Universidad de Concepción
Dirección de Postgrado
Facultad de Ciencias Físicas y Matemáticas - Programa Doctorado en Matemática

Soluciones radiales positivas para ecuaciones cuasilineales.

Positive radial solutions for quasilinear equations.

Tesis para optar al grado de Doctor en Matemática

JUAN CARLOS GUAJARDO BRAVO
CONCEPCIÓN-CHILE
2020

Profesor Guía: Rajesh Mahadevan
Profesor Co-Guía: Sebastián Lorca
Departamento de Matemática, Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción

Universidad de Concepción
Dirección de Postgrado
Facultad de Ciencias Físicas y Matemáticas - Programa Doctorado en Matemática

Soluciones radiales positivas para ecuaciones cuasilineales.

Positive radial solutions for quasilinear equations.

Tesis para optar al grado de Doctor en Matemática

JUAN CARLOS GUAJARDO BRAVO
CONCEPCIÓN-CHILE
2020

Comisión evaluadora:
Leonelo Iturriaga (UTFSM, Chile)
Dhanya Rajendran (IISER-Trivandrum, India)
Robert Stanczy (Uniwersytet Wroclawskiego, Poland)
Sebastián Lorca (U. Tarapacá, Chile)
Rajesh Mahadevan (UdeC, Chile)

Contents

Agradecimientos

Quiero comenzar agradeciendo a Rajesh Mahadevan y Sebastián Lorca por todo el apoyo que brindaron hacia mi persona, por su aporte en fortalecer mis conocimientos e indicarme el camino a seguir. También, quiero agradecer a Leonelo Iturriaga por su ayuda en cada oportunidad que la requerí. Mis agradecimientos son además para Dhanya Rajendran y Robert Stanczy por su buena disposición para evaluar este texto. Agradezco a mis amigos y compañeros por su agradable compañía y consejos.

Asimismo, quiero agradecer de manera muy especial a mi familia, la que me ha apoyado desde el comienzo en este camino; este texto va con mucho cariño dedicado a ellos.

Chapter 1

Introduction

The study of existence, multiplicity and non-existence of positive solutions to semi-linear and quasi-linear elliptic equations is relevant to many applications ranging from thermal iginition of gases [14], quantum field theory and statistical mehcanics [7], gravitational equilibrium of stars [19] etc.

This work aims to study the existence, multiplicity and no-existence of positive radial solutions(other than the 0 solution) to the problem

$$\operatorname{div}(A(|\nabla u|)\nabla u) + \lambda k(|x|)f(u) = 0, \quad x \in \Omega. \tag{1.0.1}$$

in symmetric exterior domains $\Omega \subset \mathbb{R}^n$ (complements of balls centered at the origin) for $n \geq 2$. The non negative functions A, k and f satisfy certain properties that we will specify later and $\lambda > 0$ is a parameter.

The class of functions A which we consider will include $A(|p|) = |p|^{m-2}, m > 1$ associated to the m-laplacian operator $\operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right)$ (non linear if $m \neq 2$ and coincides with the Laplacian for $m = 2$) applicable to diffusion problems. The class will also include slight perturbations of

$$A(|p|) = \left(1 + |p|^2\right)^{-1/2}.$$

which correspond to some perturbations of the mean curvature operator

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right).$$

Different behaviours of the non-linearity f at 0 and ∞ will also be considered in relation to the behaviour of A. The weight k could also turn out to be singular.

From an extensive study of such equations, it is well-known that the existence, non-existence or multiplicity of solutions depend significantly on several factors: the non-linearity f, for example, whether it is sub-linear or superlinear at infinity in the case of the Laplacian; the magnitude of the parameter λ; the domain, namely, whether it is bounded or unbounded, simply connected or not etc. There is a vast literature on these problems and these problems have been studied using different methods-variational, topological, using comparison principles etc.

We will broadly separate the problems that we study into two kinds depending on the boundary condition.

Problems in an exterior domain with classical boundary conditions

Our first objective is to study the equation

$$\operatorname{div}(A(|\nabla u|)\nabla u) + \lambda k(|x|)f(u) = 0, \qquad |x| > d \tag{1.0.2}$$

under some classical Dirichlet or Neumann type boundary conditions corresponding to one of the following conditions:

$$\left.\begin{array}{c} u = 0 \quad \text{on} \quad |x| = d \quad \text{and} \quad u \to 0 \text{ when } |x| \to \infty\,, \\ \partial u/\partial r = 0 \quad \text{on} \quad |x| = d \quad \text{and} \quad u \to 0 \text{ when } |x| \to \infty\,, \\ u = 0 \quad \text{on} \quad |x| = d \quad \text{and} \quad \partial u/\partial r \to 0 \text{ when } |x| \to \infty. \end{array}\right\} \tag{1.0.3}$$

We will prove existence and non-existence of positive radial solutions in these problems depending on the range of behavior of the functions A, f, k and the range of values of the parameter $\lambda > 0$.

We provide a brief discussion of earlier results related to this problem. The study of the existence and regularity of solutions to semi-linear and quasi-linear elliptic equations has a long history. If the domain is bounded, connected and the non-linearities have a suitable behaviour, for example, sub-critical then there is usually existence and uniqueness of regular solutions. Non-existence results show up for super-critical non-linearities or in certain domains. Ni and Serrin [27, 28] established nonexistence of singular radial solutions of quasi-linear equations of the form (1.0.1) in $\mathbb{R}^n$ for f which are superlinear at infinity. In the last few decades, several authors have also studied such problems in annular domains (see , for example, [26, 1, 5, 2, 24, 8, 13, 23, 25, 31, 34, 35]):

$$\triangle u + \lambda k(|x|)f(u) = 0 \qquad R_1 < |x| < R_2, \tag{1.0.4}$$

with one of the following boundary conditions:

$$\left.\begin{array}{c} u = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad |x| = R_2 \\ \partial u/\partial r = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad u = 0 \quad \text{on } |x| = R_2 \\ u = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad \partial u/\partial r = 0 \quad \text{on } |x| = R_2\,. \end{array}\right\} \tag{1.0.5}$$

In particular, when f is non-negative, Bandle, Coffman and Marcus [1], Coffman and Marcus [8] and Lin [24] have established the existence of positive radial solutions of $(1.0.4)-(1.0.5)$ for super-linear nonlinearities f, that is, $\lim\limits_{x\to 0} \frac{f(x)}{x} = 0$ and $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \infty$. For this the shooting method along with Sturm comparison results were used.

On the other hand, H. Wang [31] established, using fixed point methods, the existence of positive radial solutions of $(1.0.4) - (1.0.5)$ for sub-linear nonlinearities f for which $\lim\limits_{x\to 0} \frac{f(x)}{x} = \infty$ and $\lim\limits_{x\to\infty} \frac{f(x)}{x} = 0$.

Some of the results on radial solutions obtained for the case of semi-linear elliptic equations for the Laplacian($m = 2$) are also valid in the case of the m-Laplacian $m > 1$. In this direction we also mention previous works on the p-Laplacian equation in one dimension on $(0, 1)$ by Wang [32], Kong and Wang [21] and Sánchez [29]. In this more general case, superlinearity or sublinearity of f refers to the behavior of the quotient $\frac{f(x)}{x^{m-1}}$ at zero and infinity.

Subsequently, the same author Wang [33] studied, also using fixed point methods, the existence of positive radial solutions for the quasilinear equation

$$\operatorname{div}(A(|\nabla u|)\nabla u) + \lambda k(|x|)f(u) = 0, \quad R_1 < |x| < R_2, \tag{1.0.6}$$

on an annular domain with boundary conditions of the type (1.0.5) for more general classes of A, which include the m-Laplacian. The results obtained by Wang in the quasilinear problem in an annular domain encapsulates various of the previously mentioned results in an annular domain. So, we state Wang's hypothesis and results.

(H1) $\varphi(t) := A(|t|)t$ is an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and there exists two increasing homeomorphisms ψ_1 and ψ_2 from $(0, \infty)$ onto $(0, \infty)$ such that

$$\psi_1(\sigma)\varphi(t) \leq \varphi(\sigma t) \leq \psi_2(\sigma)\varphi(t) \quad \text{for all } \sigma, t > 0.$$

(H2) $k : [R_1, R_2] \to [0, \infty)$ is continuous and $k(t) \not\equiv 0$ on any subinterval of $[R_1, R_2]$.

(H3) $f : [0, \infty) \to [0, \infty)$ is continuous.

(H4) $f(u) > 0$ for $u > 0$.

Now, superlinearity or sublinearity of f refers to the behavior of the quotient $\dfrac{f(x)}{\varphi(x)}$ at zero and infinity. Fixing the notation

$$f_0 := \lim_{u \to 0} \frac{f(u)}{\varphi(u)} \quad \text{and} \quad f_\infty := \lim_{u \to \infty} \frac{f(u)}{\varphi(u)}$$

we state his main results.The boundary condition is one of the conditions in (1.0.5).

Theorem 1.0.1. *Assume that **(H1)-(H2)-(H3)** hold.*

1. *If $f_0 = 0$ and $f_\infty = \infty$, then for all $\lambda > 0$ the problem (1.0.6)-(1.0.5) has a positive radial solution.*
2. *If $f_0 = \infty$ and $f_\infty = 0$, then for all $\lambda > 0$ the problem (1.0.6)-(1.0.5) has a positive radial solution.*

Theorem 1.0.2. *Assume that **(H1)-(H2)-(H3)-(H4)** hold.*

1. *If $f_0 = 0$ or $f_\infty = 0$, then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ the problem (1.0.6)-(1.0.5) has a positive radial solution.*
2. *If $f_0 = \infty$ or $f_\infty = \infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ the problem (1.0.6)-(1.0.5) has a positive radial solution.*
3. *If $f_0 = f_\infty = 0$, then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ the problem (1.0.6)-(1.0.5) has two positive radial solutions.*
4. *If $f_0 = f_\infty = \infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ the problem (1.0.6)-(1.0.5) has two positive radial solutions.*
5. *If $f_0 < \infty$ and $f_\infty < \infty$, then there exists a $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$ the problem (1.0.6)-(1.0.5) has no positive radial solution.*
6. *If $f_0 > 0$ and $f_\infty > 0$, then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ the problem (1.0.6)-(1.0.5) has no positive radial solution.*

In the case of an exterior domain, semilinear problems involving the Laplacian have been studied using different methods by several authors: see for example Stańczy [30], do Ó et al. [11], Dhanya et al. [10], Hai and Shivaji [18], Chhetri et. al [3, 4].

Taking as reference the theorems by Wang [33] stated above, in the present work we are able to obtain similar results for positive radial solutions of the quasi-linear problem (1.0.2)-(1.0.3) in an exterior domain using the fixed point method while adapting the techniques used by Wang. We refer the reader to the introduction of Chapter 3 for precise statements of the main results obtained in the thesis on this problem. The results of this chapter appear in an article which is accepted for publication [16].

Problems in an exterior domain with nonlinear boundary conditions
Another of our objectives is to study the existence of positive radial solutions for a class of non-linear equations in an exterior domain $\mathbb{R}^n$ under non-linear boundary conditions

$$\begin{cases} -\operatorname{div}(A(|\nabla u|)\nabla u) = \lambda k(|x|)f(u), & |x| > r_0 \\ \dfrac{\partial u}{\partial \eta} + c(u)u = 0 & |x| = r_0 \\ u \to 0 \text{ when } |x| \to \infty \end{cases} \tag{1.0.7}$$

where $k : [r_0, \infty) \to (0, \infty)$ is a continuous function such that $\lim\limits_{r\to\infty} k(r) = 0$, $\frac{\partial}{\partial \eta}$ denotes the normal derivative, and $c : [0, \infty) \to (0, \infty)$ is a continuous function.

Previously, the corresponding problem for the Laplacian

$$\begin{cases} -\Delta u = \lambda k(|x|)f(u), & |x| > r_0 \\ \dfrac{\partial u}{\partial \eta} + c(u)u = 0, & |x| = r_0 \\ u \to 0 \ \text{ when } |x| \to \infty \end{cases}$$

has been addressed by Butler, Ko, Lee and Shivaji [6] and the study of steady state solutions in bounded domains by Gordon, Ko and Shivaji [17] in a thermal explosion problem. They considered a reaction term $f : [0, \infty) \to \mathbb{R}$ of class C^1 which is *sublinear* at ∞ (i.e., $\lim\limits_{s\to\infty} \frac{f(s)}{s} = 0$) and prove existence, uniqueness and multiplicity of solutions for the cases (i) $f(0) > 0$, (ii)$f(0) = 0$ and (iii) $f(0) < 0$. The results in these three cases were obtained using the method of sub and super solutions. The uniqueness results use additional hypothesis on c(strictly increasing) and f (for non-increasing $f(s)/s$ or under stronger conditions). They also show existence of three positive solutions for certain other behaviour of f. Subsequently, Dhanya, Morris and Shivaji [10] studied the existence of positive radial solutions for a similar problem while assuming that $f \in C^1([0, \infty), \mathbb{R})$ is nondecreasing, *superlinear* at ∞ (that is, $\lim_{s\to\infty} \frac{f(s)}{s} = \infty$) and $f(0) < 0$ (semipositone) and $k : [R_0, \infty) \to (0, \infty)$ is continuous satisfying $k(r) \leq \dfrac{1}{r^{n+\mu}}; \mu > 0$ for $r >> 1$. They were able to establish the existence of a positive radial solution for small values of the parameter λ using variational methods.

In Chapter 4, we set ourselves two objectives: firstly, to complement and enrich the results obtained by Butler, Ko, Lee and Shivaji [6] for various possible behaviours of f at ∞, that is, not only consider f which are sublinear at ∞. Secondly, prove these results for a class of quasi-linear problems. Many of the results are proved using the fixed-point approach and this requires a careful choice of the operator to which to apply the Krasnosel'skii fixed point theorem. The choice of this operator and establishing it's fundamental properties is our main focus. Once this is done it is immediate to obtain results for the problem with the non-linear boundary condition under similar hypothesis and following the same arguments as in Chapter 3. These results resemble, in spirit, the results obtained for the classical boundary conditions. We also prove a theorem using the sub- and super- solution techniques used in Butler, Ko, Lee and Shivaji [6] but do not explore this in more depth as we are able to prove this result only for the class of positively homogeneous operators, like the p-Laplacian. We do not discuss the semi-positone case, that is, we consider only $f(0) \geq 0$. We refer to the introduction of Chapter 4 for precise statements of the main results obtained in the thesis on quasi-linear problems in the presence of non-linear boundary conditions.

The main kind of results we have obtained for quasi-linear problems in exterior domains (*involving the classical or non-linear boundary conditions*) are sketched below without listing the complete set of hypothesis on A, f, k or c.

Existence for the complete range of parameter:

- If $f_0 = 0$ and $f_\infty = \infty$, then for all $\lambda > 0$ the problem has a positive radial solution.
- If $f_0 = \infty$ and $f_\infty = 0$, then for all $\lambda > 0$ the problem has a positive radial solution.

Existence for a limited range of parameter:

- If $f_0 = \infty$ or $f_\infty = \infty$, then for any $R > 0$ there exists $\lambda_R > 0$ such that for all $0 < \lambda \leq \lambda_R$ the problem has at least one positive radial solution u with $0 < \|u\| < R$ or $R < \|u\|$ respectively.
- If $f_0 = 0$ or $f_\infty = 0$, then for any $L > 0$ there exists $\lambda_L > 0$ such that for all $\lambda \geq \lambda_L$ the problem has at least one positive radial solution u with $0 < \|u\| < L$ or $L < \|u\|$ respectively.

Multiplicity of solutions for a limited range of parameter:

- If $f_0 = \infty$ and $f_\infty = \infty$, then for any $R > 0$ there exists λ_R such that for all $0 < \lambda \leq \lambda_R$, the problem has at least two positive radial solutions u_1, u_2 with $0 < \|u_1\| < R < \|u_2\|$.
- If $f_0 = 0$ and $f_\infty = 0$, then for any $L > 0$ there exists $\lambda_L > 0$ such that for all $\lambda \geq \lambda_L$, the problem has at least two positive radial solutions u_1, u_2 with $0 < \|u_1\| < L < \|u_2\|$.

Non-existence of solutions for a range of parameters:

- If $f_0 < \infty$ and $f_\infty < \infty$, then there exists a positive number $\underline{\lambda}$ such that the problem has no positive radial solutions for all $\lambda < \underline{\lambda}$.
- If $f_0 > 0$ and $f_\infty > 0$, then there exists a positive number $\overline{\lambda}$ such that the problem has no positive radial solutions for all $\lambda > \overline{\lambda}$.

Chapter 2

Preliminaries

The Krasnosel'skii fixed-point theorem, as well as its various generalizations, have been successfully applied in the resolution of various boundary value problems in PDE. This theorem will be our main tool. We will see some of it's versions and a sketch their proofs. Along with that, we will also need some elementary comparison results. For more details on the notions which are discussed in Sections 2.1-2.4 we refer to Zeidler [36], Kesavan [20], Guo and Lakshmikantham [15] or Deimling [9].

We start first with a set of definitions that are necessary.

2.1 Basic definitions

2.1.1 Ordered cone and partial ordering

Definition 2.1.1. *Let X be a Banach space. A subset, $\mathcal{K} \subset X$, is called a* ***cone*** *if it satisfies the following conditions:*

K1 *$\mathcal{K}$ is closed and nonempty.*

K2 *If $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$, $x, y \in \mathcal{K}$ then $\alpha x + \beta y \in \mathcal{K}$.*

K3 *If $x \in \mathcal{K}$ and $-x \in \mathcal{K}$ then $x = 0$.*

Examples of cones

- In $\mathbb{R}$, the set of positive real numbers $\mathbb{R}^+$.
- The set $\mathcal{K} = \{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} \leq z\}$ in the euclidean space $\mathbb{R}^3$.
- The set consisting of all non-negative n-tuples in $\mathbb{R}^n$: $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \quad \text{for all } i\}$.
- The set consisting of all real-valued continuous nonnegative functions defined on the interval J: $\mathcal{C}^+_{\mathbb{R}}(J) = \{x \in \mathcal{C}_{\mathbb{R}}(J) : x(t) \geq 0 \text{ on } J\}$.

Partial Ordering: Given an ordered cone $\mathcal{K} \subset X$ in a Banach space, we can define a relation of order in the following way:

$$x \leq y \quad \text{iff} \quad y - x \in \mathcal{K}\,.$$

We also say

$$x \nleq y \quad \text{iff} \quad x \leq y \quad \text{is false}\,.$$

2.1.2 Fixed points of completely continuous operators in conical shells

Definition 2.1.2. *Let X, Y Banach spaces, $\Omega \subset X$ and an application $F : \Omega \to Y$. We will say that F is* ***completely continuous or compact*** *if it is continuous and compact (namely, sends bounded subsets of Ω into relatively compact sets).*

Definition 2.1.3. *Given a Banach space with an ordered cone $\mathcal{K}$, a* ***cone operator*** *or a* ***positive operator*** *is a completely continuous operator $T : \mathcal{K} \to \mathcal{K}$.*

Definition 2.1.4. *A point $x \in \mathcal{K}$ is a* ***fixed point*** *of a cone operator T if $T(x) = x$.*

2.1.3 Compression and expansion of a conical shell

Let $\mathcal{K}$ be an ordered cone. Let $0 < R < \overline{R}$ be given. The following set

$$\mathcal{K}(R, \overline{R}) = \{x \in \mathcal{K} : R \leq \|x\| \leq \overline{R}\}$$

will be referred to as a **conical shell** whose inner and outer boundaries are, respectively, $\mathcal{K}_R = \{x \in \mathcal{K} : \|x\| = R\}$ and $\mathcal{K}_{\overline{R}} = \{x \in \mathcal{K} : \|x\| = \overline{R}\}$.

Definition 2.1.5. *If the conditions*

$$Tx \not\leq x \qquad \forall x \in \mathcal{K}_R \tag{2.1.1}$$
$$\text{and} \quad Tx \not\geq x \qquad \forall x \in \mathcal{K}_{\overline{R}} \tag{2.1.2}$$

hold, then we say that the conical shell is under ***compression*** *and if the conditions*

$$Tx \not\geq x \qquad \forall x \in \mathcal{K}_R \tag{2.1.3}$$
$$\text{and} \quad Tx \not\leq x \qquad \forall x \in \mathcal{K}_{\overline{R}} \tag{2.1.4}$$

hold, then we say that the conical shell is under ***expansion****.*

Example 2.1.6. *An illustration of this in dimension 2 is depicted in the following figures, where $X = \mathbb{R}^2$ and the cone $\mathcal{K}$ is the wedge-shaped region AOB.*

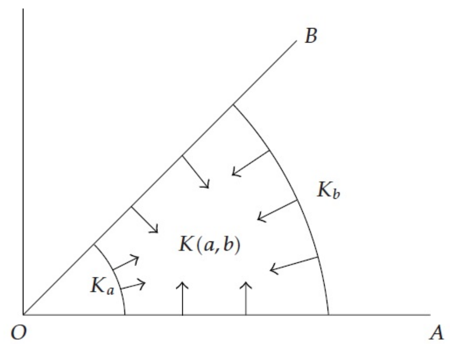

Figure 2.1: Cone compression in $\mathbb{R}^2$

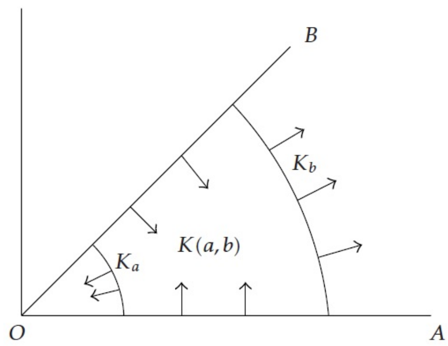

Figure 2.2: Cone expansion in $\mathbb{R}^2$

2.2 Some fixed point theorems

Now, we state a few fixed point theorems which will be used in the thesis. First we state the classical Schauder fixed point theorem for whose proof we refer directly to Kesavan [20]

Theorem 2.2.1 (Schauder Fixed Point Theorem)**.** *Let X be a real Banach space and K a closed, bounded and convex subset of X. Suppose the operator $T : K \to K$ is completely continuous. Then, T has a fixed point x in K.*

Now we state a couple of versions of Krasnosel'skii fixed point theorems whose proofs will be presented in Section 2.4.

Theorem 2.2.2 (Krasnosel'skii, 1962)**.** *Let X be a real Banach space with ordered cone $\mathcal{K}$. Suppose the operator $T : \mathcal{K} \to \mathcal{K}$ is completely continuous and the conical shell $\mathcal{K}(R, \overline{R})$ is either under compression or under expansion in the sense of Definition 2.1.5. Then, T has a fixed point x in $\mathcal{K}(R, \overline{R})$.*

An earlier results states the following.

Theorem 2.2.3 (Krasnosel'skii,1960)**.** *Let X be a Banach space and $\mathcal{K} \subset X$ be a cone in X. Let the operator $T : \mathcal{K} \to \mathcal{K}$ be completely continuous. For $0 < a < b$,*

1. *(Compressive form) T has a fixed point in the conical shell $\mathcal{K}(a, b)$ if*

$$\begin{aligned} \|Tx\| &\geq \|x\| \; \forall x \in \mathcal{K}_a & (2.2.1) \\ \textit{and} \quad \|Tx\| &\leq \|x\| \; \forall x \in \mathcal{K}_b. & (2.2.2) \end{aligned}$$

2. *(Expansive form) T has a fixed point in the conical shell $\mathcal{K}(a, b)$ if*

$$\begin{aligned} \|Tx\| &\leq \|x\| \; \forall x \in \mathcal{K}_a & (2.2.3) \\ \textit{and} \quad \|Tx\| &\geq \|x\| \; \forall x \in \mathcal{K}_b. & (2.2.4) \end{aligned}$$

In order to present a proof of the Krasnosel'skii theorems, we need to introduce the important concept of the fixed point index.

2.3 The Leray-Schauder fixed-point Index

We refer the reader to Zeidler [36] and Kesavan [20] for the notions discussed in this section.

A wide variety of mathematical models in science lead to equations of the type $Ax = y$. In particular, many kinds of differential equations, integral equations, integro-differential equations, etc. can be formulated in this way, usually on spaces of infinite dimension. The topological degree and the fixed point index are important in obtaining existence theorems for solutions of such equations. We will see, based on the concept of Leray-Schauder's degree for compact perturbations of the identity, that it is possible to define in a natural way the fixed point index of compact maps, which we then use to prove the Krasnosel'skii theorems mentioned above.

2.3.1 Leray-Schauder degree

Let X a real Banach space, $\Omega \subset X$ open and bounded, $F \in \mathfrak{K}(\Omega, X)$ (class of all compact maps) and $y \notin (I-F)(\partial\Omega)$. On these admissible triplets $(I-F, \Omega, y)$, Leray and Schauder, define a $\mathbb{Z}-$ valued function D which extends the notion of the Brouwer degree, and satisfies the following:

(D1) $D(I, \Omega, y) = 1$ for $y \in \Omega$;

(D2) $D(I-F, \Omega, y) = D(I-F, \Omega_1, y) + D(I-F, \Omega_2, y)$ whenever Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \notin (I-F)(\overline{\Omega} - (\Omega_1 \cup \Omega_2))$;

(D3) $D(I - H(t,\cdot), \Omega, y(t))$ is independent of $t \in [0,1]$ whenever $H : [0,1] \times \overline{\Omega} \to X$ is compact, $y : [0,1] \to X$ is continuous and $y(t) \notin (I - H(t,\cdot))(\partial\Omega)$ on $[0,1]$.

If $\Omega \subset X$ is an open bounded set, $F : \overline{\Omega} \to X$ is compact, and $y \notin (I-F)(\partial\Omega)$, the *Leray-Schauder degree* $D(I-F, \Omega, y)$ of $I-F$ in Ω over y is constructed from the Brouwer degree by approximating the compact mapping F over $\overline{\Omega}$ by mappings F_ε with range in a finite-dimensional subspace X_ε (containing y) of X, and showing that the Brouwer degrees $deg_B((I-F_\varepsilon)|_{X_\varepsilon}, \Omega \cap X_\varepsilon, y)$ stabilize for sufficiently small positive ε to a common value defining $D(I-F, \Omega, y)$. This topological degree 'algebraically counts" the number of zeros of $I-F-y$ in Ω. Furthermore, for F of class C^1 and $I - F'(x_0)$ invertible and for each zero of x_0 of $I-F-y$ in Ω, Leray and Schauder show that

$$D(I-F, \Omega, y) = \sum_{x_0 \in (I-F)^{-1}(y)} (-1)^{\sigma_j(x_0)}$$

where $\sigma_j(x_0)$ is the sum of the algebraic multiplicities of the eigenvalues of $F'(x_0)$ contained in $(1, \infty)$.

This Leray-Schauder degree has the following important properties:

(D4) $D(I-F, \Omega, y) \neq 0$ implies $(I-F)^{-1}(y) \neq \emptyset$.

(D5) *(Excision property)* $D(I-F, \Omega, y) = D(I-F, \Omega_1, y)$ for every open subset Ω_1 of Ω such that $y \notin (I-F)(\overline{\Omega} - \Omega_1)$

2.3.2 Fixed point index

Using the definition of Leray-Schauder degree, we can define the fixed point index, as given below, in relation to the fixed points of a compact map.

$$F(x) = x, \qquad x \in \overline{\Omega},$$

one defines the fixed-point index for $F \in \mathfrak{K}(\Omega, X)$ (class of all compact maps) by

$$i(F, \Omega) := D(I - F, \Omega, 0)\,.$$

The fixed-point index is a measure of the number of fixed points of F on Ω.

The following properties are satisfied.

(A_1) (Normalization) If $F(x) = x_0$ for all $x \in \overline{\Omega}$, and some fixed $x_0 \in \Omega$, then $i(F, \Omega) = 1$.

(A_2) (Kronecker existence principle) If $i(F, \Omega) \neq 0$, then there exists an $x \in \Omega$ such that $F(x) = x$.

(A_3) (Additivity) We have
$$i(F, \Omega) = \sum_{j=1}^{n} i(F, \Omega_j)$$
whenever $F \in \mathfrak{K}(\Omega, X)$ and $F \in \mathfrak{K}(\Omega_j, X)$ for all j, where $\{\Omega_j\}$ is a regular partition of Ω, i.e., the Ω_j are parwise disjoint and $\overline{\Omega} = \bigcup_{j=1}^{n} \overline{\Omega}_j$ and F has no fixed points on $\partial\Omega_j$ or $\partial\Omega$.

(A_4) (Homotopy invariance) $i(F, \Omega) = i(G, \Omega)$ whenever F and G are homotopically equivalent, that is, there exists a map H with the following properties:

1. $H : \overline{\Omega} \times [0, 1] \to X$ is compact;
2. $H(x, t) \neq x$ for all $(x, t) \in \partial\Omega \times [0, 1]$;
3. $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$ on $\overline{\Omega}$.

The map H is called a *compact homotopy.*

Note: If we are interested in the fixed points of a compact map $T : \mathcal{K} \to \mathcal{K}$ on a closed convex cone $\mathcal{K}$ then this can be achieved by extending $T : X \to \mathcal{K}$ to a compact map (which exists by a result of Dugundji [12]) and by studying the fixed points of the extended map on X using the fixed point index, by the permanence property $i(T, \Omega) = i(T, \mathcal{K} \cap \Omega)$.

For a proof of the Krasnosel'skii theorem (1960), we will also need the following lemma from Guo and Lakshmikantham [15] (cf. Lemma 2.3.2). Let $\mathcal{K}$ be a cone in a real Banach space X. Let Ω be a bounded open set of X.

Lemma 2.3.1. *Let $A : \mathcal{K} \cap \overline{\Omega} \to \mathcal{K}$ be completely continuous and $B : \mathcal{K} \cap \partial\Omega \to \mathcal{K}$ be completely continuous. Suppose that*

1. $\inf_{x \in \mathcal{K} \cap \partial\Omega} \|Bx\| > 0$ *and*

2. $x - Ax \neq tBx$, $\forall x \in \mathcal{K} \cap \partial\Omega$, $t \geq 0$.

Then, we have $i(A, \mathcal{K} \cap \Omega) = 0$.

2.3.3 General strategy of fixed point theorems.

It is worthwhile to keep in mind the following general strategy while trying to prove existence of fixed points for a certain map F on a domain Ω.

1. Relate a given map F by a homotopy to a simpler map G, for which $i(G,\Omega)$ is known and $i(G,\Omega) \neq 0$.
2. Apply (A_4) to get $i(F,\Omega) = i(G,\Omega) \neq 0$, and then (A_2) to conclude that the map F has a fixed point on Ω.

2.4 Proof of Theorem 2.2.3

We now present a proof of the Krasnosel'skii theorem (1960) using the Leray-Schauder fixed point index.

Proof: If T has a fixed point on the boundary of the conical shell then we are done. So, without loss of generality, we assume that $Tx \neq x$ for any $x \in \mathcal{K}$ with $\|x\| = a$ or $\|x\| = b$.

We set

$$U = \{x \in \mathcal{K} : \|x\| < a\}, \qquad V = \{x \in \mathcal{K} : \|x\| < b\}\,.$$

By (A_3) we have

$$i(T, V - \overline{U}) = i(T,V) - i(T,U)\,. \tag{2.4.1}$$

Assuming that (2.2.1) and (2.2.2) hold (that is, in the *compressive case*), we show below that

$$i(T,U) = 0, \qquad i(T,V) = 1. \tag{2.4.2}$$

Step 1: First we show that $i(T,U) = 0$. In fact, we apply Lemma 2.3.1 with $A = B = T$ and $\Omega = B(0,a)$. By (2.2.1) we have that $\|Tx\| \geq a > 0$ and thus $\inf_{x \in \mathcal{K} \cap \partial U} \|Tx\| > 0$, fulfilling the hypothesis (1) of the Lemma 2.3.1. We note that the hypothesis (2) is equivalent to $Tx \neq \mu x, \quad \forall \mu \leq 1$ and $x \in \partial U$. This holds, since we have assumed that $Tx \neq x$ for any x with $\|x\| = a$ and also, if $Tx = \mu x, \quad \mu < 1$ for any x with $\|x\| = a$, then $\|Tx\| = \mu \|x\| < \|x\|$ which contradicts the hypothesis (2.2.1) of the theorem. Therefore, by Lemma 2.3.1, we conclude that $i(T,U) = 0$.

Step 2: We show $i(T,V) = 1$ by showing that T is homotopic to 0. Consider the homotopy $H(x,t) = tTx$. If $H(x,t) = x$ for some $(x,t) \in \partial V \times [0,1]$, then $t \neq 0$ and $x \in K$, so that $\|Tx\| = \dfrac{\|x\|}{t} \geq \|x\|$, which is not possible by (2.2.2) and also since $Tx \neq x$ for $x \in \partial V$. So, we conclude that $i(T,V) = i(0,V) = 1$. So, we have $i(T, V - \overline{U}) = 1 - 0 \neq 0$.

In the *expansive case*, assuming that (2.2.3) and (2.2.4) holds we obtain

$$i(T, U - \overline{V}) = i(T,U) - i(T,V) = -1\,. \qquad \square$$

Note: The proof of Theorem 2.2.2 is even simpler since we do not need Lemma 2.3.1 for it's proof. It is enough to make the following modifications to the step 1 above, whereas, step 2 remains the same. To show $i(T,U) = 0$ we argue by contradiction. Suppose instead that $i(T,U) \neq 0$ and choose an α with $\|Tx\| \leq \alpha$ on $\overline{U}$ and an $x_0 \in K$ with $\|x_0\| > R + \alpha$ and define

$$H(x,t) = Tx + tx_0\,.$$

Now observe that $H(x,t) \neq x$ for any $(x,t) \in \partial U \times [0,1]$, for otherwise, we shall have $Tx + tx_0 = x$, which implies $(x - Tx) \in K$, that is, $Tx \leq x$ for some $x \in U$. But, (2.1.1) says that this cannot happen. Thus H is a valid homotopy and $i(H(\cdot,1),U) = i(T,U) \neq 0$. Then there exists an $x \in U$ with $Tx + x_0 = x$ which contradicts the choice $\|x_0\| > \alpha + R$.

2.5 Comparison principles for monotone operators

In this section we provide some basic comparison principles in the context of monotone operators in one-dimension involving Dirichlet, Neumann or Robin boundary conditions. These will be useful in Chapters 3 and 4 for the setting up of the problems as a fixed point problem for suitably defined cone operators.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

Lemma 2.5.1. *Let $[a,b] \subseteq [0,1]$; f, g be continuous functions on $[a,b]$ with $f \geq g$ on $[a,b]$ and p, q be strictly positive continuous functions on (a,b). Let w be a C^1 variational supersolution on $[a,b]$ of*

$$-(q(t)\varphi(p(t)w'(t)))' \geq f\,, \quad a < t < b \tag{2.5.1}$$

and z be a C^1 variational subsolution on $[a,b]$ of

$$-(q(t)\varphi(p(t)z'(t)))' \leq g \quad a < t < b \tag{2.5.2}$$

with

$$w(a) \geq z(a) \text{ and } w(b) \geq z(b)\,. \tag{2.5.3}$$

Then $w \geq z$ in $[a,b]$.

Proof: In order to prove that $w \geq z$ in $[a,b]$, it's enough to show that $\mu(\{w < z\}) = 0$, that is, $(w-z)_- = 0$ almost everywhere. Multiplying (2.5.1) by $(w-z)_-$ (note that this vanishes outside the set $\{w < z\}$ and at a and b due to the boundary inequalities (2.5.3)) and integrating on $[a,b] \cap \{w < z\}$ we obtain:

$$\begin{aligned}
\int\limits_{[a,b]\cap\{w<z\}} f(s)(w-z)_-(s)ds &\leq -\int\limits_{[a,b]\cap\{w<z\}} (q(s)\varphi(p(s)w'(s)))'(w-z)_-(s)ds \\
&= \int\limits_{[a,b]\cap\{w<z\}} q(s)\varphi(p(s)w'(s))(w-z)'_-(s)ds \\
&= -\int\limits_{[a,b]\cap\{w<z\}} q(s)\varphi(p(s)w'(s))(w'-z')(s)ds
\end{aligned}$$

where the last equality is due to the fact that $(w-z)_- = -(w-z)$ on $\{w < z\}$. Note that in the penultimate equality, the integration by parts on $[a,b] \cap \{w < z\}$ can be done after decomposing the set as a countable union of disjoint intervals and the boundary terms drop out since on the end-points of each of these intervals $(w-z)_- = 0$. In conclusion, we get

$$\int\limits_{[a,b]\cap\{w<z\}} q(s)\varphi(p(s)w'(s))(w'-z')(s)ds \leq -\int\limits_{[a,b]\cap\{w<z\}} f(s)(w-z)_-(s)ds\,. \tag{2.5.4}$$

Analogously from (2.5.2) we have:

$$-\int\limits_{[a,b]\cap\{w<z\}} q(s)\varphi(p(s)z'(s))(w'-z')(s)ds \leq \int\limits_{[a,b]\cap\{w<z\}} g(s)(w-z)_-(s)ds\,. \tag{2.5.5}$$

Adding the expressions (2.5.4) and (2.5.5)

$$\begin{aligned} 0 &\geq \int_{[a,b]\cap\{w<z\}} q(s)[\varphi(p(s)w'(s))-\varphi(p(s)z'(s))](w'-z')(s)ds \\ &= \int\limits_{[a,b]\cap\{w<z\}} \frac{q(s)}{p(s)}[\varphi(p(s)w'(s))-\varphi(p(s)z'(s))](pw'-pz')(s)ds\,. \end{aligned}$$

Since, φ is monotone increasing, we have $0 \leq (\varphi(x)-\varphi(y))(x-y) \quad \forall x, y$, which implies that the integrand in the last expression is non-negative and so,

$$\int\limits_{[a,b]\cap\{w<z\}} \frac{q(s)}{p(s)}[\varphi(p(s)w'(s))-\varphi(p(s)z'(s))](pw'-pz')(s)ds = 0\,.$$

From this again, since the integrand above is non-negative, φ is strictly increasing and p, q are non-vanishing, we conclude that

$$w'(t) \;=\; z'(t) \quad \text{a.e. in } [a,b]\cap\{w<z\}\,.$$

Therefore, $w-z$ is constant on each connected component of in $[a,b]\cap\{w<z\}$ and therefore equal to 0 there, since $(w-z)_-=0$ at the end points of each of these connected components. Therefore, we have $(w-z)_-=0$ in $[a,b]$ and consequently, $w \geq z$ in $[a,b]$. □

In the third chapter, while dealing with a Neumann type condition on the finite boundary or at infinity we will need the following two lemma.

Lemma 2.5.2. *Suppose that u is a C^1 variational supersolution of*

$$\left.\begin{array}{c} -(q(t)\varphi(p(t)u'(t)))' \geq 0 \quad 0<t<1 \\ u'(0)=0, u(1)=0\,. \end{array}\right\} \tag{2.5.6}$$

Let us suppose that u is continuous and $t^* \in [0,1]$ is such that $u(t^*)=\|u\|$. Then $u=\|u\|$ in $[0,t^*]$. Suppose that z is a solution of:

$$\left.\begin{array}{c} -(q(t)\varphi(p(t)z'(t)))' = 0 \quad t^*<t<1 \\ z(t^*)=\|u\|\,, z(1)=0\,. \end{array}\right\} \tag{2.5.7}$$

Then $u \geq z$ in $[t^*,1]$.

Proof: The assertion in $[t^*,1]$ follows by applying Lemma 2.5.1 to u and z in this interval. For the proof of the assertion in $[0,t^*]$, the arguments are similar as in that of Lemma 2.5.1. It suffices to multiply (2.5.6) by $\|u\|-u$ and integrate on $[0,t^*]\cap\{u<\|u\|\}$ to obtain:

$$\int_{[0,t^*]\cap\{u<\|u\|\}} q(s)\varphi(p(s)u'(s))u'(s)ds \leq 0\,. \tag{2.5.8}$$

On the other hand,

$$q(s)\varphi(p(s)u'(s))u'(s) = \frac{q(s)}{p(s)}\varphi(p(s)u'(s))p(s)u'(s) \geq 0\,, \tag{2.5.9}$$

since φ is monotone increasing and $\varphi(0) = 0$. From (2.5.8) and (2.5.9) we have

$$q(s)\varphi(p(s)u'(s))u'(s)ds = 0 \qquad \text{on } [0,t^*] \cap \{u < \|u\|\}\,.$$

From this again, since φ is strictly increasing and q and p are strictly positive, we conclude that $u'(t) = 0$ a.e. in $[0,t^*] \cap \{u < \|u\|\}$. Since $u(t^*) = \|u\|$ we conclude that $u = \|u\|$ in $[0,t^*]$.

□

Analogously we have the following lemma.

Lemma 2.5.3. *Suppose that u is a C^1 variational supersolution of*

$$\left.\begin{array}{c} -(q(t)\varphi(p(t)u'(t)))' \geq 0 \quad 0 < t < 1 \\ u(0) = 0, u'(1) = 0\,. \end{array}\right\} \tag{2.5.10}$$

Let us suppose that u is continuous and $t^* \in [0,1]$ is such that $u(t^*) = \|u\|$. Then $u = \|u\|$ in $[t^*,1]$. Suppose that w is a solution of:

$$\left.\begin{array}{c} -(q(t)\varphi(p(t)w'(t)))' = 0 \quad 0 < t < t^* \\ w(0) = 0, w(t^*) = \|u\| \end{array}\right\} \tag{2.5.11}$$

where w is continuous. Then $u \geq w$ in $[0,t^*]$.

Finally, in connection with the problem involving non-linear boundary condition we prove the following lemma.

Lemma 2.5.4. *Let f,g be continuous functions on $[a,b]$ with $f \geq g$ on $[a,b]$. If w and z are C^1 and satisfy the following inequalities in the variational sense*

$$\left.\begin{array}{c} -(q(t)\varphi(p(t)w'(t)))' \geq f \quad 0 < t < 1 \\ w'(0) = cw(0), w(1) = 0 \end{array}\right\} \tag{2.5.12}$$

and

$$\left.\begin{array}{c} -(q(t)\varphi(p(t)z'(t)))' \leq g \quad 0 < t < 1 \\ z'(0) = cz(0), z(1) = 0 \end{array}\right\}, \tag{2.5.13}$$

respectively, where c is a positive constant. Then $w \geq z$ in $[0,1]$.

Proof: The arguments are similar as in Lemma 2.5.1. Multiplying (2.5.12) by $(w-z)_-$ and integrating on $[0,1]$ we obtain:

$$\int_A f(s)(w-z)_-(s)ds \;\; \leq \;\; -\int_A q(s)\varphi(p(s)w'(s))(w'-z')(s)ds + q(0)\varphi(p(0)w'(0))(w-z)_-(0)$$

where $A = [0,1] \cap \{w < z\}$. Equivalently,

$$\int_A q(s)\varphi(p(s)w'(s))(w'-z')(s)ds - q(0)\varphi(p(0)w'(0))(w-z)_-(0) \leq -\int_A f(s)(w-z)_-(s)ds$$

Analogously, starting from (2.5.13) we obtain:

$$-\int_A q(s)\varphi(p(s)z'(s))(w'-z')(s)ds + q(0)\varphi(p(0)z'(0))(w-z)_-(0) \leq \int_A g(s)(w-z)_-(s)ds$$

Adding the previous two inequalities we have

$$\int_A q(s)[\varphi(p(s)w'(s)) - \varphi(p(s)z'(s))](w'-z')(s)ds + q(0)[\varphi(p(0)z'(0)) - \varphi(p(0)w'(0))](w-z)_-(0)$$
$$\leq \int_A (g(s)-f(s))(w-z)_-(s)ds \leq 0$$

Then the proof can be completed as in Lemma 2.5.1 while observing that

$$q(0)[\varphi(p(0)z'(0)) - \varphi(p(0)w'(0))](w-z)_-(0) \geq 0\,.$$

Indeed, if $w(0) \geq z(0)$ then $(w-z)_-(0) = 0$. In the remaining case, $w(0) \leq z(0)$, we have $(w-z)_-(0) = z(0) - w(0)$ and we use the relations $w'(0) = cw(0)$ and $z'(0) = cz(0)$ to get

$$q(0)[\varphi(p(0)\,c\,z(0)) - \varphi(p(0)c\,w(0))](z-w)(0) \geq 0$$

after rewriting the left hand side expression as

$$\frac{q(0)}{p(0)c}[\varphi(p(0)cz(0)) - \varphi(p(0)cw(0))](p(0)cz(0) - p(0)cw(0))$$

and using the monotony of φ. Then, proceeding as in the previous lemma we conclude that $w \geq z$ in $[0,1]$. $\square$

Chapter 3

Positive radial solutions of a quasilinear problem in an exterior domain with vanishing boundary conditions

3.1 Introduction

In this chapter we study the existence and non-existence results of positive radial solutions, given $\lambda > 0$ and $d > 0$, for the quasilinear equation:

$$\operatorname{div}(A(|\nabla u|)\nabla u) + \lambda k(|x|)f(u) = 0, \quad |x| > d, x \in \mathbb{R}^N, N \geq 2 \tag{3.1.1}$$

in conjunction with one of the following boundary condition on the exterior of the ball $B(0, d)$

$$u = 0 \text{ on } |x| = d \text{ and } u \to 0 \text{ when } |x| \to \infty \tag{3.1.2}$$
$$\partial u/\partial r = 0 \text{ on } |x| = d \text{ and } u \to 0 \text{ when } |x| \to \infty \tag{3.1.3}$$
$$u = 0 \text{ on } |x| = d \text{ and } \partial u/\partial r \to 0 \text{ when } |x| \to \infty\,. \tag{3.1.4}$$

To begin with, we set

$$\varphi(t) = A(|t|)t\,.$$

Looking for a radially symmetric solution $u(x) \equiv v(|x|)$ to the problem is equivalent to solving the ordinary differential equation:

$$-(r^{N-1}\varphi(v'(r)))' = \lambda r^{N-1}k(r)f(v(r)) \qquad \text{in} \qquad (d, \infty)\,.$$

By the change of variables $t = 1 - \dfrac{d}{r}$ and setting $w(t) = v\left(\dfrac{d}{1-t}\right)$, this equation leads to

$$-(q(t)\,\varphi(p(t)\,w'(t)))' = \lambda\tilde{k}(t)f(w(t)), \qquad 0 < t < 1, \tag{3.1.5}$$

where

$$q(t) := \left(\frac{d}{1-t}\right)^{N-1} \qquad p(t) := \frac{(1-t)^2}{d}\,, \qquad \tilde{k}(t) := \frac{d^N}{(1-t)^{N+1}}k\left(\frac{d}{1-t}\right)\,. \tag{3.1.6}$$

The boundary conditions in (3.1.2)-(3.1.4) correspond, respectively, to

$$\begin{aligned} w(0) &= w(1) = 0 && (3.1.7)\\ w'(0) &= w(1) = 0 && (3.1.8)\\ w(0) &= w'(1) = 0. && (3.1.9)\end{aligned}$$

Note that q has a singularity at $t = 1$ while $\tilde{k}$ is also possibly singular at $t = 1$. We mention that, in comparison, the study of such problems over annular domains [33] do not give rise to such singularities. This is very important as it affects the choice of the cone in which we can obtain a fixed point and the strategy required for showing the cone preserving property. It is also worth keeping in mind that q is increasing, $\lim_{t\to 1^-} q(t) = +\infty$, p is decreasing and $\lim_{t\to 1^-} p(t) = 0$. Observe also that $\tilde{k}(t) : [0, 1) \to [C_1 d^N, \infty)$ is continuous and $\tilde{k}(t) \neq 0$ on any subinterval of $[0, 1)$. The precise hypotheses on the non-linearities f and φ and on the weight k will be given at the beginning of the next section.

We shall now state our main theorems. The behaviour of the non-linearity f in comparison with φ at 0 and at ∞ will be important for the analysis and for this we set:

$$f_0 := \lim_{x\to 0} \frac{f(x)}{\varphi(x)} \quad \text{and} \quad f_\infty := \lim_{x\to\infty} \frac{f(x)}{\varphi(x)}.$$

We shall assume that the conditions from (H_1) to (H_6), stated in the next section, hold in the case of the problems (3.1.1)-(3.1.2)/ (3.1.1)-(3.1.3) and assume, additionally, that the condition (H_7) holds in the case of the problem (3.1.1)-(3.1.4).

Theorem 3.1.1.

1. *If $f_0 = \infty$ and $f_\infty = \infty$, then for any $R > 0$ there exists λ_R such that for all $0 < \lambda \le \lambda_R$, the problem (3.1.1)-(3.1.2)/ (3.1.1)-(3.1.3)/ (3.1.1)-(3.1.4) has at least two positive solutions u_1 and u_2 with $0 < \|u_1\| < R < \|u_2\|$.*

   If $f_0 = \infty$ or $f_\infty = \infty$, then for any $R > 0$ there exists $\lambda_R > 0$ such that for all $0 < \lambda \le \lambda_R$ the problem (3.1.1)-(3.1.2)/ (3.1.1)-(3.1.3)/ (3.1.1)-(3.1.4) has at least one positive solution u with $0 < \|u\| < R$ or $R < \|u\|$ respectively.

2. *If $f_0 = 0$ and $f_\infty = 0$, then for any $L > 0$ there exists $\lambda_L > 0$ such that for all $\lambda \ge \lambda_L$, the problem (3.1.1)-(3.1.2)/ (3.1.1)-(3.1.3)/ (3.1.1)-(3.1.4) has at least two positive solutions u_1 and u_2 with $0 < \|u_1\| < L < \|u_2\|$.*

   If $f_0 = 0$ or $f_\infty = 0$, then for any $L > 0$ there exists $\lambda_L > 0$ such that for all $\lambda \ge \lambda_L$ the problem (3.1.1)-(3.1.2)/ (3.1.1)-(3.1.3)/ (3.1.1)-(3.1.4) has at least one positive solution u with $0 < \|u\| < L$ or $L < \|u\|$ respectively.

Theorem 3.1.2.

1. *If $f_0 = 0$ and $f_\infty = \infty$, then for all $\lambda > 0$ the problem (3.1.1)-(3.1.2)/ (3.1.1)-(3.1.3)/ (3.1.1)-(3.1.4) has a positive solution.*

2. *If $f_0 = \infty$ and $f_\infty = 0$, then for all $\lambda > 0$ the problem (3.1.1)-(3.1.2)/ (3.1.1)-(3.1.3)/ (3.1.1)-(3.1.4) has a positive solution.*

Theorem 3.1.3.

1. *If $f_0 > 0$ and $f_\infty > 0$, then there exists a positive number $\overline{\lambda}$ such that the problem (3.1.1)-(3.1.2)/ (3.1.1)-(3.1.3)/ (3.1.1)-(3.1.4) has no positive solutions for all $\lambda > \overline{\lambda}$.*

2. *If $f_0 < \infty$ and $f_\infty < \infty$, then there exists a positive number $\underline{\lambda}$ such that the problem (3.1.1)-(3.1.2)/ (3.1.1)-(3.1.3)/ (3.1.1)-(3.1.4) has no positive solutions for all $\lambda < \underline{\lambda}$.*

The organization of the chapter is as follows. In Section 3.2, we provide the operator setting for solving the problem using fixed point methods. In Section 3, we give the proofs of the main results after establishing some preliminary results.

3.2 Setting up of the fixed point problem

In this section we establish the basic abstract framework for solving the problem.

3.2.1 The hypotheses on the non-linearities

We shall make the following assumptions on the non-linearities and the weight k:

(H_1) $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and is *pseudo-homogeneous* in the following sense. There exists two increasing and surjective homeomorphisms $\psi_1, \psi_2 : (0,\infty) \to (0,\infty)$ such that:

$$\psi_1(a)\varphi(b) \leq \varphi(ab) \leq \psi_2(a)\varphi(b) \quad \text{for all } a > 0\,, b \in \mathbb{R}\,.$$

Note: Necessarily, $\varphi(0) = 0$ and $\psi_1(s) \to 0$ as $s \to 0$. The hypothesis (H_1) is satisfied whenever φ is homogeneous or positively homogeneous like in the case of Laplacian for which $\varphi(t) = t$ and $\psi_1(a) = \psi_2(a) = a$ and, in the case of the p-Laplacian operator for which $\varphi(t) = |t|^{p-2}t$ and $\psi_1(a) = \psi_2(a) = a^{p-1}$. In Example 3.2.1 we provide other classes of operators not covered by these operators.

(H_2) For any constant C we have $\displaystyle\int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{C}{q(s)}\right) ds < \infty.$

(H_3) For any constant C we have $\displaystyle\int_0^1 \frac{1}{p(s)}\psi_i^{-1}\left(\frac{C}{q(s)}\right) ds < \infty\,, \quad i = 1, 2.$

Note: The hypothesis (H_2)-(H_3) are necessary for the operators defined in (3.2.3), (3.2.6) and (3.2.9) to be valid and for the finiteness of the constants defined in (3.2.2), (3.2.5) and (3.2.8). These conditions introduce restrictions on the dimension. In the case of Laplacian these conditions are satisfied for $N > 2$ and in the case of the p-Laplacian operator it is satisfied for $N > p$. We now provide an example of another class of operators which are not homogeneous for which the hypotheses (H_1)-(H_3) hold.

Example 3.2.1. *For $1 < \alpha < N-1$, the odd extension of the function $\varphi_\alpha(t) = \dfrac{t^\alpha}{\sqrt{1+t^2}}$ defined for $t \geq 0$ satisfies the hypotheses (H_1)-(H_3). Indeed, for the choice*

$$\psi_1(t) = \begin{cases} t^\alpha, & 0 \leq t < 1 \\ t^{\alpha-1}, & t \geq 1 \end{cases} \qquad \textit{and} \qquad \psi_2(t) = \begin{cases} t^{\alpha-1}, & 0 \leq t < 1 \\ t^\alpha, & t \geq 1 \end{cases}$$

it can be checked that these hypotheses are verified (see the Appendix for the verification). Note that the case $\alpha = 1$ is not included which would correspond to the mean curvature equation but α can be arbitrarily close to 1.

We now give some of the basic hypotheses on the non-linearity f and on the weight k.

(H_4) $f : [0, \infty) \to [0, \infty)$ is continuous and $f(s) > 0$ for $s > 0$.

Note: Since $\varphi(0) = 0$, if $f(0) = 0$, the constant function $u = 0$ is always a non-negative radial solution to the problem (3.1.5) together with any of the boundary conditions (3.1.7), (3.1.8) or (3.1.9).

(H_5) $k : [d, \infty) \to [0, \infty)$ is continuous and $k(t) \neq 0$ on any subinterval of $[d, \infty)$.

(H_6) $0 < \displaystyle\int_0^1 \tilde{k}(t)dt < \infty$.

Note: Recalling the expression for $\tilde{k}$ from (3.1.6), the condition (H_6) requires that k vanishes at infinity at a certain rate to neutralize the singularity introduced by the factor $\dfrac{1}{(1-t)^{N+1}}$. Such a condition is common while studying semilinear problems in exterior domains (see for example Dhanya et al [10]). Examples of k for which (H_6) holds are $k(r) = Cr^{-N-\mu}$ with $\mu > 0$.

(H_7) $q(1-\delta)\psi_2(p(1-\delta))$ is bounded on $0 \leq \delta < 1$.

Note: The last condition is needed for meeting the boundary condition at infinity in (3.1.4) and in the proof of the comparison principles in the next section where integration by parts in intervals of the form $[t, 1]$ is needed.

3.2.2 The function space setting

The setting for obtaining the results will be the Banach space $C[0, 1]$ equipped with the supremum norm $\|u\| = \sup\limits_{t\in[0,1]} |u(t)|$. We shall denote this space by $\mathcal{X}$. Let $0 < \delta < \frac{1}{2}$.

For solving the boundary value problem (3.1.5) along with the boundary condition (3.1.7) we will consider the cone in $\mathcal{X}$ defined by

$$\mathcal{K} := \{u \in \mathcal{X} : u(t) \geq 0,\ u(t) \geq \rho_\delta \|u\|,\ \forall t \in [\delta, 1-\delta]\} \tag{3.2.1}$$

where $0 < \rho_\delta < 1$ (guaranteed by hypothesis (H_3)) which is fixed below

$$\rho_\delta := \frac{1}{\displaystyle\int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{1}{q(s)}\right) ds} \min\left\{\int_0^\delta \frac{1}{p(s)}\psi_2^{-1}\left(\frac{1}{q(s)}\right) ds, \int_{1-\delta}^1 \frac{1}{p(s)}\psi_2^{-1}\left(\frac{1}{q(s)}\right) ds\right\}. \tag{3.2.2}$$

Given $\lambda > 0$, the solutions to the problem (3.1.5) along with the boundary condition (3.1.7) will be obtained as fixed points of an operator S_λ on $\mathcal{K}$ whose definition requires establishing the following assertion.

Assertion 3.2.2. *Given u in $\mathcal{X}$ and non-negative, let us denote by $g(t) := \lambda\tilde{k}(t)f(u(t))$. Then, there exists $\sigma \in (0,1)$ such that $Z_1(\sigma) = Z_2(\sigma)$ where*

$$Z_1(t) = \int_0^t \frac{1}{p(s)}\varphi^{-1}\left(\frac{1}{q(s)}\int_s^t g(\eta)d\eta\right)ds,\ 0 \le t \le 1$$

and

$$Z_2(t) = \int_t^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{1}{q(s)}\int_t^s g(\eta)d\eta\right)ds,\ 0 \le t \le 1\,.$$

Proof: The functions Z_1 and Z_2 are finite valued and continuous guaranteed by the hypothesis (H_2).
If $u = 0$, we may choose any $\sigma \in (0,1)$. Otherwise, $Z_1(1) > 0, Z_2(0) > 0$ and since $Z_2(1) = 0 = Z_1(0) = 0$ we have $H(0) < 0$, $H(1) > 0$ where H is the continuous function $H(t) = Z_1(t) - Z_2(t)$. Furthermore, since g is non-negative, the assumptions on φ imply that Z_1 is increasing and Z_2 is decreasing. These, imply that H is increasing and so, by the Intermediate Value Theorem, we have that there exists $\sigma \in (0,1)$ such that $H(\sigma) = 0$. □

Remark 3.2.3. *Notice that σ depends on u and it may be non-unique.*

We now define the nonlinear operator S_λ from $\mathcal{K}$ to $\mathcal{X}$ as follows. Given $u \in \mathcal{K}$ and for σ (which depends on u) as in the previous assertion we define $S_\lambda u$ similarly as in Wang [33] as follows:

$$S_\lambda u(t) := \begin{cases} \displaystyle\int_0^t \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_s^\sigma \tilde{k}(\eta)f(u(\eta))d\eta\right)ds, & 0 \le t \le \sigma \\ \displaystyle\int_t^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_\sigma^s \tilde{k}(\eta)f(u(\eta))d\eta\right)ds & \sigma \le t \le 1\,. \end{cases} \tag{3.2.3}$$

Remark 3.2.4. *Through Assertion 3.2.6 we will see that this operator is well defined, that is, it depends only on u and is independent of the choice of σ coming from Assertion 3.2.2.*

For solving the boundary value problem (3.1.5) along with the boundary condition (3.1.8) we will consider the cone in $\mathcal{X}$ defined by

$$\mathcal{C} := \{u \in \mathcal{X} : u(t) \ge 0,\ u(t) \ge \varrho_\delta\|u\|,\ \forall t \in [0, 1-\delta]\} \qquad \text{where} \tag{3.2.4}$$

$$\varrho_\delta := \frac{1}{\displaystyle\int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{1}{q(s)}\right)ds}\int_{1-\delta}^1 \frac{1}{p(s)}\psi_2^{-1}\left(\frac{1}{q(s)}\right)ds\,. \tag{3.2.5}$$

Given $\lambda > 0$, the solutions to the problem will be obtained as fixed points of the operator T_λ (which we define below) on the cone $\mathcal{C}$

$$T_\lambda u(t) := \int_t^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_0^s \tilde{k}(\eta)f(u(\eta))d\eta\right)ds, 0 \le t \le 1\,. \tag{3.2.6}$$

For solving the boundary value problem (3.1.5) along with the boundary condition (3.1.9) we will consider the cone in $\mathcal{X}$ defined by

$$\mathcal{P} := \{u \in \mathcal{X} : u(t) \ge 0,\ u(t) \ge \kappa_\delta\|u\|,\ \forall t \in [\delta, 1]\} \tag{3.2.7}$$

where the value of κ_δ is fixed below (note that $0 < \kappa_\delta < 1$)

$$\kappa_\delta := \frac{1}{\displaystyle\int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{1}{q(s)}\right) ds} \int_0^\delta \frac{1}{p(s)}\psi_2^{-1}\left(\frac{1}{q(s)}\right) ds\,. \tag{3.2.8}$$

Given $\lambda > 0$, the solutions to the problem will be obtained as fixed points of the operator V_λ (which we define below) on the cone $\mathcal{P}$

$$V_\lambda u(t) := \int_0^t \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_s^1 \tilde{k}(\eta) f(u(\eta)) d\eta\right) ds, 0 \le t \le 1\,. \tag{3.2.9}$$

Remark 3.2.5. *For every non-negative function u in $\mathcal{X}$, we observe that, by definition, the function $S_\lambda u(\cdot)$ is non-negative and differentiable separately on $[0,\sigma)$ and $(\sigma,1]$. Moreover, from the choice of σ in Assertion 3.2.2, the function $S_\lambda u(\cdot)$ is continuous at σ. It can be seen that the derivative of $S_\lambda u(\cdot)$ at σ is also continuous by using the fact that φ is odd. Thus, for each non-negative continuous function u, the function $v(\cdot) := S_\lambda u(\cdot)$ is a C^1 function on $[0,1]$ and it can be checked that v satisfies the equation*

$$-(q(t)\varphi(p(t)v'(t)))' = \lambda\tilde{k}(t) f(u(t))\,, \qquad 0 < t < 1 \tag{3.2.10}$$

along with the boundary condition (3.1.7)*. This follows from the pointwise relation*

$$v'(t) = \frac{1}{p(t)}\varphi^{-1}\left(\frac{\lambda}{q(t)}\int_t^\sigma \tilde{k}(\eta) f(u(\eta)) d\eta\right)$$

which can be seen to hold on $[0,1]$ whereas the boundary condition (3.1.7) *clearly holds.*

The definition of the operators T_λ and V_λ are much simpler compared to that of S_λ and is valid for all $u \in C[0,1]$. It is seen, similarly, as above that the function $T_\lambda u(\cdot)$ is differentiable on $[0,1]$ satisfies the equation (3.2.10) *along with the boundary condition* (3.1.8)*. In the same way, for each $u \in C[0,1]$ the function $V_\lambda u(\cdot)$ is a C^1 function on $[0,1]$ and satisfies the equation* (3.2.10) *along with the boundary condition* (3.1.9) *(the hypothesis (H_7) plays a role here). Although the operators defined here resemble that in Wang [33], the singulairty of the weights p and q for the problem in an exterior domain require new ways of handling the operators.*

Assertion 3.2.6. *$S_\lambda u$ is independent of the choice of σ which appears in Assertion 3.2.2.*

Proof: Suppose that σ_1, σ_2 with $\sigma_1 < \sigma_2$ are such that

$$H(\sigma_1) = H(\sigma_2) = 0\,.$$

Then, we observe that $g(\eta) \equiv 0$ in $[\sigma_1, \sigma_2]$ where $g(t) := \lambda\tilde{k}(t) f(u(t))$. Indeed, since $\sigma_1 < \sigma_2$, and Z_1 is increasing and Z_2 is decreasing

$$H(\sigma_1) - H(\sigma_2) = Z_1(\sigma_1) - Z_1(\sigma_2) - (Z_2(\sigma_1) - Z_2(\sigma_2)) = Z_1(\sigma_1) - Z_1(\sigma_2) + (Z_2(\sigma_2) - Z_2(\sigma_1))$$

is the sum of two non-positive quantities and is equal to zero, we have in particular that

$$Z_1(\sigma_1) = Z_1(\sigma_2) \text{ and } Z_2(\sigma_2) = Z_2(\sigma_1)\,. \tag{3.2.11}$$

This implies that $g(\eta) \equiv 0$ in $[\sigma_1, \sigma_2]$) by the way that Z_1 and Z_2 are defined and due to the non-negativity of the integrands. This observation is enough to reach our conclusions. Let us denote by $S_\lambda^{\sigma_1} u$ and $S_\lambda^{\sigma_2} u$ the two choices of the image of u under the operation defined in (3.2.3). We shall show that these two choices define the same function. Indeed, if $0 \leq t \leq \sigma_1$, we see that

$$\begin{aligned} S_\lambda^{\sigma_1} u(t) &= \int_0^t \frac{1}{p(s)} \varphi^{-1} \left(\frac{1}{q(s)} \int_s^{\sigma_1} g(\eta) d\eta \right) ds + \int_0^t \frac{1}{p(s)} \varphi^{-1} \left(\frac{1}{q(s)} \int_{\sigma_1}^{\sigma_2} g(\eta) d\eta \right) ds \\ &= \int_0^t \frac{1}{p(s)} \varphi^{-1} \left(\frac{1}{q(s)} \int_s^{\sigma_2} g(\eta) d\eta \right) ds = S_\lambda^{\sigma_2} u(t) \, . \end{aligned}$$

The case $\sigma_2 \leq t \leq 1$ is similar. Finally, if $\sigma_1 \leq t \leq \sigma_2$, we have

$$\begin{aligned} S_\lambda^{\sigma_1} u(t) &= \int_t^1 \frac{1}{p(s)} \varphi^{-1} \left(\frac{1}{q(s)} \int_{\sigma_1}^{s} g(\eta) d\eta \right) ds \\ &= \int_{\sigma_2}^1 \frac{1}{p(s)} \varphi^{-1} \left(\frac{1}{q(s)} \int_{\sigma_2}^{s} g(\eta) d\eta \right) ds = Z_2(\sigma_2) \, . \end{aligned}$$

Similarly,

$$\begin{aligned} S_\lambda^{\sigma_2} u(t) &= \int_0^t \frac{1}{p(s)} \varphi^{-1} \left(\frac{1}{q(s)} \int_s^{\sigma_2} g(\eta) d\eta \right) ds \\ &= \int_0^{\sigma_1} \frac{1}{p(s)} \varphi^{-1} \left(\frac{1}{q(s)} \int_s^{\sigma_1} g(\eta) d\eta \right) ds = Z_1(\sigma_1) \, . \end{aligned}$$

The conclusion $S_\lambda^{\sigma_1} u(t) = S_\lambda^{\sigma_2} u(t)$ follows from (3.2.11) and the fact that $Z_1(\sigma_1) = Z_2(\sigma_1)$. □

3.2.3 Some properties of the operators

We provide now the main properties of the operators S_λ, T_λ and V_λ.

Completely continuity

We first show that the operator S_λ is *completely continuous*. The arguments to show that the operators $T_\lambda : \mathcal{C} \to \mathcal{X}$ and $V_\lambda : \mathcal{P} \to \mathcal{X}$ are completely continuous form particular cases due to the definition of these operators.

Lemma 3.2.7. *The operator $S_\lambda : \mathcal{K} \to \mathcal{X}$ is completely continuous.*

Proof: We need to prove that S_λ is continuous and maps bounded sets to relatively compact sets.
Given $M > 0$, let $\theta = \max_{0<s\leq M} f(s) > 0$. We shall show that $S_\lambda(B(0, M) \cap \mathcal{K})$ is relatively compact in X, as a consequence of the Arzela-Ascoli theorem, by establishing the following:
BOUNDEDNESS: For any $u \in B(0, M) \cap \mathcal{K}$, let σ be as in the definition of $S_\lambda u$ and consider, first, $t \in [0, \sigma]$. We have

$$S_\lambda u(t) = \int_0^t \frac{1}{p(s)} \varphi^{-1} \left(\frac{\lambda}{q(s)} \int_s^{\sigma} \tilde{k}(\eta) f(u(\eta)) d\eta \right) ds$$

$$\leq \int_0^t \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda\theta}{q(s)}\int_0^1 \tilde{k}(\eta)d\eta\right)ds \quad (\text{since } \varphi^{-1} \text{ is increasing by } (H_1)),$$
$$\leq \int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda\theta}{q(s)}\int_0^1 \tilde{k}(\eta)d\eta\right)ds\,.$$

Similarly, the above estimate holds for all $t \in [\sigma, 1]$ by starting from the second expression for $S_\lambda u$ in (3.2.3). This, implies that $S_\lambda(B(0, M) \cap \mathcal{K})$ is bounded.

EQUICONTINUITY: We now will prove that $S_\lambda(B(0, M \cap \mathcal{K})) = \{S_\lambda u : \|u\| \leq M,\ u \in \mathcal{K}\}$ is an equicontinuous family in $\mathcal{X}$. Given, $\varepsilon > 0$, using the finiteness hypothesis on the integral (H_2), choose $\delta > 0$ such that

$$\int_{t_2}^{t_1} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda\theta}{q(s)}\int_0^1 \tilde{k}(\eta)d\eta\right)ds < \varepsilon\,. \tag{3.2.12}$$

Then, for $t_1, t_2 \in [0, \sigma]$ and $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}|S_\lambda u(t_1) - S_\lambda u(t_2)| &= \int_{t_2}^{t_1} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_s^\sigma \tilde{k}(\eta)f(u(\eta))d\eta\right)ds \\ &\leq \int_{t_2}^{t_1} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda\theta}{q(s)}\int_0^1 \tilde{k}(\eta)d\eta\right)ds \qquad (\text{by} \quad (H_1)) \\ &< \varepsilon \qquad (\text{by } (3.2.12))\,.\end{aligned}$$

Similarly for $t_1, t_2 \in [\sigma, 1]$ and $|t_1 - t_2| < \delta$.

In the case that $0 < t_1 < \sigma < t_2 < 1$ we have:

$$\begin{aligned}|S_\lambda u(t_2) - S_\lambda u(t_1)| &= |S_\lambda u(t_2) - S_\lambda u(\sigma) + S_\lambda u(\sigma) - S_\lambda u(t_1)| \\ &\leq |S_\lambda u(t_2) - S_\lambda u(\sigma)| + |S_\lambda u(\sigma) - S_\lambda u(t_1)| < 2\varepsilon \ \ (\text{by above two steps})\,.\end{aligned}$$

Therefore, by Ascoli-Arzelá theorem, S_λ is a compact operator.

S_λ IS CONTINUOUS: Let $w_n, w \in \mathcal{K}$ be such that $w_n \to w$ in $C[0, 1]$. We would like to prove that $S_\lambda w_n \to S_\lambda w$. The sequence w_n is bounded in $C[0, 1]$ and since S_λ is a compact operator we know that $\{S_\lambda w_n\}_{n=1}^\infty$ is an equicontinuous family. Then, for a subsequence for which we still use the same indices, $S_\lambda w_n$ converges to some $u \in X$. Since u and $S_\lambda w$ are continuous functions it is enough to show that $u = S_\lambda w$ on a dense set for which it suffices to prove the following pointwise convergence

$$S_\lambda w_n(t) \overset{n\to\infty}{\longrightarrow} S_\lambda w(t) \text{ for all } t \in [0, 1] \setminus \{\sigma\}\,. \tag{3.2.13}$$

For each of the w_n let $\sigma_n \in [0, 1]$ be as in the definition of $S_\lambda w_n$ similar as in (3.2.3) guaranteed by Assertion 3.2.2. Then, for a subsequence, we have $\sigma_n \to \sigma$.

We now prove (3.2.13). Consider $t \in [0, 1]$ and let us consider the case $0 \leq t < \sigma$. In the present case, since $\sigma_n \to \sigma$ and $t < \sigma$ it follows that $t < \sigma_n$ for all n large enough. Then, by definition

$$S_\lambda w_n(t) = \int_0^t \frac{1}{p(s)}\left(\varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_n} w_n(\eta)d\eta\right)\right)ds\,. \tag{3.2.14}$$

We note that

$$\left|\int_s^{\sigma_n} w_n(\eta)d\eta - \int_s^\sigma w(\eta)d\eta\right| \ = \ \left|\int_s^\sigma w_n(\eta)d\eta - \int_{\sigma_n}^\sigma w_n(\eta)d\eta - \int_s^\sigma w(\eta)d\eta\right|$$

$$\leq \int_s^{\sigma} |w_n(\eta) - w(\eta)| d\eta + \left| \int_{\sigma_n}^{\sigma} w_n(\eta) d\eta \right| \overset{n\to\infty}{\longrightarrow} 0$$

since w_n converges to w uniformly, $\sigma_n \to \sigma$ and w_n is uniformly bounded. The continuity of φ^{-1} implies the pointwise convergence of the integrands in (3.2.14). Then applying dominated convergence theorem to (3.2.14) we conclude that

$$S_\lambda w_n(t) = \int_0^t \frac{1}{p(s)} \left(\varphi^{-1} \left(\frac{1}{q(s)} \int_s^{\sigma_n} w_n(\eta) d\eta \right) \right) ds \to \int_0^t \frac{1}{p(s)} \left(\varphi^{-1} \left(\frac{1}{q(s)} \int_s^{\sigma} w(\eta) d\eta \right) \right) ds = S_\lambda w(t)$$

proving (3.2.13) for $t < \sigma$. The case $\sigma < t \leq 1$ can be argued similarly. □

Cone invariance

Proposition 3.2.8. *Let $\mathcal{K}$ defined by (3.2.1). Then the operator S_λ defined by* (3.2.3) *preserves the cone $\mathcal{K}$, that is, if $v \in \mathcal{K}$ then $S_\lambda v \in \mathcal{K}$.*

Proof: Let $v \in \mathcal{K}$ and so we have $\lambda \tilde{k}(t) f(v(t)) \geq 0$. Then, $S_\lambda v = u$ satisfies

$$\left. \begin{array}{c} -(q(t)\varphi(p(t)u'(t)))' = \lambda \tilde{k}(t) f(v(t)) \geq 0 \quad 0 < t < 1 \\ u(0) = 0, u(1) = 0 \end{array} \right\} \tag{3.2.15}$$

and let $t^* \in [0,1]$ be such that $u(t^*) = \|u\|$. Now, we consider an auxiliary function w which satisfies:

$$\left. \begin{array}{c} -(q(t)\varphi(p(t)w'(t)))' = 0 \quad 0 < t < t^* \\ w(0) = 0\,, w(t^*) = \|u\|\,. \end{array} \right\} \tag{3.2.16}$$

On the one hand, it follows that $w(t) = \displaystyle\int_0^t \frac{1}{p(s)} \varphi^{-1} \left(\frac{C}{q(s)} \right) ds$ with C constant. So,

$$\begin{aligned} w(t^*) = \|u\| &= \int_0^{t^*} \frac{1}{p(s)} \varphi^{-1} \left(\frac{C}{q(s)} \right) ds \\ &\leq \int_0^1 \frac{1}{p(s)} \psi_1^{-1} \left(\frac{1}{q(s)} \right) ds \ \varphi^{-1}(C) \quad \text{(from Lemma 3.3.1)} \end{aligned}$$

leading to the inequality,

$$\varphi^{-1}(C) \geq \frac{\|u\|}{\displaystyle\int_0^1 \frac{1}{p(s)} \psi_1^{-1} \left(\frac{1}{q(s)} \right) ds}\,. \tag{3.2.17}$$

On the other hand, by applying Lemma 2.5.1 to u and w in $[0, t^*]$, we have that

$$\begin{aligned} u(t) &\geq w(t) = \int_0^t \frac{1}{p(s)} \varphi^{-1} \left(\frac{C}{q(s)} \right) ds \quad \forall\, t \in [0, t^*] \\ &\geq \int_0^t \frac{1}{p(s)} \psi_2^{-1} \left(\frac{1}{q(s)} \right) ds \ \varphi^{-1}(C) \\ &\geq \frac{\|u\|}{\displaystyle\int_0^1 \frac{1}{p(s)} \psi_1^{-1} \left(\frac{1}{q(s)} \right) ds} \int_0^t \frac{1}{p(s)} \psi_2^{-1} \left(\frac{1}{q(s)} \right) ds \qquad \text{using (3.2.17)}\,. \end{aligned}$$

The estimate in $[t^*,1]$ is obtained similarly using the auxiliary function z which solves

$$\left.\begin{array}{c}-(q(t)\varphi(p(t)z'(t)))'=0 \quad t^*<t<1\\ z(t^*)=\|u\|\,, z(1)=0\,.\end{array}\right\} \tag{3.2.18}$$

By applying Lemma 2.5.1 to u and z in $[t^*,1]$, we have

$$\begin{aligned} u(t) &\geq z(t)=\int_t^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{C}{q(s)}\right)ds \quad \forall t\in[t^*,1]\\ &\geq \int_t^1 \frac{1}{p(s)}\psi_2^{-1}\left(\frac{1}{q(s)}\right)ds\,\varphi^{-1}(C)\\ &\geq \frac{\|u\|}{\displaystyle\int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{1}{q(s)}\right)ds}\int_t^1 \frac{1}{p(s)}\psi_2^{-1}\left(\frac{1}{q(s)}\right)ds \qquad \text{using (3.2.17)}\,. \end{aligned}$$

Thus, for $\delta\leq t\leq 1-\delta$, we have

$$u(t)\geq \frac{\|u\|}{\displaystyle\int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{1}{q(s)}\right)ds}\min\left\{\int_0^\delta \frac{1}{p(s)}\psi_2^{-1}\left(\frac{1}{q(s)}\right)ds,\int_{1-\delta}^1 \frac{1}{p(s)}\psi_2^{-1}\left(\frac{1}{q(s)}\right)ds\right\}=\|u\|\rho_\delta\,. \tag{3.2.19}$$

This completes the proof. □

We now show that T_λ preserves the cone $\mathcal{C}$ and V_λ preserves the cone $\mathcal{P}$.

Proposition 3.2.9. *Let $\mathcal{C}$ defined by (3.2.4). Then the operator T_λ defined by* (3.2.6) *preserves the cone $\mathcal{C}$, that is, if $v\in\mathcal{C}$ then $T_\lambda v\in\mathcal{C}$.*

Proof: Let $v\in\mathcal{C}$ and so we have $\lambda\tilde{k}(t)f(v(t))\geq 0$. Then, $T_\lambda v=U$ satisfies

$$\left.\begin{array}{c}-(q(t)\varphi(p(t)U'(t)))'=\lambda\tilde{k}(t)f(v(t))\geq 0 \quad 0<t<1\\ U'(0)=0, U(1)=0\end{array}\right\} \tag{3.2.20}$$

and let $t^*\in[0,1]$ be such that $U(t^*)=\|U\|$. Then, by Lemma 2.5.2, we obtain $U(t)=\|U\|$ for all $t\in[0,t^*]$.
The following estimate in $[t^*,1]$

$$U(t)\geq \frac{\|U\|}{\displaystyle\int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{1}{q(s)}\right)ds}\int_{1-\delta}^1 \frac{1}{p(s)}\psi_2^{-1}\left(\frac{1}{q(s)}\right)ds=\|U\|\varrho_\delta \tag{3.2.21}$$

is obtained like in the second part of the previous proposition using the auxiliary function z which is solution of (3.2.18). □

Proposition 3.2.10. *Let $\mathcal{P}$ defined by (3.2.7). Then the operator V_λ defined by* (3.2.9) *preserves the cone $\mathcal{P}$, that is, if $v\in\mathcal{P}$ then $V_\lambda v\in\mathcal{P}$.*

Proof: By similar arguments as in the previous propositions and using Lemma 2.5.3 we obtain the result. □

Remark 3.2.11. *We see that the proof of the cone preserving property requires a different approach from that used in Wang [33] and is based on comparison principles (Lemmas 2.5.1, 2.5.2 and 2.5.3).*

3.3 Proofs of the Main Results

Before we prove the main results we provide some preliminary results which will be used frequently in the proofs.

Lemma 3.3.1. *Assuming that (H_1) holds for all $a \in (0,\infty), b \in \mathbb{R}$ we have that*

$$\psi_2^{-1}(a)b \leq \varphi^{-1}(a\varphi(b)) \leq \psi_1^{-1}(a)b\,.$$

Proof: This is an elementary consequence of (H_1). For the proof see Lemma 2.6 [33]. $\square$

Lemma 3.3.2. *There exists $C_1 > 0$ such that*

$$\int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{C_1}{q(s)}\int_0^1 \tilde{k}(\eta)d\eta\right)ds < 1\,. \tag{3.3.1}$$

Proof: By the fact that $\psi_1(s) \to 0$ as $s \to 0$ and is homeomorphic on $(0,\infty)$ we have $\psi_1^{-1}(s) \to 0$ as $s \to 0$. Now, the finiteness of the integral given by (H_6) allows us to apply the Dominated Convergence theorem to conclude that the integral in (3.3.1) tends to 0 as C_1 tends to 0. In particular, C_1 can be chosen so that (3.3.1) holds. $\square$

For $0 < \delta < \frac{1}{2}$ fixed and $\lambda > 0$, $M > 0$, we define the following quantity:

$$y_{M,\lambda}(\delta) := \min\left\{\int_\delta^{\frac{1}{2}} \frac{1}{p(s)}\psi_2^{-1}\left(\frac{\lambda M}{q(s)}\int_s^{\frac{1}{2}}\tilde{k}(\eta)d\eta\right)ds, \int_{\frac{1}{2}}^{1-\delta}\frac{1}{p(s)}\psi_2^{-1}\left(\frac{\lambda M}{q(s)}\int_{\frac{1}{2}}^{s}\tilde{k}(\eta)d\eta\right)ds\right\}. \tag{3.3.2}$$

Note that by the definition of $y_{M,\lambda}(\delta)$ its dependence on λ and M is only through λM.

Lemma 3.3.3. *Let $c_\delta > 0$ be any constant (which, for our applications, shall be taken to be ρ_δ, ϱ_δ or κ_δ depending on the cone). There exists $A > 0$ such that whenever $\lambda M \geq A$ we have*

$$y_{M,\lambda}(\delta)\, c_\delta > 1\,. \tag{3.3.3}$$

Proof: Since, ψ_2^{-1} is monotone increasing and $\psi_2 : (0,\infty) \to (0,\infty)$ is homeomorphic, we have $\psi_2^{-1}(s) \to \infty$ as $s \to \infty$. So, if we let $a = \lambda M \to \infty$, then $y_{M,\lambda}(\delta) \to \infty$ by the Monotone Convergence Theorem. So, we can find $A > 0$ such that for all λ and M with $\lambda M \geq A$ the inequality (3.3.3) holds. $\square$

Remark 3.3.4. *In particular, given $\lambda > 0$ we can choose $M_\lambda = \dfrac{A}{\lambda}$ so that for all $M \geq M_\lambda$ we have*

$$y_{M,\lambda}(\delta)\, c_\delta > 1\,. \tag{3.3.4}$$

Also given, $M > 0$ we can choose $\lambda_M = \dfrac{A}{M}$ such that for all $\lambda \geq \lambda_M$ we have

$$y_{M,\lambda}(\delta)\, c_\delta > 1\,. \tag{3.3.5}$$

The following lemma will be useful in establishing that contraction holds under certain hypotheses.

Lemma 3.3.5. *Consider $C > 0$, $u \in C[0,1]$ be non-negative and assume that $f(u(t)) \leq C\varphi(u(t))$ for all $t \in [0,1]$. Then, for any $\lambda > 0$, and $\mathcal{A}$ denoting any of the operators S_λ, T_λ or V_λ we have*

$$\|\mathcal{A}u\| \leq \int_0^1 \frac{1}{p(s)} \psi_1^{-1}\left(\frac{\lambda C}{q(s)} \int_0^1 \tilde{k}(\eta) d\eta\right) ds \, \|u\| \,. \tag{3.3.6}$$

Proof: We prove the statement for $\mathcal{A} = S_\lambda$. The proof for the choices $\mathcal{A} = T_\lambda$ or $\mathcal{A} = V_\lambda$ is similar. Let $u \in C[0,1]$ be non-negative. We then obtain, using (3.2.3), that

$$\|S_\lambda u\| \leq \int_0^1 \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda}{q(s)} \int_0^1 \tilde{k}(\eta) f(u(\eta)) d\eta\right) ds \,.$$

Therefore,

$$\begin{aligned}
\|S_\lambda u\| &\leq \int_0^1 \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda C}{q(s)} \int_0^1 \tilde{k}(\eta) \varphi(u(\eta)) d\eta\right) ds \\
&\leq \int_0^1 \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda C}{q(s)} \varphi(\|u\|) \int_0^1 \tilde{k}(\eta) d\eta\right) ds \\
&\leq \int_0^1 \frac{1}{p(s)} \psi_1^{-1}\left(\frac{\lambda C}{q(s)} \int_0^1 \tilde{k}(\eta) d\eta\right) ds \, \|u\| \qquad \text{(by Lemma 3.3.1)} \,.
\end{aligned}$$

This completes the estimate. □

The following will be useful in showing that expansion holds under certain hypotheses.

Lemma 3.3.6. *Let $\mathcal{D}$ denote one of the cones $\mathcal{K}$, $\mathcal{C}$ or $\mathcal{P}$ and let c_δ be the corresponding constant ρ_δ, ϱ_δ or κ_δ, respectively, depending on whether we consider the operator $\mathcal{A}$ equal to S_λ, T_λ or V_λ, respectively. Consider $M > 0$ and $u \in \mathcal{D}$ and assume that $f(u(t)) \geq M\varphi(u(t))$ for all $t \in [\delta, 1-\delta]$ with $0 < \delta < \frac{1}{2}$ as in the definition of $\mathcal{D}$. Then, for any $\lambda > 0$ and $y_{M,\lambda}(\delta)$ as defined in (3.3.2) we have*

$$\|\mathcal{A}u\| \geq c_\delta \|u\| \, y_{M,\lambda}(\delta) \,. \tag{3.3.7}$$

Proof: We prove the statement for $\mathcal{A} = S_\lambda$. Let $u \in \mathcal{K}$. Given $u \in \mathcal{K}$ let σ be as in Assertion 3.2.2. If $\sigma \leq \frac{1}{2}$ then, we have

$$\begin{aligned}
\|S_\lambda u\| &\geq S_\lambda u(\sigma) = \int_\sigma^1 \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda}{q(s)} \int_\sigma^s \tilde{k}(\eta) f(u(\eta)) d\eta\right) ds \\
&\geq \int_{\frac{1}{2}}^{1-\delta} \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda}{q(s)} \int_{\frac{1}{2}}^s \tilde{k}(\eta) f(u(\eta)) d\eta\right) ds \\
&\geq \int_{\frac{1}{2}}^{1-\delta} \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda M}{q(s)} \int_{\frac{1}{2}}^s \tilde{k}(\eta) \varphi(u(\eta)) d\eta\right) ds \\
&\geq \int_{\frac{1}{2}}^{1-\delta} \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda M}{q(s)} \int_{\frac{1}{2}}^s \tilde{k}(\eta) d\eta \, \varphi(\rho_\delta \|u\|)\right) ds \qquad \text{(since } u \in \mathcal{K}\text{)}
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\frac{1}{2}}^{1-\delta} \frac{1}{p(s)}\psi_2^{-1}\left(\frac{\lambda M}{q(s)}\int_{\frac{1}{2}}^{s}\tilde{k}(\eta)d\eta\right)\rho_\delta\|u\|ds \qquad \text{(by Lemma 3.3.1)}\\
&= \rho_\delta\|u\|\, y_{M,\lambda}(\delta)\,.
\end{aligned}
$$

Similarly, if $\sigma \geq \frac{1}{2}$ then, we have

$$
\begin{aligned}
\|S_\lambda u\| &\geq S_\lambda u(\sigma) = \int_0^{\sigma} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_s^{\sigma}\tilde{k}(\eta)f(u(\eta))d\eta\right)ds\\
&\geq \int_\delta^{\frac{1}{2}} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_s^{\frac{1}{2}}\tilde{k}(\eta)f(u(\eta))d\eta\right)ds\\
&\geq \int_\delta^{\frac{1}{2}} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda M}{q(s)}\int_s^{\frac{1}{2}}\tilde{k}(\eta)\varphi(u(\eta))d\eta\right)ds\\
&\geq \int_\delta^{\frac{1}{2}} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda M}{q(s)}\int_s^{\frac{1}{2}}\tilde{k}(\eta)d\eta\,\varphi(\rho_\delta\|u\|)\right)ds \qquad \text{(by } u\in\mathcal{K})\\
&\geq \int_\delta^{\frac{1}{2}} \frac{1}{p(s)}\psi_2^{-1}\left(\frac{\lambda M}{q(s)}\int_s^{\frac{1}{2}}\tilde{k}(\eta)d\eta\right)\rho_\delta\|u\|ds \qquad \text{(by Lemma 3.3.1)}\\
&= \rho_\delta\|u\|\, y_{M,\lambda}(\delta)\,.
\end{aligned}
$$

When $\mathcal{A} = T_\lambda$ or $\mathcal{A} = V_\lambda$ the argument is more simple. Indeed, if $\mathcal{A} = T_\lambda$, we have

$$
\begin{aligned}
\|T_\lambda u\| &\geq T_\lambda u(1/2) = \int_{\frac{1}{2}}^{1} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_0^{s}\tilde{k}(\eta)f(u(\eta))d\eta\right)ds\\
&\geq \int_{\frac{1}{2}}^{1-\delta} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_{\frac{1}{2}}^{s}\tilde{k}(\eta)f(u(\eta))d\eta\right)ds\\
&\geq \int_{\frac{1}{2}}^{1-\delta} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda M}{q(s)}\int_{\frac{1}{2}}^{s}\tilde{k}(\eta)\varphi(u(\eta))d\eta\right)ds\\
&\geq \int_{\frac{1}{2}}^{1-\delta} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda M}{q(s)}\int_{\frac{1}{2}}^{s}\tilde{k}(\eta)d\eta\,\varphi(\varrho_\delta\|u\|)\right)ds \qquad \text{(since } \ u\in\mathcal{C})\\
&\geq \int_{\frac{1}{2}}^{1-\delta} \frac{1}{p(s)}\psi_2^{-1}\left(\frac{\lambda M}{q(s)}\int_{\frac{1}{2}}^{s}\tilde{k}(\eta)d\eta\right)\varrho_\delta\|u\|ds \qquad \text{(by Lemma 3.3.1)}\\
&= \varrho_\delta\|u\|\, y_{M,\lambda}(\delta)\,.
\end{aligned}
$$

Again if $\mathcal{A} = V_\lambda$ then, we have,

$$
\begin{aligned}
\|V_\lambda u\| &\geq V_\lambda u(1/2) = \int_0^{\frac{1}{2}} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_s^{1}\tilde{k}(\eta)f(u(\eta))d\eta\right)ds\\
&\geq \int_\delta^{\frac{1}{2}} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_s^{\frac{1}{2}}\tilde{k}(\eta)f(u(\eta))d\eta\right)ds\\
&\geq \int_\delta^{\frac{1}{2}} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda M}{q(s)}\int_s^{\frac{1}{2}}\tilde{k}(\eta)\varphi(u(\eta))d\eta\right)ds
\end{aligned}
$$

$$
\begin{aligned}
&\geq \int_\delta^{\frac{1}{2}} \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda M}{q(s)}\int_s^{\frac{1}{2}}\tilde{k}(\eta)d\eta\,\varphi(\kappa_\delta\|u\|)\right)ds \qquad (\text{by } u\in\mathcal{P})\\
&\geq \int_\delta^{\frac{1}{2}} \frac{1}{p(s)}\psi_2^{-1}\left(\frac{\lambda M}{q(s)}\int_s^{\frac{1}{2}}\tilde{k}(\eta)d\eta\right)\kappa_\delta\|u\|ds \qquad (\text{by Lemma 3.3.1})\\
&= \kappa_\delta\|u\|\,y_{M,\lambda}(\delta)\,.
\end{aligned}
$$

This completes the proof of the lemma. □

3.3.1 Proof of Theorem 3.1.1

We shall provide the proof only in the case of the boundary condition (3.1.2) which leads to the choice of the operator S_λ defined on the cone $\mathcal{K}$. The proofs in the other cases go through identically by working with the operator T_λ on the cone $\mathcal{C}$ or V_λ on the cone $\mathcal{P}$ with the corresponding version of the Lemmas 3.3.3, 3.3.5 and 3.3.6 at the beginning of this section.

First, we prove part 1 of Theorem 3.1.1.

STEP 1: Consider any $R>0$. Let $\overline{R}$ be a point where f reaches its maximum in the interval $[0,R]$. So, for $u\in\partial\Omega_R$ where $\Omega_R=\{u\in\mathcal{K}:\|u\|<R\}$ we have

$$
\begin{aligned}
\|S_\lambda u\| &\leq \int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_0^1\tilde{k}(\eta)f(u(\eta))d\eta\right)ds\\
&\leq \int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}f(\overline{R})\int_0^1\tilde{k}(\eta)d\eta\right)ds\\
&= \int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\frac{f(\overline{R})}{\varphi(\overline{R})}\varphi(\overline{R})\int_0^1\tilde{k}(\eta)d\eta\right)ds\\
&\leq \int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{\lambda}{q(s)}\frac{f(\overline{R})}{\varphi(\overline{R})}\int_0^1\tilde{k}(\eta)d\eta\right)ds\,\overline{R} \qquad (\text{by Lemma 3.3.1})\\
&\leq \int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{\lambda}{q(s)}\frac{f(\overline{R})}{\varphi(\overline{R})}\int_0^1\tilde{k}(\eta)d\eta\right)ds\,R\\
&= \int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{\lambda}{q(s)}\frac{f(\overline{R})}{\varphi(\overline{R})}\int_0^1\tilde{k}(\eta)d\eta\right)ds\,\|u\| \qquad (\text{ since } \|u\|=R)\,.
\end{aligned}
$$

If we now choose λ_R such that $\lambda_R\dfrac{f(\overline{R})}{\varphi(\overline{R})}\leq C_1$ with C_1 as in Lemma 3.3.2, then we have

$$
\int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{\lambda}{q(s)}\frac{f(\overline{R})}{\varphi(\overline{R})}\int_0^1\tilde{k}(\eta)d\eta\right)ds<1 \quad \text{for all} \quad 0<\lambda\leq\lambda_R
$$

which gives

$$
\|S_\lambda u\|<\|u\| \quad \text{for} \quad u\in\partial\Omega_R \quad \text{and for all} \quad 0<\lambda\leq\lambda_R\,. \tag{3.3.8}
$$

At this point, if $f(0)>0$ then we can immediately obtain a non-trivial positive radial solution by applying, for example, the Schauder fixed point theorem in Ω_R. However, if $f(0)=0$, it is not possible to distinguish the solution in Ω_R from the zero solution. So, the next step allows us to apply Krasnosel'skii theorem in a conical shell and thus obtain a non-trivial positive radial solution.

STEP 2: Now, let us fix $0 < \lambda \leq \lambda_R$ from the previous step and M_λ be chosen using Lemma 3.3.3 so that (3.3.4) holds. Now since we assume $f_0 = \infty$, given $M_\lambda > 0$ as above, there exists $0 < r_{M_\lambda} < R$ such that $f(x) \geq M_\lambda \varphi(x)$ if $x \leq r_{M_\lambda}$. Thus, for $u \in \partial\Omega_{r_{M_\lambda}}$, where $\Omega_{r_{M_\lambda}} = \{u \in \mathcal{K} : \|u\| < r_{M_\lambda}\}$ we have $u(t) \leq r_{M_\lambda}$ on $[0,1]$ and this gives that $f(u(t)) \geq M_\lambda \varphi(u(t))$ on $[0,1]$. So, by Lemma 3.3.6 and (3.3.4),

$$\|S_\lambda u\| > \|u\| \quad \text{for} \quad u \in \partial\Omega_{r_{M_\lambda}}\,. \tag{3.3.9}$$

As an immediate conclusion, applying Theorem 2.2.3 using (3.3.9) and (3.3.8) above, it follows that there exists at least one fixed point u_1 in $\mathcal{K} \cap (\overline{\Omega_R} \setminus \Omega_{r_{M_\lambda}})$ (with $r_{M_\lambda} < \|u_1\| < R$).

STEP 3: Now, for $0 < \lambda < \lambda_R$ and $M_\lambda > 0$ as in the previous step, since $f_\infty = \infty$ we can find a $R_{M_\lambda} > R$ such that $f(x) \geq M_\lambda \varphi(x)$ for all $x \geq R_{M_\lambda}$. Setting $N_\lambda = R_{M_\lambda}/\rho_\delta$, we observe that for $u \in \partial\Omega_{N_\lambda}$ where $\Omega_{N_\lambda} = \{u \in \mathcal{K} : \|u\| < N_\lambda\}$, we have $u(t) \geq R_{M_\lambda}$ on $[\delta, 1-\delta]$ and this implies that $f(u(t)) \geq M_\lambda \varphi(u(t))$ on $[\delta, 1-\delta]$. So, by Lemma 3.3.6 and (3.3.4), we conclude that

$$\|S_\lambda u\| > \|u\| \quad \text{for} \quad u \in \partial\Omega_{N_\lambda}\,. \tag{3.3.10}$$

Then, since we have $N_\lambda \geq R_{M_\lambda} > R$, applying again Theorem 2.2.3 using (3.3.10) and (3.3.8) above, it holds that there exists at least one fixed point u_2 in $\mathcal{K} \cap (\overline{\Omega_{N_\lambda}} \setminus \Omega_R)$ and, in fact, $R < \|u_2\| < N_\lambda$.

The conclusions of Step 2 and Step 3 together prove the first part of part 1 of Theorem 3.1.1.

STEP 4: We observe that if $f_0 = \infty$, the arguments in Steps 1 and 2 show that for any $R > 0$ there exists $\lambda_R > 0$ such that for all $0 < \lambda \leq \lambda_R$ the problem (3.1.5)-(3.1.7) has at least one positive radial solution u with $0 < \|u\| < R$.

STEP 5: If $f_\infty = \infty$, the arguments in Steps 1 and 3 show that for any $R > 0$ there exists $\lambda_R > 0$ such that for all $0 < \lambda \leq \lambda_R$ the problem (3.1.5)-(3.1.7) has at least one positive radial solution u with $R < \|u\|$.

So, the proof of part 1 of Theorem 3.1.1 is complete. Now, we prove part 2 of Theorem 3.1.1.

STEP 1: Fix $L > 0$ and let

$$m(L) := \min_{\rho_\delta L \leq s \leq L} \frac{f(s)}{\varphi(s)}\,. \tag{3.3.11}$$

We note that $m(L) > 0$ for $L > 0$ and then for $M = m(L)$ choose λ_M as in Lemma 3.3.3 so that for all $\lambda \geq \lambda_{m(L)}$ we have

$$y_{m(L),\lambda}\, \rho_\delta > 1\,. \tag{3.3.12}$$

Let us denote $\lambda_{m(L)}$ by λ_L. Therefore, for any fixed $\lambda \geq \lambda_L$, using Lemma 3.3.6 and (3.3.12), we conclude that

$$\|S_\lambda u\| > \|u\| \quad \text{for} \quad u \in \partial\Omega_L\,. \tag{3.3.13}$$

To be precise, for $u \in \mathcal{K}$ with $\|u\| = L$ we have $u(t) \geq \rho_\delta L$ for all $t \in [\delta, 1-\delta]$ and this means that $f(u(t)) \geq m(L)\varphi(u(t))$ for all $t \in [\delta, 1-\delta]$ and so Lemma 3.3.6 can be applied.

STEP 2: For L and λ as in the previous step. By Lemma 3.3.2, there exists $\varepsilon_\lambda > 0$ (for example take $\varepsilon_\lambda = \frac{C_1}{\lambda}$) such that

$$\int_0^1 \frac{1}{p(s)} \psi_1^{-1}\left(\frac{\lambda\,\varepsilon_\lambda}{q(s)} \int_0^1 \tilde{k}(\tau)d\tau\right) ds < 1\,.$$

Using the condition $f_0 = 0$, we can find a l with $0 < l < L$ such that $f(x) \leq \varepsilon_\lambda \varphi(x)$ for all $0 \leq x \leq l$. Then for $u \in \partial\Omega_l$, where $\Omega_l = \{u \in \mathcal{K} : \|u\| < l\}$ we have $f(u(s)) \leq \varepsilon_\lambda \varphi(u(s))$ for all $s \in [0,1]$. So, using Lemma 3.3.5, we have

$$\|S_\lambda u\| \;\leq\; \int_0^1 \frac{1}{p(s)} \psi_1^{-1}\left(\frac{\lambda\,\varepsilon_\lambda}{q(s)} \int_0^1 \tilde{k}(\tau)d\tau\right) ds\;\|u\|\,.$$

Therefore, we have

$$\|S_\lambda u\| < \|u\| \quad \text{for} \quad u \in \partial\Omega_l\,. \tag{3.3.14}$$

Then applying Theorem 2.2.3, using (3.3.13) and (3.3.14), S_λ has at least one fixed point u_1 in $\overline{\Omega_L} \setminus \Omega_l$ and, in fact, $l < \|u_1\| < L$.

Step 3: We introduce the non-decreasing function f^*

$$f^*(t) := \max_{0 \leq s \leq t} \{f(s)\}\,.$$

Note by $f^*_\infty = \lim_{x\to\infty} \frac{f^*(x)}{\varphi(x)}$. It can be seen that $f^*_\infty = f_\infty$ (view [33], Lemma 2.8) and since $f_\infty = 0$ it follows that $f^*_\infty = 0$. Then, for the same $\varepsilon_\lambda > 0$ fixed as above, we find a $L_\lambda > L$ such that $f^*(x) \leq \varepsilon_\lambda \varphi(x)$ for all $x \geq L_\lambda$.Then, for $u \in \partial\Omega_{L_\lambda}$, with $\Omega_{L_\lambda} = \{u \in \mathcal{K} : \|u\| < L_\lambda\}$ we have $f^*(S_\lambda) \leq \varepsilon_\lambda \varphi(L_\lambda)$. Then,

$$\begin{aligned}
\|S_\lambda u\| \;&\leq\; \int_0^1 \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda}{q(s)} \int_0^1 \tilde{k}(\tau) f^*(u(\tau))d\tau\right) ds \\
&\leq\; \int_0^1 \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda}{q(s)} \int_0^1 \tilde{k}(\tau) f^*(L_\lambda)d\tau\right) ds \quad \text{(since } f^* \text{ is increasing)} \\
&\leq\; \int_0^1 \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda\,\varepsilon_\lambda}{q(s)} \varphi(L_\lambda) \int_0^1 \tilde{k}(\tau)d\tau\right) ds \\
&\leq\; \int_0^1 \frac{1}{p(s)} \psi_1^{-1}\left(\frac{\lambda\,\varepsilon_\lambda}{q(s)} \int_0^1 \tilde{k}(\tau)d\tau\right) ds\;\; L_\lambda \qquad \text{(by Lemma 3.3.1)} \\
&\leq\; \int_0^1 \frac{1}{p(s)} \psi_1^{-1}\left(\frac{\lambda\,\varepsilon_\lambda}{q(s)} \int_0^1 \tilde{k}(\tau)d\tau\right) ds\;\|u\| \qquad (\text{ since } \|u\| = L_\lambda)\,.
\end{aligned}$$

Since, ε_λ has been fixed such that $\displaystyle\int_0^1 \frac{1}{p(s)} \psi_1^{-1}\left(\frac{\lambda\,\varepsilon_\lambda}{q(s)} \int_0^1 \tilde{k}(\tau)d\tau\right) ds < 1$, we have

$$\|S_\lambda u\| < \|u\| \quad \text{for} \quad u \in \partial\Omega_{L_\lambda}\,. \tag{3.3.15}$$

Then appliying Theorem 2.2.3 using (3.3.13) and (3.3.15), S_λ has at least one fixed point u_2 in $\overline{\Omega_{L_\lambda}} \setminus \Omega_L$ and, in fact, $L < \|u_2\| < L_\lambda$. Thus, we have proved the first part of part 2 of Theorem 3.1.1.

Step 4: We observe that if $f_0 = 0$, the arguments in Steps 1 and 2 show that for any $L > 0$ there exists $\lambda_L > 0$ such that for all $\lambda \geq \lambda_L$ the problem (3.1.5)-(3.1.7) has at least one positive radial solution u with $0 < \|u\| < L$.

Step 5: If $f_\infty = 0$, the arguments in Steps 1 and 3 show that for any $L > 0$ there exists $\lambda_L > 0$ such that for all $\lambda \geq \lambda_L$ the problem (3.1.5)-(3.1.7) has at least one positive radial solution u with $L < \|u\|$. □

3.3.2 Proof of Theorem 3.1.2

We shall provide the proof only in the case of the boundary condition (3.1.2) which leads to the choice of the operator S_λ defined on the cone $\mathcal{K}$. The proofs in the other cases go through identically by working with the operator T_λ on the cone $\mathcal{C}$ or V_λ on the cone $\mathcal{P}$ with the corresponding version of the Lemmas 3.3.3, 3.3.5 and 3.3.6 at the beginning of this section.

We prove part 1 of the theorem (part 2 of the theorem can be obtained by similar arguments).

STEP 1: Let $\lambda > 0$ be arbitrary. Then, by Lemma 3.3.2, there exists $\varepsilon_\lambda > 0$ (for example take $\varepsilon_\lambda = \frac{C_1}{\lambda}$) such that

$$\int_0^1 \frac{1}{p(s)} \psi_1^{-1} \left(\frac{\lambda \, \varepsilon_\lambda}{q(s)} \int_0^1 \tilde{k}(\tau) d\tau \right) ds < 1 \, .$$

Using the condition $f_0 = 0$, we can find a l_λ with $0 < l_\lambda$ such that $f(x) \leq \varepsilon_\lambda \varphi(x)$ for all $0 \leq x \leq l_\lambda$. Then for $u \in \partial\Omega_{l_\lambda}$, where $\Omega_{l_\lambda} = \{u \in \mathcal{K} : \|u\| < l_\lambda\}$ we have $f(u(s)) \leq \varepsilon_\lambda \varphi(u(s))$ for all $s \in [0, 1]$. So, using Lemma 3.3.5, we have

$$\|S_\lambda u\| \;\; \leq \;\; \int_0^1 \frac{1}{p(s)} \psi_1^{-1} \left(\frac{\lambda \, \varepsilon_\lambda}{q(s)} \int_0^1 \tilde{k}(\tau) d\tau \right) ds \; \|u\| \, .$$

Therefore, we have

$$\|S_\lambda u\| < |u\| \quad \text{for} \quad u \in \partial\Omega_{l_\lambda} \, . \tag{3.3.16}$$

STEP 2: Now we argue as in Step 3 of Theorem 3.1.1. For $\lambda > 0$ (arbitrary) as in the previous step, let M_λ be chosen as in Lemma 3.3.3 so that (3.3.4) holds.

Since by hypothesis $f_\infty = \infty$ we can find a $R_{M_\lambda} > 0$ such that $f(x) \geq M_\lambda \varphi(x)$ for all $x \geq R_{M_\lambda}$. Setting $N_\lambda = R_{M_\lambda}/\rho_\delta$, we observe that for $u \in \partial\Omega_{N_\lambda}$ where $\Omega_{N_\lambda} = \{u \in \mathcal{K} : \|u\| < N_\lambda\}$, we have $u(t) \geq R_{M_\lambda}$ on $[\delta, 1-\delta]$ and this implies that $f(u(t)) \geq M_\lambda \varphi(u(t))$ on $[\delta, 1-\delta]$. So, by Lemma 3.3.6 and (3.3.4), we conclude that

$$\|S_\lambda u\| > \|u\| \quad \text{for} \quad u \in \partial\Omega_{N_\lambda} \, . \tag{3.3.17}$$

Then, applying again Theorem 2.2.3 using (3.3.17) and (3.3.16) above, we obtain the existence of one fixed point u_1 in $\mathcal{K} \cap (\overline{\Omega_{N_\lambda}} \setminus \Omega_{l_\lambda})$ and in fact, $l_\lambda < \|u_1\| < N_\lambda$. This proves part 1 of the theorem. $\square$

3.3.3 Proof of Theorem 3.1.3

We shall provide the proof only in the case of the boundary condition (3.1.2) which leads to the choice of the operator S_λ defined on the cone $\mathcal{K}$. The proofs in the other cases go through identically by working with the operator T_λ on the cone $\mathcal{C}$ or V_λ on the cone $\mathcal{P}$ with the corresponding version of the Lemmas 3.3.3, 3.3.5 and 3.3.6 at the beginning of this section.

First, we prove part 1 of Theorem 3.1.3. Whenever $f_0 > 0$ and $f_\infty > 0$, we have

$$\inf_{s>0} \frac{f(s)}{\varphi(s)} > 0$$

and so there exists $M > 0$ such that

$$f(s) \geq M\varphi(s) \quad \text{for all} \quad s > 0\,. \tag{3.3.18}$$

We now suppose that there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$, with $\lambda_n > n$ for all n, such that for each n the problem has a positive solution $u_n \in \mathcal{K}$. The property (3.3.18) allows us to use Lemma 3.3.6 and, since $S_{\lambda_n} u_n = u_n$, we have

$$\|u_n\| = \|S_{\lambda_n} u_n\| \;\; \geq \;\; \rho_\delta \|u_n\|\, y_{M,\lambda_n}(\delta) \quad \text{for all } n\,.$$

This leads to a contradiction since $y_{M,\lambda_n}(\delta) \to \infty$ as $n \to \infty$. This proves part 1.

Now, we prove part 2. Whenever $f_0 < \infty$ and $f_\infty < \infty$, we have

$$\sup_{s>0} \frac{f(s)}{\varphi(s)} < \infty$$

and so there exists $C > 0$ such that

$$f(s) \leq C\varphi(s) \quad \text{for all} \quad s > 0\,. \tag{3.3.19}$$

Suppose that there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$, with $\lambda_n \in (0, 1/n)$ such that for each n problem has a positive solution $u_n \in \mathcal{K}$. The property (3.3.19) allows us to use Lemma 3.3.5 and since, since $S_{\lambda_n} u_n = u_n$, we have

$$\|u_n\| = \|S_{\lambda_n} u_n\| \;\; \leq \;\; \int_0^1 \frac{1}{p(s)} \psi_1^{-1}\left(\frac{\lambda_n C}{q(s)} \int_0^1 \tilde{k}(\tau) d\tau \right) ds \;\; \|u_n\|\,.$$

This leads to a contradiction since $\displaystyle\int_0^1 \frac{1}{p(s)} \psi_1^{-1}\left(\frac{\lambda_n C}{q(s)} \int_0^1 \tilde{k}(\tau) d\tau \right) ds \to 0$ as $\lambda_n \to 0$. Thus, part 2 of the theorem is proved and the proof is complete. □

Chapter 4

Positive radial solutions of a quasilinear problem in an exterior domain with non-linear boundary conditions

4.1 Introduction

In this chapter we study the existence and non-existence results of positive radial solutions, given $\lambda > 0$ and $r_0 > 0$, for the quasilinear equation:

$$\operatorname{div}(A(|\nabla u|)\nabla u) + \lambda k(|x|)f(u) = 0, \quad |x| > r_0, x \in \mathbb{R}^N, N \geq 2 \tag{4.1.1}$$

in conjunction with the following boundary conditions on the exterior of a ball

$$\frac{\partial u}{\partial \eta} + c(u)u = 0 \quad \text{on} \quad |x| = r_0 \quad \text{and} \quad u \to 0 \text{ when } |x| \to \infty\,. \tag{4.1.2}$$

Like in the previous chapter, by setting $\varphi(t) = A(|t|)t$, looking for a radially symmetric solution $u\,(x) \equiv v\,(|x|)$ leads to the following differential equation for $w(t) = v\,(r) = v\left(\frac{r_0}{1-t}\right)$

$$-(q\,(t)\,\varphi(p\,(t)\,w'\,(t)))' = \lambda\tilde{k}(t)f(w(t)), \qquad 0 < t < 1, \tag{4.1.3}$$

where $q(t) := \left(\dfrac{r_0}{1-t}\right)^{N-1}$, $p(t) := \dfrac{(1-t)^2}{r_0}$ and $\tilde{k}(t) := \dfrac{r_0^N}{(1-t)^{N+1}}k\left(\dfrac{r_0}{1-t}\right)$.

The boundary conditions in (4.1.2) become

$$-\frac{1}{r_0}w'\,(0) + c(w(0))w(0) = 0 \quad \text{and} \quad w\,(1) = 0\,. \tag{4.1.4}$$

Without loss of generality, we will assume that $r_0 = 1$.

The above discussion motivates us to study positive radial solutions in $C[0,1]$ to the following boundary value problem

$$\begin{cases} -(q\,(t)\,\varphi(p\,(t)\,u'\,(t)))' = \lambda\tilde{k}(t)f(u(t)), & 0 < t < 1, \\ -u'\,(0) + c(u(0))u(0) = 0 \\ u\,(1) = 0\,. \end{cases} \tag{4.1.5}$$

The precise hypotheses on the non-linearities φ, f, c and the weight k, which are all essentially the same as in the previous chapter except a few of them, will be given at the beginning of the next section. As in the previous chapter, the behaviour of the non-linearity f in comparison with φ at 0 and at ∞ will be important for the analysis and for this we set:

$$f_0 := \lim_{x\to 0} \frac{f(x)}{\varphi(x)} \quad \text{and} \quad f_\infty := \lim_{x\to\infty} \frac{f(x)}{\varphi(x)} \,.$$

We now state our main theorems. The first three theorems are modelled on Theorems 3.1.1, 3.1.2 and 3.1.3 of the previous chapter and will be proved using a fixed point argument. For this, we assume that the conditions from (H_1) to (H_7), stated in the next section, hold. The main new difficulties in obtaining these theorems lie in the proper choice of an operator and obtaining crucial estimates. These will be developed in Subsection 4.2.2.

Theorem 4.1.1.

1. *If $f_0 = \infty$ and $f_\infty = \infty$, then for any $R > 0$ there exists λ_R such that for all $0 < \lambda \leq \lambda_R$, the problem has at least two positive solutions u_1 and u_2 with $0 < \|u_1\| < R < \|u_2\|$.*

   If $f_0 = \infty$ or $f_\infty = \infty$, then for any $R > 0$ there exists $\lambda_R > 0$ such that for all $0 < \lambda \leq \lambda_R$ the problem has at least one positive solution u with $0 < \|u\| < R$ or $R < \|u\|$ respectively.

2. *If $f_0 = 0$ and $f_\infty = 0$, then for any $L > 0$ there exists $\lambda_L > 0$ such that for all $\lambda \geq \lambda_L$, the problem has at least two positive solutions u_1 and u_2 with $0 < \|u_1\| < L < \|u_2\|$.*

   If $f_0 = 0$ or $f_\infty = 0$, then for any $L > 0$ there exists $\lambda_L > 0$ such that for all $\lambda \geq \lambda_L$ the problem has at least one positive solution u with $0 < \|u\| < L$ or $L < \|u\|$ respectively.

Theorem 4.1.2.

1. *If $f_0 = 0$ and $f_\infty = \infty$, then for all $\lambda > 0$ the problem has a positive solution.*
2. *If $f_0 = \infty$ and $f_\infty = 0$, then for all $\lambda > 0$ the problem has a positive solution.*

Theorem 4.1.3.

1. *If $f_0 > 0$ and $f_\infty > 0$, then there exists a positive number $\overline{\lambda}$ such that the problem has no positive solutions for all $\lambda > \overline{\lambda}$.*
2. *If $f_0 < \infty$ and $f_\infty < \infty$, then there exists a positive number $\underline{\lambda}$ such that the problem has no positive solutions for all $\lambda < \underline{\lambda}$.*

We also prove the second part of Theorem 4.1.2 while assuming $f(0) > 0$ (which, of course, implies $f_0 = \infty$) and f is sublinear at ∞ using the method of sub- and super- solutions similarly as in Buttler, Ko, Lee and Shivaji [6]. However, we can do this only for φ which are positively homogeneous like in the case of p-Laplacian(we replace hypotheses (H_1) and (H_2) by (H_8)). So, we do not explore fully the scope of this method in this thesis.

Theorem 4.1.4. *Assume that the conditions $f(0) > 0$ and $f_\infty = 0$ hold. Then, the boundary value problem (4.1.5) has at least one positive solution for all $\lambda > 0$.*

The following sections are organized as follows: in Section 4.2 we provide some preliminaries which include: some definitions to use, a brief discussion of the hypothesis, a few useful lemmas and a discussion of the operator setting for obtaining the majority of the main results. In Subsection 4.2.2 we provide the preliminaries used for the fixed point method and in Subsection 4.2.3, we provide those for the sub-super solutions method. We deal with the main results in Section 4.3.

4.2 Setting up the problem

In this section we establish the basic notations and the abstract framework for solving the problem.

4.2.1 The hypotheses on the non-linearities

Let $\varphi(t) = A(|t|)t$. We list a set of assumptions on the non-linearities φ, f, c and the weight k that are used in the proof of theorems:

(H_1) $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and is *pseudo-homogeneous* in the following sense. There exists two increasing and surjectives homeomorphisms $\psi_1, \psi_2 : (0, \infty) \to (0, \infty)$ such that:

$$\psi_1(a)\varphi(b) \le \varphi(ab) \le \psi_2(a)\varphi(b) \quad \text{for all } a > 0, b \in \mathbb{R}.$$

As a consequence, for all $a \in (0, \infty), b \in \mathbb{R}$, we have that

$$\psi_2^{-1}(a)b \le \varphi^{-1}(a\varphi(b)) \le \psi_1^{-1}(a)b\,. \tag{4.2.1}$$

(H_2) For any constant C we have $\displaystyle\int_0^1 \frac{1}{p(s)}\psi_i^{-1}\left(\frac{C}{q(s)}\right) ds < \infty\,, \quad i = 1, 2.$

(H_3) For any constant C we have $\displaystyle\int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{C}{q(s)}\right) ds < \infty.$

We now give some of the basic hypotheses on the non-linearity f and on the weight k.

(H_4) $f : [0, \infty) \to [0, \infty)$ is continuous and $f(s) > 0$ for all $s > 0$.

(H_5) $c : [0, \infty) \to (0, \infty)$ is a continuous function.

(H_6) $k : [r_0, \infty) \to [0, \infty)$ is continuous and $k(t) \ne 0$ on any subinterval of $[r_0, \infty)$.

(H_7) $0 < \displaystyle\int_0^1 \tilde{k}(t)dt < \infty.$

(H_8) $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and is *positively homogeneous of degree* $\alpha \ge 1$ in the following sense.

$$\varphi(ab) = a^\alpha\varphi(b) \quad \text{for all } a > 0, b \in \mathbb{R}\,.$$

4.2.2 Construction of the fixed point operator

We aim to prove Theorems 4.1.1, 4.1.2 and 4.1.3 in Section 4.3 through a fixed point argument. The construction of the operator whose fixed points will provide the solution to the problem (4.1.5) is not so explicit as in the previous chapter due to the non-linear boundary condition and requires a few steps.

We start by studying the existence of a non-negative solution to the following boundary value problem for a given non-negative continuous function u on $[0,1]$ which itself is obtained by a fixed point argument.

$$\begin{cases} -(q\,(t)\,\varphi(p\,(t)\,w'\,(t)))' = \lambda\tilde{k}(t)f(u(t)), & 0<t<1, \\ -w'\,(0) + c(u(0))w(0) = 0 \\ w\,(1) = 0. \end{cases} \tag{4.2.2}$$

We have the following proposition.

Proposition 4.2.1. *Given a continuous function $u \geq 0$ on $[0,1]$ the boundary value problem*

$$\begin{cases} -(q\,(t)\,\varphi(p\,(t)\,w'\,(t)))' = \lambda\tilde{k}(t)f(u(t)), & 0<t<1, \\ -w'\,(0) + c(u(0))w^+(0) = 0 \\ w\,(1) = 0. \end{cases} \tag{4.2.3}$$

has a unique solution w_u obtained as a fixed point of the operator $R_u : C[0,1] \to C[0,1]$ defined by

$$R_u(v)(t) := \int_t^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_0^s \widetilde{k}(\eta)f(u(\eta))d\eta - \frac{q(0)}{q(s)}\varphi(p(0)c(u(0))v^+(0))\right)\,ds \ \ \text{for all } t\in[0,1]\,.$$

Furthermore, $w_u \geq 0$ and, if $u \neq 0$ then $w_u(0) > 0$.

Proof: Existence: It is easy to see the one-one correspondence between the fixed points of the operator and the solutions of (4.2.3). By arguments like in Section 3.2.3 it can be checked that R_u is a completely continuous operator on $C[0,1]$. Note that, due to the positivity of $\dfrac{q(0)}{q(s)}\varphi(p(0)c(u(0))v^+(0))$ and the monotonicity of φ, for all $t \in (0,1)$, we have

$$\begin{aligned} |R_u(v)(t)| \ &\leq \ \int_t^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_0^s \widetilde{k}(\eta)f(u(\eta))d\eta\right)\,ds \\ &\leq \ \int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_0^s \widetilde{k}(\eta)f(u(\eta))d\eta\right)\,ds := C\,. \end{aligned}$$

For C taken as above, it is clear, therefore that $R_u : B_C \to B_C$ where B_C is the closed unit ball of radius C in $C[0,1]$. Therefore, by Schauder's fixed point theorem there exists a fixed point $w_u \in B_C$.

Uniqueness: The uniqueness follows by a direct application of Lemma 2.5.4.

Positivity: If w_u is a fixed point of R_u, then it satisfies (4.2.3) pointwise in $(0,1)$. Therefore, $(q\,(t)\,\varphi(p\,(t)\,w'\,(t)))$ is non-increasing on $[0,1]$. From the fact that $q(\cdot)^{-1}$ is non-increasing and the monotonicity of φ, we obtain that $p(\cdot)w'(\cdot)$ is non-increasing on $[0,1]$.

Now, we would like to show that $w_u \geq 0$ in $[0,1]$. We claim first that $w(0) \geq 0$. Otherwise, we would have $w(0) < 0$ and so $w^+(0) = 0$ which implies, by the boundary condition that $w'(0) = 0$. By the monotonicity of $p(\cdot)w'(\cdot)$ obtained in the previous paragraph, it follows that $w'(\cdot) \leq 0$ on $[0,1]$. So, w is non-increasing on $[0,1]$ while by hypothesis $w(1) = 0$. So, clearly it cannot be true that $w(0) < 0$.

So, we are able to conclude that $w(0) \geq 0$. Therefore, by the boundary condition, either $w'(0) > 0$ or $w'(0) = 0$. On the one hand, if $w'(0) = 0$ arguing as in the last paragraph, $w'(t) \leq 0$ for all $t \in (0,1)$ and from $w(1) = 0$ we obtain that $w \geq 0$ in $(0,1)$. On the other hand, if $w'(0) > 0$, then w is increasing in a neighbourhood of 0 and so w is strictly positive in a neighbourhood of 0 with a strictly positive local maximum at some point t_0 there. At t_0, since $0 < t_0 < 1$, we would have $w'(t_0) = 0$. So, once again $w'(t) \leq 0$ for all $t_0 \leq t \leq 1$ and, together with $w(1) = 0$, this implies that $w(t) \geq 0$ for all $t \in [t_0, 1]$ and therefore, $w(t) \geq 0$ for all $t \in [0,1]$.

Now, if $u \neq 0$ we claim that $w(0) > 0$. Otherwise, similarly as in the previous argument we will obtain $w \equiv 0$ on $[0,1]$ which's impossible if $u \neq 0$. □

Choice of the fixed point operator S

In the light of Proposition 4.2.1, we define the following operator

$$Su := w_u \tag{4.2.4}$$

where w_u is the unique solution of (4.2.3) given u in $C[0,1]$ and $u \geq 0$. Then, $Su = w_u$ is obtained implicitly from the equation

$$w_u(t) = \int_t^1 \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda}{q(s)} \int_0^s \widetilde{k}(\eta) f(u(\eta)) d\eta - \frac{q(0)}{q(s)} \varphi(p(0)c(u(0))w_u(0)) \right) \, ds \;\forall t \in [0,1]\,. \tag{4.2.5}$$

Remark 4.2.2. *As a consequence, for $u \geq 0$, we have the following inequalities which follows from the fact that φ is monotone increasing*

$$Su(t) \leq \int_t^1 \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda}{q(s)} \int_0^s \tilde{k}(\eta) f(u(\eta)) d\eta \right) ds \textit{ for all } t \in [0,1]\,, \tag{4.2.6}$$

$$|Su(t_1) - Su(t_2)| \leq \int_{t_1}^{t_2} \frac{1}{p(s)} \varphi^{-1}\left(\frac{\lambda}{q(s)} \int_0^s \tilde{k}(\eta) f(u(\eta)) d\eta \right) ds \textit{ for all } t_1, t_2 \in [0,1]\,. \tag{4.2.7}$$

Now, it is clear that if u is a fixed point of S then u is a non-negative solution of (4.1.5). Conversely, if u is a non-negative solution of (4.1.5), then u itself is a solution of (4.2.3) and so by the uniqueness proved in Proposition 4.2.1 we conclude that $w_u = u$ and so u is a fixed point of S. Therefore, we shall look for solutions to (4.1.5) as fixed points of the operator S defined in (4.2.4).

The function space setting

The function space setting is nearly the same as in Subsection 3.2.2 but showing some of the properties of S requires different arguments.

We consider the Banach space $C[0,1]$ equipped with the supremum norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$ and the subspace $\mathcal{X} = \{u \in C[0,1] : u(1) = 0\}$ and consider $0 < \delta < \frac{1}{2}$. For solving the boundary value problem (4.1.5) we will consider the cone in $\mathcal{X}$ defined by

$$\mathcal{K} := \{u \in \mathcal{X} : u(t) \geq 0,\ u(t) \geq \rho_\delta \|u\|,\ \forall t \in [\delta, 1-\delta]\} \tag{4.2.8}$$

where the value of ρ_δ is fixed below (note that $0 < \rho_\delta < 1$)

$$\rho_\delta := \frac{1}{\displaystyle\int_0^1 \frac{1}{p(s)}\psi_1^{-1}\left(\frac{1}{q(s)}\right) ds} \min\left\{\int_0^\delta \frac{1}{p(s)}\psi_2^{-1}\left(\frac{1}{q(s)}\right) ds, \int_{1-\delta}^1 \frac{1}{p(s)}\psi_2^{-1}\left(\frac{1}{q(s)}\right) ds\right\}. \tag{4.2.9}$$

Given $\lambda > 0$, the solutions to the problem (4.1.5) will be obtained as fixed points of the operator S, defined in (4.2.4), on the cone $\mathcal{K}$.

Complete continuity

The complete continuity of the operator $S : \mathcal{K} \to \mathcal{X}$ requires one to prove that S is continuous and also that it maps bounded sets to relatively compact sets. For the latter, it is enough to apply Arzela-Ascoli theorem. Indeed, the boundedness and the equicontinuity of Su for u bounded in $\mathcal{X}$ can be obtained starting from (4.2.6) and (4.2.7), respectively, by arguing similarly as in Lemma 3.2.7.

The continuity of Su requires showing that the solution w_u of (4.2.2) depends continuously on u. Let $u_n, u \in \mathcal{K}$ be such that $u_n \to u$ in $C[0,1]$. We would like to prove that $w_{u_n} \to w_u$. As a consequence of Remark 4.2.2 applied to the bounded sequence u_n, we can conclude that the family w_{u_n} is equicontinuous and without loss of generality converges to a function w in X. We need to show that $w_u = w$.
By Proposition 4.2.1 w_{u_n} are non-negative and satisfy the fixed point equations

$$w_{u_n}(t) = \int_t^1 \frac{1}{p(s)}\left(\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_0^s \widetilde{k}(\eta) f(u_n(\eta))d\eta - \frac{q(0)}{q(s)}\varphi(p(0)c(u_n(0))w_{u_n}(0))\right)\right) ds\, \forall t \in [0,1]$$

By passing to the limit in the above, while using the uniform convergence of w_{u_n} to w and u_n to u, we obtain

$$w(t) = \int_t^1 \frac{1}{p(s)}\left(\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_0^s \widetilde{k}(\eta) f(u(\eta))d\eta - \frac{q(0)}{q(s)}\varphi(p(0)c(u(0))w(0))\right)\right) ds$$

by an application of the dominated convergence theorem. This means that, for this $u \in \mathcal{K}$, the function w being the limit of non-negative functions is a non-negative solution of (4.2.3). By the uniqueness of the solution proved in Proposition 4.2.1, it follows that $w = w_u$. □

Cone invariance

Proposition 4.2.3. *Let $\mathcal{K}$ defined by (4.2.8). Then the operator S defined by* (4.2.4) *preserves the cone $\mathcal{K}$, that is, if $v \in \mathcal{K}$ then $Sv \in \mathcal{K}$.*

Proof: It is enough to consider $v \in \mathcal{K} \setminus \{0\}$. The arguments of the proof are exactly the same as in Proposition 3.2.8 after observing that for such v, by the conclusions of Proposition 4.2.1, $Sv = u$ satisfies

$$\left.\begin{array}{c} -(q(t)\varphi(p(t)u'(t)))' = \lambda\tilde{k}(t)f(v(t)) \geq 0 \quad \text{in} \quad 0 < t < 1 \\ u(0) > 0, u(1) = 0 \end{array}\right\} . \tag{4.2.10}$$

□

Some useful estimates

Lemma 4.2.4. *Consider $C > 0$, $u \in C[0,1]$ be non-negative and let us suppose that $f(u(t)) \leq C\varphi(u(t))$ for all $t \in [0,1]$. Then, for any $\lambda > 0$, and the operator S we have*

$$\|Su\| \leq \int_0^1 \frac{1}{p(s)} \psi_1^{-1}\left(\frac{\lambda C}{q(s)} \int_0^1 \tilde{k}(\eta)d\eta\right) ds\, \|u\| . \tag{4.2.11}$$

Proof: It is easy to prove this starting from the inequality (4.2.6). □

The following lemma helps us to make a comparison between two operators, on the one hand, the operator S_λ of the previous chapter in connection with Dirichlet condition and, on the other hand, the operator S.

Lemma 4.2.5. *For $u \geq 0$, suppose that w is a C^1 variational supersolution of*

$$\left.\begin{array}{c} -(q(t)\varphi(p(t)w'(t)))' = \lambda\widetilde{k}(t)f(u(t)) \quad 0 < t < 1 \\ -w'(0) + c(u(0))w(0) = 0 \\ w(1) = 0\,. \end{array}\right\} \tag{4.2.12}$$

Suppose that z is a solution of:

$$\left.\begin{array}{c} -(q(t)\varphi(p(t)z'(t)))' = \lambda\widetilde{k}(t)f(u(t)) \quad 0 < t < 1 \\ z(0) = z(1) = 0 \end{array}\right\} \tag{4.2.13}$$

Then $w \geq z$ in $[0,1]$.

Proof: By Proposition 4.2.1, we know that $w \geq 0$. So, we observe that $w(0) \geq z(0)$ and $w(1) \geq z(1)$. So, we are in condition to apply Lemma 2.5.1 to obtain the desired conclusion. □

Lemma 4.2.6. *Let the cone $\mathcal{K}$ and let the constant ρ_δ, and we consider the operator S. Consider $M > 0$ and $u \in \mathcal{K}$ and assume that $f(u(t)) \geq M\varphi(u(t))$ for all $t \in [\delta, 1-\delta]$ with $0 < \delta < \frac{1}{2}$ as in the definition of $\mathcal{K}$. Then, for any $\lambda > 0$ and $y_{M,\lambda}(\delta)$ as defined in (3.3.2) we have*

$$\|Su\| \geq \rho_\delta \|u\|\, y_{M,\lambda}(\delta)\,. \tag{4.2.14}$$

Proof: Let S_λ the operator associated to the problem with Dirichlet condition defined given by (3.2.3). In Lemma 3.3.6, we have proved that $\|S_\lambda(u)\| \geq \rho_\delta \|u\| y_{M,\lambda}(\delta)$. Now, we have by Lemma 4.2.5 that $\forall u \in \mathcal{K}, S(u) \geq S_\lambda(u)$. Thus, $\|Su\| \geq \|S_\lambda(u)\| \geq \rho_\delta \|u\| y_{M,\lambda}(\delta)$. □

4.2.3 Preliminary results for applying the method of sub- and super- solutions

We will need the following notions for the proof of Theorem 4.1.4 given in Section 4.3.2 and has as it's model a result from [6] involving the Laplacian operator.

Definition 4.2.7. *By a subsolution of (4.1.5), we mean a function $\psi \in C^1[0,1]$ that satisfies*

$$\begin{cases} -(q\,(t)\,\varphi(p\,(t)\,\psi'\,(t)))' \le \lambda\tilde{k}(t)f(\psi(t)), & 0<t<1, \\ -\psi'\,(0)+c(\psi(0))\psi(0) \le 0 & \\ \psi\,(1) \le 0\,. & \end{cases} \tag{4.2.15}$$

and by a supersolution of (4.1.5), we mean a function $Z \in C^1[0,1]$ that satisfies

$$\begin{cases} -(q\,(t)\,\varphi(p\,(t)\,Z'\,(t)))' \ge \lambda\tilde{k}(t)f(Z(t)), & 0<t<1, \\ -Z'\,(0)+c(Z(0))Z(0) \ge 0 & \\ Z\,(1) \ge 0\,. & \end{cases} \tag{4.2.16}$$

We have the following result.

Proposition 4.2.8. *If ψ is a subsolution of (4.1.5) and Z is a supersolution of (4.1.5) such that $\psi \le Z$ then (4.1.5) has a solution u such that $\psi \le u \le Z$.*

Proof: We consider the auxiliary problem

$$\begin{cases} -(q\,(t)\,\varphi(p\,(t)\,u'\,(t)))' = \lambda\tilde{k}(t)f(\gamma(t,u(t))), & 0<t<1, \\ -u'\,(0)+\overline{c}(u(0)) = 0 & \\ u\,(1) = 0\,. & \end{cases} \tag{4.2.17}$$

where $\gamma : (0,1)\times\mathbb{R} \to \mathbb{R}$ is defined by

$$\gamma(t,a) := \begin{cases} Z(t) & \text{if} \quad a > Z(t) \\ a & \text{if} \quad \psi(t) \le a \le Z(t) \\ \psi(t) & \text{if} \quad a < \psi(t) \end{cases} \tag{4.2.18}$$

and

$$\overline{c}(a) := \begin{cases} c(Z(0))Z(0) & \text{if} \quad a > Z(0) \\ c(a)a & \text{if} \quad \psi(0) \le a \le Z(0) \\ c(\psi(0))\psi(0) & \text{if} \quad a < \psi(0)\,. \end{cases} \tag{4.2.19}$$

Also, we define $\overline{T} : \mathcal{X} \to \mathcal{X}$ by

$$\overline{T}(u)(t) := \int_t^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{\lambda}{q(s)}\int_0^s \tilde{k}(\eta)f(\gamma(\eta,u(\eta)))d\eta - \frac{q(0)}{q(s)}\varphi(p(0)\overline{c}(u(0)))\right)ds \tag{4.2.20}$$

This operator is easily seen to be completely continuous and bounded in $\mathcal{X}$ (let's then assume $\sup_{\mathcal{X}} \|\overline{T}u\|_\infty < C$). Now, fix $\overline{C}$ such that $C \le \overline{C}$. So, by the Schauder fixed point theorem (Theorem 2.2.1), $\overline{T} : \overline{C} \to \overline{C}$ has a fixed point theorem and thus, (4.2.17) has a solution u.

To obtain that it is indeed a solution to (4.1.5) it is enough to show that $\psi \le u \le Z$.

PROVING $\psi \le u$: Let us suppose that there exists $t_0 \in [0,1)$ such that $u(t_0) < \psi(t_0)$. Then, since $\psi(1) \le u(1) = 0$, one of the following two situations must hold. Either,

(i) there exists $(a,b) \subset (0,1)$ such that $\psi(t) > u(t)$, $\psi(a) = u(a)$ and $\psi(b) = u(b)$ or

(ii) there exists $b \in (0,1)$ such that $\psi(t) > u(t)$ in $[0,b)$ and $\psi(b) = u(b)$.

If (i) holds, then define $w(t) := \psi(t) - u(t)$ which implies $w > 0$ in (a,b) and $w(a) = w(b) = 0$. Then, there exists $\tilde{t} \in (a,b)$ such that $w'(\tilde{t}) = \psi'(\tilde{t}) - u'(\tilde{t}) = 0$ and $t \in (\tilde{t}, b)$ such that $w'(t) < 0$. From $w'(t) = \psi'(t) - u'(t) < 0$ we deduce that,

$$p(t)\psi'(t) < p(t)u'(t) \implies \varphi(p(t)\psi'(t)) < \varphi(p(t)u'(t)) \implies q(t)\varphi(p(t)\psi'(t)) < q(t)\varphi(p(t)u'(t)) \,.$$

Therefore,

$$\begin{aligned}
0 &< q(t)\varphi(p(t)u'(t)) - q(t)\varphi(p(t)\psi'(t)) \\
&= \int_{\tilde{t}}^{t} [q(t)\varphi(p(t)u'(t)) - q(t)\varphi(p(t)\psi'(t))]' \, ds \\
&= \int_{\tilde{t}}^{t} [(q(t)\varphi(p(t)u'(t)))' - (q(t)\varphi(p(t)\psi'(t)))'] \, ds \\
&\leq \int_{\tilde{t}}^{t} \left[-\lambda\tilde{k}(s)f(\gamma(s,u(s)) + \lambda\tilde{k}(s)f(\psi(s)))\right] ds \\
&= \int_{\tilde{t}}^{t} \left[-\lambda\tilde{k}(s)f(\psi(s)) + \lambda\tilde{k}(s)f(\psi(s))\right] ds = 0 \qquad (\text{since } u < \psi)\,.
\end{aligned}$$

This leads to a contradiction.

Now, if (ii) holds then define $w(t) := \psi(t) - u(t)$ which implies $w > 0$ in $[0,b)$ and $w(b) = 0$. Then, we have,

$$w'(0) = \psi'(0) - u'(0) \geq c(\psi(0))\psi(0) - \overline{c}(u(0)) = c(\psi(0))\psi(0) - c(\psi(0))\psi(0) = 0$$

where for the penultimate equality we use the fact that $\psi(0) > u(0)$ and the definition of $\overline{c}$.

So we have $w'(0) \geq 0$ and $w(b) = 0$ which implies that there exists $\tilde{t} \in [0,b)$ such that $w'(\tilde{t}) = 0$ and $w'(t) < 0$ for $t \in (\tilde{t}, b)$. Then, by repeating the arguments of case (i) in $(\tilde{t}, t)$ we obtain, once again, a contradiction.

PROVING $u \leq Z$: We use arguments similar to those used in proving $\psi \leq u$.
In fact, if we suppose that there exists $t_0 \in [0,1)$ such that $u(t_0) > Z(t_0)$, then, since $Z(1) \geq u(1) = 0$, one of the following two situations must hold. Either,

(i) there exists $(a,b) \subset (0,1)$ such that $u(t) > Z(t)$, $u(a) = Z(a)$ and $u(b) = Z(b)$ or

(ii) there exists $b \in (0,1)$ such that $u(t) > Z(t)$ in $[0,b)$ and $u(b) = Z(b)$.

We now let $w(t) := u(t) - Z(t)$.

If (i) holds, then $w > 0$ in (a,b) and $w(a) = w(b) = 0$. Then, there exists $\tilde{t} \in (a,b)$ such that $w'(\tilde{t}) = u'(\tilde{t}) - Z'(\tilde{t}) = 0$ and $t \in (\tilde{t}, b)$ such that $w'(t) < 0$. From $w'(t) = u'(t) - Z'(t) < 0$ we obtain $q(t)\varphi(p(t)u'(t)) < q(t)\varphi(p(t)Z'(t))$ from which

$$\begin{aligned}
0 &< [q(t)\varphi(p(t)Z'(t)) - q(t)\varphi(p(t)u'(t))] \\
&= \int_{\tilde{t}}^{t} [q(t)\varphi(p(t)Z'(t)) - q(t)\varphi(p(t)u'(t))]' \, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\tilde{t}}^{t}\left[-\lambda\tilde{k}(s)f(Z(s))+\lambda\tilde{k}(s)f(\gamma(s,u(s)))\right]ds \\
&= \int_{\tilde{t}}^{t}\left[-\lambda\tilde{k}(s)f(Z(s))+\lambda\tilde{k}(s)f(Z(s))\right]ds = 0 \qquad (\text{since } u > Z)\,.
\end{aligned}
$$

This leads to a contradiction.

Now, if (ii) holds, then $w > 0$ in $[0, b)$ and $w(b) = 0$. Then, we have,

$$w'(0) = u'(0) - Z'(0) \geq \overline{c}(u(0)) - c(Z(0))Z(0) = c(Z(0))Z(0) - c(Z(0))Z(0) = 0$$

where for the penultimate equality we use the fact that $u(0) > Z(0)$ and the definition of $\overline{c}$.

So we have $w'(0) \geq 0$ which implies that there exists $\tilde{t} \in [0, b)$ such that $w'(\tilde{t}) = 0$ and $w'(t) < 0$ for $t \in (\tilde{t}, b)$. Then, arguing as before we obtain a contradiction. □

4.3 Proofs of the Main Results

4.3.1 Proofs of Theorems 4.1.1, 4.1.2, 4.1.3

The proofs of Theorems 4.1.1, 4.1.2 and 4.1.3 can be obtained by arguing exactly in the same way as in the proofs of Theorems 3.1.1, 3.1.2 and 3.1.3 while using the estimates (4.2.11) and (4.2.14) in place of (3.3.6) and (3.3.7), respectively, for applying the fixed point theorem of Krasnosel'skii.

4.3.2 Proof of Theorem 4.1.4

The proof of Theorem 4.1.4 is based on the method of sub- and super- solutions as in [6]. The existence result follows immediately from Proposition 4.2.8 established in Subsection 4.2.3 once we are able to provide a sub- and super- solution to (4.1.5) satisfying the hypotheses of Proposition 4.2.8.

First, since we assume that $f(0) > 0$ and (H_6) hold, we note that $\psi_1 \equiv 0$ is a trivial *subsolution* of (4.1.5) because we have $-(q(t)\varphi(p(t)\psi_1'(t)))' < \lambda\widetilde{k}(t)f(\psi_1(t))$ and $-\psi_1'(0) + c(\psi_1(0))\psi_1(0) = 0$ along with $\psi_1(0) = 0$.

Now, we will construct a positive *supersolution* Z_2. First, let $\widetilde{f}(x) := \max_{[0,x]} f(t)$. We note that since $f_\infty = \lim_{s\to\infty} \dfrac{\widetilde{f}(s)}{\varphi(s)} = 0$, given $C > 0$, there exists $M_\lambda \gg 1$ such that $\dfrac{\widetilde{f}(M_\lambda\|e\|_\infty)}{\varphi(M_\lambda\|e\|_\infty)} \leq C$. We will choose a suitable C later. Let e the unique positive solution of

$$
\begin{cases}
-(q\,(t)\,\varphi(p\,(t)\,e'\,(t)))' = \tilde{k}(t), & 0 < t < 1, \\
e'\,(0) = 0 \\
e\,(1) = 0\,.
\end{cases}
\tag{4.3.1}
$$

which exists by direct calculations. Then, by the α-homogeneity of φ, we have

$$\frac{1}{C}\widetilde{f}(M_\lambda\|e\|_\infty) \leq \varphi(M_\lambda\|e\|_\infty) = M_\lambda^\alpha\varphi(\|e\|_\infty)\,.$$

Thus,

$$M_\lambda^\alpha \geq \frac{\widetilde{f}(M_\lambda \|e\|_\infty)}{C\,\varphi(\|e\|_\infty)}\,.$$

If we set $Z_2 = M_\lambda e$ then by the α-homogeneity of φ, we have

$$\begin{aligned}
-(q\,(t)\,\varphi(p\,(t)\,Z_2'\,(t)))' &= -(q\,(t)\,\varphi(p\,(t)\,M_\lambda e'\,(t)))' \\
&= -(q\,(t)\,\varphi(p\,(t)\,e'\,(t)))' M_\lambda^\alpha \\
&= \widetilde{k}(t) M_\lambda^\alpha \\
&\geq \widetilde{k}(t)\frac{\widetilde{f}(M_\lambda \|e\|_\infty)}{C\,\varphi(\|e\|_\infty)} \\
&\geq \widetilde{k}(t)\frac{\widetilde{f}(M_\lambda e)}{C\,\varphi(\|e\|_\infty)} \\
&\geq \widetilde{k}(t)\frac{f(M_\lambda e)}{C\,\varphi(\|e\|_\infty)} \\
&= \lambda \widetilde{k}(t) f(Z_2)
\end{aligned}$$

while choosing $C = \dfrac{1}{\lambda\,\varphi(\|e\|_\infty)}$. Also,

$$-Z_2'(0) + c(Z_2(0))Z_2(0) = -M_\lambda e'(0) + c(M_\lambda e(0))M_\lambda e(0) = c(M_\lambda e(0))M_\lambda e(0) \geq 0$$

and $Z_2(1) = M_\lambda e(1) = 0$. Hence, $Z_2 = M_\lambda e$ is a supersolution of (4.1.5). $\square$

Appendix

Calculations to show that, for $1 < \alpha < N-1$, the odd extension of the function $\varphi_\alpha(t) := \dfrac{t^\alpha}{\sqrt{1+t^2}}$ defined for $t \geq 0$ satisfies the hypotheses (H1), (H2) and (H3) in Chapter 3.

Let $\varphi(t) := \dfrac{t^\alpha}{\sqrt{1+t^2}}$; our desire is to find two increasing homeomorphisms $\psi_1, \psi_2 : (0,+\infty) \to (0,+\infty)$ such as in the (H1) assumption.

The inequality given on the assumption (H1) is equivalent to:

$$\psi_1(t) \leq \frac{\varphi(ts)}{\varphi(s)} \leq \psi_2(t);\ t,s>0$$

Hence, we start by analyzing the quotient:

$$\frac{\varphi(ts)}{\varphi(s)} = \frac{t^\alpha s^\alpha}{\sqrt{1+t^2s^2}} \cdot \frac{\sqrt{1+s^2}}{s^\alpha} = \sqrt{\frac{t^{2\alpha}(1+s^2)}{1+t^2s^2}};\ t,s>0$$

Let $g_t(s) := \dfrac{t^{2\alpha}(1+s^2)}{1+t^2s^2}$ where $g_t'(s) = \dfrac{2st^{2\alpha}(1-t^2)}{(1+t^2s^2)^2}$

We note that if $t<1$, $g_t'(s)>0$ then $g_t(s)$ is increasing; and when $t \geq 1$, $g_t'(s) < 0$ then $g_t(s)$ is decreasing.

Furthermore, $\lim\limits_{s\to 0} g_t(s) = t^{2\alpha}$ and $\lim\limits_{s\to\infty} g_t(s) = t^{2(\alpha-1)}$. This leads us to define ψ_1 and ψ_2 by

$$\psi_1(t) = \begin{cases} t^\alpha, & 0 \leq t < 1 \\ t^{\alpha-1}, & t \geq 1 \end{cases} \qquad \text{and} \qquad \psi_2(t) = \begin{cases} t^{\alpha-1}, & 0 \leq t < 1 \\ t^\alpha, & t \geq 1 \end{cases}$$

We realize in order to fulfill the (H1) hypothesis it is necessary that $\boxed{\alpha > 1}$

In addition, we need the (H2) and (H3) hypotheses to be fulfilled, then

For H2: We analize $I := \displaystyle\int_0^1 \frac{1}{p(s)} \psi_1^{-1}\left(\frac{C}{q(s)}\right) ds$

The quotient $\dfrac{C}{q(s)} < 1$ when:

$$\frac{C(1-s)^{N-1}}{R_1^{N-1}} < 1 \Leftrightarrow s > \beta \quad \text{with } \beta < 1 \text{ such that } \beta := 1 - \frac{R_1}{\sqrt[N-1]{C}}$$

Therefore, the integral is descomposed as:

$$
\begin{aligned}
I &= \int_0^\beta \frac{R_1}{(1-s)^2}\left(\frac{C(1-s)^{N-1}}{R_1^{N-1}}\right)^{\frac{1}{\alpha-1}} ds + \int_\beta^1 \frac{R_1}{(1-s)^2}\left(\frac{C(1-s)^{N-1}}{R_1^{N-1}}\right)^{\frac{1}{\alpha}} ds \\
&= A\int_0^\beta (1-s)^{\frac{N-1}{\alpha-1}-2} ds + B\int_\beta^1 (1-s)^{\frac{N-1}{\alpha}-2} ds\,;\ A, B \text{ constants}
\end{aligned}
$$

The first integral is always finite and the second one is also provided $\dfrac{N-1}{\alpha} - 2 > -1$, i.e. $\boxed{\alpha < N-1}$.

Similarly, for the integral $J := \displaystyle\int_0^1 \frac{1}{p(s)}\psi_2^{-1}\left(\frac{C}{q(s)}\right) ds$ is descomposed as below, where there exists $\gamma < 1$ and thus:

$$
J = D\int_0^\gamma (1-s)^{\frac{N-1}{\alpha}-2} ds + E\int_\gamma^1 (1-s)^{\frac{N-1}{\alpha-1}-2} ds;\ D, E \text{ constants}
$$

and J is finite if $\dfrac{N-1}{\alpha-1} - 2 > -1$, i.e. $\boxed{\alpha < N}$.

For H3: Let $K := \displaystyle\int_0^1 \frac{1}{p(s)}\varphi^{-1}\left(\frac{C}{q(s)}\right) ds.$

Now, it is not clear an expression explicit for φ^{-1} but the following shows a type of behavior:

$$
\varphi(t) = \frac{C}{q(s)} \Leftrightarrow \frac{t^\alpha}{\sqrt{1+t^2}} = \frac{C(1-s)^{N-1}}{R_1^{N-1}} \Leftrightarrow \frac{t}{\sqrt{1+t^2}} = C_1(1-s)^{\frac{N-1}{\alpha}}
$$

.

This tells us that $\varphi^{-1}\left(\frac{C}{q(s)}\right)$ has a behavior as $(1-s)^{\frac{N-1}{\alpha}}$.

Thus, as before there exists $b < 1$ such that K is descomposed as:

$$
K = F\int_0^b (1-s)^{\frac{N-1}{\alpha}-2} ds + G\int_b^1 (1-s)^{\frac{N-1}{\alpha-1}-2} ds;\ F, G \text{ constants}
$$

where K is finite if $\boxed{\alpha < N}$

Hence, for all the above $\boxed{1 < \alpha < N-1}$ is the range.

Bibliography

[1] C. Bandle, C.V. Coffman, and M.Marcus, *Nonlinear elliptic problems in annular domains*, J. Differential Equations **69** (1987), 322–345.

[2] C. Bandle and M.K. Kwong, *Semilinear elliptic problems in annular domains*, J. Appl. Math. Phys. (ZAMP) **40** (1989), 245–257.

[3] M. Chhetri, P. Drábek and R. Shivaji, *Analysis of positive solutions for classes of quasilinear singular problems on exterior domains*, Adv. Nonlinear Anal. **6** (2017), 447-459.

[4] M. Chhetri, P. Drábek and R. Shivaji, *S-shaped bifurcation diagrams in exterior domains*, Positivity **23** (2019), 1147–1164.

[5] C. Bandle and L.A. Peletier, *Nonlinear elliptic problems with critical exponent in shrinking annuli*, Math. Ann. **280** (1988), 1-19.

[6] D. Butler, E. Ko, E. K. Lee and R. Shivaji, *Positive radial solutions for elliptic equations on exterior domains with nonlinear boundary conditions*, Comm. Pure and Appl. Analysis **13** (2014), 2713-2731.

[7] S. Coleman, V. Glazer and A. Martin, *Action minima among solutions to a class of Euclidean scalar field equations*, Comm. Math. Phys. **58** (1978), 211-221.

[8] C.V. Coffman and M. Marcus, *Existence and uniqueness results for semilinear Dirichlet problems in annuli*, Arch. Rational Mech. Anal. **108** (1989), 293-307.

[9] K. Deimling, Nonlinear Functional Analysis, Springer, 1995.

[10] R. Dhanya, Q. Morris and R. Shivaji, *Existence of positive radial solutions for superlinear, semipositone problems on the exterior of a ball*, J. Math. Anal. Appl. **434** (2016), 1533-1548.

[11] J.M. do O, S. Lorca, J. Sánchez and P. Ubilla, *Non-homogeneous elliptic equations in exterior domain*, Proc. Roy. Soc. Edinburgh Ser. A **136** (2006) 139–147.

[12] J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. **1** (1951), 353–367.

[13] X. Garaizar, *Existence of positive radial solutions for semilinear elliptic equations in the annulus*, J. Differential Equations **70** (1987), 69-92.

[14] I.M. Gelfand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Tansl. Ser. 2 **29** (1963), 295–381.

[15] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, 1988.

[16] J.C. Guajardo, S. Lorca and R. Mahadevan, *Positive radial solutions of a quasilinear problem in an exterior domain with vanishing boundary conditions*, Topological Meth. Nonlin. Anal., accepted.

[17] P.V. Gordon, E. Ko and R. Shivaji *Multiplicity and uniqueness of positive solutions for elliptic equations with nonlinear boundary conditions arising in a theory of thermal explosion*, Nonlinear Anal. Real World Appl. **15** (2014), 51-57.

[18] D.D. Hai and R. Shivaji, *Positive radial solutions of a class os singular superlinear problems on the exterior of a ball with nonlinear boundary conditions*, J. Math. Anal. Appl. **456** (2017), 872-881.

[19] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1973), 241-269.

[20] S. Kesavan, Nonlinear Functional Analysis - A First Course, Text and Readings in Mathematics, Hindustan Publishing Society, 2004.

[21] L. Kong and J. Wang, *Multiple positive solutions for the one-dimensional p-Laplacian*, Nonlinear Anal. **42** (2000), 1327-1333.

[22] M.A. Krasnosel'skii, *Positive solutions of operator equations*, Noordhoff, Groningen, 1964.

[23] M. Kwong and L. Zhang, *Uniqueness of the positive solutions of* $\Delta u + f(u) = 0$ *in an annulus*, Differential and Integral Equations **4** (1991), 583-599.

[24] S.S. Lin, *On the existence of positive radial solutions for semilinear elliptic equations in annular domains*, J. Differential Equations **81** (1989), 221-233.

[25] S.S. Lin and F. M. Pai, *Existence and multiplicity of positive radial solutions for semilinear elliptic equations in annular domains*, SIAM J. Math. Anal. **22** (1991), 1500-1515.

[26] W.-M. Ni and R.D. Nussbaum,*Uniqueness and nouniqueness for positive radial solutions of* $\Delta u + f(u,r) = 0$, Comm. Pure Appl. Math. **38** (1985), 67-108.

[27] W.-M. Ni and Serrin, *Non-existence theorems for singular solutions of quasilinear partial differential equations*, Comm. Pure Appl. Math. **39** (1986), 379-399.

[28] W.-M. Ni and Serrin, *Non-existence theorems for quasilinear partial differential equations*, Rend. Circ. Mat. Palermo Suppl. **5** (1986), 171-185.

[29] J. Sánchez, *Multiple positive solutions of singular eigenvalue type problems involving the one-dimensional p-laplacian*, J. Math. Anal. Appl. **292** (2004), 401-414.

[30] R. Stańczy, *Positive solutions for superlinear elliptic equations*, J. Math. Anal. Appl. **283** (2003), 159-166.

[31] H. Wang, *On the existence of positive radial solutions for semilinear elliptic equations in the annulus*, J. Differential Equations **109** (1994), 1-7.

[32] J. Wang, *The existence of positive solutions for the one-dimensional p-Laplacian*, Proc. Amer. Math. Soc. **125** (1997), 2275-2283.

[33] H.Wang, *On the structure of positive radial solutions for quasilinear equations in annular domains*, Advances in Differential Equations **8** (2003), 111-128.

[34] S. Yadava, *Uniqueness of positive solutions of the Dirichlet problems $-\Delta u = u^p \pm u^q$ in an annulus*, J. Differential Equations **139** (1997), 194-217.

[35] S. Yadava, *Existence and exact multiplicity of positive radial solutions of semilinear elliptic problems in annuli*, Adv. Differential Equations **6** (2001), 129-154.

[36] Zeidler, Non-linear functional analysis and it's Applicactions- I, Springer, New York, 1985.