contraction of the AdS-Lorentz superalgebra applying the rescaling presented in [70],

$$Z_{ab} \to \sigma^2 Z_{ab}, \quad P_a \to \sigma P_a \ \text{ and } \ Q_\alpha \to \sigma Q_\alpha. \tag{6.19}$$

Then the limit $\sigma \to \infty$ provides us with the following (anti)commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{6.20}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{6.21}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{6.22}$$

$$[Z_{ab}, Z_{cd}] = 0, \qquad [Z_{ab}, P_c] = 0, \tag{6.23}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2} \left(\Gamma_{ab} Q\right)_\alpha, \tag{6.24}$$

$$[Z_{ab}, Q_\alpha] = 0, \qquad [P_a, Q_\alpha] = 0, \tag{6.25}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2} \left(\Gamma^{ab} C\right)_{\alpha\beta} Z_{ab}. \tag{6.26}$$

The new superalgebra obtained after a resonant $S$-expansion of $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$ and an Inönü-Wigner contraction corresponds to the Maxwell superalgebra $s\mathcal{M}$ in $D = 3$. This superalgebra contains the Maxwell algebra $\mathcal{M} = \{J_{ab}, P_a, Z_{ab}\}$ and the Lorentz type algebra $\mathcal{L}^{\mathcal{M}} = \{J_{ab}, Z_{ab}\}$ as subalgebras. In particular, the study of a 3-dimensional gravity using the Maxwell algebra was considered in [72], [73].

Let us observe that the Maxwell superalgebra $s\mathcal{M}$ does not contain a necessary relation in supergravity, expressing momenta as bilinears of supercharges. Indeed, from relation (6.26) we see that it supersymmetrizes only tensorial central charges. As we will see later, this situation is completely different in the case of a minimal Maxwell superalgebra. Before presenting the construction of a CS supergravity action for a minimal Maxwell superalgebra, let us first consider an algebraic construction of a three-dimensional supersymmetric action for the non-standard Maxwell superalgebra $s\mathcal{M}$.

### 6.1.2 Three-dimensional Maxwell CS supersymmetric action

Here we present a geometrical construction of a CS supersymmetric action using the Maxwell superalgebra and the properties of the $S$-expansion procedure. As seen from the definition of a CS Lagrangian (see (2.46)), a fundamental ingredient in the construction of a CS action is the existence of symmetric invariant tensors for the corresponding gauge group. As we have discussed in previous chapters a useful property of the $S$-expansion method is that it provides us with an invariant tensor for the $S$-expanded algebra. In fact, by Theorem VII.2 of [18], the invariant tensor of an $S$-expanded (super)algebra $\mathfrak{G}$ is given in terms of an invariant tensor of the original (super)algebra $\mathfrak{g}$ as follows

$$\left\langle T_{(A,\alpha)} T_{(B,\beta)} \right\rangle_{\mathfrak{G}} = \tilde{\alpha}_\gamma K_{\alpha\beta}{}^{\gamma} \left\langle T_A T_B \right\rangle_{\mathfrak{g}}, \tag{6.27}$$

where $\tilde{\alpha}_\gamma$ are arbitrary constants and $K_{\alpha\beta}{}^{\gamma}$ corresponds to a 2-selector. Starting from the AdS superalgebra $(2.41-2.45)$ and using the Theorem VII.2, it is possible to show that the non-vanishing components of an invariant tensor for the 3-dimensional AdS-Lorentz superalgebra are given by

$$
\begin{aligned}
&\langle J_{ab}J_{cd}\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_1 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \quad && \langle Z_{ab}P_{c}\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{J}_{ab}\tilde{P}_{c}\right\rangle, \\
&\langle J_{ab}Z_{cd}\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \quad && \langle P_{a}P_{b}\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{P}_{a}\tilde{P}_{b}\right\rangle, \\
&\langle Z_{ab}Z_{cd}\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \quad && \langle Q_{\alpha}Q_{\beta}\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta}\right\rangle, \\
&\langle J_{ab}P_{c}\rangle_{AdS-\mathcal{L}} = \tilde{\alpha}_0 \left\langle \tilde{J}_{ab}\tilde{P}_{c}\right\rangle,
\end{aligned} \tag{6.28}
$$

where $\left\langle \tilde{J}_{ab}\tilde{J}_{cd}\right\rangle, \left\langle \tilde{J}_{ab}\tilde{P}_{c}\right\rangle, \left\langle \tilde{P}_{a}\tilde{P}_{b}\right\rangle$ and $\left\langle \tilde{Q}_{\alpha}\tilde{Q}_{\beta}\right\rangle$ are the components of an invariant tensor for the $\mathfrak{osp}\left(2|1\right)\oplus\mathfrak{sp}\left(2\right)$ superalgebra [see eqs. $(2.49)-(2.52)]$.

It seems natural to derive a Chern Simons action for the Maxwell superalgebra by combining this result with the corresponding Inönü-Wigner contraction in the generators (6.19). Nevertheless, the rescaling in the generators leads to trivial invariant tensors for the Maxwell superalgebra and consequently to a trivial Chern Simons action. A possible way to avoid this problem is to generalize the Inönü-Wigner contraction by considering the rescaling not only of the generators but also of the invariant tensors. Interestingly, there is just one rescaling that preserves the structure of curvatures in the action and is given by

$$
\beta_0 \to \sigma^2\beta_0, \qquad \alpha_0 \to \sigma\alpha_0, \qquad \beta_1 \to \beta_1. \tag{6.29}
$$

where

$$
\beta_0 \equiv \tilde{\alpha}_0\mu_0, \qquad \alpha_0 \equiv \tilde{\alpha}_0\mu_1, \qquad \beta_1 \equiv \tilde{\alpha}_1\mu_0.
$$

Then, considering the rescaling of both generators (6.19) and constants (6.29) in (6.28), one can see that the limit $\sigma\to\infty$ leads to the non-trivial non-vanishing components of an invariant tensor for the Maxwell superalgebra $s\mathcal{M}$,

$$
\langle J_{ab}J_{cd}\rangle_{s\mathcal{M}} = \beta_1\left(\eta_{bc}\eta_{ad}-\eta_{ac}\eta_{bd}\right), \tag{6.30}
$$

$$
\langle J_{ab}Z_{cd}\rangle_{s\mathcal{M}} = \beta_0\left(\eta_{bc}\eta_{ad}-\eta_{ac}\eta_{bd}\right), \tag{6.31}
$$

$$
\langle J_{ab}P_{c}\rangle_{s\mathcal{M}} = \alpha_0\epsilon_{abc}, \tag{6.32}
$$

$$
\langle P_{a}P_{b}\rangle_{s\mathcal{M}} = \beta_0\eta_{ab}, \tag{6.33}
$$

$$
\langle Q_{\alpha}Q_{\beta}\rangle_{s\mathcal{M}} = \beta_0 C_{\alpha\beta}. \tag{6.34}
$$

In order to write down a CS action for the $s\mathcal{M}$ superalgebra we start from the one-form gauge connection

$$A = A^A T_A = \frac{1}{2}\omega^{ab} J_{ab} + \frac{1}{l} e^a P_a + \frac{1}{2} k^{ab} Z_{ab} + \frac{1}{\sqrt{l}} \psi^\alpha Q_\alpha, \tag{6.35}$$

where $e^a$, $\omega^{ab}$, $k^{ab}$ and $\psi$ are respectively the vielbein, the spin connection, a "matter" bosonic field and the gravitino field. These one-forms are the corresponding expanded fields of the $\mathfrak{osp}\left(2|1\right) \oplus \mathfrak{sp}\left(2\right)$ gauge fields $\left\{\tilde{\omega}^{ab}, \tilde{e}^a\,,\ \tilde{\psi}^\alpha\right\}$,

$$\begin{aligned} \omega^{ab} &= \omega^{(ab,1)} = \lambda_1 \tilde{\omega}^{ab}, \qquad e^a = e^{(a,0)} = \lambda_0 \tilde{e}^a, \\ k^{ab} &= \omega^{(ab,0)} = \lambda_0 \tilde{\omega}^{ab}, \qquad \psi^\alpha = \psi^{(\alpha,0)} = \lambda_0 \tilde{\psi}^\alpha, \end{aligned} \tag{6.36}$$

The associated curvature two-form is,

$$F = F^A T_A = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} R^a P_a + \frac{1}{2} F^{ab} Z_{ab} + \frac{1}{\sqrt{l}} \Psi^\alpha Q_\alpha, \tag{6.37}$$

where

$$\begin{aligned} R^{ab} &= d\omega^{ab} + \omega^a_{\ c}\omega^{cb}, \\ R^a &= de^a + \omega^a_{\ b} e^b = T^a, \\ F^{ab} &= dk^{ab} + \omega^a_{\ c} k^{cb} - \omega^b_{\ c} k^{ca} + \frac{1}{l^2} e^a e^b + \frac{1}{2l}\bar{\psi}\Gamma^{ab}\psi, \\ \Psi &= d\psi + \frac{1}{4}\omega_{ab}\Gamma^{ab}\psi = D\psi. \end{aligned}$$

Then, when we insert the one-form connection (6.35) into the general expression of the CS action (2.46) and using the invariant tensor $(6.30 - 6.34)$, we can write the CS supersymmetric action for the Maxwell superalgebra $s\mathcal{M}$. Explicitly, it is given by

$$\begin{aligned} S_{CS}^{(2+1)} = \kappa \int_M \Bigg[ & \frac{1}{2}\beta_1 \left( \omega^a_{\ b} d\omega^b_{\ a} + \frac{2}{3}\omega^a_{\ b}\omega^b_{\ c}\omega^c_{\ a} \right) + \frac{\alpha_0}{l}\left(\epsilon_{abc} R^{ab} e^c\right) \\ & + \beta_0 \left( R^a_{\ b} k^b_{\ a} + \frac{1}{l^2} e^a T_a + \frac{1}{l}\bar{\psi}\Psi \right) - \frac{1}{2} d\left( \beta_0 \omega^a_{\ b} k^b_{\ a} + \frac{\alpha_0}{l}\epsilon_{abc}\omega^{ab} e^c \right) \Bigg], \end{aligned} \tag{6.38}$$

The action (6.38) is split into three independent pieces proportional to $\beta_1$, $\alpha_0$ and $\beta_0$. The term proportional to $\beta_1$ corresponds to the exotic Lagrangian [8], [38]. The piece proportional to $\alpha_0$ is invariant under Poincaré and corresponds to the Einstein-Hilbert term. On the other hand, the term proportional to $\beta_0$ contains the torsional term, the fermionic term and the coupling between the new gauge field $k^{ab}$ and the Lorentz curvature $R^{ab}$. The gauge field $k^{ab}$ associated to the $Z_{ab}$ generator appears also in the boundary term. Let us note that the cosmological constant term $\epsilon_{abc}e^a e^b e^c$ does not appear in the action.

Up to boundary terms, the full action is invariant under gauge transformations of the Maxwell supergroup and under supersymmetry,

$$\delta_\epsilon \omega^{ab} = 0, \qquad \delta_\epsilon k^{ab} = -\frac{1}{l}\bar{\epsilon}\Gamma^{ab}\psi, \tag{6.39}$$

$$\delta_\epsilon e^a = 0, \qquad \delta_\epsilon \psi = D\epsilon, \tag{6.40}$$

As no field equations are requiered in order to prove this invariance, we said that it is an off-shell SUSY. Furthermore, we can see that the bosonic part of the action (6.38) corresponds to the CS gravity action found in [72] and [73] for the Maxwell algebra. Clearly, when we consider $\sigma = 1$ in the rescalings (6.19) and (6.29) we obtain the CS supergravity action for the AdS-Lorentz superalgebra presented in [69].

## 6.2 Maxwell-Chern-Simons Supergravity

Let us now consider the construction of a Chern-Simons supergravity action for the minimal $D = 3$ Maxwell superalgebra $s\mathcal{M}^g$. As we will see, this superalgebra can be derived as an $S$-expansion of $\mathfrak{osp}\,(2|1) \oplus \mathfrak{sp}\,(2)$ using an appropriate semigroup.

As in the previous section we will consider the splitting of the AdS superalgebra into subspaces $\mathfrak{g} = V_0 \oplus V_1 \oplus V_2$, where $V_0 = \left\{\tilde{J}_{ab}\right\}$, $V_1 = \left\{\tilde{Q}_\alpha\right\}$ and $V_2 = \left\{\tilde{P}_a\right\}$. The next step consists in finding a subset decomposition of a semigroup $S$ which is "resonant" with respect to the subspace structure (6.1). Let us consider $S_E^{(4)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ as the relevant abelian semigroup whose elements obey the multiplication law (3.22). Let us consider a subset decomposition $S_E^{(4)} = S_0 \cup S_1 \cup S_2$, with

$$S_0 = \{\lambda_0, \lambda_2, \lambda_4, \lambda_5\}\,, \tag{6.41}$$

$$S_1 = \{\lambda_1, \lambda_3, \lambda_5\}\,, \tag{6.42}$$

$$S_2 = \{\lambda_2, \lambda_4, \lambda_5\}\,. \tag{6.43}$$

This subset decomposition is said to be "resonant" since it satisfies the same structure as the subspaces $V_p$ [compare with eqs. (6.1)]

$$\begin{aligned} S_0 \cdot S_0 &\subset S_0, \quad S_1 \cdot S_1 \subset S_0 \cap S_2, \\ S_0 \cdot S_1 &\subset S_1, \quad S_1 \cdot S_2 \subset S_1, \\ S_0 \cdot S_2 &\subset S_2, \quad S_2 \cdot S_2 \subset S_0. \end{aligned} \tag{6.44}$$

Imposing the $0_S$-reduction condition $\lambda_5 T_A = 0_s$, we find a new Lie superalgebra generated by $\left\{J_{ab}, P_a, \tilde{Z}_{ab}, Z_{ab}, \tilde{Z}_a, Q_\alpha, \Sigma_\alpha\right\}$ where these new generators can be written as

$$
\begin{aligned}
J_{ab} &= \lambda_0 \tilde{J}_{ab}, & \tilde{Z}_a &= \lambda_4 \tilde{P}_a, \\
\tilde{Z}_{ab} &= \lambda_2 \tilde{J}_{ab}, & Q_\alpha &= \lambda_1 \tilde{Q}_\alpha, \\
Z_{ab} &= \lambda_4 \tilde{J}_{ab}, & \Sigma_\alpha &= \lambda_3 \tilde{Q}_\alpha. \\
P_a &= \lambda_2 \tilde{P}_a, & &
\end{aligned}
\tag{6.45}
$$

and satisfy the following (anti)commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{6.46}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \qquad [P_a, P_b] = Z_{ab}, \tag{6.47}$$

$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{6.48}$$

$$[P_a, Q_\alpha] = -\frac{1}{2} \left(\Gamma_a \Sigma\right)_\alpha, \tag{6.49}$$

$$[J_{ab}, Q_\alpha] = -\frac{1}{2} \left(\Gamma_{ab} Q\right)_\alpha, \tag{6.50}$$

$$[J_{ab}, \Sigma_\alpha] = -\frac{1}{2} \left(\Gamma_{ab} \Sigma\right)_\alpha, \tag{6.51}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2} \left[\left(\Gamma^{ab} C\right)_{\alpha\beta} \tilde{Z}_{ab} - 2 \left(\Gamma^a C\right)_{\alpha\beta} P_a\right], \tag{6.52}$$

$$\{Q_\alpha, \Sigma_\beta\} = -\frac{1}{2} \left[\left(\Gamma^{ab} C\right)_{\alpha\beta} Z_{ab} - 2 \left(\Gamma^a C\right)_{\alpha\beta} \tilde{Z}_a\right] \tag{6.53}$$

$$\left[J_{ab}, \tilde{Z}_{ab}\right] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \tag{6.54}$$

$$\left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{6.55}$$

$$\left[J_{ab}, \tilde{Z}_c\right] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \qquad \left[\tilde{Z}_{ab}, P_c\right] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \tag{6.56}$$

$$\left[\tilde{Z}_{ab}, Q_\alpha\right] = -\frac{1}{2} \left(\gamma_{ab} \Sigma\right)_\alpha, \tag{6.57}$$

$$\text{others} = 0 \tag{6.58}$$

where we have used the multiplication law of the semigroup (3.22) and the commutation relations of the AdS superalgebra (2.41 − 2.45). The new superalgebra obtained after a $0_S$-reduced resonant $S$-expansion of $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$ corresponds to the minimal Maxwell superalgebra $s\mathcal{M}^g$. This superalgebra can be seen as the supersymmetric extension of the generalized Maxwell algebra $\mathcal{M}^g$ in $D = 3$ dimensions [60] .

### 6.2.1 Three-dimensional Maxwell CS supergravity action

In order to write down an CS action for the minimal Maxwell superalgebra $s\mathcal{M}^g$ we start from the one-form gauge connection

$$A=\frac{1}{2}\omega^{ab}J_{ab}+\frac{1}{2}\tilde{k}^{ab}\tilde{Z}_{ab}+\frac{1}{2}k^{ab}Z_{ab}+\frac{1}{l}e^{a}P_{a}+\frac{1}{l}\tilde{h}^{a}\tilde{Z}_{a}+\frac{1}{\sqrt{l}}\psi^{\alpha}Q_{\alpha}+\frac{1}{\sqrt{l}}\xi^{\alpha}\Sigma_{\alpha}, \tag{6.59}$$

where the 1-form gauge fields are given in terms of the components of the $\mathfrak{osp}\left(2|1\right)\otimes\mathfrak{sp}\left(2\right)$ connection $\tilde{e}^{a},\tilde{\omega}^{ab}$ and $\tilde{\psi}$:

$$\begin{aligned}
&\omega^{ab}=\lambda_0\tilde{\omega}^{ab}, && \tilde{k}^{ab}=\lambda_2\tilde{\omega}^{ab} && k^{ab}=\lambda_4\tilde{\omega}^{ab},\\
&e^{a}=\lambda_2\tilde{e}^{a}, && \tilde{h}^{a}=\lambda_4\tilde{e}^{a}, && \psi^{\alpha}=\lambda_1\tilde{\psi}^{\alpha},\\
&\xi^{\alpha}=\lambda_3\tilde{\psi}^{\alpha}.
\end{aligned}$$

The associated curvature two-form is given by

$$\begin{aligned}
F=F^{A}T_{A}&=\frac{1}{2}R^{ab}J_{ab}+\frac{1}{l}R^{a}P_{a}+\frac{1}{2}\tilde{F}^{ab}\tilde{Z}_{ab}+\frac{1}{2}F^{ab}Z_{ab}\\
&+\frac{1}{l}\tilde{H}^{a}\tilde{Z}_{a}+\frac{1}{\sqrt{l}}\Psi^{\alpha}Q_{\alpha}+\frac{1}{\sqrt{l}}\Xi^{\alpha}\Sigma_{\alpha},
\end{aligned} \tag{6.60}$$

where

$$\begin{aligned}
R^{ab}&=d\omega^{ab}+\omega_{\ c}^{a}\omega^{cb},\\
R^{a}&=de^{a}+\omega_{\ b}^{a}e^{b}-\frac{1}{2}\bar{\psi}\Gamma^{a}\psi,\\
\tilde{H}^{a}&=d\tilde{h}^{a}+\omega_{\ b}^{a}\tilde{h}^{b}+\tilde{k}_{\ c}^{a}e^{c}-\bar{\xi}\Gamma^{a}\psi,\\
\tilde{F}^{ab}&=d\tilde{k}^{ab}+\omega_{\ c}^{a}\tilde{k}^{cb}-\omega_{\ c}^{b}\tilde{k}^{ca}+\frac{1}{2l}\bar{\psi}\Gamma^{ab}\psi,\\
F^{ab}&=dk^{ab}+\omega_{\ c}^{a}k^{cb}-\omega_{\ c}^{b}k^{ca}+\tilde{k}_{\ c}^{a}\tilde{k}^{cb}+\frac{1}{l^{2}}e^{a}e^{b}+\frac{1}{l}\bar{\xi}\Gamma^{ab}\psi,\\
\Psi&=d\psi+\frac{1}{4}\omega_{ab}\Gamma^{ab}\psi,\\
\Xi&=d\xi+\frac{1}{4}\omega_{ab}\Gamma^{ab}\xi+\frac{1}{4}\tilde{k}_{ab}\Gamma^{ab}\psi+\frac{1}{2l}e^{a}\Gamma_{a}\psi.
\end{aligned} \tag{6.61}$$

Considering (6.27) it is possible to show that the only non-vanishing components of a symmetric invariant tensor for the Maxwell superalgebra $s\mathcal{M}^g$, can be found in terms of the invariant tensors for $\mathfrak{osp}(2|1) \otimes \mathfrak{sp}(2)$ [see eqs. (2.49) – (2.52)]

$$\langle J_{ab} J_{cd} \rangle_{s\mathcal{M}^g} = \alpha_0 \left( \eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd} \right) \tag{6.62}$$

$$\left\langle J_{ab} \tilde{Z}_{cd} \right\rangle_{s\mathcal{M}^g} = \alpha_2 \left( \eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd} \right) \tag{6.63}$$

$$\left\langle \tilde{Z}_{ab} \tilde{Z}_{cd} \right\rangle_{s\mathcal{M}^g} = \langle J_{ab} Z_{cd} \rangle = \alpha_4 \left( \eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd} \right) \tag{6.64}$$

$$\langle J_{ab} P_c \rangle_{s\mathcal{M}^g} = \alpha_1 \epsilon_{abc} \tag{6.65}$$

$$\left\langle \tilde{Z}_{ab} P_c \right\rangle_{s\mathcal{M}^g} = \left\langle J_{ab} \tilde{Z}_c \right\rangle = \alpha_3 \epsilon_{abc} \tag{6.66}$$

$$\langle P_a P_b \rangle_{s\mathcal{M}^g} = \alpha_4 \eta_{ab} \tag{6.67}$$

$$\langle Q_\alpha Q_\beta \rangle_{s\mathcal{M}^g} = \left( \alpha_2 - \alpha_1 \right) C_{\alpha\beta} \tag{6.68}$$

$$\langle Q_\alpha \Sigma_\beta \rangle_{s\mathcal{M}^g} = \left( \alpha_4 - \alpha_3 \right) C_{\alpha\beta} \tag{6.69}$$

where we have used the following definitions

$$\alpha_0 \equiv \tilde{\alpha}_0 \mu_0, \quad \alpha_1 \equiv \tilde{\alpha}_2 \mu_1, \quad \alpha_2 \equiv \tilde{\alpha}_2 \mu_0$$
$$\alpha_3 \equiv \tilde{\alpha}_4 \mu_1, \quad \alpha_4 \equiv \tilde{\alpha}_4 \mu_0.$$

Considering (6.62) – (6.69) and the one-form connection (6.59) in the general expression for the CS action (2.46), we find that the CS supergravity action for the minimal Maxwell superalgebra $s\mathcal{M}^g$ is given explicitly by

$$\begin{aligned} S_{CS}^{(2+1)} = k \int_M \Bigg[ & \frac{\alpha_0}{2} \left( \omega^a_{\ b} d\omega^b_{\ a} + \frac{2}{3} \omega^a_{\ c} \omega^c_{\ b} \omega^b_{\ a} \right) + \frac{\alpha_1}{l} \left( \epsilon_{abc} R^{ab} e^c - \bar{\psi} \Psi \right) \\ & + \alpha_2 \left( R^a_{\ b} \tilde{k}^b_{\ a} + \frac{1}{l} \bar{\psi} \Psi \right) + \frac{\alpha_3}{l} \left( \epsilon_{abc} \left( R^{ab} \tilde{h}^c + D \tilde{k}^{ab} e^c \right) - \bar{\xi} \Psi - \bar{\psi} \Xi \right) \\ & + \alpha_4 \left( R^a_{\ b} k^b_{\ a} + \frac{1}{l^2} e^a T_a + \frac{1}{l} \bar{\xi} \Psi + \frac{1}{l} \bar{\psi} \Xi \right) \\ & - d \left( \frac{\alpha_1}{2l} \epsilon_{abc} \omega^{ab} e^c + \frac{\alpha_3}{2l} \epsilon_{abc} \left( \tilde{k}^{ab} e^c + \omega^{ab} \tilde{h}^c \right) + \frac{\alpha_2}{2} \omega^a_{\ b} \tilde{k}^b_{\ a} + \frac{\alpha_4}{2} \omega^a_{\ b} k^b_{\ a} \right) \Bigg]. \end{aligned} \tag{6.70}$$

where $T^a = De^a$ is the torsion 2-form. This is the most general supergravity action in $(2+1)$ dimensions invariant under the minimal Maxwell superalgebra $s\mathcal{M}^g$. The first term corresponds to the so called exotic Lagrangian and it is Lorentz invariant [8]. The second term describes pure supergravity without cosmological constant. The terms proportional to $\alpha_2$, $\alpha_3$ and $\alpha_4$ contain the coupling of the spin connection to the new gauge fields $\tilde{k}^{ab}$, $k^{ab}$

and $\tilde{h}^c$. In particular the new Majorana spinor field $\xi$ appears in the terms proportional to $\alpha_3$ and $\alpha_4$. This action can be seen as a supersymmetric extension of [72, 73] where new extra fields have been added in order to have well defined $S$-expanded invariant tensors.

Furthermore, note that the new fields appear also in the boundary term. The inclusion of boundary contributions to (super)gravity models has been extensively studied in [13], [76], [77], [78].

Up to boundary terms, the full action (6.70) is invariant under local gauge transformations of the Maxwell supergroup and also under both supersymmetries, the one associated to the $Q$ generator

$$\begin{array}{lll} \delta\omega^{ab} = 0, & \delta\tilde{k}^{ab} = -\frac{1}{l}\bar{\epsilon}\gamma^{ab}\psi, & \delta k^{ab} = -\frac{1}{l}\bar{\epsilon}\gamma^{ab}\xi, \\ \delta e^{a} = \bar{\epsilon}\gamma^{a}\psi, & \delta\tilde{h}^{a} = \bar{\epsilon}\gamma^{a}\xi, & \delta\xi = \frac{1}{2l}e^{a}\gamma_{a}\epsilon + \frac{1}{4}\tilde{k}^{ab}\gamma_{ab}\epsilon, \\ \delta\psi = D\epsilon. & & \end{array} \tag{6.71}$$

and the other associated to the $\Sigma$ generator

$$\begin{array}{lll} \delta\omega^{ab} = 0, & \delta\tilde{k}^{ab} = 0, & \delta k^{ab} = -\frac{1}{l}\bar{\varrho}\gamma^{ab}\psi \\ \delta e^{a} = 0, & \delta\tilde{h}^{a} = \bar{\varrho}\gamma^{a}\psi & \delta\xi = d\varrho + \frac{1}{4}\omega^{ab}\gamma_{ab}\varrho, \\ \delta\psi = 0. & & \end{array}$$

In summary, in this chapter we have derived the $D = 3$ Chern-Simons supersymmetric action from the non-standard Maxwell superalgebra $s\mathcal{M}$. We have shown that the superMaxwell symmetries can be obtained from the $\mathfrak{osp}\,(2|1)\otimes\mathfrak{sp}\,(2)$ superalgebra combining the semigroup expansion procedure with the Inönü-Wigner contraction. This procedure allowed to obtain the invariant tensors for the Maxwell superalgebra and to build the most general $D = 3$ CS supersymmetric action invariant under the Maxwell supergroup. However, since in this superalgebra the four-momentum generators $P_a$ are not expressed as bilinears expressions of fermionic generators $Q$, we have that the supersymmetric action constructed out of the non-standard Maxwell superalgebra, does not describe a supergravity action but an exotic alternative supersymmetric action.

The CS supergravity action from a minimal Maxwell superalgebra $s\mathcal{M}^g$ has also been constructed. We have shown that this superalgebra can be derived from the $\mathfrak{osp}\,(2|1)\otimes\mathfrak{sp}\,(2)$ superalgebra using the semigroup expansion method.

The CS formalism used here represents a toy model in order to approach problems present in higher dimensions or in higher $\mathcal{N}$-extended supergravity theories.

# Part III

# $\mathcal{N} = 2$ Supergravity Theory

# Chapter 7

# Observations on BI from $\mathcal{N} = 2$ supergravity and the General Ward Identity

## 7.1 Introduction

Recently, there has been a particular dedication to the study of Born-Infeld (BI) theory and its generalization to multi-vectors, in relation to supersymmetric theories. This theory describes a non-linear electrodynamics in four dimensions and enjoys of relevant features, such as electric-magnetic duality simmetry. In particular, the supersymmetric version of the BI Lagrangian was constructed in [79],[80]. These non-linear theories emerges as a low-energy limit of partially broken $U(1)^n$ rigid $\mathcal{N} = 2$ supersymmetric theory [81], in which the supersymmetric breaking scale is sent to infinity [82]. As shown in [83], this mechanism requires the introduction of *magnetic* Fayet-Iliopoulos (FI) terms besides the electric ones, with the condition that the dual FI terms be not mutually local. On the other hand, the rigid partially broken $\mathcal{N} = 2$ theory with one vector multiplet of [83] (APT model), was also obtained as a flat limit of a suitable $\mathcal{N} = 2$ supergravity in [84]. This defines a $\mathcal{N} = 2$ supergravity origin of the original one-vector BI theory.

In the original rigid limit of [84], the gauging was electric and partial supersymmetry breaking required the use of a specific choice of symplectic frame in which the prepotential of the special geometry does not exist. More general, partially broken $\mathcal{N} = 2$ supergravities were constructed in [85] using an analogous choice of symplectic frame. This restriction, which is forced within the framework of standard (i.e. electric) gaugings by some no-go theorems [86], can be avoided in the context of dyonic gaugings. In fact, as shown in [87] partial supersymmetry breaking can occur in any symplectic frame (and in particular in one in which the prepotential does exist) using an *embedding tensor* [88, 89, 90] with both electric and magnetic components. Consistency of such gaugings requires the introduction of antisymmetric tensor fields dual to scalars [91, 92, 93, 94, 95].

In this chapter we present the results obtained in [96], where we have generalized the results of [84] to the case of $n$ vector multiplets. Our starting point is the construction of an appropriate dyonic gauging of an $\mathcal{N}=2$ supergravity coupled to $n$ vector multiplets and to hypermultiplets allowing for a well-defined rigid limit to a multi-vector APT model, and thus generalizing [84]. This would clarify the supergravity origin of the multifield BI of [82] and, in particular, to understand the origin of the dyonic FI as deriving from electric and magnetic charges in the supergravity gauged model.

A crucial part of our analysis is the definition of the rigid limit: Rescalings of the fields and of the embedding tensor by powers of $\mu = M_{Pl}/\Lambda$ (where $M_{Pl}$ is the Planck mass $M_{Pl}$ and $\Lambda$ is the supersymmetry breaking scale) have to be devised in order for the original supersymmetries to survive the limit $\mu \to \infty$.

Although they decouple for $M_{Pl} \to \infty$, the gravitini and the hyperini (the fermion fields in the hypermultiplets) have a role in defining the general features of the resulting partially broken rigid supersymmetry: Their supersymmetry transformation laws survive the rigid limit and contribute a non-trivial traceless constant matrix $C_A{}^B$ to the scalar potential Ward identity of the final supersymmetric theory:

$$\mathcal{V}\delta_A^B + C_A{}^B = \sum_{i=1}^{n} \delta\lambda^{iB}\delta\lambda_{iA}\,, \tag{7.1}$$

where $\mathcal{V}$ is the scalar potential and $\lambda^{iA}$ and $\lambda_{iA} \equiv g_{i\bar{\jmath}}\lambda^{\bar{\jmath}}_A$ are the chiral and anti-chiral components of the gaugini. The constant matrix $C_A{}^B$, is an essential ingredient in order for the partial supersymmetry breaking to occur in the rigid theory. In [84] it was shown that (7.1) originates from the supergravity Ward identity. We show the same feature in our generalized dyonic setting.

Eventually, we give in a self-contained form, all the relevant identities related to the most general gauging of special Kähler and quaternionic Kähler isometries in a generic $\mathcal{N}=2$ model, including the potential Ward-identity [97]. The general proof of the Ward-identity for generic dyonic gaugings is a further result of our work.

## 7.2 General $\mathcal{N} = 2$ Gauging Identities

In the present section we give some identities which hold for the most general gauging of $\mathcal{N} = 2$ supergravity involving both electric and magnetic charges. In particular, the Ward identity [97] which is required by the supersymmetry invariance of the gauged Lagrangian, is considered. Here we shall work in Poincaré supergravity using the symplectic covariant description of the special Kähler manifold and generalize the identities given in [31] to electric-magnetic gaugings and the analysis in [91] to non-abelian gauge groups. In the later sections these results will be applied to the very specific electric-magnetic abelian gauging, in which the rigid limit of spontaneously broken $\mathcal{N} = 2$ supergravity is discussed.

We start from an $\mathcal{N} = 2$ supergravity coupled to $n$ vector multiplets and $n_H$ hypermultiplets. The scalar sector consists of $n$ complex scalars $z^i$ and $4n_H$ hyperscalars $q^u$ parametrizing a special Kähler manifold $\mathcal{M}_{SK}$ [98, 99, 100] and a quaternionic Kähler manifold $\mathcal{M}_{QK}$ [101, 102, 103], respectively, so that the scalar manifold has the form:

$$\mathcal{M}_{scalar} = \mathcal{M}_{SK}\,(n) \times \mathcal{M}_{QK}\,(n_H)\,. \tag{7.2}$$

A deep and self-contained study of the properties of special Kähler and quaternionic Kähler manifolds can be found in [31]. The main concepts are reviewed in Appendix D.

### 7.2.1 Some useful relations on the sigma-model geometry.

A special Kähler manifold is locally described by a choice of complex coordinates $z^i$ and a section of the flat holomorphic bundle defined on it:

$$\Omega^M(z) = \begin{pmatrix} X^\Lambda(z) \\ F_\Lambda(z) \end{pmatrix}\ ,\ \ \Lambda = 0, \dots, n\,, \quad M = 1, ..., 2n+2 \tag{7.3}$$

in terms of which the Kähler potential reads:

$$\mathcal{K}(z,\bar{z}) = -\log[i\,\overline{\Omega}(\bar{z})^T \mathbb{C}\Omega(z)]\,, \tag{7.4}$$

In terms of $\Omega$ and $\mathcal{K}$ one defines the covariantly holomorphic section $V^M \equiv e^{\frac{\mathcal{K}}{2}}\,\Omega^M$ (see Appendix D).

A holomorphic function $f_g(z)$ and a symplectic matrix $\mathbb{M}[g] = (\mathbb{M}[g]_M{}^N)$ are associated with each element $g$ of the identity-connected component $G_{SK}$ of the isometry group of $\mathcal{M}_{SK}$ such that, if $g : z^i \to z'^i = z'^i(z)$:

$$\Omega\,(z') = e^{f_g(z)}\,\mathbb{M}[g]^{-T}\,\Omega(z) \;\;\Leftrightarrow\;\; \mathcal{K}(z',\bar{z}') = \mathcal{K}(z,\bar{z}) - f_g(z) - \bar{f}_g(\bar{z})\,, \tag{7.5}$$

where $\mathbb{M}^{-T} \equiv (\mathbb{M}^{-1})^T$.

If $\{t_a\}$ are the infinitesimal generators of $G_{SK}$ and $k_a = k_a^i(z)\partial_i + k_a^{\bar{\imath}}(\bar{z})\partial_{\bar{\imath}}$ are the corresponding Killing vectors satisfying the closure conditions:

$$[t_a,\, t_b] = f_{ab}{}^c\, t_c \ , \ \ [k_a,\, k_b] = -f_{ab}{}^c\, k_c\,, \tag{7.6}$$

then equations (7.5) imply:

$$\begin{aligned} \ell_a\Omega^M &= k_a^i\partial_i\Omega^M = -t_{aN}{}^M\,\Omega^N + f_a(z)\Omega^M \ , \ \ \ell_a\mathcal{K} = k_a^i\partial_i\mathcal{K} + k_a^{\bar{\imath}}\partial_{\bar{\imath}}\mathcal{K} = -(f_a+\bar{f}_a)\,\mathcal{K}, & (7.7)\\ \ell_a V^M &= (k_a^i\partial_i + k_a^{\bar{\imath}}\partial_{\bar{\imath}})V^M = -t_{aN}{}^M\, V^N + \frac{f_a-\bar{f}_a}{2}\, V^M\,, & (7.8)\end{aligned}$$

where $f_a = \partial_i f k_a^i$ and $t_{aN}{}^M$ is the symplectic matrix representation of the generator $t_a$ on covariant vectors: $t_{a[N}{}^P\mathbb{C}_{M]P} = 0\,,\ (t_a\Omega)^M = -t_{aN}{}^M\,\Omega^N$.

Denote by $\mathcal{P}_a(z,\bar{z})$ the moment map corresponding to $k_a$, defined as follows [99]:

$$k_a^i = i\, g^{i\bar{\jmath}}\,\partial_{\bar{\jmath}}\mathcal{P}_a\,,\ \ k_a^{\bar{\imath}} = -i\, g^{\bar{\imath}i}\,\partial_i\mathcal{P}_a\,, \tag{7.9}$$

and satisfying, under general assumptions on $G_{SK}$,

$$i g_{i\bar{\jmath}}\, k_{[a}^i\, k_{b]}^{\bar{\jmath}} = -\frac{1}{2}\, f_{ab}{}^c\,(\mathcal{P}_c - C_c)\,, \tag{7.10}$$

where $C_c$ is constant vector in the adjoint of $G_{SK}$ which can be reabsorbed by the redefinition $\mathcal{P}_c - C_c \to \mathcal{P}_c$.

Eqs. (7.9) are solved by:

$$\begin{aligned} \mathcal{P}_a &= -\frac{i}{2}\left(k_a^i\partial_i\mathcal{K} - k_a^{\bar{\imath}}\partial_{\bar{\imath}}\mathcal{K}\right) + \mathrm{Im}(f_a) = \\ &= i\, k_a^{\bar{\imath}}\partial_{\bar{\imath}}\mathcal{K} + i\,\bar{f}_a = -i\, k_a^i\partial_i\mathcal{K} - i\, f_a\,, \end{aligned} \tag{7.11}$$

where we have used the second of (7.7) and (7.10). On the other hand, using (7.8) and (7.11) we find:

$$k_a^i\, U_i^M = -t_{aN}{}^M\, V^N + i\,\mathcal{P}_a\, V^M\,. \tag{7.12}$$

Contracting the above equation with $\mathbb{C}\overline{V}$ and using the special geometry relations $V^T\mathbb{C}\overline{V} = i$, $V^T\mathbb{C}U_i = 0$, (see Appendix D), we find:

$$\mathcal{P}_a = -V^N\, t_{aNM}\overline{V}^M = -\overline{V}^N\, t_{aNM}\, V^P\,, \tag{7.13}$$

where we have defined $t_{aNM} \equiv t_{aN}{}^P\mathbb{C}_{PM} = t_{aMN}$.

Moreover, we have the general property:

$$t_{a\,MN}\Omega^M\Omega^N = 0\ ,\quad \forall t_a\,. \tag{7.14}$$

which follows by contracting (7.7) with $\mathbb{C}\Omega$ and using the third of (D.10), i.e. $V^T\mathbb{C}U_i = 0$, which implies

$$\Omega^T\mathbb{C}\partial_i\Omega = 0\,. \tag{7.15}$$

The geometry of the quaternionic Kähler manifold is recalled in Appendix D, where the general properties of the quaternionic isometries $t_m$ and their description in terms of Killing vectors $k_m$ and tri-holomorphic momentum maps $\mathcal{P}^x_m$ are reviewed.

### 7.2.2 Symplectically-covariant gaugings of $\mathcal{N} = 2$ supergravity.

Let us consider the gauging of a gauge group $\mathcal{G}$ in the isometry group of the scalar manifold $\mathcal{M}_{scalar}$. The gauge generators are conveniently written as components of an electric-magnetic vector $X_M = (X_\Lambda,\, X^\Lambda)$, according to the notation of [93] and expanded in the generators $\{t_a,\, t_m\}$ of the isometry groups of $\mathcal{M}_{SK}$ and $\mathcal{M}_{QK}$ through the embedding tensor:

$$X_M = \Theta_M{}^a\, t_a + \Theta_M{}^m\, t_m\,. \tag{7.16}$$

The symplectic electric-magnetic duality action of $X_M$ is described by the symplectic matrices: $X_{MN}{}^P = \Theta_M{}^a\, t_{a\,N}{}^P$. Consistency of the gauging is guaranteed by the following set of linear and quadratic constraints on the embedding tensor:

$$X_{(MNP)} \equiv X_{(MN}{}^Q\mathbb{C}_{Q|P)} = 0\,, \tag{7.17}$$

$$\Theta_M{}^a\Theta_N{}^b f_{ab}{}^c + X_{MN}{}^P\,\Theta_P{}^c = 0\,, \tag{7.18}$$

$$\Theta_M{}^m\Theta_N{}^n f_{mn}{}^p + X_{MN}{}^P\,\Theta_P{}^p = 0\,, \tag{7.19}$$

$$\Theta_M{}^a\mathbb{C}^{MN}\Theta_N{}^b = \Theta_M{}^a\mathbb{C}^{MN}\Theta_N{}^n = \Theta_M{}^m\mathbb{C}^{MN}\Theta_N{}^n = 0\ . \tag{7.20}$$

Conditions (7.18), (7.19) are closure constraints, i.e. are equivalent to

$$[X_M,\, X_N] = -X_{MN}{}^P\, X_P\,. \tag{7.21}$$

The first two equalities in (7.20) follow from (7.17) and (7.18), (7.19) while the last one has to be imposed independently [93]. We can define gauge Killing vectors and momentum maps as follows:

$$k_M \equiv \Theta_M{}^a\, k_a\ ,\ \ \mathcal{P}_M \equiv \Theta_M{}^a\,\mathcal{P}_a\ ,\ \ \mathcal{P}^x_M \equiv \Theta_M{}^m\,\mathcal{P}^x_m\,. \tag{7.22}$$

From the quadratic constraints and Eqs.(7.10) and (D.36) we find the equivariance conditions:

$$
\begin{aligned}
i g_{i\bar{\jmath}}\, k^i_{[M}\, k^{\bar{\jmath}}_{N]} &= \frac{1}{2}\, X_{MN}{}^P\, \mathcal{P}_P\,, && (7.23)\\
2\, K^x_{uv}\, k^u_M\, k^v_N + \epsilon^{xyz}\, \mathcal{P}^y_M\, \mathcal{P}^z_N &= X_{MN}{}^P\, \mathcal{P}^x_P\,, && (7.24)
\end{aligned}
$$

where we have used $\lambda = -1$.

Using the linear constraint we can prove the following identities:

$$\mathcal{P}_M \Omega^M = 0\ ,\ \ k^i_M\, \Omega^M = 0\,. \tag{7.25}$$

To prove the first one we write (7.13) for the gauge-momentum maps:

$$\mathcal{P}_M = -e^{\mathcal{K}}\, X_{MNP} \overline{\Omega}^N \Omega^P\,. \tag{7.26}$$

Contracting both sides with $\Omega^M$ we find:

$$\Omega^M \mathcal{P}_M = -e^{\mathcal{K}}\, \Omega^M X_{MNP} \overline{\Omega}^N \Omega^P = \frac{e^{\mathcal{K}}}{2}\, \overline{\Omega}^N X_{NMP} \Omega^M \Omega^P = 0\,, \tag{7.27}$$

where we have used the linear constraint (7.17) and the symplectic property of the matrices $X_{MN}{}^P$:

$$2X_{(MP)N} = -X_{NMP}\,, \tag{7.28}$$

being $X_{MNP} \equiv X_{MN}{}^Q \mathbb{C}_{QP}$. Last equality in (7.27) then follows from (7.14).

Let us now prove the second of (7.25)

$$\Omega^M\, k^i_M = i\, g^{i\bar{\jmath}}\, \Omega^M\, \partial_{\bar{\jmath}} \mathcal{P}_M = i\, g^{i\bar{\jmath}}\, \partial_{\bar{\jmath}}(\Omega^M\, \mathcal{P}_M) = 0\,, \tag{7.29}$$

where we have used the first of (7.25).

From (7.25) we can deduce the following relations:

$$D_i(V^M \mathcal{P}_M) = 0\ \Rightarrow\ U^M_i \mathcal{P}_M + V^M \partial_i \mathcal{P} = 0\ \Rightarrow\ U^M_i \mathcal{P}_M + i\, g_{i\bar{\jmath}}\, k^{\bar{\jmath}}_M V^M = 0\,. \tag{7.30}$$

Contracting (7.12) with the embedding tensor we find:

$$k^i_M\, U^P_i = -X_{MN}{}^P\, V^N + i\, \mathcal{P}_M\, V^P\,. \tag{7.31}$$

Contracting both sides with $\overline{V}^M$ and using the first of (7.25) we find:

$$\overline{V}^M k^i_M\, U^P_i = -X_{MN}{}^P\, \overline{V}^M V^N\,. \tag{7.32}$$

Next we contract both sides with $\Theta_P$, where $\Theta_P$ can be either ${\Theta_P}^a$ or ${\Theta_P}^n$ and use the quadratic constraints (7.21) which imply that the *generalized structure constants* ${X_{MN}}^P$ are antisymmetric in the first two indices only if contracted to the right by $\Theta_P$: ${X_{MN}}^P\Theta_P = -{X_{NM}}^P\Theta_P$. By virtue of this feature we find:

$$\overline{V}^M k^i_M\, U^P_i \Theta_P = -{X_{MN}}^P\, \overline{V}^M V^N \Theta_P = {X_{NM}}^P\, \overline{V}^M V^N \Theta_P = -V^M k^{\bar{\imath}}_M\, \overline{U}^P_{\bar{\imath}}\, \Theta_P\,. \tag{7.33}$$

The identities (7.25) and (7.33) were proven in the electric case in [99]. Here, for the first time, we give a general, compact proof of their generalization to a generic dyonic gauging, showing that they directly follow from the linear constraint on the embedding tensor.

### 7.2.3 The general Ward identity

The supersymmetry Ward identity [97] is required by the cancelation of the supersymmetry variation terms of the gauged Lagrangian, which are quadratic in the embedding tensor. It expresses a relation between the fermion shift matrices and the scalar potential $\mathcal{V}(z, \bar{z}, q)$ and has the following form:

$$g_{i\bar{\jmath}}\, W^{i\,AC}\overline{W}^{\bar{\jmath}}_{BC} + 2\, {N_\alpha}^A\, {N^\alpha}_B - 12\, S^{AC} S_{BC} = \delta^B_A\, \mathcal{V}(z, \bar{z}, q)\,, \tag{7.34}$$

where $W^{i\,AC}$, $N^\alpha_B$, $S_{AB}$ are the supersymmetry shift-matrices of the gaugini $\lambda^i$, hyperini $\zeta^\alpha$ and gravitini $\psi_A$, respectively[1]. In this case we have that these fermion shifts have the following symplectically-invariant expressions:

$$\begin{aligned} S_{AB} &= \frac{i}{2}\, {(\sigma^x)_A}^C \epsilon_{BC}\, \mathcal{P}^x_M\, V^M\,, & (7.35)\\ W^{i\,AB} &= \epsilon^{AB}\, k^i_M\, \overline{V}^M - i\, {(\sigma^x)_C}^B\, \epsilon^{CA} \mathcal{P}^x_M\, g^{i\bar{\jmath}} \overline{U}^M_{\bar{\jmath}}\,, & (7.36)\\ {N_\alpha}^A &= 2\mathcal{U}^A_{u\,\alpha}\, k^u_M\, \overline{V}^M\,, \quad {N^\alpha}_A \equiv ({N_\alpha}^A)^* = -2\, {\mathcal{U}_{u\,A}}^\alpha\, k^u_M\, V^M\,. & (7.37) \end{aligned}$$

Let us now prove the Ward identity [97] for the generic dyonic gauging of $\mathcal{N} = 2$ supergravity. We shall evaluate each term in the left hand side of (7.34) separately.

Let us firt evaluate the square of the gaugini shifts:

$$\begin{aligned} W^{i\,AC}\overline{W}^{\bar{\jmath}}_{BC} g_{i\bar{\jmath}} &= \delta^A_B\, k^i_M k^{\bar{\jmath}}_N g_{i\bar{\jmath}} \overline{V}^M\, V^N - i\, {(\sigma^x)_B}^A \left(k^{\bar{\jmath}}_M\, V^M\, \overline{U}^N_{\bar{\jmath}} - k^i_M\, \overline{V}^M\, U^N_i\right) \mathcal{P}^x_N + \\ &\quad + {(\sigma^x\sigma^y)_B}^A\, \mathcal{P}^x_M \mathcal{P}^y_N U^{MN}\,, \end{aligned} \tag{7.38}$$

[1]We use the following convention for rising and lowering symplectic indices:

$$v_A = \epsilon_{AB}\, v^B\,,\;\; v^A = \epsilon^{BA}\, v_B\,,\;\; v_\alpha = \mathbb{C}_{\alpha\beta}\, v^\beta\,,\;\; v^\alpha = \mathbb{C}^{\beta\alpha}\, v_\beta\,.$$

where $U^{MN} \equiv U_i^N \, g^{i\bar{\jmath}} \, \overline{U}_{\bar{\jmath}}^N$, see (D.20). On the r.h.s of the above expression we split the terms proportional to $\delta_B^A$ from those proportional to $(\sigma^x)_B{}^A$ and use Eq. (7.33) to find:

$$\begin{aligned} W^{i\,AC}\overline{W}^{\bar{\jmath}}_{BC} g_{i\bar{\jmath}} \;=\;& \delta_B^A \left(k^i_M k^{\bar{\jmath}}_N g_{i\bar{\jmath}} \overline{V}^M \, V^N + \mathcal{P}^x_N \mathcal{P}^x_M U^{MN}\right) + i\,(\sigma^x)_B{}^A \, \Big(-2\,X_{MN}{}^P \overline{V}^M \, V^N \, \mathcal{P}^x_P + \\ & + \epsilon^{xyz} \, \mathcal{P}^y_M \mathcal{P}^z_N U^{[MN]}\Big) \; . \end{aligned} \tag{7.39}$$

Now using Eqs. (D.20) and the locality constraint (7.20) we can write:

$$\mathcal{P}^y_M \mathcal{P}^z_N U^{[MN]} = -\frac{i}{2} \, \mathcal{P}^y_M \mathcal{P}^z_N \mathbb{C}^{MN} - \mathcal{P}^y_M \mathcal{P}^z_N \overline{V}^{[M} V^{N]} = -\mathcal{P}^y_M \mathcal{P}^z_N \overline{V}^{[M} V^{N]} \, , \tag{7.40}$$

so that we finally find:

$$\begin{aligned} W^{i\,AC}\overline{W}^{\bar{\jmath}}_{BC} g_{i\bar{\jmath}} \;=\;& \delta_B^A \left(k^i_M k^{\bar{\jmath}}_N g_{i\bar{\jmath}} \overline{V}^M \, V^N + \mathcal{P}^x_N \mathcal{P}^x_M U^{MN}\right) + i\,(\sigma^x)_B{}^A \, \Big(-2\,X_{MN}{}^P \overline{V}^M \, V^N \, \mathcal{P}^x_P + \\ & - \epsilon^{xyz} \, \mathcal{P}^y_M \mathcal{P}^z_N \, \overline{V}^M V^N\Big) \end{aligned} \tag{7.41}$$

Let us now consider the evaluation of the square of the hyperini shifts:

$$2\, N_\alpha{}^A \, N^\alpha{}_A = 8\, \mathcal{U}^{A\alpha}_u \mathcal{U}_{v\,B\alpha} \, k^u_M \, k^v_N \, \overline{V}^M V^N = 4\left(\delta^A_B h_{uv} + i\,(\sigma^x)_B{}^A \, K^x_{uv}\right) k^u_M \, k^v_N \, \overline{V}^M V^N \, . \tag{7.42}$$

where we have used Eq. (D.31). Finally let us consider the square of the gravitini shifts:

$$-12\, S^{AC} \, S_{BC} = -3\,(\sigma^x\sigma^y)_B{}^A \, \mathcal{P}^x_M \mathcal{P}^y_N \, V^M \overline{V}^N = -3\,\mathcal{P}^x_M \mathcal{P}^x_N \, V^M \overline{V}^N + 3i\,\epsilon^{xyz} \, \mathcal{P}^y_M \mathcal{P}^z_N \, \overline{V}^M V^N (\sigma^x)_B{}^A \, . \tag{7.43}$$

In this way, we find the following expression:

$$g_{i\bar{\jmath}} \, W^{i\,AC} \overline{W}^{\bar{\jmath}}_{BC} + 2\, N_\alpha{}^A \, N^\alpha{}_B - 12\, S^{AC} S_{BC} = \delta^A_B \, V(z,\bar{z},q) + i\, Z^x \, (\sigma^x)_B{}^A \, , \tag{7.44}$$

where

$$V(z,\bar{z},q) = (k^i_M k^{\bar{\jmath}}_N g_{i\bar{\jmath}} + 4\, h_{uv} k^u_M k^v_N) \overline{V}^M \, V^N + (U^{MN} - 3\, V^M \overline{V}^N) \mathcal{P}^x_N \mathcal{P}^x_M \, , \tag{7.45}$$

is the general symplectic invariant expression of the scalar potential given in [93] as a generalization of [31] to the case of dyonic gaugings, and

$$Z^x = (-2\, X_{MN}{}^P \, \mathcal{P}^x_P + 2\,\epsilon^{xyz} \, \mathcal{P}^y_M \mathcal{P}^z_N + 4\, K^x_{uv} k^u_M \, k^v_N) \overline{V}^M V^N \, . \tag{7.46}$$

From the equivariance condition (7.24) it follows that $Z^x = 0$, so that the Ward identity is proven.

### 7.2.4 Abelian gauging of quaternionic isometries

The previous discussion holds for the gauging of a gauge group $\mathcal{G}$ in the isometry group of the scalar manifold $\mathcal{M}_{scalar}$. In what follows, we will consider a gauging which involves an abelian group of quaternionic isometries. In this way, being only quaternionic isometries gauged, the generalized structure constants vanish: $X_{MN}{}^P = 0$. Then, (7.24) implies

$$K^x_{uv}\, k^u_M\, k^v_N = -\frac{1}{2}\epsilon^{xyz}\,\mathcal{P}^y_M\,\mathcal{P}^z_N.$$

Using this identity, it is easy to show that in this case the three fermion-shift contribute to $Z^x$ and show that they cancel against one another:

$$g_{i\bar{\jmath}}\, W^{i\,AC}\overline{W}^{\bar{\jmath}}_{BC} \;\rightarrow\; -\epsilon^{xyz}\,\mathcal{P}^y_M\mathcal{P}^z_N\overline{V}^M V^N\,, \tag{7.47}$$

$$2\,N_\alpha{}^A\,N^\alpha{}_B \;\rightarrow\; -2\,\epsilon^{xyz}\,\mathcal{P}^y_M\mathcal{P}^z_N\overline{V}^M V^N\,, \tag{7.48}$$

$$-12\,S^{AC}S_{BC} \;\rightarrow\; 3\,\epsilon^{xyz}\,\mathcal{P}^y_M\mathcal{P}^z_N\overline{V}^M V^N\,. \tag{7.49}$$

In what follows, we will be interested in the limit of a gauged $\mathcal{N} = 2$ supergravity of this kind to a rigid supersymmetric theory of $n$ vector multiplets [81] (rigid limit), along the lines of [84]. In particular, the rigid limit of the Ward identity (7.34) [83, 84, 104, 105] will be a crucial point in our analysis.

The Ward identity of an $\mathcal{N} = 2$ (abelian) rigid supersymmetric theory of $n$ vector multiplets is given by the general expression [83, 84, 105]:

$$\mathring{g}_{i\bar{\jmath}}\,\mathring{W}^{i\,AC}\overline{\mathring{W}}^{\bar{\jmath}}_{BC} = \delta^A_B\,\mathcal{V}^{(APT)}_{\mathcal{N}=2}(z,\bar{z}) + C_B{}^A\,, \tag{7.50}$$

where $\mathcal{V}^{(APT)}_{\mathcal{N}=2}(z,\bar{z})$ is the $\mathcal{N} = 2$ scalar potential in the spontaneously broken rigid theory, which reproduces the APT scalar potential in the case of one-vector multiplet, $C_B{}^A$ is a $\mathfrak{su}(2)$-traceless matrix, $\mathring{g}_{i\bar{\jmath}}$ is the metric of the rigid special Kähler manifold describing the scalar fields $z^i$ in the vector multiplets and $\mathring{W}^{i\,AC}$ are the gaugini shift-matrices.

As shown in [83, 84], partial breaking of supersymmetry can occur only if $C_B{}^A \neq 0$. This happens in the presence of mutually non-local electric and magnetic Fayet-Iliopoulos terms [83].

The symplectically-covariant relations (7.47),(7.48),(7.49) allow to elucidate the meaning of the matrix $C_B{}^A$ by relating the rigid Ward identity (7.50) to the supergravity one (7.34). In fact, let us rewrite the Ward identity in the form:

$$g_{i\bar{\jmath}}\,W^{i\,AC}\overline{W}^{\bar{\jmath}}_{BC} = \delta^B_A\,\mathcal{V}(z,\bar{z},q) - 2\,N_\alpha{}^A\,N^\alpha{}_B + 12\,S^{AC}S_{BC}\,, \tag{7.51}$$

As we will see in the next section, all squared fermion-shift matrices in (7.51) survive in the rigid limit ($M_{Pl} \to \infty$). In particular the left-hand-side of (7.51) reproduces that of (7.50), while the constant matrix $C_B{}^A$ receives contribution from the terms in $N_\alpha{}^A \, N^\alpha{}_B$, $S^{AC} S_{BC}$ proportional to $\sigma^x$, which are given in (7.48), (7.49). More specifically we will find that:

$$C_B{}^A = \lim_{M_{Pl}\to\infty} \frac{M_{Pl}^4}{\Lambda^4} \left( -i\,\epsilon^{xyz}\,\mathcal{P}^y_M \mathcal{P}^z_N \overline{V}^M V^N (\sigma^z)_B{}^A \right) , \tag{7.52}$$

where $\Lambda$ is the supersymmetry-breaking scale. The same hyperini and gravitini shift-matrices also contribute terms proportional to $\delta^A_B$ which affect the form of the scalar potential in the resulting rigid theory. These terms were explicitly computed in (7.42) and (7.43) so that we can identify:

$$\mathcal{V}^{(APT)}_{\mathcal{N}=2} = \lim_{M_{Pl}\to\infty} \frac{M_{Pl}^4}{\Lambda^4} \left[ \mathcal{V}(z, \bar{z}, q) - (4\, h_{uv}\, k^u_M k^v_N - 3\, \mathcal{P}^x_M \mathcal{P}^x_N) \overline{V}^M V^N \right] . \tag{7.53}$$

As we shall prove in the next section, in the rigid limit, the leading order terms in $\Theta_N{}^n V^N$ are independent of $z^i$, $\bar{z}^i$, but only depend on the hyperscalars $q^u$, so that:

$$\mathcal{V}^{(APT)}_{\mathcal{N}=2} = \lim_{M_{Pl}\to\infty} \frac{M_{Pl}^4}{\Lambda^4} \left[ \mathcal{V}(z, \bar{z}, q) \right] + A(q) \, . \tag{7.54}$$

Since the fluctuations of $q^u$ are suppressed by a factor $M_{Pl}^{-1}$, in the rigid theory the hyperscalars are non-dynamical, i.e. constants. As a consequence of this the $\mathcal{N} = 2$ scalar potential of the rigid theory $\mathcal{V}^{(APT)}$ is given by the rigid limit of the supergravity potential $\mathcal{V}$ modulo an *unphysical additive constant*. This was already observed in [84] in a particular model.

## 7.3 Multi-vector generalization of the APT model

In this section, we present a supergravity model with partial breaking of $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry which, in the low energy limit, gives rise to a rigid supersymmetric theory corresponding to the generalization of the APT model [83] to a generic number $n$ of vector multiplets. As we will see, this procedure admits a well defined limit to many-vectors supersymmetric Born-Infeld theory.

The minimal underlying supergravity model consists of $\mathcal{N} = 2$ supergravity coupled to $n$ vector multiplets and a single charged hypermultiplet, whose scalars parametrize the quaternionic manifold

$$\mathcal{M}_{QK} = \frac{SO\,(4,1)}{SO\,(4)} . \tag{7.55}$$

Following the procedure adopted in [84], let us consider a special geometry symplectic section

$$\Omega^M\left(z^i\right)=\begin{pmatrix}X^\Lambda\left(z^i\right)\\ F_\Lambda\left(z^i\right)\end{pmatrix}\qquad \Lambda=0,I,\qquad I,i=1,\ldots,n, \tag{7.56}$$

(where $i$ are holomorphic-coordinate indices) in a symplectic frame where a holomorphic prepotential exists. Using special coordinates $z^i=\delta^i_I X^I/X^0$, it takes the form:

$$F\left(X^\Lambda\right)=-i\left(X^0\right)^2 f\left(X^i/X^0\right)\ , \tag{7.57}$$

so that, choosing:

$$X^\Lambda=\begin{cases} X^0=1\\ X^i=z^i\end{cases}\ , \tag{7.58}$$

we found

$$F_\Lambda=\begin{cases} F_0=\partial F/\partial X^0=-i\left(2f-z^i\partial_i f\right)\\ F_i=\partial F/\partial X^i=-i\partial_i f\end{cases}\ , \tag{7.59}$$

and

$$\Omega^M=\begin{pmatrix}1\\ z^i\\ -i\left(2f-z^i\partial_i f\right)\\ -i\partial_i f\end{pmatrix}. \tag{7.60}$$

In terms of the holomorphic sections the Kähler potential reads

$$\begin{aligned}\mathcal{K}&=-\ln\left[i\left(\bar{X}^\Lambda F_\Lambda-X^\Lambda\bar{F}_\Lambda\right)\right],\\ &=-\ln\left[2\left(f+\bar{f}\right)-\left(z-\bar{z}\right)^i\left(\partial_i f-\overline{\partial_i f}\right)\right].\end{aligned} \tag{7.61}$$

In order to generalize the procedure in [84] to the case of $n$ vector multiplets, we should consider a rigid limit ($\mu=M_{Pl}/\Lambda\to\infty$, where $M_{Pl}$ denotes the Planck scale and $\Lambda$ the supersymmetry breaking scale), leading to partial breaking $\mathcal{N}=2\to\mathcal{N}=1$ in a rigid supersymmetric theory. In the derivation of [84] for partial breaking $\mathcal{N}=2\to\mathcal{N}=1$, an essential point was the presence of a linear term (in the holomorphic special coordinate $z$) in the expansion of the prepotential $f(z)$ in powers of $\frac{1}{\mu}$:

$$f\left(z\right)=\frac{1}{4}+\frac{z}{2\mu}+\frac{\phi(z)}{2\mu^2}+O\left(\frac{1}{\mu^3}\right). \tag{7.62}$$

In this way, for the case of many vector multiplets we need to introduce a set of $n$ constant parameters $\eta_i$, so that the holomorphic prepotential takes the form

$$f\left(z^i\right)=\frac{1}{4}+\frac{\eta_i z^i}{2\mu}+\frac{\phi(z^i)}{2\mu^2}+O\left(\frac{1}{\mu^3}\right). \tag{7.63}$$

Using the standard formula for the Kähler potential (7.61) one derives, up to order $\mu^{-3}$

$$\begin{aligned}\mathcal{K} &= \frac{\mathring{\mathcal{K}}^{(1)}}{\mu}+\frac{\mathring{\mathcal{K}}}{\mu^2}r \\ &= -\frac{\eta_i\left(z+\bar{z}\right)^i}{\mu}-\frac{1}{\mu^2}\left[\phi+\bar{\phi}-(z-\bar{z})^i\left(\frac{\partial_i\phi-\overline{\partial_i\phi}}{2}\right)-\frac{\left(\eta_i\left(z+\bar{z}\right)^i\right)^2}{2}\right].\end{aligned}$$

so that

$$\begin{aligned}g_{i\bar{\jmath}} &= \partial_i\partial_{\bar{\jmath}}\mathcal{K} \\ &= \frac{1}{\mu^2}\mathring{g}_{i\bar{\jmath}}=\frac{1}{\mu^2}\left\{\eta_i\eta_j-\frac{1}{2}\left(\overline{\partial_{ij}\phi}+\partial_{ij}\phi\right)\right\},\end{aligned} \tag{7.64}$$

where $\mathring{g}_{i\bar{\jmath}}$ corresponds to the rigid special Kähler metric. Let us note that the rigid special Kähler metric can be found, in terms of the (rigid) $Sp(2n)$-symplectic section

$$\hat{\Omega}^{\mathcal{M}}=\left(\begin{array}{c} z^i \\ \partial_i\mathcal{F}\end{array}\right)=\left(\begin{array}{c} z^i \\ \frac{i}{2}(\eta_i\eta_j z^j-\partial_i\phi)\end{array}\right),\quad \mathcal{M}=1,\cdots,2n\,, \tag{7.65}$$

from the (rigid) prepotential

$$\mathcal{F}=\frac{i}{4}\left[\left(\eta_i z^i\right)^2-2\phi\right]. \tag{7.66}$$

In fact,

$$\begin{aligned}\mathcal{F}_{i\bar{\jmath}} &= \partial_i\partial_{\bar{\jmath}}\mathcal{F}=\frac{i}{2}\left(\eta_i\eta_{\bar{\jmath}}-\partial_i\partial_{\bar{\jmath}}\phi\right) \\ &= \frac{i}{4}\left(\overline{\partial_i\partial_{\bar{\jmath}}\phi}-\partial_i\partial_{\bar{\jmath}}\phi\right)+\frac{i}{2}\left(\eta_i\eta_{\bar{\jmath}}-\frac{1}{2}\left(\overline{\partial_i\partial_{\bar{\jmath}}\phi}+\partial_i\partial_{\bar{\jmath}}\phi\right)\right) \\ &= \frac{i}{4}\left(\overline{\partial_i\partial_{\bar{\jmath}}\phi}-\partial_i\partial_{\bar{\jmath}}\phi\right)+\frac{i}{2}\mathring{g}_{i\bar{\jmath}},\end{aligned}$$

which can be written as

$$\mathcal{F}_{i\bar{\jmath}}=\tau_{1i\bar{\jmath}}+i\tau_{2i\bar{\jmath}},$$

and where we have defined

$$\tau_{1i\bar{\jmath}} \equiv \frac{i}{4}\left(\overline{\partial_i\partial_{\bar{\jmath}}\phi}-\partial_i\partial_{\bar{\jmath}}\phi\right), \tag{7.67}$$

$$\tau_{2i\bar{\jmath}} \equiv \frac{\mathring{g}_{i\bar{\jmath}}}{2}. \tag{7.68}$$

The covariantly holomorphic symplectic section $V^M \equiv e^{\mathcal{K}/2}\Omega^M$ has the following expansion

$$V^M = \left( \begin{array}{c} 1 - \frac{1}{2\mu}\eta_i \left(z+\bar{z}\right)^i + O\left(1/\mu^2\right) \\ z^j - \frac{1}{2\mu}\eta_i \left(z+\bar{z}\right)^i z^j + O\left(1/\mu^2\right) \\ -i\left[\frac{1}{2} + \frac{1}{2\mu}\left\{\eta_i z^i - \frac{1}{2}\eta_i \left(z+\bar{z}\right)^i\right\}\right] + O\left(1/\mu^2\right) \\ -\frac{i}{2\mu}\eta_j + O\left(1/\mu^2\right) \end{array} \right) . \tag{7.69}$$

Furthermore, the Kähler-covariant derivative of the symplectic section defined by,

$$U_i^M = D_i V^M = \partial_i V^M + \frac{\partial_i \mathcal{K}}{2} V^M , \tag{7.70}$$

takes the form

$$U_i^M = \left( \begin{array}{c} -\frac{\eta_i}{\mu} + \frac{1}{2\mu^2}\left(-\left[\partial_i\phi + \overline{\partial_i\phi}\right] + \partial_{ij}\phi\left[z-\bar{z}\right]^j + 3\eta_i\eta_j\left[z+\bar{z}\right]^j\right) + O\left(1/\mu^3\right) \\ \delta_i^j - \frac{1}{\mu}\left(\frac{1}{2}\eta_k\left(z+\bar{z}\right)^k \delta_i^j + \eta_i z^j\right) + O\left(1/\mu^3\right) \\ -\frac{i}{4\mu^2}\left(\left[\partial_i\phi - \overline{\partial_i\phi}\right] - \partial_{ij}\phi\left[z+\bar{z}\right]^j + 2\eta_i\eta_j z^j\right) + O\left(1/\mu^3\right) \\ -\frac{i}{2\mu^2}\left(\partial_{ij}\phi - \eta_i\eta_j\right) + O\left(1/\mu^3\right) \end{array} \right) . \tag{7.71}$$

As we will see in the following subsection, a natural interpretation of the constant parameters $\eta_i$ appearing in the symplectic section $\hat{\Omega}^{\mathcal{M}}$ and in the metric $\mathring{g}_{i\bar{\jmath}}$ of the rigid theory, can be given in supergravity as charges associated with the gauging procedure, when a different choice of symplectic frame is considered.

Let us now consider the gauging of two translational isometries in the hypermultiplet sector involving both electric and magnetic charges [91, 92]. This gauging can be described in terms of a (redundant) symplectic vector of gauge generators $X_M \equiv (X_\Lambda,\, X^\Lambda)$, expressed as linear combinations of the isometry generators $t_m$, $m = 1, \ldots, \dim\mathcal{G}$, of the quaternionic Kähler manifold through an embedding tensor [90, 93]:

$$X_M = \Theta_M{}^m\, t_m \, . \tag{7.72}$$

We choose the gauging involving only two translational isometries $t_m$ $(m = 1, 2)$ and the embedding tensor $\Theta_M^m$

$$\Theta_M^\alpha = \left(\Theta_M^1, \Theta_M^2\right) = \begin{pmatrix} \Theta_0{}^1 & \Theta_0{}^2 \\ \Theta_i{}^1 & \Theta_i{}^2 \\ \Theta^{0\,1} & \Theta^{0\,2} \\ \Theta^{i\,1} & \Theta^{i\,2} \end{pmatrix} = \begin{pmatrix} e/\mu^2 & \sigma/\mu^2 \\ 0 & 0 \\ 0 & 0 \\ m^i/\mu & 0 \end{pmatrix} , \tag{7.73}$$

depending on constant charges $e, \sigma, m^i$, and satisfying the locality condition

$$\mathbb{C}^{MN}\Theta_M{}^m\Theta_N{}^n = 0\,, \text{ where } \mathbb{C}^{MN} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7.74}$$

The embedded Killing vectors $k_M{}^u = \left(k_\Lambda{}^u, k^{\Lambda\, u}\right)$ are related to the geometrical Killing vectors $k_\alpha{}^u$ ($\alpha = 1, \ldots, \dim \mathcal{G}$) generating the isometry group $\mathcal{G}$ of $\mathcal{M}_{QK}$ by:

$$k_M{}^u = \Theta_M{}^m k_m{}^u. \tag{7.75}$$

The introduction of the embedding tensor allows to write the fermion shifts $\delta_\epsilon^{(\Theta)}$ of the supersymmetry transformation laws in a symplectic covariant way. For $\mathcal{N} = 2$ supergravity, they are given by

$$\delta_\epsilon^{(\Theta)}\lambda^{i\;A} = W^{i\;AB}\epsilon_B, \tag{7.76}$$

$$\delta_\epsilon^{(\Theta)}\psi_{A\;\mu} = iS_{AB}\gamma_\mu\epsilon^B, \tag{7.77}$$

$$\delta_\epsilon^{(\Theta)}\zeta^\alpha = N_A^\alpha\epsilon^A, \tag{7.78}$$

where the fermion shifts are given by [see (7.35) - (7.37)]:

$$W^{i\;AB} = ig^{i\bar{\jmath}}\left(\sigma^x\right)_C{}^B\epsilon^{CA}U_{\bar{\jmath}}{}^M\Theta_M{}^m\mathcal{P}^x_m, \tag{7.79}$$

$$S_{AB} = \frac{i}{2}\left(\sigma^x\right)_A{}^C\epsilon_{BC}V^M\Theta_M{}^m\mathcal{P}^x_m, \tag{7.80}$$

$$N_A^\alpha = -2\mathcal{U}^\alpha_{A|u}k_m{}^uV^M\Theta_M{}^m. \tag{7.81}$$

where we have set $k^i_M = 0$, since our gauging does not involve special Kähler isometries.

Denoting by $\varphi$ and $\vec{q} \equiv \{q^1, q^2, q^3\}$ the four hyper-scalars in the solvable parametrization, the metric of the quaternionic Kähler manifold has the following form

$$ds^2 = \frac{1}{2}\left(d\varphi^2 + e^{2\varphi}d\vec{q}\cdot d\vec{q}\right), \tag{7.82}$$

and the corresponding vielbein $\mathcal{U}^\alpha_{A|u}$, appearing in the supersymmetry shift-matrices of the hyperini, reads [84]:

$$\mathcal{U}^\alpha_A = \mathcal{U}^\alpha_{A|u}dq^u = -\frac{1}{2}\epsilon^{\alpha\beta}\left[d\varphi + ie^\varphi d\vec{q}\cdot\vec{\sigma}\right]_{A\;\beta}, \tag{7.83}$$

where $\left(\sigma^x\right)_A{}^C$ are the standard Pauli matrices and $\mathcal{P}^x_m$ are the quaternionic momentum maps associated with the quaternionic isometries via the relation:

$$\mathcal{P}^x_m = -k_m{}^u\omega^x_u, \tag{7.84}$$

where $\omega_u^x$ denotes the $SU(2)$-connection on $\mathcal{M}_{QK}$. The metric (7.82) is invariant under constant translation of the three axions: $\vec{q} \to \vec{q} + \vec{c}$. We shall choose to gauge the two translations $t_n$ acting on $q^2, q^3$.

The gauging under consideration (7.73) involves two traslational isometries $t_n$ whose momentum maps can be chosen as follows

$$\mathcal{P}_m^x = (\mathcal{P}_1^x, \mathcal{P}_2^x) = \delta_m^x e^{\varphi},$$

with

$$\mathcal{P}_1^x = (0,1,0)\, e^{\varphi}, \tag{7.85}$$

$$\mathcal{P}_2^x = (0,0,1)\, e^{\varphi}. \tag{7.86}$$

In the next section, the two hyperscalars $q^2, q^3$ will be dualized into antisymmetric tensor fields $B_{n|\mu\nu}$.

### 7.3.1 Partial supersymmetry breaking and rigid limit

Here we will consider the prescription of [84][2]. The partial supersymmetry breaking is recovered considering the limit $\mu = \frac{M_{Pl}}{\Lambda} \to \infty$. Since the fermionic shifts are written in natural units $c = \hbar = M_{Pl} = 1$, and in order to explicitly perform the limit, it is convenient to reintroduce the appropriate dependence on the Planck Mass and on the supersymmetry breaking scale $\Lambda$, due to the gauging, in the supergravity expressions. Since the scale $\Lambda$ is related to the gravitino mass by $\Lambda^2 = M_{Pl}\, m_{\frac{3}{2}}$, and that the special-Kähler sigma-model metric rescales according to (7.64), then the canonically normalized kinetic terms are recovered by the rescaling [84]:

$$\begin{aligned} &x^\mu \to M_{Pl} x^\mu, \qquad \epsilon \to M_{Pl}^{1/2}\epsilon, \\ &\psi_\mu \to M_{Pl}^{-3/2}\psi_\mu, \qquad \lambda \to \left(M_{Pl}\Lambda^2\right)^{-1/2}\lambda, \qquad \zeta^\alpha \to M_{Pl}^{-3/2}\zeta^\alpha. \end{aligned} \tag{7.87}$$

[2]In the next Section, we will consider the low energy limit of the Lagrangian starting from a different, $\mu$-independent, symplectic frame of the supergravity theory, and thus we will approach the rescaling in a different way.

If we use the above rescaling we find that the shifts of the fermions read

$$
\begin{aligned}
\delta\lambda^{i\,A} &= -i\Lambda^2\epsilon^{CA}\left[\mathring{g}^{i\bar{\jmath}}\left(e^x_{\bar{\jmath}} - \tau_{1\bar{\jmath}k}m^{k\,x}\right) + \frac{i}{2}m^{i\,x}\right](\sigma^x)_C{}^B\, e^{\varphi}\epsilon_B,\\
\delta\psi_{A\,\mu} &= -\frac{\Lambda^2}{2}\epsilon_{BC}\left[e^x - i\frac{\eta_j}{2}m^{j\,x}\right](\sigma^x)_A{}^C\, e^{\varphi}\epsilon^B,\\
\delta\zeta^{\alpha} &= -i\Lambda^2\epsilon^{\alpha\beta}\left[e^x - i\frac{\eta_j}{2}m^{j\,x}\right](\sigma^x)_{\beta\,A}\, e^{\varphi}\epsilon^A,
\end{aligned}
\tag{7.88}
$$

where the following definitions have been used:

$$
\begin{aligned}
e^x &= (0, e, \sigma) = (0, e^m)\,,\\
m^{i\,x} &= \left(0, m^i, 0\right) = (0, m^{im})\,,\\
e^x_i &= \eta_i e^x.
\end{aligned}
\tag{7.89}
$$

Let us note that, as we will see in detail by the analysis of the Lagrangian in the rigid limit, the hypermultiplet decouple in the rigid theory and the momentum maps $\mathcal{P}^x_M$ reduce to constant Fayet-Iliopoulos terms $\mathbb{P}^x_{\mathcal{M}} = (m^{ix}, e^x_i)$. The relation between them can be read explicitly from the gaugino shift:

$$
\mathring{g}^{i\bar{\jmath}}\bar{U}^M_{\bar{\jmath}}\mathcal{P}^x_M = \left[\mathring{g}^{i\bar{\jmath}}\left(e^x_{\bar{\jmath}} - \tau_{1\bar{\jmath}k}m^{k\,x}\right) + \frac{i}{2}m^{i\,x}\right] = \mathring{g}^{i\bar{\jmath}}\bar{U}^{\mathcal{M}}_{\bar{\jmath}}\mathbb{P}^x_{\mathcal{M}}\,,
\tag{7.90}
$$

where $U^{\mathcal{M}}_i$ are related to the rigid symplectic sections introduced in (7.65) by $U^{\mathcal{M}}_i = \partial_i\hat{\Omega}^{\mathcal{M}}$. We emphasize here that in this formulation of the rigid limit, the FI terms are expressed not only in terms of the parameters $e, \sigma, m_i$ defining the embedding tensor (the gauging parameters), but also in terms of the parameters $\eta_i$ characterizing the special geometry through the choice of the prepotential (7.63). In the next subsection we shall discuss a different formulation in which the FI terms only descend from the supergravity gauging parameters.

For the case of one vector multiplet, $n = 1$, eq. (7.88) reproduces the results of [84] leading to the APT model.

### 7.3.2 Some comments on the interpretation of the constant parameters $\eta_i$

It is well known that partial breaking of rigid supersymmetry crucially requires, in order to evade previously stated no-go theorems[108, 109], that the quantity $\xi^x$, defined by

$$
\xi^x \equiv \frac{1}{2}\epsilon^{xyz}\mathbb{P}^{y\mathcal{M}}\mathbb{P}^{z\mathcal{N}}\mathbb{C}_{\mathcal{M}\mathcal{N}} = \epsilon^{xyz}e^y_i m^{zi}\,,
\tag{7.91}
$$

with $e_i^y, m^{zi}$ given by (7.89), be different from zero [3]. This relation looks like a non-locality condition. Nevertheless, the choice of embedding tensor as in (7.73) implies that the locality condition

$$\Theta^m_{\mathcal{M}}\Theta^n_{\mathcal{N}}\mathbb{C}_{\mathcal{M}\mathcal{N}} = 2\Theta^{i[m}\Theta_i^{n]} = 0 \tag{7.92}$$

is satisfied in the rigid theory so that, the condition $\epsilon^{xyz}\mathcal{P}^{y\mathcal{M}}\mathcal{P}^{z\mathcal{N}}\mathbb{C}_{\mathcal{M}\mathcal{N}} = 0$, with $\mathcal{P}^x_{\mathcal{M}} = \mathcal{P}^x_m\Theta^m_{\mathcal{M}}$, is satisfied in the chosen frame. This is not in contradiction with (7.91) since the FI parameters $\mathbb{P}^x_{\mathcal{M}}$ of the rigid theory are not the simple restriction of the supergravity momentum maps to the $Sp\,(2n,\mathbb{R})$-index $\mathcal{M}$. In fact, the momentum maps in supergravity and the Fayet-Iliopoulos terms of the rigid theory are related through (7.90), which non-trivially involves the contribution from the index 0 of the symplectic section, keeping a memory of the graviphoton. On the other hand, as eq.s (7.64) and (7.65) show, the geometry of the rigid theory *in the chosen coordinate frame* depends in a non trivial way on the constant parameters $\eta_i$, also appearing in (7.91) through the charges $e_i^y = e^y\eta_i$.

As we will see, the embedding of the theory in supergravity allows to clarify the topological role of all the constant parameters involved in the gauging, showing that the $\eta_i$ required in the special geometry of the rigid theory in order to have partial supersymmetry breaking (with its BI low energy limit), can be traded with charges via a symplectic rotation involving a redefinition of the special coordinates in the *underlying supergravity theory*.
Indeed, consider the (electric) symplectic transformation in supergravity:

$$S(\eta,\mu) = \begin{pmatrix} 1 & \eta_i/\mu & 0 & 0 \\ 0 & \frac{1}{\mu}\mathbf{1_n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\eta_i & \mu\mathbf{1_n} \end{pmatrix} \tag{7.93}$$

which induces the following rotation in the symplectic section (7.69):

$$\tilde{\Omega}^M = S\cdot\Omega^M = \begin{pmatrix} X^0 + \frac{1}{\mu}\eta_i X^i \\ \frac{1}{\mu}X^i \\ F_0 \\ \mu F_i - \eta_i F_0 \end{pmatrix} = \begin{pmatrix} \tilde{X}^0 \\ \tilde{X}^i \\ \tilde{F}_0 \\ \tilde{F}_i \end{pmatrix}. \tag{7.94}$$

The new holomorphic prepotential is $\tilde{F}(\tilde{X}) = F(X)$. Since the new special coordinates $\tilde{z}^i$

[3] As shown in [105], this condition is also necessary to achieve, in the low energy limit, a multi-field generalization of the Born-Infeld Lagrangian.

are related to the old ones by

$$\tilde{z}^i = \frac{z^i}{\mu + \eta_j z^j} = \frac{1}{\mu}\omega^i \,, \tag{7.95}$$

then the reduced prepotential $\tilde{f}(\tilde{z})$ is related to $f(z)$ by (see (7.57)):

$$\tilde{f}(\tilde{z}) = (1 + \frac{1}{\mu}\eta_j z^j)^{-2} f(z)$$

that is

$$\tilde{f}(\tilde{z}) = \left(\frac{1}{4} + \frac{1}{2\mu^2}\tilde{\phi}(\tilde{z}) + O(\frac{1}{\mu^3})\right) \tag{7.96}$$

where $\tilde{\phi}(\tilde{z}) = \phi(z) - \frac{1}{2}(\eta_i \tilde{z}^i)^2 \equiv \Phi(\omega)$. Note that in the new frame the linear term in $\tilde{z}$ has disappeared from (7.96). Moreover, after the symplectic rotation, the covariantly holomorphic symplectic sections $\tilde{V}^M = e^{\frac{\mathcal{K}}{2}}\tilde{\Omega}^M$ and $\tilde{U}_i^M = \partial_i \tilde{V}^M$ can be written in a *generic* coordinate frame and behave, in the rigid limit $\mu \to \infty$, as:

$$\tilde{V}^M \;=\; \begin{pmatrix} X^0 \\ 0 \\ F_0 \\ 0 \end{pmatrix} + \frac{1}{\mu}\begin{pmatrix} 0 \\ \mathring{X}^I(\omega) \\ 0 \\ \mathring{F}_I(\omega) \end{pmatrix} + O\left(1/\mu^2\right) \,, \tag{7.97}$$

$$\tilde{U}_i^M \;=\; \frac{1}{\mu}\begin{pmatrix} 0 \\ \partial_i \mathring{X}^I \\ 0 \\ \partial_i \mathring{F}_I \end{pmatrix} + O\left(1/\mu^2\right) \,, \tag{7.98}$$

where $\mathring{\Omega}^{\mathcal{M}} \equiv (\mathring{X}^I, \mathring{F}_I)$ $(I = 1, \cdots n)$ denotes the symplectic section or the rigid theory (in special coordinates $\mathring{X}^I(\omega) = \omega^i$, $\mathring{F}_I(\omega) = \frac{\partial \Phi}{\partial \omega^i}$). In the new frame the symplectic structure $Sp(2n+2)$ of the supergravity theory flows in the rigid limit to a manifest $Sp(2n)$ structure. In particular, the 0-directions have a different $\mu$-rescaling with respect to the $\mathcal{M}$-directions. They are then directly associated to the Hodge-bundle of the local special geometry (that is to the graviphoton direction) and are projected out in the low energy limit. Still, the special-geometry sigma-model metric in supergravity is related to its counterpart $\mathring{g}_{i\bar{\jmath}}$ in the rigid limit by:

$$g_{i\bar{\jmath}} = \frac{1}{\mu^2}\mathring{g}_{i\bar{\jmath}} \,, \tag{7.99}$$

while the relations of special geometry imply a low-energy rescaling of the vector-kinetic-matrix $\mathcal{N}_{\Lambda\Sigma}$ corresponding to the following identification of the matrix $\mathring{\mathcal{N}}_{\Lambda\Sigma}$ of the rigid theory:

$$\mathcal{N}_{00} = \mathring{\mathcal{N}}_{00}\,, \quad \mathcal{N}_{IJ} = \mathring{\mathcal{N}}_{IJ}\,, \quad \mathcal{N}_{0I} = \frac{1}{\mu}\mathring{\mathcal{N}}_{0I}\,. \tag{7.100}$$

The symplectic transformation (7.93) acts on the embedding tensor (7.73) as follows

$$\tilde{\Theta}^m_M = \Theta^m_N \cdot (S^{-1})^N{}_M = \frac{1}{\mu^2}\left(e^m, -\eta_i e^m, \eta_i m^{im}, m^{im}\right) = \frac{1}{\mu^2}\mathring{\Theta}^m_M\,, \tag{7.101}$$

where $\mathring{\Theta}^m_M$ is the embedding tensor of the rigid theory. In this way, in the new frame the parameters $\eta_i$ play the role of charges, since $\tilde{\Theta}^m_i = \eta_i e^m$ are the electric charges associated with the vector multiplets and $\tilde{\Theta}^{0m} = \eta_i m^{im}$ is the magnetic charge associated with the graviphoton. Note that in the old frame both of them were zero.

As a consequence, the new embedding tensor (7.101) satifies the same locality condition (7.74) as the old one, but now

$$\tilde{\Theta}^{\Lambda[m}\tilde{\Theta}^{n]}_\Lambda = 0 \quad \Rightarrow \quad \tilde{\Theta}^{0[m}\tilde{\Theta}^{n]}_0 = -\tilde{\Theta}^{i[m}\tilde{\Theta}^{n]}_i = \frac{1}{\mu^4}e^m\eta_i m^{in} \neq 0\,. \tag{7.102}$$

This expresses a sort of "non-locality" of the rigid theory, and hints toward a high-energy interpretation of it in terms of a non-triviality of the fiber bundle associated with the graviphoton. In the new frame the graviphoton is identified with the 0 direction of the vector field strengths, what is not true in the old frame. More specifically, if we denote by $A^\Lambda_\mu = (A^0_\mu,\, A^I_\mu)$, the $n+1$ supergravity vector fields, in the new symplectic frame, $A^0_\mu$ is consistently identified with the graviphoton while $A^I_\mu$ with the vector fields of the resulting rigid theory. Since in the rigid limit the graviphoton decouples from the spectrum, we find that the *rigid supersymmetric theory found as low energy limit of supergravity in the new frame is actually non local. However, as we are going to discuss, the non-locality only affects the fermionic directions of superspace, while it does not emerge as a non-locality on space-time.* This clarifies the meaning of (7.91), which expresses indeed the non locality of the rigid theory, when all the constant parameters needed for the partial breaking of supersymmetry are expressed as electric and magnetic charges in the embedding tensor.

Moreover, this non-locality poses no obstruction to a correct definition of the vector fields $A^I_\mu$ in the rigid theory, by virtue of an interesting mechanism which is at work in the rigid limit: A generic feature of magnetic gaugings in supergravity is the fact that the vector fields $A^\Lambda_\mu$ corresponding to non-vanishing magnetic components $\Theta^{\Lambda m}$ of the embedding tensor, are not well defined since the corresponding field strengths $F^\Lambda_{\mu\nu}$ are not covariantly closed

$$DF^\Lambda \propto \Theta^{\Lambda m}\, dB_m + \cdots \neq 0\,, \tag{7.103}$$

$B_{m|\,\mu\mu}$ being antisymmetric tensor fields. This poses no problem because such vector fields, in a vacuum, are "eaten" by the tensor ones $B_m$ and become their longitudinal components by virtue of the "anti-Higgs" mechanism [107]. This is the case of the vectors $A^I_\mu$ which are thus not well defined in the chosen supergravity gauging. In the rigid limit however, as we shall show, the antisymmetric tensor fields decouple, thus preventing the anti-Higgs mechanism from taking place, so that the vectors $A^I_\mu$ survive and, at the same time, become well defined. As we shall illustrate, the *magnetic* character of the FI parameters $\Theta^{I\,m}$ in the rigid theory can be also related, besides to their position within the $Sp(2n,\mathbb{R})$-covariant parameter vectors $(\Theta_I{}^m,\, \Theta^{I\,m})$, to the following feature of the vector field strengths: While $dF^I$ vanish in space-time, they *do not* vanish in superspace since:

$$dF^I = \frac{i}{2}\Theta^{Im}\mathcal{P}^x_m\,(\sigma^x)_A{}^B\,\bar{\psi}_B \wedge \gamma_a \psi^A \wedge V^a \neq 0\,. \tag{7.104}$$

In other words, the magnetic FI terms parametrize a non-locality only along the fermionic directions of superspace, thus not affecting the well-definiteness of $A^I_\mu$.

The effects of the non-locality (7.102) are directly related to the supersymmetric structure of the theory. As said before, the non locality of the rigid theory is related to the non-triviality of the fiber bundle associated with the graviphoton in the rigid limit. Because of this and as already noted in [84], the supergravity modes associated with the underlying $\mathcal{N}=2$ supergravity theory still freely propagate in the rigid theory (see (7.88)) even if decoupled from the visible sector. As a consequence, the $SU(2)$-Lie algebra valued term $C_A{}^B$ appearing in the supersymmetry Ward-identity of the spontaneously broken rigid theory can be understood as the contribution to the Ward identity from gravitini and hyperini, still propagating in the rigid theory.

On the other hand, it is known from [91, 93, 95, 106] that, in the presence of magnetic charges $m^{\Lambda n}$ in supersymmetric theories, the natural symplectic frame to deal with them is rotated with respect to the purely electric frame, allowing for the presence of antisymmetric tensors $B_{n|\mu\nu}$, coupled to the gauge fields $A^\Lambda$ in the combinations $\hat{F}^\Lambda_{\mu\nu} = F^\Lambda_{\mu\nu} + 2m^{\Lambda n}B_{n\mu\nu}$ [4]. The $\mathcal{N}=2$ supersymmetric Free Differential Algebra in four dimensions contains in particular, in the case where the antisymmetric tensors dualize scalars in the quaternionic

[4] The fermionic shifts found in [84] and generalized to $n$ vector multiplets in this work are in fact naturally recovered in the symplectic frame where some of the hyper-scalars are dualized to tensor fields, as one can explicitly check by comparison with Section 3 of [91], and in particular eqs. (3.13) - (3.15) there.

sector

$$\hat{F}^{(2)\Lambda} \equiv dA^\Lambda + 2m^{\Lambda n}B_n + (L^\Lambda(z)\bar{\psi}_A \wedge \psi_B \epsilon^{AB} + h.c.) \tag{7.105}$$

$$H_n^{(3)} \equiv dB_n + \frac{i}{2}\mathcal{P}_n^x \,(\sigma^x)_A^{\ B}\, \bar{\psi}_B \wedge \gamma_a \psi^A \wedge V^a \tag{7.106}$$

where $L^\Lambda$ are the upper-part of the special geometry symplectic sections $V^M$ and $\mathcal{P}_n^x$ are functions of the hyperscalars [92]. From (7.105) and (7.106) we obtain

$$d\hat{F}^\Lambda = \Theta^{\Lambda n}\left(2H_n - i\mathcal{P}_n^x \,(\sigma^x)_A^{\ B}\, \bar{\psi}_B \wedge \gamma_a \psi^A \wedge V^a\right), \tag{7.107}$$

where we have identified $m^{\Lambda n}$ with $\Theta^{\Lambda n}$. In the low energy limit the hyperscalars are not suppresed but tend to constants, in such a way that $\Theta_M^{\ n}\mathcal{P}_n^x$ becomes constants $\Theta_M^{\ n}\mathbb{P}_n^x \neq 0$ whose restriction to the non-zero indices $\Theta_{\mathcal{M}}^{\ n}\mathbb{P}_n^x$ yield the FI parameters. Then, from the expression (7.107), taking account the decoupling of the tensor fields, the clousure of the free differential algebra gives

$$d\hat{F}^I \propto i\Theta^{I\ n}\mathbb{P}_n^x \,(\sigma^x)_A^{\ B}\, \bar{\psi}_B \wedge \gamma_a \psi^A \wedge V^a + \cdots \neq 0\,. \tag{7.108}$$

From (7.108) we see that the non-locality only affects the fermionic directions of superspace, while it does not emerge as a non-locality on space-time.

## 7.4 Rigid limit of the $\mathcal{N} = 2$ supergravity Lagrangian

In this section, we consider the rigid limit of the $\mathcal{N} = 2$ supergravity Lagrangian corresponding to partial breaking of supersymmetry, and whose gauge structure has been discussed in the previous section.

We shall work in the symplectic frame where the gauging structure of the theory is unveiled and shown to involve the presence of magnetic charges. In this way, the natural framework to perform the limit is the version of the Lagrangian where some of the scalars of the hypermultiplets are Hodge-dualized to antisymmetric tensors $B_{m\mu\nu}$[91, 92, 93, 95, 106].

In order to perform the rigid limit, it is convenient to reintroduce in the Lagrangian (usually written in natural units $c = \hbar = 1$, but with also $M_{Pl} = 1$) the appropriate scale dimensions. We will consider the limit process in two main steps: We will first explicitly write the correct Planck-mass dependence of the physical fields in the supergravity Lagrangian and then, after considering the low energy ($\mu \to \infty$) behavior of the special-geometry sigma-model sector, we will get the appropriate redefinitions of the physical fields appearing in the rigid supersymmetric theory.

The canonical scale dimensions of the fields in natural units $c = \hbar = 1$ are:

$$[x^\mu] = M^{-1}\,, \quad [\partial_\mu] = M\,, \quad [A^\Lambda_\mu] = [B_{m\mu\nu}] = M\,, \quad [z^i_{(can.)}] = [q^u_{(can.)}] = M,$$
$$[\psi^A_\mu] = [\lambda^A] = [\zeta^\alpha] = M^{3/2}\,, \quad [\epsilon^A] = M^{-1/2}\,,$$

while the embedding tensor is adimensional. For the embedding tensor we will consider its symplectic-covariant expression (7.101). Since the scalars $z^i, q^u$ appear in the theory through non-linear sigma-models, we will keep them adimensional (that is we will consider $z^i \equiv z^i_{(can.)}/M_{Pl}$, $q^u \equiv q^u_{(can.)}/M_{Pl}$).

Following this prescription, the Lagrangian in [92] can be split in terms of Planck-scale powers and reads, up to four fermions terms:

$$\mathcal{L} = \mathcal{L}_{(4)} + \mathcal{L}_{(2)} + \mathcal{L}_{(1)} + \mathcal{L}_{(0)} + \mathcal{L}_{(-1)} \tag{7.109}$$

where

$$\mathcal{L}_{(4)} = M^4_{Pl}\mathcal{V}(z,q) \tag{7.110}$$

$$\mathcal{L}_{(2)} = M^2_{Pl}\left(-\frac{R}{2} + g_{i\bar{\jmath}}\partial^\mu z^i\partial_\mu \bar{z}^{\bar{\jmath}} + h_{uv}\partial_\mu q^u \partial^\mu q^v\right) \tag{7.111}$$

$$\begin{aligned}\mathcal{L}_{(1)} = {} & M_{Pl}\Big\{(-\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}})\left[2\mathcal{H}_{m|\nu\rho\sigma}A^m_u\partial_\mu q^u + \frac{1}{2}B_{m|\mu\nu}\Theta_\Lambda^{\;m}\left(\hat{\mathcal{F}}^\Lambda_{\rho\sigma} - M_{Pl}\frac{1}{2}\Theta^{\;\Lambda n}B_{n|\rho\sigma}\right)\right] + \\ & + \big(2S_{AB}\bar{\psi}^A_\mu\gamma^{\mu\nu}\psi^B_\nu + ig_{i\bar{\jmath}}W^{iAB}\bar{\lambda}^{\bar{\jmath}}_A\gamma_\mu\psi^\mu_B + 2iN^A_\alpha\bar{\zeta}^\alpha\gamma_\mu\psi^\mu_A \\ & + \mathcal{M}^{\alpha\beta}\bar{\zeta}_\alpha\zeta_\beta + \mathcal{M}^\alpha_{iB}\bar{\zeta}_\alpha\lambda^{iB} + \mathcal{M}_{iAjB}\bar{\lambda}^{iA}\lambda^{jB} + \text{h.c.}\big)\Big\}\end{aligned} \tag{7.112}$$

$$\begin{aligned}\mathcal{L}_{(0)} = {} & i\left(\bar{\mathcal{N}}_{\Lambda\Sigma}\hat{\mathcal{F}}^{-\Lambda}_{\mu\nu}\hat{\mathcal{F}}^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\hat{\mathcal{F}}^{+\Lambda}_{\mu\nu}\hat{\mathcal{F}}^{+\Sigma\mu\nu}\right) + 6\mathcal{M}^{mn}\mathcal{H}_{m\mu\nu\rho}\mathcal{H}_n^{\;\mu\nu\rho} + \\ & + \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\bar{\psi}^A_\mu\gamma_\nu\rho_{A|\lambda\sigma} - \bar{\psi}_{A|\mu}\gamma_\nu\rho^A_{\lambda\sigma}\right) - \frac{i}{2}g_{i\bar{\jmath}}\left(\bar{\lambda}^{iA}\gamma^\mu\nabla_\mu\lambda^{\bar{\jmath}}_A + \bar{\lambda}^{\bar{\jmath}}_A\gamma^\mu\nabla_\mu\lambda^{iA}\right) + \\ & - i\left(\bar{\zeta}^\alpha\gamma^\mu\nabla_\mu\zeta_\alpha + \bar{\zeta}_\alpha\gamma^\mu\nabla_\mu\zeta^\alpha\right) + \\ & - g_{i\bar{\jmath}}\partial_\mu\bar{z}^{\bar{\jmath}}\left(\bar{\psi}^\mu_A\lambda^{iA} - \bar{\lambda}^{iA}\gamma^{\mu\nu}\psi_{A\nu} + h.c.\right) - 2\mathcal{U}^{\alpha A}_u\partial_\mu q^u\left(\bar{\psi}^\mu_A\zeta_\alpha - \bar{\zeta}_\alpha\gamma^{\mu\nu}\psi_{A\nu} + \text{h.c.}\right)\end{aligned} \tag{7.113}$$

$$\begin{aligned}\mathcal{L}_{(-1)} = {} & M^{-1}_{Pl}\Big\{\hat{\mathcal{F}}^{-\Lambda}_{\mu\nu}I_{\Lambda\Sigma}\Big[L^\Sigma\bar{\psi}^{A\mu}\psi^{B\nu}\epsilon_{AB} - 4if^\Sigma_{\bar{\imath}}\bar{\lambda}^{\bar{\imath}}_A\gamma^\nu\psi^\mu_B\epsilon^{AB} + \frac{1}{2}\nabla_i f^\Sigma_j\bar{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\epsilon_{AB} + \\ & - L^\Sigma\bar{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta\mathbb{C}^{\alpha\beta}\Big] + \text{h.c.} + \\ & + 2\mathcal{M}^{mn}\mathcal{H}_m^{\;\mu\nu\rho}\left[\mathcal{U}_n^{\;A\alpha}\left(3i\bar{\psi}_{A\mu}\gamma_{\nu\rho}\zeta_\alpha + \bar{\psi}_{A\mu}\zeta_\alpha\right) + i\Delta_{n\alpha}^{\;\;\beta}\,\zeta_\beta\gamma_{\mu\nu\rho}\zeta^\alpha\right]\Big\}\,,\end{aligned} \tag{7.114}$$

where $h_{uv}$, $A^m_u$, $\mathcal{M}^{mn}$ are the components of the quaternionic metric after dualizition of the scalars $q^m$ to antisymmetric tensors $B_{m|\mu\nu}$, $\hat{\mathcal{F}}^\Lambda_{\mu\nu} := \mathcal{F}^\Lambda + \frac{1}{2}M_{Pl}\,\Theta^{\Lambda m}B_{\mu\nu m}$ are the gauge field-strengths undergoing the anti-Higgs mechanism introduced in (7.105) (in our case $\Theta^{\Lambda m} =$

$m^{\Lambda m} = \frac{1}{\mu^2}\eta_i m^{im}$), and $\mathcal{F}^{\pm\Lambda}_{\mu\nu} = \frac{1}{2}\left(\mathcal{F}^{\Lambda}_{\mu\nu} \pm \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda\rho\sigma}\right)$ denotes projection on (anti)self-dual part [5]. Furthermore, the mass-matrices are given by [31, 92]

$$\mathcal{M}^{\alpha\beta} = -\mathcal{U}^{\alpha A}_{u}\mathcal{U}^{\beta B}_{v}\epsilon_{AB}\Theta_M{}^m\nabla^{[u}k^{v]}_m V^M, \tag{7.116}$$

$$\mathcal{M}^{\alpha}_{iB} = -4\mathcal{U}^{\alpha}_{Bu}\Theta_M{}^m k^u_m U_i{}^M, \tag{7.117}$$

$$\mathcal{M}_{iAjB} = \frac{i}{3}\left(\sigma_x\epsilon^{-1}\right)_{AB}\Theta_M{}^m\mathcal{P}^x_m\nabla_j U_i{}^M. \tag{7.118}$$

To perform the rigid limit $\frac{M_{Pl}}{\Lambda} \equiv \mu \to \infty$ of the Lagrangian, we must first consider the limit of the various couplings in the Lagrangian, and clarify the relation between supergravity fields and their rigid counterparts correspondingly. We will identify the fields of the rigid supersymmetric theory with a ring, to distinguish them from the supergravity fields.

From the previous section we know that the special-Kähler metric rescales as (7.99), so that the kinetic terms of scalars and spinors in the vector multiplets in the rigid limit read (from (7.111) and (7.113):

$$\frac{1}{\mu^2}\mathring{g}_{i\bar{\jmath}}\left[M^2_{Pl}\partial^\mu z^i\partial_\mu\bar{z}^{\bar{\jmath}} - \frac{i}{2}\left(\bar{\lambda}^{iA}\gamma^\mu\nabla_\mu\lambda^{\bar{\jmath}}_A + \bar{\lambda}^{\bar{\jmath}}_A\gamma^\mu\nabla_\mu\lambda^{iA}\right)\right].$$

This implies that the gaugini of the rigid theory should be related to their supegravity relatives as follows:

$$\mathring{\lambda}^{iA} = \frac{1}{\mu}\lambda^{iA}, \tag{7.119}$$

while the holomorphic scalars should not be rescaled

$$\mathring{z}^i = z^i.$$

Thus, we have that

$$\mathcal{L}_{rig} = \cdots\mathring{g}_{i\bar{\jmath}}\left[\Lambda^2\partial^\mu\mathring{z}^i\partial_\mu\bar{\mathring{z}}^{\bar{\jmath}} - \frac{i}{2}\left(\bar{\mathring{\lambda}}^{iA}\gamma^\mu\nabla_\mu\mathring{\lambda}^{\bar{\jmath}}_A + \bar{\mathring{\lambda}}^{\bar{\jmath}}_A\gamma^\mu\nabla_\mu\mathring{\lambda}^{iA}\right)\right] + \cdots$$

Furthermore, since the components of the gauge kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ rescale as (7.100), then the gauge vector should not be redefined:

$$\mathring{A}^\Lambda_\mu = A^\Lambda_\mu\,, \tag{7.120}$$

[5] In a symplectic frame, where the gauge fields undergo the standard Higgs-mechanism by coupling to the scalars in the quaternionic sector (not dualized to antisymmetric tensors), the gauge-covariant derivative in the quaternionic sector is defined as

$$\nabla_\mu q^u = \partial_\mu q^u + M^{-1}_{Pl}A^\Lambda_\mu\Theta_\Lambda{}^\alpha k_\alpha{}^u\,. \tag{7.115}$$

and the gauge kinetic term reads, at low energies:

$$I_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma|\mu\nu} = \mathring{I}_{00} F^{0}_{\mu\nu} F^{0|\mu\nu} + \mathring{I}_{IJ} F^{I}_{\mu\nu} F^{J|\mu\nu} + \frac{2}{\mu} \mathring{I}_{0I} F^{0}_{\mu\nu} F^{I|\mu\nu} + \mathcal{O}(1/\mu^2)$$

where $I_{\Lambda\Sigma} \equiv \mathrm{Im}(\mathcal{N}_{\Lambda\Sigma})$.

Given (7.97),(7.98),(7.101) and (7.120), we can identify the low energy limit of the self-dual components of the graviphoton $T^-_{\mu\nu}$ and of the matter vectors $G^{-i}_{\mu\nu}$. We find that

$$T^-_{\mu\nu} \equiv I_{\Lambda\Sigma} L^{\Lambda} F^{-\Sigma}_{\mu\nu} \to \mathring{I}_{00} \mathring{X}^0 \mathring{F}^{-0}_{\mu\nu} + O(\frac{1}{\mu}) \tag{7.121}$$

$$g_{i\bar{\jmath}} G^{-i}_{\mu\nu} \equiv \frac{i}{2} I_{\Lambda\Sigma} f^{\Lambda}_{\bar{\jmath}} F^{-\Sigma}_{\mu\nu} \to \frac{i}{2\mu} \mathring{I}_{IJ} \mathring{f}^{I}_{i} \mathring{F}^{-J}_{\mu\nu} + O(\frac{1}{\mu^2}) \tag{7.122}$$

showing that, in the rigid limit, the gauge-index 0 corresponds to the graviphoton direction, while the gauge-index $I$ to the matter-vectors directions.

The rescalings of the fermion shifts and spinor mass matrices follow from the low energy limit of the symplectic sections and embedding tensor discussed in the previous section. They are:

$$W^{i\,AB} = \frac{1}{\mu} \mathring{W}^{i\,AB}\,, \qquad \mathcal{M}^{\alpha\beta} = \frac{1}{\mu^2} \mathring{\mathcal{M}}^{\alpha\beta}\,, \tag{7.123}$$

$$S_{AB} = \frac{1}{\mu^2} \mathring{S}_{AB}\,, \qquad \mathcal{M}^{\alpha}_{iB} = \frac{1}{\mu^3} \mathring{\mathcal{M}}^{\alpha}_{iB}\,, \tag{7.124}$$

$$N^{\alpha}_{A} = \frac{1}{\mu^2} \mathring{N}^{\alpha}_{A}\,, \qquad \mathcal{M}_{iAjB} = \frac{1}{\mu^3} \mathring{\mathcal{M}}_{iAjB}\,. \tag{7.125}$$

As a consequence, the scalar potential rescales as

$$\mathcal{V} = \frac{1}{\mu^4} \mathring{\mathcal{V}} \tag{7.126}$$

In this way, the different contributions to the Lagrangian (7.109), when written in terms of the rescaled fields, read:

$$\mathcal{L}_{(4)} = \Lambda^4 \mathring{\mathcal{V}}(z,q) \tag{7.127}$$

$$\mathcal{L}_{(2)} = M_{Pl}^2\left(-\frac{R}{2}+h_{uv}\partial_\mu q^u\partial^\mu q^v\right)+\Lambda^2\mathring{g}_{i\bar{\jmath}}\partial^\mu \mathring{z}^i\partial_\mu \mathring{\bar{z}}^{\bar{\jmath}} \tag{7.128}$$

$$\begin{aligned}\mathcal{L}_{(1)} &= M_{Pl}\Bigg\{(-\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}})\left[2\mathcal{H}_{m|\nu\rho\sigma}A^m_u\partial_\mu q^u+\frac{1}{2\mu^2}B_{m|\mu\nu}\mathring{\Theta}^m_\Lambda\left(\hat{\mathcal{F}}^\Lambda_{\rho\sigma}-\frac{M_{Pl}}{\mu^2}\frac{1}{2}\mathring{\Theta}^{\ \Lambda n}B_{n|\rho\sigma}\right)\right]+\\ &\quad+\frac{1}{\mu^2}\left(2\mathring{S}_{AB}\bar{\psi}^A_\mu\gamma^{\mu\nu}\psi^B_\nu+i\mathring{g}_{i\bar{\jmath}}\mathring{W}^{iAB}\mathring{\bar{\lambda}}^{\bar{\jmath}}_A\gamma_\mu\psi^\mu_B+2i\mathring{N}^A_\alpha\bar{\zeta}^\alpha\gamma_\mu\psi^\mu_A+\text{h.c.}\right)+\\ &\quad+\frac{1}{\mu^2}\left(\mathring{\mathcal{M}}^{\alpha\beta}\bar{\zeta}_\alpha\zeta_\beta+\mathring{\mathcal{M}}^\alpha_{iB}\bar{\zeta}_\alpha\mathring{\lambda}^{iB}+\text{h.c.}\right)\Bigg\}+\\ &\quad+\Lambda\left(\mathring{\mathcal{M}}_{iAjB}\mathring{\bar{\lambda}}^{iA}\mathring{\lambda}^{jB}+\text{h.c.}\right).\end{aligned} \tag{7.129}$$

$$\begin{aligned}\mathcal{L}_{(0)} &= i\left(\bar{\mathcal{N}}_{\Lambda\Sigma}\hat{\mathcal{F}}^{-\Lambda}_{\mu\nu}\hat{\mathcal{F}}^{-\Sigma\mu\nu}-\mathcal{N}_{\Lambda\Sigma}\hat{\mathcal{F}}^{+\Lambda}_{\mu\nu}\hat{\mathcal{F}}^{+\Sigma\mu\nu}\right)+6\mathcal{M}^{mn}\mathcal{H}_{m|\mu\nu\rho}\mathcal{H}_n^{\ \mu\nu\rho}+\\ &\quad+\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\bar{\psi}^A_\mu\gamma_\nu\rho_{A|\lambda\sigma}-\bar{\psi}_{A|\mu}\gamma_\nu\rho^A_{\lambda\sigma}\right)-\frac{i}{2}\mathring{g}_{i\bar{\jmath}}\left(\mathring{\bar{\lambda}}^{iA}\gamma^\mu\nabla_\mu\mathring{\lambda}^{\bar{\jmath}}_A+\mathring{\bar{\lambda}}^{\bar{\jmath}}_A\gamma^\mu\nabla_\mu\mathring{\lambda}^{iA}\right)+\\ &\quad-i\left(\bar{\zeta}^\alpha\gamma^\mu\nabla_\mu\zeta_\alpha+\bar{\zeta}_\alpha\gamma^\mu\nabla_\mu\zeta^\alpha\right)+\\ &\quad-\frac{1}{\mu}\mathring{g}_{i\bar{\jmath}}[\partial_\mu\bar{z}^{\bar{\jmath}}\left(\bar{\psi}^\mu_A\mathring{\lambda}^{iA}-\mathring{\bar{\lambda}}^{iA}\gamma^{\mu\nu}\psi_{A\nu}\right)+h.c.]-2\mathcal{U}^{\alpha A}_u\partial_\mu q^u\left(\bar{\psi}^\mu_A\zeta_\alpha-\bar{\zeta}_\alpha\gamma^{\mu\nu}\psi_{A\nu}+h.c.\right)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{(-1)} &= \Lambda^{-1}\mathcal{F}^{-I}_{\mu\nu}\mathring{I}_{IJ}\left[\frac{1}{2}\nabla_i\mathring{f}^J_j\mathring{\bar{\lambda}}^{iA}\gamma^{\mu\nu}\mathring{\lambda}^{jB}\epsilon_{AB}\right]-M_{Pl}^{-1}\left[4i\mathring{\bar{f}}^J_{\bar{\imath}}\mathring{\bar{\lambda}}^{\bar{\imath}}_A\gamma^\nu\psi^\mu_B\epsilon^{AB}+\text{h.c.}\right]+\\ &\quad+M_{Pl}^{-1}\Big\{\mathcal{F}^{-0}_{\mu\nu}\mathring{I}_{00}\mathring{L}^0\left[\bar{\psi}^{A\mu}\psi^{B\nu}\epsilon_{AB}-\bar{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta\mathbb{C}^{\alpha\beta}+\text{h.c.}\right]+\\ &\qquad+2\mathcal{M}^{mn}\mathcal{H}_m^{\ \mu\nu\rho}\left[\mathcal{U}_n^{\ A\alpha}\left(3i\bar{\psi}_{A\mu}\gamma_{\nu\rho}\zeta_\alpha+\bar{\psi}_{A\mu}\zeta_\alpha\right)+i\Delta_{n\alpha}^{\ \ \beta}\zeta_\beta\gamma_{\mu\nu\rho}\zeta^\alpha\right]\Big\},\end{aligned} \tag{7.130}$$

and it reduces, in the limit $\mu \to \infty$, to:

$$\mathcal{L}_{(4)} = \Lambda^4 \mathring{\mathcal{V}}(z,q) \tag{7.131}$$

$$\mathcal{L}_{(2)} = M_{Pl}^2 \left( -\frac{R}{2} + h_{uv}\partial_\mu q^u \partial^\mu q^v \right) + \Lambda^2 \mathring{g}_{i\bar{\jmath}} \partial^\mu z^i \partial_\mu \bar{z}^{\bar{\jmath}} \tag{7.132}$$

$$\mathcal{L}_{(1)} = -2\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} M_{Pl} \mathcal{H}_{m|\nu\rho\sigma} A_u^m \partial_\mu q^u + \Lambda \left( \mathring{\mathcal{M}}_{iAjB} \mathring{\bar{\lambda}}^{iA} \mathring{\lambda}^{jB} + \text{h.c.} \right). \tag{7.133}$$

$$\begin{aligned}\mathcal{L}_{(0)} = {} & i\left( \mathring{\bar{N}}_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}^{-\Lambda} \mathcal{F}^{-\Sigma\mu\nu} - \mathring{\mathcal{N}}_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}^{+\Lambda} \mathcal{F}^{+\Sigma\mu\nu} \right) + 6\mathcal{M}^{mn} \mathcal{H}_{m\mu\nu\rho} \mathcal{H}_n^{\ \mu\nu\rho} + \\ & + \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}} \left( \bar{\psi}_\mu^A \gamma_\nu \rho_{A|\lambda\sigma} - \bar{\psi}_{A|\mu} \gamma_\nu \rho_{\lambda\sigma}^A \right) - \frac{i}{2} \mathring{g}_{i\bar{\jmath}} \left( \mathring{\bar{\lambda}}^{iA} \gamma^\mu \nabla_\mu \mathring{\lambda}_A^{\bar{\jmath}} + \mathring{\bar{\lambda}}_A^{\bar{\jmath}} \gamma^\mu \nabla_\mu \mathring{\lambda}^{iA} \right) + \\ & - i \left( \bar{\zeta}^\alpha \gamma^\mu \nabla_\mu \zeta_\alpha + \bar{\zeta}_\alpha \gamma^\mu \nabla_\mu \zeta^\alpha \right) - 2\mathcal{U}_u^{\alpha A} \partial_\mu q^u \left( \bar{\psi}_A^\mu \zeta_\alpha - \bar{\zeta}_\alpha \gamma^{\mu\nu} \psi_{A\nu} + \text{h.c.} \right)\end{aligned} \tag{7.134}$$

$$\mathcal{L}_{(-1)} = \Lambda^{-1} \mathring{\mathcal{F}}_{\mu\nu}^{-I} \mathring{I}_{IJ} \left[ \frac{1}{2} \nabla_i \mathring{f}_j^J \mathring{\bar{\lambda}}^{iA} \gamma^{\mu\nu} \mathring{\lambda}^{jB} \epsilon_{AB} + \text{h.c.} \right]. \tag{7.135}$$

Note that after the appropriate rescalings and the low energy limit, the supergravity Lagrangian reduces to an observable sector corresponding to the rigid Lagrangian of [83], undergoing spontaneous breaking to $\mathcal{N}=1$ supersymmetry, plus a hidden sector, still propagating but fully decoupled from the observable sector:

$$\mathcal{L}_{sugra} \to \mathcal{L}_{APT} + \mathcal{L}_{hidden} \tag{7.136}$$

where

$$\begin{aligned}\mathcal{L}_{APT} = {} & \Lambda^2 \mathring{g}_{i\bar{\jmath}} \partial^\mu z^i \partial_\mu \bar{z}^{\bar{\jmath}} - \frac{i}{2} \mathring{g}_{i\bar{\jmath}} \left( \mathring{\bar{\lambda}}^{iA} \gamma^\mu \nabla_\mu \mathring{\lambda}_A^{\bar{\jmath}} + \mathring{\bar{\lambda}}_A^{\bar{\jmath}} \gamma^\mu \nabla_\mu \mathring{\lambda}^{iA} \right) + \\ & + i \left( \mathring{\bar{N}}_{IJ} \mathcal{F}_{\mu\nu}^{-I} \mathcal{F}^{-J\mu\nu} - \mathring{\mathcal{N}}_{IJ} \mathcal{F}_{\mu\nu}^{+I} \mathcal{F}^{+J\mu\nu} \right) + \\ & + \Lambda^4 \mathring{\mathcal{V}} + \Lambda \left( \mathring{\mathcal{M}}_{iAjB} \mathring{\bar{\lambda}}^{iA} \mathring{\lambda}^{jB} + \text{h.c.} \right) + \\ & + \Lambda^{-1} \mathring{\mathcal{F}}_{\mu\nu}^{-I} \mathring{I}_{IJ} \left[ \frac{1}{2} \nabla_i \mathring{f}_j^J \mathring{\bar{\lambda}}^{iA} \gamma^{\mu\nu} \mathring{\lambda}^{jB} \epsilon_{AB} + h.c. \right]\end{aligned} \tag{7.137}$$

$$\begin{aligned}\mathcal{L}_{hidden} = {} & M_{Pl}^2 \left( -\frac{R}{2} + h_{uv} \partial_\mu q^u \partial^\mu q^v \right) + i \left( \mathring{\bar{N}}_{00} \mathcal{F}_{\mu\nu}^{-0} \mathcal{F}^{-0\mu\nu} - \mathring{\mathcal{N}}_{00} \mathcal{F}_{\mu\nu}^{+0} \mathcal{F}^{+0\mu\nu} \right) + \\ & + 6\mathcal{M}^{mn} \mathcal{H}_{m|\mu\nu\rho} \mathcal{H}_n^{\ \mu\nu\rho} - 2\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} M_{Pl} \mathcal{H}_{m|\nu\rho\sigma} A_u^m \partial_\mu q^u + \\ & + \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}} \left( \bar{\psi}_\mu^A \gamma_\nu \rho_{A|\lambda\sigma} - \bar{\psi}_{A|\mu} \gamma_\nu \rho_{\lambda\sigma}^A \right) - i \left( \bar{\zeta}^\alpha \gamma^\mu \nabla_\mu \zeta_\alpha + \bar{\zeta}_\alpha \gamma^\mu \nabla_\mu \zeta^\alpha \right) + \\ & - 2\mathcal{U}_u^{\alpha A} \partial_\mu q^u \left( \bar{\psi}_A^\mu \zeta_\alpha - \bar{\zeta}_\alpha \gamma^{\mu\nu} \psi_{A\nu} + h.c. \right)\end{aligned} \tag{7.138}$$

Let us note that in the low energy limit the space-time metric, the graviphoton, the anti-symmetric tensors and the scalars of the hypermultiplet sector, together with their fermionic super partners obey the field equations of free waves not interacting with the rest. In particular, the metric can be chosen as a constant background, the hyperscalars can be set to constant values.

In conclusion, in this chapter we have investigated the supergravity origin of a $U(1)^n$, rigid, partially-broken $\mathcal{N} = 2$ supersymmetric theory whose infra-red limit is described by the multi-field BI action of [82].

The high-energy supergravity is characterized by a visible sector described by the $n$ vector multiplets surviving the rigid limit, and by a hidden one consisting of the gravitational multiplet and by a hypermultiplet, which decouple as the Planck mass is sent to infinity.

# Conclusions

In this thesis, we studied pure and matter coupled supergravity theories in different frameworks. Standard supergravity was extended to incorporate other interesting features like enlarged symmetries, matter couplings and cosmological constant. In particular, we constructed different supergravity Lagrangians in three and four dimensions following a geometrical approach, and using the useful properties of the $S$-expansion procedure. Moreover, we presented the multi-vector generalization of a rigid, partially broken $\mathcal{N}=2$ supersymmetric theory as a rigid limit of a gauged $\mathcal{N}=2$ supergravity with electric and magnetic charges.

In Chaper 3, we presented supersymmetric extensions of the Maxwell type algebras in $D=4$ dimensions. Using the properties of the $S$-expansion method we showed that inequivalent Maxwell superalgebras can be obtained when different semigroups are chosen. Thus, we obtained a family of Maxwell superalgebras having the Maxwell type algebras as subalgebras. In particular, the $S$-expansion of $\mathfrak{osp}\left(4|1\right)$ allowed us to obtain the minimal Maxwell superalgebra $s\mathcal{M}_4$. Then choosing different semigroups we defined new minimal $D=4$ Maxwell type superalgebras $s\mathcal{M}_{m+2}$, which can be seen as a generalization of the D'Auria-Fré superalgebra and the Green algebras introduced in [48], [59] respectively.

We also showed that the $D=4$, $\mathcal{N}$-extended Maxwell superalgebra $s\mathcal{M}^{(\mathcal{N})}$ derived initially as a MC expansion in [57], can be alternatively obtained as an $S$-expansion of $\mathfrak{osp}\left(4|\mathcal{N}\right)$. Choosing bigger semigroups presented new $D=4$ $\mathcal{N}$-extended Maxwell type superalgebras. The method considered here could play an important role in the context of supergravity in higher dimensions.

In Chapter 4, we presented a geometric formulation of $\mathcal{N}=1$ supergravity in four dimensions, where the relevant gauge fields of the theory are those corresponding to the minimal Maxwell superalgebra $s\mathcal{M}_4$. We showed that $\mathcal{N}=1$, $D=4$ pure supergravity can be derived alternatively as the MacDowell-Mansouri like action, which is constructed exclusively in terms of the curvatures of the Maxwell type superalgebra $s\mathcal{M}_4$. Then we obtained the minimal supergravity action in four dimensions from the $s\mathcal{M}_{m+2}$ superalgebra. The invariance under supersymmetry was also discussed. A future work could be consider the $\mathcal{N}$-extended Maxwell superalgebras and the construction of $\mathcal{N}$-extended supergravities in diverse dimensions in a very similar way to the one shown here.

In Chapter 5, we analyzed the physical consequences of considering the supersymmetric extension of the AdS-Lorentz algebra in the construction of a minimal supergravity theory. Based on the AdS-Lorentz superalgebra $sAdS - \mathcal{L}_4$ we built the minimal $D = 4$ supergravity action which includes a generalized supersymmetric cosmological constant term. In this way, an alternative way of introducing the supersymmetric cosmological constant in supergravity was presented. We also derived the equations of motion of and the supersymmetry transformations.

In Chapter 6, we derived the $D = 3$ Chern-Simons supersymmetric action from the (standard) Maxwell superalgebra $s\mathcal{M}$. We showed that the Maxwell superalgebra can be obtained from the $\mathfrak{osp}\left(2|1\right) \otimes \mathfrak{sp}\left(2\right)$ superalgebra combining the semigroup expansion procedure with the Inönü-Wigner contraction. This procedure allowed to obtain the non-vanishing components of an invariant tensor for the Maxwell superalgebra and to build the most general $D = 3$ CS supersymmetric action invariant under the Maxwell supergroup. The action describes an "exotic" supersymmetric theory without cosmological constant in three dimensions. The CS supergravity action from a generalized minimal Maxwell superalgebra $s\mathcal{M}^g$ was also constructed. We showed that this generalized minimal Maxwell superalgebra can be derived from the $\mathfrak{osp}\left(2|1\right) \otimes \mathfrak{sp}\left(2\right)$ superalgebra using the semigroup expansion method and choosing a particular semigroup.

Eventually, in Chapter 7 we presented the multi-vector generalization of a rigid, partially broken $\mathcal{N} = 2$ supersymmetric theory as a rigid limit of a suitable gauged $\mathcal{N} = 2$ supergravity with electric and magnetic charges. We considered a new frame in which, in the rigid limit, manifest symplectic invariance is preserved and the electric and magnetic Fayet-Iliopoulos terms are fully originated from the components of the embedding tensor. Furthermore, we gave a general proof of the Ward identity for generic dyonic gaugings.

# Appendix

# Appendix A

# S-expansion method

In this appendix, we review the principal aspects of the $S$-expansion method introduced in [18]. The $S$-expansion procedure consists in combining the inner multiplication law of a semigroup $S$ with the structure constants of a Lie (super)algebra $\mathfrak{g}$. This approach is entirely based on operations performed on the (super)algebra generators, and thus differs from the expansion method introduced in [44], where the dual Maurer-Cartan formalism was used.

Let $S = \{\lambda_\alpha\}$ be a finite abelian semigroup with 2-selector $K_{\alpha\beta}{}^{\gamma}$ defined by

$$K_{\alpha\beta}{}^{\gamma} = \begin{cases} 1, & \text{when } \lambda_\alpha \lambda_\beta = \lambda_\gamma, \\ 0, & \text{otherwise,} \end{cases} \tag{A.1}$$

and $\mathfrak{g}$ a Lie (super)algebra with basis $\{\mathbf{T}_A\}$ and structure constants $C_{AB}{}^{C}$,

$$[\mathbf{T}_A, \mathbf{T}_B] = C_{AB}{}^{C} \mathbf{T}_C. \tag{A.2}$$

Then, the direct product $\mathfrak{G} = S \times \mathfrak{g}$ is also a Lie (super)algebra with structure constants $C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} = K_{\alpha\beta}{}^{\gamma} C_{AB}{}^{C}$, given by

$$\left[\mathbf{T}_{(A,\alpha)}, \mathbf{T}_{(B,\beta)}\right] = C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} \mathbf{T}_{(C,\gamma)}. \tag{A.3}$$

The Lie algebra $\mathfrak{G}$ defined by $\mathfrak{G} = S \times \mathfrak{g}$ is called $S$-expanded algebra of $\mathfrak{g}$.

When the semigroup has a zero element $0_S \in S$, it plays a somewhat peculiar role in the $S$-expanded algebra. The algebra obtained by imposing the condition $0_S \mathbf{T}_A = 0$ on $\mathfrak{G}$ is called $0_S$-reduced algebra of $\mathfrak{G}$.

There are different ways of extracting smaller algebras from $\mathfrak{G} = S \times \mathfrak{g}$. Nevertheless, before extracting smaller algebras it is necessary to apply a decomposition of the original algebra $\mathfrak{g}$. Let $\mathfrak{g} = \bigoplus_{p \in I} V_p$ be a decomposition of $\mathfrak{g}$ in subspaces $V_p$, where $I$ is a set of indices. Then for each $p, q \in I$ it is always possible to define $i_{(p,q)} \subset I$ such that

$$[V_p, V_q] \subset \bigoplus_{r \in i_{(p,q)}} V_r. \tag{A.4}$$

Now, let $S = \bigcup_{p \in I} S_p$ be a subset decomposition of the abelian semigroup $S$ such that

$$S_p \cdot S_q \subset \bigcup_{r \in i_{(p,q)}} S_p. \tag{A.5}$$

When such subset decomposition exists, then we say

$$\mathfrak{G}_R = \bigoplus_{p \in I} S_p \times V_p, \tag{A.6}$$

is a resonant subalgebra of $\mathfrak{G} = S \times \mathfrak{g}$.

Another case of smaller algebra can be obtained when the semigroup has a zero element $0_S \in S$. The algebra obtained after imposing the condition $0_S \mathbf{T}_A = 0$ on $\mathfrak{G}$ is called $0_S$-reduced algebra of $\mathfrak{G}$. Interestingly, there is a way to extract a reduced algebra from a resonant subalgebra. Let $\mathfrak{G}_R = \bigoplus_p S_p \times V_p$ be a resonant subalgebra of $\mathfrak{G} = S \times \mathfrak{g}$. Let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$ such that

$$\hat{S}_p \cap \check{S}_p = \varnothing, \tag{A.7}$$

$$\check{S}_p \cdot \hat{S}_q \subset \bigcap_{r \in i_{(p,q)}} \hat{S}_r. \tag{A.8}$$

Then, these conditions induce the decomposition

$$\check{\mathfrak{G}}_R = \bigoplus_{p \in I} \check{S}_p \times V_p, \tag{A.9}$$

$$\hat{\mathfrak{G}}_R = \bigoplus_{p \in I} \hat{S}_p \times V_p, \tag{A.10}$$

with

$$\left[\check{\mathfrak{G}}_R, \hat{\mathfrak{G}}_R\right] \subset \hat{\mathfrak{G}}_R, \tag{A.11}$$

and therefore $\left|\check{\mathfrak{G}}_R\right|$ corresponds to a reduced algebra of $\mathfrak{G}_R$.

Finding the invariant tensors for an arbitrary (super)algebra is not only an interesting mathematical problem, but also a physical one. As we have seen in the previous chapters an invariant tensor is a crucial ingredient in the construction of supergravity Lagrangians in odd and even dimensions.

A useful property of the $S$-expansion procedure is that it provides us with an invariant tensor for the $S$-expanded algebra $\mathfrak{G} = S \times \mathfrak{g}$ in terms of an invariant tensor for $\mathfrak{g}$. As was shown in [18] the theorem VII.1 provides a general expression for an invariant tensor for an expanded algebra.

**Theorem VII.1:** Let $S$ be an abelian semigroup, $\mathfrak{g}$ a Lie (super)algebra of basis $\{\mathbf{T}_A\}$, and let $\langle \mathbf{T}_{A_n} \cdots \mathbf{T}_{A_n} \rangle$ be an invariant tensor for $\mathfrak{g}$. Then, the expression

$$\langle \mathbf{T}_{(A_1,\alpha_1)} \cdots \mathbf{T}_{(A_n,\alpha_n)} \rangle = \alpha_\gamma K_{\alpha_1 \cdots \alpha_n}{}^{\gamma} \langle \mathbf{T}_{A_1} \cdots \mathbf{T}_{A_n} \rangle \tag{A.12}$$

where $\alpha_\gamma$ are arbitrary constants and $K_{\alpha_1 \cdots \alpha_n}{}^{\gamma}$ is the $n$-selector for $S$, corresponds to an invariant tensor for the $S$-expanded algebra $\mathfrak{G} = S \times \mathfrak{g}$.

Furthermore, as was pointed out in [18] we can find the components of an invariant tensor for the resonant subalgebra $\mathfrak{G}_R = \bigoplus_p S_p \times V_p$. In fact, the $\mathfrak{G}_R$-valued components of $(A.12)$ are given by

$$\langle \mathbf{T}_{(a_{p_1},\alpha_{p_1})} \cdots \mathbf{T}_{(a_{p_n},\alpha_{p_n})} \rangle = \alpha_\gamma K_{\alpha_{p_1} \cdots \alpha_{p_n}}{}^{\gamma} \langle \mathbf{T}_{a_{p_1}} \cdots \mathbf{T}_{a_{p_n}} \rangle, \quad \text{with } \lambda_{\alpha_p} \in S_p$$

It is important to note that since the $0_S$-reduced algebra is not a subalgebra, in general the $0_S$-reduced algebra-valued components of $(A.12)$ do not lead to an invariant tensor. In [18] it was announced a theorem providing a general expression for an invariant tensor for a $0_S$-reduced algebra.

**Theorem VII.2:** Let $S$ be an abelian semigroup with nonzero elements $\lambda_i$, $i = 0, \ldots, N$, and $\lambda_{N+1} = 0_S$. Let $\mathfrak{g}$ be a Lie (super)algebra of basis $\{\mathbf{T}_A\}$, and let $\langle \mathbf{T}_{A_n} \cdots \mathbf{T}_{A_n} \rangle$ be an invariant tensor for $\mathfrak{g}$. The expression

$$\langle \mathbf{T}_{(A_1,i_1)} \cdots \mathbf{T}_{(A_n,i_n)} \rangle = \alpha_j K_{i_1 \cdots i_n}{}^{j} \langle \mathbf{T}_{A_1} \cdots \mathbf{T}_{A_n} \rangle \tag{A.13}$$

where $\alpha_j$ are arbitrary constants, corresponds to an invariant tensor for the $0_S$-reduced algebra obtained from $\mathfrak{G} = S \times \mathfrak{g}$.

The proof to these definitions and Theorems can be found in [18].

# Appendix B

# Generalized Maxwell algebra

In this appendix we show how to obtain the $D$-dimensional generalized Maxwell algebra $\mathcal{M}^{g}$ from $\mathfrak{so}\left(D-1,2\right)$, using the $S$-expansion procedure. As in previous cases, we have to consider a subspaces decomposition of the original algebra $\mathfrak{so}\left(D-1,2\right)$,

$$\mathfrak{g}=\mathfrak{so}\left(D-1,2\right)=\mathfrak{so}\left(D-1,1\right)\oplus\frac{\mathfrak{so}\left(D-1,2\right)}{\mathfrak{so}\left(D-1,1\right)}=V_{0}\oplus V_{1}, \tag{B.1}$$

where $V_0$ is generated by the Lorentz generator $\tilde{J}_{ab}$ and $V_1$ is generated by the $AdS$ boost generator $\tilde{P}_a$. The $\tilde{J}_{ab}$, $\tilde{P}_a$ generators satisfy the commutations relations (3.7) – (3.9), thus the subspace structure can be written as

$$[V_0,V_0]\subset V_0,\qquad [V_0,V_1]\subset V_1,\qquad [V_1,V_1]\subset V_0. \tag{B.2}$$

Let $S_E^{(2)}=\{\lambda_0,\lambda_1,\lambda_2,\lambda_3\}$ be a finite abelian semigroup whose elements are dimensionless and obey the multiplication law (3.14). Let us consider a subset decomposition $S_E^{(2)}=S_0\cup S_1$, with

$$S_0=\{\lambda_0,\lambda_1,\lambda_2,\lambda_3\}, \tag{B.3}$$

$$S_1=\{\lambda_1,\lambda_2,\lambda_3\}, \tag{B.4}$$

This subset decomposition is said to be "resonant" because it satisfies [compare with eqs. $(B.2)$]

$$S_0\cdot S_0\subset S_0,\quad S_0\cdot S_1\subset S_1,\quad S_1\cdot S_1\subset S_{0.} \tag{B.5}$$

Imposing the $0_S$-reduction condition $\lambda_3 T_A=0$,we find a new Lie algebra generated by $\left\{J_{ab},P_a,Z_{ab},\tilde{Z}_{ab},\tilde{Z}_a\right\}$. These generators are defined in terms of the AdS generators as follows

$$J_{ab} = J_{ab,0} = \lambda_0 \tilde{J}_{ab}, \quad P_a = P_{a,1} = \lambda_1 \tilde{P}_a, \tag{B.6}$$
$$\tilde{Z}_{ab} = J_{ab,1} = \lambda_1 \tilde{J}_{ab}, \quad \tilde{Z}_a = P_{a,2} = \lambda_2 \tilde{P}_a, \tag{B.7}$$
$$Z_{ab} = J_{ab,2} = \lambda_2 \tilde{J}_{ab}, \tag{B.8}$$

and satisfy the commutation relations

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \tag{B.9}$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \tag{B.10}$$
$$[P_a, P_b] = Z_{ab}, \tag{B.11}$$
$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{B.12}$$
$$\left[J_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} \tilde{Z}_{ad} - \eta_{ac} \tilde{Z}_{bd} - \eta_{bd} \tilde{Z}_{ac} + \eta_{ad} \tilde{Z}_{bc}, \tag{B.13}$$
$$\left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \tag{B.14}$$
$$\left[J_{ab}, \tilde{Z}_c\right] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \tag{B.15}$$
$$\left[\tilde{Z}_{ab}, P_c\right] = \eta_{bc} \tilde{Z}_a - \eta_{ac} \tilde{Z}_b, \tag{B.16}$$
$$\text{others} = 0, \tag{B.17}$$

where we have used the multiplication law of the semigroup (3.14) and the commutation relations of the original algebra. The new algebra obtained after a $0_S$-reduced resonant $S$-expansion of $\mathfrak{so}(3,2)$ corresponds to a generalized Maxwell algebra $\mathcal{M}^g$ [57] in $D$-dimensions, and contains the Maxwell algebra $\mathcal{M}$ as a subalgebra . It is interesting to observe that the $\mathcal{M}^g$ algebra is very similar to the Maxwell type algebra $\mathcal{M}_6$ introduced in [12, 17]. In fact, one could identify $Z_{ab}$, $\tilde{Z}_{ab}$ and $\tilde{Z}_a$ with $Z_{ab}^{(1)}$, $Z_{ab}^{(2)}$ and $Z_a$ of $\mathcal{M}_6$ respectively. However, the commutation relations $(B.11)$, $(B.14)$ and $(B.16)$ are subtly different of those of Maxwell type algebra $\mathcal{M}_6$.

# Appendix C

# Notations and conventions

In this Appendix we summarize our notation and conventions used in Chapters 2, 3, 4 and 5 for the gamma matrices in $D = 4$.

$$\begin{aligned}
\eta_{ab} &= (-1, 1, 1, 1)\,, \qquad \{\gamma_a, \gamma_b\} = -2\eta_{ab}, \qquad [\gamma_a, \gamma_b] = 2\gamma_{ab} \\
\gamma_5 &\equiv -\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4, \qquad \gamma_5^2 = -1, \qquad \{\gamma_5, \gamma_a\} = \quad [\gamma_5, \gamma_{ab}] = 0
\end{aligned}$$

We are working with Majorana spinors, satisfying $\bar{\psi} = \psi^T C$, where $C$ is the charge conjugation matrix.

Furthermore, we are using that $C\gamma_a$ and $C\gamma_{ab}$ are symmetric, while $C$, $C\gamma_5$ and $C\gamma_5\gamma_a$ are antisymmetric gamma matrices.

## C.1 Useful identities

$$\gamma_{ab}\gamma_5 = -\frac{1}{2}\epsilon_{abcd}\gamma^{cd}, \tag{C.1}$$

$$\gamma_a\gamma_b = \gamma_{ab} - \eta_{ab}, \tag{C.2}$$

$$\gamma^{ab}\gamma_{cd} = \epsilon^{ab}{}_{cd}\gamma_5 - 4\delta^{[a}_{[c}\gamma^{b]}_{d]} - 2\delta^{ab}_{cd}, \tag{C.3}$$

$$\gamma^{ab}\gamma^c = 2\gamma^{[a}\delta^{b]}_c - \epsilon^{abcd}\gamma_5\gamma_d, \tag{C.4}$$

$$\gamma^c\gamma^{ab} = -2\gamma^{[a}\delta^{b]}_c - \epsilon^{abcd}\gamma_5\gamma_d. \tag{C.5}$$

$$\psi\bar{\psi} = \frac{1}{2}\gamma_a\bar{\psi}\gamma^a\psi - \frac{1}{8}\gamma_{ab}\bar{\psi}\gamma^{ab}\psi, \tag{C.6}$$

$$\gamma_a\psi\bar{\psi}\gamma^a\psi = 0, \tag{C.7}$$

$$\gamma_{ab}\psi\bar{\psi}\gamma^{ab}\psi = 0. \tag{C.8}$$

# Appendix D

# Special Kähler and Quaternionic Kähler Manifolds

In this appendix we review the main properties of special Kähler and quaternionic Kähler manifolds. We shall consider a $\mathcal{N}=2$ supergravity theory that contains $2n+4n_H$ scalar fields interacting through a $\sigma$-model based on the following scalar manifold:

$$\mathcal{M}_{scalar}=\mathcal{M}_{SK}\left(n\right)\times\mathcal{M}_{QK}\left(n_H\right),$$

where $\mathcal{M}_{SK}\left(n\right)$ is a special Kähler manifold with $n$ complex dimensions and $\mathcal{M}_{QK}\left(n_H\right)$ is a quaternionic manifold with $n_H$ quaternionic dimensions.

## D.1 Special Kähler Manifolds

The special Kähler geometry arises in the coupling of vector multiplets to $\mathcal{N}=2$, $D=4$ supergravity. In this case the complex scalar fields sitting in the vector multiplets span a manifold $\mathcal{M}_{SK}$ which is not only Kählerian but also special Kählerian.

A special Kähler manifold is a Kähler manifold of restricted type (Hodge manifold) endowed with a flat, symplectic, holomorphic bundle, and with a hermitian metric

$$ds^2=g_{i\bar{\jmath}}\left(z,\bar{z}\right)dz^i\otimes d\bar{z}^{\bar{\jmath}}, \tag{D.1}$$

such that the $(1,1)$-form

$$K=ig_{i\bar{\jmath}}\left(z,\bar{z}\right)dz^i\wedge d\bar{z}^{\bar{\jmath}} \tag{D.2}$$

is closed ($dK = 0$). As in all Kähler manifolds the metric has the form:

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K} \tag{D.3}$$

On a special Kähler manifold one can always introduce a tensor bundle whose holomorphic section will be denoted by $\Omega\left(z\right) = \left(\Omega^M\left(z\right)\right)$, $M = 1, ..., 2n+2$, and will have the following structure

$$\Omega(z) = \begin{pmatrix} X^\Lambda(z) \\ F_\Lambda(z) \end{pmatrix}\ , \quad \Lambda = 0, \ldots, n\,, \tag{D.4}$$

The Kähler potential can be written in terms of this holomorphic section as follows

$$\mathcal{K}(z, \bar{z}) = -\log[i\,\overline{\Omega}(\bar{z})^T \mathbb{C}\Omega(z)] \tag{D.5}$$

where $\mathbb{C} = (\mathbb{C}_{MN})$ is the $\mathrm{Sp}(2(n+1), \mathbb{R})$-invariant matrix;

$$\mathbb{C} \equiv \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}\ . \tag{D.6}$$

The transition functions connecting overlapping coordinate patches $U_{\mathtt{m}}, U_{\mathtt{n}}$ on $\mathcal{M}_{SK}$, act on $\Omega\left(z\right)$ as follows

$$\Omega_{\mathtt{m}} = e^{f_{\mathtt{mn}}} \mathbb{M}_{\mathtt{mn}} \Omega_{\mathtt{n}}$$

where $f_{\mathtt{mn}} = f_{\mathtt{mn}}\left(z\right)$ is a holomorphic function and $\mathbb{M}_{\mathtt{mn}}$ is a constant $\mathrm{Sp}(2(n+1), \mathbb{R})$ matrix. Moreover, the action on $\mathcal{K}$ amounts to a Kähler transformation:

$$\mathcal{K}_{\mathtt{m}} \to \mathcal{K}_{\mathtt{n}} - f_{\mathtt{mn}} - \bar{f}_{\mathtt{mn}}, \tag{D.7}$$

We can define a *covariantly holomorphic* section $V(z, \bar{z})$ as follows

$$V(z, \bar{z}) = (V^M(z, \bar{z})) = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \equiv e^{\mathcal{K}/2} \Omega\left(z\right). \tag{D.8}$$

satisfying

$$1 = i\left\langle V | \bar{V} \right\rangle = i\left(\bar{L}^\Lambda M_\Lambda - \bar{M}_\Sigma L^\Sigma\right), \qquad \nabla_{\bar{i}} V = 0.$$

The last equality implies that $V$ is covariantly holomorphic. The action of the transition functions on $V$ amounts to a constant symplectic transformation combined with a $U(1)$-phase related to the Kähler transformation:

$$V_{\mathtt{m}} = e^{i \mathrm{Im}(f_{\mathtt{mn}})} \mathbb{M}_{\mathtt{mn}} V_{\mathtt{n}}$$

We define the $U(1)$-covariant derivative on $V$ as follows:

$$U_i = \nabla_i V = \left(\partial_i + \frac{1}{2}\partial_i \mathcal{K}\right) V \equiv \begin{pmatrix} f_i^\Lambda \\ h_{\Sigma|i} \end{pmatrix}. \tag{D.9}$$

Furthermore,

$$\nabla_i U_j = i C_{ijk} g^{k\bar{l}} \bar{U}_{\bar{l}}$$

where $C_{ijk}$ is a covariantly holomorphic symmetric three-tensor. Thus, in a special Kähler manifold the section $V$ and its covariant derivative $U_i$ need to satisfy the following properties:

$$\nabla_i U_j \equiv \partial_i U_j + \frac{\partial_i \mathcal{K}}{2} U_j - \Gamma^k_{ij} U_k = i C_{ijk} g^{k\bar{l}} \bar{U}_{\bar{l}}, \quad \nabla_i \bar{U}_{\bar{j}} = g_{i\bar{j}} \bar{V}, \quad V^T \mathbb{C} \mathbb{U}_i = 0, \quad V^T \mathbb{C} \bar{U}_{\bar{k}} = 0 \tag{D.10}$$

Now we can introduce the period matrix via the relations

$$\bar{M}_\Lambda = \bar{\mathcal{N}}_{\Lambda\Sigma} \bar{L}^\Sigma, \qquad h_{\Sigma|i} = \bar{\mathcal{N}}_{\Sigma\Lambda} f_i^\Lambda \tag{D.11}$$

which can be solved introducing the vectors

$$f_I^\Lambda = \begin{pmatrix} f_i^\Lambda \\ \bar{L}^\Lambda \end{pmatrix}, \qquad h_{\Lambda|I} = \begin{pmatrix} h_{\Lambda|i} \\ \bar{M}_\Lambda \end{pmatrix}$$

and setting

$$\bar{\mathcal{N}}_{\Lambda\Sigma} = h_{\Lambda|I} \circ \left(f^{-1}\right)^I_\Sigma. \tag{D.12}$$

Using $V$ and its covariant derivatives, we can construct the following matrix:

$$\mathbb{L}(z,\bar{z})^M{}_{\underline{N}} \equiv \left(V, \mathsf{e}_{\bar{I}}{}^{\bar{\imath}} \overline{U}^M_{\bar{\imath}}, \overline{V}^M, \mathsf{e}_I{}^i U_i^M\right), \tag{D.13}$$

where $\mathsf{e}_I{}^i$ are the inverse vielbein matrices $g_{i\bar{j}} = \sum_{I=\bar{I}=1}^n \mathsf{e}_i{}^I \bar{\mathsf{e}}_{\bar{j}}{}^{\bar{I}}$, and $\underline{N}$ is a holonomy group index. Eqs. (D.10) imply the following property of $\mathbb{L}$:

$$\mathbb{L}^\dagger \mathbb{C} \mathbb{L} = \varpi, \tag{D.14}$$

where

$$\varpi \equiv -i \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}. \tag{D.15}$$

If we change the complex index $\underline{N}$ into a real one by means of the Cayley matrix $\mathcal{A}$, thus defining:

$$\mathbb{L}_{\mathrm{Sp}} \equiv \mathbb{L}\mathcal{A}, \quad \mathcal{A} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & i\,\mathbf{1} \\ \mathbf{1} & -i\,\mathbf{1} \end{pmatrix}, \tag{D.16}$$

Eq. (D.14) expresses the condition that the real matrix $\mathbb{L}_{\rm Sp}$ be symplectic since $\varpi = \mathcal{A}\mathbb{C}\mathcal{A}^\dagger$. As a consequence of this also $\mathbb{L}_{\rm Sp}^T$ is symplectic and this implies an other set of identities which can be cast in the following compact form:

$$\mathbb{L}\varpi\mathbb{L}^\dagger = \mathbb{C}\,. \tag{D.17}$$

Let us define in terms of $\mathbb{L}$ the following symmetric, negative-definite, symplectic matrix encoding all the information about the coupling of the vector fields to the scalars:

$$\begin{aligned} \mathcal{M}(z,\bar{z}) &= (\mathcal{M}_{MN}) \equiv \mathbb{C}\mathbb{L}\mathbb{L}^\dagger\mathbb{C} = \mathcal{M}(z,\bar{z})^T\,,\\ \mathcal{M}\mathbb{C}\mathcal{M} &= \mathbb{C}\,. \end{aligned} \tag{D.18}$$

Furthermore, under an isometry transformation $g:\ z \to z'$ in $G_{SK}$, using (7.5), we find that $\mathcal{M}$ transforms linearly:

$$\mathcal{M}(z,\bar{z}) \ \to\ \mathcal{M}(z',\bar{z}') = \mathbb{M}[g]^T\mathcal{M}(z,\bar{z})\mathbb{M}[g]\,. \tag{D.19}$$

From the previous properties of $V$ and $U_i$ we find the following general symplectic covariant relation:

$$U^{MN} \equiv g^{i\bar{\jmath}}\,U_i^M U_{\bar{\jmath}}^N = -\frac{1}{2}\mathcal{M}^{MN} - \frac{i}{2}\,\mathbb{C}^{MN} - \overline{V}^M V^N\,, \tag{D.20}$$

where $\mathcal{M}^{MN}$ are the components of $\mathcal{M}^{-1} = -\mathbb{L}\mathbb{L}^\dagger$.

If $k_a$ is the Killing vector defining an infinitesimal isometry, then the invariance of the Kähler form $K$, $\ell_a K = 0$, implies

$$\ell_a K = d(\iota_a K) = 0 \ \Rightarrow\ \ \iota_a K = -d\mathcal{P}_a\,, \tag{D.21}$$

where $\iota_a$ denotes the contraction of $K$ with $k_a$. The last equation defines the momentum maps and is equivalent to Eqs. (7.9).

## D.2 Quaternionic Kähler manifolds

Let us now consider the hypermultiplet sector of a $\mathcal{N}=2$ theory. Here there are four real scalar fields for each hypermultiplet and, at least locally, they can be seen as the four components of a quaternion. In this sector the scalar manifold $\mathcal{M}_{QK}\,(n_H)$ has dimension multiple of four, dim$\mathcal{M}_{QK} = 4n_H$.

A quaternionic manifold is a $4n_H$-dimensional real manifold endowed with a metric $h$:

$$ds^2 = h_{uv}\,(q)\,dq^u \otimes dq^v, \qquad u,v = 1,\dots,4n_H \tag{D.22}$$

and three complex structures $(J^x)_v^u$ , $x = 1, 2, 3$ that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy} + \epsilon^{xyz} J^z. \tag{D.23}$$

The triplet of two-forms $K^x$

$$K^x = K^x_{uv} dq^u \wedge dq^v; \quad K^x_{uv} = h_{uw} (J^x)_v^w , \tag{D.24}$$

is covariantly closed with respect to an $SU(2) \simeq Sp(2)$ connection $\omega^x$

$$\nabla K^x \equiv dK^x + \epsilon^{xyz} \omega^y \wedge K^z = 0,$$

with curvature given by

$$\Omega^x \equiv d\omega^x + \frac{1}{2} \epsilon^{xyz} \omega^y \wedge \omega^z = \lambda K^x, \tag{D.25}$$

where $\lambda = -1$ is fixed by supersymmetry, together with appropriate normalizations for the kinetic terms in the Lagrangian.

Equations $(D.24)$ and $(D.23)$ imply the following relation

$$K^x_{uw} h^{ws} K^y_{sv} = -\delta^{xy} h_{uv} + \epsilon^{xyz} K^z_{uv}, \tag{D.26}$$

where $h^{ws}$ are the components of the inverse metric.

As a consequence of this structure the manifold $\mathcal{M}_{QK}(n_H)$ has a holonomy group

$$H = Hol (Q (n_H)) = SU (2) \otimes H', \tag{D.27}$$

where $H' \in Sp(2n_H, \mathbb{R})$. Then, introducing flat indices $\{A, B, C = 1, 2\}$ , $\{\alpha, \beta, \gamma = 1, ..., 2n_H\}$ that run, respectively, in the fundamental representations of $SU(2)$ and $Sp(2n_H)$, we can introduce a vielbein 1-form

$$\mathcal{U}^{A\alpha} = \mathcal{U}^{A\alpha}_u (q)\, dq^u \tag{D.28}$$

such that

$$h_{uv} = \mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v \mathbb{C}_{\alpha\beta} \epsilon_{AB} \tag{D.29}$$

where $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ and $\epsilon_{AB} = -\epsilon_{BA}$ are, respectively, the flat $Sp(2n_H)$ and $Sp(2) \sim SU(2)$ invariant metrics. The vielbein $\mathcal{U}^{A\alpha}$ is covariantly closed with respect to the $SU(2)$-connection $\omega^x$ and to some $Sp(2n_H)$-Lie algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$ :

$$\nabla \mathcal{U}^{A\alpha} \equiv d\mathcal{U}^{A\alpha} + \frac{i}{2} \omega^x (\epsilon \sigma_x \epsilon)^A{}_B \wedge \mathcal{U}^{B\alpha} + \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma} \mathbb{C}_{\beta\gamma} = 0 \tag{D.30}$$

where $(\sigma_x)^A{}_B$ are the standard Pauli matrices. Furthermore the 1-forms $\mathcal{U}^{A\alpha}$ satisfy the following relations:

$$
\begin{aligned}
\mathcal{U}_{A\alpha} &\equiv \left(\mathcal{U}^{A\alpha}\right)^* = \epsilon_{AB}\mathbb{C}_{\alpha\beta}\mathcal{U}^{B\beta} \\
\mathcal{U}_{A\alpha u}\mathcal{U}_v^{B\alpha} &= \frac{1}{2}h_{uv}\delta_A^B - \frac{i}{2}K^x_{uv}\,(\sigma^x)_A{}^B\,.
\end{aligned}
\tag{D.31}
$$

Let us now consider infinitesimal isometries generated by $t_m$, whose action on the scalar fields is described by Killing vectors $k_m = k_m^u\,\partial_u$, closing the isometry algebra:

$$
[t_m,\, t_n] = f_{mn}{}^p\, t_p \quad , \quad [k_m,\, k_n] = -f_{mn}{}^p\, k_p\,, \tag{D.32}
$$

and leaving the 4-form $\sum_{x=1}^3 K^x \wedge K^x$ invariant [99]. This condition amounts to require:

$$
\ell_n K^x = \epsilon^{xyz}\, K^y\, W_n^z\,, \tag{D.33}
$$

where $W_n^z$ is an $SU(2)$-compensator. This equation is solved by writing the Killing vectors $k_n$ in terms of *tri-holomorphic momentum maps* $\mathcal{P}_n^x$ as follows [99]:

$$
\iota_n K^x = -\nabla\mathcal{P}_n^x = -(d\mathcal{P}_n^x + \epsilon^{xyz}\omega^y\,\mathcal{P}_n^z)\,, \tag{D.34}
$$

provided

$$
\mathcal{P}_n^x = \lambda^{-1}(\iota_n\omega^x - W_n^x) = W_n^x - \iota_n\omega^x\,, \tag{D.35}
$$

where we have used $\lambda = -1$. For those isometries with vanishing compensator, $W_n^x = 0$, the momentum maps have the simple expression:

$$
\mathcal{P}_n^x = -k_n^u\,\omega_u^x.
$$

Just as for the special Kähler manifolds, the momentum maps satisfy Poisson brackets described by the following equivariance condition:

$$
2\,K_{uv}\,k_n^u\,k_m^v - \lambda\,\epsilon^{xyz}\,\mathcal{P}_n^y\,\mathcal{P}_m^z = -f_{mn}{}^p\,\mathcal{P}_p^x\,. \tag{D.36}
$$

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