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Termodinámica de agujeros negros con campos de materia: formulación hamiltoniana y cargas conservadas

Hairy black holes thermodynamics: Hamiltonian formulation and conserved charges

Tesis para optar al grado de Doctor en Ciencias Físicas

por

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Resumen

Esta tesis tiene como principal objetivo explorar desde un punto de vista termodinámico la interacción de la gravedad con campos de materia. En particular, estudiamos el acoplamiento de un campo escalar y un campo de gauge bajo diferentes escenarios. En tres dimensiones, obtenemos nuevas soluciones desde una acción de gravedad con un campo escalar y campo de gauge conformalmente acoplados. En cuatro dimensiones, consideramos una solución diónica en presencia de un campo escalar dilatónico, la cual sirve de laboratorio para aclarar el rol de la carga magnética en la primera ley de la termodinámica. Para cumplir el objetivo de esta tesis, algunos conocimientos previos son de relevancia. Primero explicamos un método hamiltoniano para calcular cargas conservadas, el cual es considerado a lo largo de todo su desarrollo. Este es el llamado método de Regge-Teitelboim. Otro tema de estudio es considerar la presencia de campos de materia y cómo su presencia influye en el comportamiento de la métrica en infinito. Esto es de mucha importancia al calcular las cargas globales del sistema ya que estas son sensibles a las condiciones asintóticas de los campos. Con estas herramientas verificamos que la variación de la carga magnética aparece como consecuencia de que la acción hamiltoniana tenga un extremo, aun cuando esta no proviene de ninguna simetría de la acción. Cuando la gravedad es formulada como una teoría Chern-Simons, la termodinámica de agujeros negros puede también ser obtenida exclusivamente en término de los campos de gauge y la topología de la variedad. Mostramos que esto también es posible cuando hay campos escalares acoplados minimal y conformalmente, pudiéndose aplicar las mismas técnicas que para gravedad pura en este tipo de agujeros negros. En particular, esto es posible ya que esta clase de soluciones pueden escribirse en término de conexiones de gauge. Luego, las condiciones de regularidad son impuestas sobre las holonomías a lo largo del ciclo termal del toro en el horizonte de eventos, fijando sus potenciales químicos. La entropía es calculada de dos maneras, desde la energía libre de Gibbs y luego usando la fórmula de la entropía para una teoría Chern-Simons. Ambas dan como resultado la ley del área modificada para la entropía.

Esta tesis contiene los detalles y marco teórico de los resultados publicados en los siguientes artículos:

1.- "*Three-dimensional black holes with conformally coupled scalar and gauge fields*", M. Cárdenas, O. Fuentealba and C. Martínez, Phys. Rev. D **90**, no. 12, 124072 (2014).

2.- "*Thermodynamics of three-dimensional hairy black holes in terms of gauge fields*", M. Cárdenas, O. Fuentealba, C. Martínez and R. Troncoso. Trabajo en preparación.

3.-"*On conserved charges and thermodynamics of the AdS_4 dyonic black hole*", M. Cárdenas, O. Fuentealba and J. Matulich, JHEP **1605**, 001 (2016).

Abstract

The principal goal of this thesis is to explore the interaction among matter fields and gravity from a thermodynamic point of view. In particular, we explore the presence of a scalar field and a gauge field, coupled to gravity in different scenarios. In three dimensions, we obtain new solutions from an action of gravity with a conformally coupled scalar and gauge field. In four dimensions, we consider a dyonic black hole with a dilatonic scalar field which serves as a laboratory to study the role of the magnetic charge in the first law of black hole thermodynamics. Some ingredients are relevant for accomplishing the objetive. First, we explain a Hamiltonian method for computing charges which is considered along the whole thesis. This is the Regge-Teitelboim approach. In this context, we note that the global charges of a system are sensitive to the asymptotic conditions of the fields; this issue is studied when matter fields are included since they can backreact on the asymptotic behavior of the metric at infinity. With these tools we verify that the variation of the magnetic charge appears as a consequence that the Hamiltonian action has an extremum, although it is not originated by any symmetry of the action. Whenever it is possible to describe gravity as a Chern-Simons theory, the thermodynamics of black holes can be obtained exclusively in terms of gauge fields and the topology of the manifold. It is shown that when we have scalar fields conformally and minimally coupled to gravity, the same technics can be applied to hairy black holes even though we are no longer in a topological theory. In particular, this is possible since the hairy black holes can be written in terms of gauge connections. Then, regularity conditions are imposed on the holonomies along the thermal cycle of the torus at event horizon, fixing the chemical potentials of the solutions. The entropy is derived in two ways; the first one from the Gibbs free energy and the second one from a general formula for the entropy in terms for a Chern-Simons theory. Both give the same result compared with the modified area law.

In this thesis, the work and results of the following publications are detailed:

1.- "*Three-dimensional black holes with conformally coupled scalar and gauge fields*", M. Cárdenas, O. Fuentealba and C. Martínez, Phys. Rev. D **90**, no. 12, 124072 (2014).

2.-"*Thermodynamics of three-dimensional hairy black holes in terms of gauge fields*", M. Cárdenas, O. Fuentealba, C. Martínez and R. Troncoso. Work in preparation.

3.-"*On conserved charges and thermodynamics of the* AdS_4 *dyonic black hole*", M. Cárdenas, O. Fuentealba and J. Matulich, JHEP **1605**, 001 (2016).

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Chapter 1

Introducción

A lo largo de los últimos años, los agujeros negros han estado involucrados en un gran número de aplicaciones físicas. De forma más reciente, la colisión de dos agujeros negros llevaron a la confirmación final de la existencia de las ondas gravitacionales [1]. Es así como desde un punto de vista fenomenológico, estos objetos son de extrema importancia dado que representan la evolución final de un cuerpo muy masivo que ha colapsado gravitacionalmente. Cuando esto ocurre, el espaciotiempo está tan fuertemente curvado, que ningún cuerpo, incluyendo la luz, pueden salir de él. Aparentemente, todo lo que entra en él pierde su identidad, preservando solo su masa, carga y momentum angular. De hecho, esos son los únicos parámetros desde los cuales podemos caracterizar agujeros negros hasta el momento.

Por otro lado, desde el punto de vista teórico, los agujeros negros también han tenido un rol protagónico dado que podrían exhibir aspectos cuánticos de la gravedad. Los trabajos seminales de Bekenstein y Hawking [2, 3] mostraron que los agujeros negros tiene propiedades parecidas a las termodinámicas y que en efecto, radían. A partir de esto, nos hemos convencido de que la primera ley de la termodinámica para agujeros negros,

$$\delta M = T\delta S - \Omega\delta J,$$

es más que una relación mecánica de sus parámetros, si no que también una pieza fundamental en la búsqueda de una teoría cuántica de la gravedad. La entropía S del agujero negro no sería una cantidad térmica, si no más bien una medida de la información perdida cuando un objeto entra en él. Además, la existencia de esta cantidad tiene directa relación con la mecánica cuántica, dado que al tomar el límite $\hbar \to 0$ no hay ninguna interpretación clásica para la entropía de Bekenstein-Hawking

$$S = \frac{A}{4G\hbar}.$$

En este contexto, gravedad en tres dimensiones ha sido un lugar fructífero para estudiar aspectos cuánticos de la gravedad dado que no hay grados de libertad propagantes, lo que hace posible formularla como una teoría de Chern-Simons, al ser puramente topológica. Otro aspecto que ha potenciado este estudio es que el comportamiento asintótico de la métrica de anti-de Sitter (AdS) en tres dimensiones es invariante bajo las transformaciones del grupo conforme en dos dimensiones [7]. Esto influyó en la formulación de la correspondecia AdS_3/CFT_2 [8], que implica que una teoría conforme de campos correspondería a la teoría cuántica de AdS en tres dimensiones. Aquí, la existencia de una solución, el agujero negro de BTZ (Bañados-Teitelboim-Zanelli) [9, 10], también ha servido como laboratorio para probar tales aspectos cuánticos. De hecho, la entropía de este agujero negro pudo ser obtenida desde el conteo microscopico de microestados a través de la fórmula de Cardy [11].

Otro aspecto muy activo de tres dimensiones ha sido el estudio de soluciones asintóticamente AdS provistas de un campo escalar. Notablemente, su comportamiento asintótico, aun en presencia de materia, es invariante bajo el grupo conforme en dos dimensiones, lo que ha servido para que estas soluciones hayan sido relacionadas con superconductores a través de la conjetura de dualidad gravedad/teoría de gauge [12]. Adicionalmente, en una área paralela, existen esfuerzos recientes para medir el teorema de no-pelo desde observaciones astronómicas [13, 14]. La amplia literatura acerca de estos agujeros negros y su aplicaciones demuestran su relevancia física. Esto empezó con una solución precursora de agujero negro dotada de un campo escalar conformalmente acoplado [15] y luego minimálmente acoplado [16], dando paso a un gran número de soluciones con similares características [17]-[28]. Como fue demostrado primero en [16], y luego discutido vastamente en la literatura (ver [29] por ejemplo), el campo escalar puede tener un decaimiento en el infinito tan lento que termina contribuyendo a la masa del agujero negro. Así mismo, otro resultado interesante en esta área, es que el cálculo de la entropía de este tipo de soluciones fue obtenida exitosamente a través de la fórmula de Cardy [19, 20, 21, 28]. En este contexto, es posible considerar otros tipo de acoplamientos de materia igualmente sencillos, entre los cuales están los campos de gauge. Por ejemplo, el BTZ cargado fue introducido en [9] y la versión rotante cargada en [24]. En ambos casos, la dinámica del campo de gauge estuvo determinada por el lagrangiano de Maxwell usual, que mostraba un campo eléctrico con un comportamiento logarítmico y dependiente de la coordenada radial. Es en este escenario de soluciones y aplicaciones en el cual se enmarcan los primeros dos trabajos de esta tesis. Ambas son nuevas soluciones de gravedad con campos de materia acoplados en tres dimensiones, como los anteriormente descritos y son además objetos de interés termodinámico. La primera de ellas se estudia siguiendo la formulación métrica usual, y la segunda, en términos de conexiones siguiendo las técnicas conocidas para gravedad como una teoría Chern-Simons.

Como ya fue mencionado, el modelo más simple para estudiar la interacción de campos de materia con la gravedad es a través del acoplamiento de campos escalares. Tales modelos no solo poseen soluciones de agujeros negros si no que también evaden la conjetura de no-pelo. El primer ejemplo de este hecho fue dado a través de una solución de gravedad en un espacio asintóticamente plano en cuatro dimensiones con un campo escalar y un campo electromagnético [30] [31], [32]. Desafortunadamente, el campo escalar diverge en el horizonte y la métrica corresponde al caso de un agujero negro Reissner-Nordström extremo. Este no es el caso de la solución que consideramos en esta tesis, la cual cumple con presentar estos campos [33]; un campo eléctrico, un campo magnético y un campo escalar dilatónico. Esta solución es un agujero negro asintóticamente AdS debido a la presencia de un potencial de autointeracción. En su análisis termodinámico [33], se afirmó que la primera ley de la termodinámica no se satisfacía a menos que se agregara una nueva carga, cantidad que los autores interpretaron como una carga escalar [33], [34]. Sin embargo, esto conflictúa con el hecho que no existen cargas escalares asociadas a la solución, al no estar ésta originada en ninguna simetría de la acción (ya sea de Noether o topológica). Además, ya se ha identificado claramente cómo los campos escalares contribuyen a la variación de la masa con términos no integrables genéricamente. Esto ha sido demostrado a través de condiciones asintóticas generales a través del método Hamiltoniano [29], además de otros métodos [36], [37]. Una solución en tres dimensiones que exhibe también este comportamiento, se presenta también en esta tesis [38].

Otra motivación para estudiar la solución [33] tiene que ver con el cálculo de su energía libre de Gibbs. Al hacerlo, se debe proceder desde una acción bien definida que además presente algunas características termodinámicas, como por ejemplo exhibir las cargas del sistema con sus respectivos potenciales químicos. La energía libre de Gibbs presentada en [33] falla en este aspecto dado que no puede recobrar la contribución magnética, ni tampoco incluye la supuesta carga escalar. Como se verá, la adición de un nuevo término hará que se obtenga la contribución magnética a la acción euclidea.

Esta tesis está organizada de la siguiente manera:

En el primer capitulo, damos el marco teórico en el cual se desarrolla. La formulación Hamiltoniana de la gravedad será nuestra principal herramienta para calcular cargas conservadas y verificar la validez de la primera ley de la termodinámica para agujeros negros, al requerir que la acción Hamiltoniana tenga un extremo. Después, en el segundo capítulo, destacamos la importancia de las simetrías asintóticas en gravedad y damos un ejemplo, gravedad en tres dimensiones con constante cosmológica negativa ($\Lambda < 0$). Destacamos también cómo la presencia de materia puede alterar el comportamiento asintótico de la métrica si el campo escalar tiene una expansión asintótica lo suficientemente lenta en el infinito.

En el tercer capítulo, consideramos gravedad con constante cosmológica negativa en la presencia de un campo escalar y un campo de gauge abeliano. Esta fuente de materia compuesta está caracterizada por el hecho que ambos campos están conformalmente acoplados a gravedad. La teoría admite soluciones de agujero negro las cuales son descritas a través de sus propiedades termodinámicas. Además, calculamos sus cargas conservadas e imponemos condiciones de borde sobre la variación de la masa para que sea integrable, de la misma forma que se explica en el segundo capítulo de esta tesis.

En el capítulo cuatro, explicamos cómo gravedad en tres dimensiones puede ser vista como una teoría Chern-Simons y cómo es posible desarrollar el análisis termodinámico exclusivamente en términos de los campos de gauge y la topología de la variedad, sin hacer referencia a la métrica del espaciotiempo. El capítulo cinco sigue este procedimiento, pero para una acción que tiene un campo escalar acoplado conformalmente, donde la solución de agujero negro asociada es expresada en términos de conexiones. La versión euclídea de esta acción es usada para analizar la termodinámica de una solución rotante de agujero negro en el ensemble gran canónico. La condición de regularidad es impuesta pidiendo que la holonomía evaluada en el horizonte sea trivial. Esto fija las constantes de integración del sistema en términos de los potenciales químicos. La masa y el momentum angular son también calculados, las cuales coiciden con las cantidades obtenidas por el método hamiltoniano. La entropía en cambio, es calculada de dos maneras, la primera desde la energía libre de Gibbs y la segunda desde la ley general para la entropía en una teoría Chern-Simons. Ambas coinciden con la ley para la entropía del área modificada, la cual es aplicada en este tipo de acciones con acoplamiento no minimal de materia. La parte final de este capítulo está dedicado a explicar cómo esta formulación puede también aplicarse cuando el campo escalar está minimalmente acoplado a la gravedad.

El capítulo final de esta tesis intenta aclara los roles de ciertos campos de materia en la primera ley de la termódinámica para agujeros negros. En particular, usando una solución diónica en presencia de una campo escalar dilatónico, determinamos el rol de la carga magnética como también la contribución del campo escalar al requerir que la acción Hamiltoniana tenga un extremo. Para probar lo anterior, formulamos un principio de acción bien definido y finito para el sistema y probamos que existe un término adicional debido a la existencia del monopolo magnético. Concluimos que la carga magnética, a pesar de tener un origen topológico, aparece en la primera ley de la termodinámica para agujeros negros, mientras que el campo escalar aparece como una contribución a la masa.

Chapter 2

Introduction

In different contexts during the last years, a great number of physical applications involving black holes have emerged. Most recently, the merge of two black holes led to the final confirmation of the gravitational waves existance [1]. From a phenomenological point of view, black holes are extremely important because they are what remains after an extremely massive object has suffered gravitational collapse. The spacetime is so strongly curved that no kind of object, including the light, can come out. Apparently, everything that falls into the black hole, loses its features, preserving only its mass, charge, angular momentum. In fact, these are the only parameters that we have from which we can characterize black holes.

On the other hand, from a theoretical point of view, black holes have an intriguing role since they could exhibit quantum aspects of gravity. From the seminal works of Bekenstein [2, 3] that show thermodynamic-like features of black holes and Hawking's work which proves that black holes radiate [4], we have convinced ourselves that the fist law of black hole thermodynamics

$$\delta M = T\delta S - \Omega\delta J,$$

is more than a mechanical relation, but a fundamental piece in the search of a quantum theory of gravity. The entropy of the black hole would not be a thermal entropy but a quantity related with the information lost after an object enters into it. Moreover, it would be senseless if there were no quantum theory of gravity because taking the limit $\hbar \to 0$ does not have any physical interpretation by virtue of Bekenstein-Hawking entropy

$$S = \frac{A}{4G\hbar}.$$

In this context, three-dimensional gravity has been a fruitful arena for quantum gravity since it is devoid of degrees of freedom and it has a topological formulation as a Chern-Simons theory [5, 6] . In particular the asymptotic conditions of three-dimensional anti-de Sitter spaces (AdS) are left invariant under the conformal group in two dimensions [7]. This means that using the AdS/CFT correspondence [8], this conformal field theory should correspond to the quantum AdS gravity in three dimensions. Here, the existance of a black hole solution, the Bañados-Teitelboim-Zanelli (BTZ) black hole [9, 10], has served as a laboratory to test quantum issues. In fact, the entropy of this black hole could be obtained from microscopical computation of the semiclassical black hole entropy by means of Cardy formula [11].

Asymptotically anti-de Sitter black holes endowed with a scalar field have been also a very active topic in three dimensions. Their asymptotic conditions even in the presence of matter are left invariant under the conformal group in two dimensions. For this reason, they have been related to superconductors by means of the gravity/gauge duality [12]. Additionally, in a different scenario, efforts towards testing the no-hair theorem from astronomical observations have been recently developed [13, 14]. The extensive literature about hairy black holes and the broad applications confirm their physical relevance. It started with the precursory hairy black holes dressed with a conformally [15] and a minimally coupled scalar field [16], and they were followed by a number of other exact three-dimensional hairy black holes [17]-[28]. As was proved first in [16], and then vastly discussed in the literature (see for instance [29]), the scalar field can exhibit a slow fall-off at infinity in such a way that it contributes to the mass of the black hole. Other interesting subject is the computation of the entropy of the three-dimensional hairy black holes from the asymptotic growth of the number of states by means of Cardy formula [19, 20, 21, 28]. On the other hand, another kind of simple matter couplings can be considered, in particular gauge fields. For example, the electrically charged BTZ black hole was introduced in [9] and its rotation version was presented in [24]. In both cases the dynamics of the gauge field was defined by the usual Maxwell Lagrangian, and consequently, the gauge field exhibited a logarithmic dependence on the radial coordinate, as expected in $2+1$ dimensions. This is the context in which the two first works of this thesis are presented. Both of them show novel hairy black hole solutions in three dimensions and study their properties from a themodynamic point of view. The first one from the usual metric formulation and the second one, in terms of gauge fields using the technics developed for a Chern-Simons theory.

As was mentioned, the simplest model that studys the interaction among matter fields and gravity is given through the coupling of scalar fields. Remarkably, such a model has shown to possess black hole solutions circumventing the no-hair conjecture. The first example of this fact is the four-dimensional solution obtained by Bocharova, Bronnikov, and Melnikov [30] in the oriental part of the world and then by Bekenstein [31], [32] in occident. The black hole has a conformally coupled scalar field (including electromagnetism) with vanishing cosmological constant. Unfortunately the scalar field diverges on the horizon and the metric corresponds to the extreme Reissner-Nordström metric. The black hole solution that will be treated in this thesis in [33] has the ingredients above mentioned; an electric field, a magnetic field and also a scalar field with a dilatonic coupling. This black hole is asymptotically AdS provided the system is endowed with a self-interacting potential. In the thermodynamical analysis of [33], the authors claimed that the first law of black hole thermodynamics is not satisfied unless one adds a term which is interpreted as a scalar charge [33], [34]. However, this argument conflicts with the fact that there is no a scalar charge associated to the solution because it has neither a symmetric origin nor a topological one. Moreover, it has been clearly identified that the scalar field contributes to the mass variation, generically, with non-integrable terms. This has been proven with general asymptotic conditions through Hamiltonian [29] and other methods [36], [37], and even with an explicit black hole example in presence of gauge fields in three dimensions [38].

Another motivation for studying the black hole solution presented in [33] has to do with the computation of its Gibbs free energy. To do so, one has to have a well-defined action and also to present other thermodynamic features, e.g. to match all the charges of the solution with their respective chemical potentials. The Gibbs free energy presented in [33] fails to do that since it cannot recover the magnetic contribution to the Euclidean action, not even if one includes the additional scalar charge. The value of the action in [33] was obtained through the holographic renormalization method described in [39], by adding counterterms, which only include radial surface terms, to get a finite action principle. As we will see in this thesis, it is necessary to add an additional term to obtain the magnetic contribution to the Euclidean action.

This thesis is organized as follows:

In the first chapter, we give the theoretical frame in which this thesis is developed. The Hamiltonian formulation of gravity will be our main tool for computing conserved charges and verifiyng the validity of the first law of black hole thermodynamics, by requiring the Hamiltonian action to attain an extremun. Later on, in the second chapter we stress the importance of the asymptotic symmetrys and give an example, three-dimensional gravity with negative cosmological constant ($\Lambda < 0$). We claim also that the presence of matter can change the asymptotic behavior of the metric if the scalar field has a sufficiently slow fall-off at infinity.

In the third chapter we consider three-dimensional gravity with negative cosmological constant in the presence of a single real scalar field and an Abelian gauge field. This composite matter source is characterized by the fact that both fields are conformally coupled to gravity. The theory admits black hole solutions wich are characterized through their thermodynamic properties. Boundary conditions are neccesary for having an integrable mass, in the sense of what is explained in the second chapter .

In chapter four, we explain how gravity in three dimensions can be seen as a

Chern-Simon theory and how we are able to perform a themodynamic analysis of the BTZ black hole exclusively in terms of gauge fields and the topology of the manifold, without making any reference to the spacetime metric. Chapter five follows the former approach but now, applied to hairy black holes. We present a procedure to describe these solutions for gravity with a conformally coupled scalar field in terms of connections and formulate the action as a Chern-Simons-like action. The Euclidean version of this action is used for analyzing the thermodynamics of the rotating hairy black hole solution in the grand canonical ensemble. Regularity conditions on the holonomy at the horizon fix the integration constants of the solution in terms of the chemical potentials. The mass and angular momentum are computed and they coincide with the global charges obtained from the Hamiltonian approach. The entropy is derived in two ways; the first one from the Gibbs free energy and the second one from a general formula for the entropy in terms of the on-shell holonomies. Both give the same result compared with the modified area law. The final part of this chapter is devoted to explain how this formulation can be also applied to gravity with a minimally coupled scalar field.

The final chapter of this thesis addresses the question about the role of certain fields in the first law of black hole thermodynamics. In particular, by using a dyonic solution in the presence of a dilatonic scalar field, we settle the role of the magnetic charge and the scalar field contribution in thermodynamics by requiring the Hamiltonian action to attain an extremun. To prove the latter, we formulate a well-defined and finite Hamiltonian action principle for the system and we prove that there is an additional term coming from a total derivative in the polar angle which appears due to existence of a magnetic monopole. We conclude that the magnetic charge, despite of having a topological root, appears in the first law of black hole thermodynamics, while the scalar field appears as a contribution to the mass.

Chapter 3

Hamiltonian formalism for General Relativity

At first sight, the utility of the Hamiltonian formalism for General relativity could be unclear for who are used to the covariance of the theory. The Einstein equations written in a covariant way

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.1}$$

are extremely beautiful since they represent the fact that the laws of physics are independent of the observer. However, because of the Bianchi identities,

$$\nabla_{\mu} G^{\mu\nu} = 0, \tag{3.2}$$

gravity is actually a constrained system. Then, there are parts of the Einstein equations which are dynamical and parts which are constraints. To elucidate this issue forward to the quantization of the theory, fundamental work was done during the fifty's by P. Dirac [40], and then by R. Arnowitt, S. Deser and C.W. Misner which resume their work of years in [41].

The dynamics of General Relativity is considered a Cauchy problem, where the evolution of a $(d-1)$ surface (where the fields are defined) is considered. Let Σ a spacelike hypersurface of a manifold $\mathcal{M}$ and $\xi^{\perp}$, ξ^{i} deformations on it. The generator of the hypersurface deformation is given by the Hamiltonian of the system $H[\xi^{\perp}, \xi^{i}]$, where

$$H[\xi^{\perp}, \xi^{i}] = \int dx^{d-1} \left(\xi^{\perp} \mathcal{H}_{\perp} + \xi^{i} \mathcal{H}_{i} \right) + Q[\xi^{\perp}, \xi^{i}]. \tag{3.3}$$

The Hamiltonian is a linear combination of the constraints $\mathcal{H}_{\perp}$. They are explicitly given by

$$\mathcal{H}_\perp = \frac{2\kappa}{\sqrt{\gamma}}\left(\pi^{ij}\pi_{ij} - \frac{1}{(d-2)}\left(\pi^i{}_i\right)^2\right) - \frac{\sqrt{\gamma}}{2\kappa}\left(R-2\Lambda\right), \tag{3.4}$$

$$\mathcal{H}_i = 2\nabla_j\pi^j{}_i. \tag{3.5}$$

The dynamical variables of the system are given by the set $\{\gamma_{ij}, \pi^{ij}\}$, where γ_{ij} is the spatial metric of hypersurface, obtained from ADM decomposition of the spacetime metric $g_{\mu\nu}$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{0i} & g_{ij} \end{pmatrix} \tag{3.6}$$

$$= \begin{pmatrix} -N_\perp^2 + N^i N_i & N_j \\ N_i & \gamma_{ij} \end{pmatrix}, \tag{3.7}$$

and

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{0i} & g^{ij} \end{pmatrix} \tag{3.8}$$

$$= \begin{pmatrix} -\frac{1}{N_\perp^2} & \frac{N^j}{N_\perp^2} \\ \frac{N^i}{N_\perp^2} & \gamma^{ij} - \frac{N^i N^j}{N_\perp^2} \end{pmatrix}. \tag{3.9}$$

We identify $N_\perp$ as the *Lapse function* and N_i as the *Shift fuction* which are related with the deformations of the spacetime. On the other hand, π^{ij} is the momentum conjugated to the three-dimensional metric γ_{ij},

$$\pi^{ij} = -\frac{\sqrt{\gamma}}{2\kappa}\left(K^{ij} - \gamma^{ij}K\right), \tag{3.10}$$

where the extrinsic curvature K^{ij} is given by

$$K_{ij} = \frac{1}{2N_\perp}\left(\nabla_i N_j + \nabla_j N_i - \dot{\gamma}_{ij}\right). \tag{3.11}$$

In (3.4), R stands for the scalar curvature of the $(d-1)$-dimensional spatial metric γ_{ij}, Λ is the cosmological constant and κ is the gravitational constant. In (3.3) Q is the surface integral at spatial infinity needed to have a well defined functional derivative, as was pointed out in the work of Regge and Teitelboim [42].

When (3.3) is varied, the total derivatives that appear are transformed into surface terms. Those surface terms must be canceled by δQ, which is the criteria for determining its value, i.e. $\delta H = 0$ on the constraint surface. The explicit expression for the surface integral in any dimension is given by

$$\begin{aligned}\delta Q &= \int d^{d-2} S_l G^{ijkl} \left(\xi^{\perp} \nabla_k \delta\gamma_{ij} - \partial_k \xi^{\perp} \delta\gamma_{ij}\right) \\ &\quad + \int d^{d-2} S_l \left[2\xi_k \delta\pi^{kl} + \left(2\xi^k \pi^{jl} - \xi^l \pi^{kj}\right) \delta\gamma_{jk}\right], \end{aligned} \tag{3.12}$$

with

$$G^{ijkl} = \frac{1}{2}\sqrt{\gamma}\left(\gamma^{ik}\gamma^{jl} + \gamma^{il}\gamma^{jk} - 2\gamma^{ij}\gamma^{kl}\right). \tag{3.13}$$

In the canonical formalism, the transformation law of the fields (γ_{ij}, π^{ij}) are obtained after performing the Poisson bracket of the field with the canonical Hamiltonian. The basic non-vanishing Poisson brackets are given by

$$\left\{\gamma_{ij}(x), \pi^{lm}(x')\right\} = \frac{1}{2}\left(\delta_i^l \delta_j^m + \delta_i^m \delta_j^l\right) \delta(x - x'), \tag{3.14}$$

Then we can compute the following Poisson brackets,

$$\begin{aligned}\delta\gamma_{ij} &= \left\{\gamma_{ij}(x), \int d^{d-1}x' \left(\xi^{\perp}\mathcal{H}_{\perp} + \xi^i \mathcal{H}_i\right)\right\} \\ &= \xi^{\perp}\frac{4\kappa}{\sqrt{\gamma}}\left(\pi_{ij} - \frac{1}{(d-2)}\gamma_{ij}\pi\right) + \nabla_i \xi_j + \nabla_j \xi_i, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\delta\pi^{ij} &= \left\{\pi^{ij}(x), \int d^{d-1}x' \left(\xi^{\perp}\mathcal{H}_{\perp} + \xi^i \mathcal{H}_i\right)\right\} \\ &= -\frac{\xi^{\perp}\sqrt{\gamma}}{2\kappa}\left(R^{ij} - \frac{1}{2}\gamma^{ij}R + \Lambda\gamma^{ij}\right) + \frac{\kappa\xi^{\perp}}{\sqrt{\gamma}}\gamma^{ij}\left(\pi^{kl}\pi_{kl} - \frac{1}{(d-2)}\pi^2\right) \\ &\quad -\frac{4\xi^{\perp}\kappa}{\sqrt{\gamma}}\left({\pi^i}_l \pi^{jl} - \frac{1}{(d-2)}\pi\pi^{ij}\right) + \frac{\sqrt{\gamma}}{2\kappa}\left(\left(\nabla^{(i}\nabla^{j)}\xi^{\perp}\right) - \gamma^{ij}\left(\nabla^k \nabla_k \xi^{\perp}\right)\right) \\ &\quad + \left(\nabla_k \xi^k\right)\pi^{ij} - \left(\nabla_k \xi^i\right)\pi^{kj} - \left(\nabla_k \xi^j\right)\pi^{ki} + \xi^k\left(\nabla_k \pi^{ij}\right). \end{aligned} \tag{3.16}$$

The Hamiltonian evolution coincides with the Lie derivative of the fields when we are on-shell, i.e the equations of motion are satisfied. For a further discussion of the relation, see [43].

Note that δQ in (3.12) stands for the surface term after taking the functional derivatives with respect to the canonical variables in the phase space of the Hamiltonian generator H. This surface term determines the conserved charges in the

Regge-Teitelboim approach [42]. A priori, for a generic configuration δQ is a non-integrable quantity and one must then also provide the asymptotic behaviour of the fields representing the space of solutions at infinity. In some cases the latter is not enough for integrating δQ and some additional integrability conditions must be imposed on the phase space (for all practical purposes, on the integration constants of the solution). We shall also need the variation of the canonical variables under surface deformations, which are given by the Poisson brackets of the phase space variables and the Hamiltonian generators. The importance of the asymptotic conditions, preserved under asymptotic symmetries, will be detailed in the next Chapter.

Chapter 4

Asymptotic structure of three-dimensional gravity with negative cosmological constant

We define the asymptotic symmetries of a system by all the surface deformations $\left(\xi^{\perp}, \xi^{i}\right)$ that preserve the asymptotic conditions making the corresponding boundary terms (3.12)–that we identify as the global charges of the system– integrable and finite. By asymptotic conditions, we refer to the behavior of the dynamic fields at infinity (an example will be see further in detail). The asymptotic symmetries of gravity are in general non trivial because they can be given by a greater group than the group of isometries of the space-time. For example, for flat spaces in three and four dimensions the group of asymptoyic symmetries is the infinite dimensional BMS group (Bondi-van der Burg-Metzner-Sachs) [44, 45]. On the contrary, in four dimensional gravity with a negative cosmological constant, the isometry group is the same as the group of the asymptotic symetries, the $SO(3,2)$ group [46].

In this context, there is a crucial example that helped to promote the AdS/CFT conjecture. In the work of Brown and Henneaux [47], it was claimed that the asymptotic symmetries of gravity with negative cosmological constant in three dimensions is the conformal group in two dimensions with the presence of a central charge. The boundary conditions considered for obtaining that result are the following

$$f_{tt} = \mathcal{O}\left(1\right), \quad f_{tr} = \mathcal{O}\left(\tfrac{1}{r^3}\right), \quad f_{t\theta} = \mathcal{O}\left(1\right), \tag{4.1}$$

$$f_{rr} = \mathcal{O}\left(\tfrac{1}{r^4}\right), \quad f_{r\theta} = \mathcal{O}\left(\tfrac{1}{r^3}\right), \quad f_{\theta\theta} = \mathcal{O}\left(1\right), \tag{4.2}$$

where $f_{\mu\nu}$ corresponds to the deviation with respect to *background* $\bar{g}_{\mu\nu}$, i.e. $g_{\mu\nu} = \bar{g}_{\mu\nu} + f_{\mu\nu}$, which is the AdS spacetime in three dimensiones

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^{\mu}dx^{\nu} = -\left(\frac{r^2}{l^2}+1\right)dt^2 + \left(\frac{r^2}{l^2}+1\right)^{-1}dr^2 + r^2 d\theta^2, \tag{4.3}$$

and l is the AdS radius. The group of transformations that left the asymptotic conditions (4.1) y (4.2) invariant, is generated by the vectorial field,

$$\begin{aligned}
\xi^{+} &= Y^{+} + \frac{l^2}{2r^2}\partial_{-}^2 Y^{-} + \ldots, \\
\xi^{-} &= Y^{-} + \frac{l^2}{2r^2}\partial_{+}^2 Y^{+} + \ldots, \\
\xi^{r} &= -\frac{r}{2}\left(\partial_{+} Y^{+} + \partial_{-} Y^{-}\right) + \ldots,
\end{aligned} \tag{4.4}$$

where it has been used the change of variables $x^{\pm} = t/l \pm \theta$ and $Y^{\pm}$ are arbitrary functions of $x^{\pm}$.

The conditions (4.1) and (4.2) can be satisfied even when localized matter fields are present. More precisely, this happens when their fall-off is sufficiently fast at infinity, so as to give no contributions to the surface integrals defining the generators of the asymptotic symmetries.

When 2+1 gravity minimally coupled to a self-interacting scalar field is consider,

$$I[g,\phi] = \int dx^3 \sqrt{-g}\left[\frac{R}{2\kappa} - \frac{1}{2}\left(\nabla\phi\right)^2 - V\left(\phi\right)\right], \tag{4.5}$$

there are solutions of the theory that decay at infinity as $\phi \sim r^{1/2}$. When this occurs, the generators of the asymptotic symmetries acquire a contribution from the scalar field, but the algebra of the canonical generators possesses the standard central extension [16]. The asymptotic fall-off of these solutions read as

$$\phi = \frac{\chi\left(t,\theta\right)}{r^{1/2}} - \frac{2}{3}\frac{\chi\left(t,\theta\right)^3}{r^{3/2}} + \mathcal{O}\left(r^{-5/2}\right), \tag{4.6}$$

and the asymptotic decay of the $g_{r\mu}$ must go slower as compared with (4.1) and (4.2),

$$\begin{aligned}
g_{rr} &= \frac{l^2}{r^2} - \frac{4l^2\chi\left(t,\theta\right)^2}{r^3} + \mathcal{O}\left(r^{-4}\right), \\
g_{r\theta} &= \mathcal{O}\left(r^{-2}\right), \\
g_{rt} &= \mathcal{O}\left(r^{-2}\right).
\end{aligned} \tag{4.7}$$

In this way, when the contributions of the scalar field and the asymptotic decayment of the metrics are considered in the definition of the charges, using the Regge-Teitelboim approach, we get that

$$\delta Q = \delta Q^{G} + \delta Q^{\phi}, \tag{4.8}$$

where δQ^G is given in (3.12) and

$$\delta Q^{\phi} = -\int dS_i \left(\xi^{\perp} \sqrt{\gamma} \partial^i \phi \delta \phi + \xi^i \pi_{\phi} \delta \phi \right). \tag{4.9}$$

In particular, when the decayment of the fields are considered as (4.6) and(4.7), it is possible to see that there are a divergent pieces appearing in the gravitational part and also in the scalar field part. Remarkably, they cancel each other when g_{rr} is considered dependent of χ in the way it was. In this context, the asymptotic expansion for the scalar field can be considered more general, i.e. where the leading an sub-leading terms of the expansion are independet, such that

$$\phi = \frac{\chi(t,\theta)}{r^{1/2}} - \frac{\beta(t,\theta)}{r^{3/2}} + \mathcal{O}\left(r^{-5/2}\right), \tag{4.10}$$

however, invariance of the asymptotic conditions under the Virasoro symmetry implies

$$\beta = \alpha \chi^3. \tag{4.11}$$

The only example of this generic behavoir (4.10) for an exact solution was obtained in a work that is part of this thesis in Chapter 5.

There are other interesting cases, for example, when the mass saturates the Breitenlohner-Freedman bound $m_{BF} = -(d-1)^2/4l^2$. This bound for the mass of the scalar feld is a lower bound for having an stable system in the anti-de Sitter space [49]. Here, the asymptotic behavior of the metric has a slower fall–off than that of pure gravity with a localized distribution of matter due to the backreaction of the scalar field, which has a logarithmic branch decreasing as $r^{-(d-1)/2} ln\,(r)$ at spatial infinity. This behavior does not affect the asymptotic symmetry group as compared with pure gravity with negative cosmological constant. There is also a general analysis for $d \geq 4$ [29]. In there, it is noticed that when the mass of the scalar field belongs to the range $m^2_{BF} \leq m^2 \leq m^2_{BF} + l^{-2}$, it has a slow fall-off at infinity which back reacts on the metric. This fact modifies its standard asymptotic behavior and forces to impose some conditions for having well defined Hamiltonian generators, for all elements of the anti-de Sitter algebra. A physical condition is to left the scalar field invariant under asymptotic AdS symetry ξ,

$$\phi \rightarrow \phi + \mathcal{L}_{\xi} \phi. \tag{4.12}$$

As a result, the above implies to impose a functional relationship on the coefficients of the leading and subleading term of the scalar field fall-off, as it will be seen in the example presented in Chapter 7.

Chapter 5

Three-dimensional black holes with conformally coupled scalar and gauge fields

In this Chapter we consider three-dimensional gravity with negative cosmological constant in the presence of a single real scalar field and an Abelian gauge field. This composite matter source is characterized by the fact that both fields are conformally coupled to gravity, in contrast with some recently proposed models [25, 26]. The action for the scalar field contains, in addition to the kinetic term, an interacting term with the curvature and a sixth-power self interaction potential. With these ingredients, this non-minimal action for the scalar field becomes conformal invariant. On the other hand, it is well known that the Maxwell action is invariant under conformal transformations of the metric only in four dimensions. This symmetry is recovered in any spacetime dimension n if the Maxwell Lagrangian is raised to the $(n/4)^{\text{th}}$ power [27]. Therefore, a Lagrangian of this form describes the Abelian gauge field considered in this work. Remarkably, this conformal invariant action for the gauge field may provide a Coulomb-like electric field in arbitrary dimensions. We introduce the action and the corresponding field equations, which are solved using a circularly symmetric ansatz and the black hole solutions are identified. Since the solutions are given by simple expressions, the search for black holes is greatly simplified. The geometries asymptotically approach anti-de Sitter spacetime, and the scalar fields are regular on and outside of the corresponding horizons. The mass and electric charge of the black holes are determined using the Regge-Teitelboim method [42]. Boundary conditions over the leading and sub-leading terms in the asymptotic form of the scalar field are required for obtaining the mass. Since the scalar field is defined for two independent integration constants, a wide class of boundary conditions are allowed, even those that spoil the asymptotic AdS invariance. It is also performed the thermodynamic analysis of the solution. The temperature, electric potential and entropy are determined. The entropy is not automatically a positive definite quantity in this non-minimal frame and additional conditions must be imposed on the integration and self-interacting coupling constants in order to ensure a positive entropy.

5.1 Black hole solutions

We consider three-dimensional gravity with negative cosmological constant in presence of a scalar and an electromagnetic field, being both fields conformally coupled to gravity. The action is given by

$$I[g_{\mu\nu},\phi,A_\mu]=\int d^3x\sqrt{-g}\left[\frac{R+2l^{-2}}{2\kappa}-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{16}R\phi^2-\lambda\phi^6\right. \tag{5.1}$$

$$\left.+\sigma\left(-F^{\mu\nu}F_{\mu\nu}\right)^{3/4}\right], \tag{5.2}$$

where κ is the gravitational constant and l is the AdS radius. Moreover, λ and σ are the coupling constants of the self-interaction potential and the nonlinear electromagnetic term, respectively.

The equations of motion are

$$E_{\mu\nu}\equiv G_{\mu\nu}+\Lambda g_{\mu\nu}-\kappa\left(T^{\phi}_{\mu\nu}+T^{A}_{\mu\nu}\right) = 0, \tag{5.3a}$$

$$\Box\phi-\frac{1}{8}R\phi-6\lambda\phi^5 = 0, \tag{5.3b}$$

$$\partial_\mu\left(\sqrt{-g}\mathcal{F}^{-1/4}F^{\mu\nu}\right) = 0, \tag{5.3c}$$

with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $\mathcal{F}=-F^{\mu\nu}F_{\mu\nu}$.

The energy-momentum tensor of the scalar field is given by

$$T^{\phi}_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}\partial^\lambda\phi\partial_\lambda\phi-\lambda g_{\mu\nu}\phi^6+\frac{1}{8}\left[g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu+G_{\mu\nu}\right]\phi^2, \tag{5.4}$$

and

$$T^{A}_{\mu\nu}=\sigma\left(3F_{\lambda\mu}F^{\lambda}{}_{\nu}\mathcal{F}^{-1/4}+g_{\mu\nu}\mathcal{F}^{3/4}\right) \tag{5.5}$$

is the corresponding one for the nonlinear electromagnetic field.

It is worth noticing that the negative sign inside the nonlinear electromagnetic term in the action (5.1) ensures that purely electric configurations remain real. Furthermore, the coupling constant σ is chosen to be positive[1] in order to keep

[1]Without lost of generality, σ is chosen to be $2^{1/4}$ just for simplifying numerical factors.

the energy density of the electromagnetic field –the $T^{A}_{\hat{0}\hat{0}}$ component of the energy-momentum tensor in the orthonormal frame– positive for this class of configurations.

Since the fields are conformally coupled, their corresponding stress tensors are traceless on-shell, so that Einstein's equations (5.3a) imply

$$R = -6l^{-2}. \tag{5.6}$$

We will deal with asymptotically AdS spacetimes. In this context, potentials unbounded from below, for instance the case for $\lambda < 0$ in the action (5.1), do not generate the sort of instabilities as in asymptotically flat spacetimes, provided the mass of the scalar field is bounded from below by the Breitenlohner-Freedman one m^2_{BF} [50, 51, 52]. In three dimensions, $m^2_{BF} = -l^{-2}$, and because of (5.6), in our case the mass of the scalar field is $3/4l^{-2}$, which satisfies the mentioned bound.

We look for static and circularly symmetric configurations described by the line element

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\theta^2, \tag{5.7}$$

a scalar field depending just on the radial coordinate, and a gauge field of the form $A = A_t(r)dt$, which generates a purely radial electric field. The coordinates range as $-\infty < t < \infty, 0 \leq r < \infty, 0 \leq \theta < 2\pi$.

From the subtraction $E^t_t - E^r_r$ in (5.3a), a second-order differential equation for the scalar field is obtained, whose integration yields

$$\phi(r) = \sqrt{\frac{b}{r+c}}, \tag{5.8}$$

where b and c are integration constants. Moreover, from the nonlinear Maxwell equation (5.3c) the gauge field is easily obtained (modulo gauge transformations)

$$A = -\frac{q}{r}dt. \tag{5.9}$$

The constant q is related with the electric charge as we will show below in Subsec. 5.2. Finally, the metric function $F(r)$ can be directly obtained from equation (5.6), which gives

$$F(r) = \frac{r^2}{l^2} + a_1 + \frac{a_2}{r}, \tag{5.10}$$

where a_1 and a_2 are integration constants. It is clear, from the line element (5.7) and the radial dependence of F shown in (5.10), that these solutions are asymptotically anti-de Sitter spacetimes whose asymptotic behaviors match the well-known Brown-Henneaux conditions [47]. However, as is discussed in Sec. 5.2, boundary conditions on the matter fields could spoil the conformal invariance of the full configuration.

The case of vanishing scalar field ($b = 0$) was studied in [53], and we will not consider it here. The remaining equations of motion give relations among the integration constants a_1, a_2, b, c and q. Since we are interested in the case of nonvanishing scalar field we assume $b \neq 0$ in these equations, which give rise to two different branches: i) $c = 0$ and ii) $c \neq 0$. Furthermore, it is convenient to address the case without self-interaction potential ($\lambda = 0$) in a separate section.

5.1.1 Case $c = 0$: Black hole dressed with a stealth composite matter source

The solution is determined by the metric function

$$F(r) = \frac{r^2}{l^2} + 24\lambda b^2, \tag{5.11}$$

the scalar field

$$\phi(r) = \sqrt{\frac{b}{r}}, \tag{5.12}$$

and the gauge field given by (5.9) with

$$|q|^{3/2} = -\lambda b^3. \tag{5.13}$$

The scalar field is real provided $b > 0$. Moreover, in order to ensure a real q it is necessary to fix the coupling constant $\lambda \leq 0$ as one can see from the r.h.s. of (5.13). In this case, the spacetime corresponds to a black hole, whose horizon is located at $r_+^2 = -24\lambda l^2 b^2$. It should be noticed that this black hole has the same metric as the static and uncharged BTZ black hole. However, it possesses a nonvanishing electric charge and is dressed with a conformal scalar field. This occurs because the total energy-momentum tensor vanishes, i.e., the scalar field and gauge field contributions cancel out. Therefore, the above solution can be considered as a stealth configuration [54, 55, 56, 57, 58, 59] produced by two different matter sources. The metric is free of singularities and the matter fields diverge at the origin, $r = 0$.

5.1.2 Black holes in the general case $c, \lambda \neq 0$

First, it is convenient to redefine the coupling constant as $\lambda = \kappa^2\alpha/(512 l^2)$, where now α plays the role of the coupling constant associated to the self-interaction potential. Additionally, we also define $b = 8ac/\kappa$, where a is an integration constant.

In this way, the solution with a nonvanishing scalar field is given by the metric function

$$F(r) = \frac{r^2}{l^2} - \frac{(1-\alpha a^2)}{l^2}\left(\frac{2c^3}{r} + 3c^2\right), \tag{5.14}$$

the scalar field

$$\phi(r) = \sqrt{\frac{8}{\kappa}}\sqrt{\frac{ac}{r+c}}, \tag{5.15}$$

and the gauge field shown in (5.9), with an integration constant q satisfying

$$|q|^{3/2} = -\frac{c^3(1-\alpha a^2)(1-a)}{\kappa l^2}. \tag{5.16}$$

This expression indicates that the gauge field vanishes for two particular values of a, which allow to rediscover previous *uncharged* solutions. The case $a = 1$ corresponds to the scalar hairy black hole found in [15, 16], and the case $a = 1/\sqrt{\alpha}$ is the massless hairy solution reported in [18]. Hereafter, we focus our attention in new black hole configurations with $q \neq 0$.

The horizons are located at the positive roots of the cubic equation $F(r) = 0$. By replacing $r = cx$, this problem is reduced to solve

$$x^2 - (1-\alpha a^2)\left(\frac{2}{x}+3\right) = 0. \tag{5.17}$$

In the case $c > 0$, we are interested in the positive roots of (5.17), and for $c < 0$ the relevant roots correspond to the negative ones. Since we are dealing with a cubic equation, it is possible to write down their exact roots x_i in the following form

$$x_i = z_i^+ z_i^- \left(z_i^+ + z_i^-\right), \qquad i = 1, 2, 3 \tag{5.18}$$

with

$$z_i^+ = \gamma_i(1+\sqrt{\alpha a^2})^{1/3} \qquad \text{and} \qquad z_i^- = \bar{\gamma}_i(1-\sqrt{\alpha a^2})^{1/3}. \tag{5.19}$$

Here γ_i represent the roots of unity $\gamma_i^3 = 1$, and $\bar{\gamma}_i$ are their complex conjugates . These are

$$\gamma_1 = 1, \qquad \gamma_2 = -\left(\frac{1+i\sqrt{3}}{2}\right), \qquad \gamma_3 = -\left(\frac{1-i\sqrt{3}}{2}\right). \tag{5.20}$$

For $\alpha \leq 0$, $z_i^- = \bar{z}_i^+$ so that all the roots (5.18) are real. In the opposite case $\alpha > 0$, we note that x_1 is always a real root of (5.17) and x_2, x_3 are complex.

The qualitative behavior of the roots is illustrated in figures (5.1) and (5.2). The real roots correspond to the intersection of a parabola with a hyperbola as is shown in (5.17). This can be described as follow:

- If $1 - \alpha a^2 > 0$, the root x_1 is positive and the nature of x_2 and x_3 depend on the sign of α. If $\alpha < 0$, x_2 and x_3 are negative. On contrary, if $\alpha > 0$, x_2 and x_3 are complex numbers (see Fig. (5.1)).

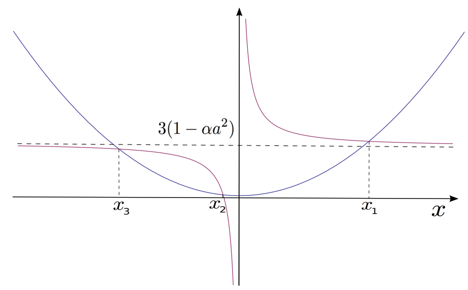

Figure 5.1: **Case** $(1-\alpha a^2)>0$: The roots x_1, x_2, x_3 are shown for the case $(1-\alpha a^2)>0$. The root x_1 is positive and the roots x_2 and x_3 depend on the sign of α. If $\alpha<0$, x_2 and x_3 are negative. Alternatively, if $\alpha>0$, x_2 and x_3 are both complex numbers, since the hyperbola does not intersect the parabola for $x<0$.

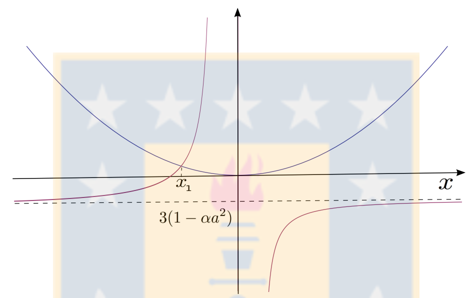

Figure 5.2: **Case** $(1-\alpha a^2)<0$: The roots x_1, x_2, x_3 are shown for the case $(1-\alpha a^2)<0$. The root x_1 is negative and x_2 and x_3 are both complex roots.

- If $1-\alpha a^2<0$, the root x_1 is negative and x_2 and x_3 are complex roots (see Fig. (5.2)).

After capturing the general properties of the roots of (5.17), we are in position to analyze the existence of black hole solutions. The analysis requires to study both signs of the integration constant c as is shown below. Note that, for thermodynamic considerations explained in Sec. 5.3, the presence of a horizon is not enough to ensure physically sensible black hole solutions.

5.1.2.1 Event horizon for $c>0$

The previous analysis indicates that only for $1-\alpha a^2>0$ there is a positive root, x_1. Moreover, the condition $a>1$ appears by demanding positivity of the r.h.s of (5.16). The intersection of these two inequalities, $1-\alpha a^2>0$ and $a>1$, implies that: **(A)** there is no restriction for any $\alpha<0$, or **(B)** for a positive self-interacting coupling parameter α, it is required to be bounded from above such that $0<\alpha\leq 1$, in conjunction with a bounded integration constant $1 < a < 1/\sqrt{\alpha}$.

Under the conditions **(A)** or **(B)** there exists an event horizon located at $r_+ = x_1 c$. Additionally, from the analytic expression of x_1 it is possible to determine bounds for the event horizon according to the sign of the self-coupling parameter. Under the conditions **(A)** we have $r_+ > 2c$, while **(B)** provides the bounds $0 < r_+ < 2c$.

Since r and c are positive, the scalar field is regular everywhere. The gauge field and metric are singular at the origin $r = 0$, as one can read from (5.9) and from the Kretschmann invariant, $12l^{-4}(1 + 2c^6(1 - \alpha a^2)^2 r^{-6})$, respectively.

5.1.2.2 Event horizon for $c < 0$

We are now interested in the negative roots of (5.17). First, the root $x_1 < 0$ can be discarded since it requires the condition $1 - \alpha a^2 < 0$ and also, from (5.16), $a > 1$. This last requirement is incompatible with the necessary condition $a < 0$ to ensure a real scalar field (5.15). Therefore, x_1 does not produce an event horizon. Second, it is possible to consider the roots x_2 and x_3, which become negative real numbers provided $1 - \alpha a^2 > 0$ and $\alpha < 0$ (conditions labeled by **(C)**). From the definitions of x_2 and x_3 one can to extract the following properties: $2/3 < |x_2| < 1$ and $|x_3| > 1$. Then, since $|x_3| > |x_2|$ the event horizon is located at $r_+ = x_3 c$, provided conditions **(C)** are satisfied. The root x_2 gives rise an inner horizon. Since we are considering α and $a \neq 0$, the root x_2 cannot equal x_3, then an extreme black hole does not occur. Due to the inequality $r_+ > -c$, the singularity of the scalar field at $r = -c$ is hidden by the event horizon r_+. As in the previous case, the metric and gauge field are singular only at the origin.

5.1.3 Black hole in absence of self-interaction potential ($\lambda = 0$)

A particularly simple solution is obtained in absence of self-interaction potential. The metric function $F(r)$ reduces to

$$F(r) = \frac{(r + c)^2(r - 2c)}{rl^2}, \tag{5.21}$$

and the gauge and scalar fields are given by (5.9) and (5.15), respectively. Now, the constant q satisfies

$$|q|^{3/2} = \frac{c^3(a - 1)}{\kappa l^2}. \tag{5.22}$$

Although it is possible to consider $c < 0$, the double zero of $F(r)$ at $r = -c$ is not suitable to be promoted to event horizon because the scalar field (5.15) is singular there. We adopt a conservative point of view saying that the singularity of the scalar field prevents the existence of an extreme black hole. Thus, we consider only the simple root $r = 2c$, which becomes an event horizon for $c > 0$. The condition $a > 1$ arises from (5.22). As in the previous case with $c > 0$, the gauge and metric fields are singular at the origin, and the scalar field is regular everywhere.

5.2 Mass and electric charge

The aim of this section is to determine the conserved charges of the black holes introduced above. For this goal we consider the hamiltonian Regge-Teitelboim method [42]. In this approach the charges $Q[\xi, \xi^A]$ are the surface terms added to the Hamiltonian generator in order to ensure well-defined functional derivatives. The bulk piece of the canonical generator

$$H[\xi, \xi^A] = \int dx^2 \left(\xi^{\perp}\mathcal{H}_{\perp} + \xi^i \mathcal{H}_i + \xi^A \mathcal{G}\right) + Q[\xi, \xi^A], \tag{5.23}$$

is a linear combination of the constrains $\mathcal{H}_{\perp}$ and $\mathcal{H}_i$, where i denotes the two spatial dimensions, and $\mathcal{G}$ is the Gauss constraint associated to the Abelian gauge field. The charge corresponds to the canonical generator for vanishing constraints. The vector $\xi = (\xi^{\perp}, \xi^i)$ represents the asymptotic symmetries of the spacetime, and ξ^A is the parameter associated to the Abelian gauge symmetry.

For the class of solutions we are dealing with, it is sufficient to consider a minisuperspace of circularly symmetric configurations defined by the line element

$$ds^2 = -\left(N^{\perp}(r)\right)^2 dt^2 + F(r)^{-1} dr^2 + r^2 d\theta^2, \tag{5.24}$$

a scalar field $\phi(r)$ and a gauge field $A = A_t(r)dt$. In this case, the only nontrivial canonical momentum is that corresponding to the gauge field $\mathcal{E}(r)$, which is given by

$$\mathcal{E}(r) = 3r \left(\frac{F(r)}{\left(N^{\perp}(r)\right)^2}\right)^{1/4} |F_{tr}|^{1/2} \operatorname{sign}(F_{tr}). \tag{5.25}$$

Since all the canonical momenta associated to the gravitational field and the scalar field vanish, the constraint $\mathcal{H}_i$ is identically zero, $\mathcal{H}_{\perp}$ takes the form

$$\mathcal{H}_{\perp} = \frac{1}{\sqrt{F}} \left(\frac{F'}{2\kappa}\left(1 - \frac{\kappa\phi^2}{8}\right) + \frac{rF}{4}(\phi'^2 - \phi\phi'') - \frac{\phi\phi'}{8}(rF' + 2F) - \frac{r}{l^2\kappa}\right. \tag{5.26}$$

$$\left. + r\lambda\phi^6 + \frac{\mathcal{E}^3}{27r^2}\right), \tag{5.27}$$

and the Gauss constraint reduces to $\mathcal{G} = -\partial_r \mathcal{E}$.

The variation of surface term δQ is chosen so that $\delta H = 0$ on the vanishing constraints. In this case, the boundary is a circle S^1 of infinite radius. Integrating over the angular coordinate, we obtain

$$\begin{aligned}\delta Q(\xi^{\perp}, \xi^{A}) &= \left[\frac{\pi\xi^{\perp}\left(-8+\kappa\phi\left(\phi+2r\phi'\right)\right)}{8\kappa\sqrt{F}}\delta F - \frac{1}{2}\pi r\sqrt{F}\left(\phi\left(\xi^{\perp}\right)' + 3\xi^{\perp}\phi'\right)\delta\phi\right. \\ &+ \left.\frac{1}{2}\pi r\sqrt{F}\xi^{\perp}\phi\delta\phi' + 2\pi\xi^{A}\delta\mathcal{E}\right]_{r\to\infty}. \end{aligned} \tag{5.28}$$

The integration of δQ requires to choose suitable asymptotic conditions for all fields. These conditions should allow for the asymptotic behavior of the exact solutions found in the previous section. These conditions, specified up to the order that contributes to the charge, are given by

$$F(r) = \frac{r^2}{l^2} + F_0 + \mathcal{O}\left(\frac{1}{r}\right), \tag{5.29}$$

$$\phi(r) = \frac{\phi_0}{r^{1/2}} + \frac{\phi_1}{r^{3/2}} + \mathcal{O}\left(\frac{1}{r^{5/2}}\right), \tag{5.30}$$

$$\mathcal{E}(r) = \mathcal{E}_0 + \mathcal{O}\left(\frac{1}{r}\right), \tag{5.31}$$

$$\xi^{\perp}(r) = \frac{r}{l}\xi_0 + \mathcal{O}\left(\frac{1}{r}\right), \tag{5.32}$$

$$\xi^{A}(r) = \xi_0^{A} + \mathcal{O}\left(\frac{1}{r}\right), \tag{5.33}$$

where the quantities labeled with subscripts 0 and 1 are arbitrary constants. Under these asymptotic conditions the variation ((5.28)) reduces to

$$\delta Q = \xi_0\left(-\frac{\pi\delta F_0}{\kappa} + \frac{\pi}{2l^2}\left(3\phi_1\delta\phi_0 - \phi_0\delta\phi_1\right)\right) + 2\pi\xi_0^{A}\delta\mathcal{E}_0. \tag{5.34}$$

The mass M is the conserved charge associated to time translation symmetry, parametrized here by ξ_0, and the electric charge Q_{e} is that coming from the U(1) gauge invariance, represented by the gauge parameter ξ_0^{A}. From (5.34) we can read directly

$$\delta M = -\frac{\pi\delta F_0}{\kappa} + \frac{\pi}{2l^2}\left(3\phi_1\delta\phi_0 - \phi_0\delta\phi_1\right), \tag{5.35}$$

$$\delta Q_{\mathrm{e}} = -2\pi\delta\mathcal{E}_0. \tag{5.36}$$

The minus sign in (5.36) comes from the sign difference between the electric field density and the canonical momentum of the gauge field. The electric charge can be immediately integrated, and is given by the leading term of the canonical momentum of the gauge field:

$$Q_{\rm e} = -2\pi\mathcal{E}_0. \tag{5.37}$$

It is clear that the second term in (5.35), which takes into account the contribution of the scalar field to the mass, provided $\phi_0 \neq 0$ and $\phi_1 \neq 0$, needs a boundary condition for integrating it, i. e., a functional relation $\phi_1 = \phi_1(\phi_0)$. In simple words, the mass is determined after imposing boundary conditions, and is given by[2]

$$M = -\frac{\pi F_0}{\kappa} + \frac{\pi}{2l^2}\int\left(3\phi_1 - \phi_0\frac{d\phi_1}{d\phi_0}\right)d\phi_0. \tag{5.38}$$

Apart from the boundary conditions $\phi_0 = 0$ or $\phi_1 = 0$, there is only one additional case which also leads a vanishing contribution from scalar field to the mass: the functional relation

$$\phi_1 = \gamma\phi_0^3, \tag{5.39}$$

where γ is a constant without variation. These three boundary conditions share a same feature: they do not spoil the conformal invariance of a scalar field approaching to infinity in the form (5.30), as pointed out in [16] (for a recent related discussion in four dimensions see [60]). Any other functional relation $\phi_1 = \phi_1(\phi_0)$, in the way of Designer Gravity [61], breaks the conformal invariance of the scalar field and consequently, the asymptotic AdS symmetry of the whole configuration.

We can now compute the mass and electric charge for the black holes found in Section 7.11. The first case is the black hole with stealth matter described in section 5.1.1. In this case, $F_0 = 24\lambda b^2, \phi_0 = \sqrt{b}, \phi_1 = 0$ and $\mathcal{E}_0 = 3\lambda^{1/3}b\,\mathrm{sign}(q)$. Then, evaluating (5.38) and (5.37), the corresponding mass and electric charge are

$$M = -\frac{24\pi\lambda b^2}{\kappa}, \quad \text{and} \quad Q_{\rm e} = 6\pi(-\lambda)^{1/3}b\,\mathrm{sign}(q), \tag{5.40}$$

respectively.

For the black holes found in section 5.1.2, $\phi_0, \phi_1 \neq 0$ and a boundary condition is required in order to determine the mass. For instance, the boundary condition

$$\phi_1 = \gamma\phi_0^n, \tag{5.41}$$

where $\gamma, n \neq -1$ are constants without variation, yields a mass

$$M = -\frac{\pi F_0}{\kappa} + \frac{\pi\gamma(3-n)}{2l^2(n+1)}\phi_0^{n+1}. \tag{5.42}$$

[2]Two arbitrary additive constants (but fixed, i. e. without variation) appear in the integration of (5.35) and (5.36). They will be set to zero in order that the massless BTZ has a vanishing mass, and in absence of the gauge field, the solution be electrically uncharge.

Then, using the asymptotic values for this class of black holes,

$$F_0 = -\frac{3c^2(1-\alpha a^2)}{l^2}, \qquad \phi_0 = \sqrt{\frac{8ac}{\kappa}}, \qquad \phi_1 = -\sqrt{\frac{2ac^3}{\kappa}}, \tag{5.43}$$

the mass and the electric charge can be written as

$$M = \frac{3\pi c^2(1-\alpha a^2)}{\kappa l^2} + \frac{\pi\gamma(3-n)}{2l^2(n+1)}\left(\frac{8ac}{\kappa}\right)^{\frac{n+1}{2}}, \tag{5.44}$$

$$Q_{\mathrm{e}} = 6\pi|q|^{1/2}\,\mathrm{sign}(q), \tag{5.45}$$

where q is given in Eq. (5.16). Note that the boundary condition (5.41) fixes a relation between the integration constants a and c. Finally, the limit $\alpha \to 0$ in (5.44) and (5.45) gives the mass and electric charge of the black hole without self-interaction potential described in Section 5.1.3.

5.3 Thermodynamics

This section is devoted to study thermodynamic properties of the charged hairy black holes shown in Subsec. 7.11. The conjugate variables associated to the conserved charges, mass and electric charge, are the temperature and the electric potential, respectively. The temperature can be obtained by means of the surface gravity κ_{H}

$$T = \frac{\kappa_{\mathrm{H}}}{2\pi}, \tag{5.46}$$

which is given by $\kappa_{\mathrm{H}}^2 = -1/2\nabla^\mu\chi_\nu\nabla_\mu\chi^\nu$ with $\chi^\mu = (1,0,0)$. Additionally, the electric potential is defined as

$$\Phi := A_0(r_+) - A_0(\infty) = -\frac{q}{r_+}. \tag{5.47}$$

The entropy can be found using the modified Bekenstein-Hawking area formula,

$$S = \Omega(r_+)\frac{4\pi^2 r_+}{\kappa}, \tag{5.48}$$

where the factor $\Omega(r_+) = 1 - \kappa\phi(r_+)^2/8$ comes from the nonminimally coupling term in the action [62, 63]. Since this factor is not positive definite, the entropy could become negative. In order to avoid such a non-well-behaved thermodynamic situation, solutions in which $\Omega(r_+)$ is negative must be discarded as black holes. For this reason, it is necessary to impose additional constrains on the integration constants and α as discussed in detail below.

We start examining the validity of the first law for the black holes introduced in Subsec. 7.11. Using the variation of the global charges (5.35) and (5.36), and the expressions for the temperature (5.46), entropy (5.48) and electric potential (5.47), evaluated on each particular black hole, it is possible to prove that the first law of black hole thermodynamics

$$\delta M = T\delta S + \Phi\delta Q \tag{5.49}$$

holds in all the cases. It can be seen as follow. In the general case $c \neq 0$ the expressions for each member of the above equation are given by

$$\delta M = -\frac{2\pi c^2}{l^2\kappa}(1+3\alpha a)\,\delta a + \frac{6\pi c}{l^2\kappa}\left(1-\alpha a^2\right)\delta c \tag{5.50}$$

$$\begin{aligned}\delta Q_e = &-\frac{2^{5/6}\pi \text{sign}(q)\sigma^{2/3} c\,(3\alpha a^2 - 2\alpha a - 1)\,\delta a}{\kappa l^2 (1-a)^{2/3}\left(1-\alpha a^2\right)^{2/3}} \\ &-\frac{3\ 2^{5/6}\pi \text{sign}(q)\sigma^{2/3}\left(1-a\right)^{1/3}\left(1-\alpha a^2\right)^{1/3}\delta c}{\kappa l^2}\end{aligned} \tag{5.51}$$

$$\delta S = -\frac{4\pi^2 x\delta a}{\kappa(1+x)} - \frac{4\pi^2(-1+a-x)x\delta c}{\kappa(1+x)} + \frac{4\pi^2 c(-a+(1+x)^2)\delta x}{\kappa(1+x)^2}. \tag{5.52}$$

After applying repeatedly the following identities which comes from $F(r_+) = 0$

$$x^3 = (1-\alpha a^2)(2+3x), \qquad \delta x = -\frac{2\alpha a(2+3x)\delta a}{3(-1+\alpha a^2+x^2)}, \tag{5.53}$$

it is possible to show that the first law is satisfied. Note that this property holds regardless a relation between a and c, i. e., the first law is satisfied for any boundary condition. For $c = 0$ the check is easier, we have

$$\delta M = -\frac{48\pi\lambda b\delta b}{\kappa}, \tag{5.54}$$

$$\delta Q_e = 6\pi(-\lambda)^{1/3}\text{sign}(q)\delta b, \tag{5.55}$$

$$\delta S = \frac{8l\sqrt{6|\lambda|}\delta b}{\kappa}. \tag{5.56}$$

Now, we analyze the thermodynamic behavior of the black hole solutions according to the different values of c. In the case $c = 0$, discussed in Subsec. 5.1.1, there exists an event horizon only if the coupling constant λ is negative. Then, the temperature, electric potential and entropy are

$$T = \frac{r_+}{2\pi l^2}, \qquad \Phi = -\frac{\text{sign}(q) r_+}{24(-\lambda)^{1/3} l^2}, \qquad S = \left(1 - \frac{\kappa}{8l\sqrt{24(-\lambda)}}\right)\frac{4\pi^2 r_+}{\kappa}, \tag{5.57}$$

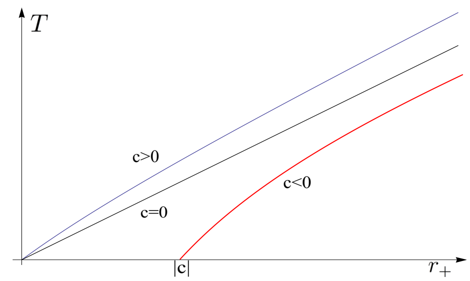

Figure 5.3: **Temperature vs horizon**: The behavior of the temperature T as a function of the horizon radius r_+ is shown. For all possible values of the integration constant c, the temperature is a monotonically increasing function of r_+. For large values of r_+, T approaches a linear function of r_+ with a slope $1/(2\pi l^2)$, which matches the straight line describing the case $c = 0$. Note that for a given temperature, three possible black hole configurations of different sizes can exist.

respectively. We can see that these quantities are linear functions of $r_+ = 2\sqrt{6|\lambda|}lb$. However, the entropy is positive only if $\sqrt{-\lambda} < \kappa/8l\sqrt{24}$. Thus, this physical requirement on the entropy yields an upper bound for the coupling constant λ.

For the case $c \neq 0$, studied in Sec. 5.1.2, the expressions for temperature, electric potential, and entropy computed from (5.46), (5.48), and (5.47) are

$$T = \frac{3r_+}{2\pi l^2}\left(\frac{r_+ + c}{3r_+ + 2c}\right), \; \Phi = -\frac{\text{sign}(q)c^2(1-\alpha a^2)^{2/3}(1-a)^{2/3}}{(\kappa l^2)^{2/3} r_+}, \tag{5.58}$$

$$S = \left(1 - \frac{ac}{r_+ + c}\right)\frac{4\pi^2 r_+}{\kappa}, \tag{5.59}$$

respectively.

First, we analyse the temperature behavior. Since $r_+ + c > 0$, the temperature is a positive, monotonically increasing function of r_+ as shown in Fig. 5.3. For large values of r_+, the temperature approaches a linear function of r_+ with the same slope, $1/(2\pi l^2)$, as for that in the case $c = 0$, which coincides with the temperature of the static BTZ black hole.

Now, we focus the attention on the entropy (5.59) for $c \neq 0$. As mentioned, the entropy derived from the action (5.1) is not a positive definite quantity. Then, the conditions that guarantee black holes with positive entropy must be determined.

A conformal transformation maps the action (5.1) to the Einstein frame (EF), where the scalar field is minimally coupled to gravity. The entropy in the Einstein frame follows the Bekenstein-Hawking area law, and hence is positive definite quantity. Naively, one may think that negative entropy configurations, now mapped into the Einstein frame could have a positive entropy. Remarkably, as shown in [64], for a similar class of solutions in four dimensions, they are not mapped into black holes but naked singularities in the new frame. The mechanism acts as follow. In three dimensions the conformal transformation is given by

$$g_{\mu\nu}^{\mathrm{EF}} = \Omega(\phi)^2 g_{\mu\nu} = \left(1 - \frac{\kappa}{8}\phi^2\right)^2 g_{\mu\nu} \tag{5.60}$$

First, we note the hypersurfaces where the conformal factor $\Omega(\phi)$ vanishes are mapped into curvature singularities of the corresponding image in the EF. For the solutions presented in this work, $\Omega(\phi)$ is a monotonously increasing function of r approaching 1 for a large r, but it is not a positive definite function. For configurations where $\Omega(\phi(r_+)) \leq 0$, the conformal factor necessarily vanishes in a hypersurface located at $r_0 \geq r_+$ generating a naked singularity in the EF. On contrary, for those configurations with $\Omega(\phi(r_+)) > 0$, the curvature singularity occurs at $r_0 < r_+$. Only in the latter case black holes in the conformal frame are mapped into black holes in the EF. In consequence, since $\Omega(\phi(r_+)) > 0$ is the same condition for ensuring a positive entropy, only black holes having a well-defined entropy in the conformal frame generate black holes in the EF. Therefore one concludes that the conditions for the black holes parameters in the conformal frame, ensuring the entropy to be positive, exactly coincide with the ones that guarantee cosmic censorship in Einstein frame.

When the gauge potential is turned off, we get hairy black holes solutions. Some of them are already known in the literature. In Chapter 7 we give a novel way to obtain the thermodynamic parameters for this kind of solutions in three dimensions. Actually, the tools considered so far, are used for connections in a Chern-Simons theory. That is the reason because next Chapter will be a preparation for presenting our work.

Chapter 6

General Relativity in three spacetime dimensions as a Chern-Simons theory

Three-dimensional gravity has proven to be a remarkably fertile ground for the study of gravity. The theory is topological and since there are no propagating degrees of freedom, the theory can be expressed as a Chern-Simons theory. In this scheme, the only fields are gauge connections, where we lost the sense of causality present in the metric formulation. This issue affects directly the way in wich we understand black holes and how we characterize them. The next chapter is focused on presenting the Chern-Simons action and how it is related with gravity. For pure $(2+1)$ gravity, it consists of one copy of the Poincaré algebra and for positive cosmological constant one copy of $SO(3,1)$. For negative cosmological constant, in particular, there is a black hole solution, the BTZ black hole and the action can be reformulated in terms of two Chern-Simons connections for $sl(2,\mathbb{R})$. In this case, we characterize the solution through its thermodynamics by regularazing the connnection along its contractible cycles. [1]

6.1 Action and field equations

The Chern-Simons action is given by following

$$I_{CS} = \frac{k}{4\pi} \int\limits_{M} \left\langle AdA + \frac{2}{3}A^3 \right\rangle \,, \tag{6.1}$$

[1]This Chapter is based on the lecture notes [68].

where M is a manifold, such that $M = \Sigma \times R$, where Σ is a spacelike surface and R is a real timelike line. Here k is a constant called level, relevant at quantum regime and A is the gauge field spanned in the algebra such that $A = A_{\mu}^{I} T_{I} dx^{\mu}$ where T_I stand for the generators of a Lie algebra $\mathfrak{g}$. This algebra is assumed to admit an invariant nondegenerate bilinear form $g_{IJ} = \langle T_I, T_J \rangle$.

The Chern-Simons field equations can be readly obtain taking the variation of (6.1)

$$\begin{aligned} \delta I_{CS} &= \frac{k}{4\pi}\delta \int_{M_3} \left\langle AdA + \frac{2}{3}A^3 \right\rangle = \frac{k}{4\pi}\delta \int_{M_4} \left\langle F^2 \right\rangle & (6.2) \\ &= \frac{k}{2\pi} \int_{M_4} \langle FD\delta A \rangle = \frac{k}{2\pi} \int_{M_4} \langle d\left(F\delta A\right) \rangle & (6.3) \\ &= \frac{k}{2\pi} \int_{M_3} \langle F\delta A \rangle \,, & (6.4) \end{aligned}$$

where

$$F = dA + A^2 = 0\,. \tag{6.5}$$

Then, the field equations for a Chern-Simons theory implies that the connection is locally flat on-shell.

6.2 Canonical formulation

The Chern-Simons action (6.1) can be written in a Hamiltonian form, where

$$I_H = -\frac{k}{4\pi} \int_{\Sigma \times R} dt d^2 x \varepsilon^{ij} \left\langle A_i \dot{A}_j - A_t F_{ij} \right\rangle + B_H \,. \tag{6.6}$$

This is straightforward when it is considered that the connection is split as $A = A_i dx^i + A_t dx^t$. In the above equation, B_H is a boundary term included in the Chern-Simons action for having gauge invariance in the action and depends on the boundary conditions. From (6.6), it is possible to see also that A_t is a Lagrange multiplier and that A_j are the dynamical fields. They satisfy the following Poisson bracket

$$\left\{ A_i^I\left(x\right), A_j^J\left(x'\right) \right\} = \frac{2\pi}{k} g^{IJ} \varepsilon_{ij} \delta\left(x - x'\right) \,, \tag{6.7}$$

where the constraint associated to A_t is given by

$$G = \frac{k}{4\pi}\varepsilon^{ij} F_{ij} \,. \tag{6.8}$$

As a consequence, the smeared generator (see e. g. [65, 66, 67]) reads,

$$G(\Lambda) = \int\limits_{\Sigma} d^2x \, \langle \Lambda G \rangle \ . \tag{6.9}$$

With the definition of the generator, we are able to compute the infinitesimal gauge transformation on the dynamical fields, which is given by $\delta A_i = \{A_i, G(\Lambda)\} = \partial_i \Lambda + [A_i, \Lambda]$, where the parameter Λ is Lie algebra valued.

When geometry has a boundary $\partial\Sigma \neq 0$ at spatial infinity, then $G(\Lambda)$ should be supplemented by a boundary term $Q(\Lambda)$ according to the Regge-Teitelboim approach [42],

$$\bar{G}(\Lambda) = G(\Lambda) + Q(\Lambda) \ . \tag{6.10}$$

This term improves the generator such that its functional variation is well-defined everywhere, where the variation of the conserved charge associated to the asymptotic gauge symmetry spanned by Λ is given by

$$\delta Q(\Lambda) = -\frac{k}{2\pi} \int\limits_{\Sigma} \langle \Lambda \delta A_\theta \rangle \, d\theta \, , \tag{6.11}$$

which is determined by the dynamical fields at a fixed time slice at the boundary Σ.

When the system is on-shell $F_{\nu\mu} = 0$, by virtue of $\mathcal{L}_\xi A_\mu = \nabla_\mu (\xi^\nu A_\nu) + \xi^\nu F_{\nu\mu}$, the diffeomorphisms $\delta_\xi A_\mu = -\mathcal{L}_\xi A_\mu$ are the same as the gauge transformations with parameter $\Lambda = -\xi^\mu A_\mu$. As a result, the variation of the asymptotic symmetry generator spanned by an asymptotic Killing vector reads

$$\delta Q(\xi) = \frac{k}{2\pi} \int\limits_{\partial\Sigma} \xi^\mu \langle A_\mu \delta A_\theta \rangle \, d\theta \, . \tag{6.12}$$

On the other hand, the transformation of the Lagrange multiplier A_t is given by

$$\delta A_t = \partial_t \Lambda + [A_t, \Lambda] \ . \tag{6.13}$$

This variation is recovered when one requires that the improved action is invariant under gauge transformations.

In order to integrate the variation of the canonical generators (6.11) a precise set of asymptotic conditions must be given. In the next section, we will explicitly show the relation between three-dimensional gravity and a Chern-Simons theory.

6.3 General Relativity with negative cosmological constant in three dimensions

General Relativity in vacuum can be described in terms of a Chern-Simons action [5, 6]. In particular, when we are considering gravity with $\Lambda < 0$ the actions reads as ,

$$I = I_{CS}\left[A^{+}\right] - I_{CS}\left[A^{-}\right] \ . \tag{6.14}$$

Here the connection A is defined in the Lie algebra $\mathfrak{g} = \mathfrak{g}_{+} + \mathfrak{g}_{-}$, where $\mathfrak{g}_{\pm}$ stand for two independent copies of $sl\left(2,\mathbb{R}\right)$ (details of the change of basis from $so(2,2)$ are give in Appendix 11). We assume that the algebra is described by the same set of matrices L_i, with $i = -1, 0, 1$, given by

$$L_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad ; \quad L_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad ; \quad L_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \ , \tag{6.15}$$

so that the $sl\left(2,\mathbb{R}\right)$ algebra reads

$$\left[L_i, L_j\right] = \left(i - j\right) L_{i+j} \ . \tag{6.16}$$

The connection then splits in two independent $sl\left(2,\mathbb{R}\right)$-valued gauge fields, according to $A = A^{+} + A^{-}$. On the other hand, the invariant bilinear forms correspond to the trace for the representation (6.15) where its nonvanishing components are given by $\langle L_1, L_{-1}\rangle = -1$ and $\langle L_0, L_0\rangle = \frac{1}{2}$.

For making the relation with the fields of gravity in first order, let us consider that the gauge fields are written in terms of the spacetime geometry fields e, ω as

$$A^{\pm} = \omega \pm \frac{e}{l} \ , \tag{6.17}$$

where the field $e^a = e^a_\mu dx^\mu$ is the dreibein 1-form and $\omega^a = \omega^a_\mu dx^\mu$ is the dualized spin connection 1-form, which defines the dualized curvature 2-form $R^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c$. Then, using (6.17) and (6.14), it is possible to recover

$$I_{GR} = \frac{1}{2\kappa}\int\left(2R^a e_a - \frac{\Lambda}{3}\epsilon_{abc}e^a e^b e^c\right) . \tag{6.18}$$

The above action is the first order formulation of gravity in three dimensions with cosmological constant Λ. Here, the metric is recovered from $g_{\mu\nu} = 2\mathrm{tr}\left(e_\mu e_\nu\right)$ and the equation of motion associated to ω gives rise to a torsionless condition, leading to a Riemannian spacetime.

6.3.1 BTZ black hole and its thermodynamics

The asymptotic behaviour of gravity with negative cosmological constant can be aslo reformulated in terms of the gauge fields $A^{\pm}$. In this formalism, it is interesting to notice that the radial dependence is entirely captured by the group elements $g_{\pm} = e^{\pm\rho L_0}$, so that the asymptotic form of the connections is given by

$$A^{\pm} = g_{\pm}^{-1} a^{\pm} g_{\pm} + g_{\pm}^{-1} dg_{\pm} \ , \tag{6.19}$$

where $a^{\pm} = a_{\theta}^{\pm} d\theta + a_t^{\pm} dt$. If we consider the change of variables, $x^{\pm} = \frac{t}{l} \pm \theta$, then the connection is given by

$$a^{\pm} = \pm \left(L_{\pm 1} - \frac{2\pi}{k} \mathcal{L}_{\pm} L_{\mp 1} \right) dx^{\pm} \ , \tag{6.20}$$

where the functions $\mathcal{L}_{\pm} = \mathcal{L}_{\pm}(t, \theta)$. The asymptotic form of the dynamical fields $a_{\theta}^{\pm}$ is preserved under gauge transformations, $\delta a_{\theta}^{\pm} = \partial_{\theta} \Lambda^{\pm} + \left[a_{\theta}^{\pm}, \Lambda^{\pm} \right]$, generated by

$$\Lambda^{\pm}(\varepsilon_{\pm}) = \varepsilon_{\pm} L_{\pm 1} \mp \varepsilon'_{\pm} L_0 + \frac{1}{2} \left(\varepsilon''_{\pm} - \frac{4\pi}{k} \varepsilon_{\pm} \mathcal{L}_{\pm} \right) L_{\mp 1} \ , \tag{6.21}$$

where the prime indicates the derivative with respect to θ. Here $\varepsilon_{\pm} = \varepsilon_{\pm}(t, \theta)$ provided the functions $\mathcal{L}_{\pm}$ transform as

$$\delta \mathcal{L}_{\pm} = \varepsilon_{\pm} \mathcal{L}'_{\pm} + 2 \mathcal{L}_{\pm} \varepsilon'_{\pm} - \frac{k}{4\pi} \varepsilon'''_{\pm} \ . \tag{6.22}$$

It is also necessary to impose on $a_t^{\pm}$ to be mapped into themselves under the same gauge transformations, considering also the transformation laws in (6.22). This impose some chirality conditions on $\mathcal{L}_{\pm}$ and the parameters $\varepsilon_{\pm}$,

$$\partial_{\mp} \mathcal{L}_{\pm} = 0 \ \ , \ \ \partial_{\mp} \varepsilon_{\pm} = 0 \ . \tag{6.23}$$

In the connection (6.20), when $\mathcal{L}_{\pm}$ are nonnegative constants, the asymptotic conditions contain also the BTZ black hole solution .

Now we are in position to study the BTZ thermodynamics as shown in [69]. The Euclidean black hole has the topology of a solid torus that corresponds to $\mathbb{R}^2 \times S^1$, where $\mathbb{R}^2$ stands for the one of the $\rho - \tau$ plane. Here, $\tau = -it$ is the Euclidean time, fulfilling $0 \leq \tau < \beta$, where $\beta = T^{-1}$ is the inverse of the Hawking temperature.

An important objet to consider here is the holonomy of the gauge field around a closed cycle $\mathcal{C}$, defined as

$$\mathcal{H}_{\mathcal{C}} = P \exp \left(\int_{\mathcal{C}} A_{\mu} dx^{\mu} \right) \ . \tag{6.24}$$

$\mathcal{H}$ is an element of the gauge group and it is sensitive to the global properties of the manifold. For the solution we are considering, the gauge group corresponds to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, the holonomy around $\mathcal{C}$ is

$$\mathcal{H}_{\mathcal{C}}^{\pm} = P \exp\left(\int_{\mathcal{C}} A_{\mu}^{\pm} dx^{\mu}\right) \ . \tag{6.25}$$

As the topology of the manifold is the one of a solid torus, there are contractible cycles along the thermal coordinate. We say they are trivial because satisfy,

$$\mathcal{H}_{\mathcal{C}}^{\pm} = -1 \ , \tag{6.26}$$

where the negative sign is due to the fact that, according to (6.15), we are dealing with the fundamental (spinorial) representation of $SL(2,\mathbb{R})$. For simplicity, here we computate the static case, i.e., for $\mathcal{L} := \mathcal{L}_{\pm}$. Since the holonomies around the thermal cycle of the BTZ black hole are trivial, the conditions in (6.26) reduce to

$$\mathcal{H}_{\tau}^{\pm} = e^{\beta a_{\tau}^{\pm}} = e^{i\beta a_{t}^{\pm}} = -1 \ . \tag{6.27}$$

The eigenvalues of $i\beta a_t$ are given by $\pm in\pi$, where n is a positive integer, then

$$\beta^2 \mathrm{tr}\left[\left(a_t^{\pm}\right)^2\right] = 2n^2\pi^2 \ . \tag{6.28}$$

The above ecuation fix the Euclidean time as

$$\beta = l\sqrt{\frac{\pi k}{2\mathcal{L}}} \ , \tag{6.29}$$

choosing $n = 1$, so as to make contact with the metric formulation, where β is fixed demanding the absence of conical singularities at the horizon. The value (6.29) coincides with the Hawking temperature.

Chapter 7

Gravity with a scalar field in 3D from a Chern-Simons form: thermodynamics of hairy black holes in terms of gauge fields

In this section, we present the theory of gravity with a conformally coupled scalar field and a novel rotating black hole solution. We generalized the result of 1970 obtained by Deser in d dimensions, that it is posible to recast the conformally invariant scalar field action as a gravity action. After that, we formulate the theory in terms of one-forms, so that we can redifine them as connections. We express the action as a Chern-Simons form and compute the global charges of the theory from the boundary terms of Chern-Simon. We study the thermodynamics of a hairy black hole solution in terms of gauge fields using the regularity conditions for connections in a Chern-Simons theory. The same analysis can be done in the case of Einstein gravity with a minimally coupled self-interacting real scalar field in three spacetime dimensions. We present the connections and equations of motion of this formulation.

7.1 Gravity with a conformally coupled scalar field

7.1.1 Metric formulation and black hole solution

We are interested in a scalar field conformally coupled to gravity in presence of a self-interaction potential $\lambda\phi^6$, which does not spoil the conformal invariance, described by the matter action

$$I^{\phi}\left[g_{\mu\nu},\phi\right]=\int d^3x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{16}R\phi^2-\lambda\phi^6\right), \tag{7.1}$$

and the gravitational theory to be considered is standard gravity with a cosmological constant Λ. Then, the full action that describes this theory is given by

$$I^{(2)}\left[g_{\mu\nu},\phi\right]=\int d^3x\sqrt{-g}\left(\frac{R-2\Lambda}{2\kappa}-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{16}R\phi^2-\lambda\phi^6\right), \tag{7.2}$$

where κ is the gravitational constant and λ is the self-interaction parameter. The corresponding field equations are

$$G_{\mu\nu}+\Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{7.3}$$

$$\Box\phi-\frac{1}{8}R\phi-6\lambda\phi^5 = 0. \tag{7.4}$$

The stress tensor

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi-g_{\mu\nu}\lambda\phi^6+\frac{1}{8}\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu+G_{\mu\nu}\right)\phi^2, \tag{7.5}$$

is traceless by virtue of (7.4), so that the scalar curvature is constant, $R=-6\ell^{-2}$.

The above field equations admit an asymptotically anti-de Sitter rotating black hole solution

$$ds^2=-N^\perp(r)^2\,dt^2+F(r)^{-2}\,dr^2+H(r)^2\left(d\theta+N^\theta(r)\,dt\right)^2, \tag{7.6}$$

with

$$N^\perp(r)^2 = \frac{r^2F(r)^2}{H(r)^2}N^2(\infty), \tag{7.7}$$

$$F(r)^2 = \frac{r^2}{\ell^2}-\frac{(1-\alpha)}{\ell^2}\left(\frac{2c^3}{r}+3c^2\right), \tag{7.8}$$

$$H(r)^2 = r^2+\frac{(1-\alpha)\,\omega^2}{1-\omega^2}\left(\frac{2c^3}{r}+3c^2\right), \tag{7.9}$$

$$N^\theta(r) = N^\theta(\infty)+\frac{(1-\alpha)\,\omega}{\ell\left(1-\omega^2\right)H(r)^2}\left(\frac{2c^3}{r}+3c^2\right)N(\infty), \tag{7.10}$$

dressed with a scalar field

$$\phi(r)=\sqrt{\frac{8c}{\kappa(r+c)}}, \tag{7.11}$$

where c, ω, $N(\infty)$, $N^\theta(\infty)$ are integration constants and $\alpha=512\ell^2\lambda/\kappa^2$ is a convenient redefinition of self-interaction constant λ. The coordinates range as $-\infty<t<\infty$, $0\leq r\leq\infty$ and $0\leq\theta<2\pi$. Here $rN(\infty)/\ell$ and $N^\theta(\infty)$ are the values of the lapse and shift functions at $r\to\infty$, respectively. Following the standard normalization of the time-like Killing vector ∂_t at infinity, we choose $N(\infty) = 1$. The scalar field remains real in the asymptotic region $r \to \infty$ provided $c > 0$. Under this condition the scalar field is regular everywhere for $r \geq 0$.

This black hole corresponds to the spinning version of that introduced in [16] and can be obtained from the static one performing a suitable boost in the $t - \theta$ plane. The event horizon is located at $r_+ = cx$, where

$$x = (1-\alpha)^{1/3} \left[\left(1+\sqrt{\alpha}\right)^{1/3} + \left(1-\sqrt{\alpha}\right)^{1/3} \right], \tag{7.12}$$

provided by $\alpha < 1$ (otherwise there is no horizon). For $\alpha < 0$, $r_+ > 2c$ and for $0 \leq \alpha < 1$, the horizon ranges as $2c \geq r_+ > 0$. The limit cases $\alpha = 1$ and $c = 0$ produce a massless BTZ dressed with a stealth scalar field, i. e., a non-trivial scalar field with a vanishing stress tensor, and they will be discarded hereafter.

The areal function $H(r)^2$ remains positive on and outside the horizon, if apart from the conditions previously demanded $(c > 0, \alpha < 1)$ is also required that $\omega^2 < 1$. In Sec. 7.2.1, the integration constant ω will be related with the angular velocity.

The metric is singular at the origin $r = 0$ for $\alpha \neq 1$, as one can see from the Kretschmann invariant

$$R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} = \frac{12}{\ell^4} \left(1 - \frac{2c^6 (1-\alpha)^2}{r^6} \right), \tag{7.13}$$

and its asymptotic behavior fullfills the Brown-Henneaux boundary conditions [47] because the scalar field has a sufficiently fast falloff at infinity (see Appendix 12).

7.1.2 Recasting the conformally invariant scalar field action as a gravity action

In 1970, Deser [70] proved that the four-dimensional version of (7.1) can be recast as the Einstein-Hilbert action by using a conformal transformation of the metric. Indeed, this can be done in arbitrary dimensions including also a conformally invariant self-interaction potential. Let us consider the conformal transformation

$$\bar{g}_{\mu\nu}(x) = \phi(x)^{\frac{4}{n-2}} g_{\mu\nu}(x), \tag{7.14}$$

in $n > 2$ dimensions. The Ricci scalar transforms as

$$\sqrt{-\bar{g}}\bar{R} = \sqrt{-g} \left(R\phi^2 - \frac{4(n-1)}{n-2} \phi \Box \phi \right) \tag{7.15}$$

$$= \sqrt{-g} \left(R\phi^2 + \frac{4(n-1)}{n-2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) \tag{7.16}$$

$$-\frac{4(n-1)}{n-2} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \phi \partial_\nu \phi \right), \tag{7.17}$$

and therefore

$$-\int d^n x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\frac{n-2}{8(n-1)}R\phi^2+\lambda\phi^{\frac{2n}{n-2}}\right)=\frac{1}{2\bar{\kappa}}\int dx^n\sqrt{-\bar{g}}\left(\bar{R}-2\bar{\Lambda}\right), \tag{7.18}$$

up to a boundary term, where

$$\bar{\kappa}=-\frac{4(n-1)}{n-2}\quad\text{and}\quad\bar{\Lambda}=\bar{\kappa}\lambda. \tag{7.19}$$

This simple computation shows that the action for a conformally coupled scalar field can be written as an Einstein-Hilbert action with a cosmological constant for the rescaled metric, where the gravitational constant, which depends on the dimension, is negative and the cosmological constant is proportional to the self-interaction constant, but with opposite sign. Note that in absence of a self-interaction potential ($\lambda=0$), we have $\bar{\Lambda}=0$.

In consequence, in three dimensions ($n=3$) we can write (7.2) as

$$I^{(2)}\left[g_{\mu\nu},\phi\right]=\frac{1}{2\kappa}\int dx^3\sqrt{-g}\left(R-2\Lambda\right)+\frac{1}{2\bar{\kappa}}\int dx^3\sqrt{-\bar{g}}\left(\bar{R}-2\bar{\Lambda}\right), \tag{7.20}$$

with $\bar{\kappa}=-8$ and $\bar{\Lambda}=-8\lambda$. We have expressed the total action $I^{(2)}\left[g_{\mu\nu},\phi\right]$ as the sum of two standard gravity action with different gravitational and cosmological constants. This last statement has a fundamental role for the upcoming analysis.

7.1.3 Formulation in terms of one-forms

Inspired by (7.20) of the action (7.2) the following action is proposed,

$$I^{(1)}\left[e,\phi\right]=\frac{1}{2\kappa}\int\left(2R^a e_a-\frac{\Lambda}{3}\epsilon_{abc}e^a e^b e^c\right)+\frac{1}{2\bar{\kappa}}\int\left(2\bar{R}^a\bar{e}_a-\frac{\bar{\Lambda}}{3}\epsilon_{abc}\bar{e}^a\bar{e}^b\bar{e}^c\right). \tag{7.21}$$

The 1-form field $e^a=e^a_\mu dx^\mu$ is the dreibein and $\omega^a=\omega^a_\mu dx^\mu$ is the dualized 1-form spin connection, which defines the dualized 2-form curvature $R^a=d\omega^a+\frac{1}{2}\epsilon^{abc}\omega_b\omega_c$. Here, ω^a is the torsionless spin connection associated to e^a that is obtained by solving the equation of motion associated to ω^a,

$$T^a=de^a+\epsilon^a_{bc}\omega^b e^c=0. \tag{7.22}$$

From the vanishing torsion condition we obtain

$$\omega^a_\mu=\left(\frac{1}{2}e^a_\mu\epsilon^{bcd}-\epsilon^{abc}e^d_\mu\right)e^\nu_b e^\lambda_c\partial_\nu e_{d\lambda}, \tag{7.23}$$

The field $\bar{e}^a = \bar{e}^a_\mu dx^\mu$ is a function of the scalar field ϕ and e^a through the relation

$$\bar{e}^a_\mu = \phi^2 e^a_\mu, \tag{7.24}$$

and the 2-form $\bar{R}^a$ is defined in terms of $\bar{\omega}^a = \bar{\omega}^a_\mu dx^\mu$ as $\bar{R}^a = d\bar{\omega}^a + \frac{1}{2}\epsilon^{abc}\bar{\omega}_b\bar{\omega}_c$. From the equation of motion associated to $\bar{\omega}$, $d\bar{e}^a + \epsilon^a_{bc}\bar{\omega}^b\bar{e}^c = 0$, we fix the value of $\bar{\omega}^a$ as

$$\bar{\omega}^a = \omega^a + 2\phi^{-1} * (e^a d\phi) . \tag{7.25}$$

The Hodge dual means $*(e^a d\phi) = \epsilon^{abc}\partial_\nu \phi e^\nu_b e_c$.As a result, the Euler-Lagrange equations for each field in the action (7.21) are given by

$$\delta e^a \;:\; 2R_a - \Lambda\epsilon_{abc}e^b e^c = -\frac{\kappa}{\bar{\kappa}}\phi^2\left(2\bar{R}_a - \bar{\Lambda}\epsilon_{abc}\bar{e}^b\bar{e}^c\right), \tag{7.26}$$

$$\delta\phi \;:\; \phi e^a\left(2\bar{R}_a - \bar{\Lambda}\epsilon_{abc}\bar{e}^b\bar{e}^c\right) = 0, \tag{7.27}$$

where the variations $\delta I/\delta\omega^a$ and $\delta I/\delta\bar{\omega}^a$ vanish by virtue of (10.5) and(10.6).The equations of motion obtained from the metric formalism (7.3)-(7.4) and those coming from the action, (7.26) and (7.27), are respectively equivalent, as is shown in Appendix 10. The equivalence can be established using the following correspondence between the dreibein and the metric

$$\eta_{ab}e^a_\mu e^b_\nu = g_{\mu\nu}, \tag{7.28}$$

in conjunction with the one relating the curvature 2-form and the Riemann curvature

$$R^{ab}{}_{\mu\nu}(\omega) = e^a_\lambda e^b_\rho R^{\lambda\rho}{}_{\mu\nu}(\Gamma) . \tag{7.29}$$

In a similar way the equivalence at the level of the action is obtained (see the details in Appendix 10).

7.2 The action from a Chern-Simons form

Considering the action proposed in the previous section (7.21), appearing as a sum of two gravity actions with different gravitational and cosmological constants, it is possible to give a description of gravity in the presence of a conformally coupled scalar field by means of a Chern-Simons form, which is the main goal of this section. In fact, based on the well-known results [5] and [6], the action (7.21) can be written as

$$I^{(1)}\left[A[e], \bar{A}[e,\phi]\right] = I_{CS}[A] + I_{CS}\left[\bar{A}\right], \tag{7.30}$$

with

$$I_{CS}\left[A\right] = \frac{k}{4\pi}\int\limits_{\Sigma}\left\langle AdA+\frac{2}{3}A^{3}\right\rangle, \tag{7.31}$$

$$I_{CS}\left[\bar{A}\right] = \frac{\bar{k}}{4\pi}\int\limits_{\Sigma}\left\langle \bar{A}d\bar{A}+\frac{2}{3}\bar{A}^{3}\right\rangle, \tag{7.32}$$

where $k=2\pi/\kappa$ and $\bar{k}=2\pi/\bar{\kappa}$. These Chern-Simons forms are defined on a manifold Σ of topology $\Sigma=\mathbb{R}\times\partial\Sigma$, where $\partial\Sigma$ stands for the space-like section. This formulation for serve for

The gauge connection A can be valued in the (anti-)de Sitter or Poincaré algebras, depending on the signs of the effective cosmological constant Λ, where the generators J_a and P_a are such that

$$A=\omega^{a}J_{a}+e^{a}P_{a}, \tag{7.33}$$

whose the invariant bilinear form is given by

$$\langle J_a, J_b\rangle=\langle P_a, P_b\rangle=0, \qquad \langle J_a, P_b\rangle=\eta_{ab}, \tag{7.34}$$

and $\eta_{ab}=\text{diag}\left(-1,1,1\right)$. The commutation relations of $so(2,2)$ generators read as

$$[J_a, J_b]=\epsilon_{abc}J^{c} \quad , \quad [J_a, P_b]=\epsilon_{abc}P^{c} \quad , \qquad [P_a, P_b]=-\Lambda\epsilon_{abc}J^{c}. \tag{7.35}$$

On the other hand, the gauge connection $\bar{A}$ is written as

$$\bar{A}=\bar{\omega}^{a}\bar{J}_{a}+\bar{e}^{a}\bar{P}_{a}, \tag{7.36}$$

where $\bar{e}=\bar{e}\left(e,\phi\right)$ and $\bar{\omega}=\bar{\omega}\left(e,\phi\right)$ are given in (7.24) and (7.25), respectively. Since $\bar{\Lambda}$ is defined according to the value of the self-interaction parameter λ, the generators $\bar{J}_a$ and $\bar{P}_a$ span three different algebras

$$\left[\bar{J}_a, \bar{J}_b\right]=\epsilon_{abc}\bar{J}^{c} \quad , \quad \left[\bar{J}_a, \bar{P}_b\right]=\epsilon_{abc}\bar{P}^{c} \quad , \qquad \left[\bar{P}_a, \bar{P}_b\right]=-\bar{\Lambda}\epsilon_{abc}\bar{J}^{c}. \tag{7.37}$$

For a vanishing λ, $\bar{\Lambda}=0$ and the above algebra corresponds to the Poincaré algebra. For $\lambda>0$, we have an anti-de-Sitter-like case because $\bar{\Lambda}<0$ which implies the $so(2,2)$ algebra. Finally, we have de-Sitter-like case for $\lambda<0$ due to $\bar{\Lambda}>0$ and the above commutation relations represent the $so(3,1)$ algebra.

The suitable invariant bilinear form for these algebras is given by

$$\left\langle\bar{J}_a, \bar{J}_b\right\rangle=\left\langle\bar{P}_a, \bar{P}_b\right\rangle=0 \quad , \quad \left\langle\bar{J}_a, \bar{P}_b\right\rangle=\eta_{ab}, \tag{7.38}$$

Since the action (7.30) written in terms of the dynamical fields e,ϕ exactly coincides with the action (7.21), the variation of (7.30) with respect to the dynamical fields leads to the same equations of motion found in section 2.2., i.e. equations (7.26), (7.27). The action (7.21) as a Chern-Simons form offers an useful and simple formalism for studying the thermodynamical properties of black-hole solutions of this theory as is explained in the next section.

7.2.1 Black hole thermodynamics

The thermodynamical properties of the hairy rotating black hole (7.6)-(7.11) will be studied by means of the Euclidean approach. In this scheme, $\tau = -it$ is the Euclidean time with period β, where β is the inverse of the temperature T. The Euclidean version of a stationary non-extremal three-dimensional black hole has a topology $\mathbb{R}^2 \times S^1$, where S^1 is the circle parametrized by θ and $\mathbb{R}^2$ is the plane $r - \tau$ described in polar coordinates centered at $r = r_+$. Identifying the points (r, τ, θ) and $(r, \tau + \beta, \theta + 2\pi)$ the black hole becomes a solid torus [69]. On the other hand, the Euclidean Hamiltonian action I_E is a linear combination of the Hamiltonian constraints with an additional surface term B that makes the Euclidean action a well-defined functional [42]. Once the constraints are fulfilled and considering stationary configurations —required for holding the thermodynamical equilibrium— the Euclidean action is just given by the surface term

$$I_E = \int_{r_+}^{\infty} dr \partial_r B = B(\infty) - B(r_+), \tag{7.39}$$

so that

$$\delta I_E = \delta B(\infty) - \delta B(r_+). \tag{7.40}$$

For the Euclidean version of (7.30), the variation of the surface term is given by [65, 66]

$$\delta B = -\frac{k\beta}{2\pi} \int_{S^1} \langle A_\tau \delta A_\theta \rangle \, d\theta - \frac{k\beta}{2\pi} \int_{S^1} \left\langle \bar{A}_\tau \delta \bar{A}_\theta \right\rangle d\theta, \tag{7.41}$$

Note that since the fields are stationary the integration along τ yields the factor β in (7.41).

7.2.2 Grand canonical ensemble: mass, angular momentum and entropy

Since we are interested in obtaining I_E, then it is necessary to integrate each piece of (7.40). This integration requires to impose boundary conditions at infinity and at the horizon. In thermodynamics, this procedure is equivalent to fix an ensemble. In our case we choose the grand canonical ensemble, where the on-shell action is related to the Gibbs free energy $\mathcal{G}$ as

$$I_E = \beta \mathcal{G}. \tag{7.42}$$

The Gibbs free energy depends on the mass M, angular momentum J and entropy S of the system, and is given by

$$\mathcal{G} = M + \Omega J - TS. \tag{7.43}$$

In equilibrium, the extensive parameters M, J and the entropy S are functions of the chemical potentials $\beta = T^{-1}$ and the angular velocity Ω, which are fixed. Thus, we can write

$$I_E(\beta, \Omega) = \beta M + \beta\Omega J - S. \tag{7.44}$$

Therefore, from the equilibrium condition $\delta I_E = 0$ we can obtain

$$M = \left(\frac{\partial I_E}{\partial \beta}\right)_\Omega - \frac{\Omega}{\beta}\left(\frac{\partial I_E}{\partial \Omega}\right)_\beta, \tag{7.45}$$

$$J = \frac{1}{\beta}\left(\frac{\partial I_E}{\partial \Omega}\right)_\beta, \tag{7.46}$$

$$S = \beta\left(\frac{\partial I_E}{\partial \beta}\right)_\Omega - I_E. \tag{7.47}$$

The above expressions allow us to determine the global charges and entropy from the Euclidean action.

7.2.3 Euclidean action computation

This section is focused on the calculation of I_E for the hairy rotating black hole introduced in section 2. This solution is asymptotically AdS, then the symmetry algebra for the first C-S is $so(2,2)$. We start with the variation of the surface term at infinity $\delta B(\infty)$. From (7.41), we get

$$\delta B(\infty) = \beta\delta\left(\frac{3\pi(1-\alpha)c^2(1+\omega^2)}{\ell^2\kappa(1-\omega^2)}\right) + \beta N^\theta(\infty)\,\delta\left(\frac{6\pi(1-\alpha)c^2\omega}{\ell\kappa(1-\omega^2)}\right). \tag{7.48}$$

Because the chemical potentials β and $\Omega = N^\theta(\infty)$ are fixed, the integration of (7.48) yields

$$B(\infty) = \beta\left(\frac{3\pi(1-\alpha)c^2(1+\omega^2)}{\ell^2\kappa(1-\omega^2)}\right) + \beta N^\theta(\infty)\left(\frac{6\pi(1-\alpha)c^2\omega}{\ell\kappa(1-\omega^2)}\right). \tag{7.49}$$

Now, we turn to determine the contribution of the surface term to the Euclidean action at the horizon. Here it is necessary to demand the regularization of the connection A on the horizon r_+ as has been explained in detail for the BTZ black hole in [69], and for higher spin black holes, for instance, in [72]. The regularity condition dictates that the holonomy $\mathcal{H}$ along the thermal cycle at the horizon must be trivial.

$$\mathcal{H} = \exp\left[\int A_\mu dx^\mu\right]_{r_+} = \exp\left[\beta A_\tau\right]_{r_+} = -\mathbb{I}. \tag{7.50}$$

Such regularization implies that the integration constants c and ω of the black hole are set in terms of β and $N^{\theta}(\infty)$, in agreement with the definition of the grand canonical ensemble.

The (A)dS algebras have a suitable matrix representation, from which one can recover the invariant bilinear form required for constructing the proper Chern-Simons action. Then, such regularization process can be performed through the direct diagonalization of the holonomy $\mathcal{H}$ at the horizon. As mentioned, the action (7.21) is written as the sum of a Chern-Simons action depending on the gauge connection A spanned in the $so(2,2)$ algebra, and a second part, which is a Chern-Simons-like action for a connection $\bar{A}$ that can be spanned in different algebras depending on the sign of $\bar{\Lambda}$. In absence of a self-interacting potential, $\bar{A}$ is defined on the Poincaré algebra that needs a different method to implement the regularity conditions. This is because of the lack of a suitable matrix representation from which one can recover the invariant bilinear form required for constructing the proper action. Then, despite it is possible to proceed for the $so(2,2)$ and $so(3,1)$ algebras following the process mentioned above (by solving (7.50) using a matrix representation) the regularity conditions for $\bar{A}$ will be implemented following an alternative method presented in [71]. This procedure covers the Poincaré case and also resume all the computations including the case $\bar{\Lambda} \neq 0$. First, the method requires to find an adequate gauge transformation g that permits to gauge away the τ- components of the dreibein $\bar{e}_{\tau}$. In general, such gauge transformation is generated by the group element

$$g = e^{p^{a}P_{a}} e^{\rho^{b}J_{b}}, \tag{7.51}$$

so that $g\bar{A}g^{-1} \equiv \bar{a}$ is the gauge transformation of $\bar{A}$. For our purpose, it not necessary to find a suitable group element, this is because of the form of the chosen frame for describing the black hole (11.1). Since the holonomy condition (7.50) is evaluated at the event horizon, where $F(r_{+})^{2} = 0$, the time component of the vielbein on the horizon vanishes $e_{\tau}(r_{+}) = 0$ only if $N^{\theta}(r_{+}) = 0$. This can be seen explicitly

$$\bar{a}_{\tau} = \frac{8(\alpha-1)(3x+2)c(N^{\theta}(\infty)\ell+\omega)}{\kappa(x+1)\ell\left(\omega^{2}-1\right)\sqrt{x\left(\frac{(\alpha-1)(3x+2)\omega^{2}}{\omega^{2}-1}+x^{3}\right)}}P_{2} \tag{7.52}$$

$$+\frac{3(\alpha-1)(x+1)c(N^{\theta}(\infty)\ell\omega+N^{\infty})}{\ell^{2}\left(\omega^{2}-1\right)\sqrt{x\left(\frac{(\alpha-1)(3x+2)\omega^{2}}{\omega^{2}-1}+x^{3}\right)}}J_{2}, \tag{7.53}$$

fixing $N^{\theta}(\infty)$ as

$$N^{\theta}(\infty) = -\frac{\omega}{\ell}, \tag{7.54}$$

if the removal of the P_2 component is demanded.

The second step is to calculate the holonomy of the spin connection $\bar{\omega}$ along the thermal cycle at the horizon and to request to be trivial. Then the regularity condition is stated as

$$\mathcal{H} = \exp\left[\int \bar{\omega}_\mu dx^\mu\right]_{r_+} = \exp\left[\beta\bar{\omega}_\tau\right]_{r_+} = -\mathbb{I}, \tag{7.55}$$

The characteristic polynomial of the $sl(2,\mathbb{R})$ matrix is given by

$$(\xi)^2 + \det\left[\beta\omega_\tau\right] = 0. \tag{7.56}$$

A more convenient form of the above condition can be obtained as follows. The regularity condition (7.50) impose that the eigenvalue ξ must be $\pm i\pi$, and using the Cayley-Hamilton theorem, we can get an equivalent condition

$$\mathrm{tr}\left[(\beta\bar{\omega}_\tau)^2\right] + 2\pi^2 = 0, \tag{7.57}$$

which is solved for β^2, yielding

$$\beta^2 = \frac{4\pi^2 x^4 \ell^4}{9c^2 (1+x)^2 (1-\alpha)^2 (1-\omega^2)}, \tag{7.58}$$

after Eq. (7.54) is used.

Now, we proceed to impose the regularity condition on the gauge connection A. Since $so(2,2) \simeq so(2,1)^+ \oplus so(2,1)^-$, the latter can be written in terms of two copies of $sl(2,\mathbb{R})$ (to see more details, go to appendix 11). Thus, for this case the action (7.31) is split in two terms as,

$$I_{CS}[A] = I_{CS}\left[A^+\right] - I_{CS}\left[A^-\right], \tag{7.59}$$

where $k = 2\pi\ell/\kappa$. The 1-forms $A^\pm = A^n_\pm L^\pm_n$ are valued on the $sl(2,\mathbb{R})$ algebra. The generators for each copy are described by the same set of matrices as above L_n. Taking this into account, the holonomy condition (7.50) becomes

$$\mathcal{H}^\pm = \exp\left[\int A^\pm_\mu dx^\mu\right]_{r_+} = \exp\left[\beta A^\pm_\tau\right]_{r_+} = -\mathbb{I}, \tag{7.60}$$

The characteristic polynomials of the $sl(2,\mathbb{R})$ matrices in this case are given by

$$\left(\xi^\pm\right)^2 + \det\left[\beta A^\pm_\tau\right] = 0. \tag{7.61}$$

By using that the eigenvalues $\xi^\pm$ must be $\pm i\pi$, and using the Cayley-Hamilton theorem, the equations (7.61) turn out to be

$$\mathrm{tr}\left[\left(\beta A^\pm_\tau\right)^2\right] + 2\pi^2 = 0. \tag{7.62}$$

The chemical potential $N^{\theta}(\infty)$ already computed (7.54) is replaced in (7.62). Then, it is obtained that the chemical potential β takes the form given in (7.58). By choosing its positive branch, we get that

$$\beta = \frac{2\pi x^2 \ell^2}{3c\,(1+x)\,(1-\alpha)\sqrt{1-\omega^2}}. \tag{7.63}$$

By replacing the value of the chemical potentials (7.54) and (7.63) in the variation the surface term δB (7.41) evaluated at horizon, this surface term can be integrated yielding

$$B\left(r_+\right) = \frac{4\pi^2 c x^2}{\kappa\,(1+x)\sqrt{1-\omega^2}}. \tag{7.64}$$

Once the variation of the surface term is integrated at infinity (7.49) and at the horizon (7.64), the Euclidean action is determined as the subtraction of both contributions (7.39),

$$I_E = \beta\left(\frac{3\pi(1-\alpha)c^2\left(1+\omega^2\right)}{\ell^2\kappa\left(1-\omega^2\right)}\right) + \beta N^{\theta}\left(\infty\right)\left(\frac{6\pi(1-\alpha)c^2\omega}{\ell\kappa\left(1-\omega^2\right)}\right) - \frac{4\pi^2 c x^2}{\kappa\,(1+x)\sqrt{1-\omega^2}}. \tag{7.65}$$

Considering the equations (7.54) and (7.63), it is possible to replace c and ω as functions of β and $N^{\theta}\left(\infty\right)$ in the Euclidean action (7.65) as required in the grand canonical ensemble. This yields

$$I_E(\beta, N^{\theta}(\infty)) = -\frac{4\pi^3 x^4 \ell^2}{3\beta(1+x)^2\left(1-N^{\theta}\left(\infty\right)^2\ell^2\right)(1-\alpha)\kappa}. \tag{7.66}$$

Using the thermodynamical relations defined in (7.45), the mass M, angular momentum J and entropy S are determined to be

$$M = \frac{4\pi^3 x^4 \ell^2\left(1+N^{\theta}\left(\infty\right)^2\ell^2\right)}{3\beta^2\kappa(1+x)^2\left(1-N^{\theta}\left(\infty\right)^2\ell^2\right)^2(1-\alpha)} = \frac{3\pi\left(1-\alpha\right)c^2\left(1+\omega^2\right)}{\kappa\ell^2\left(1-\omega^2\right)}, \tag{7.67}$$

$$J = \frac{8N^{\theta}\left(\infty\right)\pi^3 x^4 \ell^4}{3(1+x)^2\left(1-N^{\theta}\left(\infty\right)^2\ell^2\right)^2(\alpha-1)\beta^2\kappa} = \frac{6\pi\left(1-\alpha\right)c^2\omega}{\kappa\ell\left(1-\omega^2\right)}, \tag{7.68}$$

$$S = \frac{8\pi^3 x^4 \ell^2}{3(1+x)^2\left(1-N^{\theta}\left(\infty\right)^2\ell^2\right)(1-\alpha)\beta\kappa} = \frac{4\pi^2 c x^2}{\kappa\,(1+x)\sqrt{1-\omega^2}}. \tag{7.69}$$

The mass and angular momentum coincide with those computed following the Regge-Teitelboim method [42], as is shown in the Appendix 12. Also, the entropy matches with the one found through the modified Bekenstein-Hawking formula [62, 63],

$$S = \left(1-\pi G\phi(r_+)^2\right)\frac{A}{4G}, \tag{7.70}$$

where A denotes the area of the horizon and $\kappa = 8\pi G$.

We also verify the general formula for the entropy of the black hole in terms of the on-shell holonomies proposed in [72]. Here we have to consider the contribution of both gauge connections $\left\{A, \bar{A}\right\}$ such that

$$S = -k\beta\langle A_\tau A_\varphi\rangle_{\text{on-shell}} - \bar{k}\beta\langle \bar{A}_\tau \bar{A}_\varphi\rangle_{\text{on-shell}}, \tag{7.71}$$

which gives as a result (7.70).

7.3 Changing the frame

In this section we want to show how it is possible to have the same kind of formulation in term of connections for gravity with a minimally coupled scalar field. In fact, what is helpful for getting this result is to consider that the action for gravity with a conformally coupled scalar scalar field and a self-interacting potential,

$$I^{(2)}\left[\tilde{g}_{\mu\nu}, \tilde{\phi}\right] = \int d^3x\sqrt{-\tilde{g}}\left(\frac{\tilde{R} - 2\Lambda}{2\kappa} - \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} - \frac{1}{16}\tilde{R}\tilde{\phi}^2 - \lambda\tilde{\phi}^6\right), \tag{7.72}$$

can be mapped by the scale transformation $\bar{g}_{\mu\nu} = \Omega^{-2}g_{\mu\nu}$ to the action of Einstein gravity with a self-interacting minimally coupled scalar field in three spacetime dimensions

$$I\left[\phi, g_{\mu\nu}\right] = \int d^3x\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V\left(\phi\right)\right), \tag{7.73}$$

with the potential

$$V\left(\phi\right) = \frac{\Lambda}{\kappa}\cosh^6\left(\sqrt{\frac{\kappa}{8}}\phi\right) - \lambda\sinh^6\left(\sqrt{\frac{\kappa}{8}}\phi\right). \tag{7.74}$$

For obtaining the kinetic term of the scalar field, we have to fix

$$\Omega = \left(1 - \tilde{\phi}^2\right), \tag{7.75}$$

redefining $\tilde{\phi} = tanh\left(\phi\right)$. In this sense –and following the same spirit of the conformally coupled action and gravity– we can apply the transformation(7.75) to both vielbains e and $\bar{e}$ presented above. The result of this transformation is shown in the following section.

7.4 Gravity with a minimally coupled scalar field

In this section we show that the action of Einstein gravity with a self-interacting minimally coupled scalar field in three spacetime dimensions

$$I\left[\phi, g_{\mu\nu}\right] = \int d^3x\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V\left(\phi\right)\right) , \tag{7.76}$$

with the potential

$$V\left(\phi\right) = \frac{\Lambda}{\kappa}\cosh^6\left(\sqrt{\frac{\kappa}{8}}\phi\right) - \lambda\sinh^6\left(\sqrt{\frac{\kappa}{8}}\phi\right) , \tag{7.77}$$

can be recast using a Chern-Simons form built from the direct sum of the algebras g^+ and g^-, where $g^\pm$ can be the (anti-)de Sitter or Poincaré algebras, depending on the signs of the effective cosmological constant Λ and the self-interacting coupling constant λ.

The potential $V(\phi)$ represents a two-parameter family of the self-interactions for the scalar field that admits hairy black holes solutions. In the case of $\Lambda < 0$ and $\lambda \geq \Lambda/\kappa$, static black holes were found [16] and further analyzed in [18, 19]. A rotating black hole with $\lambda = 0$ was presented in [21]. These black holes are dressed with a scalar field, whose fall-off at infinity is slow enough as it can contributes to mass, relaxing in this way the usual Brown-Henneaux asymptotic conditions in pure gravity.

Let us consider the action

$$I\left[\phi, e, \omega^+, \omega^-\right] = \frac{k^+}{4\pi}\int\left\langle A^+dA^+ + \frac{2}{3}A_+^3\right\rangle + \frac{k^-}{4\pi}\int\left\langle A^-dA^- + \frac{2}{3}A_-^3\right\rangle , \tag{7.78}$$

where the gauge connections read as follow

$$A^+ = \cosh^2\left(\sqrt{\frac{\kappa}{8}}\phi\right)e^aP_a^+ + \omega_+^aJ_a^+ , \tag{7.79}$$

$$A^- = \frac{8}{\kappa}\sinh^2\left(\sqrt{\frac{\kappa}{8}}\phi\right)e^aP_a^- + \omega_-^aJ_a^- . \tag{7.80}$$

The generators of the gauge connections obey the following algebras

$$\left[J_a^\pm, J_b^\pm\right] = \epsilon_{abc}J_\pm^c \quad , \quad \left[J_a^\pm, P_b^\pm\right] = \epsilon_{abc}P_\pm^c \quad , \quad \left[P_a^\pm, P_b^\pm\right] = -\Lambda^\pm\epsilon_{abc}J_\pm^c . \tag{7.81}$$

which correspond to $so(2,2)$ if $\Lambda^\pm < 0$, $so(3,1)$ if $\Lambda^\pm > 0$, and $iso(2,1)$ for $\Lambda^\pm = 0$.

The nonvanishing components of the brackets are $\left\langle J_a^\pm, P_b^\pm\right\rangle = \eta_{ab}$, where η_{ab} stands for the Minkowski metric. The levels are $k^+ = 2\pi/\kappa$ and $k^- = -\pi/4$.

The independent fields are $\left\{\phi, e^a, \omega_{\pm}^a\right\}$, however the field equations associated to $\omega_{\pm}^a$,

$$d\left(\cosh^2\left(\sqrt{\frac{\kappa}{8}}\phi\right) e_a\right) + \cosh^2\left(\sqrt{\frac{\kappa}{8}}\phi\right) \epsilon_{abc}\omega_+^b e^c = 0\,, \tag{7.82}$$

$$d\left(\sinh^2\left(\sqrt{\frac{\kappa}{8}}\phi\right) e_a\right) + \sinh^2\left(\sqrt{\frac{\kappa}{8}}\phi\right) \epsilon_{abc}\omega_-^b e^c = 0\,, \tag{7.83}$$

are algebraic for $\omega_{\pm}^a$:

$$\omega_{\pm}^a\left(e,\phi\right) = \omega^a\left(e\right) - 2\sqrt{\frac{\kappa}{8}}\tanh^{\pm 1}\left(\sqrt{\frac{\kappa}{8}}\phi\right) * \left(e^a d\phi\right)\,. \tag{7.84}$$

where ω^a is the torsionless spin connection associated to e^a (therefore the theory is defined on a Riemannian geometry).

A second order action $I(e^a,\phi)$ is obtained by replacing (7.84) in the Chern-Simons action (7.78). This action is equivalent to (7.76), up to a boundary term, with a self-interaction potential

$$V\left(\phi\right) = \frac{\Lambda^+}{\kappa}\cosh^6\left(\sqrt{\frac{\kappa}{8}}\phi\right) - \frac{\Lambda^-}{8}\left(\frac{8}{\kappa}\right)^3\sinh^6\left(\sqrt{\frac{\kappa}{8}}\phi\right)\,, \tag{7.85}$$

where $\Lambda^+ = \Lambda$ and $\Lambda^- = 8\lambda\left(\frac{\kappa}{8}\right)^3$. This fact can be explicitly shown using that

$$\eta_{ab}e_\mu^a e_\nu^b = g_{\mu\nu} \quad , \quad R^{ab}{}_{\mu\nu} = e_\lambda^a e_\rho^b R^{\lambda\rho}{}_{\mu\nu}\,, \tag{7.86}$$

where $R^{ab}{}_{\mu\nu}$ are the components of the curvature two-form associated to the torsionless spin connection ω^a, and $R^{\lambda\rho}{}_{\mu\nu}$ are the components of the Riemann tensor.

7.5 Field equations

In this section we will obtain the field equations from the Chern-Simons action (7.78). It is well known that the variation of a generic Chern-Simons action is given by

$$\delta I_{CS} = \frac{k}{2\pi}\int\left\langle F\delta A\right\rangle\,, \tag{7.87}$$

up to boundary terms, where $F = dA + A^2$ stands for the curvature two-form associated to the gauge connection A.

Using this result for the action (7.78), we obtain that its variation becomes

$$\begin{aligned}\delta I &= \frac{k^{+}}{2\pi}\int\left\langle F^{+}\delta A^{+}\right\rangle+\frac{k^{-}}{2\pi}\int\left\langle F^{-}\delta A^{-}\right\rangle \\ &= \frac{1}{2\pi}\int\left\langle\left(k^{+}F^{+}\frac{\delta A^{+}}{\delta e^{a}}+k^{-}F^{-}\frac{\delta A^{-}}{\delta e^{a}}\right)\delta e^{a}+\left(k^{+}F^{+}\frac{\delta A^{+}}{\delta\phi}+k^{-}F^{-}\frac{\delta A^{-}}{\delta\phi}\right)\delta\phi\right\rangle .\end{aligned}$$

In the above computation, the variations $\delta I/\delta\omega_{\pm}^{a}$ have been considered to vanish by virtue of (7.84).

Taking into account the definition of $A^{\pm}$ given in (7.79) and (7.80), beside of the fact that the components of $F^{\pm}$ stand exclusively along the Lorentz generators by virtue of (7.84), the field equations related to the variations of e^{a} and ϕ turn out to be

$$\cosh^{2}\left(\sqrt{\frac{\kappa}{8}}\phi\right)F_{a}^{+}-\sinh^{2}\left(\sqrt{\frac{\kappa}{8}}\phi\right)F_{a}^{-} = 0\,, \tag{7.88}$$

$$\left(F_{a}^{+}-F_{a}^{-}\right)e^{a} = 0\,, \tag{7.89}$$

respectively, with

$$F_{+}^{a}=R_{+}^{a}-\frac{1}{2}\Lambda^{+}\cosh^{4}\left(\sqrt{\frac{\kappa}{8}}\phi\right)\epsilon^{abc}e_{b}e_{c}\,, \tag{7.90}$$

$$F_{-}^{a}=R_{-}^{a}-\frac{4}{\kappa}\Lambda^{-}\sinh^{4}\left(\sqrt{\frac{\kappa}{8}}\phi\right)\epsilon^{abc}e_{b}e_{c}\,, \tag{7.91}$$

where $R_{\pm}^{a}=d\omega_{\pm}^{a}+\frac{1}{2}\epsilon^{abc}\omega_{b}^{\pm}\omega_{c}^{\pm}$ is the curvature two-form associated to $\omega_{\pm}^{a}$. By using (7.84) and the relations (7.86), it is possible to show that equations (7.88) and (7.89) reduce respectively to the Einstein and the scalar field equations,

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}\,, \tag{7.92}$$

$$\Box\phi-\frac{dV(\phi)}{d\phi} = 0\,, \tag{7.93}$$

where the potential $V(\phi)$ is given in (7.77) and the resulting energy-momentum tensor has the expected form

$$T_{\mu\nu}=\partial_{\mu}\phi\partial_{\nu}\phi-\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_{\alpha}\phi\partial_{\beta}\phi-g_{\mu\nu}V(\phi)\,. \tag{7.94}$$

Chapter 8

Conserverd charges and thermodynamics of the AdS_4 dyonic black hole

We present the Lagrangian and the AdS_4 dyonic dilatonic black hole solution of [33]. We focus on the Hamiltonian analysis and the corresponding conserved charges. The mass and the electric charge are computed using the Regge-Teitelboim Hamiltonian approach. There are two contributions in the variation of the mass, the gravitational part and the scalar field part (already identified in [73]). Integrability conditions have to be imposed because the presence of the scalar field leads to a non-integrable term. Suitable boundary conditions are chosen in order to preserve the AdS symmetry of the scalar field fall-off. This implies a precise relation among the coefficients of the leading and subleading terms of the scalar field, as was noted in [29]. We also perform the thermodynamic analysis of the solution and introduce the Hamiltonian Euclidean action. For simplicity the calculations are done in a suitable Euclidean minisuperspace. To obtain the Gibbs free energy we compute the value of the Euclidean Hamiltonian action endowed with a suitable radial boundary term and an additional term. These terms have to be added in order to have a well-defined and finite Hamiltonian action principle. It is possible to identify the variation of the Hamiltonian conserved charges of the system from the variation of the boundary term at infinity, which are the mass and the electric charge. On the other hand, the variation of the magnetic charge comes from the additional term. This term has to be considered due to the presence of a magnetic monopole. The chemical potentials associated to the Noether charges are the Lagrange multipliers of the system at infinity. Unlike the magnetic potential, they are obtained through regularity conditions at the horizon. Remarkably, the magnetic potential is already determined by the variation of the additional term, together with the magnetic charge. It is worth noting that the first law of black hole thermodynamics is satisfied independently of the integrability conditions on the mass, since the relation only involves the variation of the conserved charges. Once the Gibbs free energy is obtained the value of the mass, the electric charge, the magnetic charge and the entropy are verified using the known thermodynamic relations.

8.1 AdS$_4$ dyonic black hole solution

We consider four-dimensional gravity with negative cosmological constant in the presence of an Abelian gauge field and a dilatonic scalar field with a self-interacting potential. The action reads

$$I[g_{\mu\nu}, A_\mu, \phi] = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}e^{-\sqrt{3}\phi}F^{\mu\nu}F_{\mu\nu} - V(\phi)\right). \quad (8.1)$$

Hereafter the gravitational constant is chosen as $\kappa = 1/2$.[1] The self-interacting potential of the scalar field is given by

$$V(\phi) = -6g^2\cosh\left(\frac{\phi}{\sqrt{3}}\right), \quad (8.2)$$

where the coupling constant g determines the AdS radius as $\ell^2 = g^{-2}$. The theory given by (8.1) corresponds to the bosonic sector of two possible dimensional reductions, which depend on the coupling constant g in the following way. In the case of vanishing g the action is obtained after a S^1 reduction of five-dimensional pure gravity. On the other hand, if $g \neq 0$ the action can be obtained after a S^7 reduction of eleven-dimensional supergravity [35].

The gravitational field equations for the action (8.1) are

$$G_{\mu\nu} = T^\phi_{\mu\nu} + T^A_{\mu\nu}, \quad (8.3)$$

where the contributions to the energy-momentum tensor of the dilatonic scalar field and the gauge field are given by

$$T^\phi_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}\partial^\lambda\phi\partial_\lambda\phi + \frac{1}{2}g_{\mu\nu}V(\phi), \quad (8.4)$$

$$T^A_{\mu\nu} = \frac{1}{2}e^{-\sqrt{3}\phi}\left(F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4}g_{\mu\nu}F^{\lambda\rho}F_{\lambda\rho}\right), \quad (8.5)$$

respectively. The equation for the scalar field is

[1] The vacuum permeability constant located in front of the Maxwell-like action in (8.1) turns out to be normalized to one after the dimensional reduction.

$$\Box\phi + \frac{\sqrt{3}}{4}e^{-\sqrt{3}\phi}F^{\mu\nu}F_{\mu\nu} - \frac{dV}{d\phi} = 0, \tag{8.6}$$

and the equation for the gauge field reads

$$\nabla_\mu\left(e^{-\sqrt{3}\phi}F^{\mu\nu}\right) = 0. \tag{8.7}$$

This system admits an AdS dyonic black hole which is static and spherically symmetric [33]. The line element of this configuration can be written as

$$ds^2 = -\left(H_1H_2\right)^{-1/2}fdt^2 + \frac{dr^2}{\left(H_1H_2\right)^{-1/2}f} + \left(H_1H_2\right)^{1/2}r^2\left(d\theta^2 + \sin^2\left(\theta\right)d\varphi^2\right), \tag{8.8}$$

where the functions H_1, H_2 and f are given by

$$f\left(r\right) = f_0\left(r\right) + g^2r^2H_1\left(r\right)H_2\left(r\right), \quad f_0\left(r\right) = 1 - \frac{2\mu}{r}, \tag{8.9}$$

$$H_1\left(r\right) = \gamma_1^{-1}\left(1 - 2\beta_1 f_0\left(r\right) + \beta_1\beta_2 f_0\left(r\right)^2\right), \tag{8.10}$$

$$H_2\left(r\right) = \gamma_2^{-1}\left(1 - 2\beta_2 f_0\left(r\right) + \beta_1\beta_2 f_0\left(r\right)^2\right), \tag{8.11}$$

with $\gamma_1 = 1 - 2\beta_1 + \beta_1\beta_2$, and $\gamma_2 = 1 - 2\beta_2 + \beta_1\beta_2$. The dilatonic scalar field is given by

$$\phi\left(r\right) = \frac{\sqrt{3}}{2}\log\left(\frac{H_2\left(r\right)}{H_1\left(r\right)}\right), \tag{8.12}$$

whereas the one-form gauge field has the following form

$$A = A_t(r)dt + A_\varphi(\theta)d\varphi. \tag{8.13}$$

The time component of (8.13) is

$$A_t\left(r\right) = \frac{\sqrt{2}\left(1 - H_1\left(r\right) - \beta_1\left(f_0 - H_1\left(r\right)\right)\right)}{\sqrt{\beta_1\gamma_2}H_1\left(r\right)}, \tag{8.14}$$

while the definition of the angular component of the gauge potential depends on the hemisphere, in order to avoid the Dirac string [74]. Hence,

$$A_\varphi\left(\theta\right) = \begin{cases} p(1+\cos\left(\theta\right)) & , \quad 0 \leq \theta < \frac{\pi}{2} - \delta, \\ p\left(-1+\cos\left(\theta\right)\right) & , \quad \frac{\pi}{2} + \delta < \theta \leq \pi, \end{cases} \tag{8.15}$$

where $p = 2\sqrt{2}\mu\gamma_2^{-1}\sqrt{\beta_2\gamma_1}$ and $\delta \to 0$ (Wu-Yang monopole [75], [76]). In this solution the coordinate ranges are $0 < r < \infty$, $-\infty < t < \infty$, $0 \leq \theta < \pi$ and

$0 \leq \varphi < 2\pi$. All the integration constants (μ, β_1, β_2 , γ_1, γ_2) are restricted to be positive.

In the case of $\beta_1 = \beta_2$, the dilatonic scalar field is decoupled and the solution turns out to be an AdS dyonic Reissner-Nordström black hole where the electric and magnetic charges have the same value. If $\beta_1 = 0$ the solution is purely magnetic and in the case of $\beta_2 = 0$ the configuration becomes purely electric. If $\mu = 0$ the solution turns out to be AdS spacetime.

8.2 Hamiltonian generator and surface integrals

The Hamiltonian generator for the Lagrangian (8.1) reads

$$H\left[\xi,\xi^{A}\right]=\int d^{3}x\left(\xi^{\perp}\mathcal{H}_{\perp}+\xi^{i}\mathcal{H}_{i}-\xi^{A}\mathcal{G}\right)+Q\left[\xi,\xi^{A}\right], \tag{8.16}$$

where the boundary term $Q\left[\xi,\xi^{A}\right]$, which corresponds to the conserved charges in the Regge-Teitelboim approach, ensures that the Hamiltonian generator has well-defined functional derivatives [42]. The bulk term appearing in (8.16) is a linear combination of the constraints $\mathcal{H}_{\perp}$, $\mathcal{H}_i$ and $\mathcal{G}$, where the first two are the energy and momentum densities and the last one corresponds to the Gauss constraint associated to the Abelian gauge field. The asymptotic surface deformations of the spacetime are given by the vector $\xi = \left(\xi^{\perp},\xi^{i}\right)$ and ξ^A is the gauge parameter of the Abelian symmetry. The constraints are explicitly given by

$$\begin{aligned}
\mathcal{H}_{\perp} &= \frac{1}{\sqrt{\gamma}}\left(\pi^{ij}\pi_{ij}-\frac{1}{2}\left(\pi^{i}{}_{i}\right)^{2}\right)-\sqrt{\gamma}R \\
&\quad +\frac{\pi_{\phi}^{2}}{2\sqrt{\gamma}}+\sqrt{\gamma}\left(\frac{1}{2}\partial^{i}\phi\partial_{i}\phi+V(\phi)\right)+e^{\sqrt{3}\phi}\frac{\pi^{i}\pi_{i}}{2\sqrt{\gamma}}+\frac{1}{4}\sqrt{\gamma}e^{-\sqrt{3}\phi}F^{ij}F_{ij}, \\
\mathcal{H}_{i} &= 2\nabla_{j}\pi^{j}{}_{i}+\pi_{\phi}\partial_{i}\phi+\pi^{j}F_{ij}, \\
\mathcal{G} &= \partial_{i}\pi^{i}.
\end{aligned} \tag{8.17}$$

The dynamical variables of the system are the spatial components of the fields $\{\gamma_{ij},\, A_i,\, \phi\}$, where γ_{ij} is the spatial metric of the ADM decomposition. Here R stands for the scalar curvature of the three-dimensional spatial metric γ_{ij} and the self-interacting potential of the scalar field $V(\phi)$ is defined in eq. (8.2). The momentum conjugated to the three-dimensional metric γ_{ij} is

$$\pi^{ij}=-\sqrt{\gamma}\left(K^{ij}-\gamma^{ij}K\right), \tag{8.18}$$

where the extrinsic curvature is given by

$$K_{ij}=\frac{1}{2N^{\perp}}\left(\nabla_{i}N_{j}+\nabla_{j}N_{i}-\dot{\gamma}_{ij}\right). \tag{8.19}$$

The momentum for the dilatonic field ϕ reads

$$\pi_\phi = \frac{\sqrt{\gamma}}{N^\perp}\left(\dot{\phi} - N^i \partial_i \phi\right), \tag{8.20}$$

and for the gauge field A_i,

$$\pi^i = -\frac{\sqrt{\gamma} e^{-\sqrt{3}\phi}}{N^\perp}\left(-\gamma^{ij} F_{0j} + N^j \gamma^{ik} F_{jk}\right). \tag{8.21}$$

The variation of the surface term gets different contributions according to the field content of the theory, such that

$$\delta Q\left[\xi, \xi^A\right] = \delta Q^G\left[\xi, \xi^A\right] + \delta Q^\phi\left[\xi, \xi^A\right] + \delta Q^A\left[\xi, \xi^A\right], \tag{8.22}$$

where δQ was obtained after demanding that $\delta H = 0$ on the constraint surface. The explicit expressions for the surface integrals are given by

$$\begin{aligned} \delta Q^G &= \int dS_l G^{ijkl}\left(\xi^\perp \nabla_k \delta\gamma_{ij} - \partial_k \xi^\perp \delta\gamma_{ij}\right) \\ &\quad + \int dS_l\left[2\xi_k \delta\pi^{kl} + \left(2\xi^k \pi^{jl} - \xi^l \pi^{kj}\right)\delta\gamma_{jk}\right], \end{aligned} \tag{8.23}$$

$$\delta Q^\phi = -\int dS_i\left(\xi^\perp \sqrt{\gamma}\partial^i \phi \delta\phi + \xi^i \pi_\phi \delta\phi\right), \tag{8.24}$$

$$\delta Q^A = -\int dS_i\left[\xi^\perp \sqrt{\gamma} e^{-\sqrt{3}\phi} F^{ij} \delta A_j + \left(\xi^i \pi^j - \pi^j \xi^i\right)\delta A_j - \xi^A \delta\pi^i\right], \tag{8.25}$$

with

$$G^{ijkl} = \frac{1}{2}\sqrt{\gamma}\left(\gamma^{ik}\gamma^{jl} + \gamma^{il}\gamma^{jk} - 2\gamma^{ij}\gamma^{kl}\right). \tag{8.26}$$

8.2.1 Conserved charges of the AdS$_4$ dyonic black hole

In order to obtain the above surface integrals let us consider a static and spherically symmetric minisuperspace in which the AdS$_4$ dyonic black hole (8.8) is included. For simplicity we perform the following change of variable in the radial coordinate

$$\rho^2 = \sqrt{H_1(r) H_2(r)} r^2. \tag{8.27}$$

The line element then reads

$$ds^2 = -N^\perp(\rho)^2 dt^2 + \frac{d\rho^2}{F(\rho)} + \rho^2\left(d\theta^2 + \sin^2(\theta) d\varphi^2\right). \tag{8.28}$$

The gauge field ansatz is given by

$$A = A_t(\rho)\, dt + A_\varphi(\theta)\, d\varphi, \tag{8.29}$$

and the scalar field also depends on the radial coordinate $\phi = \phi(\rho)$. Taking this into consideration the only nonvanishing momentum in the minisuperspace is the radial component of the electromagnetic one, where $\pi^\rho = p^\rho(\rho, \theta)$. Therefore, the value of the Hamiltonian charges, computed on the sphere S^2 of infinite radius, is given by

$$\begin{aligned} \delta Q &= \Bigg[-\xi^t \left(\frac{8\pi\rho N^\perp \delta F}{\sqrt{F}} + 4\pi\sqrt{F} N^\perp \rho^2 \partial_\rho \phi \delta\phi \right) \\ &\quad -\xi^t \pi \left[\left(\int \frac{N^\perp e^{-\sqrt{3}\phi}}{\sqrt{F}\rho^2}\, d\rho \right) \csc(\theta) \delta A_\varphi \partial_\theta A_\varphi \right]_{\theta=0}^{\theta=\pi} + 2\pi\xi^A \int_0^\pi \delta p^\rho d\theta \Bigg]_{\rho\to\infty} \end{aligned} \tag{8.30}$$

Here, we have applied the definition of the deformation vectors $\xi^\perp$ and ξ^i in terms of the Killing vectors ξ^t and $\bar{\xi}^i$, which read

$$\xi^\perp = N^\perp \xi^t, \tag{8.31}$$

$$\xi^i = \bar{\xi}^i + N^i \xi^t. \tag{8.32}$$

In order to compute and perform a proper analysis of the charges, we must give suitable asymptotic conditions that determine the behavior of the fields at infinity. These conditions are specified up to the orders that contribute to the charges, such that

$$F(\rho) = g^2\rho^2 + 1 + F_0 + \frac{F_1}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right), \tag{8.33}$$

$$N^\perp(\rho) = g\rho + \mathcal{O}\left(\frac{1}{\rho}\right), \tag{8.34}$$

$$\phi(\rho) = \frac{\phi_1}{\rho} + \frac{\phi_2}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^3}\right), \tag{8.35}$$

$$p^\rho(\rho, \theta) = p_0 \sin(\theta) + \mathcal{O}\left(\frac{1}{\rho^1}\right), \tag{8.36}$$

$$\xi^A = \xi_0^A + \mathcal{O}\left(\frac{1}{\rho}\right). \tag{8.37}$$

The coefficients in the expansions given above are parameters that depend on the integration constants of the corresponding solution. The variation of the charge obtained after inserting the proposed asymptotic behavior in (8.30) is given by[2]

[2] It has to be noted that a divergent term appears in the variation of the charge but it vanishes once it is evaluated on the solution. This is because the divergent part of the gravitational contribution is cancelled by the divergent part of the scalar field contribution by virtue of the relation $\delta F_0 = \frac{g^2}{2}\phi_1\delta\phi_1$.

$$\delta Q = \xi^t \left[-8\pi\delta F_1 + 4\pi g^2 \left(2\phi_2\delta\phi_1 + \phi_1\delta\phi_2 \right) \right] + 4\pi\xi_0^A \delta p_0. \tag{8.38}$$

The mass is the conserved charge associated to time translations, which in this approach is obtained from $\delta M = \delta Q\left[\xi^t\right]$, while the electric charge is the charge associated to the Abelian gauge transformations, where $\delta Q_e = \delta Q\left[\xi^A\right]$. Then, the variations of the mass and the electric charge read

$$\delta M = -8\pi\delta F_1 + 4\pi g^2 \left(2\phi_2\delta\phi_1 + \phi_1\delta\phi_2 \right), \tag{8.39}$$

$$\delta Q_e = 4\pi\delta p_0. \tag{8.40}$$

The electric charge can be directly integrated for the AdS_4 dyonic black hole, which in terms of the integration constants of the solution is written as

$$Q_e = \frac{16\pi\sqrt{2}\mu\sqrt{\beta_1\gamma_2}}{\gamma_1}. \tag{8.41}$$

In contrast, the mass is generically non-integrable and its variation is explicitly given by

$$\delta M = \delta\left(\frac{16\pi\left(1+\beta_1\right)\left(1-\beta_2\right)\left(1-\beta_1\beta_2\right)\mu}{\gamma_1\gamma_2} + \frac{64\pi g^2\mu^3\left(1-\beta_1\beta_2\right)\left(\beta_1-\beta_2\right)^2\gamma}{\gamma_1^3\gamma_2^3} \right) + \Phi, \tag{8.42}$$

with

$$\gamma = \beta_1 + \beta_2 - 8\beta_1\beta_2 + 6\beta_1^2\beta_2 + 6\beta_1\beta_2^2 - 8\beta_1^2\beta_2^2 + \beta_1^3\beta_2^2 + \beta_1^2\beta_2^2\,. \tag{8.43}$$

Note that the variation of the mass coincides with the one computed in [73], which has the non-integrable term Φ that comes from the scalar field part of the energy density. This term is given by

$$\Phi = 4\pi g^2 \left(2\phi_2\delta\phi_1 + \phi_1\delta\phi_2 \right), \tag{8.44}$$

where the leading and subleading terms of the scalar field fall-off are respectively

$$\phi_1 = \frac{2\sqrt{3}\left(\beta_2\left(1+\beta_1{}^2\right) - \beta_1\left(1+\beta_2{}^2\right)\right)\mu}{\gamma_1\gamma_2}, \tag{8.45}$$

$$\begin{aligned} \phi_2 = & \frac{2\sqrt{3}\left(-\beta_2^2\left(1-\beta_1^4\right) - 2\beta_1\beta_2^2(-4+3\beta_2)\right)\mu^2}{\gamma_1^2\gamma_2^2} \\ & + \frac{2\sqrt{3}\left(-2\beta_1^3\beta_2(-3+4\beta_2) - \beta_1{}^2\left(-1+8\beta_2-8\beta_2^3+\beta_2^4\right)\right)\mu^2}{\gamma_1^2\gamma_2^2}. \end{aligned} \tag{8.46}$$

The presence of a non-integrable term Φ in the variation of the mass (8.39) forces us to impose relations among the fall-off coefficients of the scalar field. If the variations are treated as exterior derivatives, the condition $\delta^2 M = 0$ is a sufficient condition to ensure the existence of M. Indeed, this condition is equivalent to requiring that the second derivatives of the functional M with respect to the integration constants commute. Then,

$$\begin{aligned} \delta^2 M &= \delta\Phi && (8.47)\\ &= 4\pi g^2 \left(2\delta\phi_2 \wedge \delta\phi_1 + \delta\phi_1 \wedge \delta\phi_2\right) && (8.48)\\ &= 4\pi g^2 \delta\phi_2 \wedge \delta\phi_1 \equiv 0\,. && (8.49) \end{aligned}$$

This implies the functional relation $\phi_2 = \phi_2(\phi_1)$. Hence, the mass generically takes the form

$$M = -8\pi F_1 + 4\pi g^2 \int \left(2\phi_2 + \phi_1 \frac{d\phi_2}{d\phi_1}\right) d\phi_1. \tag{8.50}$$

At this point it is necessary to impose a boundary condition that fixes a precise relation between the leading and subleading terms of the scalar field behavior at infinity. One possible condition is to demand preservation of the AdS symmetry of the scalar field's asymptotic fall-off, which can be done since the AdS_4 dyonic dilatonic black hole of [33] is within the asymptotic conditions for AdS spacetimes analyzed in [37], [49], [29]. These references construct a set of boundary conditions for having well-defined and finite Hamiltonian generators for all the elements of the AdS algebra in the case of gravity minimally coupled to scalar fields. We are allowed to impose certain relations on the leading and subleading terms of the scalar field fall-off provided the scalar field does not break the AdS symmetry at infinity. These boundary conditions are $(\phi_1 = 0, \quad \phi_2 \neq 0)$, $(\phi_1 \neq 0, \quad \phi_2 = 0)$ and $\phi_2 = c\phi_1^2$, where c is not allowed to vary. In terms of the integration constants the relation $\phi_2 = c\phi_1^2$ becomes

$$\begin{aligned} -2(\beta_1-\beta_2)\mu^2 \Big[&-\left(\sqrt{3}+6c\right)\beta_2 - \left(\sqrt{3}-6c\right){\beta_1}^3{\beta_2}^2 \\ &-\beta_1\left(\sqrt{3}-6c-8\sqrt{3}\beta_2+6\left(\sqrt{3}-2c\right){\beta_2}^2\right) \\ &-{\beta_1}^2\beta_2\left(6\left(\sqrt{3}+2c\right)-8\sqrt{3}\beta_2+\left(\sqrt{3}+6c\right){\beta_2}^2\right)\Big] = 0. \end{aligned} \tag{8.51}$$

From eq. (8.51) we observe three cases, two of them being nontrivial. When $\mu = 0$ the mass, the electric charge and the magnetic charge vanish giving rise to the vacuum solution which turns out to be AdS_4 spacetime. The other two cases imply that $\beta_1 = \beta_1(\beta_2)$ in such a way, that they force the terms in (8.42) that are proportional to g^2 to vanish. Hence, the mass becomes the AMD mass [81], [82] obtained in [33],

$$M = \frac{16\pi(1-\beta_1)(1-\beta_2)(1-\beta_1\beta_2)\mu}{\gamma_1\gamma_2}. \tag{8.52}$$

This fact is in agreement with [60], where it was pointed out that some holographic prescriptions are suitable for computing the mass for hairy spacetimes when the scalar field respects the AdS invariance at infinity. In this context, different kinds of boundary conditions were considered in [83], [84], [22].

8.3 Thermodynamics of the AdS$_4$ dyonic black hole

The thermodynamic analysis of the AdS$_4$ dyonic dilatonic black hole is performed in this section. We define the Euclidean Hamiltonian action of the theory including a surface term and an additional polar boundary term to have a finite action principle. The presence of the latter is due to the existence of a magnetic monopole in the solution. For simplicity, we take a minisuperspace in which the AdS$_4$ dyonic black hole is included. The variation of the Euclidean Hamiltonian action is computed in the grand canonical ensemble, where the chemical potentials are fixed. Remarkably, the magnetic charge emerges from the additional term accompanied by its respective chemical potential. The value of the temperature and the electric potential, on the other hand, are fixed by imposing regularity conditions. When the variations of the additional surface and polar boundary terms are determined, as was mentioned above, integrability conditions are needed to be imposed to determine the value of the Euclidean Hamiltonian action leading to the Gibbs free energy.

8.3.1 Hamiltonian action and Euclidean minisuperspace

Let us consider spacetimes with a manifold of topology $\mathbb{R}^2 \times S^2$. The plane $\mathbb{R}^2$ is centered at the event horizon r_+ and is parametrized by the periodic Euclidean time τ and the radial coordinate r. These plane coordinates range as

$$0 \leq \tau < \beta, \tag{8.53}$$

$$r_+ \leq r < \infty, \tag{8.54}$$

with β the inverse of the Hawking temperature and the 2-sphere S^2 stands for the topology of the base manifold. The Hamiltonian Euclidean action for the system is given by

$$I^E = \int\limits_0^\beta d\tau \int\limits_\Sigma d^3x \left[\dot{\gamma}_{ij}\pi^{ij} + \dot{A}_i\pi^i + \dot{\phi}\pi_\phi - \left(N^\perp \mathcal{H}_\perp + N^i\mathcal{H}_i - A_\tau \mathcal{G}\right)\right] + B, \tag{8.55}$$

where $\Sigma = \mathbb{R} \times S^2$ is the spatial section of the manifold. Note that the additional term B in (8.55) needs to be added to the action in order to have a well-defined variational principle, and it is crucial for determining the value of the action for stationary configurations.

The Euclidean continuation of the AdS_4 dyonic black hole (8.8) is considered. The line element reads

$$ds^2 = N^{\perp}(r)^2 d\tau^2 + \frac{dr^2}{F(r)} + H(r)\left(d\theta^2 + \sin^2(\theta)\, d\varphi^2\right), \tag{8.56}$$

where the gauge field ansatz and the scalar field are given by

$$A = A_\tau(r)\, d\tau + A_\varphi(\theta)\, d\varphi, \tag{8.57}$$

$$\phi = \phi(r). \tag{8.58}$$

The radial component of the electromagnetic field momentum is $\pi^r = p^r(r,\theta)$ (all the other momenta of the fields vanish). Hence, it is possible to obtain the following reduced action

$$I^E = -2\pi\beta \int_{r_+}^{\infty} dr \int_0^{\pi} d\theta \left(N^{\perp}(r)\,\mathcal{H}_\perp - A_\tau(r)\,\mathcal{G}\right) + B, \tag{8.59}$$

from (8.55), where the reduced constraints take the form

$$\begin{aligned}
\mathcal{H}_\perp \;&=\; -\frac{e^{-\sqrt{3}\phi}\sin(\theta)}{2\sqrt{FH}}\Big[-\csc^2(\theta)\left(\partial_\theta A_\varphi\right)^2 - 2e^{\sqrt{3}\phi}H\left(\partial_r F\partial_r H + 2F\partial_r^2 H - 2\right) \qquad (8.60)\\
&+ e^{\sqrt{3}\phi}H^2\left(12g^2\cosh\left(\frac{\phi}{\sqrt{3}}\right) - F\left(\partial_r\phi\right)^2\right)\\
&e^{\sqrt{3}\phi}F\left(\partial_r H\right)^2 + \csc^2(\theta)e^{2\sqrt{3}\phi}\left(p^r\right)^2\Big], \qquad (8.61)\\
\mathcal{G} \;&=\; \partial_r p^r. \qquad (8.62)
\end{aligned}$$

The variation of the reduced action (8.59) with respect to the Lagrange multipliers $N^\perp$ and A_τ indicates that the constraints have to vanish

$$\mathcal{H}_\perp = 0, \qquad \mathcal{G} = 0. \tag{8.63}$$

These equations define the constraint surface. On the other hand, the variation of (8.59) with respect to the independent functions of the dynamical fields in the minisuperspace leads to the field equations. The field equations related to $F(r)$ and

$H(r)$ are given by

$$\frac{e^{-\sqrt{3}\phi}\sin(\theta)}{4F^{3/2}H}\left(N^{\perp}\left(-\csc^{2}(\theta)\left(\partial_{\theta}A_{\varphi}\right)^{2}-F\left(\partial_{r}H\right)^{2}e^{\sqrt{3}\phi}+\csc^{2}(\theta)e^{2\sqrt{3}\phi}\left(p^{r}\right)^{2}\right)\right.$$
$$\left.+H^{2}N^{\perp}e^{\sqrt{3}\phi}\left(F\left(\partial_{r}\phi\right)^{2}+12g^{2}\cosh\left(\frac{\phi}{\sqrt{3}}\right)\right)+4He^{\sqrt{3}\phi}\left(N^{\perp}-F\partial_{r}H\partial_{r}N^{\perp}\right)\right)=0, \tag{8.64}$$

$$\frac{e^{-\sqrt{3}\phi}\sin(\theta)}{2\sqrt{F}H^{2}}\left(N^{\perp}\left(-\csc^{2}(\theta)\left(\partial_{\theta}A_{\varphi}\right)^{2}-F\left(\partial_{r}H\right)^{2}e^{\sqrt{3}\phi}+\csc^{2}(\theta)e^{2\sqrt{3}\phi}\left(p^{r}\right)^{2}\right)\right.$$
$$-H^{2}e^{\sqrt{3}\phi}\left(N^{\perp}\left(12g^{2}\cosh\left(\frac{\phi}{\sqrt{3}}\right)-F\left(\partial_{r}\phi\right)^{2}\right)-2\left(\partial_{r}F\partial_{r}N^{\perp}+2F\partial_{r}^{2}N^{\perp}\right)\right)$$
$$\left.+He^{\sqrt{3}\phi}\left(N^{\perp}\left(\partial_{r}F\partial_{r}H+2F\partial_{r}^{2}H\right)+2F\partial_{r}H\partial_{r}N^{\perp}\right)\right)=0, \tag{8.65}$$

respectively. The field equations associated to $A_{\varphi}(r,\theta)$ and $p^{r}(r,\theta)$ are

$$\frac{N^{\perp}e^{-\sqrt{3}\phi}\csc(\theta)}{\sqrt{F(r,s)}H(r,s)}\left(\partial_{\theta}A_{\varphi}\cot(\theta)-\partial_{\theta}^{2}A_{\varphi}\right)=0,\qquad \partial_{r}A_{\tau}+\frac{\csc(\theta)N^{\perp}e^{\sqrt{3}\phi}p^{r}}{\sqrt{F}H}=0, \tag{8.66}$$

and finally the scalar field equation reads

$$-\frac{e^{-\sqrt{3}\phi}\sin(\theta)}{2\sqrt{F}H}\left(N^{\perp}\left(\sqrt{3}\csc^{2}(\theta)\left(\partial_{\theta}A_{\varphi}\right)^{2}+2FHe^{\sqrt{3}\phi}\partial_{r}H\partial_{r}\phi\right.\right.$$
$$+H^{2}e^{\sqrt{3}\phi}\left(\partial_{r}F\partial_{r}\phi+2F\partial_{r}^{2}\phi+4\sqrt{3}g^{2}\sinh\left(\frac{\phi}{\sqrt{3}}\right)\right)$$
$$\left.\left.+\sqrt{3}\csc^{2}(\theta)e^{2\sqrt{3}\phi}\left(p^{r}\right)^{2}\right)+2FH^{2}e^{\sqrt{3}\phi}\partial_{r}\phi\partial_{r}N^{\perp}\right)=0. \tag{8.67}$$

Then, the variation of the reduced action (8.59) on the constraint surface, evaluated on-shell (i.e. eqs. (8.64) to (8.67) have to be satisfied), becomes

$$\begin{aligned}\delta I^{E}\Big|_{on-shell} &= -2\pi\beta\int_{0}^{\pi}d\theta\left[N^{\perp}\sin(\theta)\left(\frac{\partial_{r}H\delta F+\partial_{r}F\delta H}{\sqrt{F}}-\frac{\sqrt{F}\partial_{r}H\delta H}{H}\right.\right.\\ &\quad\left.\left.+\frac{\sqrt{F}H\partial_{r}\phi\delta\phi}{2}+2\sqrt{F}\partial_{r}\delta H\right)-\partial_{r}\left(2N^{\perp}\sin(\theta)\sqrt{F}\right)\delta H-A_{\tau}\delta p^{r}\right]_{r_{+}}^{\infty}\\ &\quad-2\pi\beta\int_{r+}^{\infty}dr\left[\frac{N^{\perp}e^{-\sqrt{3}\phi}}{H\sqrt{F}\sin\theta}\partial_{\theta}A_{\varphi}\delta A_{\varphi}\right]_{0}^{\pi}+\delta B.\end{aligned} \tag{8.68}$$

If we demand that the action has an extremum, i.e., $\delta I^E\Big|_{on-shell} = 0$, the variation of the additional term δB must necessarily be given by

$$\begin{aligned}\delta B &= 2\pi\beta\int_0^\pi d\theta\left[N^\perp \sin(\theta)\left(\frac{\partial_r H\delta F + \partial_r F\delta H}{\sqrt{F}} - \frac{\sqrt{F}\partial_r H\delta H}{H}\right.\right.\\&\left.\left.+\frac{\sqrt{F}H\partial_r\phi\delta\phi}{2} + 2\sqrt{F}\partial_r\delta H\right) - \partial_r\left(2N^\perp\sin(\theta)\sqrt{F}\right)\delta H - A_\tau\delta p^r\right]_{r_+}^{\infty}\\&+2\pi\beta\int_{r+}^{\infty} dr\left[\frac{N^\perp e^{-\sqrt{3}\phi}}{H\sqrt{F}\sin(\theta)}\partial_\theta A_\varphi\delta A_\varphi\right]_0^\pi .\end{aligned}\tag{8.69}$$

It is possible to recognize two kinds of terms in this expression. The surface term comes from a total derivative in the radial coordinate and a boundary term that comes from a total derivative in the polar angle. The latter is clearly not vanishing because of the presence of an angular component depending on the polar angle in the gauge field. The analysis of the variation of the term B and the evaluation on the AdS_4 dyonic black hole (8.69) will be performed in the following subsection.

8.3.2 Gibbs free energy and first law

From (8.69) we can identify different contributions, depending on whether the term comes from a total derivative in the radial coordinate, or whether the term comes from a total derivative in the polar angle, which will be identified as a polar boundary term. The surface term evaluated at infinity will be denoted by $\delta B(\infty)$ while $\delta B(r_+)$ will stand for the surface term at the horizon. The polar boundary term will be denoted by δB_θ. Hence, the variation of B, see (8.69), can be written as

$$\delta B = \delta B(\infty) + \delta B(r_+) + \delta B_\theta, \tag{8.70}$$

where the surface term at infinity is given by

$$\begin{aligned}\delta B(\infty) &= 2\pi\beta\int_0^\pi d\theta\left[N^\perp \sin(\theta)\left(\frac{\partial_r H\delta F + \partial_r F\delta H}{\sqrt{F}} - \frac{\sqrt{F}\partial_r H\delta H}{H}\right.\right.\\&\left.\left.+\frac{\sqrt{F}H\partial_r\phi\delta\phi}{2} + 2\sqrt{F}\partial_r\delta H\right) - \partial_r\left(2N^\perp\sin(\theta)\sqrt{F}\right)\delta H - A_\tau\delta p^r\right]_{\infty},\end{aligned}\tag{8.71}$$

the surface term at the horizon is

$$
\begin{aligned}
\delta B(r_+) &= -2\pi\beta \int_0^\pi d\theta \left[N^\perp \sin(\theta) \left(\frac{\partial_r H \delta F + \partial_r F \delta H}{\sqrt{F}} - \frac{\sqrt{F}\partial_r H \delta H}{H} \right.\right. \\
&\quad \left.\left. + \frac{\sqrt{F} H \partial_r \phi \delta\phi}{2} + 2\sqrt{F}\partial_r \delta H \right) - \partial_r \left(2N^\perp \sin(\theta)\sqrt{F} \right) \delta H - A_\tau \delta p^r \right]_{r_+} ,
\end{aligned}
\tag{8.72}
$$

and the polar boundary term reads

$$
\delta B_\theta = 2\pi\beta \int_{r+}^\infty dr \left[\frac{N^\perp e^{-\sqrt{3}\phi}}{H\sqrt{F}\sin(\theta)} \partial_\theta A_\varphi \delta A_\varphi \right]_0^\pi . \tag{8.73}
$$

Once the different contributions to the variation of B are identified one can analyze their physical content. It is possible to find the variation of the charges coming from symmetries of the action together with their respective chemical potentials from the surface term at infinity $\delta B(\infty)$. The chemical potentials correspond to the Lagrange multipliers of the respective symmetry at infinity (as was shown in Section 8.2). This is because at the end of the day the term (8.71) is obtained from the boundary term of the Hamiltonian, which ensures that the canonical generators have well-defined functional derivatives [42]. The variations of the mass and the electric charge of the AdS_4 dyonic dilatonic black hole will be identified from $\delta B(\infty)$. The entropy of the black hole, which corresponds to the Bekenstein-Hawking entropy, will be obtained from the surface term at the horizon $\delta B(r_+)$. Finally, the contribution of the topological charge of the system, leading to the variation of the magnetic charge multiplied by the magnetic potential, can be identified from the polar boundary term δB_θ .

Let us introduce the Euclidean continuation of the AdS_4 dyonic dilatonic black hole that satisfies the field equations (8.64)-(8.67) and the constraints (8.63). This is obtained after performing the identifications $t \to -i\tau$ and $\beta_1 \to -\beta_1$ in the Lorentzian solution. Then the black hole functions take the form

$$
H_1(r) = \gamma_1^{-1} \left(1 + 2\beta_1 f_0(r) - \beta_1\beta_2 f_0(r)^2 \right), \tag{8.74}
$$

$$
H_2(r) = \gamma_2^{-1} \left(1 - 2\beta_2 f_0(r) - \beta_1\beta_2 f_0(r)^2 \right), \tag{8.75}
$$

where $\gamma_1 = 1 + 2\beta_1 - \beta_1\beta_2$ and $\gamma_2 = 1 - 2\beta_2 - \beta_1\beta_2$. The functions $F(r)$ and $H(r)$ in the line element (8.56) are

$$
F(r) = \frac{f(r)}{\sqrt{H_1(r) H_2(r)}}, \qquad H(r) = \sqrt{H_1(r) H_2(r)} r^2, \tag{8.76}
$$

where the function $f(r)$ is the same as the one given in (8.9). The lapse function is $N^{\perp}(r) = \sqrt{F(r)}$. The scalar field is defined in (8.12) and the temporal component of the gauge field is given by

$$A_{\tau}(r) = -\frac{\sqrt{2}\left(1 - H_1(r) + \beta_1\left(f_0(r) - H_1(r)\right)\right)}{\sqrt{\beta_1\gamma_2}H_1(r)} + \Phi_e. \tag{8.77}$$

Note that the possibility of adding a constant Φ_e allows one to have a regular gauge field at the horizon. This constant is related to the electrostatic potential of the solution when the regularity conditions on the black hole horizon are established. The angular component of the gauge field takes the same definition as given in (8.15).

Inserting the Euclidean continuation of the AdS_4 dyonic dilatonic black hole in the surface term at infinity $\delta B(\infty)$, given in eq. (8.71), we get

$$\delta B(\infty) = -\beta\delta M - \beta\Phi_e\delta Q_e, \tag{8.78}$$

where the variations of the mass and the electric charge are given by

$$\delta M = \delta\left(\frac{16\pi\left(1+\beta_1\right)\left(1-\beta_2\right)\left(1+\beta_1\beta_2\right)\mu}{\gamma_1\gamma_2}\right) + \Theta, \tag{8.79}$$

$$\delta Q_e = 4\pi\delta\left(\frac{2\sqrt{2}\mu\sqrt{\beta_1\gamma_2}}{\gamma_1}\right). \tag{8.80}$$

The above variations coincide with the values computed in (8.42) and (8.41). In the variation of the mass we clearly obtain a contribution

$$\begin{aligned} \Theta &= \frac{64\pi g^2\mu^3\left(1+\beta_1\beta_2\right)\left(\beta_1+\beta_2\right)^2\gamma}{\gamma_1^3\gamma_2^3} + \Phi^E \\ &= -\frac{32\pi g^2\mu^3\left(\beta_1+\beta_2\right)}{\gamma_1^2\gamma_2^2}\left(\beta_2\left(1-2\beta_1-2\beta_2+\beta_1\beta_2\right)\delta\beta_1\right., \\ &\quad \left.-\beta_1\left(1+2\beta_1+2\beta_2+\beta_1\beta_2\right)\delta\beta_2\right) \end{aligned} \tag{8.81}$$

where Φ^E is the Euclidean continuation of Φ. Here Θ corresponds to the new scalar charge term in the context of [33].

The inverse of the temperature β and the electrostatic potential Φ_e are determined through the regularity conditions at the horizon. Indeed, we find

$$\beta = \frac{4\pi\sqrt{H_1(r_+)H_2(r_+)}}{f'(r_+)}, \qquad \Phi_e = -\sqrt{\frac{2}{\beta_1\gamma_2}}\left(1+\beta_1 - \frac{1+\beta_1 f_0(r_+)}{H_1(r_+)}\right). \tag{8.82}$$

The value of the temperature is obtained by demanding absence of conical singularities around the event horizon, while the electrostatic potential comes from the trivial holonomy condition of the gauge field around a temporal cycle on the plane $r-\tau$ at the event horizon. Inserting the values of the chemical potentials (8.82) into the surface term at the horizon $\delta B(r_+)$, we get that this term exactly coincides with the Bekenstein-Hawking entropy

$$\delta B(r_+) = \delta\left(16\pi^2\sqrt{H_1(r_+)H_2(r_+)}r_+^2\right) = \delta S. \tag{8.83}$$

The polar boundary term δB_θ has to be carefully computed using the definition of the angular component of the gauge field given in (8.15). Then,

$$\begin{aligned}
\delta B_\theta &= 2\pi\beta\left(\int_{r+}^{\infty} dr\frac{e^{-\sqrt{3}\phi}}{H}\right)\left(\left[\frac{\partial_\theta A_\varphi \delta A_\varphi}{\sin(\theta)}\right]_0^{\pi/2-\delta} + \left[\frac{\partial_\theta A_\varphi \delta A_\varphi}{\sin(\theta)}\right]_{\pi/2+\delta}^{\pi}\right)_{\delta\to 0} \\
&= -2\pi\beta\left(\int_{r+}^{\infty} dr\frac{e^{-\sqrt{3}\phi}}{H}\right)\left([p\delta p(1+\cos(\theta))]_0^{\pi/2-\delta} + [p\delta p(-1+\cos(\theta))]_{\pi/2+\delta}^{\pi}\right)_{\delta\to 0} \\
&= 4\pi\beta\left(\int_{r+}^{\infty} dr\frac{e^{-\sqrt{3}\phi}}{H}\right)p\delta p.
\end{aligned} \tag{8.84}$$

This term can be conveniently written as

$$\delta B_\theta = -\beta\Phi_m\delta Q_m, \tag{8.85}$$

where we can identify the magnetic potential

$$\Phi_m = -\sqrt{\frac{2}{\beta_2\gamma_1}}\left(1-\beta_2-\frac{1-\beta_2 f_0(r_+)}{H_2(r_+)}\right), \tag{8.86}$$

and also the value of variation of the magnetic charge

$$\delta Q_m = 4\pi\delta\left(\frac{2\sqrt{2}\mu\sqrt{\beta_2\gamma_1}}{\gamma_2}\right). \tag{8.87}$$

As a consequence, the variation of the boundary term B is given by

$$\delta B = \delta S - \beta\delta M - \beta\Phi_e\delta Q_e - \beta\Phi_m\delta Q_m. \tag{8.88}$$

Note that once this term is integrated, the value of B corresponds to the Euclidean Hamiltonian action I^E evaluated on stationary configurations and on the constraint surface. In the grand canonical ensemble I^E is related to the Gibbs free energy by $I^E = -\beta G$. It is also worth to point out that since the first law of black hole thermodynamics,

$$\delta M = T\delta S - \Phi_e\delta Q_e - \Phi_m\delta Q_m, \tag{8.89}$$

is a consequence of the Euclidean action having an extremum, (8.89) is identically satisfied independently of the boundary conditions on the mass. This is because (8.89) is a relation that only involves the variation of the conserved charges. This can be shown explicitly by introducing the value for the charge variations (8.79), (8.80), (8.87) and the chemical potentials obtained by using the regularity conditions (8.82) into (8.89).

Once the mass is integrated using arbitrary boundary conditions (see Section 8.2), it is possible to find the value of the Gibbs free energy which is equivalent to the Euclidean Hamiltonian action evaluated on-shell,

$$I^E = S - \beta M - \beta \Phi_e Q_e - \beta \Phi_m Q_m. \tag{8.90}$$

Recalling that we have chosen the grand canonical ensemble and taking the Euclidean action as our thermodynamic potential, the values of the extensive quantities, the mass, the electric charge, the magnetic charge and the entropy are obtained through the following thermodynamic relations

$$M = -\left(\frac{\partial I^E}{\partial \beta}\right)_{\Phi_e,\Phi_m} + \frac{\Phi_e}{\beta}\left(\frac{\partial I^E}{\partial \Phi_e}\right)_{\beta,\Phi_m} + \frac{\Phi_m}{\beta}\left(\frac{\partial I^E}{\partial \Phi_m}\right)_{\beta,\Phi_e}, \tag{8.91}$$

$$Q_e = -\frac{1}{\beta}\left(\frac{\partial I^E}{\partial \Phi_e}\right)_{\beta,\Phi_m}, \tag{8.92}$$

$$Q_m = -\frac{1}{\beta}\left(\frac{\partial I^E}{\partial \Phi_m}\right)_{\beta,\Phi_e}, \tag{8.93}$$

$$S = I_E - \beta\left(\frac{\partial I^E}{\partial \beta}\right)_{\Phi_e,\Phi_m}. \tag{8.94}$$

The values of the charges and the entropy computed above coincide with (8.50), (8.41), (8.87) and (8.83), respectively.

Chapter 9

Conclusions

Throughout this thesis, we have studied the interaction among matter fields and gravity from a thermodynamic point of view. In particular, we explored the presence of a scalar field and a gauge field, coupled to gravity in different scenarios.

In Chapter 5, we have obtained exact, circularly symmetric, three-dimensional black holes, which are regular on and outside their event horizons, endowed with conformally coupled scalar and gauge fields. The black holes are described by means of very simple expressions, even in the presence of a self-interaction potential compatible with the conformal invariance. For this reason, their geometries and thermodynamic properties can be easily explored, and consequently, the physical meaning of them becomes clear. In general, the integration of the field equations provides two arbitrary constants which parametrize the solutions in conjunction with the self-interaction coupling constant. The black holes can be classified in three groups. The first group, discussed in Sec. 5.1.1, includes those with a stealth composite matter source, where the contributions of both fields to the energy-momentum tensor cancel out. The case in which the three parameters do not vanish defines the second group treated in Sec. 5.1.2. Here two black holes appear, one with a single horizon, and another one having an inner horizon, which can not become extreme, keeping a nontrivial scalar field. The third group is defined by the absence of the self-interaction potential (Sec. 5.1.3). This class contains the electrically charged version of the black hole found in [15]. Additionally, an extreme black hole emerges if the condition of regularity for the fields at the horizon is removed.

It is worth noting that the asymptotic behavior of the metrics satisfies the Brown-Henneaux asymptotic conditions even in the case of a nontrivial scalar and gauge fields. This means that these are asymptotically AdS spacetimes. However, the entire configuration is endowed with the asymptotic AdS invariance only if the scalar field allows it.

The conserved charges, mass and the electric charge, were determined under the

Regge-Teitelboim approach. It was found that boundary conditions on the leading and sub-leading terms of the asymptotic form of the scalar field are necessary in order to obtain the mass. This fact is in accordance with the physical statement which says that the mass is well defined after boundary conditions are imposed.

Remarkably, the scalar fields presented in sections 5.1.2 and 5.1.3 have an asymptotic behavior allowing to analyze a wide class of boundary conditions, even including those that break the asymptotic AdS symmetry. This is possible because the scalar field contains two independent integration constants unlike other exact solutions as far we know, which are defined with only one integration constant and hence no other boundary condition is required. These black holes could be considered in the context of the so-called Designer Gravity theories [61], in which general boundary conditions were numerically studied. However, since the black holes shown here are exact solutions, these could be very useful for those models.

The temperature of the black holes is a monotonically increasing function of the horizon radius r_+, which approaches the linear one for large r_+ as it happens in general for the AdS black holes. On the other hand, the factor appearing in the modified entropy area law is not necessarily positive definite. Hence the positiveness of the entropy requires extra conditions on the integration constants and the coupling parameter α. We note that for a large enough negative coupling constant the entropy is positive without other conditions apart of those necessary for the existence of black holes. Since in the Einstein frame the entropy is a positive definite quantity, one may think that negative entropy configurations could have a well-defined thermodynamic description in that frame as well. However, this class of solutions are mapped to naked singularities in the Einstein frame. It is worth pointing out the exact correspondence between the the positiveness of the entropy in the conformal frame and the cosmic censorship principle in the Einstein frame. The black holes in the Einstein frame, and their geometrical and thermodynamic properties deserve further attention and they are interesting enough as to be considered in a future work. Finally, in three dimensions adding angular momentum is not a difficult task, and it would be interesting to study the spinning versions of the black holes introduced here.

In Chapter 7, we have discussed how an action in terms of one-forms naturally arises after the matter action, given by a conformally coupled scalar field with a self-interaction $\lambda\phi^6$ –that does not spoil the conformal invariance– is mapped to pure gravity by means of conformal transformation involving the scalar field. The resulting gravity action has a negative gravitational constant and a cosmological constant -8λ. This leads to a Chern-Simons-like description for this system. This is possible since the metric can be written in terms of gauge connections. In this context, a new hairy black hole was introduced, which corresponds to a rotating solution. The thermodynamics of this black hole was analyzed in the grand canonical ensemble, where the free energy, defined as a surface term, was derived from the Euclidean version of the Chern-Simons-like action. The advantage of this formulation comes for the simplicity of this surface term in comparison with the one associated to second-order metric formalism. The regularity conditions were imposed on the holonomies along the thermal cycle of the torus at event horizon, fixing the chemical potentials of the solutions. The entropy was derived in two ways; the first one from the Gibbs free energy and the second one from a general formula for the entropy in terms for a Chern-Simons theory. Both gave the same result compared with the modified area law. We showed how this aproach is also valid for gravity with a minimally coupled scalar field. An interesting problem related to this work is to analyze the dynamical structure of the proposed action in terms of one-forms . The canonical formulation would shed light about its symmetries and propagating degrees of freedom. Moreover, it would be also worth to explore a different three-dimensional gravity action, for instance, the so-called new massive gravity with a conformally coupled scalar field.

In Chapter 8, we have carried out the thermodynamic analysis of a new class of AdS_4 dyonic dilatonic black holes recently proposed in [33], which are solutions of the bosonic sector of a Kaluza-Klein reduction of eleven-dimensional supergravity. The conserved Noether charges were computed using the Regge-Teitelboim Hamiltonian approach. These correspond to the mass, which acquires contributions from the scalar field and the electric charge. It was also shown that the mass acquires non-integrable contributions from the scalar field, in which case it was necessary to impose integrability conditions to have a definite mass. These conditions are generically solved by imposing boundary conditions that relate the leading and subleading terms of the scalar field fall-off. A possible physical condition to establish the arbitrary functions coming from the integrability condition is to preserve the AdS symmetry of the scalar field behavior at infinity as was established in [16], [37], [49], [29]. The Hamiltonian Euclidean action was computed by demanding that the action has an extremum, where its value was given by the corresponding radial boundary term plus an additional term, because of the presence of a magnetic monopole. The computation was performed in the grand canonical ensemble. The conserved charges were identified from the thermodynamic analysis. The Noether charges, the mass and the electric charge, were obtained from the radial boundary term at infinity, unlike the magnetic charge. The latter one comes from the additional term. Remarkably, the magnetic potential appeared already in the variation of the boundary term, unlike the chemical potentials associated to the Noether charges which are the Lagrange multipliers of the system at infinity. They are obtained by imposing regularity conditions at the horizon. Considering the above, it is possible to verify that the first law of black hole thermodynamics is identically satisfied. This is a consequence of having a well-defined and finite Hamiltonian action principle.

A different way to deal with the thermodynamics of dyonic black holes is to consider a manifestly duality invariant action that involves two U(1) symmetries, producing the appearance of electric and magnetic Gauss constraints [86]. The dyonic Reissner-Nordström black hole is a solution of the system proposed in [86], however in that case the magnetic and the electric fields appear as Coulomb potentials, hence the solution is devoid of stringy singularities. In this case, all the conserved charges that appear in the first law come from symmetries of the action.

It would be interesting to analyze the existence of phase transitions between the dyonic dilatonic black hole solution and the dyonic Reissner-Nordström black hole, i.e. studying the probability that below a critical temperature the dyonic Reissner-Nordström black hole spontaneously changes to a state that is dressed with a dilaton scalar field. This kind of results have been reproduced, for instance, in the case of four-dimensional topological black holes dressed with a scalar field in [87].

Appendices

Chapter 10

Appendix A: Action and equations of motion: from the action in term of one forms to the second-order formulation

In this appendix we show how the action

$$I^{(1)}\left[e,\phi\right]=\frac{1}{2\kappa}\int\left(2R^{a}e_{a}-\frac{\Lambda}{3}\epsilon_{abc}e^{a}e^{b}e^{c}\right)+\frac{1}{2\bar{\kappa}}\int\left(2\bar{R}^{a}\bar{e}_{a}-\frac{\bar{\Lambda}}{3}\epsilon_{abc}\bar{e}^{a}\bar{e}^{b}\bar{e}^{c}\right) \quad (10.1)$$

can be mapped to (7.2). The first part of the above action is mapped to

$$\frac{1}{2\kappa}\int d^{3}x\sqrt{-g}\left(R-2\Lambda\right)\,, \quad (10.2)$$

just using the following relations

$$\epsilon_{abc}e_{\mu}^{a}e_{\nu}^{b}e_{\rho}^{c}=e\epsilon_{\mu\nu\rho},\quad R^{a}{}_{\mu\nu}\left(\omega\right)=\frac{1}{2}\epsilon^{a}{}_{bc}e_{\lambda}^{b}e_{\rho}^{c}R^{\lambda\rho}{}_{\mu\nu}\left(\Gamma\right), \quad (10.3)$$

As claimed, the second part

$$\frac{1}{2\bar{\kappa}}\int\left(2\bar{R}^{a}\bar{e}_{a}-\frac{\bar{\Lambda}}{3}\epsilon_{abc}\bar{e}^{a}\bar{e}^{b}\bar{e}^{c}\right) \quad (10.4)$$

is mapped to (7.1) provided $\bar{\Lambda}=\bar{\kappa}\lambda$, $\bar{\kappa}=-8$. This can be seen as follows. From the vanishing torsion condition, $de_{a}+\epsilon_{abc}\omega^{b}e^{c}=0$, we impose

$$\omega_{\mu}^{a}=\left(\frac{1}{2}e_{\mu}^{a}\epsilon^{bcd}-\epsilon^{abc}e_{\mu}^{d}\right)e_{b}^{\nu}e_{c}^{\lambda}\partial_{\nu}e_{d\lambda}, \quad (10.5)$$

so $\bar{\omega}$ reduces to

$$\bar{\omega}^a = \omega^a + 2\phi^{-1}\epsilon^{abc} e_b e_{c\nu}\partial^\nu \phi. \tag{10.6}$$

Replacing the above in

$$\bar{R}^a = d\bar{\omega}^a + \frac{1}{2}\epsilon^{abc}\bar{\omega}_b\bar{\omega}_c, \tag{10.7}$$

and considering the conformal transformation $\bar{e}^a = \phi^2 e^a$, as well as the relations (10.3), the action (10.4) becomes the second-order one (7.1).

Now, we will show that the equations of motion

$$2R_a - \Lambda\epsilon_{abc}e^b e^c = -\frac{\kappa}{\bar{\kappa}}\left(2\bar{R}_a - \bar{\Lambda}\epsilon_{abc}\bar{e}^b\bar{e}^c\right), \tag{10.8}$$

$$\phi e^a \left(2\bar{R}_a - \bar{\Lambda}\epsilon_{abc}\bar{e}^b\bar{e}^c\right) = 0, \tag{10.9}$$

are equivalent to the second-order equations for the conformally coupled scalar field. First, the l.h.s of (10.8) can be easily identified with

$$G_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{10.10}$$

just after considering that the relation between the 2-form curvature and the Rieman tensor (10.3) holds and recalling that $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor. Now, for computing the r.h.s. of (10.8) one must solve consider the value fir ω (10.5), which implies (10.6). At last, replacing (10.6) in (10.7), one gets that the r.h.s of (10.8) is reduced to the conformal coupling stress-energy tensor (7.5), i.e.

$$\frac{\kappa}{8}\left(\bar{G}_{\mu\nu} - 8\lambda\bar{g}_{\mu\nu}\right)\phi^2 = \kappa T_{\mu\nu}, \tag{10.11}$$

where $\bar{g}_{\mu\nu} = \phi^4 g_{\mu\nu}$ and

$$\bar{G}_{\mu\nu} = G_{\mu\nu} - 2\nabla_\mu\nabla_\nu \log\phi + 2g_{\mu\nu}\nabla_\alpha\nabla^\alpha \log\phi + 4\nabla_\mu \log\phi\nabla_\nu \log\phi, \tag{10.12}$$

obtaining the well-known Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{10.13}$$

The scalar field equation (7.4) can be recovered from (10.9). Following the same steps required for obtaining (10.11), Eq. (10.9) reduces to

$$2e\bar{g}^{\mu\nu}\left(\bar{G}_{\mu\nu} - 8\lambda\bar{g}_{\mu\nu}\right)\phi^5 = 0, \tag{10.14}$$

and using (10.12), the scalar field equation

$$\Box\phi - \frac{1}{8}R\phi - 6\lambda\phi^5 = 0. \tag{10.15}$$

is easily recovered.

Chapter 11

Appendix B : Black hole gauge connection

The dreibein for the line element (7.6) is chosen as

$$e^0 = N^{\perp}(r)\,dt, \quad e^1 = \frac{dr}{F(r)}, \quad e^2 = H(r)\left(d\theta + N^{\theta}(r)\,dt\right), \tag{11.1}$$

where the Lorentz metric is $\eta_{ab} = \mathrm{diag}(-1,1,1)$. Then, the components of the 1-form field $\bar{e}^a$ are

$$\bar{e}^0 = \phi^2 N^{\perp}(r)\,dt, \quad e^1 = \frac{\phi^2 dr}{F(r)}, \quad e^2 = \phi^2 H(r)\left(d\theta + N^{\theta}(r)\,dt\right), \tag{11.2}$$

with the scalar field

$$\phi(r) = \sqrt{\frac{8c}{\kappa(r+c)}}. \tag{11.3}$$

For obtaining the spin connection, one solves the torsion equation $de^a + \epsilon^{abc}\omega_b e_c = 0$, whose general solution is

$$\omega^a = \left[\left(\frac{1}{2}e^a_\mu \epsilon^{bcd} - \epsilon^{abc}e^d_\mu\right) e^\nu_b e^\lambda_c \partial_\nu e_{d\lambda}\right] dx^\mu. \tag{11.4}$$

Thus, the components reads as

$$\omega^{0} = \frac{1}{2}\left[2N^{\theta}\left(r\right)H\left(r\right)' + H\left(r\right)N^{\theta}\left(r\right)'\right]dt + H\left(r\right)'F\left(r\right)d\theta, \tag{11.5}$$

$$\omega^{1} = \frac{H\left(r\right)^{2}N^{\theta}\left(r\right)'}{2rF\left(r\right)}dr, \tag{11.6}$$

$$\begin{aligned}\omega^{2} &= \left[\frac{rF\left(r\right)F\left(r\right)'}{H\left(r\right)} + \frac{F\left(r\right)^{2}\left(H\left(r\right) - rH\left(r\right)'\right)}{H\left(r\right)^{2}} - \frac{H\left(r\right)^{3}N^{\theta}\left(r\right)N^{\theta}\left(r\right)'}{2r}\right]dt \\ &- \frac{H\left(r\right)^{3}N^{\theta}\left(r\right)'}{2r}dr.\end{aligned} \tag{11.7}$$

From (10.6), one can write $\bar{\omega}^{a} = \omega^{a} + \tilde{\omega}^{a}$, with

$$\tilde{\omega}^{a} = 2\phi^{-1}\epsilon^{abc}e_{b}e_{c\nu}\partial^{\nu}\phi, \tag{11.8}$$

whose components are

$$\tilde{\omega}^{0} = 2\phi\left(r\right)^{-1}F\left(r\right)H\left(r\right)\left[d\theta + N^{\theta}\left(r\right)dt\right]\phi\left(r\right)', \tag{11.9}$$

$$\tilde{\omega}^{1} = 0, \tag{11.10}$$

$$\tilde{\omega}^{2} = 2\phi\left(r\right)^{-1}F\left(r\right)N^{\perp}\left(r\right)\phi\left(r\right)'dt. \tag{11.11}$$

The gauge connection A is defined as

$$A = e^{a}P_{a} + \omega^{a}J_{a}, \tag{11.12}$$

where P_a and J_a are the generators of the anti-de Sitter algebra $so\left(2,2\right)$ in three dimensions, which reads as

$$\left[J_{a},J_{b}\right] = \epsilon_{abc}J^{c}, \quad \left[J_{a},P_{b}\right] = \epsilon_{abc}P^{c}, \qquad \left[P_{a},P_{b}\right] = -\Lambda\epsilon_{abc}J^{c}, \tag{11.13}$$

with $\Lambda = -1/\ell^{2}$. But $so(2,2) \simeq so\left(2,1\right)^{+} \oplus so\left(2,1\right)^{-}$, where the generators for each copy of the Lorentz algebra in three dimensions will be denoted by the set $j_{a}^{\pm}$. The isomorphism between the $so(2,2)$ and the $so\left(2,1\right)^{\pm}$ generators is given by

$$\ell P_{a} = j_{a}^{+} - j_{a}^{-}, \quad J_{a} = j_{a}^{+} + j_{a}^{-}, \tag{11.14}$$

and the $so\left(2,1\right)^{\pm}$ algebras are

$$\left[j_{a}^{\pm}, j_{b}^{\pm}\right] = \epsilon_{abc}j^{\pm\,c}. \tag{11.15}$$

Considering the above explanation, the hairy black hole gauge connection can be written as a direct sum of gauge connections for the Lorentz algebra $so(2,1)$, i.e., $A = A^{+} \oplus A^{-}$, where

$$A^{\pm} = \left(\omega^{a} \pm \frac{e^{a}}{\ell}\right)j_{a}^{\pm}. \tag{11.16}$$

Finally, one has to consider that $so\left(2,1\right)^{\pm}\simeq sl\left(2,\mathbb{R}\right)^{\pm}$, where the generators for each copy of the special linear group in two dimensions are $L_n^{\pm}$ with $n=-1,0,1$ and its algebra reads $[L_n^{\pm},L_m^{\pm}]=\left(n-m\right)L_{n+m}^{\pm}$. The change of basis is

$$j_0^{\pm}=\frac{1}{2}\left(L_{-1}^{\pm}+L_1^{\pm}\right),\quad j_1^{\pm}=\frac{1}{2}\left(L_{-1}^{\pm}-L_1^{\pm}\right),\quad j_2^{\pm}=L_0^{\pm}. \tag{11.17}$$

Chapter 12

Appendix C : The hairy black hole charges in the Regge-Teitelboim approach

We will obtain the global charges of the black hole solution (7.6)-(7.11) as surface integrals using the Regge-Teitelboim approach [42]. For this purpose, we consider the following stationary minisuperspace

$$ds^2 = -N^{\perp}(r)^2 dt^2 + F(r)^{-2} dr^2 + H(r)^2 \left(d\theta + N^{\theta}(r)\, dt\right)^2, \tag{12.1}$$

where the scalar field just depends on the radial coordinate $\phi = \phi(r)$.

The Hamiltonian generator of the asymptotic symmetry ξ is determined by a linear combinations of the canonical constraints, $\mathcal{H}_{\perp}$ and $\mathcal{H}_i$, and by a surface term $Q[\xi]$,

$$H[\xi] = \int d^2x \left(\xi^{\perp}\mathcal{H}_{\perp} + \xi^i \mathcal{H}_i\right) + Q[\xi], \tag{12.2}$$

where the surface term is chosen such that $H[\xi]$ has well defined functional derivatives. The components $\xi^{\perp}, \xi^{\theta}$ of the allowed surface deformation vector.

The only non-vanishing canonical momentum in the proposed minisuperspace is $\pi_{\theta}^{r} \equiv P(r)$. Therefore, the constraints for the action (7.2) are given by

$$\begin{aligned} \mathcal{H}_{\perp} = \frac{1}{\kappa}\Bigg[& \frac{4\kappa^2 P^2}{F\Omega(\phi)H^3} + \Omega(\phi)F'H' - \frac{\kappa}{4}FH'\phi\phi' + \Omega(\phi)FH'' \\ & + \frac{H}{F}\left(\Lambda + \kappa\lambda\phi^6 - \frac{\kappa}{4}F\phi F'\phi' + \frac{\kappa}{4}F^2\left(\phi'^2 - \phi\phi''\right)\right)\Bigg], \end{aligned} \tag{12.3}$$

$$\mathcal{H}_{\theta} = -2P', \tag{12.4}$$

with

$$P(r) = -\frac{FH^3\Omega(\phi)(N^\theta)'}{4\kappa N^\perp}, \tag{12.5}$$

and $\Omega(\phi) = 1 - \kappa\phi^2/8$.

Then, the variation of the surface term at infinity is

$$\begin{aligned}\delta Q[\xi] = \frac{2\pi}{\kappa}\Big[&\xi^\perp\left(\frac{\kappa}{4}FH\left(\phi\delta\phi' - 3\phi'\delta\phi\right) - \Omega(\phi)F\delta H' + \left(\frac{\kappa}{4}H\phi\phi' - \Omega(\phi)H'\right)\delta F\right)\\ &+ \xi^{\perp\prime}F\left(\Omega(\phi)\delta H - \frac{\kappa}{4}H\phi\delta\phi\right) + 2\kappa\xi^\theta\delta P\Big]_{r\to\infty},\end{aligned} \tag{12.6}$$

where the integration over θ, which provides the factor 2π in (12.6), has been done.

On the other hand, the asymptotic behavior of the fields at infinity that accommodates the black hole solution is given by

$$F^2(r) = \frac{r^2}{\ell^2} + F_1 + \mathcal{O}\left(\frac{1}{r}\right), \tag{12.7}$$

$$H^2(r) = r^2 + H_1 + \mathcal{O}\left(\frac{1}{r}\right), \tag{12.8}$$

$$P(r) = P_0 + \mathcal{O}\left(\frac{1}{r}\right), \tag{12.9}$$

$$\phi(r) = \frac{\phi_0}{r^{1/2}} + \frac{\phi_1}{r^{3/2}} + \mathcal{O}\left(\frac{1}{r^{5/2}}\right), \tag{12.10}$$

and the asymptotic symmetries behave as

$$\xi^\perp(r) = \frac{r}{\ell} + \mathcal{O}\left(\frac{1}{r}\right), \tag{12.11}$$

$$\xi^\theta(r) = \xi_0^\theta + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{12.12}$$

Above, the quantities labelled with subscript 0 and 1 are constants.

The mass is the charge associated to the gauge parameter $\xi^\perp$, then $M = Q(\xi^\perp = 1, \xi^\theta = 0) = Q(\partial_t)$, and the angular momentum $J = Q(\xi^\perp = 0, \xi^\theta = 1) = Q(\partial_\theta)$ is the charge related to ξ^θ.

In consequence, for the given asymptotic conditions (12.7) and the asymptotic symmetries (12.11), the variation of the global charges are

$$\delta M = \delta Q[\partial_t] = -\frac{\pi\delta F_1}{\kappa} + \frac{2\pi\delta H_1}{\kappa\ell^2} + \frac{\pi}{2\ell^2}\left(3\phi_1\delta\phi_0 - \phi_0\delta\phi_1\right), \tag{12.13}$$

$$\delta J = \delta Q[\partial_\theta] = 4\pi\delta P_0. \tag{12.14}$$

The integration of the scalar field contribution requires a functional relation between ϕ_0 and ϕ_1. This contribution vanishes[1] if the scalar field is invariant against rescaling of the radial variable at infinity, which demands that ϕ_1 is proportional to ϕ_0^3, as occurs for the scalar field considered here. Therefore, we have

$$M = -\frac{\pi F_1}{\kappa} + \frac{2\pi H_1}{\kappa \ell^2}, \quad J = 4\pi P_0. \tag{12.15}$$

The arbitrary additive constants, coming from the integration of each variation, are set to be zero in order to obtain vanishing mass and angular momentum for the massless BTZ black hole.

For the particular rotating black hole solution presented in Sec. 2, we have

$$F_1 = -\frac{3c^2(1-\alpha)}{\ell^2}, \quad H_1 = \frac{3c^2\omega^2(1-\alpha)}{1-\omega^2}, \quad P_0 = \frac{3c^2\omega\,(1-\alpha)}{2\kappa\ell\,(1-\omega^2)}, \quad \phi_1 = -\frac{\kappa}{16}\phi_0^3, \tag{12.16}$$

then, the mass and angular momentum for the hairy rotating black hole are

$$M = \frac{3\pi c^2\,(1-\alpha)\,(1+\omega^2)}{\kappa\ell^2\,(1-\omega^2)}, \qquad J = \frac{6\pi\,(1-\alpha)\,c^2\omega}{\kappa\ell\,(1-\omega^2)}. \tag{12.17}$$

[1] A recent discussion about this contribution to the mass and the asymptotically AdS symmetry can be found in [38] and [60], for three and four spacetime dimensions, respectively. A review based on the Hamiltonian approach, which includes higher dimensions, is available in [29].

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