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AN HDG METHOD FOR STOKES FLOW ON DISSIMILAR NON-MATCHING MESHES

POR

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AN HDG METHOD FOR STOKES FLOW ON DISSIMILAR NON-MATCHING MESHES

UN MÉTODO DE GALERKIN DISCONTINUO HIBRIDIZABLE PARA EL PROBLEMA DE STOKES CON MALLAS NO CONFORMES

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"What is a man? A miserable little pile of secrets."

— Dracula, *Symphony of the Night (1997)*

A mi familia: Jaime, Doris, Ximena, Felipe y Roger.

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Abstract

In this work we propose and analyze an HDG method for the Stokes equation whose domain is discretized by two independent polygonal subdomains with different meshsizes. This causes a non-conformity at the intersection of the subdomains or might even leave a gap (unmeshed region) between them. In order to appropiately couple these two different discretizations, we propose suitable transmission conditions such that we preserve the high order of the scheme. On the other hand, stability estimates are established in order to show the well-posedness of the method and the error estimates. In particular, for smooth enough solutions, the L^2 norm of the errors associated to the approximations of the velocity gradient, the velocity and the pressure are of order h^{k+1}, where h is the meshsize and k is the polynomial degree of the local approximation spaces. Moreover, the method presents superconvergence of the velocity trace and the divergence-free postprocessed velocity. Finally, we show numerical experiments that validate our theory and the capacities of the method.

Resumen

En esta tesis se propone y analiza un método HDG para la ecuación de Stokes cuyo dominio es discretizado por dos subdominios poligonales independientes entre sí y de diferente tamaño de malla. Esto produce una no-conformidad en la intersección de ambos dominios e incluso puede generar un gap (región no discretizada) entre ambos. Para acoplar de forma adecuada estas dos diferentes discretizaciones, se proponen condiciones de transmisión apropiadas de manera de preservar el alto orden del esquema. Por otro lado, se establecen estimaciones de estabilidad para demostrar unisolvencia del esquema propuesto y estimaciones del error. En particular, para soluciones suficientemente suaves, la norma L^2 de los errores asociados a las aproximaciones de la velocidad, su gradiente y la presión son de orden h^{k+1} donde h es el tamaño de malla y k el grado polinomial de los espacios locales de aproximación. Además, se demuestra superconvergencia tanto de la aproximación de la traza de la velocidad, como del postproceso de la velocidad que satisface la condición de de divergencia nula. Por último, se presentan experimentos numéricos avalando la teoría y mostrando las capacidades del método.

CHAPTER 1

Introduction

In different applications, interfaces divide the domain of interest $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) into several subdomains. For instance, in the case of solid-fluid interactions, the boundary of the solid corresponds to an interface where the equations of motion for the fluid are coupled with the elasticity equations of the former via appropiate transmission conditions, namely the no-slip condition (continuity of the velocities) and the balance of forces (or, more specifically, of the stresses). Nonetheless, as the geometrical complexity and the required spatial sampling of the subdomains increases, e.g. when the region occupied by the fluid requires a finer mesh compared to the meshsize of the discretization of the solid, it is not uncommon to mesh these regions separately, leading to a mismatch between the two discretizations as one can observe in Figure 1.1. As mentioned in [42], it is possible to identify in the literature two configurations where the domain is discretized by the union of different computational subdomains. In the first one, subdomains are independently meshed, originating a discretization of the domain made of *dissimilar meshes*, where the discrete interfaces of neighboring subdomains need to be properly "tied" (cf. [27]), as is the case in Figure 1.2 (left), which can lead to gaps and overlaps between the two meshes; in the second configuration, an interface is endowed with two different grids originating from *non-matching* interfaces as happens, for example, in the domain decomposition method like in [44]. An example of this can be seen in Figure 1.2 (right), where both grids follow a linear interpolation of the physical interface. We note that, while gaps can occur when that interface is not polygonal, overlaps between the domains are ruled out in this case.

In general, methods that deal with smooth boundaries and interfaces can be classified as *fitted* or *unfitted*, where the former adjusts the discretization to the not necessarily polygonal boundaries and interfaces (cf. [3], [4]), as is the case with isoparametric [2] or isogeometric [26] elements, while the latter considers discretizations that are relatively independent of the curved side (we refer the reader to [20, Sect. 1] for a thorough review of unfitted methods). Although fitted methods present easy implementation of previously known methods with high accuracy, the construction of such a mesh might result too difficult for complex geometries; in the case of isoparametric elements, the complexity relies on the computation of non-linear maps which result in the need of higher-order quadratures for the basis functions. Furthermore, the use of curvilinear maps might not eliminate the spatial mismatch of the discretizations unless the parametrization of the interface can be represented exactly by these mappings. On the other hand, unfitted methods present a more simple geometric approach, but at the cost of presenting a higher difficulty to devise high-order methods as the variational crime is much higher in this case. In the context of finite differences, the Immersed Boundary method (IB) has been shown to obtain first order accuracy for the velocity [35] and, in the case of Finite Element methods, Mortar methods have been used to impose the transmission conditions using Lagrange multipliers, but with sub-optimal convergence rates [28, 29]. Higher-order results have been obtained with the Cut Finite Element method (CutFEM) [5, 6], which uses a Nitsche-type approach by adding pressure stabilization and ghost penalty terms, although the results remain quasi-optimal as inf-sup stable spaces must still be chosen in this case. Thus, with the aim of devising a high-order method that takes into account the very complex geometries usually present in fluid-solid interaction problems, while remaining computationally modest, we present a high-order unfitted

HDG method for the Stokes problem with an interface.

In [11], a high order method for problems with curved interfaces is shown to be optimally convergent. It draws upon the ideas of the polynomial extension finite element method (PE-FEM), originally developed for problems with smooth boundaries [12], where instead of adjusting the mesh to the curved domain, a polynomial extrapolation of the approximate solution is used to match the prescribed Neumann or Dirichlet boundary condition. This approach has been successfully applied in the context of HDG methods during the last decade in works such as [20] and [40].

Let us briefly describe the historic perspective of the development of HDG methods. The main criticism of Discontinuous Galerkin (DG) methods is due to the fact that they have too many globally coupled degrees of freedom. In order to overcome this drawback, [16] introduced a unifying framework for hybridization of DG methods for diffusion problems, where the only globally coupled degrees of freedom are those of the numerical traces on the inter-element boundaries. The remaining unknowns are then obtained by solving local problems on each element. To be more precise, at the continuous level, the intra-element variables can be written in terms of the inter-element unknowns by solving local problems on each element. These problems, called local-solvers, can be discretized by a DG method, generating a family of methods named HDG methods. Furthermore, the analysis of general geometries and the estimates for meshes with hanging nodes for these methods is carried out in [9, 10].

We will now discuss the HDG method and its applications related to our context. In addition to diffusion equations, in the context of fluid mechanics, HDG methods have been developed for a wide variety of problems such as Stokes flow equations [15, 19, 36, 17, 21], quasi-Newtonian Stokes flow [31, 32], Stokes–Darcy coupling [33], Brinkman problem [1, 30] and Navier–Stokes equations [39, 41, 8, 37], just to name a few. In particular, we take note of [15], where a class of HDG methods for the Stokes problem considering a vorticity-velocity-pressure formulation was derived. Furthermore, it was shown that the method can be hybridized in four different ways including tangential velocity/pressure and velocity/average pressure hybridizations.

Hybridization for DG methods for Stokes was initially introduced in [7] as a technique that allowed the use of globally divergence-free velocity spaces without having to actually carry out their almost-impossible constructions. The technique was then further developed, with a similar intention, in the framework of mixed methods in [13, 14]. Later on, the analysis of an HDG method based on a velocity gradient-velocity-pressure formulation was analyzed in [17], where an element-by-element postprocess of the velocity was introduced in order to achieve a globally divergence-free condition.

This work is closely related to the technique developed in [18, 20, 23] to handle curved domains, which is based on transferring the boundary condition to the computational boundary using a line integration of the extrapolated approximation of the gradient. This technique has been successfully applied and analyzed in different contexts [24, 40, 43, 22]. In our case, based on the ideas of [26], which extend the transferring technique of [20] from the curved boundary case to the curved interface one, this work proposes an extension of the ideas presented in [43] to the context of a Stokes interface problem, which, to the best of our knowledge, has not been introduced as of yet. To that end, we consider the following formulation:

$$\mathbb{L} - \nabla \boldsymbol{u} = 0 \qquad \text{on } \Omega, \tag{1.0.1a}$$

$$-\nabla \cdot (\nu \mathbb{L} - p\mathbb{I}) = \boldsymbol{f} \qquad \text{on } \Omega, \tag{1.0.1b}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{on } \Omega, \tag{1.0.1c}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma, \tag{1.0.1d}$$

$$[\![\boldsymbol{u}]\!] = 0 \qquad \text{on } \mathcal{I}, \tag{1.0.1e}$$

$$[\![(\nu \mathbb{L} - p\mathbb{I})\boldsymbol{n}]\!] = 0 \qquad \text{on } \mathcal{I}, \tag{1.0.1f}$$

$$\int_\Omega p = 0, \tag{1.0.1g}$$

where $\Omega \subset \mathbb{R}^d$ will be a polygonal ($d = 2$) or polyhedral ($d = 3$) domain with boundary $\Gamma := \partial\Omega$; $\Omega^1, \Omega^2 \subset \Omega$ are two disjoint open subsets such that $\mathcal{I} := \overline{\Omega^1} \cap \overline{\Omega^2}$ is the interface that separates them, $\nu > 0$ is a constant viscosity, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is the volumetric force acting on the fluid, $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\Omega)$ is the boundary data that satisfies $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} = 0$ ($\boldsymbol{n}$ being the outward unit normal to Ω), and $[\![\star]\!] := \star \cdot \boldsymbol{n}^+ + \star \cdot \boldsymbol{n}^-$ is the usual jump operator. While the method presented in [43] allows us to deal with curved boundaries and thus removing the restriction for Ω to be polygonal or polyhedral, for the sake of simplicity we choose to avoid this in order to focus on the treatment of the transmission conditions on the interface.

The remainder of this work is organized as follows. In Chapter 2 we will introduce the HDG scheme along with a postprocess for the recovery of the pressure and the construction of a divergence-free approximation of the velocity, as well as the necessary notation related to the discretization and transferring segments. In Chapter 3 we carry out the analysis of the a priori analysis of the method, first presenting the main results and later their respective proofs. Then, in Chapter 4 we present numerical results to validate our theoretical estimates. Finally, in Chapter 5 we discuss some applications of the method that were left out of the analysis.

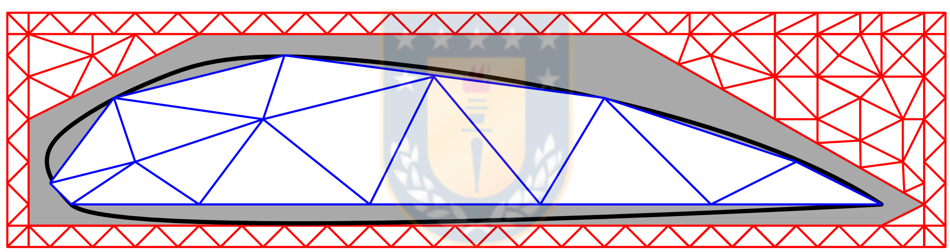

Figure 1.1: Example of dissimilar meshes for a complex geometry, where the difficulty to approximate the non-polygonal interface (black solid line) leads to gaps (shaded region).

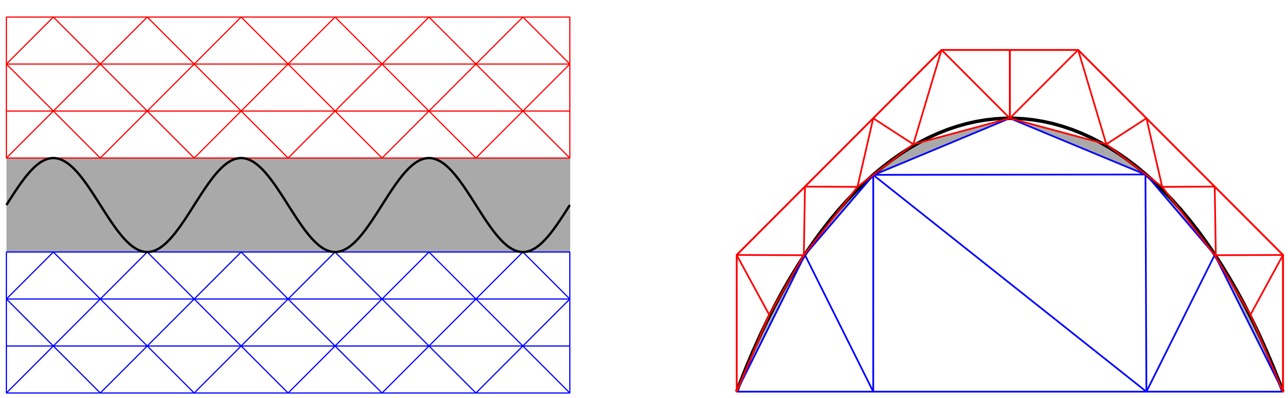

Figure 1.2: Example of dissimilar meshes (left) and non-matching meshes (right). In the first case, two independent meshes that approach the interface must be properly "tied", while in the latter the boundaries of each discretization respect the interface, leading to gaps in the non-polygonal case.

CHAPTER 2

The HDG method

2.1 Notation and preliminaries

2.1.1 Computational domain

The physical domain Ω consists of two disjoint open subdomains, Ω^1 and Ω^2, with outward unit normal vectors $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$, respectively, such that $\mathcal{I} = \overline{\Omega^1} \cap \overline{\Omega^2}$ represents the interface between the two subdomains. For $i \in \{1, 2\}$ and $h_i > 0$, let $\Omega^i_{h_i} = \{K\}$ denote a $(\Gamma \cap \partial\Omega^i)$-conforming triangulation of Ω^i, with boundary $\Gamma^i_{h_i}$, made of polyhedral elements K of size proportional to h_i. Without loss of generality, suppose $h_2 \geq h_1$. We assume each element K is a simplex, a quadrilateral ($d = 2$) or a hexahedron ($d = 3$). Also, to simplify notation, we will just write h instead of h_i when there is no confusion, i.e., when the label h indicates the size of the triangulation Ω^1_h or Ω^2_h. In Figure 2.1, we can observe the physical domain (left) and its corresponding independent discretizations (right), where the physical interface $\mathcal{I}$ is approximated via two discrete interfaces, $\mathcal{I}^1_h$ and $\mathcal{I}^2_h$, that are far from each other.

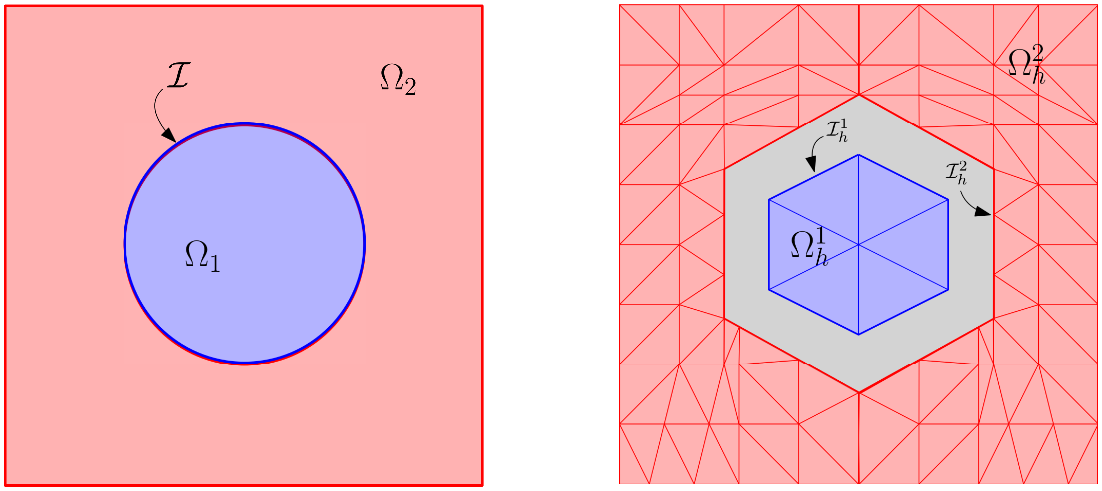

Figure 2.1: The physical domains (left) and the corresponding discretizations (right).

The family of triangulations $\{\Omega_h^i\}_{h>0}$ is assumed to be shape-regular, i.e., there exists $\kappa_i > 0$ such that for all elements $K \in \Omega_h^i$ and all $h > 0$, $h_K/\rho_K \leq \kappa_i$, where h_K is the diameter of K and ρ_K is the diameter of the largest ball contained in K. For every element K, we will denote by $\boldsymbol{n}_K$ the outward unit normal vector to K, writing $\boldsymbol{n}$ instead of $\boldsymbol{n}_K$ when there is no confusion. The set of all the faces e of Ω_h^i is denoted by $\mathcal{E}_h^i$.

2.1.2 Connecting segments

We introduce a mapping $\boldsymbol{\varphi} : \mathcal{I}_h^2 \to \mathcal{I}_h^1$ such that for each point $\boldsymbol{x}^2 \in \mathcal{I}_h^2$, we associate a point $\boldsymbol{x}^1 = \boldsymbol{\varphi}(\boldsymbol{x}^2) \in \mathcal{I}_h^2$. We denote by $\ell(\boldsymbol{x}^2)$ the segment starting at $\boldsymbol{x}^2$ and ending at $\boldsymbol{x}^1$, with unit tangent vector $\boldsymbol{m}$ and length $|\ell(\boldsymbol{x}^2)|$. The segment $\sigma(\boldsymbol{x}^2)$ is referred as the *connecting segment* associated to $\boldsymbol{x}^2$ and is assumed to satisfy two conditions: it does not intersect the interior of another segment and its length $|\ell(\boldsymbol{x}^2)|$ is of order at most $\max\{h_1, h_2\} = h_2$.

Now, let $\boldsymbol{v}^2$ be a vertex of $\mathcal{I}_h^2$. We assume that the point $\boldsymbol{v}^1$ associated to $\boldsymbol{v}^2$ is a vertex of $\mathcal{I}_h^1$. Thus, $\mathcal{I}_h^1$ induces a partition of $\mathcal{I}_h^2$ that we denote by $\mathcal{F}_h^2 = \{F\}$, where every face F of the partition corresponds to the *opposing face* to some $e \in \mathcal{I}_h^1$, i.e., $F = \boldsymbol{\varphi}^{-1}(e)$. An example of such a partition can be seen in Figure 2.2, where each face on the bottom mesh is divided into two faces corresponding to the opposing side of some face in the upper mesh. This partition is induced by the connecting segments between the vertices (solid black arrows) and an example of the mapping φ for an arbitrary point $\boldsymbol{x}^2$ on the bottom interface (dashed arrow).

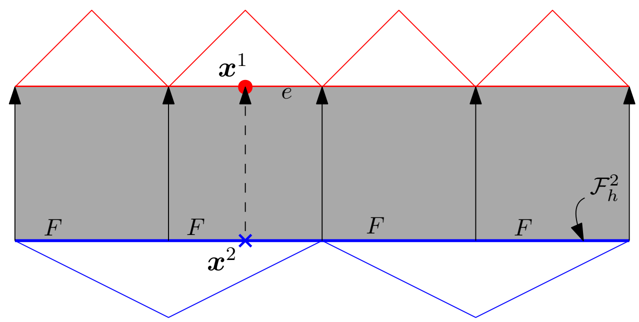

Figure 2.2: Partition $\mathcal{F}_h^2$ of $\mathcal{I}_h^2$ induced by $\mathcal{I}_h^1$ via φ.

Let $i \in \{1, 2\}$. Given a face $e \in \mathcal{I}_h^i$ belonging to the element $K_e \in \Omega_h^i$, we define the *extrapolation patch* as

$$K_e^{\text{ext}} := \{\boldsymbol{x} + \boldsymbol{n}^i t : 0 \leq t \leq |\ell(\boldsymbol{x})|\,,\ \boldsymbol{x} \in e\}.$$

We denote by $h_e^\perp$ (resp. δ_e) the largest distance of a point inside K_e^1 (resp. K_e^{ext}) to the plane determined by the face e. In other words,

$$h_e^\perp = \max_{\boldsymbol{x} \in K_e} |\text{dist}(\boldsymbol{x}, e)|\,, \qquad \delta_e := \max_{\boldsymbol{x} \in e} |\ell(\boldsymbol{x})|\,,$$

where $\text{dist}(\boldsymbol{x}, e)$ denotes the distance from $\boldsymbol{x}$ to the face e. We note that δ_e is a measure to the local size of the gap and $\delta := \max_e \delta_e$ is an upper bound of the gap. We define the ratio $r_e := \delta_e / h_e^\perp$ and its maximum $R_i := \max\limits_{e \in \mathcal{I}_h^i} r_e$. An illustration of this notation can be seen in Figure 2.3 for the leftmost element of the upper mesh.

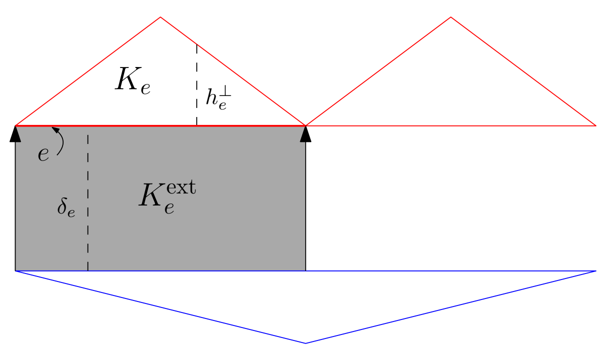

Figure 2.3: Extrapolation region and gap for the leftmost element of the upper mesh.

2.1.3 Spaces and norms

Given an element K and a non-negative integer k, $\mathbb{P}_k(K)$ denotes the space of polynomials of total degree at most k on K. For any face e, $\mathbb{P}_k(e)$ denotes the space of polynomials of total degree at most k on e. Then, we define $\mathbb{P}_k(\partial K) := \bigcup_{e\in\partial K}\mathbb{P}_k(e)$ as the space of total degree at most k on ∂K. Given a region $D\subset\mathbb{R}^d$, we denote by $(\cdot,\cdot)_D$ and $\langle\cdot,\cdot\rangle_{\partial D}$ the $L^2(D)$ and $L^2(\partial D)$ inner products, respectively. The L^2-norms over D and ∂D will be denoted by $\|\cdot\|_D$ and $\|\cdot\|_{\partial D}$. We use the standard notation for Sobolev spaces and their associated norms and seminorms, where vector-valued functions and their corresponding spaces are denoted in bold face, and blackboard bold for the tensor-valued case.

For a given polynomial of degree k and $i\in\{1,2\}$, we introduce the finite dimensional spaces

$$\begin{aligned}
\mathbb{V}_h^i &:= \{\mathbb{G}\in\mathbb{L}^2(\Omega_h^i): \quad \mathbb{G}\big|_K \in [\mathbb{P}_k(K)]^{d\times d}, \quad \forall K\in\Omega_h^i\},\\
\boldsymbol{V}_h^i &:= \{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega_h^i): \quad \boldsymbol{v}\big|_K \in [\mathbb{P}_k(K)]^{d}, \quad \forall K\in\Omega_h^i\},\\
V_h^i &:= \{q\in L^2(\Omega_h^i): \quad q\big|_K \in \mathbb{P}_k(K), \quad \forall K\in\Omega_h^i\},\\
\boldsymbol{M}_h^i &:= \{\boldsymbol{\mu}\in\boldsymbol{L}^2(\mathcal{E}_h^i): \quad \boldsymbol{\mu}\big|_e \in [\mathbb{P}_k(e)]^{d}, \quad \forall e\in\mathcal{E}_h^i\},\\
\boldsymbol{M}_h(\mathcal{I}_h^i) &:= \{\boldsymbol{\mu}\in\boldsymbol{L}^2(\mathcal{I}_h^i): \quad \boldsymbol{\mu}\big|_e \in [\mathbb{P}_k(e)]^{d}, \quad \forall e\in\mathcal{I}_h^i\}.
\end{aligned}$$

The inner products for the triangulation Ω_h^i, ($i=1,2$) are given by

$$(\cdot,\cdot)_{\Omega_h^i} := \sum_{K\in\Omega_h^i}(\cdot,\cdot)_K, \qquad \langle\cdot,\cdot\rangle_{\partial\Omega_h^i} := \sum_{K\in\Omega_h^i}\langle\cdot,\cdot\rangle_{\partial K}, \quad \text{and} \quad \langle\cdot,\cdot\rangle_{\mathcal{I}_h^i} := \sum_{e\in\mathcal{I}_h^i}\langle\cdot,\cdot\rangle_e,$$

and their corresponding norms will be denoted, respectively, by

$$\|\cdot\|_{\Omega_h^i} := \left(\sum_{K\in\Omega_h^i}\|\cdot\|_K^2\right)^{1/2}, \ \|\cdot\|_{\partial\Omega_h^i} := \left(\sum_{K\in\Omega_h^i}\|\cdot\|_{\partial K}^2\right)^{1/2}, \text{and } \|\cdot\|_{\mathcal{I}_h^i} := \left(\sum_{e\in\mathcal{I}_h^i}\|\cdot\|_e^2\right)^{1/2}.$$

To avoid proliferation of unimportant constants, we will use the terminology $a\lesssim b$ whenever $a\le Cb$ and C is a positive constant independent of h.

2.1.4 Extrapolation operator

The region enclosed by Ω_h^1 and Ω_h^2 will be denoted by Ω_h^{ext}. We notice that Ω_h^{ext} is not meshed. As a consequence, we don't have an HDG approximation in there. That is why the HDG approximation of the velocity gradient $\mathbb{L}$ and the pressure field $\tilde{p}$ will be locally extrapolated from the computational domain $\Omega_h^1 \cup \Omega_h^2$ to Ω_h^{ext}. More precisely, let $\mathbb{G}|_K : [\mathbb{P}_k(K)]^{d\times d} \to \mathbb{R}$ be a tensor-valued polynomial function which is defined on an element K in $\Omega_h^1 \cup \Omega_h^2$ such that $\overline{K} \cap \overline{\Omega_h^{\text{ext}}} \neq \emptyset$. We will define its extension to Ω_h^{ext} as

$$\boldsymbol{E}_{\mathbb{G}|_K}(\boldsymbol{y}) := \mathbb{G}|_K(\boldsymbol{y}) \qquad \forall \boldsymbol{y} \in \Omega_h^{\text{ext}}. \tag{2.1.1}$$

Note that the extended function $\boldsymbol{E}_{\boldsymbol{p}|_K}$ is either a tensor-valued whose support inclued Ω_h^{ext}. Each element K will have its own extended function.

2.1.5 The HDG projection

In the analysis, we will employ the HDG projection defined in [17]. Let $i \in \{1,2\}$ and $(\mathbb{L}, \boldsymbol{u}, \tilde{p}) \in \mathbb{H}^1(\Omega_h^i) \times \boldsymbol{H}^1(\Omega_h^i) \times H^1(\Omega_h^i)$, we take its projection $\Pi_h^i(\mathbb{L}, \boldsymbol{u}, \tilde{p}) := (\Pi_{\mathbb{V}^i}\mathbb{L}, \Pi_{\boldsymbol{V}^i}\boldsymbol{u}, \Pi_{V^i}\tilde{p})$ as the element of $\mathbb{V}_h^i \times \boldsymbol{V}_h^i \times V_h^i$ defines as follows. On an arbitrary element K of Ω_h^i, the values of the projected function on K are determined by requiring that

$$(\Pi_{\mathbb{V}^i}\mathbb{L}, \mathbb{G})_K = (\mathbb{L}, \mathbb{G})_K \qquad \forall \mathbb{G} \in [\mathbb{P}_{k-1}(K)]^{d\times d} \tag{2.1.2a}$$

$$(\Pi_{\boldsymbol{V}^i}\boldsymbol{u}, \boldsymbol{v})_K = (\boldsymbol{u}, \boldsymbol{v})_K \qquad \forall \boldsymbol{v} \in [\mathbb{P}_{k-1}(K)]^d \tag{2.1.2b}$$

$$(\Pi_{V^i}\tilde{p}, q)_K = (\tilde{p}, q)_K \qquad \forall q \in \mathbb{P}_{k-1}(K) \tag{2.1.2c}$$

$$(\operatorname{tr}\Pi_{\mathbb{V}^i}\mathbb{L}, q)_K = (\operatorname{tr}\mathbb{L}, q)_K \qquad \forall \mathbb{G} \in \mathbb{P}_k(K) \tag{2.1.2d}$$

$$\langle \nu\Pi_{\mathbb{V}^i}\mathbb{L}\boldsymbol{n} - \Pi_{V^i}\tilde{p}\boldsymbol{n} - \tau\nu\Pi_{\boldsymbol{V}^i}\boldsymbol{u}, \boldsymbol{\mu}\rangle_e = \langle \nu\mathbb{L}\boldsymbol{n} - \tilde{p}\boldsymbol{n} - \tau\nu\boldsymbol{u}, \boldsymbol{\mu}\rangle_e \qquad \forall \boldsymbol{\mu} \in [\mathbb{P}_k(e)]^d \tag{2.1.2e}$$

for all faces e of the simplex K, where $\tau > 0$ is the stabilization parameter of the HDG method. Furthermore, if $(\mathbb{L}, \boldsymbol{u}, \tilde{p}) \in \mathbb{H}^{l_\sigma+1}(\Omega_h^i) \times \boldsymbol{H}^{l_{\boldsymbol{u}}+1}(\Omega_h^i) \times H^{l_\sigma+1}(\Omega_h^i)$, for $l_\sigma, l_{\boldsymbol{u}} \in [0,k]$, we have that the above defined projection satisfies (cf. [17, Theorem 2.1]) the following properties:

$$\|\boldsymbol{u}-\Pi_{\boldsymbol{V}^i}\boldsymbol{u}\|_K \lesssim h_K^{l_{\boldsymbol{u}}+1}\,|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(K)} + h_K^{l_\sigma+1}(\tau\nu)^{-1}\,|\nabla\cdot(\nu\mathbb{L}-\tilde{p}\mathbb{I})|_{\mathbb{H}^k(K)}\,, \tag{2.1.3a}$$

$$\begin{aligned}\|\nu(\mathbb{L}-\Pi_{\mathbb{V}^i}\mathbb{L})\|_K + \|\tilde{p}-\Pi_{V^i}\tilde{p}\|_K \lesssim h_K^{l_\sigma+1}\,|\nu\mathbb{L}-\tilde{p}\mathbb{I}|_{\mathbb{H}^{k+1}(K)} + h_K^{l_{\boldsymbol{u}}+1}\tau\nu\,|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(K)}\\ + \tau\nu\,\|\boldsymbol{u}-\Pi_{\boldsymbol{V}^i}\boldsymbol{u}\|_K\,,\end{aligned} \tag{2.1.3b}$$

where $\mathbb{I}$ is the identity tensor.

2.2 The HDG scheme

Since the computational domain $\Omega_h^1\cup\Omega_h^2$ does not necessarily coincide with the physical domain Ω, we introduce the following decomposition to impose the zero-mean of the pressure

$$p=\overline{p}^{\Omega_h}+\tilde{p},$$

where $\overline{p}^{\Omega_h} := \dfrac{1}{|\Omega_h^1\cup\Omega_h^2|}\displaystyle\int_{\Omega_h^1\cup\Omega_h^2} p$ is the mean of p on the computational domain and $\widetilde{p}$ is a function that belongs to $L_0^2(\Omega_h^1\cup\Omega_h^2) := \{q\in L^2(\Omega_h^1\cup\Omega_h^2) : (q,1)_{\Omega_h^1\cup\Omega_h^2}=0\}$; we write $\tilde{p}^i := \tilde{p}|_{\Omega_h^i}$.

For $i\in\{1,2\}$, we are looking for the approximation $(\mathbb{L}_h^i,\boldsymbol{u}_h^i,\tilde{p}_h^i,\widehat{\boldsymbol{u}}_h^i)\in\mathbb{V}_h^i\times\boldsymbol{V}_h^i\times V_h^i\times\boldsymbol{M}_h^i$ of $(\mathbb{L}|_{\Omega_h^i},\boldsymbol{u}|_{\Omega_h^i},\tilde{p}_{\Omega_h^i},\boldsymbol{u}|_{\mathcal{E}_h^i})$ that satisfies

$$(\mathbb{L}_h^i,\mathbb{G})_{\Omega_h^i} + (\boldsymbol{u}_h^i,\nabla\cdot\mathbb{G})_{\Omega_h^i} - \langle\widehat{\boldsymbol{u}}_h^i,\mathbb{G}\boldsymbol{n}\rangle_{\partial\Omega_h^i} = 0, \tag{2.2.1a}$$

$$(\nu\mathbb{L}_h^i,\nabla\boldsymbol{v})_{\Omega_h^i} - (\tilde{p}_h^i,\nabla\cdot\boldsymbol{v})_{\Omega_h^i} - \langle\widehat{\sigma}_h^i\boldsymbol{n}^i,\boldsymbol{v}\rangle_{\partial\Omega_h^i} = (\boldsymbol{f},\boldsymbol{v})_{\Omega_h^i}, \tag{2.2.1b}$$

$$-(\boldsymbol{u}_h^i,\nabla q)_{\Omega_h^i} + \langle\widehat{\boldsymbol{u}}_h^i\cdot\boldsymbol{n},q\rangle_{\partial\Omega_h^i} = 0, \tag{2.2.1c}$$

$$\langle\widehat{\boldsymbol{u}}_h^i,\boldsymbol{\mu}\rangle_{\Gamma_h^i\setminus\mathcal{I}_h^i} = \langle\boldsymbol{g},\boldsymbol{\mu}\rangle_{\Gamma_h^i\setminus\mathcal{I}_h^i}, \tag{2.2.1d}$$

$$\langle\widehat{\sigma}_h^i\boldsymbol{n}^i,\boldsymbol{\mu}\rangle_{\partial\Omega_h^i\setminus\Gamma_h^i} = 0, \tag{2.2.1e}$$

for all $(\mathbb{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathbb{V}_h^i \times \boldsymbol{V}_h^i \times V_h^i \times \boldsymbol{M}_h^i$, where

$$\widehat{\sigma}_h^i \boldsymbol{n}^i := \nu \mathbb{L}_h^i \boldsymbol{n}^i - \tilde{p}_h^i \boldsymbol{n}^i - \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i) \qquad \text{on } \partial\Omega_h^i. \tag{2.2.1f}$$

and τ is a positive stabilization function defined in $\partial\Omega_h^1 \cup \partial\Omega_h^2$, assumed to be uniformly bounded. The previous system is coupled via a global *uniqueness condition* of the pressure across both subdomains

$$(\tilde{p}_h^1, 1)_{\Omega_h^1} + (\tilde{p}_h^2, 1)_{\Omega_h^2} = 0, \tag{2.2.1g}$$

and the imposition of suitable *transmission conditions*,

$$\langle \widehat{\boldsymbol{u}}_h^1 - \tilde{u}_h^2, \boldsymbol{\mu} \rangle_{\mathcal{I}_h^1} = 0, \qquad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h(\mathcal{I}_h^1), \tag{2.2.1h}$$

$$\langle \widehat{\sigma}_h^2 \boldsymbol{n}^2 + \tilde{\sigma}_h^1, \boldsymbol{\mu} \rangle_{\mathcal{I}_h^2} = 0, \qquad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h(\mathcal{I}_h^2), \tag{2.2.1i}$$

where $\tilde{u}_h^2$ y $\tilde{\sigma}_h^1$ are approximations of $\boldsymbol{u}|_{\mathcal{I}_h^1}$ and $-(\nu\mathbb{L} - p\mathbb{I})\boldsymbol{n}^2|_{\mathcal{I}_h^2}$ on the opposite interface, respectively, defined as

$$\tilde{u}_h^2(\boldsymbol{x}^1) = \widehat{\boldsymbol{u}}_h^2(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \int_0^1 \boldsymbol{E}_{\mathbb{L}_h^2}(\boldsymbol{x}(s))\boldsymbol{m}(\boldsymbol{x}(s))\mathrm{d}s \tag{2.2.1j}$$

and

$$\tilde{\sigma}_h^1(\boldsymbol{x}^2) = -\nu \boldsymbol{E}_{\mathbb{L}_h^1}(\boldsymbol{x}^2)\boldsymbol{n}^2 + \boldsymbol{E}_{\tilde{p}_h^1}(\boldsymbol{x}^2)\boldsymbol{n}^2 - \tau\nu(\boldsymbol{u}_h^1(\boldsymbol{x}^1) - \widehat{\boldsymbol{u}}_h^1(\boldsymbol{x}^1)), \tag{2.2.1k}$$

where $\boldsymbol{x}^1 \in \mathcal{I}_h^1$ is the corresponding point to $\boldsymbol{x}^2 \in \mathcal{I}_h^2$ under the mapping $\boldsymbol{\varphi}$ and $\boldsymbol{x}(s) := \boldsymbol{x}^1 + (\boldsymbol{x}^1 - \boldsymbol{x}^2)s$ for $s \in [0,1]$. Here $\boldsymbol{E}$ denotes the local extrapolation defined in subsection 2.1.4.

Remark 1. *Instead of the transmission conditions* (2.2.1h) *and* (2.2.1i)*, it is also possible to alternatively choose*

$$\langle \widehat{\boldsymbol{u}}_h^2 - \tilde{u}_h^1, \boldsymbol{\mu} \rangle_{\mathcal{I}_h^2} = 0, \qquad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h(\mathcal{I}_h^2), \tag{2.2.2}$$

$$\langle \widehat{\sigma}_h^1 \boldsymbol{n}^1 + \tilde{\sigma}_h^2, \boldsymbol{\mu} \rangle_{\mathcal{I}_h^1} = 0, \qquad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h(\mathcal{I}_h^1) \tag{2.2.3}$$

where, for $\boldsymbol{x}^1 \in \mathcal{I}_h^1$*, its corresponding point* $\boldsymbol{x}^2 \in \mathcal{I}_h^2$ *and* $\widetilde{\boldsymbol{x}}(s) = \boldsymbol{x}^2 + (\boldsymbol{x}^1 - \boldsymbol{x}^2)s$*, for* $s \in [0,1]$*, we define*

$$\tilde{u}_h^1(\boldsymbol{x}^2) = \widehat{\boldsymbol{u}}_h^1(\boldsymbol{x}^1) + \left|\ell(\boldsymbol{x}^2)\right| \int_0^1 \boldsymbol{E}_{\mathbb{L}_h^1}(\widetilde{\boldsymbol{x}}(s))\boldsymbol{m}(\widetilde{\boldsymbol{x}}(s))\mathrm{d}s \tag{2.2.4a}$$

and

$$\tilde{\sigma}_h^2(\boldsymbol{x}^1) = -\nu \boldsymbol{E}_{\mathbb{L}_h^2}(\boldsymbol{x}^1)\boldsymbol{n}^1 + \boldsymbol{E}_{\tilde{p}_h^2}(\boldsymbol{x}^1)\boldsymbol{n}^1 - \tau\nu(\boldsymbol{u}_h^2(\boldsymbol{x}^2) - \widehat{\boldsymbol{u}}_h^2(\boldsymbol{x}^2)). \tag{2.2.4b}$$

2.2.1 Recovery of p_h

In this section, we follow the method proposed in [43] with slight differences in order to incorporate the two approximations $\tilde{p}_h^1$ and $\tilde{p}_h^2$.

Since $0 = \int_\Omega p = |\Omega|\, \overline{p}^{\Omega_h} + \int_{\Omega_h^{\text{ext}}} \tilde{p}$, a natural approximation is

$$\overline{p}^{\Omega_h} \approx \overline{p_h}^{\Omega_h} := -\frac{1}{|\Omega|}\int_{\Omega_h^{\text{ext}}} \tilde{p}_h, \tag{2.2.5}$$

but it is not clear how to approximate $\tilde{p}_h$ on Ω_h^{ext} since we do not have an HDG approximation in the unmeshed area. Thus, we propose

$$\tilde{p}_h|_{\Omega_h^{\text{ext}}} := \frac{1}{2}\left(\boldsymbol{E}_{\tilde{p}_h^1} + \boldsymbol{E}_{\tilde{p}_h^2}\right)$$

and then

$$\overline{p_h}^{\Omega_h} = -\frac{1}{2\,|\Omega|}\left(\int_{\Omega_h^{\text{ext}}} \tilde{p}_h^1 + \int_{\Omega_h^{\text{ext}}} \tilde{p}_h^2\right),$$

from which we set

$$p_h := \overline{p_h}^{\Omega_h} + \tilde{p}_h \tag{2.2.6}$$

as our approximation of p.

2.2.2 Divergence-free postprocessing

For $i \in \{1,2\}$, we consider the divergence-free postprocessed solution $(\boldsymbol{u}_h^\star)^i$ defined in [17], which in the two dimensional case is the piecewise polynomial function satisfying, for all

$K \in \Omega_h^i$ and $e \in \partial K$,

$$(\boldsymbol{u}_h^\star)^i \in \mathbb{P}_{k+1}(K)$$

$$((\boldsymbol{u}_h^\star)^i - \boldsymbol{u}_h^i, \nabla w)_K = 0 \qquad \forall w \in \mathbb{P}_k(K), \tag{2.2.7a}$$

$$(\nabla \times (\boldsymbol{u}_h^\star)^i - \mathsf{w}_h^i, w b_K)_K = 0 \qquad \forall w \in \mathbb{P}_{k-1}(K), \tag{2.2.7b}$$

$$\langle ((\boldsymbol{u}_h^\star)^i - \widehat{\boldsymbol{u}}_h^i) \cdot \boldsymbol{n}, \mu \rangle_e = 0 \qquad \forall \mu \in \mathbb{P}_k(e), \tag{2.2.7c}$$

$$\langle \partial_{\boldsymbol{t}}((\boldsymbol{u}_h^\star)^i \cdot \boldsymbol{n}) - \boldsymbol{n} \times \left(\{\mathbb{L}_h^t\} \boldsymbol{n} \right), \partial_{\boldsymbol{t}} \mu \rangle_e = 0 \qquad \forall \mu \in \mathbb{P}_{k+1}(e)^\perp, \tag{2.2.7d}$$

where $\boldsymbol{n} \times \boldsymbol{a} := n_1 a_2 - n_2 a_1$, $\partial_{\boldsymbol{t}} := n_2 \partial_1 + n_1 \partial_2$ is the tangential derivative, $\nabla \times \boldsymbol{u} := \partial_1 u_2 - \partial_2 u_1$ is the two-dimensional curl, $\mathsf{w}_h^i := (\mathbb{L}_h^i)_{21} - (\mathbb{L}_h^i)_{12}$ is an approximation of the vorticity, b_K is the usual bubble function associated to the simplex K and $\{\star\} := \frac{1}{2}(\star^+ + \star^-)$ is the usual average operator.

We note that, while [17] shows that the postprocessed solution is divergence-free in Ω when it coincides with the computational domain, in our context we would have that $\boldsymbol{u}_h^\star$ is divergence free in $\Omega_h^1 \cup \Omega_h^2$ as we do not have an approximation of $\boldsymbol{u}_h$ in the non-meshed region.

CHAPTER 3

Analysis of the HDG method

We will divide this chapter in two sections: first, we will state all the main results and estimates that derived from the analysis of the method mostly without proof so as not to distract the reader too much with technical details and provide a cleaner, less confusing outline; later, in the next section, we will show the derivation of the previously stated results in a more detailed fashion.

In order to provide a more streamlined explanation of the results discussed in this chapter, all which rely on smallness assumptions that depend on multiple constants, we open with a small section dedicated to the auxiliary estimates that we use to prove our estimates and where some of these constants come from.

3.1 Auxiliary results

3.1.1 Extrapolation estimates

For $e \in \mathcal{I}_h^1 \cup \mathcal{I}_h^2$, we define

$$\mathbb{V}^k := \left\{ \mathbb{G} \in [\mathbb{P}_k(K_e^{\text{ext}})]^{d\times d} : \mathbb{G}\boldsymbol{n}_e \neq 0 \text{ on each } e \subset \partial K_e^{\text{ext}} \right\}$$

where we denoted by $\boldsymbol{n}_e$ the interior normal vector to K_e^{ext} along the face e, i.e. the exterior normal vector to K_e pointing in the direction of K_e^{ext}. We can then introduce the constants

$$C_e^{\text{ext}} := \frac{1}{\sqrt{r_e}} \sup_{\mathbb{G}\in\mathbb{V}^k} \frac{\|\mathbb{G}\boldsymbol{n}_e\|_{K_e^{\text{ext}}}}{\|\mathbb{G}\boldsymbol{n}_e\|_{K_e}}, \qquad C_e^{\text{inv}} := h_e^{\perp} \sup_{\mathbb{G}\in\mathbb{V}^k} \frac{\|\partial_{\boldsymbol{n}_e}\mathbb{G}\|_{K_e}}{\|\mathbb{G}\boldsymbol{n}_e\|_{K_e}} \tag{3.1.1}$$

Adapting the result in Lemma A.2 of [20] to the tensor-valued case, we have that these constants are independent of the meshsize, but depend on the polynomial degree k.

On the other hand, following the ideas in [20] adapted to our case, it will be useful to introduce the following auxiliary functions. Let $e \in \mathcal{I}_h^2$ that belongs to K_e and K_e^{ext}. For a polynomial function $\mathbb{G}$ on K_e, we define

$$\Lambda^i_{\mathbb{G}|_{K_e}}(\boldsymbol{x}^2) := \frac{1}{|\ell(\boldsymbol{x}^2)|} \int_0^{|\ell(\boldsymbol{x}^2)|} \left(\boldsymbol{E}_{\mathbb{G}|_{K_e}}(\boldsymbol{x}^2 + \boldsymbol{n}^2 s) - \boldsymbol{E}_{\mathbb{G}|_{K_e}}(\boldsymbol{x}^i) \right) \boldsymbol{n}^2 \mathrm{d}s, \tag{3.1.2}$$

for $i \in \{1, 2\}$, where we recall that $\boldsymbol{x}^2 \in e$ and $\boldsymbol{x}^1 \in \mathcal{I}_h^1$ are connected by the segment $\ell(\boldsymbol{x}^2)$. Adapting Lemma 5.2 in [20], we have that

$$\left\| |\ell|^{1/2} \Lambda^i_{\mathbb{G}|_{K_e}} \right\|_e \leq \frac{1}{\sqrt{3}} r_e^{3/2} C_e^{\text{ext}} C_e^{\text{inv}} \|\mathbb{G}\|_{K_e} \qquad \forall \mathbb{G} \in \mathbb{V}(K_e) \tag{3.1.3a}$$

$$\left\| |\ell|^{1/2} \Lambda^i_{\mathbb{G}|_{K_e}} \right\|_e \leq \frac{1}{\sqrt{3}} r_e \left\| h_e^{\perp} \partial_{\boldsymbol{n}} \mathbb{G}\boldsymbol{n} \right\|_{K_e^{\text{ext}}} \qquad \forall \mathbb{G} \in \mathbb{H}^1(K_e^{\text{ext}}) \tag{3.1.3b}$$

The following lemma, adapted from [42] to the tensor-valued case, will be useful in the error analysis of the method and, specifically, in the development of a duality argument later on.

Lemma 1. *Suppose that $\boldsymbol{\varphi} : \mathcal{I}_h^2 \to \mathcal{I}_h^1$ is a bijection. The following assertions hold true: If $\boldsymbol{\phi} \in$*

$\boldsymbol{H}^2(\Omega)$ *and* $\boldsymbol{\Phi} := \nabla\boldsymbol{\phi}$, *then*

$$\left\| |\ell|^{-1/2} (\boldsymbol{\phi} - \boldsymbol{\phi} \circ \boldsymbol{\varphi}) - |\ell|^{1/2} (\boldsymbol{\Phi} \circ \boldsymbol{\varphi})\boldsymbol{n}^2 \right\|_{\mathcal{I}_h^2} \lesssim \delta \left\| \boldsymbol{\phi} \right\|_{\boldsymbol{H}^2(\Omega)} \tag{3.1.4a}$$

$$\left\| |\ell|^{-1/2} (\boldsymbol{\phi} - \boldsymbol{\phi} \circ \boldsymbol{\varphi}) \right\|_{\mathcal{I}_h^2} \lesssim \delta^{1/2} \left\| \boldsymbol{\phi} \right\|_{\boldsymbol{H}^2(\Omega)} \tag{3.1.4b}$$

If $\boldsymbol{\Phi} \in \mathbb{H}^1(\Omega)$, *then*

$$\left\| |\ell|^{-1/2} (\boldsymbol{\Phi} - \boldsymbol{\Phi} \ \circ \boldsymbol{\varphi})\boldsymbol{n}^2 \right\|_{\mathcal{I}_h^2} \lesssim \left\| \boldsymbol{\Phi} \right\|_{\mathbb{H}^1(\Omega)} \tag{3.1.4c}$$

Let $F \in \mathcal{F}_h$, $e = \boldsymbol{\varphi}(F)$ *and* K_e *the element* e *belongs to. If* $p \in \mathbb{P}_k(K_e)$, *then*

$$\left\| p - p \circ \boldsymbol{\varphi} \right\|_F \lesssim C_e^{\text{ext}} \delta_e h_e^{-3/2} \left\| p \right\|_{K_e} \tag{3.1.4d}$$

Finally, we recall the discrete trace inequality ([38, Lemma 1.21]): if ϕ is a scalar, vector or tensor-valued polynomial in K_e, then

$$\left\| \phi \right\|_e \leq C_e^{\text{tr}} h_e^{-1/2} \left\| \phi \right\|_{K_e}, \tag{3.1.5}$$

where C_e^{tr} is independent of the meshsize but depends on the polynomial degree.

3.2 Main results

In this section we will show, under certain assumptions, a stability estimate associated to the HDG method. Namely, we take the polynomial degree $k \geq 1$ to avoid technicalities and assume that

(A.1) $\Omega_h^1 \cap \Omega_h^2 = \emptyset$, i.e. there is no overlap between the subdomains,

(A.2) the mapping $\boldsymbol{\varphi} : \mathcal{I}_h^2 \to \mathcal{I}_h^2$ is a bijection,

(A.3) for each $e \in \mathcal{I}_h^1$, $\boldsymbol{m} = \boldsymbol{n}^2$ and $\boldsymbol{m} = -\boldsymbol{n}^1$,

(A.4) there are no hanging nodes, i.e. $\mathcal{F} = \mathcal{I}_h^2$, (A.5) $\tau^{-1/2}\max_{e\in\mathcal{I}_h^1}\left((C_e^1C_e^{\text{ext}})^2(\delta_e^2+\delta_e^{3/2})h_e^{-3}\right)+\max_{e\in\mathcal{I}_h^1}\left((C_e^1C_e^{\text{tr}}C_e^{\text{ext}})^2\delta_e^2h_e^{-4}\right)+\max_{e\in\mathcal{I}_h^2}r_e^3(C_e^{\text{ext}}C_e^{\text{inv}})^2\leq\dfrac{1}{4}$

(A.6) $\dfrac{1}{2}\max_{e\in\mathcal{I}_h^1}\delta^{1/2}\tau^{-1/2}+\max_{e\in\mathcal{I}_h^2}\delta\tau\leq\dfrac{1}{2}$

(A.7) $C_2\delta\max_{e\in\mathcal{I}_h^2}h_e^{-1}(C_e^{\text{tr}})^2\leq\dfrac{1}{4}$, where C_2 is a positive constant, independent of the meshsize, that will appear in Lemma 7.

(A.8) $C_2\left\{\delta^2\left(\tau+\nu^2+\max_{e\in\mathcal{I}_h^2}h_e^{-3}(C_e^{\text{ext}})^2\right)+\delta\left(\tau^{-1}+1+\delta\max_{e\in\mathcal{I}_h^2}r_e^3(C_e^{\text{ext}}C_e^{\text{inv}})^2\right)+2\nu^2h^2\right\}\leq\dfrac{1}{16}$, where $h:=\max\{h_1,h_2\}$.

(A.9) $8\nu^2C_3\left(C_{\tilde{p}}^{\mathbb{L}}+C_{\tilde{p}}^{\boldsymbol{u}}\right)\leq\dfrac{1}{16}$, where $C_{\tilde{p}}^{\mathbb{L}}:=\dfrac{1}{\nu^2}\Big(\tau^{-1/2}\max_{e\in\mathcal{I}_h^1}\left((C_e^1C_e^{\text{ext}})^2(\delta_e^2+\delta_e^{3/2})h_e^{-3}\right)$ $+\max_{e\in\mathcal{I}_h^1}\left((C_e^1C_e^{\text{tr}}C_e^{\text{ext}})^2\delta_e^2h_e^{-4}\right)+2\max_{e\in\mathcal{I}_h^2}\delta_eh_e^{-1}(C_e^{\text{tr}})^2+\dfrac{1}{256C_3}\Big)$ is a constant that will appear in Lemma 4, $C_{\tilde{p}}^{\boldsymbol{u}}:=2C_2\max_{e\in\mathcal{I}_h^2}\left\{\delta_eh_e^{-1}(C_e^{\text{tr}})^2+(\delta_e^2h_e^{-3}(C_e^{\text{ext}})^2)\right\}$ is a constant that will appear in Corollary 3, and C_3 is a positive constant, independent of the meshsize, that will appear in 8.

(A.10) $\beta\widehat{C}_{\text{tr}}^1\max_{e\in\mathcal{I}_h^1}C_e^1C_e^{\text{ext}}\delta_eh_e^{-3/2}+\beta\widehat{C}_{\text{ext}}^2\max_{e\in\mathcal{I}_h^2}\delta_eh_e^{-1/2}\leq1-2^{-1/2}$, where β is the constant associated to an inf-sup condition that will appear in the proof of Lemma 8.

Note that the purpose of (A.1) is to simplify the analysis, but our method still works, without any modification, when there are overlaps, as long as the other assumptions are satisfied. An example of this is given in Chapter 5. As discussed in [42], assumption (A.2) is not too strong when $\mathcal{I}_h^1$ and $\mathcal{I}_h^2$ share the same topological properties, which is expected when both meshes are built from the same CAD geometry. These two assumptions will be assumed to hold along the manuscript without explicitly mentioning them.

Assumption (A.3) means that the direction of the connecting segments must be parallel to the normals computed at its ends. Fortunately, the estimates of this work are also true if, instead of (A.3), we assume $1+\boldsymbol{m}\cdot\boldsymbol{n}^1$ and $1-\boldsymbol{m}\cdot\boldsymbol{n}^2$ are small enough, i.e. the direction of the connecting segments does not deviate too much from the normals at its ends.

Assumption (A.4) is related to the fact that, under the presence of hanging nodes, the map $\boldsymbol{\varphi}$ fails to preserve polynomials on $e\in\mathcal{I}_h^2$ to polynomials on the corresponding faces $\{F\}\subset$

$\mathcal{I}_h^2$. This would force us to use the $\boldsymbol{L}^2$ projection onto $\boldsymbol{M}_h^2$ in our arguments, which will ultimately lead us to the appearance of lower powers of h in the error estimates as seen in [42]. Nevertheless, it is possible to prove that the method is still optimal in all the variables, except in $\mathbb{L}$ where the order of convergence loses half a power of h (see Chapter 4).

Finally, (A.5)-(A.10) refer to smallness assumptions regarding the meshsize and the size of the gap. We note that these assumptions are satisfied for small enough h and if δ is proportional to at least a power of 2 of h, which is the case when $\mathcal{I}_h^1$ and $\mathcal{I}_h^2$ correspond to piecewise linear interpolations of the interface $\mathcal{I}$. We end this section by mentioning that, while most constants present in assumptions (A.5)-(A.10) vanish when there is no gap, this is not the case for (A.9). In fact, we see that as δ goes to 0, the term $8\nu^2 C_3 \left(C_{\tilde{p}}^{\mathbb{L}} + C_{\tilde{p}}^{\boldsymbol{u}}\right)$ goes to $\dfrac{1}{32}$. This is coherent with [43], where a similar inf-sup argument is used in order to prove an estimate of the pressure when there is no such gap in the mesh. We also observe that the left-hand sides in all the assumptions are bounded even for small viscosity.

3.2.1 Stability estimates

We introduce a modified version of (2.2.1) with artificial source terms that will help us to be more explicit with our estimates in order to reuse these results both for the well-posedness of the scheme and the error bounds. Therefore, we consider the same problem (2.2.1), but (2.2.1a) is replaced by

$$(\mathbb{L}_h^i, \mathbb{G})_{\Omega_h^i} + (\boldsymbol{u}_h^i, \nabla \cdot \mathbb{G})_{\Omega_h^i} - \langle \widehat{\boldsymbol{u}}_h^i, \mathbb{G}\boldsymbol{n}\rangle_{\partial\Omega_h^i} = (\mathbb{H}^i, \mathbb{G})_{\Omega_h^i}, \tag{3.2.1a}$$

where $\mathbb{H}^i \in \mathbb{L}^2(\Omega_h^i)$ is a given function such that it is orthogonal to polynomials of degree $k-1$, and (2.2.1h) and (2.2.1i) are replaced by

$$\langle \widehat{\boldsymbol{u}}_h^1 - \tilde{u}_h^2, \boldsymbol{\mu}\rangle_{\mathcal{I}_h^1} = \langle \boldsymbol{T}_1, \boldsymbol{\mu}\rangle_{\mathcal{I}_h^1}, \qquad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h(\mathcal{I}_h^1) \tag{3.2.1b}$$

$$\langle \widehat{\sigma}_h^2 \boldsymbol{n}^2 + \tilde{\sigma}_h^1, \boldsymbol{\mu}\rangle_{\mathcal{I}_h^2} = \langle \boldsymbol{T}_2, \boldsymbol{\mu}\rangle_{\mathcal{I}_h^2}, \qquad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h(\mathcal{I}_h^2), \tag{3.2.1c}$$

where $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are given functions belonging to $\boldsymbol{L}^2(\mathcal{I}_h^1)$ and $\boldsymbol{L}^2(\mathcal{I}_h^2)$, respectively. In particular, to show well-posedness, $\mathbb{H}^i$ and $\boldsymbol{T}_i$ will be zero, whereas $\mathbb{H}^i$ and $\boldsymbol{T}_i$ will be related to projection errors when proving the error bounds.

For our convenience, let us define

$$\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h) := \left(\sum_{i=1}^{2} \left\|\mathbb{L}_h^i\right\|_{\Omega_h^i}^2 + \sum_{i=1}^{2} \left\|\tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)\right\|_{\partial\Omega_h^i}^2 + \left\||\ell|^{-1/2}\left(\tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2\right)\right\|_{\mathcal{I}_h^2}^2 \right)^{1/2}. \tag{3.2.2}$$

We are now ready to state the main results of this section.

Theorem 1. *Suppose Assumptions A hold true. If τ is of order one, $k \geq 1$ and $h < 1$, then there exists $h_0 \in (0,1)$ such that, for all $h < h_0$, it holds*

$$\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)^2 \lesssim \sum_{i=1}^{2} \left\|\mathbb{H}^i\right\|_{\Omega_h^i}^2 + \frac{1}{\nu^2}\|\boldsymbol{f}\|_{\Omega}^2 + \|\boldsymbol{g}\|_{\Gamma}^2 + \left\|h_e^{-1/2}\boldsymbol{T}_1\right\|_{\mathcal{I}_h^1}^2 + \left\|h_e^{-1/2}\boldsymbol{T}_2\right\|_{\mathcal{I}_h^2}^2 \tag{3.2.3}$$

Morover, if elliptic regularity of the dual problem (3.3.7) *holds true, it holds*

$$\begin{aligned}\sum_{i=1}^{2} \left\|\boldsymbol{u}_h^i\right\|_{\Omega_h^i}^2 \lesssim &\left(\delta + \nu\delta^2 + \max_{e\in\mathcal{I}_h^2} \delta_e^2 h_e^{-3} + h_1^2 + h_2^2\right) \mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)^2 \\ &+ \sum_{i=1}^{2} \left\|\mathbb{H}^i\right\|_{\Omega_h^i}^2 + \|\boldsymbol{f}\|_{\Omega}^2 + \|\boldsymbol{g}\|_{\Gamma}^2 + \|\boldsymbol{T}_1\|_{\mathcal{I}_h^1}^2 + \|\boldsymbol{T}_2\|_{\mathcal{I}_h^2}^2\end{aligned} \tag{3.2.4}$$

and

$$\sum_{i=1}^{2} \left\|\tilde{p}_h^i\right\|_{\Omega_h^i}^2 \lesssim \nu^2 \mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)^2 + \|\boldsymbol{f}\|_{\Omega}^2. \tag{3.2.5}$$

Since the $\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)$ is present in upper bounds for $\|\mathbb{L}_h^i\|_{\Omega_h^i}$ (a trivial result that follows from its definition), $\|\boldsymbol{u}_h^i\|_{\Omega_h^i}$ and $\|\tilde{p}_h^i\|_{\Omega_h^i}$, for $i \in \{1, 2\}$, the fact that the estimate (3.2.3) depends solely on the source data will provide an easy proof of the well-posedness of the scheme and, later on, of the error estimates of the scheme.

3.2.2 Well-posedness of the scheme

Theorem 2. *The HDG scheme* (2.2.1) *has a unique solution.*

Proof. Let all sources to be equal to zero, i.e., $\mathbb{H}^1 \equiv 0$, $\mathbb{H}^2 \equiv 0$, $\boldsymbol{f} \equiv 0$, $\boldsymbol{g} \equiv 0$, $\boldsymbol{T}_1 \equiv 0$ and $\boldsymbol{T}_2 \equiv 0$. Since (2.2.1) is a square system, we just need to show that it has the trivial solution. From (3.2.2) and Theorem 1, it is clear that

$$\sum_{i=1}^{2} \left\|\mathbb{L}_h^i\right\|^2 + \sum_{i=1}^{2} \left\|\tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)\right\|_{\partial\Omega_h^i}^2 = 0,$$

and, from (3.2.4) and (3.2.5),

$$\sum_{i=1}^{2} \left\|\mathbb{L}_h^i\right\|^2 + \sum_{i=1}^{2} \left\|\boldsymbol{u}_h^i\right\|_{\Omega_h^i} + \sum_{i=1}^{2} \left\|\tilde{p}_h^i\right\|_{\Omega_h^i}^2 + \sum_{i=1}^{2} \left\|\tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)\right\|_{\partial\Omega_h^i}^2 = 0 \tag{3.2.6}$$

Thus, for $i \in \{1, 2\}$, we have that $\mathbb{L}_h^i \equiv 0$, $\boldsymbol{u}_h^i \equiv 0$ and $\tilde{p}_h^i \equiv 0$. Since we also have that $\boldsymbol{u}_h^i = \widehat{\boldsymbol{u}}_h^i$ on $\partial\Omega_h^i$, it follows that $\widehat{\boldsymbol{u}}_h^i \equiv 0$, too.

$\square$

3.2.3 Error estimates

Theorem 3. *Suppose Assumptions A and elliptic regularity hold true. If τ is of order one, $k \geq 1$ and $(\mathbb{L}, \boldsymbol{u}, \tilde{p}) \in \mathbb{H}^{l_\sigma+1}(\Omega) \times \boldsymbol{H}^{l_{\boldsymbol{u}}+1}(\Omega) \times H^{l_\sigma+1}(\Omega)$ for $l_\sigma, l_{\boldsymbol{u}} \in [0, k]$, then there exists $h_0 \in (0, 1)$ such that for all $h < h_0$, it holds*

$$\begin{aligned}
&\left(\sum_{i=1}^{2} \left\|\mathbb{L} - \mathbb{L}_h^i\right\|_{\Omega_h^i}^2\right)^{1/2} + \left(\sum_{i=1}^{2} \left\|\boldsymbol{u} - \boldsymbol{u}_h^i\right\|_{\Omega_h^i}^2\right)^{1/2} + \left(\sum_{i=1}^{2} \left\|\tilde{p} - \tilde{p}_h^i\right\|_{\Omega_h^i}^2\right)^{1/2} \\
&\lesssim (1 + \nu^{-1})^{1/2} h^{l_\sigma+1} \left|\nu\mathbb{L} - \tilde{p}\mathbb{I}\right|_{\mathbb{H}^{l_\sigma+1}(\Omega)} + (1 + \nu)^{1/2} h^{l_{\boldsymbol{u}}+1} \left|\boldsymbol{u}\right|_{\boldsymbol{H}^{l_{\boldsymbol{u}}+1}(\Omega)}
\end{aligned} \tag{3.2.7}$$

Now, considering the postprocess of the pressure detailed in subsection 2.2.1, Theorem 3 in [43] alongside Theorem 3 allows us to deduce the following corollary.

Corollary 1. *Suppose the same assumptions of Theorem 3 hold and* $(\mathbb{L}, \boldsymbol{u}, \tilde{p}) \in \mathbb{H}^{k+1}(\Omega)\times\boldsymbol{H}^{k+1}(\Omega)\times H^{k+1}(\Omega)$. *Let* $h = \max\{h_1, h_2\}$. *It holds that*

$$\left(\sum_{i=1}^{2}\left\|\mathbb{L}-\mathbb{L}_h^i\right\|^2_{\Omega_h^i}\right)^{1/2}+\left(\sum_{i=1}^{2}\left\|\boldsymbol{u}-\boldsymbol{u}_h^i\right\|^2_{\Omega_h^i}\right)^{1/2}+\left(\sum_{i=1}^{2}\left\|p-p_h^i\right\|^2_{\Omega_h^i}\right)^{1/2}\lesssim h^{k+1} \tag{3.2.8}$$

Next we show the error estimates for the divergence-free postprocessing of $\boldsymbol{u}_h^i$, for $i \in \{1, 2\}$, defined in subsection 2.2.2.

Corollary 2. *If the assumptions of Corollary 1 hold true and* $\boldsymbol{u} \in \boldsymbol{H}^{k+2}(\Omega)$, *if* $(\boldsymbol{u}_h^*)^i$ *satisfies* (2.2.7), *then it holds*

$$\left(\sum_{i=1}^{2}\left\|\boldsymbol{u}-(\boldsymbol{u}_h^*)^i\right\|^2_{\Omega_h^i}\right)^{1/2}\lesssim h^{k+2} \tag{3.2.9}$$

3.3 Proofs

3.3.1 An energy argument

We will now prove the estimate found in Theorem 1. But first, in order to showcase an important technique that will be used constantly throughout the analysis of the terms associated to the mismatch between $\mathcal{I}_h^1$ and $\mathcal{I}_h^2$, we prove the following lemma.

Lemma 2. *Let* $i \in \{1, 2\}$. *We define* $\mathbb{T}^i := \langle\widehat{\sigma}_h^i\boldsymbol{n}^i, \widehat{\boldsymbol{u}}_h^i\rangle_{\mathcal{I}_h^i}$. *It holds that*

$$\begin{aligned}\mathbb{T}^1+\mathbb{T}^2 =\ & \langle\sigma_h^1\boldsymbol{n}^1+\boldsymbol{E}_{\sigma_h^1\boldsymbol{n}^2}\circ\boldsymbol{\varphi}^{-1}, \widehat{\boldsymbol{u}}_h^1-\boldsymbol{u}_h^1\rangle_{\mathcal{I}_h^1}+\langle\sigma_h^1\boldsymbol{n}^1+\boldsymbol{E}_{\sigma_h^1\boldsymbol{n}^2}\circ\boldsymbol{\varphi}^{-1}, \boldsymbol{u}_h^1\rangle_{\mathcal{I}_h^1}\\ &-\nu\left\||\ell|^{-1/2}\left(\tilde{u}_h^2\circ\boldsymbol{\varphi}-\widehat{\boldsymbol{u}}_h^2\right)\right\|_{\mathcal{I}_h^2}+\nu\langle\Lambda^2_{\mathbb{L}_h^2}(\boldsymbol{x}^2), \tilde{u}_h^2\circ\boldsymbol{\varphi}-\widehat{\boldsymbol{u}}_h^2\rangle_{\mathcal{I}_h^2}\\ &+\langle\tilde{p}_h^2\boldsymbol{n}^2, \tilde{u}_h^2\circ\boldsymbol{\varphi}-\widehat{\boldsymbol{u}}_h^2\rangle_{\mathcal{I}_h^2}+\nu\langle\tau(\boldsymbol{u}_h^2-\widehat{\boldsymbol{u}}_h^2), \tilde{u}_h^2\circ\boldsymbol{\varphi}-\widehat{\boldsymbol{u}}_h^2\rangle_{\mathcal{I}_h^2}\end{aligned}$$

Before proving this result, we point out that in the case where there is no gap, $\mathbb{T}^1+\mathbb{T}^2 = 0$ in virtue of the fact that $\mathcal{I}_h^1 = \mathcal{I}_h^2$, $\tilde{u}_h^2 = \widehat{\boldsymbol{u}}_h^2$ and (2.2.1j).

Proof. We decompose

$$\begin{aligned}\mathbb{T}^1 + \mathbb{T}^2 &= \langle \widehat{\sigma}_h^1 \boldsymbol{n}^1, \widehat{\boldsymbol{u}}_h^1 \rangle_{\mathcal{I}_h^1} + \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \widehat{\boldsymbol{u}}_h^2 \rangle_{\mathcal{I}_h^2} \\ &= \langle \widehat{\sigma}_h^1 \boldsymbol{n}^1 - \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}, \widehat{\boldsymbol{u}}_h^1 \rangle_{\mathcal{I}_h^1} + \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \widehat{\boldsymbol{u}}_h^2 - \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} + T,\end{aligned}$$

where

$$T = \langle \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}, \widehat{\boldsymbol{u}}_h^1 \rangle_{\mathcal{I}_h^1} + \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2}$$

Since $\boldsymbol{\varphi} : \mathcal{I}_h^2 \to \mathcal{I}_h^1$ is a bijection such that $\boldsymbol{x}^2 - \boldsymbol{\varphi}(\boldsymbol{x}^2) = |\ell(\boldsymbol{x}^2)|\, \boldsymbol{m}(\boldsymbol{x}^2)$ and we are under the assumption that $\boldsymbol{m} = \boldsymbol{n}^2 = -\boldsymbol{n}^1$, which implies that both interfaces are parallel, then $\boldsymbol{\varphi}$ must be affine (a translation, even) on each $F \in \mathcal{I}_h^2$. Furthermore, if we assume that there are no hanging nodes, then $\boldsymbol{\varphi}^{-1}$ is also affine for each $e \in \mathcal{I}_h^1$. Thus, both $\tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}$ and $\tilde{u}_h^2 \circ \boldsymbol{\varphi}$ are polynomial when restricted to $e \in \mathcal{I}_h^1$ and $F \in \mathcal{I}_h^2$, respectively. In other words, $\tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1} \in \boldsymbol{M}_h(\mathcal{I}_h^1)$ and $\tilde{u}_h^2 \circ \boldsymbol{\varphi} \in \boldsymbol{M}_h(\mathcal{I}_h^2)$.

Using the transmittion conditions (2.2.1i) and (2.2.1h), we have that

$$T = \langle \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}, \tilde{u}_h^2 \rangle_{\mathcal{I}_h^1} - \langle \tilde{\sigma}_h^1, \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} + \langle \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{T}_1 \rangle_{\mathcal{I}_h^1} + \langle \boldsymbol{T}_2, \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2}$$

and, since $|\mathcal{I}_h^1| = |\mathcal{I}_h^2|$ because of the previous assumptions,

$$\begin{aligned}T &= \langle \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1} \circ \boldsymbol{\varphi}, \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} - \langle \tilde{\sigma}_h^1, \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} + \langle \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{T}_1 \rangle_{\mathcal{I}_h^1} + \langle \boldsymbol{T}_2, \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\ &= \langle \boldsymbol{T}_1, \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1} \rangle_{\mathcal{I}_h^1} + \langle \boldsymbol{T}_2, \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2}.\end{aligned}$$

Adding and substracting convenient terms, we obtain

$$\begin{aligned}T &= \langle \boldsymbol{T}_1, \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1} \rangle_{\mathcal{I}_h^1} + \langle \boldsymbol{T}_2, \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\ &= -\langle \boldsymbol{T}_1, \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1} \rangle_{\mathcal{I}_h^1} + \langle h_e^{-1/2} \boldsymbol{T}_1, h_e^{1/2} \widehat{\sigma}_h^1 \boldsymbol{n}^1 \rangle_{\mathcal{I}_h^1} \\ &+ \langle |\ell|^{1/2} \boldsymbol{T}_2, |\ell|^{-1/2} (\tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2) \rangle_{\mathcal{I}_h^2} + \langle \tau^{-1/2} \boldsymbol{T}_2, \tau^{1/2} (\widehat{\boldsymbol{u}}_h^2 - \boldsymbol{u}_h^2) \rangle_{\mathcal{I}_h^2} + \langle h_e^{-1/2} \boldsymbol{T}_2, h_e^{1/2} \boldsymbol{u}_h^2 \rangle_{\mathcal{I}_h^2}\end{aligned}$$

From the definition of the flux (2.2.1f) and definition (2.2.1k), we obtain

$$\widehat{\sigma}_h^1 \boldsymbol{n}^1 - \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1} = \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1}$$

and so, replacing this into the original sum and expanding the numerical flux on the second term,

$$\begin{aligned}
\mathbb{T}^1 + \mathbb{T}^2 &= \langle \widehat{\sigma}_h^1 \boldsymbol{n}^1 - \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}, \widehat{\boldsymbol{u}}_h^1 \rangle_{\mathcal{I}_h^1} + \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \widehat{\boldsymbol{u}}_h^2 - \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\
&= \langle \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1}, \widehat{\boldsymbol{u}}_h^1 \rangle_{\mathcal{I}_h^1} + \langle \sigma_h^2 \boldsymbol{n}^2 - \tau\nu(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2), \widehat{\boldsymbol{u}}_h^2 - \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\
&= \langle \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1}, \widehat{\boldsymbol{u}}_h^1 - \boldsymbol{u}_h^1 \rangle_{\mathcal{I}_h^1} + \langle \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{u}_h^1 \rangle_{\mathcal{I}_h^1} \\
&+ \langle \sigma_h^2 \boldsymbol{n}^2, \widehat{\boldsymbol{u}}_h^2 - \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} - \langle \tau\nu(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2), \widehat{\boldsymbol{u}}_h^2 - \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} + T
\end{aligned}$$

On the other hand, from (2.2.1j) and the assumption $\boldsymbol{m} = \boldsymbol{n}^2$, we have

$$\begin{aligned}
\tilde{u}_h^2 \circ \boldsymbol{\varphi}(\boldsymbol{x}^2) &= \widehat{\boldsymbol{u}}_h^2(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \int_0^1 \boldsymbol{E}_{\mathbb{L}_h^2}(\boldsymbol{x}^2 + \boldsymbol{n}^2 \left|\ell(\boldsymbol{x}^2)\right| s) \boldsymbol{n}^2 \mathrm{d}s \\
&= \widehat{\boldsymbol{u}}_h^2(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \underbrace{\int_0^1 \boldsymbol{E}_{\mathbb{L}_h^2}(\boldsymbol{x}^2 + \boldsymbol{n}^2 \left|\ell(\boldsymbol{x}^2)\right| s) \boldsymbol{n}^2 - \mathbb{L}_h^2(\boldsymbol{x}^2)\boldsymbol{n}^2 \mathrm{d}s}_{=:\Lambda^2_{\mathbb{L}_h^2}(\boldsymbol{x}^2)} + \left|\ell(\boldsymbol{x}^2)\right| \mathbb{L}_h^2(\boldsymbol{x}^2)\boldsymbol{n}^2 \\
&= \widehat{\boldsymbol{u}}_h^2(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \Lambda^2_{\mathbb{L}_h^2}(\boldsymbol{x}^2) + \frac{|\ell(\boldsymbol{x}^2)|}{\nu} \left(\nu \mathbb{L}_h^2(\boldsymbol{x}^2) - \tilde{p}_h^2(\boldsymbol{x}^2)\mathbb{I}\right) \boldsymbol{n}^2 + \frac{|\ell(\boldsymbol{x}^2)|}{\nu} \tilde{p}_h^2(\boldsymbol{x}^2)\boldsymbol{n}^2 \\
&= \widehat{\boldsymbol{u}}_h^2(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \Lambda^2_{\mathbb{L}_h^2}(\boldsymbol{x}^2) + \frac{|\ell(\boldsymbol{x}^2)|}{\nu} \sigma_h^2 \boldsymbol{n}^2(\boldsymbol{x}^2) + \frac{|\ell(\boldsymbol{x}^2)|}{\nu} \tilde{p}_h^2(\boldsymbol{x}^2)\boldsymbol{n}^2
\end{aligned}$$

and so

$$\sigma_h^2 \boldsymbol{n}^2(\boldsymbol{x}^2) = -\frac{\nu}{|\ell(\boldsymbol{x}^2)|} \left(\widehat{\boldsymbol{u}}_h^2(\boldsymbol{x}^2) - \tilde{u}_h^2 \circ \boldsymbol{\varphi}(\boldsymbol{x}^2)\right) - \nu \Lambda^2_{\mathbb{L}_h^2}(\boldsymbol{x}^2) - \tilde{p}_h^2(\boldsymbol{x}^2)\boldsymbol{n}^2. \tag{3.3.1}$$

Replacing this in $\mathbb{T}^1 + \mathbb{T}^2$, we have

$$\begin{aligned}
\mathbb{T}^1 + \mathbb{T}^2 &= \langle \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1}, \widehat{\boldsymbol{u}}_h^1 - \boldsymbol{u}_h^1 \rangle_{\mathcal{I}_h^1} + \langle \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{u}_h^1 \rangle_{\mathcal{I}_h^1} \\
&- \left\langle \frac{\nu}{|\ell(\boldsymbol{x}^2)|} \left(\widehat{\boldsymbol{u}}_h^2(\boldsymbol{x}^2) - \tilde{u}_h^2 \circ \boldsymbol{\varphi}(\boldsymbol{x}^2)\right), \widehat{\boldsymbol{u}}_h^2 - \tilde{u}_h^2 \circ \boldsymbol{\varphi} \right\rangle_{\mathcal{I}_h^2} - \langle \nu \Lambda^2_{\mathbb{L}_h^2}(\boldsymbol{x}^2), \widehat{\boldsymbol{u}}_h^2 - \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\
&- \langle \tilde{p}_h^2 \boldsymbol{n}^2, \widehat{\boldsymbol{u}}_h^2 - \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} - \langle \tau\nu(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2), \widehat{\boldsymbol{u}}_h^2 - \tilde{u}_h^2 \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2}.
\end{aligned}$$

Flipping the signs of some terms in order to have $\tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2$ in the second argument of the inner products, we obtain the desired result. $\square$

Now we continue with the proof of Theorem 1. To that end, we provide the following lemma.

Lemma 3. *It holds that*

$$\sum_{i=1}^{2} \nu \left(\left\| \mathbb{L}_h^i \right\|_{\Omega_h^i}^2 + \left\| \tau^{1/2} (\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i) \right\|_{\partial\Omega_h^i}^2 \right) + \nu \left\| |\ell|^{-1/2} \left(\tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2 \right) \right\|_{\mathcal{I}_h^2}^2 = \sum_{i=1}^{10} I^i + \sum_{i=1}^{2} \left(\nu (\mathbb{H}, \mathbb{L}_h^i)_{\Omega_h^i} + (\boldsymbol{f}, \boldsymbol{u}_h^i)_{\Omega_h^i} + \langle \boldsymbol{g}, \widehat{\sigma} \boldsymbol{n}_h^i \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i} \right) \tag{3.3.2}$$

where

$$\begin{aligned}
I^1 &:= \langle \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1}, \widehat{\boldsymbol{u}}_h^1 - \boldsymbol{u}_h^1 \rangle_{\mathcal{I}_h^1}, & I^6 &:= -\langle \boldsymbol{T}_1, \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1} \rangle_{\mathcal{I}_h^1} \\
I^2 &:= \langle \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{u}_h^1 \rangle_{\mathcal{I}_h^1}, & I^7 &:= \langle h_e^{-1/2} \boldsymbol{T}_1, h_e^{1/2} \widehat{\sigma}_h^1 \boldsymbol{n}^1 \rangle_{\mathcal{I}_h^1} \\
I^3 &:= \nu \langle \Lambda_{\mathbb{L}_h^2}, \tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2 \rangle_{\mathcal{I}_h^2}, & I^8 &:= \langle |\ell|^{1/2} \boldsymbol{T}_2, |\ell|^{-1/2} \left(\tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2 \right) \rangle_{\mathcal{I}_h^2} \\
I^4 &:= \langle \tilde{p}^2 \boldsymbol{n}^2, \tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2 \rangle_{\mathcal{I}_h^2}, & I^9 &:= \langle \tau^{-1/2} \boldsymbol{T}_2, \tau^{1/2} (\widehat{\boldsymbol{u}}_h^2 - \boldsymbol{u}_h^2) \rangle_{\mathcal{I}_h^2} \\
I^5 &:= \nu \langle \tau (\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2), \tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2 \rangle_{\mathcal{I}_h^2}, & I^{10} &:= \langle h_e^{-1/2} \boldsymbol{T}_2, h_e^{1/2} \boldsymbol{u}_h^2 \rangle_{\mathcal{I}_h^2}.
\end{aligned}$$

Proof. In (2.2.1), for $i \in \{1, 2\}$, we test with $\mathbb{G}|_{\Omega_h^i} = \nu \mathbb{L}_h^i$, $\boldsymbol{v}|_{\Omega_h^i} = \boldsymbol{u}_h^i$, $q|_{\Omega_h^i} = \tilde{p}_h^i$ and

$$\boldsymbol{\mu} = \begin{cases} \widehat{\sigma}_h^i \boldsymbol{n}_h^i & \text{on } \Gamma_h^i \setminus \mathcal{I}_h^i, \\ \widehat{\boldsymbol{u}}_h^i & \text{on } \partial\Omega_h^i \setminus \Gamma_h^i. \end{cases}$$

The resulting system is

$$\begin{aligned}
(\mathbb{L}_h^i, \nu \mathbb{L}_h^i)_{\Omega_h^i} + (\boldsymbol{u}_h^i, \nabla \cdot \nu \mathbb{L}_h^i)_{\Omega_h^i} - \langle \widehat{\boldsymbol{u}}_h^i, \nu \mathbb{L}_h^i \boldsymbol{n} \rangle_{\partial\Omega_h^i} &= (\mathbb{H}, \nu \mathbb{L}_h^i)_{\Omega_h^i}, \\
(\nu \mathbb{L}_h^i, \nabla \boldsymbol{u}_h^i)_{\Omega_h^i} - (\tilde{p}_h^i, \nabla \cdot \boldsymbol{u}_h^i)_{\Omega_h^i} - \langle \widehat{\sigma}_h^i \boldsymbol{n}^i, \boldsymbol{u}_h^i \rangle_{\partial\Omega_h^i} &= (\boldsymbol{f}, \boldsymbol{u}_h^i)_{\Omega_h^i}, \\
-(\boldsymbol{u}_h^i, \nabla \tilde{p}_h^i)_{\Omega_h^i} + \langle \widehat{\boldsymbol{u}}_h^i \cdot \boldsymbol{n}, \tilde{p}_h^i \rangle_{\partial\Omega_h^i} &= 0, \\
\langle \widehat{\boldsymbol{u}}_h^i, \widehat{\sigma} \boldsymbol{n}_h^i \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i} &= \langle \boldsymbol{g}, \widehat{\sigma}_h^i \boldsymbol{n}^i \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i},
\end{aligned}$$

$$\langle \widehat{\sigma}_h^i \boldsymbol{n}^i, \widehat{\boldsymbol{u}}_h^i \rangle_{\partial\Omega_h^i \setminus \Gamma_h^i} = 0.$$

Noting that $\nabla \tilde{p}_h^i = \nabla \cdot (\tilde{p}_h^i \mathbb{I})$ and, reversing the integration by parts used to deduce the weak formulation,

$$\begin{aligned}
(\nu \mathbb{L}_h^i, \nabla \boldsymbol{u}_h^i)_{\Omega_h^i} - (\tilde{p}_h^i, \nabla \cdot \boldsymbol{u}_h^i)_{\Omega_h^i} &= (\nu \mathbb{L}_h^i - \tilde{p}_h^i \mathbb{I}, \nabla \boldsymbol{u}_h^i)_{\Omega_h^i} \\
&= -(\nabla \cdot (\nu \mathbb{L}_h^i - \tilde{p}_h^i \mathbb{I}), \boldsymbol{u}_h^i)_{\Omega_h^i} + \langle \nu \mathbb{L}_h^i - \tilde{p}_h^i \mathbb{I}, \boldsymbol{u}_h^i \rangle_{\partial\Omega_h^i},
\end{aligned}$$

we can sum, for a fixed i, all the equations previously stated to obtain

$$\begin{aligned}
(\boldsymbol{u}_h^i, \nabla \cdot \nu \mathbb{L}_h^i)_{\Omega_h^i} - (\nabla \cdot (\nu \mathbb{L}_h^i - \tilde{p}_h^i \mathbb{I}), \boldsymbol{u}_h^i)_{\Omega_h^i} - (\boldsymbol{u}_h^i, \nabla \tilde{p}_h^i)_{\Omega_h^i} \\
+ \langle \nu \mathbb{L}_h^i - \tilde{p}_h^i \mathbb{I}, \boldsymbol{u}_h^i \rangle_{\partial\Omega_h^i} - \langle \widehat{\sigma} \boldsymbol{n}, \boldsymbol{u}_h^i \rangle_{\partial\Omega_h^i} \\
+ \langle \widehat{\boldsymbol{u}}_h^i \cdot \boldsymbol{n}, \tilde{p}_h^i \rangle_{\partial\Omega_h^i} - \langle \widehat{\boldsymbol{u}}_h^i, \nu \mathbb{L}_h^i \boldsymbol{n} \rangle_{\partial\Omega_h^i} \\
+ \nu (\mathbb{L}_h^i, \mathbb{L}_h^i)_{\Omega_h^i} + \langle \widehat{\boldsymbol{u}}_h^i, \widehat{\sigma} \boldsymbol{n}_h^i \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i} + \langle \widehat{\sigma} \boldsymbol{n}, \widehat{\boldsymbol{u}}_h^i \rangle_{\partial\Omega_h^i \setminus \Gamma_h^i} \\
= \nu (\mathbb{H}, \mathbb{L}_h^i)_{\Omega_h^i} + (\boldsymbol{f}, \boldsymbol{u}_h^i)_{\Omega_h^i} + \langle \boldsymbol{g}, \widehat{\sigma} \boldsymbol{n}_h^i \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i}.
\end{aligned} \tag{3.3.3}$$

Notice that the first three terms cancel out and the next two terms can be written as $\langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \boldsymbol{u}_h^i \rangle_{\partial\Omega_h^i}$ in virtue of the definition of $\widehat{\sigma}\boldsymbol{n}_h^i$. On the other hand the sixth and seventh terms can be rewritten as

$$\begin{aligned}
-\langle \widehat{\boldsymbol{u}}_h^i, \nu \mathbb{L}_h^i \boldsymbol{n} \rangle_{\partial\Omega_h^i} + \langle \widehat{\boldsymbol{u}}_h^i \cdot \boldsymbol{n}, \tilde{p}_h^i \rangle_{\partial\Omega_h^i} &= -\langle \widehat{\boldsymbol{u}}_h^i, (\nu \mathbb{L}_h^i - \tilde{p}_h^i \mathbb{I}) \boldsymbol{n} \rangle_{\partial\Omega_h^i} \\
&= -\langle \widehat{\boldsymbol{u}}_h^i, \widehat{\sigma}_h^i \boldsymbol{n}^i + \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i) \rangle_{\partial\Omega_h^i} \\
&= -\langle \widehat{\boldsymbol{u}}_h^i, \widehat{\sigma}_h^i \boldsymbol{n}^i \rangle_{\partial\Omega_h^i} - \langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \widehat{\boldsymbol{u}}_h^i \rangle_{\partial\Omega_h^i}.
\end{aligned}$$

Noting that the first term of this expression is the same as the last two terms of the left-hand side of (3.3.3), albeit integrated on different domains, we write

$$-\langle \widehat{\boldsymbol{u}}_h^i, \widehat{\sigma}_h^i \boldsymbol{n}^i \rangle_{\partial\Omega_h^i} + \langle \widehat{\boldsymbol{u}}_h^i, \widehat{\sigma}_h^i \boldsymbol{n}^i \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i} + \langle \widehat{\sigma} \boldsymbol{n}_h^i, \widehat{\boldsymbol{u}}_h^i \rangle_{\partial\Omega_h^i \setminus \Gamma_h^i} = -\langle \widehat{\boldsymbol{u}}_h^i, \widehat{\sigma}_h^i \boldsymbol{n}^i \rangle_{\mathcal{I}_h^i}.$$

Thus, (3.3.3) is simplified to,

$$\nu(\mathbb{L}_h^i, \mathbb{L}_h^i)_{\Omega_h^i} + \langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i \rangle_{\partial\Omega_h^i} - \langle \widehat{\boldsymbol{u}}_h^i, \widehat{\sigma}_h^i \boldsymbol{n}^i \rangle_{\mathcal{I}_h^i}$$
$$= \nu(\mathbb{H}, \mathbb{L}_h^i)_{\Omega_h^i} + (\boldsymbol{f}, \boldsymbol{u}_h^i)_{\Omega_h^i} + \langle \boldsymbol{g}, \widehat{\sigma}_h^i \boldsymbol{n}^i \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i},$$

or, equivalently,

$$\nu\left(\left\| \mathbb{L}_h^i \right\|_{\Omega_h^i}^2 + \left\| \tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i) \right\|_{\partial\Omega_h^i}^2 \right) - \mathbb{T}^i = \nu(\mathbb{H}, \mathbb{L}_h^i)_{\Omega_h^i} + (\boldsymbol{f}, \boldsymbol{u}_h^i)_{\Omega_h^i} + \langle \boldsymbol{g}, \widehat{\sigma}_h^i \boldsymbol{n}^i \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i}.$$

We sum over $i \in \{1, 2\}$ to obtain

$$\sum_{i=1}^{2} \nu\left(\left\| \mathbb{L}_h^i \right\|_{\Omega_h^i}^2 + \left\| \tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i) \right\|_{\partial\Omega_h^i}^2 \right) - \sum_{i=1}^{2} \mathbb{T}^i$$
$$= \sum_{i=1}^{2} \left(\nu(\mathbb{H}, \mathbb{L}_h^i)_{\Omega_h^i} + (\boldsymbol{f}, \boldsymbol{u}_h^i)_{\Omega_h^i} + \langle \boldsymbol{g}, \widehat{\sigma}_h^i \boldsymbol{n}^i \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i} \right). \tag{3.3.4}$$

The result follows from (3.3.4) and Lemma 2. □

Lemma 4. *Under assumptions (A), it holds that*

$$\begin{aligned}
\frac{1}{4}\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)^2 \leq & \sum_{i=1}^{2} \left\| \mathbb{H} \right\|_{\Omega_h^i}^2 + \frac{2}{\nu^2} \left\| \boldsymbol{f} \right\|_{\Omega}^2 \\
& + \max_{e \in \Gamma_h^1 \setminus \mathcal{I}_h^1 \cup \Gamma_h^2 \setminus \mathcal{I}_h^2} \left((8 + 256\nu^{-2} C_3)(C_e^{\mathrm{tr}})^2 h_e^{-1} + 8\tau \right) \left\| \boldsymbol{g} \right\|_{\Gamma} \\
& + \sum_{i=1}^{2} \left\| \boldsymbol{u}_h^i \right\|_{\Omega_h^i}^2 + C_{\tilde{p}}^{\mathbb{L}} \sum_{i=1}^{2} \left\| \tilde{p}_h^i \right\|_{\Omega_h^i}^2 \\
& + \left\| \left\{ \tau^{1/2} + \left(8(C_e^{\mathrm{tr}})^2 + 256\nu^{-2}(C_e^{\mathrm{tr}})^2 C_3 + 8 \right) h_e^{-1} \right\}^{1/2} \boldsymbol{T}_1 \right\|_{\mathcal{I}_h^1}^2 \\
& + \left\| \nu^{-1/2}(4\delta_e + 4\tau^{-1} + 2(C_e^{\mathrm{tr}})^2 h_e^{-1})^{1/2} \boldsymbol{T}_2 \right\|_{\mathcal{I}_h^2}^2.
\end{aligned} \tag{3.3.5}$$

where we recall that

$$C^{\mathbb{L}}_{\tilde{p}} = \frac{1}{\nu^2}\Big(\tau^{-1/2}\max_{e\in\mathcal{I}_h^1}\big((C_e^1 C_e^{\text{ext}})^2(\delta_e^2+\delta_e^{3/2})h_e^{-3}\big) + \max_{e\in\mathcal{I}_h^1}\big((C_e^1 C_e^{\text{tr}} C_e^{\text{ext}})^2\delta_e^2 h_e^{-4}\big) \\ +2\max_{e\in\mathcal{I}_h^2}\delta_e h_e^{-1}(C_e^{\text{tr}})^2 + \frac{1}{256C_3}\Big),$$

and $C_3 > 0$ is a constant, independent of the meshsize, that will appear later on in the analysis. We make this last term appear by choosing $\varepsilon_{13}^p = \varepsilon_7^p = 256\nu^{-1}(C_e^{\text{tr}})^2 C_3$.

Proof. We can bound each term in the right-hand side of (3.3.2) as follows: For I^1, we can use Young's inequality with $\varepsilon_1 > 0$ to obtain

$$I^1 \leq \sum_{e\in\mathcal{I}_h^1}\left\{\frac{\varepsilon_1}{2}\left\|\tau^{-1/2}\left(\sigma_h^1\boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1\boldsymbol{n}^2}\circ\boldsymbol{\varphi}^{-1}\right)\right\|_e^2 + \frac{1}{2\varepsilon_1}\left\|\tau^{1/2}(\boldsymbol{u}_h^1-\widehat{\boldsymbol{u}}_h^1)\right\|_e^2\right\}$$

Since $\boldsymbol{\varphi}$ is an isometric bijection (under our assumptions) and $\boldsymbol{n}^1\circ\boldsymbol{\varphi} = -\boldsymbol{n}^2$, for $F = \boldsymbol{\varphi}(e)$,

$$\begin{aligned}\left\|\tau^{-1/2}\left(\sigma_h^1\boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1\boldsymbol{n}^2}\circ\boldsymbol{\varphi}^{-1}\right)\right\|_e^2 &= \left\|\tau^{-1/2}\left(\sigma_h^1\boldsymbol{n}^1\circ\boldsymbol{\varphi} + \boldsymbol{E}_{\sigma_h^1\boldsymbol{n}^2}\right)\right\|_F^2\\ &= \left\|\tau^{-1/2}\left(\nu(\boldsymbol{E}_{\mathbb{L}_h^1} - \mathbb{L}_h^1\circ\boldsymbol{\varphi})\boldsymbol{n}^2 + (\boldsymbol{E}_{\tilde{p}_h^1} - \tilde{p}_h^1\circ\boldsymbol{\varphi})\boldsymbol{n}^2\right)\right\|_F^2\\ &\leq 2\Big\{\left\|\tau^{-1/2}\left(\nu(\boldsymbol{E}_{\mathbb{L}_h^1} - \mathbb{L}_h^1\circ\boldsymbol{\varphi})\right)\right\|_F^2\\ &\qquad + \left\|\tau^{-1/2}\left(\boldsymbol{E}_{\tilde{p}_h^1} - \tilde{p}_h^1\circ\boldsymbol{\varphi}\right)\right\|_F^2\Big\}\end{aligned}$$

and so

$$I^1 \leq \sum_{e\in\mathcal{I}_h^1}\Big\{\varepsilon_1\left\|\tau^{-1/2}\left(\nu(\boldsymbol{E}_{\mathbb{L}_h^1} - \mathbb{L}_h^1\circ\boldsymbol{\varphi})\right)\right\|_F^2 + \varepsilon_1\left\|\tau^{-1/2}\left(\boldsymbol{E}_{\tilde{p}_h^1} - \tilde{p}_h^1\circ\boldsymbol{\varphi}\right)\right\|_F^2 \\ + \frac{1}{2\varepsilon_1}\left\|\tau^{1/2}(\boldsymbol{u}_h^1-\widehat{\boldsymbol{u}}_h^1)\right\|_e^2\Big\}$$

For I^2, a similar argument yields

$$I^2 \leq \sum_{e\in\mathcal{I}_h^1} \left\{ \varepsilon_2 \left\| \nu(\boldsymbol{E}_{\mathbb{L}_h^1} - \mathbb{L}_h^1 \circ \boldsymbol{\varphi}) \right\|_F^2 + \varepsilon_2 \left\| \boldsymbol{E}_{\tilde{p}_h^1} - \tilde{p}_h^1 \circ \boldsymbol{\varphi} \right\|_F^2 + \frac{1}{2\varepsilon_2} \left\| \boldsymbol{u}_h^1 \right\|_e^2 \right\}$$

for $\varepsilon_2 > 0$.

Before continuing with the remaining terms of the right-hand side, we will provide an estimate for some of the terms that had not appared before, namely $\left\| \nu(\boldsymbol{E}_{\mathbb{L}_h^1} - \mathbb{L}_h^1 \circ \boldsymbol{\varphi}) \right\|_F$ and $\varepsilon_2 \left\| \boldsymbol{E}_{\tilde{p}_h^1} - \tilde{p}_h^1 \circ \boldsymbol{\varphi} \right\|_F$. According to Lemma 1, there exists $C_e^1 > 0$ such that

$$\left\| \nu(\boldsymbol{E}_{\mathbb{L}_h^1} - \mathbb{L}_h^1 \circ \boldsymbol{\varphi}) \right\|_F \leq \nu C_e^1 C_e^{\text{ext}} \delta_e h_e^{-3/2} \left\| \mathbb{L}_h^1 \right\|_{K_e}$$

and

$$\left\| \boldsymbol{E}_{\tilde{p}_h^1} - \tilde{p}_h^1 \circ \boldsymbol{\varphi} \right\|_F \leq C_e^1 C_e^{\text{ext}} \delta_e h_e^{-3/2} \left\| \tilde{p}_h^1 \right\|_{K_e},$$

therefore

$$\begin{aligned} I^1 \leq & \left[\max_{e\in\mathcal{I}_h^1} \varepsilon_1 \tau^{-1} (C_e^1 C_e^{\text{ext}})^2 \delta_e^2 h_e^{-3} \right] \left\| \mathbb{L}_h^1 \right\|_{\Omega_h^1}^2 + \left[\max_{e\in\mathcal{I}_h^1} \varepsilon_1 \tau^{-1} (C_e^1 C_e^{\text{ext}})^2 \delta_e^2 h_e^{-3} \right] \left\| \tilde{p}_h^1 \right\|_{\Omega_h^1}^2 \\ & + \sum_{e\in\mathcal{I}_h^1} \frac{1}{2\varepsilon_1} \left\| \tau^{1/2} (\boldsymbol{u}_h^1 - \widehat{\boldsymbol{u}}_h^1) \right\|_e^2 \\ I^2 \leq & \left[\max_{e\in\mathcal{I}_h^1} \varepsilon_2 (C_e^1 C_e^{\text{ext}})^2 \delta_e^2 h_e^{-3} \right] \left\| \mathbb{L}_h^1 \right\|_{\Omega_h^1}^2 + \left[\max_{e\in\mathcal{I}_h^1} \varepsilon_2 (C_e^1 C_e^{\text{ext}})^2 \delta_e^2 h_e^{-3} \right] \left\| \tilde{p}_h^1 \right\|_{\Omega_h^1}^2 \\ & + \max_{e\in\mathcal{I}_h^1} \frac{1}{2\varepsilon_2} (C_e^{\text{tr}})^2 h_e^{-1} \left\| \boldsymbol{u}_h^1 \right\|_{\Omega_h^1}^2 . \end{aligned}$$

For I^3, using Young's inequality with $\varepsilon_3 > 0$, we have

$$\begin{aligned} \langle \Lambda_{\mathbb{L}_h^2}, \tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2 \rangle_{\mathcal{I}_h^2} & \leq \sum_{e\in\mathcal{I}_h^2} \left\{ \frac{\varepsilon_3}{2} \left\| |\ell|^{1/2} \Lambda_{\mathbb{L}_h^2} \right\|_e^2 + \frac{1}{2\varepsilon_3} \left\| |\ell|^{-1/2} (\tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2) \right\|_e^2 \right\} \\ & \leq \left[\max_{e\in\mathcal{I}_h^2} \frac{\varepsilon_3}{2} r_e^3 \left(C_e^{\text{ext}} C_e^{\text{inv}} \right)^2 \right] \left\| \mathbb{L}_h^2 \right\|_{\Omega_h^2}^2 + \sum_{e\in\mathcal{I}_h^2} \frac{1}{2\varepsilon_3} \left\| |\ell|^{-1/2} (\tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2) \right\|_e^2, \end{aligned}$$

and so

$$I^3 \leq \nu \left[\max_{e \in \mathcal{I}_h^2} \frac{\varepsilon_3}{2} r_e^3 \left(C_e^{\text{ext}} C_e^{\text{inv}} \right)^2 \right] \left\| \mathbb{L}_h^2 \right\|_{\Omega_h^2}^2 + \sum_{e \in \mathcal{I}_h^2} \frac{\nu}{2\varepsilon_3} \left\| |\ell|^{-1/2} \left(\tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2 \right) \right\|_e^2 .$$

For I^4, first we use Young's inequality with ε_4 to obtain

$$I^4 \leq \sum_{e \in \mathcal{I}_h^2} \left\{ \frac{\varepsilon_4}{2} \left\| |\ell|^{1/2} \tilde{p}_h^2 \boldsymbol{n}^2 \right\|_e^2 + \frac{1}{2\varepsilon_4} \left\| |\ell|^{-1/2} \left(\tilde{u}_h^2 \circ \phi - \widehat{\boldsymbol{u}}_h^2 \right) \right\|_e^2 \right\},$$

and then, using a discrete trace inequality, we have that

$$\left\| |\ell|^{1/2} \tilde{p}_h^2 \boldsymbol{n}^2 \right\|_e^2 \leq \delta_e \left\| \tilde{p}_h^2 \right\|_e^2 \leq \left(\delta_e h_e^{-1} C_e^{\text{tr}} \right) \left\| \tilde{p}_h^2 \right\|_{K_e}^2$$

and so

$$I^4 \leq \left[\max_{e \in \mathcal{I}_h^2} \frac{\varepsilon_4}{2} (\delta_e h_e^{-1} C_e^{\text{tr}}) \right] \left\| \tilde{p}_h^2 \right\|_{\Omega_h^2}^2 + \sum_{e \in \mathcal{I}_h^2} \frac{1}{2\varepsilon_4} \left\| |\ell|^{-1/2} \left(\tilde{u}_h^2 \circ \phi - \widehat{\boldsymbol{u}}_h^2 \right) \right\|_e^2 .$$

For I^5, once again we use Young's inequality, now with $\varepsilon_5 > 0$, to obtain

$$\begin{aligned} \langle \tau(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2), \tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2 \rangle_{\mathcal{I}_h^2} &\leq \sum_{e \in \mathcal{I}_h^2} \left\{ \frac{\varepsilon_5}{2} \left\| |\ell|^{1/2} \tau (\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2) \right\|_e^2 + \frac{1}{2\varepsilon_5} \left\| |\ell|^{-1/2} \left(\tilde{u}_h^2 \circ \phi - \widehat{\boldsymbol{u}}_h^2 \right) \right\|_e^2 \right\} \\ &\leq \left[\max_{e \in \mathcal{I}_h^2} \frac{\varepsilon_5}{2} \delta_e \tau \right] \left\| \tau^{1/2} (\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2) \right\|_{\mathcal{I}_h^2}^2 \\ &\quad + \sum_{e \in \mathcal{I}_h^2} \frac{1}{2\varepsilon_5} \left\| |\ell|^{-1/2} \left(\tilde{u}_h^2 \circ \phi - \widehat{\boldsymbol{u}}_h^2 \right) \right\|_{\mathcal{I}_h^2}^2 , \end{aligned}$$

and so

$$I^5 \leq \nu \left[\max_{e \in \mathcal{I}_h^2} \frac{\varepsilon_5}{2} \delta_e \tau \right] \sum_{i=1}^{2} \left\| \tau^{1/2} (\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2) \right\|_{\partial \Omega_h^i}^2 + \sum_{e \in \mathcal{I}_h^2} \frac{\nu}{2\varepsilon_5} \left\| |\ell|^{-1/2} \left(\tilde{u}_h^2 \circ \phi - \widehat{\boldsymbol{u}}_h^2 \right) \right\|_e^2$$

For I^6, we follow an argument similar to I^1 with $\varepsilon_6 > 0$ to arrive at

$$I^6 \leq \nu^2 \max_{e \in \mathcal{I}_h^1} \varepsilon_6 (C_e^1 C_e^{\text{ext}})^2 \delta_e^2 h_e^{-3} \left\| \mathbb{L}_h^1 \right\|_{\Omega_h^1}^2 + \max_{e \in \mathcal{I}_h^1} \varepsilon_6 (C_e^1 C_e^{\text{ext}})^2 \delta_e^2 h_e^{-3} \left\| \tilde{p}_h^1 \right\|_{\Omega_h^1}^2 + \sum_{e \in \mathcal{I}_h^1} \frac{1}{2} \left\| \varepsilon_6^{-1/2} \boldsymbol{T}_1 \right\|_e^2$$

For I^7, first we decompose it into

$$\begin{aligned}
I^7 =& \nu \langle h_e^{-1/2} \boldsymbol{T}_1, h_e^{1/2} \mathbb{L}_h^1 \boldsymbol{n}^1 \rangle_{\mathcal{I}_h^1} - \langle h_e^{-1/2} \boldsymbol{T}_1, h_e^{1/2} \tilde{p}_h^1 \boldsymbol{n}^1 \rangle_{\mathcal{I}_h^1} \\
&- \nu \langle h_e^{-1/2} \tau^{1/2} \boldsymbol{T}_1, h_e^{1/2} \tau^{1/2} (\boldsymbol{u}_h^1 - \widehat{\boldsymbol{u}}_h^1) \rangle_{\mathcal{I}_h^1} \\
=& \nu \sum_{e \in \mathcal{I}_h^1} \langle h_e^{-1/2} \boldsymbol{T}_1, h_e^{1/2} \mathbb{L}_h^1 \boldsymbol{n}^1 \rangle_e - \sum_{e \in \mathcal{I}_h^1} \langle h_e^{-1/2} \boldsymbol{T}_1, h_e^{1/2} \tilde{p}_h^1 \boldsymbol{n}^1 \rangle_e \\
&- \nu \sum_{e \in \mathcal{I}_h^1} \langle h_e^{-1/2} \tau^{1/2} \boldsymbol{T}_1, h_e^{1/2} \tau^{1/2} (\boldsymbol{u}_h^1 - \widehat{\boldsymbol{u}}_h^1) \rangle_e
\end{aligned}$$

and then we use Young's inequality with $\varepsilon_7^L, \varepsilon_7^p, \varepsilon_7^u > 0$ to each of these terms to get

$$\begin{aligned}
\langle h_e^{-1/2} \boldsymbol{T}_1, h_e^{1/2} \mathbb{L}_h^1 \boldsymbol{n}^1 \rangle_e &\leq \frac{1}{2} \left\| (\varepsilon_7^L h_e^{-1})^{1/2} \boldsymbol{T}_1 \right\|_e^2 + \frac{1}{2\varepsilon_7^L} \left\| h_e^{1/2} \mathbb{L}_h^1 \right\|_e^2 \\
&\leq \frac{1}{2} \left\| (\varepsilon_7^L h_e^{-1})^{1/2} \boldsymbol{T}_1 \right\|_e^2 + \frac{(C_e^{\mathrm{tr}})^2}{2\varepsilon_7^L} \left\| \mathbb{L}_h^1 \right\|_{K_e}^2,
\end{aligned}$$

$$\begin{aligned}
-\langle h_e^{-1/2} \boldsymbol{T}_1, h_e^{1/2} \tilde{p}_h^1 \boldsymbol{n}^1 \rangle_e &\leq \frac{1}{2} \left\| (\varepsilon_7^p h_e^{-1})^{1/2} \boldsymbol{T}_1 \right\|_e^2 + \frac{1}{2\varepsilon_7^p} \left\| h_e^{1/2} \tilde{p}_h^1 \right\|_e^2 \\
&\leq \frac{1}{2} \left\| (\varepsilon_7^p h_e^{-1})^{1/2} \boldsymbol{T}_1 \right\|_e^2 + \frac{(C_e^{\mathrm{tr}})^2}{2\varepsilon_7^p} \left\| \tilde{p}_h^1 \right\|_{K_e}^2,
\end{aligned}$$

$$\begin{aligned}
-\langle h_e^{-1/2} \tau^{1/2} \boldsymbol{T}_1, h_e^{1/2} \tau^{1/2} (\boldsymbol{u}_h^1 - \widehat{\boldsymbol{u}}_h^1) \rangle_e &\leq \frac{1}{2} \left\| (\varepsilon_7^u h_e^{-1})^{1/2} \boldsymbol{T}_1 \right\|_e^2 + \frac{1}{2\varepsilon_7^u} \left\| h_e^{1/2} \tau^{1/2} (\boldsymbol{u}_h^1 - \widehat{\boldsymbol{u}}_h^1) \right\|_e^2 \\
&\leq \frac{1}{2} \left\| (\varepsilon_7^u h_e^{-1})^{1/2} \boldsymbol{T}_1 \right\|_e^2 + \frac{(C_e^{\mathrm{tr}})^2}{2\varepsilon_7^u} \left\| \tau^{1/2} (\boldsymbol{u}_h^1 - \widehat{\boldsymbol{u}}_h^1) \right\|_{K_e}^2 .
\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}
I^7 \leq& \nu \max_{e \in \mathcal{I}_h^1} \frac{(C_e^{\mathrm{tr}})^2}{2\varepsilon_7^L} \left\| \mathbb{L}_h^1 \right\|_{\Omega_h^1}^2 + \max_{e \in \mathcal{I}_h^1} \frac{(C_e^{\mathrm{tr}})^2}{2\varepsilon_7^p} \left\| \tilde{p}_h^1 \right\|_{\Omega_h^1}^2 + \nu \max_{e \in \mathcal{I}_h^1} \frac{(C_e^{\mathrm{tr}})^2}{2\varepsilon_7^u} \left\| \tau^{1/2} (\boldsymbol{u}_h^1 - \widehat{\boldsymbol{u}}_h^1) \right\|_{\Omega_h^1}^2 \\
&+ \frac{1}{2} \left\| \left\{ \left(\nu \varepsilon_7^L + \varepsilon_7^p + \nu \varepsilon_7^u \right) h_e^{-1} \right\}^{1/2} \boldsymbol{T}_1 \right\|_{\mathcal{I}_h^1}^2
\end{aligned}$$

For I^8, Young's inequality with $\varepsilon_8 > 0$ suffices to obtain

$$I^8 \leq \sum_{e\in\mathcal{I}_h^2} \frac{1}{2\varepsilon_8} \left\| |\ell|^{-1/2} \left(\tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2\right) \right\|_e^2 + \frac{1}{2} \sum_{e\in\mathcal{I}_h^2} \left\| \varepsilon_8^{1/2} \delta_e^{1/2} \boldsymbol{T}_2 \right\|_e^2$$

For I^9, we use Young's inequality with ε_9 to get

$$I^9 \leq \frac{1}{2} \sum_{e\in\mathcal{I}_h^2} \left\| \varepsilon_9^{1/2} \tau^{-1/2} \boldsymbol{T}_2 \right\|_e^2 + \sum_{e\in\mathcal{I}_h^2} \frac{1}{2\varepsilon_9} \left\| \tau^{1/2} (\widehat{\boldsymbol{u}}_h^2 - \boldsymbol{u}_h^2) \right\|_e^2$$

For I^{10}, we use Young's inequality with ε_{10} followed by a discrete trace inequality to get

$$I^{10} \leq \frac{1}{2} \sum_{e\in\mathcal{I}_h^2} \left\| \varepsilon_{10}^{1/2} h_e^{-1/2} \boldsymbol{T}_2 \right\|_e^2 + \max_{e\in\mathcal{I}_h^2} \frac{1}{2\varepsilon_{10}} (C_e^{\text{tr}})^2 \left\| \boldsymbol{u}_h^2 \right\|_{\Omega_h^2}^2$$

For the sixth and seventh term, we use Young's inequality with $\varepsilon_{11}, \varepsilon_{12} > 0$ to obtain

- $\displaystyle\sum_{i=1}^{2} \nu(\mathbb{H}, \mathbb{L}_h^i)_{\Omega_h^i} \leq \frac{\nu\varepsilon_{11}}{2} \sum_{i=1}^{2} \|\mathbb{H}\|_{\Omega_h^i}^2 + \frac{\nu}{2\varepsilon_{11}} \sum_{i=1}^{2} \left\|\mathbb{L}_h^i\right\|_{\Omega_h^i}^2$,
- $\displaystyle\sum_{i=1}^{2} (\boldsymbol{f}, \boldsymbol{u}_h^i)_{\Omega_h^i} \leq \varepsilon_{12} \|\boldsymbol{f}\|_\Omega^2 + \frac{1}{2\varepsilon_{12}} \sum_{i=1}^{2} \left\|\boldsymbol{u}_h^i\right\|_{\Omega_h^i}^2$

Finally, for the last term we have

$$\begin{aligned}
\sum_{i=1}^{2} \langle \boldsymbol{g}, \widehat{\sigma} \boldsymbol{n}_h^i \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i} &= \sum_{i=1}^{2} \langle \boldsymbol{g}, \nu \mathbb{L}_h^i \boldsymbol{n}^i - \tilde{p}_h^i \boldsymbol{n}^i - \tau\nu (\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i) \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i} \\
&= \sum_{i=1}^{2} \langle h_e^{-1/2} \boldsymbol{g}, h_e^{1/2} \left(\nu \mathbb{L}_h^i \boldsymbol{n}^i - \tilde{p}_h^i \boldsymbol{n}^i\right) \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i} - \nu \sum_{i=1}^{2} \langle \boldsymbol{g}, \tau (\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i) \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i}
\end{aligned}$$

Using Young's inequality with $\varepsilon_{13}^L, \varepsilon_{13}^p > 0$ and a discrete trace inequality, for $i \in \{1, 2\}$ we have

$$\begin{aligned}
\langle h_e^{-1/2} \boldsymbol{g}, h_e^{1/2} \left(\nu \mathbb{L}_h^i \boldsymbol{n}^i - \tilde{p}_h^i \boldsymbol{n}^i\right) \rangle_{\Gamma_h^i \setminus \mathcal{I}_h^i} \leq \sum_{e\in\Gamma_h^i \setminus \mathcal{I}_h^i} \Bigg\{ & \frac{\nu\varepsilon_{13}^L + \varepsilon_{13}^p}{2} h_e^{-1} \|\boldsymbol{g}\|_e^2 + \nu \frac{(C_e^{\text{tr}})^2}{2\varepsilon_{13}^L} \left\|\mathbb{L}_h^i\right\|_{K_e}^2 \\
& + \frac{(C_e^{\text{tr}})^2}{2\varepsilon_{13}^p} \left\|\tilde{p}_h^i\right\|_{K_e}^2 \Bigg\}
\end{aligned}$$

and so

$$\sum_{i=1}^{2}\langle h_e^{-1/2}\boldsymbol{g}, h_e^{1/2}\left(\nu\mathbb{L}_h^i\boldsymbol{n}^i - \tilde{p}_h^i\boldsymbol{n}^i\right)\rangle_{\Gamma_h^i\setminus\mathcal{I}_h^i} \leq \max_{e\in\Gamma_h^1\setminus\mathcal{I}_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2}\left(\nu\varepsilon_{13}^L + \varepsilon_{13}^p\right)h_e^{-1}\|\boldsymbol{g}\|_\Gamma^2$$
$$+\max_{e\in\Gamma_h^1\setminus\mathcal{I}_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2}\nu\frac{(C_e^{\mathrm{tr}})^2}{\varepsilon_{13}^L}\sum_{i=1}^{2}\left\|\mathbb{L}_h^i\right\|_{\Omega_h^i}^2$$
$$+\max_{e\in\Gamma_h^1\setminus\mathcal{I}_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2}\frac{(C_e^{\mathrm{tr}})^2}{\varepsilon_{13}^p}\sum_{i=1}^{2}\left\|\tilde{p}_h^i\right\|_{\Omega_h^i}^2$$

For the second term and $\varepsilon_{14} > 0$, we have

$$-\nu\sum_{i=1}^{2}\langle\boldsymbol{g},\tau(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)\rangle_{\Gamma_h^i\setminus\mathcal{I}_h^i} \leq \nu\sum_{i=1}^{2}\sum_{e\in\Gamma_h^i\setminus\mathcal{I}_h^i}\frac{\varepsilon_{14}}{2}\left\|\tau^{1/2}\boldsymbol{g}\right\|_e^2 + \nu\sum_{i=1}^{2}\sum_{e\in\Gamma_h^i\setminus\mathcal{I}_h^i}\frac{1}{2\varepsilon_{14}}\left\|\tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)\right\|_e^2$$
$$\leq \nu\left(\max_{e\in\Gamma_h^1\setminus\mathcal{I}_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2}\varepsilon_{14}\tau\right)\|\boldsymbol{g}\|_\Gamma^2$$
$$+\nu\max_{e\in\Gamma_h^1\setminus\mathcal{I}_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2}\frac{1}{2\varepsilon_{14}}\sum_{i=1}^{2}\left\|\tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)\right\|_{\partial\Omega_h^i}^2$$

With these estimates in mind and dividing by $\nu > 0$, from (3.3.2) we obtain

$$\tilde{C}^1\sum_{i=1}^{2}\left\|\mathbb{L}_h^i\right\|_{\Omega_h^i}^2 + \tilde{C}^2\sum_{i=1}^{2}\left\|\tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)\right\|_{\partial\Omega_h^i}^2 + \tilde{C}^3\left\||\ell|^{-1/2}\left(\tilde{u}_h^2\circ\boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2\right)\right\|_{\mathcal{I}_h^2}^2$$
$$\leq \tilde{C}^4\sum_{i=1}^{2}\|\mathbb{H}\|_{\Omega_h^i}^2 + \tilde{C}^5\|\boldsymbol{f}\|_\Omega^2 + \tilde{C}^6\|\boldsymbol{g}\|_\Gamma + \tilde{C}^7\sum_{i=1}^{2}\left\|\boldsymbol{u}_h^i\right\|_{\Omega_h^i}^2 + \tilde{C}^8\sum_{i=1}^{2}\left\|\tilde{p}_h^i\right\|_{\Omega_h^i}^2$$
$$+\left\|\left\{\nu^{-1}\varepsilon_6^{-1} + \left(\varepsilon_7^L + \nu^{-1}\varepsilon_7^p + \varepsilon_7^u\right)h_e^{-1}\right\}^{1/2}\boldsymbol{T}_1\right\|_{\mathcal{I}_h^1}^2 + \left\|\nu^{-1/2}(\varepsilon_8\delta_e + \varepsilon_9\tau^{-1} + \varepsilon_{10}h_e^{-1})^{1/2}\boldsymbol{T}_2\right\|_{\mathcal{I}_h^2}^2 \tag{3.3.6}$$

where

$$\tilde{C}^1 := 1 - \nu\max_{e\in\mathcal{I}_h^1}(\varepsilon_1\tau^{-1} + \varepsilon_2 + \varepsilon_6)\left((C_e^1 C_e^{\mathrm{ext}})^2\delta_e^2 h_e^{-3}\right) - \max_{e\in\mathcal{I}_h^2}\frac{\varepsilon_3}{2}r_e^3(C_e^{\mathrm{ext}}C_e^{\mathrm{inv}})^2$$
$$-\frac{1}{2\varepsilon_{11}} - \max_{e\in\Gamma_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2}\left(\frac{1}{\varepsilon_{13}^L} + \frac{1}{\varepsilon_7^L}\right)(C_e^{\mathrm{tr}})^2$$
$$\tilde{C}^2 := 1 - \max_{e\in\mathcal{I}_h^1}\frac{1}{2\nu\varepsilon_1} - \max_{e\in\mathcal{I}_h^2}\frac{\varepsilon_5}{2}\delta_e\tau - \max_{e\in\Gamma_h^1\setminus\mathcal{I}_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2}\frac{1}{2\varepsilon_{14}} - \max_{e\in\mathcal{I}_h^2}\frac{1}{2\varepsilon_9} - \max_{e\in\Gamma_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2}\frac{1}{2\varepsilon_7^u}$$

$$\tilde{C}^3 := 1 - \max_{e\in\mathcal{I}_h^2}\left(\frac{1}{2\varepsilon_3}+\frac{1}{2\nu\varepsilon_4}+\frac{1}{2\varepsilon_5}+\frac{1}{2\varepsilon_8}\right)$$
$$\tilde{C}^4 := \frac{\varepsilon_{11}}{2}$$
$$\tilde{C}^5 := \frac{\varepsilon_{12}}{\nu}$$
$$\tilde{C}^6 := \max_{e\in\Gamma_h^1\setminus\mathcal{I}_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2}\left(\nu^{-1}\left(\nu\varepsilon_{13}^L+\varepsilon_{13}^p\right)h_e^{-1}+\varepsilon_{14}\tau\right)$$
$$\tilde{C}^7 := \max_{e\in\mathcal{I}_h^1}\frac{(C_e^{\mathrm{tr}})^2h_e^{-1}}{2\nu\varepsilon_2}+\frac{1}{2\nu\varepsilon_{12}}+\max_{e\in\mathcal{I}_h^2}\frac{(C_e^{\mathrm{tr}})^2}{2\varepsilon_{10}}$$
$$\tilde{C}^8 := \frac{1}{\nu}\max_{e\in\mathcal{I}_h^1}\left((\varepsilon_1\tau^{-1}+\varepsilon_2+\varepsilon_6)(C_e^1C_e^{\mathrm{ext}})^2\delta_e^2h_e^{-3}\right)$$
$$+\frac{1}{\nu}\max_{e\in\mathcal{I}_h^2}\frac{\varepsilon_4}{2}\delta_eh_e^{-1}(C_e^{\mathrm{tr}})^2+\frac{1}{\nu}\max_{e\in\Gamma_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2}\left(\frac{(C_e^{\mathrm{tr}})^2}{\varepsilon_{13}^p}+\frac{(C_e^{\mathrm{tr}})^2}{\varepsilon_7^p}\right)$$

In order for our estimate to work, we need $\tilde{C}^i > 0$ for $i \in \{1,2,3\}$ and have every term containing a negative power of h to be either accompanied by the gap δ or dealt appropiately via the choices of ε. For that purpose, we notice that the only negative powers of h that do not have a gap term to compensate are present in $\tilde{C}^6$ and $\tilde{C}^7$; in spite of that, $\tilde{C}^6$ is part of a source term and, in the error estimates, this power of h is irrelevant as the term will vanish. Thus, we choose $\varepsilon_2 = \nu^{-1}h_e^{-1}(C_e^{\mathrm{tr}})^2$ to get

$$\tilde{C}^1 = 1 - \nu\max_{e\in\mathcal{I}_h^1}\left((\varepsilon_1\tau^{-1}+\varepsilon_6)(C_e^1C_e^{\mathrm{ext}})^2\delta_e^2h_e^{-3}\right) - \max_{e\in\mathcal{I}_h^1}\left((C_e^1C_e^{\mathrm{tr}}C_e^{\mathrm{ext}})^2\delta_e^2h_e^{-4}\right)$$
$$-\max_{e\in\mathcal{I}_h^2}\frac{\varepsilon_3}{2}r_e^3(C_e^{\mathrm{ext}}C_e^{\mathrm{inv}})^2-\frac{1}{2\varepsilon_{11}}-\nu\max_{e\in\Gamma_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2}\left(\frac{1}{\varepsilon_{13}}+\frac{1}{\varepsilon_7^L}\right)(C_e^{\mathrm{tr}})^2$$

where the third term is roughly proportional to δ^2h^{-4}, which will limit our gap to be of at least order h^2 in order for this estimate to work.

Afterwards, we choose $\varepsilon_1 = \nu^{-1}\delta_e^{-1/2}\tau^{1/2}$, $\varepsilon_3 = 2$, $\varepsilon_6 = \nu^{-1}\tau^{-1/2}$, $\varepsilon_7^L = \varepsilon_{13}^L = 8(C_e^{\mathrm{tr}})^2$ and $\varepsilon_{11} = 2$ to obtain $\tilde{C}^1 = \dfrac{3}{4} - \mathbb{Q}^1$, where

$$\mathbb{Q}^1 := \tau^{-1/2}\max_{e\in\mathcal{I}_h^1}\left((C_e^1C_e^{\mathrm{ext}})^2(\delta_e^2+\delta_e^{3/2})h_e^{-3}\right)+\max_{e\in\mathcal{I}_h^1}\left((C_e^1C_e^{\mathrm{tr}}C_e^{\mathrm{ext}})^2\delta_e^2h_e^{-4}\right)+\max_{e\in\mathcal{I}_h^2}r_e^3(C_e^{\mathrm{ext}}C_e^{\mathrm{inv}})^2+\frac{1}{4},$$

which, according to our assumptions, is bounded by $\dfrac{1}{2}$ and hence $\tilde{C}^1 \geq \dfrac{1}{4}$.

For $\tilde{C}^2$ and $\tilde{C}^3$, we have

$$\tilde{C}^2 = 1 - \max_{e\in\mathcal{I}_h^1} \frac{1}{2}\delta_e^{1/2}\tau^{-1/2} - \max_{e\in\mathcal{I}_h^2} \frac{\varepsilon_5}{2}\delta_e\tau - \max_{e\in\Gamma_h^1\setminus\mathcal{I}_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2} \frac{1}{2\varepsilon_{14}} - \max_{e\in\mathcal{I}_h^2} \frac{1}{2\varepsilon_9} - \max_{e\in\Gamma_h^1\cup\Gamma_h^2\setminus\mathcal{I}_h^2} \frac{1}{2\varepsilon_7^u}$$

$$\tilde{C}^3 = \frac{3}{4} - \max_{e\in\mathcal{I}_h^2}\left(\frac{1}{2\nu\varepsilon_4} + \frac{1}{2\varepsilon_5} + \frac{1}{2\varepsilon_8}\right)$$

Choosing $\varepsilon_4 = 4\nu^{-1}$, $\varepsilon_5 = 2$, $\varepsilon_8 = 4$, $\varepsilon_9 = 4$, $\varepsilon_{14} = 8$, $\varepsilon_7^u = 8$, we are left with $\tilde{C}^2 = \dfrac{3}{4} - \mathbb{Q}^2$ and $\tilde{C}^3 = \dfrac{1}{4}$, where

$$\mathbb{Q}^2 := \frac{1}{2}\max_{e\in\mathcal{I}_h^1} \delta^{1/2}\tau^{-1/2} + \max_{e\in\mathcal{I}_h^2} \delta\tau$$

is bounded in our assumptions by $\dfrac{1}{2}$ to obtain $\tilde{C}^2 \geq \dfrac{1}{4}$.

Finally, we are free to choose ε_{10} and ε_{12}. In order to make some later estimates easier, we choose $\varepsilon_{10} = 2(C_e^{\text{tr}})^2$ and $\varepsilon_{12} = 2\nu^{-1}$ to obtain $\tilde{C}^7 = 1$. □

We observe that the right-hand side of (3.3.5) depends on $\sum_{i=1}^{2} \left\|\boldsymbol{u}_h^i\right\|_{\Omega_h^i}$ and $\sum_{i=1}^{2} \left\|\tilde{p}_h^i\right\|_{\Omega^i}$. To bound $\sum_{i=1}^{2} \left\|\boldsymbol{u}_h^i\right\|_{\Omega_h^i}$ we will employ a duality argument (section 3.3.2), and to bound $\sum_{i=1}^{2} \left\|\tilde{p}_h^i\right\|_{\Omega_h^i}$ we will use an inf-sup condition (section 3.3.3).

3.3.2 A duality argument

We now provide a bound for $\sum_{i=1}^{2} \left\|\boldsymbol{u}_h^i\right\|_{\Omega_h^i}$ via a duality argument.

For $\boldsymbol{\theta} \in \boldsymbol{L}^2(\Omega)$, let $(\boldsymbol{\Phi}, \boldsymbol{\phi}, \phi)$ be the solution of

$$\boldsymbol{\Phi} + \nabla\boldsymbol{\phi} = 0 \qquad \text{on } \Omega, \tag{3.3.7a}$$

$$\nabla\cdot(\nu\boldsymbol{\Phi} - \phi\mathbb{I}) = \boldsymbol{\theta} \qquad \text{on } \Omega, \tag{3.3.7b}$$

$$-\nabla\cdot\boldsymbol{\phi} = 0 \qquad \text{on } \Omega, \tag{3.3.7c}$$

$$\boldsymbol{\phi} = \mathbf{0} \qquad \text{on } \Gamma. \tag{3.3.7d}$$

Suppose that elliptic regularity holds, that is,

$$\nu \|\mathbf{\Phi}\|_{\mathbb{H}^1(\Omega)} + \nu \|\boldsymbol{\phi}\|_{\boldsymbol{H}^2(\Omega)} + \|\phi\|_{H^1(\Omega)} \lesssim \|\boldsymbol{\theta}\|_{\boldsymbol{H}^1(\Omega)}, \tag{3.3.8}$$

which is the case if Ω convex in two dimensions [34] or a convex polyhedron in the three-dimensional case [25].

Lemma 5. *It holds that*

$$\sum_{i=1}^{2}(\boldsymbol{u}_h^i, \boldsymbol{\theta})_{\Omega_h^i} = -\nu \sum_{i=1}^{2}\left\{(\mathbb{H}^i, \mathbf{\Phi} - \Pi_{\mathbb{V}^i}\mathbf{\Phi})_{\Omega_h^i} + (\mathbb{H}^i, \nabla(\boldsymbol{\phi} - \Pi_{\boldsymbol{V}^i}\boldsymbol{\phi}))_{\Omega_h^i}\right\} + \sum_{i=1}^{2}(\nu\mathbb{L}_h^i, \mathbf{\Phi} - \Pi_{\mathbb{V}^i}\mathbf{\Phi})_{\Omega_h^i} \tag{3.3.9}$$

$$+ \sum_{i=1}^{2}(\boldsymbol{f}, \Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})_{\Omega_h^i} + \sum_{i=1}^{2}\langle \boldsymbol{g}, \mathrm{P}_{\boldsymbol{M}^i}(\nu\mathbf{\Phi}\boldsymbol{n} - \phi\boldsymbol{n})\rangle_{\Gamma_h^i\setminus\mathcal{I}_h^i} + \sum_{i=1}^{2}\mathbb{T}_{\boldsymbol{u}}^i.$$

where $\mathbb{T}_{\boldsymbol{u}}^i := \langle \widehat{\sigma}_h^i \boldsymbol{n}^i, \boldsymbol{\phi}\rangle_{\mathcal{I}_h^i} + \langle \widehat{\boldsymbol{u}}_h^i, \nu\mathbf{\Phi}\boldsymbol{n}^i - \phi\boldsymbol{n}^i\rangle_{\mathcal{I}_h^i}$.

And, just like in Lemma 2, we emphasize that $\mathbb{T}_{\boldsymbol{u}}^1 + \mathbb{T}_{\boldsymbol{u}}^2$ vanishes if there is no gap.

Proof. For $i \in \{1, 2\}$, testing (3.3.7) with $(\mathbb{G}, \boldsymbol{v}, q) \in \mathbb{V}_h^i \times \boldsymbol{V}_h^i \times V_h^i$, we obtain

$$(\mathbb{G}, \mathbf{\Phi})_{\Omega_h^i} + (\mathbb{G}, \nabla\boldsymbol{\phi})_{\Omega_h^i} = 0 \tag{3.3.10a}$$

$$\nu(\boldsymbol{v}, \nabla\cdot\mathbf{\Phi})_{\Omega_h^i} - (\boldsymbol{v}, \nabla\phi)_{\Omega_h^i} = (\boldsymbol{v}, \boldsymbol{\theta})_{\Omega_h^i} \tag{3.3.10b}$$

$$-(q, \nabla\cdot\boldsymbol{\phi})_{\Omega_h^i} = 0 \tag{3.3.10c}$$

If we use the identities from (31) in [43], we have

$$\begin{aligned}
(\mathbb{G}, \mathbf{\Phi})_{\Omega_h^i} + (\mathbb{G}, \nabla\boldsymbol{\phi})_{\Omega_h^i} &= (\mathbb{G}, \mathbf{\Phi})_{\Omega_h^i} - (\nabla\cdot\mathbb{G}, \Pi_{V^i}\boldsymbol{\phi})_{\Omega_h^i} + \langle \mathbb{G}\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\partial\Omega_h^i} \\
\nu(\boldsymbol{v}, \nabla\cdot\mathbf{\Phi})_{\Omega_h^i} - (\boldsymbol{v}, \nabla\phi)_{\Omega_h^i} &= \nu(\boldsymbol{v}, \nabla\cdot\Pi_{\boldsymbol{V}^i}\mathbf{\Phi})_{\Omega_h^i} + \nu\langle \boldsymbol{v}, (\mathbf{\Phi} - \Pi_{\mathbb{V}^i}\mathbf{\Phi})\boldsymbol{n}\rangle_{\partial\Omega_h^i} \\
&\quad - (\boldsymbol{v}, \nabla\Pi_{V^i}\phi)_{\Omega_h^i} - \langle \boldsymbol{v}, (\phi - \Pi_{V^i}\phi)\boldsymbol{n}\rangle_{\partial\Omega_h^i} \\
-(q, \nabla\cdot\boldsymbol{\phi})_{\Omega_h^i} &= -(\nabla q, \Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})_{\Omega_h^i} + \langle q\boldsymbol{n}, \boldsymbol{\phi}\rangle_{\partial\Omega_h^i}
\end{aligned}$$

where $(\Pi_{\mathbb{V}^i}\mathbf{\Phi},\ \Pi_{\boldsymbol{V}^i}\boldsymbol{\phi}, \Pi_{V^i}\phi)$ is the HDG projection of $(\mathbf{\Phi}, \boldsymbol{\phi}, \phi)$.

If we add the first two equations of the dual problem and substract the third one, we obtain

$$
\begin{aligned}
(\boldsymbol{v},\boldsymbol{\theta})_{\Omega_h^i} &= (\mathbb{G},\boldsymbol{\Phi})_{\Omega_h^i} - (\nabla\cdot\mathbb{G},\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})_{\Omega_h^i} + \langle\mathbb{G}\boldsymbol{n},\boldsymbol{\phi}\rangle_{\partial\Omega_h^i} + \nu(\boldsymbol{v},\nabla\cdot\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i} \\
&\quad + \nu\langle\boldsymbol{v},(\boldsymbol{\Phi}-\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})\boldsymbol{n}\rangle_{\partial\Omega_h^i} - (\boldsymbol{v},\nabla\Pi_{V^i}\phi)_{\Omega_h^i} - \langle\boldsymbol{v},(\phi-\Pi_{V^i}\phi)\boldsymbol{n}\rangle_{\partial\Omega_h^i} \\
&\quad + (\nabla q,\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})_{\Omega_h^i} - \langle q\boldsymbol{n},\boldsymbol{\phi}\rangle_{\partial\Omega_h^i} \\
&= (\mathbb{G},\boldsymbol{\Phi}-\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i} + \langle\mathbb{G}\boldsymbol{n},\boldsymbol{\phi}\rangle_{\partial\Omega_h^i} + \langle\boldsymbol{v},\nu(\boldsymbol{\Phi}-\Pi_{\mathbb{V}^i}\boldsymbol{\Phi}) - (\phi-\Pi_{V^i}\phi)\boldsymbol{n}\rangle_{\partial\Omega_h^i} \\
&\quad - \langle q,\boldsymbol{n},\boldsymbol{\phi}\rangle_{\partial\Omega_h^i} - (\boldsymbol{v},\nabla\Pi_{V^i}\phi)_{\Omega_h^i} \\
&\quad + \left\{(\mathbb{G},\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i} + (\boldsymbol{v},\nu\nabla\cdot\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i}\right\} - \left\{(\nabla\cdot(\mathbb{G}-q\mathbb{I}),\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})_{\Omega_h^i}\right\}
\end{aligned}
$$

Now, taking $\mathbb{G}=\nu\mathbb{L}_h^i$, $\boldsymbol{v}=\boldsymbol{u}_h^i$, $q=\tilde{p}_h^i$, we have

$$
\begin{aligned}
(\boldsymbol{u}_h^i,\boldsymbol{\theta})_{\Omega_h^i} &= (\nu\mathbb{L}_h^i,\boldsymbol{\Phi}-\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i} + \langle\nu\mathbb{L}_h^i\boldsymbol{n},\boldsymbol{\phi}\rangle_{\partial\Omega_h^i} + \langle\boldsymbol{u}_h^i,\nu(\boldsymbol{\Phi}-\Pi_{\mathbb{V}^i}\boldsymbol{\Phi}) - (\phi-\Pi_{V^i}\phi)\boldsymbol{n}\rangle_{\partial\Omega_h^i} \\
&\quad - \langle\tilde{p}_h^i,\boldsymbol{n},\boldsymbol{\phi}\rangle_{\partial\Omega_h^i} - (\boldsymbol{u}_h^i,\nabla\Pi_{V^i}\phi)_{\Omega_h^i} \\
&\quad + \left\{(\nu\mathbb{L}_h^i,\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i} + (\boldsymbol{u}_h^i,\nu\nabla\cdot\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i}\right\} + \left\{(-\nabla\cdot(\nu\mathbb{L}_h^i-\tilde{p}_h^i\mathbb{I}),\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})_{\Omega_h^i}\right\}
\end{aligned}
$$

and, using (2.2.1),

$$
\begin{aligned}
(\boldsymbol{u}_h^i,\boldsymbol{\theta})_{\Omega_h^i} &= (\nu\mathbb{L}_h^i,\boldsymbol{\Phi}-\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i} + \langle\nu\mathbb{L}_h^i\boldsymbol{n},\boldsymbol{\phi}\rangle_{\partial\Omega_h^i} + \langle\boldsymbol{u}_h^i,\nu(\boldsymbol{\Phi}-\Pi_{\mathbb{V}^i}\boldsymbol{\Phi}) - (\phi-\Pi_{V^i}\phi)\boldsymbol{n}\rangle_{\partial\Omega_h^i} \\
&\quad - \langle\tilde{p}_h^i,\boldsymbol{n},\boldsymbol{\phi}\rangle_{\partial\Omega_h^i} - \langle\widehat{\boldsymbol{u}}_h^i\cdot\boldsymbol{n},\Pi_{V^i}\phi\rangle_{\partial\Omega_h^i} \\
&\quad + \left\{\nu(\mathbb{H}^i,\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i} + \nu\langle\widehat{\boldsymbol{u}}_h^i,\Pi_{\mathbb{V}^i}\boldsymbol{\Phi}\boldsymbol{n}\rangle_{\partial\Omega_h^i}\right\} \\
&\quad + \left\{(\boldsymbol{f},\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})_{\Omega_h^i} - \langle\tau\nu(\boldsymbol{u}_h^i-\widehat{\boldsymbol{u}}_h^i),\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi}\rangle_{\partial\Omega_h^i}\right\}
\end{aligned}
$$

Adding and substracting convenient terms, we obtain

$$
\begin{aligned}
(\boldsymbol{u}_h^i,\boldsymbol{\theta})_{\Omega_h^i} &= (\nu\mathbb{L}_h^i,\boldsymbol{\Phi}-\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i} + \langle\widehat{\sigma}_h^i\boldsymbol{n}^i,\boldsymbol{\phi}\rangle_{\partial\Omega_h^i} + \langle\widehat{\boldsymbol{u}}_h^i,\nu\boldsymbol{\Phi}\boldsymbol{n}-\phi\boldsymbol{n}\rangle_{\partial\Omega_h^i} + \nu(\mathbb{H}^i,\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i} \\
&\quad + (\boldsymbol{f},\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})_{\Omega_h^i} + \langle\boldsymbol{u}_h^i-\widehat{\boldsymbol{u}}_h^i,\nu(\boldsymbol{\Phi}-\Pi_{\mathbb{V}^i}\boldsymbol{\Phi})\boldsymbol{n} - (\phi-\Pi_{V^i}\phi)\boldsymbol{n} + \tau\nu(\boldsymbol{\phi}-\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})\rangle_{\partial\Omega_h^i}
\end{aligned}
$$

We have that

$$\begin{aligned}\langle\widehat{\sigma}_h^i\boldsymbol{n}^i,\boldsymbol{\phi}\rangle_{\partial\Omega_h^i} &= \langle\widehat{\sigma}_h^i\boldsymbol{n}^i,\mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{\phi}\rangle_{\partial\Omega_h^i}\\ &= \langle\widehat{\sigma}_h^i\boldsymbol{n}^i,\mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{\phi}\rangle_{\partial\Omega_h^i\setminus\Gamma_h^i} + \langle\widehat{\sigma}_h^i\boldsymbol{n}^i,\mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{\phi}\rangle_{\Gamma_h^i\setminus\mathcal{I}_h^i} + \langle\widehat{\sigma}_h^i\boldsymbol{n}^i,\mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{\phi}\rangle_{\mathcal{I}_h^i}\\ &= \langle\widehat{\sigma}_h^i\boldsymbol{n}^i,\mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{\phi}\rangle_{\mathcal{I}_h^i}\end{aligned}$$

where we used (2.2.1e) with $\boldsymbol{\mu}=\mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{\phi}$ and the fact that $\boldsymbol{\phi}|_\Gamma=\mathbf{0}$ to eliminate the first and second terms of the right-hand side, respectively.

From the definition of the HDG projection, we have that

$$\langle\boldsymbol{u}_h^i-\widehat{\boldsymbol{u}}_h^i,\nu(\mathbf{\Phi}-\Pi_{\mathbb{V}^i}\mathbf{\Phi})\boldsymbol{n}-(\phi-\Pi_{V^i}\phi)\boldsymbol{n}+\tau\nu(\boldsymbol{\phi}-\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})\rangle_{\partial\Omega_h^i}=0$$

using (2.1.2e) with $\boldsymbol{\mu}=\boldsymbol{u}_h^i-\widehat{\boldsymbol{u}}_h^i$.

On the other hand, using (2.2.1d) with $\boldsymbol{\mu}=\mathrm{P}_{\boldsymbol{M}^i}(\nu\mathbf{\Phi}\boldsymbol{n}-\phi\boldsymbol{n})$ and the fact that $\widehat{\boldsymbol{u}}_h^i$ is single valued on internal faces, it follows that

$$\begin{aligned}\langle\widehat{\boldsymbol{u}}_h^i,\nu\mathbf{\Phi}\boldsymbol{n}-\phi\boldsymbol{n}\rangle_{\partial\Omega_h^i} &= \langle\widehat{\boldsymbol{u}}_h^i,\nu\mathbf{\Phi}\boldsymbol{n}-\phi\boldsymbol{n}\rangle_{\partial\Omega_h^i\setminus\Gamma_h^i}+\langle\widehat{\boldsymbol{u}}_h^i,\mathrm{P}_{\boldsymbol{M}^i}(\nu\mathbf{\Phi}\boldsymbol{n}-\phi\boldsymbol{n})\rangle_{\Gamma_h^i\setminus\mathcal{I}_h^i}+\langle\widehat{\boldsymbol{u}}_h^i,\nu\mathbf{\Phi}\boldsymbol{n}-\phi\boldsymbol{n}\rangle_{\mathcal{I}_h^i}\\ &= \langle\boldsymbol{g},\mathrm{P}_{\boldsymbol{M}^i}(\nu\mathbf{\Phi}\boldsymbol{n}-\phi\boldsymbol{n})\rangle_{\Gamma_h^i\setminus\mathcal{I}_h^i}+\langle\widehat{\boldsymbol{u}}_h^i,\nu\mathbf{\Phi}\boldsymbol{n}-\phi\boldsymbol{n}\rangle_{\mathcal{I}_h^i}\end{aligned}$$

Furthermore, since $\boldsymbol{H}^i$ is assumed orthogonal to $[\mathbb{P}_{k-1}(\Omega)]^{d\times d}$, we have

$$\begin{aligned}(\mathbb{H}^i,\Pi_{\mathbb{V}^i}\mathbf{\Phi})_{\Omega_h^i} &= (\mathbb{H}^i,\Pi_{\mathbb{V}^i}\mathbf{\Phi}-\mathbf{\Phi})_{\Omega_h^i}+(\mathbb{H}^i,\mathbf{\Phi})_{\Omega_h^i}\\ &= (\mathbb{H}^i,\Pi_{\mathbb{V}^i}\mathbf{\Phi}-\mathbf{\Phi})_{\Omega_h^i}+(\mathbb{H}^i,\nabla\boldsymbol{\phi}-\nabla\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})_{\Omega_h^i}\\ &= -(\mathbb{H}^i,\mathbf{\Phi}-\Pi_{\mathbb{V}^i}\mathbf{\Phi})_{\Omega_h^i}+(\mathbb{H}^i,\nabla(\boldsymbol{\phi}-\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi}))_{\Omega_h^i}\end{aligned}$$

and so

$$\begin{aligned}(\boldsymbol{u}_h^i,\boldsymbol{\theta})_{\Omega_h^i} &= (\nu\mathbb{L}_h^i,\mathbf{\Phi}-\Pi_{\mathbb{V}^i}\mathbf{\Phi})_{\Omega_h^i}+\langle\widehat{\sigma}_h^i\boldsymbol{n}^i,\boldsymbol{\phi}\rangle_{\mathcal{I}_h^i}+\langle\widehat{\boldsymbol{u}}_h^i,\nu\mathbf{\Phi}\boldsymbol{n}^i-\phi\boldsymbol{n}^i\rangle_{\mathcal{I}_h^i}\\ &\quad-\nu(\mathbb{H}^i,\mathbf{\Phi}-\Pi_{\mathbb{V}^i}\mathbf{\Phi})_{\Omega_h^i}+\nu(\mathbb{H}^i,\nabla(\boldsymbol{\phi}-\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi}))_{\Omega_h^i}+(\boldsymbol{f},\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})_{\Omega_h^i}\\ &\quad+\langle\boldsymbol{g},\mathrm{P}_{\boldsymbol{M}^i}(\nu\mathbf{\Phi}\boldsymbol{n}-\phi\boldsymbol{n})\rangle_{\Gamma_h^i\setminus\mathcal{I}_h^i}.\end{aligned}$$

The result follows after summing over $i \in \{1,2\}$. $\square$

Lemma 6. *It holds that*

$$\mathbb{T}_{\boldsymbol{u}}^1 + \mathbb{T}_{\boldsymbol{u}}^2 \lesssim \|\boldsymbol{\theta}\|_\Omega \Bigg((\delta\tau^{1/2} + \delta^{1/2}\tau^{-1/2}) \left\|\tau^{1/2}(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2)\right\|_{\mathcal{I}_h^2} + \delta^{1/2} h_2^{-1/2} \left\|\boldsymbol{u}_h^2\right\|_{\Omega_h^2}$$
$$+ \delta^{1/2} \max_{e\in\mathcal{I}_h^2} r_e^{3/2} C_e^{\text{ext}} C_e^{\text{inv}} \left\|\mathbb{L}_h^2\right\|_{\Omega_h^2} + (\nu\delta + \delta^{1/2}) \left\| |\ell|^{-1/2} (\tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2) \right\|_{\mathcal{I}_h^2}$$
$$+ \delta^{1/2} \max_{e\in\mathcal{I}_h^2} h_e^{-1/2} C_e^{\text{tr}} \left\|\tilde{p}_h^2\right\|_{\Omega_h^2} + \max_{e\in\mathcal{I}_h^2} (\delta_e h_e^{-3/2} C_e^{\text{ext}}) \left\|\mathbb{L}_h^1\right\| + \max_{e\in\mathcal{I}_h^2} (\delta_e h_e^{-3/2} C_e^{\text{ext}}) \left\|\tilde{p}_h^1\right\|_{\Omega_h^1} \Bigg)$$
$$+ \|\boldsymbol{T}_1\|_{\mathcal{I}_h^1} \|\boldsymbol{\theta}\|_\Omega + \|\boldsymbol{T}_2\|_{\mathcal{I}_h^2} \|\boldsymbol{\theta}\|_\Omega \tag{3.3.12}$$

Proof. As we have done before, we will deal with these terms by expressing them in terms of the mismatch between $\mathcal{I}_h^1$ and $\mathcal{I}_h^2$.

Using (2.2.1h), and (2.2.1k) with (2.2.1f),

$$\begin{aligned}
\mathbb{T}_{\boldsymbol{u}}^1 &= \langle \widehat{\sigma}_h^1 \boldsymbol{n}^1, \boldsymbol{\phi}\rangle_{\mathcal{I}_h^1} + \langle \widehat{\boldsymbol{u}}_h^1, \nu\boldsymbol{\Phi}\boldsymbol{n}^1 - \phi\boldsymbol{n}^1\rangle_{\mathcal{I}_h^1} \\
&= \langle \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1\boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1} + \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{\phi}\rangle_{\mathcal{I}_h^1} + \langle \widehat{\boldsymbol{u}}_h^1, \mathrm{P}_{\boldsymbol{M}_h^1}(\nu\boldsymbol{\Phi}\boldsymbol{n}^1 - \phi\boldsymbol{n}^1)\rangle_{\mathcal{I}_h^1} \\
&= \langle \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1\boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1} + \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{\phi}\rangle_{\mathcal{I}_h^1} + \langle \tilde{u}_h^2, \mathrm{P}_{\boldsymbol{M}_h^1}(\nu\boldsymbol{\Phi}\boldsymbol{n}^1 - \phi\boldsymbol{n}^1)\rangle_{\mathcal{I}_h^1} \\
&\quad + \langle \boldsymbol{T}_1, \mathrm{P}_{\boldsymbol{M}_h^1}(\nu\boldsymbol{\Phi}\boldsymbol{n}^1 - \phi\boldsymbol{n}^1)\rangle_{\mathcal{I}_h^1}
\end{aligned}$$

Mapping these integrals from $\mathcal{I}_h^1$ to $\mathcal{I}_h^2$,

$$\begin{aligned}
\mathbb{T}_{\boldsymbol{u}}^1 &= \langle \tilde{u}_h^2 \circ \boldsymbol{\varphi}, \mathrm{P}_{\boldsymbol{M}_h^1}(\nu\boldsymbol{\Phi}\boldsymbol{n}^1 - \phi\boldsymbol{n}^1) \circ \boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} + \langle \tilde{\sigma}_h^1, \boldsymbol{\phi} \circ \boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} \\
&\quad + \langle \sigma_h^1 \boldsymbol{n}^1 \circ \boldsymbol{\varphi} + \boldsymbol{E}_{\sigma_h^1\boldsymbol{n}^2}, \boldsymbol{\phi}\rangle_{\mathcal{I}_h^2} + \langle \boldsymbol{T}_1, \mathrm{P}_{\boldsymbol{M}_h^1}(\nu\boldsymbol{\Phi}\boldsymbol{n}^1 - \phi\boldsymbol{n}^1)\rangle_{\mathcal{I}_h^1}.
\end{aligned}$$

Since we're omitting hanging nodes from our analysis, we have that $\tilde{\sigma}_h^1 \in \boldsymbol{M}_h(\mathcal{I}_h^2)$ and so we can use (2.2.1i) to obtain

$$\begin{aligned}
\langle \tilde{\sigma}_h^1, \boldsymbol{\phi} \circ \boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} &= \langle \tilde{\sigma}_h^1, \mathrm{P}_{\boldsymbol{M}_h^2}(\boldsymbol{\phi} \circ \boldsymbol{\varphi})\rangle_{\mathcal{I}_h^2} \\
&= -\langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \mathrm{P}_{\boldsymbol{M}_h^2}(\boldsymbol{\phi} \circ \boldsymbol{\varphi})\rangle_{\mathcal{I}_h^2} + \langle \boldsymbol{T}_2, \mathrm{P}_{\boldsymbol{M}_h^2}(\boldsymbol{\phi} \circ \boldsymbol{\varphi})\rangle_{\mathcal{I}_h^2}
\end{aligned}$$

$$= -\langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{\phi} \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} + \langle \boldsymbol{T}_2, \mathrm{P}_{\boldsymbol{M}_h^2}(\boldsymbol{\phi} \circ \boldsymbol{\varphi}) \rangle_{\mathcal{I}_h^2}.$$

Using that

$$\begin{aligned}
\langle \tilde{u}_h^2 \circ \boldsymbol{\varphi}, \mathrm{P}_{\boldsymbol{M}_h^1}(\nu \boldsymbol{\Phi} \boldsymbol{n}^1 - \phi \boldsymbol{n}^1) \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} &= \langle \tilde{u}_h^2, \mathrm{P}_{\boldsymbol{M}_h^1}(\nu \boldsymbol{\Phi} \boldsymbol{n}^1 - \phi \boldsymbol{n}^1) \rangle_{\mathcal{I}_h^1} \\
&= \langle \tilde{u}_h^2, \nu \boldsymbol{\Phi} \boldsymbol{n}^1 - \phi \boldsymbol{n}^1 \rangle_{\mathcal{I}_h^1} \\
&= \langle \tilde{u}_h^2 \circ \boldsymbol{\varphi}, (\nu \boldsymbol{\Phi} \boldsymbol{n}^1 - \phi \boldsymbol{n}^1) \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2},
\end{aligned}$$

the fact that $\boldsymbol{n}^1 \circ \boldsymbol{\varphi} = -\boldsymbol{n}^2$, and adding and substracting $\langle \widehat{\boldsymbol{u}}_h^2, (\nu \boldsymbol{\Phi} \boldsymbol{n}^2 - \phi \boldsymbol{n}^2) \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2}$, we have

$$\begin{aligned}
\mathbb{T}_{\boldsymbol{u}}^1 = &-\langle \tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2, (\nu \boldsymbol{\Phi} \boldsymbol{n}^2 - \phi \boldsymbol{n}^2) \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\
&- \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{\phi} \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} - \langle \widehat{\boldsymbol{u}}_h^2, (\nu \boldsymbol{\Phi} \boldsymbol{n}^2 - \phi \boldsymbol{n}^2) \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\
&- \langle \sigma_h^1 \boldsymbol{n}^2 \circ \boldsymbol{\varphi} - \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2}, \boldsymbol{\phi} \rangle_{\mathcal{I}_h^2} + \langle \boldsymbol{T}_1, \mathrm{P}_{\boldsymbol{M}_h^1}(\nu \boldsymbol{\Phi} \boldsymbol{n}^1 - \phi \boldsymbol{n}^1) + \langle \boldsymbol{T}_2, \mathrm{P}_{\boldsymbol{M}_h^2}(\boldsymbol{\phi} \circ \boldsymbol{\varphi}) \rangle_{\mathcal{I}_h^2} \rangle_{\mathcal{I}_h^1}.
\end{aligned}$$

Adding $\mathbb{T}_{\boldsymbol{u}}^2$, we have

$$\begin{aligned}
\mathbb{T}_{\boldsymbol{u}}^1 + \mathbb{T}_{\boldsymbol{u}}^2 = &-\langle \tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2, (\nu \boldsymbol{\Phi} \boldsymbol{n}^2 - \phi \boldsymbol{n}^2) \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\
&+ \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{\phi} - \boldsymbol{\phi} \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} + \langle \widehat{\boldsymbol{u}}_h^2, (\nu \boldsymbol{\Phi} \boldsymbol{n}^2 - \phi \boldsymbol{n}^2) - (\nu \boldsymbol{\Phi} \boldsymbol{n}^2 - \phi \boldsymbol{n}^2) \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\
&- \langle \sigma_h^1 \boldsymbol{n}^2 \circ \boldsymbol{\varphi} - \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2}, \boldsymbol{\phi} \rangle_{\mathcal{I}_h^2} + \langle \boldsymbol{T}_1, \mathrm{P}_{\boldsymbol{M}_h^1}(\nu \boldsymbol{\Phi} \boldsymbol{n}^1 - \phi \boldsymbol{n}^1) + \langle \boldsymbol{T}_2, \mathrm{P}_{\boldsymbol{M}_h^2}(\boldsymbol{\phi} \circ \boldsymbol{\varphi}) \rangle_{\mathcal{I}_h^2} \rangle_{\mathcal{I}_h^1} \\
= &-\langle \tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2, (\nu \boldsymbol{\Phi} \boldsymbol{n}^2 - \phi \boldsymbol{n}^2) \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\
&+ \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{\phi} - \boldsymbol{\phi} \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} - \langle \boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2, (\nu \boldsymbol{\Phi} \boldsymbol{n}^2 - \phi \boldsymbol{n}^2) - (\nu \boldsymbol{\Phi} \boldsymbol{n}^2 - \phi \boldsymbol{n}^2) \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\
&+ \langle \boldsymbol{u}_h^2, (\nu \boldsymbol{\Phi} \boldsymbol{n}^2 - \phi \boldsymbol{n}^2) - (\nu \boldsymbol{\Phi} \boldsymbol{n}^2 - \phi \boldsymbol{n}^2) \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\
&- \langle \sigma_h^1 \boldsymbol{n}^2 \circ \boldsymbol{\varphi} - \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2}, \boldsymbol{\phi} \rangle_{\mathcal{I}_h^2} + \langle \boldsymbol{T}_1, \mathrm{P}_{\boldsymbol{M}_h^1}(\nu \boldsymbol{\Phi} \boldsymbol{n}^1 - \phi \boldsymbol{n}^1) \rangle_{\mathcal{I}_h^1} + \langle \boldsymbol{T}_2, \mathrm{P}_{\boldsymbol{M}_h^2}(\boldsymbol{\phi} \circ \boldsymbol{\varphi}) \rangle_{\mathcal{I}_h^2}.
\end{aligned}$$

Using (3.3.1), we have

$$\begin{aligned}
\langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{\phi} - \boldsymbol{\phi} \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} = &\,\nu \langle |\ell|^{-1} (\tilde{u}_h^2 \circ \boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2), \boldsymbol{\phi} - \boldsymbol{\phi} \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} - \nu \langle \Lambda_{\mathbb{L}_h^2}^2, \boldsymbol{\phi} - \boldsymbol{\phi} \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} \\
&- \langle \tilde{p}_h^2 \boldsymbol{n}^2, \boldsymbol{\phi} - \boldsymbol{\phi} \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2} - \nu \langle \tau (\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2), \boldsymbol{\phi} - \boldsymbol{\phi} \circ \boldsymbol{\varphi} \rangle_{\mathcal{I}_h^2}
\end{aligned}$$

and so we can write

$$\mathbb{T}^1_{\boldsymbol{u}} + \mathbb{T}^2_{\boldsymbol{u}} = \sum_{i=1}^{9} \mathbb{S}^i,$$

where

$$\begin{aligned}
\mathbb{S}^1 &:= -\left\langle \tau^{1/2}(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2), \nu\tau^{1/2}(\boldsymbol{\phi} - \boldsymbol{\phi}\circ\boldsymbol{\phi}) + \tau^{-1/2}\left((\nu\boldsymbol{\Phi}\boldsymbol{n}^2 - \phi\boldsymbol{n}^2) - (\nu\boldsymbol{\Phi}\boldsymbol{n}^2 - \phi\boldsymbol{n}^2)\circ\boldsymbol{\varphi}\right)\right\rangle_{\mathcal{I}_h^2}\\
\mathbb{S}^2 &:= \langle |\ell|^{1/2}\,\boldsymbol{u}_h^2, |\ell|^{-1/2}\left((\nu\boldsymbol{\Phi}\boldsymbol{n}^2 - \phi\boldsymbol{n}^2) - (\nu\boldsymbol{\Phi}\boldsymbol{n}^2 - \phi\boldsymbol{n}^2)\circ\boldsymbol{\varphi}\right)\rangle_{\mathcal{I}_h^2}\\
\mathbb{S}^3 &:= \nu\langle |\ell|^{1/2}\,\Lambda^2_{\mathbb{L}_h^2}, |\ell|^{-1/2}\left(\boldsymbol{\phi} - \boldsymbol{\phi}\circ\boldsymbol{\varphi}\right)\rangle_{\mathcal{I}_h^2}\\
\mathbb{S}^4 &:= \nu\left\langle |\ell|^{-1/2}\left(\tilde{u}_h^2\circ\boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2\right), |\ell|^{-1/2}\left(\boldsymbol{\phi} - \boldsymbol{\phi}\circ\boldsymbol{\varphi}\right) - |\ell|^{1/2}\left(\boldsymbol{\Phi}\circ\boldsymbol{\varphi}\right)\boldsymbol{n}^2\right\rangle_{\mathcal{I}_h^2}\\
\mathbb{S}^5 &:= \left\langle |\ell|^{-1/2}\left(\tilde{u}_h^2\circ\boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2\right), |\ell|^{1/2}\left(\phi\circ\boldsymbol{\varphi}\right)\boldsymbol{n}^2\right\rangle_{\mathcal{I}_h^2}\\
\mathbb{S}^6 &:= -\left\langle |\ell|^{1/2}\,\tilde{p}_h^2\boldsymbol{n}^2, |\ell|^{-1/2}\left(\boldsymbol{\phi} - \boldsymbol{\phi}\circ\boldsymbol{\varphi}\right)\right\rangle_{\mathcal{I}_h^2}\\
\mathbb{S}^7 &:= \left\langle \boldsymbol{E}_{\sigma_h^1\boldsymbol{n}^2} - \sigma_h^1\boldsymbol{n}^2\circ\boldsymbol{\varphi}, \boldsymbol{\phi}\circ\boldsymbol{\varphi}\right\rangle_{\mathcal{I}_h^2}\\
\mathbb{S}^8 &:= \left\langle \boldsymbol{T}_1, \mathrm{P}_{\boldsymbol{M}_h^1}(\nu\boldsymbol{\Phi}\boldsymbol{n}^1 - \phi\boldsymbol{n}^1)\right\rangle_{\mathcal{I}_h^1}\\
\mathbb{S}^9 &:= \left\langle \boldsymbol{T}_2, \mathrm{P}_{\boldsymbol{M}_h^2}(\boldsymbol{\phi}\circ\boldsymbol{\varphi})\right\rangle_{\mathcal{I}_h^2}
\end{aligned}$$

Using the Cauchy-Schwartz inequality, the estimates from Lemma 1, discrete trace inequalities and the regularity assumption (3.3.8), we have

$$\begin{aligned}
\mathbb{S}^1 &\lesssim \left\|\tau^{1/2}(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2)\right\|_{\mathcal{I}_h^2}\left(\tau^{1/2}\delta\left\|\boldsymbol{\phi}\right\|_{\boldsymbol{H}^2(\Omega)} + \tau^{-1/2}\delta^{1/2}\left\|\nu\boldsymbol{\Phi} - \phi\mathbb{I}\right\|_{\mathbb{H}^1(\Omega)}\right)\\
&\lesssim (\delta\tau^{1/2} + \delta^{1/2}\tau^{-1/2})\left\|\tau^{1/2}(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2)\right\|_{\mathcal{I}_h^2}\left\|\boldsymbol{\theta}\right\|_\Omega\\
\mathbb{S}^2 &\lesssim \left\||\ell|^{1/2}\,\boldsymbol{u}_h^2\right\|_{\mathcal{I}_h^2}\left\|\nu\boldsymbol{\Phi} - \phi\mathbb{I}\right\|_{\mathbb{H}^1(\Omega)} \lesssim \max_{e\in\mathcal{I}_h^2} h_e^{-1/2} C_e^{\mathrm{tr}}\left\|\boldsymbol{u}_h^2\right\|_{\Omega_h^2}\left\|\boldsymbol{\theta}\right\|_{\Omega_h^2}\\
\mathbb{S}^3 &\lesssim \delta^{1/2}\left\||\ell|^{1/2}\,\Lambda^2_{\mathbb{L}_h^2}\right\|\left\|\boldsymbol{\phi}\right\|_{\boldsymbol{H}^2(\Omega)} \lesssim \delta^{1/2}\max_{e\in\mathcal{I}_h^2} r_e^{3/2} C_e^{\mathrm{ext}} C_e^{\mathrm{inv}}\left\|\mathbb{L}_h^2\right\|_{\Omega_h^2}\left\|\boldsymbol{\theta}\right\|_\Omega\\
\mathbb{S}^4 &\lesssim \nu\delta\left\||\ell|^{-1/2}\left(\tilde{u}_h^2\circ\boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2\right)\right\|_{\mathcal{I}_h^2}\left\|\boldsymbol{\phi}\right\|_{\boldsymbol{H}^2(\Omega)} \lesssim \nu\delta\left\||\ell|^{-1/2}\left(\tilde{u}_h^2\circ\boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2\right)\right\|_{\mathcal{I}_h^2}\left\|\boldsymbol{\theta}\right\|_\Omega
\end{aligned}$$

$$\mathbb{S}^5 \lesssim \delta^{1/2}\left\||\ell|^{-1/2}\left(\tilde{u}_h^2\circ\boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2\right)\right\|_{\mathcal{I}_h^2}\left\|\phi\circ\boldsymbol{\varphi}\right\|_{\mathcal{I}_h^2} \lesssim \delta^{1/2}\left\||\ell|^{-1/2}\left(\tilde{u}_h^2\circ\boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2\right)\right\|_{\mathcal{I}_h^2}\left\|\boldsymbol{\theta}\right\|_\Omega$$

$$\mathbb{S}^6 \lesssim \delta^{1/2} \left\| |\ell|^{1/2} \tilde{p}_h^2 \right\|_{\mathcal{I}_h^2} \|\boldsymbol{\phi}\|_{\boldsymbol{H}^2(\Omega)} \lesssim \delta^{1/2} \max_{e\in\mathcal{I}_h^2} h_e^{-1/2} C_e^{\text{tr}} \left\|\tilde{p}_h^2\right\|_{\Omega_h^2} \|\boldsymbol{\theta}\|_\Omega$$

$$\mathbb{S}^7 \lesssim \left\| \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} - \sigma_h^1 \boldsymbol{n}^2 \circ \boldsymbol{\varphi} \right\|_{\mathcal{I}_h^2} \|\boldsymbol{\phi}\circ\boldsymbol{\varphi}\|_{\mathcal{I}_h^2} \lesssim \max_{e\in\mathcal{I}_h^2} (\delta_e h_e^{-3/2} C_e^{\text{ext}}) \left\|\nu\mathbb{L}_h^1 - \tilde{p}_h^1 \mathbb{I}\right\|_{\Omega_h^1} \|\boldsymbol{\theta}\|_\Omega$$

$$\mathbb{S}^8 \lesssim \|\boldsymbol{T}_1\|_{\mathcal{I}_h^1} \|\nu\boldsymbol{\Phi} - \phi\mathbb{I}\|_{\mathcal{I}_h^1} \lesssim \|\boldsymbol{T}_1\|_{\mathcal{I}_h^1} \|\boldsymbol{\theta}\|_\Omega$$

$$\mathbb{S}^9 \lesssim \|\boldsymbol{T}_2\|_{\mathcal{I}_h^2} \|\boldsymbol{\phi}\circ\boldsymbol{\varphi}\|_{\mathcal{I}_h^2} \lesssim \|\boldsymbol{T}_2\|_{\mathcal{I}_h^2} \|\boldsymbol{\theta}\|_\Omega .$$

and so (3.3.12) follows. □

Lemma 7. *It holds that*

$$\begin{aligned}\sum_{i=1}^{2} \left\|\boldsymbol{u}_h^i\right\|_{\Omega_h^i}^2 \leq 2C_2 \Big(&\delta^2\tau + \delta\tau^{-1} + \delta \max_{e\in\mathcal{I}_h^2} r_e^3 (C_e^{\text{ext}} C_e^{\text{inv}})^2 + \nu^2\delta^2 + \delta + \max_{e\in\mathcal{I}_h^2} \delta_e^2 h_e^{-3} (C_e^{\text{ext}})^2 \\ &+ \nu^2 h_1^{2\min\{1,k\}} + \nu^2 h_2^{2\min\{1,k\}}\Big) \mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)^2 \\ &+ 2C_2 \max_{e\in\mathcal{I}_h^2} \left\{\delta_e h_e^{-1} (C_e^{\text{tr}})^2 + (\delta_e^2 h_e^{-3} (C_e^{\text{ext}})^2)\right\} \left(\sum_{i=1}^{2} \left\|\tilde{p}_h^i\right\|_{\Omega_h^i}^2\right) \\ &+ 2C_2 \left(\nu^2 \sum_{i=1}^{2} h_i^{2\min\{1,k\}} \left\|\mathbb{H}^i\right\|_{\Omega_h^i}^2 + \sum_{i=1}^{2} \|\boldsymbol{f}\|_{\Omega_h^i}^2 + \sum_{i=1}^{2} \|\boldsymbol{g}\|_{\Gamma_h^i\setminus\mathcal{I}_h^i}^2 + \|\boldsymbol{T}_1\|_{\mathcal{I}_h^1}^2 + \|\boldsymbol{T}_2\|_{\mathcal{I}_h^2}^2\right),\end{aligned} \tag{3.3.13}$$

where C_2 is a positive constant independent of the meshsize.

Proof. For the remaining terms of (3.3.9), we have the bounds

$$\begin{aligned}\sum_{i=1}^{2} (\nu\mathbb{L}_h^i, \boldsymbol{\Phi} - \Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i} &\leq \nu \left(\sum_{i=1}^{2} \left\|\mathbb{L}_h^i\right\|_{\Omega_h^i} \|\boldsymbol{\Phi} - \Pi_{\mathbb{V}^i}\boldsymbol{\Phi}\|_{\Omega_h^i}\right) \\ &\lesssim \nu \left(h_1^{\min\{1,k\}} + h_2^{\min\{1,k\}}\right) \sum_{i=1}^{2} \left\|\mathbb{L}_h^i\right\|_{\Omega_h^i} \|\boldsymbol{\theta}\|_\Omega ,\end{aligned}$$

$$\begin{aligned}-\nu \sum_{i=1}^{2} \Big\{(\mathbb{H}^i, \boldsymbol{\Phi} - \Pi_{\mathbb{V}^i}\boldsymbol{\Phi})_{\Omega_h^i} + (\mathbb{H}^i, \nabla(\boldsymbol{\phi} - \Pi_{\boldsymbol{V}^i}\boldsymbol{\phi}))_{\Omega_h^i}\Big\} &\leq \nu \Big(\sum_{i=1}^{2} \left\|\mathbb{H}^i\right\|_{\Omega_h^i} \Big\{ \|\boldsymbol{\Phi} - \Pi_{\mathbb{V}^i}\boldsymbol{\Phi}\|_{\Omega_h^i} \\ &\qquad + \|\nabla(\boldsymbol{\phi} - \Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})\|_{\Omega_h^i} \Big\}\Big) \\ &\lesssim \nu \left(\sum_{i=1}^{2} h_i^{\min\{1,k\}} \left\|\mathbb{H}^i\right\|_{\Omega_h^i}\right) \|\boldsymbol{\theta}\|_\Omega ,\end{aligned}$$

$$\sum_{i=1}^{2}(\boldsymbol{f},\Pi_{\boldsymbol{V}^i}\boldsymbol{\phi})_{\Omega_h^i} \le \left(\sum_{i=1}^{2}\|\boldsymbol{f}\|_{\Omega_h^i}\right)\|\boldsymbol{\phi}\|_\Omega \le \left(\sum_{i=1}^{2}\|\boldsymbol{f}\|_{\Omega_h^i}\right)\|\boldsymbol{\theta}\|_\Omega\,,$$

$$\begin{aligned}\sum_{i=1}^{2}\langle\boldsymbol{g},\mathrm{P}_{\boldsymbol{M}^i}(\nu\boldsymbol{\Phi}\boldsymbol{n}-\phi\boldsymbol{n})\rangle_{\Gamma_h^i\setminus\mathcal{I}_h^i} &\le \left(\sum_{i=1}^{2}\|\boldsymbol{g}\|_{\Gamma_h^i\setminus\mathcal{I}_h^i}\right)\|\nu\boldsymbol{\Phi}\boldsymbol{n}-\phi\boldsymbol{n}\|_\Gamma\\ &\lesssim \left(\sum_{i=1}^{2}\|\boldsymbol{g}\|_{\Gamma_h^i\setminus\mathcal{I}_h^i}\right)\|\boldsymbol{\theta}\|_\Omega\,.\end{aligned}$$

Since each of the terms in the right-hand side, save for the first one, has been bounded by terms that appear in estimate (3.3.6) and the projection errors, we will now provide the remaining bounds for $\mathbb{T}_{\boldsymbol{u}}^1+\mathbb{T}_{\boldsymbol{u}}^2$.

$$\begin{aligned}\sum_{i=1}^{2}(\boldsymbol{u}_h^i,\boldsymbol{\theta})_{\Omega_h^i} \lesssim&\Big(\delta\tau^{1/2}+\delta^{1/2}\tau^{-1/2}+\delta^{1/2}\max_{e\in\mathcal{I}_h^2} r_e^{3/2}C_e^{\mathrm{ext}}C_e^{\mathrm{inv}}+\nu\delta\\ &+\delta^{1/2}+\max_{e\in\mathcal{I}_h^2}\delta_e h_e^{-3/2}C_e^{\mathrm{ext}}+\nu h_1^{\min\{1,k\}}+\nu h_2^{\min\{1,k\}}\Big)\\ &\left(\sum_{i=1}^{2}\left\|\mathbb{L}_h^i\right\|_{\Omega_h^i}+\sum_{i=1}^{2}\left\|\tau^{1/2}(\boldsymbol{u}_h^i-\widehat{\boldsymbol{u}}_h^i)\right\|_{\partial\Omega_h^i}+\left\||\ell|^{-1/2}\left(\tilde{u}_h^2\circ\boldsymbol{\varphi}-\widehat{\boldsymbol{u}}_h^2\right)\right\|_{\mathcal{I}_h^2}\right)\|\boldsymbol{\theta}\|_\Omega\\ &+\delta^{1/2}\max_{e\in\mathcal{I}_h^2}h_e^{-1/2}C_e^{\mathrm{tr}}\left\|\boldsymbol{u}_h^2\right\|_{\Omega_h^2}\|\boldsymbol{\theta}\|_\Omega\\ &+\left(\delta^{1/2}\max_{e\in\mathcal{I}_h^2}h_e^{-1/2}C_e^{\mathrm{tr}}+\max_{e\in\mathcal{I}_h^2}(\delta_e h_e^{-3/2}C_e^{\mathrm{ext}})\right)\left(\sum_{i=1}^{2}\left\|\tilde{p}_h^i\right\|_{\Omega_h^i}\right)\|\boldsymbol{\theta}\|_\Omega\\ &+\left(\nu\sum_{i=1}^{2}h_i^{\min\{1,k\}}\left\|\mathbb{H}^i\right\|_{\Omega_h^i}+\sum_{i=1}^{2}\|\boldsymbol{f}\|_{\Omega_h^i}+\sum_{i=1}^{2}\|\boldsymbol{g}\|_{\Gamma_h^i\setminus\mathcal{I}_h^i}+\|\boldsymbol{T}_1\|_{\mathcal{I}_h^1}+\|\boldsymbol{T}_2\|_{\mathcal{I}_h^2}\right)\|\boldsymbol{\theta}\|_\Omega\end{aligned}$$

which, after using Young's inequality on the right side, leaves us with

$$\begin{aligned}\sum_{i=1}^{2}(\boldsymbol{u}_h^i,\boldsymbol{\theta})_{\Omega_h^i} \le C_2\Bigg\{&\Big(\delta^2\tau+\delta\tau^{-1}+\delta\max_{e\in\mathcal{I}_h^2}r_e^3(C_e^{\mathrm{ext}}C_e^{\mathrm{inv}})^2\\ &+\nu^2\delta^2+\delta+\max_{e\in\mathcal{I}_h^2}\delta_e^2h_e^{-3}(C_e^{\mathrm{ext}})^2+\nu^2h_1^{2\min\{1,k\}}+\nu^2h_2^{2\min\{1,k\}}\Big)\\ &\left(\sum_{i=1}^{2}\left\|\mathbb{L}_h^i\right\|_{\Omega_h^i}^2+\sum_{i=1}^{2}\left\|\tau^{1/2}(\boldsymbol{u}_h^i-\widehat{\boldsymbol{u}}_h^i)\right\|_{\partial\Omega_h^i}^2+\left\||\ell|^{-1/2}\left(\tilde{u}_h^2\circ\boldsymbol{\varphi}-\widehat{\boldsymbol{u}}_h^2\right)\right\|_{\mathcal{I}_h^2}^2\right)\end{aligned}$$

$$+ \delta \max_{e\in\mathcal{I}_h^2} h_e^{-1}(C_e^{\mathrm{tr}})^2 \left\|\boldsymbol{u}_h^2\right\|_{\Omega_h^2}^2 + \max_{e\in\mathcal{I}_h^2}\left(\delta_e h_e^{-1}(C_e^{\mathrm{tr}})^2 + (\delta_e^2 h_e^{-3}(C_e^{\mathrm{ext}})^2)\right)\left(\sum_{i=1}^2 \left\|\tilde{p}_h^i\right\|_{\Omega_h^i}^2\right)$$
$$+\left(\nu^2\sum_{i=1}^2 h_i^{2\min\{1,k\}}\left\|\mathbb{H}^i\right\|_{\Omega_h^i}^2 + \sum_{i=1}^2\|\boldsymbol{f}\|_{\Omega_h^i}^2 + \sum_{i=1}^2\|\boldsymbol{g}\|_{\Gamma_h^i\setminus\mathcal{I}_h^i}^2 + \|\boldsymbol{T}_1\|_{\mathcal{I}_h^1}^2 + \|\boldsymbol{T}_2\|_{\mathcal{I}_h^2}^2\right)\Bigg\}$$
$$+\frac{1}{4}\|\boldsymbol{\theta}\|_\Omega^2$$

where $C_2 > 0$ is a constant independent of the meshsize as mentioned before.

Taking $\boldsymbol{\theta} = \begin{cases}\boldsymbol{u}_h^1 & \text{in } \Omega_h^1,\\ \boldsymbol{u}_h^2 & \text{in } \Omega_h^2.\end{cases}$, we have

$$\sum_{i=1}^2\left\|\boldsymbol{u}_h^i\right\|_{\Omega_h^i}^2 \le C_2\Bigg\{\Bigg(\delta^2\tau + \delta\tau^{-1} + \delta\max_{e\in\mathcal{I}_h^2} r_e^3(C_e^{\mathrm{ext}}C_e^{\mathrm{inv}})^2$$
$$+\nu^2\delta^2 + \delta + \max_{e\in\mathcal{I}_h^2}\delta_e^2 h_e^{-3}(C_e^{\mathrm{ext}})^2 + \nu^2 h_1^{2\min\{1,k\}} + \nu^2 h_2^{2\min\{1,k\}}\Bigg)$$
$$\left(\sum_{i=1}^2\left\|\mathbb{L}_h^i\right\|_{\Omega_h^i}^2 + \sum_{i=1}^2\left\|\tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)\right\|_{\partial\Omega_h^i}^2 + \left\||\ell|^{-1/2}\left(\tilde{u}_h^2\circ\boldsymbol{\varphi} - \widehat{\boldsymbol{u}}_h^2\right)\right\|_{\mathcal{I}_h^2}^2\right)$$
$$+ \delta \max_{e\in\mathcal{I}_h^2} h_e^{-1}(C_e^{\mathrm{tr}})^2 \left\|\boldsymbol{u}_h^2\right\|_{\Omega_h^2}^2 + \max_{e\in\mathcal{I}_h^2}\left(\delta_e h_e^{-1}(C_e^{\mathrm{tr}})^2 + (\delta_e^2 h_e^{-3}(C_e^{\mathrm{ext}})^2)\right)\left(\sum_{i=1}^2 \left\|\tilde{p}_h^i\right\|_{\Omega_h^i}^2\right)$$
$$+\left(\nu^2\sum_{i=1}^2 h_i^{2\min\{1,k\}}\left\|\mathbb{H}^i\right\|_{\Omega_h^i}^2 + \sum_{i=1}^2\|\boldsymbol{f}\|_{\Omega_h^i}^2 + \sum_{i=1}^2\|\boldsymbol{g}\|_{\Gamma_h^i\setminus\mathcal{I}_h^i}^2 + \|\boldsymbol{T}_1\|_{\mathcal{I}_h^1}^2 + \|\boldsymbol{T}_2\|_{\mathcal{I}_h^2}^2\right)\Bigg\}$$
$$+\frac{1}{4}\sum_{i=1}^2\left\|\boldsymbol{u}_h^i\right\|_{\Omega_h^i}^2. \tag{3.3.14}$$

Finally, the result frollows from assumption (A.7). □

Under assumption (A.8), we can replace (3.3.13) in (3.3.5) to obtain the following corollary.

Corollary 3. *It holds that*

$$\frac{1}{8}\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)^2 \leq \left(1 + 2C_2\nu^2\left(h_1^{2\min\{1,k\}} + h_2^{2\min\{1,k\}}\right)\right)\sum_{i=1}^{2}\|\mathbb{H}\|^2_{\Omega_h^i} + \left(\frac{2}{\nu^2} + 4C_2\right)\|\boldsymbol{f}\|^2_\Omega$$
$$+ \left(\frac{4}{\nu^2}\max_{e\in\mathcal{I}_h^2}\delta_e^{-1}h_e^{-1} + 4\tau + 4C_2\right)\|\boldsymbol{g}\|_\Gamma + \left(C_{\tilde{p}}^{\boldsymbol{u}} + C_{\tilde{p}}^{\mathbb{L}}\right)\sum_{i=1}^{2}\left\|\tilde{p}_h^i\right\|^2_{\Omega_h^i}$$
$$+ \left\|(2C_2^{1/2} + 2^{-1}\nu^{-1}(\nu\tau^{1/2} + \delta^{-1}8\nu(C_e^{\mathrm{tr}})^2h_e^{-1}))^{1/2}\boldsymbol{T}_1\right\|^2_{\mathcal{I}_h^1}$$
$$+ \left\|(2C_2^{1/2} + 2^{-1}\nu^{-1}(4\delta_e + 4\tau^{-1} + 2(C_e^{\mathrm{tr}})^2h_e^{-1}))^{1/2}\boldsymbol{T}_2\right\|^2_{\mathcal{I}_h^2}, \tag{3.3.15}$$

where we recall that

$$C_{\tilde{p}}^{\boldsymbol{u}} = 2C_2\max_{e\in\mathcal{I}_h^2}\left\{\delta_e h_e^{-1}(C_e^{\mathrm{tr}})^2 + (\delta_e^2 h_e^{-3}(C_e^{\mathrm{ext}})^2)\right\}.$$

3.3.3 Estimate of the pressure

Adjusting the proof of Lemma 2 in [43] to our context, we provide a bound for $\sum_{i=1}^{2}\left\|\tilde{p}_h^i\right\|_{\Omega_h^i}$ as follows.

Lemma 8. *It holds that*

$$\sum_{i=1}^{2}\left\|\tilde{p}_h^i\right\|^2_{\Omega_h^i} \leq 8\nu^2 C_3\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)^2 + 8C_3\|\boldsymbol{f}\|^2_\Omega, \tag{3.3.16}$$

where C_3 is a positive constant independent of the meshsize.

Proof. Let

$$\tilde{\tilde{p}}_h := \begin{cases} \tilde{p}_h, & \text{on } \Omega_h^1 \cup \Omega_h^2 \\ 0, & \text{on } \Omega\setminus(\Omega_h^1\cup\Omega_h^2) \end{cases}.$$

It is clear that $\tilde{\tilde{p}}_h \in L^2(\Omega)$ and

$$\int_\Omega \tilde{\tilde{p}}_h = \int_{\Omega_h^1\cup\Omega_h^2}\tilde{p}_h = 0,$$

therefore, there exists $\beta > 0$ such that

$$\|\tilde{\tilde{p}}_h\|_\Omega \leq \beta \sup_{\substack{\boldsymbol{w}\in\boldsymbol{H}_0^1(\Omega)\\ \boldsymbol{w}\neq\boldsymbol{0}}} \frac{(\tilde{\tilde{p}}_h, \nabla\cdot\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}} \tag{3.3.17}$$

Note that, since $\tilde{\tilde{p}}_h$ is an extension by zero of $\tilde{p}$, then $\|\tilde{\tilde{p}}_h\|_\Omega = \|\tilde{p}\|_{\Omega_h^1\cup\Omega_h^2}$ and, for $\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)$, $(\tilde{\tilde{p}}_h, \boldsymbol{v})_\Omega = (\tilde{p}_h^1, \boldsymbol{v})_{\Omega_h^1} + (\tilde{p}_h^2, \boldsymbol{v})_{\Omega_h^2}$. Replacing this in (3.3.17), we get

$$\|\tilde{p}\|_{\Omega_h^1\cup\Omega_h^2} \leq \beta \sup_{\substack{\boldsymbol{w}\in\boldsymbol{H}_0^1(\Omega)\\ \boldsymbol{w}\neq\boldsymbol{0}}} \frac{(\tilde{p}_h^1, \nabla\cdot\boldsymbol{w})_{\Omega_h^1} + (\tilde{p}_h^2, \nabla\cdot\boldsymbol{w})_{\Omega_h^2}}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}} \tag{3.3.18}$$

For $i\in\{1,2\}$, let $\boldsymbol{P}^i : \boldsymbol{H}^1(\Omega) \to \boldsymbol{V}_h^i$ be any projection such that $(\boldsymbol{P}^i\boldsymbol{w}-\boldsymbol{w}, \boldsymbol{v})_K = 0$, $\forall \boldsymbol{v}\in [\mathbb{P}_{k-1}(K)]^d$ for all $K\in\Omega_h^i$.

Following (very closely) the proof of Lemma 2 in [43], we integrate by parts and use the projection $\boldsymbol{P}^i$ to get

$$(\tilde{p}_h^i, \nabla\cdot\boldsymbol{w})_{\Omega_h^i} = -(\nabla\tilde{p}_h^i, \boldsymbol{w})_{\Omega_h^i} + \langle \tilde{p}_h^i\boldsymbol{n}^i, \boldsymbol{w}\rangle_{\partial\Omega_h^i} = -(\nabla\tilde{p}_h^i, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i} + \langle \tilde{p}_h^i\boldsymbol{n}^i, \boldsymbol{w}\rangle_{\partial\Omega_h^i} \tag{3.3.19}$$

Integrating by parts again and using (2.2.1b) with $\boldsymbol{v} = \boldsymbol{P}^i\boldsymbol{w}$,

$$\begin{aligned}
-(\nabla\tilde{p}_h^i, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i} &= (\tilde{p}_h^i, \nabla\cdot\boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i} - \langle \tilde{p}_h^i\boldsymbol{n}^i, \boldsymbol{P}^i\boldsymbol{w}\rangle_{\partial\Omega_h^i}\\
&= (\nu\mathbb{L}_h^i, \nabla\boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i} - \langle \widehat{\sigma}_h^i\boldsymbol{n}^i, \boldsymbol{P}^i\boldsymbol{w}\rangle_{\partial\Omega_h^i} - (\boldsymbol{f}, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i} - \langle \tilde{p}_h^i\boldsymbol{n}^i, \boldsymbol{P}^i\boldsymbol{w}\rangle_{\partial\Omega_h^i}\\
&= (\nu\mathbb{L}_h^i, \nabla\boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i} - \langle \nu\mathbb{L}_h^i\boldsymbol{n}^i, \boldsymbol{P}^i\boldsymbol{w}\rangle_{\partial\Omega_h^i} + \langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \boldsymbol{P}^i\boldsymbol{w}\rangle_{\partial\Omega_h^i} - (\boldsymbol{f}, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i}
\end{aligned}$$

Integrating by parts and using the projection,

$$(\nu\mathbb{L}_h^i, \nabla\boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i} - \langle \nu\mathbb{L}_h^i\boldsymbol{n}^i, \boldsymbol{P}^i\boldsymbol{w}\rangle_{\partial\Omega_h^i} = -(\nu\nabla\cdot\mathbb{L}_h^i, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i} = -(\nu\nabla\cdot\mathbb{L}_h^i, \boldsymbol{w})_{\Omega_h^i}$$

Integrating by parts again, $-(\nu\nabla\cdot\mathbb{L}_h^i, \boldsymbol{w})_{\Omega_h^i} = (\nu\mathbb{L}_h^i, \nabla\boldsymbol{w})_{\Omega_h^i} - \langle \nu\mathbb{L}_h^i\boldsymbol{n}^i, \boldsymbol{w}\rangle_{\partial\Omega_h^i}$ and so

$$-(\nabla\tilde{p}_h^i, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i} = (\nu\mathbb{L}_h^i, \nabla\boldsymbol{w})_{\Omega_h^i} - \langle \nu\mathbb{L}_h^i\boldsymbol{n}^i, \boldsymbol{w}\rangle_{\partial\Omega_h^i} + \langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \boldsymbol{P}^i\boldsymbol{w}\rangle_{\partial\Omega_h^i} - (\boldsymbol{f}, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i}$$

Replacing this in (3.3.19),

$$(\tilde{p}_h^i, \nabla\cdot\boldsymbol{w})_{\Omega_h^i} = (\nu\mathbb{L}_h^i, \nabla\boldsymbol{w})_{\Omega_h^i} - \langle \nu\mathbb{L}_h^i\boldsymbol{n}^i - \tilde{p}_h^i\boldsymbol{n}^i, \boldsymbol{w}\rangle_{\partial\Omega_h^i} + \langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \boldsymbol{P}^i\boldsymbol{w}\rangle_{\partial\Omega_h^i} - (\boldsymbol{f}, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i}$$

and, after adding and substracting $\langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \boldsymbol{w}\rangle_{\partial\Omega_h^i}$,

$$(\tilde{p}_h^i, \nabla\cdot\boldsymbol{w})_{\Omega_h^i} = (\nu\mathbb{L}_h^i, \nabla\boldsymbol{w})_{\Omega_h^i} - \langle \widehat{\sigma}_h^i\boldsymbol{n}^i, \boldsymbol{w}\rangle_{\partial\Omega_h^i} + \langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \boldsymbol{P}^i\boldsymbol{w} - \boldsymbol{w}\rangle_{\partial\Omega_h^i} - (\boldsymbol{f}, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i}.$$

Since $\tilde{\sigma}_h^i\boldsymbol{n}^i$ and $\tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)$ are polynomials, we can use the $\boldsymbol{L}^2$ projection onto $\boldsymbol{M}_h^i$ to get

$$(\tilde{p}_h^i, \nabla\cdot\boldsymbol{w})_{\Omega_h^i} = (\nu\mathbb{L}_h^i, \nabla\boldsymbol{w})_{\Omega_h^i} - \langle \widehat{\sigma}_h^i\boldsymbol{n}^i, \mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{w}\rangle_{\partial\Omega_h^i} + \langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \boldsymbol{P}^i\boldsymbol{w} - \mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{w}\rangle_{\partial\Omega_h^i} - (\boldsymbol{f}, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i}.$$

and, using (2.2.1e) with $\boldsymbol{\mu} = \mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{w}$,

$$(\tilde{p}_h^i, \nabla\cdot\boldsymbol{w})_{\Omega_h^i} = (\nu\mathbb{L}_h^i, \nabla\boldsymbol{w})_{\Omega_h^i} - \langle \widehat{\sigma}_h^i\boldsymbol{n}^i, \mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{w}\rangle_{\Gamma_h^i} + \langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \boldsymbol{P}^i\boldsymbol{w} - \mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{w}\rangle_{\partial\Omega_h^i} - (\boldsymbol{f}, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i}.$$

Since $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$, whose boundary coincides with both Γ_h^i save for the interface, we have

$$(\tilde{p}_h^i, \nabla\cdot\boldsymbol{w})_{\Omega_h^i} = (\nu\mathbb{L}_h^i, \nabla\boldsymbol{w})_{\Omega_h^i} - \langle \widehat{\sigma}_h^i\boldsymbol{n}^i, \mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{w}\rangle_{\mathcal{I}_h^i} + \langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \boldsymbol{P}^i\boldsymbol{w} - \mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{w}\rangle_{\partial\Omega_h^i} - (\boldsymbol{f}, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i}.$$

Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
(\nu\mathbb{L}_h^i, \nabla\boldsymbol{w})_{\Omega_h^i} &\le \nu \left\|\mathbb{L}_h^i\right\|_{\Omega_h^i} \|\nabla\boldsymbol{w}\|_{\Omega_h^i} \le \nu \left\|\mathbb{L}_h^i\right\|_{\Omega_h^i} \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \\
\langle \tau\nu(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i), \boldsymbol{P}^i\boldsymbol{w} - \mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{w}\rangle_{\partial\Omega_h^i} &\le \nu \left\|\tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)\right\|_{\partial\Omega_h^i} \left\|\tau^{1/2}\left(\boldsymbol{P}^i\boldsymbol{w} - \mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{w}\right)\right\|_{\partial\Omega_h^i} \\
-(\boldsymbol{f}, \boldsymbol{P}^i\boldsymbol{w})_{\Omega_h^i} &\le \|\boldsymbol{f}\|_{\Omega_h^i} \left\|\boldsymbol{P}^i\boldsymbol{w}\right\|_{\Omega_h^i} \le \|\boldsymbol{f}\|_{\Omega_h^i} \|\boldsymbol{w}\|_{\Omega_h^i} \le \|\boldsymbol{f}\|_{\Omega_h^i} \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}
\end{aligned}$$

and so, since,

$$\sup_{\substack{\boldsymbol{w}\in\boldsymbol{H}_0^1(\Omega)\\ \boldsymbol{w}\neq\boldsymbol{0}}} \frac{\left\|\tau^{1/2}\left(\boldsymbol{P}^i\boldsymbol{w} - \mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{w}\right)\right\|_{\partial\Omega_h^i}}{\|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}} \le \max\left\{1, \max_{K\in\Omega_h^1\cup\Omega_h^2} (\tau h_K)^{1/2}\right\},$$

we can control all the terms on the right-hand side of (3.3.18) save for the ones with numerator $\langle \widehat{\sigma}_h^i \boldsymbol{n}^i, \mathrm{P}_{\boldsymbol{M}^i} \boldsymbol{w}\rangle_{\mathcal{I}_h^i}$.

We deal with those terms as follows: first, since the first arguments of both terms are polynomials, we can drop the $\boldsymbol{L}^2$ projections and then add and substract convenient terms as we did when expanding $\mathbb{T}^1 + \mathbb{T}^2$, leading to

$$\begin{aligned}
\langle \widehat{\sigma}_h^1 \boldsymbol{n}^1, \mathrm{P}_{\boldsymbol{M}_h^1} \boldsymbol{w}\rangle_{\mathcal{I}_h^1} + \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \mathrm{P}_{\boldsymbol{M}_h^2} \boldsymbol{w}\rangle_{\mathcal{I}_h^2} &= \langle \widehat{\sigma}_h^1 \boldsymbol{n}^1, \boldsymbol{w}\rangle_{\mathcal{I}_h^1} + \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{w}\rangle_{\mathcal{I}_h^2} \\
&= \langle \widehat{\sigma}_h^1 \boldsymbol{n}^1 - \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{w}\rangle_{\mathcal{I}_h^1} + \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{w} - \boldsymbol{w}\circ\boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} \\
&+ \langle \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{w}\rangle_{\mathcal{I}_h^1} + \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{w}\circ\boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2},
\end{aligned} \tag{3.3.20}$$

where the last two terms cancel out since

$$\begin{aligned}
\langle \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{w}\rangle_{\mathcal{I}_h^1} + \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{w}\circ\boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} &= \langle \tilde{\sigma}_h^1, \boldsymbol{w}\circ\boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} + \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{w}\circ\boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} \\
&= \langle \tilde{\sigma}_h^1 + \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{w}\circ\boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} \\
&= \langle \tilde{\sigma}_h^1 + \widehat{\sigma}_h^2 \boldsymbol{n}^2, \mathrm{P}_{\boldsymbol{M}_h^2}(\boldsymbol{w}\circ\boldsymbol{\varphi})\rangle_{\mathcal{I}_h^2} \\
&= 0.
\end{aligned}$$

Since $\widehat{\sigma}_h^1 \boldsymbol{n}^1 - \tilde{\sigma}_h^1 \circ \boldsymbol{\varphi}^1 = \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1}$ and $\widehat{\sigma}_h^2 \boldsymbol{n}^2 = \nu \mathbb{L}_h^2 \boldsymbol{n}^2 - \tilde{p}_h^2 \boldsymbol{n}^2 - \tau\nu(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2)$, we replace this in (3.3.20) and use the Cauchy-Schwartz inequality to obtain

$$\begin{aligned}
\langle \widehat{\sigma}_h^1 \boldsymbol{n}^1, \boldsymbol{w}\rangle_{\mathcal{I}_h^1} + \langle \widehat{\sigma}_h^2 \boldsymbol{n}^2, \boldsymbol{w}\rangle_{\mathcal{I}_h^2} &= \langle \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{w}\rangle_{\mathcal{I}_h^1} + \langle \nu \mathbb{L}_h^2 \boldsymbol{n}^2, \boldsymbol{w} - \boldsymbol{w}\circ\boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} \\
&- \langle \tilde{p}_h^2 \boldsymbol{n}^2, \boldsymbol{w} - \boldsymbol{w}\circ\boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} - \langle \tau\nu(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2), \boldsymbol{w} - \boldsymbol{w}\circ\boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} \\
&\leq \left\| \sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1} \right\|_{\mathcal{I}_h^1} \|\boldsymbol{w}\|_{\mathcal{I}_h^1} \\
&+ \nu \left\| h_e^{1/2} \mathbb{L}_h^2 \right\|_{\mathcal{I}_h^2} \left\| h_e^{-1/2}(\boldsymbol{w} - \boldsymbol{w}\circ\boldsymbol{\varphi}) \right\|_{\mathcal{I}_h^2} \\
&+ \left\| \tilde{p}_h^2 \right\|_{\mathcal{I}_h^2} \|\boldsymbol{w} - \boldsymbol{w}\circ\boldsymbol{\varphi}\|_{\mathcal{I}_h^2} \\
&+ \nu \left\| \tau^{1/2}(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2) \right\|_{\mathcal{I}_h^2} \left\| \tau^{1/2}(\boldsymbol{w} - \boldsymbol{w}\circ\boldsymbol{\varphi}) \right\|_{\mathcal{I}_h^2}
\end{aligned}$$

Since, for $i \in \{1,2\}$, $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega) \subset \boldsymbol{H}^1(\Omega_h^i)$, we have that

$$\begin{aligned}
\|\boldsymbol{w}\|_{\mathcal{I}_h^1} &\leq \widehat{C}_{\mathrm{tr}}^1 \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \\
\|\boldsymbol{w} - \boldsymbol{w} \circ \boldsymbol{\varphi}\|_{\mathcal{I}_h^2} &\leq \widehat{C}_{\mathrm{ext}}^2 \max_{e \in \mathcal{I}_h^2} \delta_e \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \\
\left\|h_e^{-1/2}(\boldsymbol{w} - \boldsymbol{w} \circ \boldsymbol{\varphi})\right\|_{\mathcal{I}_h^2} &\leq \widehat{C}_{\mathrm{ext}}^2 \max_{e \in \mathcal{I}_h^2} \delta_e h_e^{-1/2} \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)} \\
\left\|\tau^{1/2}(\boldsymbol{w} - \boldsymbol{w} \circ \boldsymbol{\varphi})\right\|_{\mathcal{I}_h^2} &\leq \widehat{C}_{\mathrm{ext}}^2 \max_{e \in \mathcal{I}_h^2} \delta_e \tau^{1/2} \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\Omega)}
\end{aligned}$$

where $\widehat{C}_{\mathrm{tr}}^1 > 0$ is a constant related to the trace inequality on Ω_h^1 and $\widehat{C}_{\mathrm{ext}}^2 > 0$ is a constant related to the extrapolation error.

The rest of the terms can easily be bounded as previously done when they first appeared in the analysis.

$$\begin{aligned}
\left\|\sigma_h^1 \boldsymbol{n}^1 + \boldsymbol{E}_{\sigma_h^1 \boldsymbol{n}^2} \circ \boldsymbol{\varphi}^{-1}\right\|_{\mathcal{I}_h^1} &\leq \nu \max_{e \in \mathcal{I}_h^1} C_e^1 C_e^{\mathrm{ext}} \delta_e h_e^{-3/2} \left\|\mathbb{L}_h^1\right\|_{\Omega_h^1} + \max_{e \in \mathcal{I}_h^1} C_e^1 C_e^{\mathrm{ext}} \delta_e h_e^{-3/2} \left\|\tilde{p}_h^1\right\|_{\Omega_h^1} \\
\nu \left\|\mathbb{L}_h^2\right\|_{\mathcal{I}_h^2} &\leq \nu \left\|\mathbb{L}_h^2\right\|_{\Omega_h^2} \\
\left\|\tilde{p}_h^2\right\|_{\mathcal{I}_h^2} &\leq \max_{e \in \mathcal{I}_h^2} h_e^{-1/2} \left\|\tilde{p}_h^2\right\|_{\Omega_h^2} \\
\nu \left\|\tau^{1/2}(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2)\right\|_{\mathcal{I}_h^2} &\leq \nu \left\|\tau^{1/2}(\boldsymbol{u}_h^2 - \widehat{\boldsymbol{u}}_h^2)\right\|_{\partial\Omega_h^2}
\end{aligned}$$

Finally, putting everything together, we obtain an estimate of the form

$$\|\tilde{p}_h\|_{\Omega_h^1 \cup \Omega_h^2} \leq \nu \left[\tilde{\tilde{C}}^1 \sum_{i=1}^{2} \left\|\mathbb{L}_h^i\right\|_{\Omega_h^i} + \tilde{\tilde{C}}^2 \sum_{i=1}^{2} \left\|\tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)\right\|_{\partial\Omega_h^i} \right] + \tilde{\tilde{C}}^3 \|\tilde{p}_h\|_{\Omega_h^1 \cup \Omega_h^2} + 2\beta \|\boldsymbol{f}\|_\Omega$$

or, equivalently,

$$\frac{1}{\nu}\left(1 - \tilde{\tilde{C}}^3\right) \|\tilde{p}_h\|_{\Omega_h^1 \cup \Omega_h^2} \leq \tilde{\tilde{C}}^1 \sum_{i=1}^{2} \left\|\mathbb{L}_h^i\right\|_{\Omega_h^i} + \tilde{\tilde{C}}^2 \sum_{i=1}^{2} \left\|\tau^{1/2}(\boldsymbol{u}_h^i - \widehat{\boldsymbol{u}}_h^i)\right\|_{\partial\Omega_h^i} + \frac{2}{\nu} \|\boldsymbol{f}\|_\Omega$$

where

$$\tilde{\tilde{C}}^1 := \beta\left(1 + \widehat{C}^1_{\text{tr}} \max_{e\in\mathcal{I}^1_h} C^1_e C^{\text{ext}}_e \delta_e h_e^{-3/2} + \max_{e\in\mathcal{I}^2_h} h_e^{-1/2}\delta_e\right)$$
$$\tilde{\tilde{C}}^2 = \beta \max\left\{1, \max_{K\in\Omega^1_h\cup\Omega^2_h} (\tau h_K)^{1/2}\right\} + \beta\widehat{C}^2_{\text{ext}} \max_{e\in\mathcal{I}^2_h} \delta_e \tau^{1/2}$$
$$\tilde{\tilde{C}}^3 = \beta\widehat{C}^1_{\text{tr}} \max_{e\in\mathcal{I}^1_h} C^1_e C^{\text{ext}}_e \delta_e h_e^{-3/2} + \beta\widehat{C}^2_{\text{ext}} \max_{e\in\mathcal{I}^2_h} \delta_e h_e^{-1/2}.$$

If we assume that $\tilde{\tilde{C}}^1, \tilde{\tilde{C}}^2 \leq 2$, then we can conclude that there exists $C_3 > 0$, independent of the meshsize, such that

$$\begin{aligned}\frac{1}{4\nu^2}\left(1-\tilde{\tilde{C}}^3\right)^2 \sum_{i=1}^{2} \|\tilde{p}^i_h\|^2_{\Omega^i_h} &\leq C_3\left(\sum_{i=1}^{2}\|\mathbb{L}^i_h\|^2_{\Omega^i_h} + \sum_{i=1}^{2}\|\tau^{1/2}(\boldsymbol{u}^i_h - \widehat{\boldsymbol{u}}^i_h)\|^2_{\partial\Omega^i_h}\right) + \frac{C_3}{\nu^2}\|\boldsymbol{f}\|^2_\Omega \\ &\leq C_3\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)^2 + \frac{C_3}{\nu^2}\|\boldsymbol{f}\|^2_\Omega\end{aligned}$$

and so

$$\sum_{i=1}^{2}\|\tilde{p}^i_h\|^2_{\Omega^i_h} \leq \left[\frac{4\nu^2 C_3}{\left(1-\tilde{\tilde{C}}^3\right)^2}\right]\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)^2 + \left[\frac{4C_3}{\left(1-\tilde{\tilde{C}}^3\right)^2}\right]\|\boldsymbol{f}\|^2_\Omega.$$

By assumption (A.10), we have $\dfrac{1}{(1-\tilde{\tilde{C}}^3)^2} \leq 2$ and so the result follows. □

Aditionally, we can put together all the estimates we have derived so far to obtain an explicit form of (3.2.3). In fact, if we replace the previous bound in (3.3.15), we have

$$\begin{aligned}\frac{1}{8}\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)^2 \leq\ & \left(1 + 2C_2\nu^2\left(h_1^{2\min\{1,k\}} + h_2^{2\min\{1,k\}}\right)\right)\sum_{i=1}^{2}\|\mathbb{H}\|^2_{\Omega^i_h} \\ &+ \left(\frac{2}{\nu^2} + 4C_2 + 8C_3\right)\|\boldsymbol{f}\|^2_\Omega \\ &+ \left(\frac{4}{\nu^2}\max_{e\in\mathcal{I}^2_h}\delta_e^{-1}h_e^{-1} + 4\tau + 4C_2\right)\|\boldsymbol{g}\|_\Gamma \\ &+ 8\nu^2 C_3\left(C^{\boldsymbol{u}}_{\tilde{p}} + C^{\mathbb{L}}_{\tilde{p}}\right)\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)\end{aligned}$$

$$+ \left\| (2C_2^{1/2} + 2^{-1}\nu^{-1}(\nu\tau^{1/2} + \delta^{-1}8\nu(C_e^{\mathrm{tr}})^2 h_e^{-1}))^{1/2}\boldsymbol{T}_1 \right\|_{\mathcal{I}_h^1}^2$$
$$+ \left\| (2C_2^{1/2} + 2^{-1}\nu^{-1}(4\delta_e + 4\tau^{-1} + 2(C_e^{\mathrm{tr}})^2 h_e^{-1}))^{1/2}\boldsymbol{T}_2 \right\|_{\mathcal{I}_h^2}^2,$$

Using assumption (A.9), we have the explicit estimate

$$\begin{aligned}
\frac{1}{16}\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)^2 \leq & \left(1 + 2C_2\nu^2\left(h_1^{2\min\{1,k\}} + h_2^{2\min\{1,k\}}\right)\right)\sum_{i=1}^{2}\|\mathbb{H}\|_{\Omega_h^i}^2 \\
& + \left(\frac{2}{\nu^2} + 4C_2 + 8C_3\right)\|\boldsymbol{f}\|_{\Omega}^2 \\
& + \left(\frac{4}{\nu^2}\max_{e\in\mathcal{I}_h^2}\delta_e^{-1}h_e^{-1} + 4\tau + 4C_2\right)\|\boldsymbol{g}\|_{\Gamma} \\
& + \left\| (2C_2^{1/2} + 2^{-1}\nu^{-1}(\nu\tau^{1/2} + \delta^{-1}8\nu(C_e^{\mathrm{tr}})^2 h_e^{-1}))^{1/2}\boldsymbol{T}_1 \right\|_{\mathcal{I}_h^1}^2 \\
& + \left\| (2C_2^{1/2} + 2^{-1}\nu^{-1}(4\delta_e + 4\tau^{-1} + 2(C_e^{\mathrm{tr}})^2 h_e^{-1}))^{1/2}\boldsymbol{T}_2 \right\|_{\mathcal{I}_h^2}^2. \qquad (3.3.21)
\end{aligned}$$

3.3.4 Proof of Theorem 1

It suffices to note that (3.2.3) corresponds to (3.3.21), (3.2.4) to Lemma 7, and (3.2.5) to Lemma 8. □

3.3.5 Proof of the error estimates

Let $i \in \{1, 2\}$. We define

$$\begin{aligned}
&E^{\mathbb{L}^i} := \Pi_{\mathbb{V}^i}\mathbb{L} - \mathbb{L}_h^i, \quad \boldsymbol{\varepsilon}^{\boldsymbol{u}^i} := \Pi_{\boldsymbol{V}^i}\boldsymbol{u} - \boldsymbol{u}_h^i, \quad \varepsilon^{p^i} := \Pi_{V^i}\tilde{p} - \tilde{p}_h^i \\
&\boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}^i} := \mathrm{P}_{\boldsymbol{M}^i}\boldsymbol{u} - \widehat{\boldsymbol{u}}_h^i, \quad \boldsymbol{\varepsilon}^{\widehat{\sigma}^i}\boldsymbol{n}^i := \mathrm{P}_{\boldsymbol{M}^i}\left(\nu\mathbb{L}\boldsymbol{n}^i - \tilde{p}\boldsymbol{n}^i\right) - \widehat{\sigma}_h^i\boldsymbol{n}^i
\end{aligned}$$

where we recall that $(\Pi_{\mathbb{V}^i}\mathbb{L}, \Pi_{\boldsymbol{V}^i}\boldsymbol{u}, \Pi_{V^i}\tilde{p})$ is the HDG projection of $(\mathbb{L}, \boldsymbol{u}, \tilde{p})$ and $\mathrm{P}_{\boldsymbol{M}^i}$ is the $\boldsymbol{L}^2$ projection onto $\boldsymbol{M}_h^i$.

Moreover, for $\boldsymbol{x}^1 \in \mathcal{I}_h^1$, let

$$\varepsilon^{\tilde{u}^2}(\boldsymbol{x}^1) = \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}^2}(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \int_0^1 \boldsymbol{E}_{E^{\mathbb{L}^2}}(\boldsymbol{x}(s))\boldsymbol{n}^2 \mathrm{d}s$$

and for $\boldsymbol{x}^2 \in \mathcal{I}_h^2$, let

$$\varepsilon^{\tilde{\sigma}^1}(\boldsymbol{x}^2) = -\nu \boldsymbol{E}_{E^{\mathbb{L}^1}}(\boldsymbol{x}^2)\boldsymbol{n}^2 + \boldsymbol{E}_{\varepsilon^{p^1}}(\boldsymbol{x}^2)\boldsymbol{n}^2 - \tau\nu(\boldsymbol{\varepsilon}^{\boldsymbol{u}^1} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}^1})(\boldsymbol{\varphi}(\boldsymbol{x}^2)).$$

By Lemma 3.1 in [17], it follows that, for $i \in \{1,2\}$, these projection of the errors satisfy

$$(E_h^{\mathbb{L}^i}, \mathbb{G})_{\Omega_h^i} + (\boldsymbol{\varepsilon}^{\boldsymbol{u}^i}, \nabla\cdot\mathbb{G})_{\Omega_h^i} - \langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}^i}, \mathbb{G}\boldsymbol{n}\rangle_{\partial\Omega_h^i} = -(\mathbb{L} - \Pi_{\mathbb{V}^i}\mathbb{L}, \mathbb{G})_{\Omega_h^i}, \tag{3.3.22a}$$

$$(\nu E_h^{\mathbb{L}^i}, \nabla \boldsymbol{v})_{\Omega_h^i} - (\varepsilon^{p^i}, \nabla\cdot\boldsymbol{v})_{\Omega_h^i} - \langle \widehat{\sigma}_h^i \boldsymbol{n}^i, \boldsymbol{v}\rangle_{\partial\Omega_h^i} = 0, \tag{3.3.22b}$$

$$-(\boldsymbol{\varepsilon}^{\boldsymbol{u}^i}, \nabla q)_{\Omega_h^i} + \langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}^i}\cdot\boldsymbol{n}, q\rangle_{\partial\Omega_h^i} = 0, \tag{3.3.22c}$$

$$\langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}^i}, \boldsymbol{\mu}\rangle_{\Gamma_h^i\setminus\mathcal{I}_h^i} = 0, \tag{3.3.22d}$$

$$\langle \boldsymbol{\varepsilon}^{\widehat{\sigma}^i}, \boldsymbol{\mu}\rangle_{\partial\Omega_h^i\setminus\Gamma_h^i} = 0, \tag{3.3.22e}$$

where

$$\boldsymbol{\varepsilon}^{\widehat{\sigma}^i}\boldsymbol{n}^i = \nu E^{\mathbb{L}^i}\boldsymbol{n}^i - \varepsilon^{p^i}\boldsymbol{n}^i - \tau\nu(\boldsymbol{\varepsilon}^{\boldsymbol{u}^i} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}^i}) \qquad \text{on } \partial\Omega_h^i. \tag{3.3.22f}$$

Furthermore, the uniqueness conditions reads

$$(\varepsilon^{p^1}, 1)_{\Omega_h^1} + (\varepsilon^{p^2}, 1)_{\Omega_h^2} = (\Pi_{V^1}\tilde{p} - \tilde{p})_{\Omega_h^1} + (\Pi_{V^2}\tilde{p} - \tilde{p})_{\Omega_h^2}, \tag{3.3.22g}$$

which, for $k > 0$, is simply

$$(\varepsilon^{p^1}, 1)_{\Omega_h^1} + (\varepsilon^{p^2}, 1)_{\Omega_h^2} = 0.$$

For the analogue of the transmission conditions, we have that, for $\boldsymbol{x}^1 = \boldsymbol{\varphi}(\boldsymbol{x}^2)$,

$$\begin{aligned}
\varepsilon^{\tilde{u}^2}(\boldsymbol{x}^1) &= \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}^2}(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \int_0^1 \boldsymbol{E}_{E^{\mathbb{L}^2}}(\boldsymbol{x}(s))\boldsymbol{n}^2 \mathrm{d}s \\
&= (\mathrm{P}_{\boldsymbol{M}^2}\boldsymbol{u} - \widehat{\boldsymbol{u}}_h^2)(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \int_0^1 \boldsymbol{E}_{\Pi_{\mathbb{V}^2}\mathbb{L}^2}(\boldsymbol{x}(s))\boldsymbol{n}^2 \mathrm{d}s - \left|\ell(\boldsymbol{x}^2)\right| \int_0^1 \boldsymbol{E}_{\mathbb{L}_h^2}(\boldsymbol{x}(s))\boldsymbol{n}^2 \mathrm{d}s
\end{aligned}$$

$$= \mathrm{P}_{\boldsymbol{M}^2}\boldsymbol{u}(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \int_0^1 \boldsymbol{E}_{\Pi_{\mathbb{V}^2}\mathbb{L}^2}(\boldsymbol{x}(s))\boldsymbol{n}^2 \mathrm{d}s - \tilde{u}_h^2(\boldsymbol{x}^1)$$
$$= \mathrm{P}_{\boldsymbol{M}^2}\boldsymbol{u}(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \int_0^1 \Pi_{\mathbb{V}^2}\mathbb{L}^2(\boldsymbol{x}(s))\boldsymbol{n}^2 \mathrm{d}s - \tilde{u}_h^2(\boldsymbol{x}^1)$$

where we used (2.2.1j) and the definition of the extension operator. Furthermore, since

$$\boldsymbol{u}(\boldsymbol{x}^1) = \boldsymbol{u}(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \int_0^1 \mathbb{L}(\boldsymbol{x}(s))\boldsymbol{n}^2 \mathrm{d}s,$$

we have that

$$\begin{aligned}
\boldsymbol{u}(\boldsymbol{x}^1) - \varepsilon^{\tilde{u}^2}(\boldsymbol{x}^1) &= (\boldsymbol{u} - \mathrm{P}_{\boldsymbol{M}^2}\boldsymbol{u})(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \int_0^1 (\mathbb{L} - \Pi_{\mathbb{V}^2}\mathbb{L})(\boldsymbol{x}(s))\boldsymbol{n}^2 + \tilde{u}_h^2(\boldsymbol{x}^1) \\
&= (\boldsymbol{u} - \mathrm{P}_{\boldsymbol{M}^2}\boldsymbol{u})(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| \Lambda_{(\mathbb{L}-\Pi_{\mathbb{V}^2}\mathbb{L})}(\boldsymbol{x}^2) + \left|\ell(\boldsymbol{x}^2)\right| (\mathbb{L} - \Pi_{\mathbb{V}^2}\mathbb{L})(\boldsymbol{x}^1) \\
&\quad + \tilde{u}_h^2(\boldsymbol{x}^1)
\end{aligned}$$

Thus, if we take $\boldsymbol{\mu} \in \boldsymbol{M}_h^1$, we have

$$\begin{aligned}
\langle \mathrm{P}_{\boldsymbol{M}_h^1}\boldsymbol{u} - \varepsilon^{\tilde{u}^2}, \boldsymbol{\mu}\rangle_{\mathcal{I}_h^1} &= \langle (\boldsymbol{u} - \mathrm{P}_{\boldsymbol{M}^2}\boldsymbol{u}) \circ \boldsymbol{\varphi}^{-1} + |\ell|\, \Lambda_{(\mathbb{L}-\Pi_{\mathbb{V}^2}\mathbb{L})} + |\ell|\, (\mathbb{L} - \Pi_{\mathbb{V}^2}\mathbb{L}), \boldsymbol{\mu}\rangle_{\mathcal{I}_h^1} + \langle \tilde{u}_h^2, \boldsymbol{\mu}\rangle_{\mathcal{I}_h^1} \\
&= \langle (\boldsymbol{u} - \mathrm{P}_{\boldsymbol{M}^2}\boldsymbol{u}) \circ \boldsymbol{\varphi}^{-1} + |\ell|\, \Lambda_{(\mathbb{L}-\Pi_{\mathbb{V}^2}\mathbb{L})} + |\ell|\, (\mathbb{L} - \Pi_{\mathbb{V}^2}\mathbb{L}), \boldsymbol{\mu}\rangle_{\mathcal{I}_h^1} + \langle \hat{\boldsymbol{u}}_h^1, \boldsymbol{\mu}\rangle_{\mathcal{I}_h^1}
\end{aligned}$$

Note that, since our analysis is free of hanging nodes, $\boldsymbol{\mu} \circ \boldsymbol{\varphi} \in \boldsymbol{M}_h^2$ and so

$$\langle (\boldsymbol{u} - \mathrm{P}_{\boldsymbol{M}^2}\boldsymbol{u}) \circ \boldsymbol{\varphi}^{-1}, \boldsymbol{\mu}\rangle_{\mathcal{I}_h^1} = \langle \boldsymbol{u} - \mathrm{P}_{\boldsymbol{M}^2}\boldsymbol{u}, \boldsymbol{\mu} \circ \boldsymbol{\varphi}\rangle_{\mathcal{I}_h^2} = 0.$$

Substracting the last term of the right-hand side, we obtain

$$\langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}^1} - \varepsilon^{\tilde{u}^2}, \boldsymbol{\mu}\rangle_{\mathcal{I}_h^1} = \langle |\ell|\, \Lambda_{(\mathbb{L}-\Pi_{\mathbb{V}^2}\mathbb{L})} + |\ell|\, (\mathbb{L} - \Pi_{\mathbb{V}^2}\mathbb{L}), \boldsymbol{\mu}\rangle_{\mathcal{I}_h^1},$$

which is the analogue to (2.2.1h).

On the other hand, following the proof of [42] with $\sigma := \nu\mathbb{L} - \tilde{p}\boldsymbol{I}$ taking the role of $\boldsymbol{q}$, we arrive at

$$\langle \boldsymbol{\varepsilon}^{\widehat{\sigma}^2}\boldsymbol{n}^2 + \varepsilon^{\tilde{\sigma}^1}, \boldsymbol{\mu}\rangle_{\mathcal{I}_h^2} = \langle \nu\left[(\mathbb{L} - \Pi_{\mathbb{V}^1}\mathbb{L}) - (\mathbb{L} - \Pi_{\mathbb{V}^1}\mathbb{L})\circ\boldsymbol{\varphi}\right]\boldsymbol{n}^1\rangle_{\mathcal{I}_h^2} - \langle[(\tilde{p} - \Pi_{V^1}\tilde{p}) - (\tilde{p} - \Pi_{V^1}\tilde{p})\circ\boldsymbol{\varphi}]\,\boldsymbol{n}^1\rangle_{\mathcal{I}_h^2} \tag{3.3.22h}$$

which is the analogue to (2.2.1i).

The equations that the projection of the errors satisfy are similar to those of the HDG scheme, where $(\mathbb{L} - \Pi_{\mathbb{V}^i})$ plays the role of $\mathbb{H}^i$ and $\mathbf{0}$ plays the role of $\boldsymbol{f}$ and $\boldsymbol{g}$. For the transmission conditions, we see that

$$\boldsymbol{T}_1 = |\ell|\,\Lambda_{\boldsymbol{I}^{\mathbb{L}^2}} + |\ell|\,\boldsymbol{I}^{\mathbb{L}^2}$$

and

$$\boldsymbol{T}_2 = \left\{(\nu\boldsymbol{I}^{\mathbb{L}^1} - \boldsymbol{I}^{p^1}) - (\nu\boldsymbol{I}^{\mathbb{L}^1} - \boldsymbol{I}^{p^1})\circ\boldsymbol{\varphi}\right\}\boldsymbol{n}^1,$$

in this context, where $\boldsymbol{I}^{\mathbb{L}^1} := \mathbb{L} - \Pi_{\mathbb{V}^1}\mathbb{L}$, $\boldsymbol{I}^{\mathbb{L}^2} := \mathbb{L} - \Pi_{\mathbb{V}^2}\mathbb{L}$ and $\boldsymbol{I}^{p^1} := \tilde{p} - \Pi_{V^1}\tilde{p}$.

To apply our previous estimates, we notice that

$$\begin{aligned}\left\|h_e^{-1/2}\left(|\ell|\,\Lambda_{\boldsymbol{I}^{\mathbb{L}^2}} + |\ell|\,\boldsymbol{I}^{\mathbb{L}^2}\right)\right\|_{\mathcal{I}_h^1}^2 &= \left\|h_e^{-1/2}\,|\ell|\,\Lambda_{\boldsymbol{I}^{\mathbb{L}^2}}\right\|_{\mathcal{I}_h^1}^2 + \left\|h_e^{-1/2}\,|\ell|\,\boldsymbol{I}^{\mathbb{L}^2}\right\|_{\mathcal{I}_h^1}^2 \\ &\lesssim \delta^4 h_1^{-4}\left\|\boldsymbol{I}^{\mathbb{L}^2}\right\|_{\Omega_h^1}^2 + \delta h_1^{-1}\left\|\boldsymbol{I}^{\mathbb{L}^2}\right\|_{\Omega_h^1}^2\end{aligned}$$

and

$$\begin{aligned}\left\|h_e^{-1/2}\left\{(\nu\boldsymbol{I}^{\mathbb{L}^1} - \boldsymbol{I}^{p^1}) - (\nu\boldsymbol{I}^{\mathbb{L}^1} - \boldsymbol{I}^{p^1})\circ\boldsymbol{\varphi}\right\}\boldsymbol{n}^1\right\|_{\mathcal{I}_h^2}^2 &\lesssim \delta h_2^{-1}\left\|(\nu\boldsymbol{I}^{\mathbb{L}^1} - \boldsymbol{I}^{p^1})\right\|_{\Omega_h^2}^2 \\ &\lesssim \delta h_2^{-1}\left(\left\|\nu\boldsymbol{I}^{\mathbb{L}^1}\right\|_{\Omega_h^2}^2 + \left\|\boldsymbol{I}^{p^1}\right\|_{\Omega_h^2}^2\right).\end{aligned}$$

By our assumptions, the terms of the form δh_i^{-1} are bounded for $i \in \{1, 2\}$. Therefore, by applying Theorem 1 to the context of the projection of the errors, we have the estimate

$$\mathcal{E}(E^L, \boldsymbol{\varepsilon}^{\boldsymbol{u}}, \varepsilon^p, \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})^2 \lesssim \left\|\boldsymbol{I}^{\mathbb{L}^2}\right\|_{\Omega_h^1}^2 + \left\|\nu\boldsymbol{I}^{\mathbb{L}^1}\right\|_{\Omega_h^2}^2 + \left\|\boldsymbol{I}^{p^1}\right\|_{\Omega_h^2}^2, \tag{3.3.23}$$

where $\mathcal{E}(E^L, \boldsymbol{\varepsilon}^{\boldsymbol{u}}, \varepsilon^p, \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})$ is the analogue to $\mathcal{S}(\mathbb{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, \tilde{u}_h)$ in the context of the projection of the errors. Thus, Theorem 3 follows by (2.1.3) and the estimates from Theorem 1 applied to our context. More precisely, if $(\mathbb{L}, \boldsymbol{u}, \tilde{p}) \in \mathbb{H}^{l_\sigma+1}(\Omega_h^i) \times \boldsymbol{H}^{l_{\boldsymbol{u}}+1}(\Omega_h^i) \times H^{l_\sigma+1}(\Omega_h^i)$, for $l_\sigma, l_{\boldsymbol{u}} \in [0, k]$, we have that

$$\left\| \boldsymbol{I}^{\mathbb{L}^2} \right\|_{\Omega_h^1}^2 \lesssim \nu^{-1} h_2^{2(l_\sigma+1)} \left| \nu\mathbb{L} - \tilde{p}\mathbb{I} \right|_{\mathbb{H}^{l_\sigma+1}(\Omega)}^2 + h_2^{2(l_{\boldsymbol{u}}+1)} \left| \boldsymbol{u} \right|_{\boldsymbol{H}^{l_{\boldsymbol{u}}+1}(\Omega)}^2 ,$$

and

$$\left\| \nu \boldsymbol{I}^{\mathbb{L}^1} \right\|_{\Omega_h^2}^2 + \left\| \boldsymbol{I}^{p^1} \right\|_{\Omega_h^2}^2 \lesssim h_1^{2(l_\sigma+1)} \left| \nu\mathbb{L} - \tilde{p}\mathbb{I} \right|_{\mathbb{H}^{l_\sigma+1}(\Omega)}^2 + h_1^{2(l_{\boldsymbol{u}}+1)} \nu \left| \boldsymbol{u} \right|_{\boldsymbol{H}^{l_{\boldsymbol{u}}+1}(\Omega)}^2 ,$$

and so, recalling that we set $h = \max\{h_1, h_2\}$, it follows that

$$\mathcal{E}(E^L, \boldsymbol{\varepsilon}^{\boldsymbol{u}}, \varepsilon^p, \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})^2 \lesssim (1 + \nu^{-1}) h^{2(l_\sigma+1)} \left| \nu\mathbb{L} - \tilde{p}\mathbb{I} \right|_{\mathbb{H}^{l_\sigma+1}(\Omega)}^2 + (1 + \nu) h^{2(l_{\boldsymbol{u}}+1)} \left| \boldsymbol{u} \right|_{\boldsymbol{H}^{l_{\boldsymbol{u}}+1}(\Omega)}^2 . \quad (3.3.24)$$

Adapting the proof of Theorem 2 to our context and using (3.3.24), we obtain the upper bounds found in (3). □

CHAPTER 4

Numerical results

In this chapter we present the numerical results obtained for the HDG scheme presented in (2.2.1) and implemented in MATLAB for the two dimensional case in order to validate the theoretical results. To that end, we define

$$e_{\mathbb{L}} := \left(\sum_{i=1}^{2} \left\| \mathbb{L} - \mathbb{L}_h^i \right\|_{\Omega_h^i}^2 \right)^{1/2}, \quad e_{\boldsymbol{u}} := \left(\sum_{i=1}^{2} \left\| \boldsymbol{u} - \boldsymbol{u}_h^i \right\|_{\Omega_h^i}^2 \right)^{1/2}, \quad e_p := \left(\sum_{i=1}^{2} \left\| p - p_h^i \right\|_{\Omega_h^i}^2 \right)^{1/2}$$

$$e_{\widehat{\boldsymbol{u}}} := \left(\sum_{i=1}^{2} \| \mathrm{P}_{\boldsymbol{M}^i} \boldsymbol{u} - \widehat{\boldsymbol{u}} \|_{h,\partial\Omega_h^i}^2 \right)^{1/2}, \quad e_{\boldsymbol{u}^\star} := \left(\sum_{i=1}^{2} \left\| \boldsymbol{u} - (\boldsymbol{u}_h^\star)^i \right\|_{\Omega_h^i}^2 \right)^{1/2}.$$

Denoting by q_1 and q_2 any of the previous quantities for two consecutive meshes with N_1 and N_2 elements, respectively, we define its estimated order of convergence as

$$\text{e.o.c.} := -2 \frac{\log(q_2/q_1)}{\log(N_2/N_1)}.$$

In all of the following experiments, we will consider the exact solution

$$\boldsymbol{u} = \begin{bmatrix} \sin(\pi x)\sin(\pi y) \\ \cos(\pi x)\cos(\pi y) \end{bmatrix}, \quad p = \sin(2\pi x)\sin(2\pi y).$$

with viscosities $\nu \in \{1, 10^{-3}, 10^{-6}\}$.

4.1 Flat interface

We consider the physical domain $\Omega := (0,1)^2$ divided by the flat interface $y = 0.5$ and approximated via two subdomains $\Omega_h^1 := (0,1)\times(0.5+\delta/2, 1)$ and $\Omega_h^2 := (0,1)\times(0.5-\delta/2, 1)$, i.e. two rectangular subdomains that are a distance δ apart in the vertical sense. We note that, while our method is proposed to deal with curved interfaces, the actual shape of the physical interface is irrelevant as the estimates depend of the gap between the discretizations of the subdomains.

4.1.1 No gap

We take $\delta = 0$ and, s expected, this is the best-behaved case. In Tables 4.1, 4.2 and 4.3 we observe the expected $k+1$ convergence for the first quantities and the $k+2$ convergence of the last two, as predicted by Theorem 3.

4.1.2 Positive gap of order h

This case is not covered in the theory previously discussed, since (A.5), (A.9) and (A.10) do not hold. However, in Tables 4.4, 4.5 and 4.6 we see that we have $k+1$ convergence for the first variables in all three cases, but the convergence of $e_{\widehat{\boldsymbol{u}}}$ and $e_{\boldsymbol{u}^\star}$ turns sub-optimal. For $k=1$, we observe that it is strictly greater than $k+1$ but not quite $k+2$, while for $k=2$ and $k=3$, it completely goes down to $k+1$ as the rest of the variables.

Table 4.1: History of convergence of the HDG scheme for $\nu = 1$ and $\delta = 0$.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{\boldsymbol{u}^\star}$	e.o.c.
1	32	1.53e-01	—	8.48e-02	—	8.62e-02	—	9.02e-03	—	5.50e-03	—
	120	4.12e-02	1.99	2.43e-02	1.89	2.50e-02	1.87	1.34e-03	2.89	7.77e-04	2.96
	466	1.13e-02	1.91	6.27e-03	2.00	6.11e-03	2.08	1.87e-04	2.90	1.17e-04	2.79
	1812	2.83e-03	2.03	1.59e-03	2.02	1.53e-03	2.04	2.40e-05	3.02	1.48e-05	3.04
	7186	7.25e-04	1.98	4.03e-04	1.99	3.92e-04	1.97	3.16e-06	2.94	1.95e-06	2.95
	28794	1.82e-04	1.99	1.00e-04	2.01	9.74e-05	2.01	3.96e-07	2.99	2.48e-07	2.97
2	32	2.04e-02	—	1.20e-02	—	1.93e-02	—	1.21e-03	—	5.43e-04	—
	120	2.63e-03	3.10	1.56e-03	3.09	2.11e-03	3.35	7.43e-05	4.22	3.39e-05	4.20
	466	3.83e-04	2.84	2.22e-04	2.87	3.15e-04	2.80	5.89e-06	3.74	2.67e-06	3.75
	1812	4.94e-05	3.02	2.83e-05	3.03	4.00e-05	3.04	3.81e-07	4.03	1.76e-07	4.01
	7186	6.44e-06	2.96	3.59e-06	3.00	5.15e-06	2.98	2.55e-08	3.93	1.20e-08	3.90
	28794	8.07e-07	2.99	4.47e-07	3.00	6.49e-07	2.99	1.60e-09	3.99	7.50e-10	3.99
3	32	2.30e-03	—	1.49e-03	—	2.55e-03	—	1.27e-04	—	4.14e-05	—
	120	1.86e-04	3.81	1.17e-04	3.85	2.05e-04	3.81	5.68e-06	4.70	1.52e-06	5.00
	466	1.34e-05	3.87	7.61e-06	4.03	1.37e-05	3.99	2.07e-07	4.88	6.56e-08	4.63
	1812	8.36e-07	4.09	4.89e-07	4.04	8.54e-07	4.09	6.64e-09	5.06	2.05e-09	5.10
	7186	5.85e-08	3.86	3.24e-08	3.94	5.91e-08	3.88	2.41e-10	4.82	7.58e-11	4.79
	28794	3.65e-09	3.99	2.01e-09	4.01	3.68e-09	4.00	7.53e-12	4.99	2.38e-12	4.99

Table 4.2: History of convergence of the HDG scheme for $\nu = 10^{-3}$ and $\delta = 0$.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{\boldsymbol{u}^\star}$	e.o.c.
1	32	5.74e+01	—	4.20e+01	—	7.37e-02	—	5.83e+00	—	2.08e+00	—
	120	2.00e+01	1.59	1.28e+01	1.80	2.21e-02	1.83	1.07e+00	2.57	3.72e-01	2.60
	466	5.30e+00	1.96	3.17e+00	2.05	5.43e-03	2.07	1.39e-01	3.01	5.44e-02	2.84
	1812	1.36e+00	2.00	8.15e-01	2.00	1.37e-03	2.02	1.81e-02	3.00	6.83e-03	3.06
	7186	3.56e-01	1.95	2.06e-01	2.00	3.54e-04	1.97	2.40e-03	2.93	9.38e-04	2.88
	28794	8.90e-02	2.00	5.10e-02	2.01	8.80e-05	2.01	3.00e-04	3.00	1.17e-04	2.99
2	32	1.41e+01	—	9.04e+00	—	1.87e-02	—	1.12e+00	—	3.47e-01	—
	120	1.78e+00	3.13	1.15e+00	3.12	2.04e-03	3.36	7.04e-02	4.18	2.03e-02	4.30
	466	2.77e-01	2.74	1.71e-01	2.82	3.07e-04	2.79	5.50e-03	3.76	1.76e-03	3.60
	1812	3.55e-02	3.03	2.17e-02	3.04	3.90e-05	3.04	3.56e-04	4.03	1.17e-04	4.00
	7186	4.61e-03	2.96	2.73e-03	3.01	5.01e-06	2.98	2.37e-05	3.93	7.82e-06	3.92
	28794	5.83e-04	2.98	3.41e-04	3.00	6.31e-07	2.98	1.49e-06	3.98	4.95e-07	3.98
3	32	1.91e+00	—	1.35e+00	—	2.53e-03	—	1.21e-01	—	2.93e-02	—
	120	1.68e-01	3.67	1.09e-01	3.81	2.04e-04	3.81	5.65e-03	4.64	1.24e-03	4.79
	466	1.19e-02	3.90	6.96e-03	4.05	1.36e-05	3.99	2.03e-04	4.90	5.41e-05	4.62
	1812	7.45e-04	4.08	4.49e-04	4.04	8.49e-07	4.09	6.55e-06	5.06	1.68e-06	5.12
	7186	5.27e-05	3.85	2.98e-05	3.94	5.87e-08	3.88	2.37e-07	4.82	6.37e-08	4.75
	28794	3.28e-06	4.00	1.84e-06	4.01	3.66e-09	4.00	7.43e-09	4.99	1.99e-09	4.99

Table 4.3: History of convergence of the HDG scheme for $\nu = 10^{-6}$ and $\delta = 0$.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{\boldsymbol{u}^\star}$	e.o.c.
1	32	5.74e+04	—	4.20e+04	—	7.38e-02	—	5.83e+03	—	2.08e+03	—
	120	2.00e+04	1.59	1.28e+04	1.80	2.21e-02	1.83	1.07e+03	2.57	3.72e+02	2.60
	466	5.30e+03	1.96	3.17e+03	2.05	5.43e-03	2.07	1.39e+02	3.01	5.44e+01	2.84
	1812	1.36e+03	2.00	8.15e+02	2.00	1.37e-03	2.02	1.81e+01	3.00	6.83e+00	3.06
	7186	3.56e+02	1.95	2.06e+02	2.00	3.54e-04	1.97	2.40e+00	2.93	9.38e-01	2.88
	28794	8.90e+01	2.00	5.10e+01	2.01	8.80e-05	2.01	3.00e-01	3.00	1.17e-01	2.99
2	32	1.41e+04	—	9.04e+03	—	1.87e-02	—	1.12e+03	—	3.47e+02	—
	120	1.78e+03	3.13	1.15e+03	3.12	2.04e-03	3.36	7.04e+01	4.18	2.03e+01	4.30
	466	2.77e+02	2.74	1.71e+02	2.82	3.07e-04	2.79	5.50e+00	3.76	1.76e+00	3.60
	1812	3.55e+01	3.03	2.17e+01	3.04	3.90e-05	3.04	3.56e-01	4.03	1.17e-01	4.00
	7186	4.61e+00	2.96	2.73e+00	3.01	5.01e-06	2.98	2.37e-02	3.93	7.82e-03	3.92
	28794	5.83e-01	2.98	3.41e-01	3.00	6.31e-07	2.98	1.49e-03	3.98	4.95e-04	3.98
3	32	1.91e+03	—	1.35e+03	—	2.53e-03	—	1.21e+02	—	2.93e+01	—
	120	1.68e+02	3.67	1.09e+02	3.81	2.04e-04	3.81	5.65e+00	4.64	1.24e+00	4.79
	466	1.19e+01	3.90	6.96e+00	4.05	1.36e-05	3.99	2.03e-01	4.90	5.41e-02	4.62
	1812	7.45e-01	4.08	4.49e-01	4.04	8.49e-07	4.09	6.55e-03	5.06	1.68e-03	5.12
	7186	5.27e-02	3.85	2.98e-02	3.94	5.87e-08	3.88	2.37e-04	4.82	6.37e-05	4.75
	28794	3.28e-03	4.00	1.84e-03	4.01	3.66e-09	4.00	7.43e-06	4.99	1.99e-06	4.99

Table 4.4: History of convergence of the HDG scheme for $\nu = 1$ and $\delta = \mathcal{O}(h)$.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{\boldsymbol{u}^\star}$	e.o.c.
1	32	1.43e-01	—	8.14e-02	—	7.85e-02	—	9.02e-03	—	5.46e-03	—
	120	4.10e-02	1.89	2.37e-02	1.87	2.44e-02	1.77	1.44e-03	2.77	8.40e-04	2.83
	468	1.12e-02	1.91	6.20e-03	1.97	6.14e-03	2.03	2.11e-04	2.83	1.28e-04	2.76
	1824	2.82e-03	2.02	1.58e-03	2.02	1.54e-03	2.04	3.70e-05	2.56	2.06e-05	2.69
	7204	7.25e-04	1.98	4.02e-04	1.99	3.98e-04	1.97	8.93e-06	2.07	4.67e-06	2.16
	28888	1.81e-04	2.00	9.95e-05	2.01	9.86e-05	2.01	1.95e-06	2.19	1.01e-06	2.21
2	32	2.24e-02	—	1.16e-02	—	2.48e-02	—	2.28e-03	—	1.03e-03	—
	120	2.82e-03	3.14	1.54e-03	3.06	2.72e-03	3.34	2.31e-04	3.46	1.11e-04	3.36
	468	3.93e-04	2.90	2.18e-04	2.87	3.57e-04	2.98	2.37e-05	3.34	1.18e-05	3.30
	1824	5.13e-05	2.99	2.80e-05	3.02	4.59e-05	3.02	2.95e-06	3.07	1.48e-06	3.05
	7204	6.59e-06	2.99	3.57e-06	3.00	5.94e-06	2.98	4.08e-07	2.88	2.06e-07	2.87
	28888	8.20e-07	3.00	4.45e-07	3.00	7.17e-07	3.05	4.32e-08	3.23	2.19e-08	3.23
3	32	2.52e-03	—	1.26e-03	—	2.73e-03	—	3.91e-04	—	1.64e-04	—
	120	1.93e-04	3.89	1.12e-04	3.67	2.08e-04	3.89	9.84e-06	5.57	3.88e-06	5.67
	468	1.39e-05	3.87	7.63e-06	3.94	1.43e-05	3.94	3.74e-07	4.80	1.58e-07	4.70
	1824	8.73e-07	4.07	4.87e-07	4.04	8.96e-07	4.07	2.19e-08	4.18	1.05e-08	3.99
	7204	5.88e-08	3.93	3.22e-08	3.96	6.02e-08	3.93	1.20e-09	4.22	5.95e-10	4.18
	28888	3.72e-09	3.98	2.01e-09	4.00	3.80e-09	3.98	8.04e-11	3.90	4.06e-11	3.87

Table 4.5: History of convergence of the HDG scheme for $\nu = 10^{-3}$ and $\delta = \mathcal{O}(h)$.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{\boldsymbol{u}^\star}$	e.o.c.
1	32	5.35e+01	—	3.86e+01	—	6.52e-02	—	5.92e+00	—	2.50e+00	—
	120	1.99e+01	1.50	1.25e+01	1.71	2.15e-02	1.68	1.12e+00	2.52	4.12e-01	2.73
	468	5.32e+00	1.94	3.16e+00	2.02	5.43e-03	2.02	1.49e-01	2.97	6.13e-02	2.80
	1824	1.36e+00	2.01	8.09e-01	2.00	1.37e-03	2.03	2.01e-02	2.94	8.04e-03	2.99
	7204	3.58e-01	1.94	2.05e-01	2.00	3.55e-04	1.97	3.11e-03	2.72	1.35e-03	2.60
	28888	8.92e-02	2.00	5.10e-02	2.01	8.82e-05	2.00	5.15e-04	2.59	2.42e-04	2.48
2	32	1.61e+01	—	9.02e+00	—	2.08e-02	—	1.67e+00	—	6.41e-01	—
	120	1.99e+00	3.16	1.15e+00	3.12	2.42e-03	3.25	1.79e-01	3.37	8.27e-02	3.10
	468	2.87e-01	2.84	1.67e-01	2.83	3.31e-04	2.92	1.97e-02	3.24	9.66e-03	3.15
	1824	3.73e-02	3.00	2.15e-02	3.02	4.27e-05	3.01	2.49e-03	3.04	1.25e-03	3.00
	7204	4.78e-03	2.99	2.72e-03	3.01	5.55e-06	2.97	3.58e-04	2.83	1.81e-04	2.81
	28888	5.92e-04	3.01	3.38e-04	3.00	6.75e-07	3.03	3.80e-05	3.23	1.93e-05	3.23
3	32	2.25e+00	—	1.12e+00	—	2.84e-03	—	4.12e-01	—	1.74e-01	—
	120	1.76e-01	3.86	1.04e-01	3.60	2.10e-04	3.94	1.20e-02	5.34	5.05e-03	5.35
	468	1.25e-02	3.88	7.01e-03	3.96	1.44e-05	3.94	5.08e-04	4.65	2.28e-04	4.55
	1824	7.83e-04	4.08	4.47e-04	4.05	8.93e-07	4.09	2.40e-05	4.49	1.15e-05	4.40
	7204	5.32e-05	3.92	2.96e-05	3.95	6.00e-08	3.93	1.27e-06	4.28	6.27e-07	4.23
	28888	3.36e-06	3.98	1.85e-06	3.99	3.77e-09	3.98	7.04e-08	4.17	3.53e-08	4.14

Table 4.6: History of convergence of the HDG scheme for $\nu = 10^{-6}$ and $\delta = \mathcal{O}(h)$.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{\boldsymbol{u}^\star}$	e.o.c.
1	32	5.35e+04	—	3.86e+04	—	6.52e-02	—	5.92e+03	—	2.50e+03	—
	120	1.99e+04	1.50	1.25e+04	1.71	2.15e-02	1.68	1.12e+03	2.52	4.12e+02	2.73
	468	5.32e+03	1.94	3.16e+03	2.02	5.43e-03	2.02	1.49e+02	2.97	6.13e+01	2.80
	1824	1.36e+03	2.01	8.09e+02	2.00	1.37e-03	2.03	2.01e+01	2.94	8.05e+00	2.99
	7204	3.58e+02	1.94	2.05e+02	2.00	3.55e-04	1.97	3.12e+00	2.71	1.35e+00	2.59
	28888	8.92e+01	2.00	5.10e+01	2.01	8.82e-05	2.00	5.16e-01	2.59	2.42e-01	2.48
2	32	1.61e+04	—	9.02e+03	—	2.08e-02	—	1.67e+03	—	6.41e+02	—
	120	1.99e+03	3.16	1.15e+03	3.12	2.42e-03	3.25	1.79e+02	3.37	8.26e+01	3.10
	468	2.87e+02	2.84	1.67e+02	2.83	3.31e-04	2.92	1.97e+01	3.24	9.66e+00	3.15
	1824	3.73e+01	3.00	2.15e+01	3.02	4.27e-05	3.01	2.49e+00	3.04	1.25e+00	3.00
	7204	4.78e+00	2.99	2.72e+00	3.01	5.55e-06	2.97	3.58e-01	2.83	1.81e-01	2.81
	28888	5.92e-01	3.01	3.38e-01	3.00	6.75e-07	3.03	3.80e-02	3.23	1.93e-02	3.23
3	32	2.25e+03	—	1.12e+03	—	2.84e-03	—	4.12e+02	—	1.74e+02	—
	120	1.76e+02	3.86	1.04e+02	3.60	2.10e-04	3.94	1.20e+01	5.34	5.05e+00	5.35
	468	1.25e+01	3.88	7.01e+00	3.96	1.44e-05	3.94	5.09e-01	4.65	2.29e-01	4.55
	1824	7.83e-01	4.08	4.47e-01	4.05	8.93e-07	4.09	2.41e-02	4.49	1.15e-02	4.40
	7204	5.32e-02	3.92	2.96e-02	3.95	6.00e-08	3.93	1.27e-03	4.28	6.28e-04	4.23
	28888	3.36e-03	3.98	1.85e-03	3.99	3.77e-09	3.98	7.04e-05	4.17	3.53e-05	4.14

4.1.3 Positive gap of order h^2

In Tables 4.7, 4.8 and 4.9 we observe that the theoretical results are validated as the gap being of order h^2 recovers all the superconvergent quantities of the method, much like in the no gap case.

Table 4.7: History of convergence of the HDG scheme for $\nu = 1$ and $\delta = \mathcal{O}(h^2)$.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{\boldsymbol{u}^\star}$	e.o.c.
1	32	1.47e-01	—	8.41e-02	—	7.84e-02	—	8.35e-03	—	5.35e-03	—
	120	4.11e-02	1.93	2.43e-02	1.88	2.42e-02	1.78	1.31e-03	2.81	7.73e-04	2.93
	462	1.14e-02	1.90	6.35e-03	1.99	6.19e-03	2.02	1.92e-04	2.85	1.20e-04	2.77
	1816	2.83e-03	2.04	1.59e-03	2.03	1.53e-03	2.05	2.39e-05	3.05	1.47e-05	3.06
	7170	7.33e-04	1.97	4.06e-04	1.99	3.94e-04	1.97	3.20e-06	2.93	1.99e-06	2.92
	28832	1.81e-04	2.01	9.99e-05	2.01	9.75e-05	2.01	3.97e-07	3.00	2.47e-07	3.00
2	32	2.12e-02	—	1.23e-02	—	2.12e-02	—	1.43e-03	—	6.13e-04	—
	120	2.67e-03	3.14	1.59e-03	3.09	2.23e-03	3.41	8.21e-05	4.32	3.72e-05	4.24
	462	3.86e-04	2.87	2.24e-04	2.90	3.20e-04	2.88	6.16e-06	3.84	2.80e-06	3.84
	1816	4.88e-05	3.02	2.81e-05	3.04	3.95e-05	3.06	3.84e-07	4.05	1.79e-07	4.02
	7170	6.42e-06	2.95	3.59e-06	3.00	5.15e-06	2.97	2.63e-08	3.91	1.23e-08	3.89
	28832	8.11e-07	2.97	4.48e-07	2.99	6.48e-07	2.98	1.64e-09	3.99	7.73e-10	3.98
3	32	2.03e-03	—	1.34e-03	—	2.09e-03	—	1.45e-04	—	5.51e-05	—
	120	1.79e-04	3.67	1.15e-04	3.72	1.97e-04	3.58	5.50e-06	4.95	1.53e-06	5.42
	462	1.38e-05	3.81	7.81e-06	3.99	1.41e-05	3.91	2.16e-07	4.81	6.76e-08	4.63
	1816	8.33e-07	4.10	4.88e-07	4.05	8.53e-07	4.10	6.64e-09	5.09	2.05e-09	5.11
	7170	5.88e-08	3.86	3.27e-08	3.94	5.94e-08	3.88	2.43e-10	4.82	7.67e-11	4.78
	28832	3.67e-09	3.99	2.01e-09	4.01	3.70e-09	3.99	7.62e-12	4.98	2.41e-12	4.97

Furthermore, in order to better illustrate the performance of the method, in Figures 4.1, 4.2 and 4.3, we observe the plot of the approximations of the first component of $\boldsymbol{u}$, the pressure field and the first component of the post-processed velocity for two different meshes, $\nu = 1$, and $k = 1, 2, 3$.

Table 4.8: History of convergence of the HDG scheme for $\nu = 10^{-3}$ and $\delta = \mathcal{O}(h^2)$.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{\boldsymbol{u}^\star}$	e.o.c.
1	32	5.25e+01	—	3.96e+01	—	6.45e-02	—	5.21e+00	—	2.00e+00	—
	120	1.96e+01	1.49	1.27e+01	1.72	2.15e-02	1.66	1.04e+00	2.44	3.58e-01	2.60
	462	5.37e+00	1.92	3.20e+00	2.04	5.51e-03	2.02	1.43e-01	2.94	5.53e-02	2.77
	1816	1.36e+00	2.00	8.15e-01	2.00	1.38e-03	2.03	1.80e-02	3.03	6.85e-03	3.05
	7170	3.58e-01	1.95	2.07e-01	2.00	3.56e-04	1.97	2.42e-03	2.92	9.38e-04	2.90
	28832	8.91e-02	2.00	5.11e-02	2.01	8.81e-05	2.01	3.01e-04	3.00	1.18e-04	2.98
2	32	1.53e+01	—	9.51e+00	—	2.03e-02	—	1.29e+00	—	4.12e-01	—
	120	1.85e+00	3.20	1.19e+00	3.15	2.16e-03	3.39	7.59e-02	4.28	2.39e-02	4.31
	462	2.80e-01	2.80	1.73e-01	2.86	3.11e-04	2.87	5.71e-03	3.84	1.89e-03	3.77
	1816	3.49e-02	3.04	2.15e-02	3.05	3.84e-05	3.06	3.55e-04	4.06	1.20e-04	4.03
	7170	4.60e-03	2.95	2.74e-03	3.00	5.01e-06	2.97	2.42e-05	3.91	8.24e-06	3.90
	28832	5.83e-04	2.97	3.41e-04	2.99	6.31e-07	2.98	1.52e-06	3.98	5.12e-07	3.99
3	32	1.63e+00	—	1.19e+00	—	2.08e-03	—	1.43e-01	—	5.00e-02	—
	120	1.61e-01	3.50	1.06e-01	3.66	1.96e-04	3.57	5.52e-03	4.92	1.29e-03	5.53
	462	1.22e-02	3.82	7.14e-03	4.01	1.40e-05	3.91	2.13e-04	4.83	5.57e-05	4.66
	1816	7.44e-04	4.09	4.49e-04	4.04	8.48e-07	4.10	6.55e-06	5.08	1.70e-06	5.10
	7170	5.27e-05	3.86	3.00e-05	3.94	5.90e-08	3.88	2.39e-07	4.82	6.38e-08	4.78
	28832	3.30e-06	3.98	1.85e-06	4.00	3.68e-09	3.99	7.50e-09	4.98	2.03e-09	4.96

Table 4.9: History of convergence o the HDG scheme for $\nu = 10^{-6}$ and $\delta = \mathcal{O}(h^2)$.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{\boldsymbol{u}^\star}$	e.o.c.
1	32	5.25e+04	—	3.96e+04	—	6.45e-02	—	5.21e+03	—	2.00e+03	—
	120	1.96e+04	1.49	1.27e+04	1.72	2.15e-02	1.66	1.04e+03	2.44	3.58e+02	2.60
	462	5.37e+03	1.92	3.20e+03	2.04	5.51e-03	2.02	1.43e+02	2.94	5.53e+01	2.77
	1816	1.36e+03	2.00	8.15e+02	2.00	1.38e-03	2.03	1.80e+01	3.03	6.85e+00	3.05
	7170	3.58e+02	1.95	2.07e+02	2.00	3.56e-04	1.97	2.42e+00	2.92	9.38e-01	2.90
	28832	8.91e+01	2.00	5.11e+01	2.01	8.81e-05	2.01	3.01e-01	3.00	1.18e-01	2.98
2	32	1.53e+04	—	9.51e+03	—	2.03e-02	—	1.29e+03	—	4.12e+02	—
	120	1.85e+03	3.20	1.19e+03	3.15	2.16e-03	3.39	7.59e+01	4.28	2.39e+01	4.31
	462	2.80e+02	2.80	1.73e+02	2.86	3.11e-04	2.87	5.71e+00	3.84	1.89e+00	3.77
	1816	3.49e+01	3.04	2.15e+01	3.05	3.84e-05	3.06	3.55e-01	4.06	1.20e-01	4.03
	7170	4.60e+00	2.95	2.74e+00	3.00	5.01e-06	2.97	2.42e-02	3.91	8.24e-03	3.90
	28832	5.83e-01	2.97	3.41e-01	2.99	6.31e-07	2.98	1.52e-03	3.98	5.12e-04	3.99
3	32	1.63e+03	—	1.19e+03	—	2.08e-03	—	1.43e+02	—	5.00e+01	—
	120	1.61e+02	3.50	1.06e+02	3.66	1.96e-04	3.57	5.52e+00	4.92	1.29e+00	5.53
	462	1.22e+01	3.82	7.14e+00	4.01	1.40e-05	3.91	2.13e-01	4.83	5.57e-02	4.66
	1816	7.44e-01	4.09	4.49e-01	4.04	8.48e-07	4.10	6.55e-03	5.08	1.70e-03	5.10
	7170	5.27e-02	3.86	3.00e-02	3.94	5.90e-08	3.88	2.39e-04	4.82	6.38e-05	4.78
	28832	3.30e-03	3.98	1.85e-03	4.00	3.68e-09	3.99	7.50e-06	4.98	2.03e-06	4.96

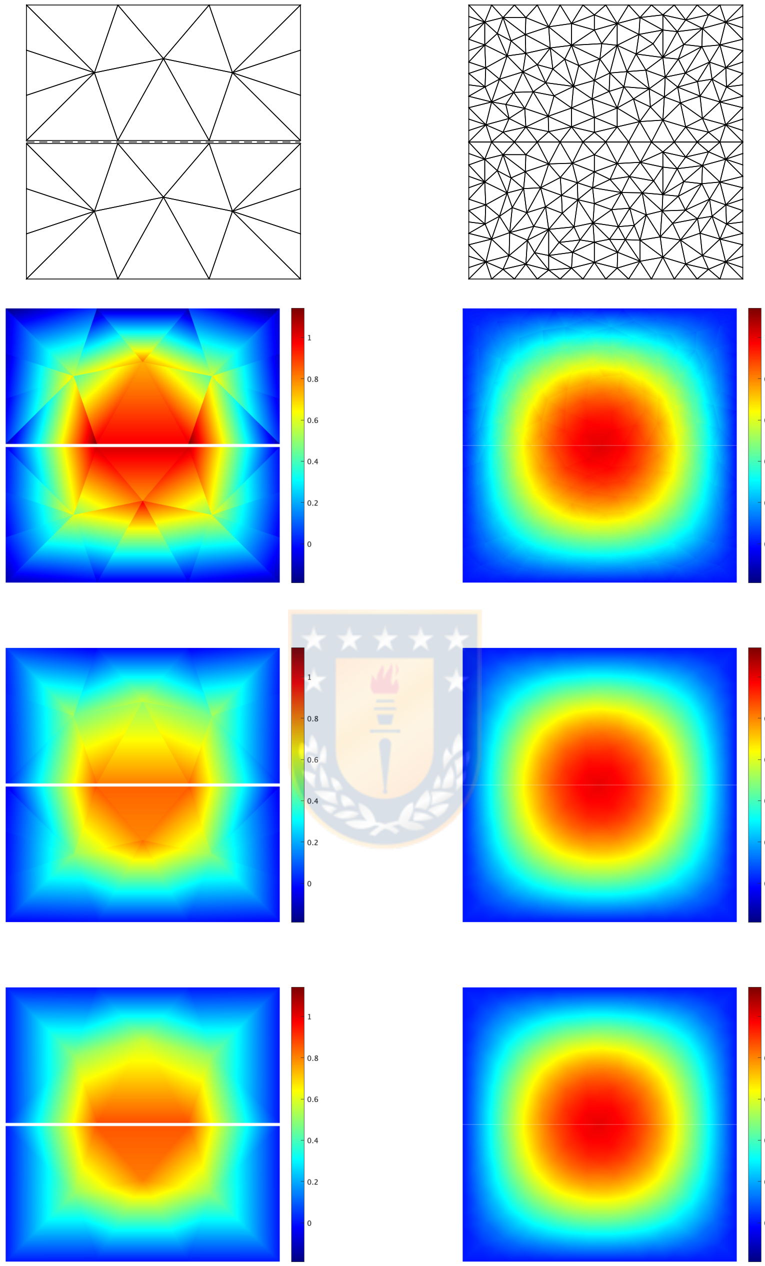

Figure 4.1: Approximation of the first component of $\boldsymbol{u}$ with $\nu = 1$ and a positive gap of order h^2. Columns: meshes with $N = 32$ and $N = 462$ elements. Topmost row illustrate the dissimilar meshes with the dashed line as the flat interface, while the rest correspond to the case $k = 1, 2, 3$, respectively.

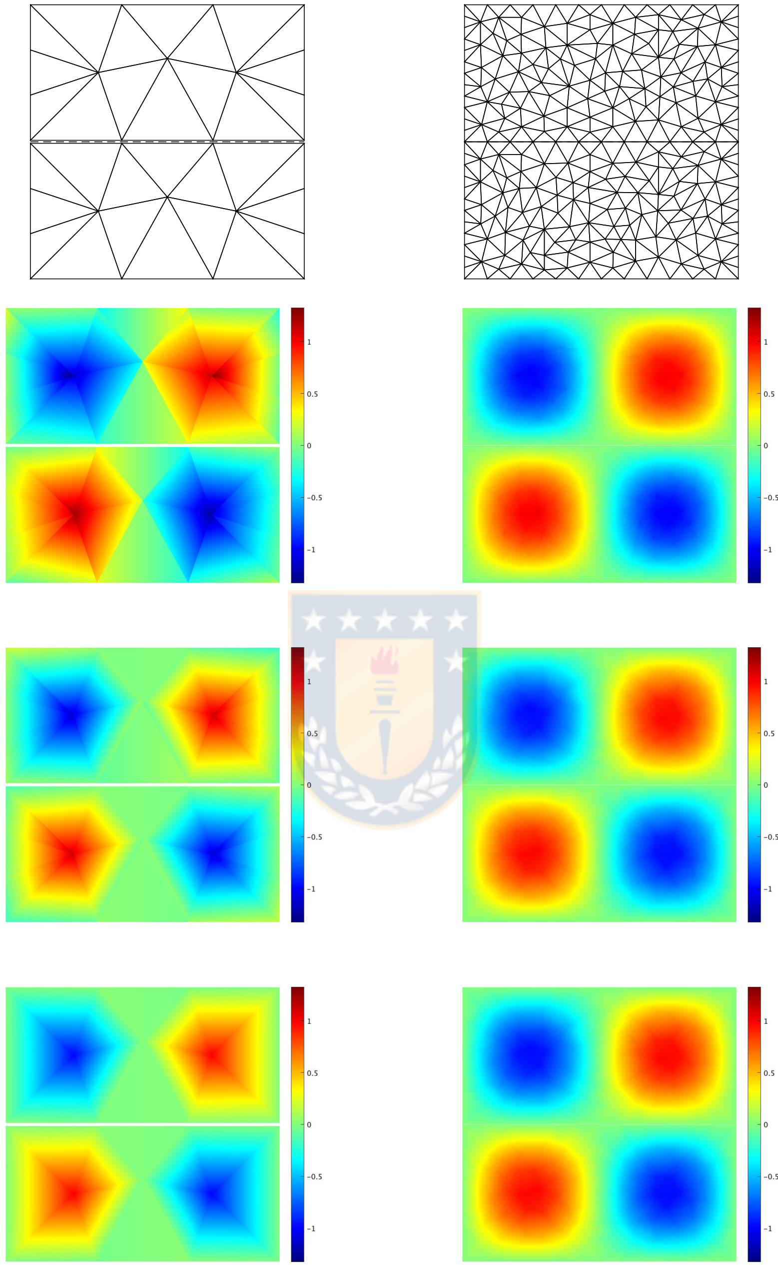

Figure 4.2: Approximation of p with $\nu = 1$ and a positive gap of order h^2. Columns: meshes with $N = 32$ and $N = 462$ elements. Topmost row illustrate the dissimilar meshes with the dashed line as the flat interface, while the rest correspond to the case $k = 1, 2, 3$, respectively.

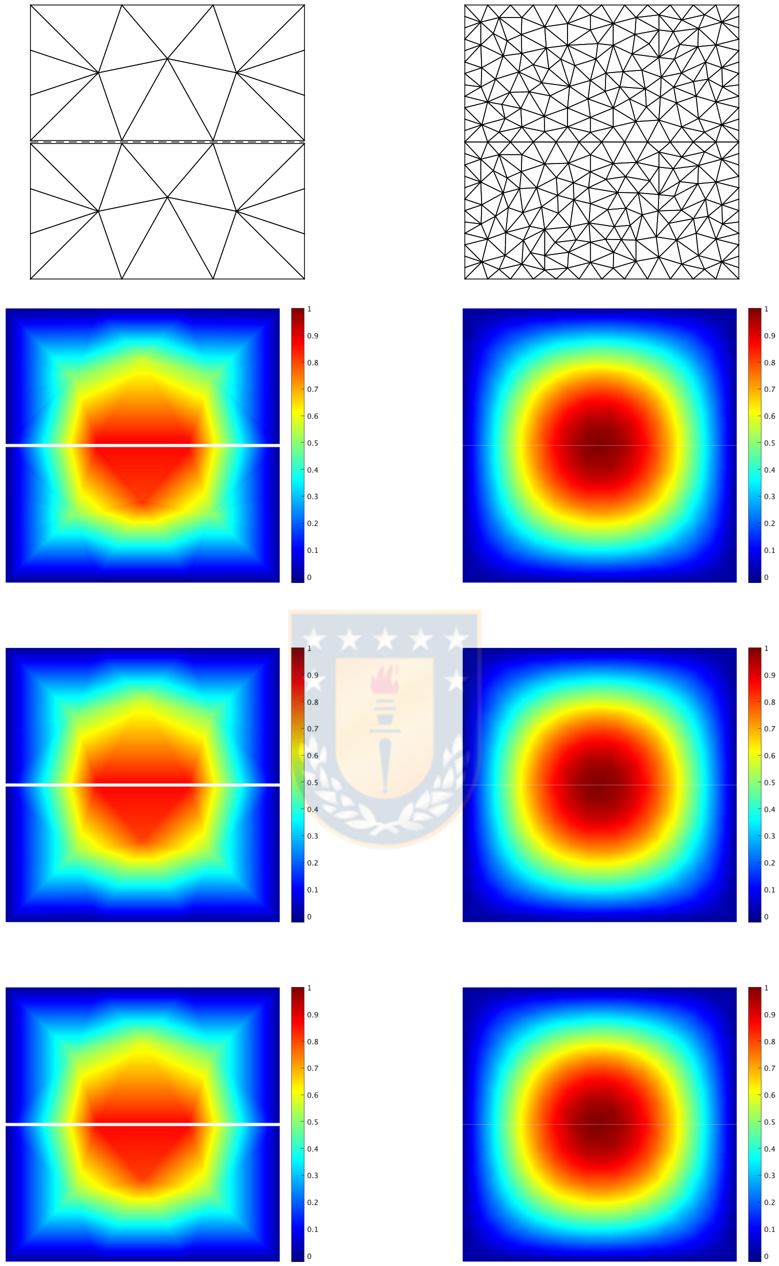

Figure 4.3: Approximation of the first component of $\boldsymbol{u}$ via a divergence-free postprocess with $\nu = 1$ and a positive gap of order h^2. Columns: meshes with $N = 32$ and $N = 462$ elements. Topmost row illustrate the dissimilar meshes with the dashed line as the flat interface, while the rest correspond to the case $k = 1, 2, 3$, respectively.

4.2 Special cases

4.2.1 Negative gap

While assumption (A.1) ruled out the case with overlaps between the meshes, as we mentioned before, the method works exactly the same as we can observe in Tables 4.10, 4.11 and 4.12. We note that, in this case, in order to carry out the pressure postprocess, we consider all overlaps as regions with "negative area".

Table 4.10: History of convergence of the HDG scheme for $\nu = 1$ and negative gap of order h^2.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{u^\star}$	e.o.c.
1	32	1.58e-01	—	8.74e-02	—	1.10e-01	—	1.11e-02	—	9.66e-03	—
	124	4.04e-02	2.02	2.32e-02	1.96	2.45e-02	2.22	1.29e-03	3.18	7.79e-04	3.72
	464	1.13e-02	1.93	6.29e-03	1.98	6.14e-03	2.10	1.89e-04	2.91	1.17e-04	2.87
	1812	2.84e-03	2.03	1.59e-03	2.02	1.53e-03	2.04	2.41e-05	3.03	1.49e-05	3.04
	7234	7.27e-04	1.97	4.02e-04	1.99	3.93e-04	1.96	3.20e-06	2.92	1.96e-06	2.93
	28870	1.82e-04	2.00	9.97e-05	2.01	9.76e-05	2.02	3.97e-07	3.02	2.48e-07	2.99
2	32	1.79e-02	—	1.02e-02	—	1.30e-02	—	9.64e-04	—	2.52e-03	—
	124	2.61e-03	2.85	1.44e-03	2.88	2.07e-03	2.72	7.65e-05	3.74	3.57e-05	6.29
	464	3.85e-04	2.90	2.24e-04	2.83	3.18e-04	2.84	6.09e-06	3.84	2.78e-06	3.87
	1812	4.95e-05	3.01	2.83e-05	3.03	4.01e-05	3.04	3.93e-07	4.02	1.82e-07	4.00
	7234	6.41e-06	2.95	3.57e-06	2.99	5.13e-06	2.97	2.57e-08	3.94	1.23e-08	3.89
	28870	8.05e-07	3.00	4.46e-07	3.01	6.44e-07	3.00	1.63e-09	3.99	7.68e-10	4.01
3	32	3.02e-03	—	1.81e-03	—	3.82e-03	—	2.34e-04	—	1.25e-03	—
	124	1.81e-04	4.15	1.09e-04	4.15	1.98e-04	4.37	5.75e-06	5.47	1.83e-06	9.64
	464	1.35e-05	3.94	7.67e-06	4.02	1.38e-05	4.04	2.11e-07	5.01	6.66e-08	5.02
	1812	8.38e-07	4.08	4.89e-07	4.04	8.55e-07	4.08	6.67e-09	5.07	2.07e-09	5.10
	7234	5.91e-08	3.83	3.26e-08	3.92	5.99e-08	3.84	2.47e-10	4.76	7.67e-11	4.76
	28870	3.68e-09	4.01	2.01e-09	4.02	3.70e-09	4.02	7.61e-12	5.03	2.42e-12	5.00

In Figure 4.4 we observe one of the meshes used in this case, where the dashed line represents the flat interface and the bottom mesh is overlapped on top of the other.

Table 4.11: History of convergence of the HDG scheme for $\nu = 10^{-3}$ and negative gap of order h^2.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{u^\star}$	e.o.c.
1	32	7.01e+01	—	4.79e+01	—	9.64e-02	—	7.63e+00	—	2.53e+00	—
	124	1.97e+01	1.88	1.22e+01	2.02	2.14e-02	2.22	1.02e+00	2.98	3.81e-01	2.80
	464	5.31e+00	1.98	3.18e+00	2.04	5.45e-03	2.08	1.40e-01	3.00	5.44e-02	2.95
	1812	1.36e+00	2.00	8.15e-01	2.00	1.38e-03	2.02	1.81e-02	3.01	6.83e-03	3.05
	7234	3.58e-01	1.93	2.05e-01	1.99	3.55e-04	1.96	2.45e-03	2.89	9.44e-04	2.86
	28870	8.92e-02	2.01	5.11e-02	2.01	8.81e-05	2.01	3.01e-04	3.03	1.18e-04	3.00
2	32	9.49e+00	—	6.58e+00	—	1.18e-02	—	7.14e-01	—	2.53e-01	—
	124	1.79e+00	2.47	1.06e+00	2.70	2.00e-03	2.62	7.07e-02	3.41	2.23e-02	3.59
	464	2.79e-01	2.82	1.72e-01	2.76	3.09e-04	2.83	5.66e-03	3.83	1.86e-03	3.76
	1812	3.56e-02	3.02	2.17e-02	3.04	3.91e-05	3.04	3.64e-04	4.03	1.23e-04	3.99
	7234	4.58e-03	2.96	2.72e-03	3.00	4.98e-06	2.97	2.38e-05	3.94	8.11e-06	3.93
	28870	5.79e-04	2.99	3.39e-04	3.01	6.27e-07	3.00	1.51e-06	3.98	5.13e-07	3.99
3	32	2.73e+00	—	1.70e+00	—	3.80e-03	—	2.37e-01	—	8.02e-02	—
	124	1.64e-01	4.15	1.01e-01	4.17	1.97e-04	4.37	5.84e-03	5.47	1.70e-03	5.69
	464	1.20e-02	3.96	7.02e-03	4.04	1.37e-05	4.04	2.09e-04	5.05	5.65e-05	5.16
	1812	7.46e-04	4.08	4.49e-04	4.03	8.50e-07	4.08	6.60e-06	5.07	1.71e-06	5.13
	7234	5.33e-05	3.81	2.99e-05	3.91	5.95e-08	3.84	2.44e-07	4.76	6.49e-08	4.73
	28870	3.31e-06	4.02	1.85e-06	4.02	3.68e-09	4.02	7.50e-09	5.03	2.03e-09	5.01

Table 4.12: History of convergence of the HDG scheme for $\nu = 10^{-6}$ and negative gap of order h^2.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{u^\star}$	e.o.c.
2	32	9.49e+03	—	6.58e+03	—	1.18e-02	—	7.13e+02	—	2.53e+02	—
	124	1.79e+03	2.47	1.06e+03	2.70	2.00e-03	2.62	7.07e+01	3.41	2.23e+01	3.59
	464	2.79e+02	2.82	1.72e+02	2.76	3.09e-04	2.83	5.66e+00	3.83	1.86e+00	3.76
	1812	3.56e+01	3.02	2.17e+01	3.04	3.91e-05	3.04	3.64e-01	4.03	1.23e-01	3.99
	7234	4.58e+00	2.96	2.72e+00	3.00	4.98e-06	2.97	2.38e-02	3.94	8.11e-03	3.93
	28870	5.79e-01	2.99	3.39e-01	3.01	6.27e-07	3.00	1.51e-03	3.98	5.13e-04	3.99
3	32	2.73e+03	—	1.70e+03	—	3.80e-03	—	2.37e+02	—	8.02e+01	—
	124	1.64e+02	4.15	1.01e+02	4.17	1.97e-04	4.37	5.84e+00	5.47	1.70e+00	5.69
	464	1.20e+01	3.96	7.02e+00	4.04	1.37e-05	4.04	2.09e-01	5.05	5.65e-02	5.16
	1812	7.46e-01	4.08	4.49e-01	4.03	8.50e-07	4.08	6.60e-03	5.07	1.71e-03	5.13
	7234	5.33e-02	3.81	2.99e-02	3.91	5.95e-08	3.84	2.44e-04	4.76	6.49e-05	4.73
	28870	3.31e-03	4.02	1.85e-03	4.02	3.68e-09	4.02	7.50e-06	5.03	2.03e-06	5.01

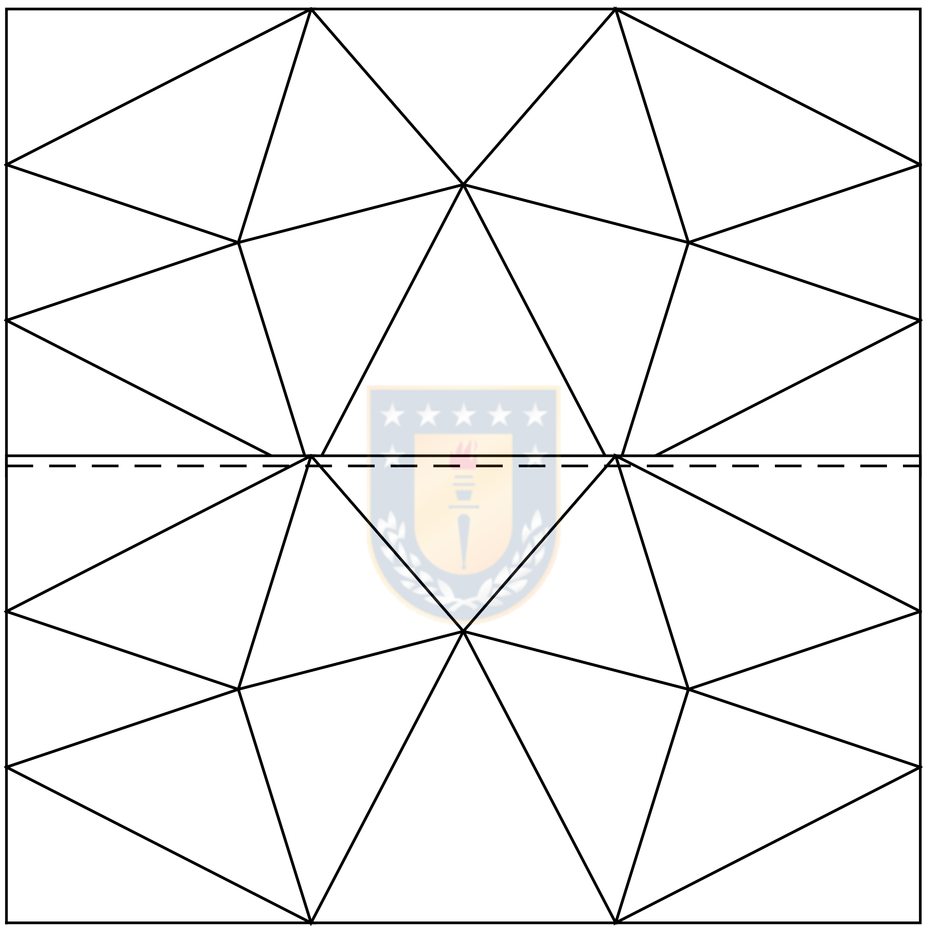

Figure 4.4: Overlapping dissimilar meshes with $N = 32$ elements with a flat interface (dashed line).

4.2.2 Presence of hanging nodes

Table 4.13: History of convergence of the HDG scheme for $\nu = 1$, $\delta = \mathcal{O}(h^2)$ and hanging nodes present on the discrete interfaces.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{u^\star}$	e.o.c.
1	76	1.85e-01	—	5.40e-02	—	1.54e-01	—	1.98e-02	—	7.70e-03	—
	290	5.71e-02	1.76	1.66e-02	1.77	3.88e-02	2.06	2.71e-03	2.97	9.42e-04	3.14
	1150	1.93e-02	1.58	4.52e-03	1.88	1.33e-02	1.55	4.53e-04	2.60	1.39e-04	2.78
	4492	6.56e-03	1.58	1.17e-03	1.98	4.49e-03	1.60	8.07e-05	2.53	2.43e-05	2.56
	17986	2.29e-03	1.52	2.97e-04	1.98	1.59e-03	1.50	1.46e-05	2.47	4.45e-06	2.45
	70030	7.90e-04	1.56	7.24e-05	2.08	5.43e-04	1.58	2.62e-06	2.52	8.23e-07	2.49
2	76	4.03e-02	—	7.94e-03	—	4.58e-02	—	8.52e-03	—	4.01e-03	—
	290	4.18e-03	3.38	1.10e-03	2.95	4.39e-03	3.50	3.75e-04	4.66	1.77e-04	4.66
	1150	6.21e-04	2.77	1.53e-04	2.87	5.07e-04	3.13	2.29e-05	4.06	1.05e-05	4.11
	4492	1.00e-04	2.68	2.01e-05	2.98	7.47e-05	2.81	1.11e-06	4.44	4.47e-07	4.63
	17986	1.73e-05	2.53	2.57e-06	2.97	1.30e-05	2.52	1.08e-07	3.37	4.52e-08	3.30
	70030	2.98e-06	2.59	3.16e-07	3.08	2.20e-06	2.62	8.96e-09	3.66	3.66e-09	3.70
3	76	3.34e-02	—	3.12e-03	—	7.20e-02	—	6.89e-03	—	3.03e-03	—
	290	6.20e-04	5.95	9.01e-05	5.29	1.25e-03	6.05	1.04e-04	6.26	4.75e-05	6.21
	1150	2.80e-05	4.50	5.67e-06	4.02	5.05e-05	4.66	4.12e-06	4.69	1.96e-06	4.63
	4492	1.50e-06	4.29	3.72e-07	4.00	2.25e-06	4.57	1.67e-07	4.70	8.26e-08	4.65
	17986	1.05e-07	3.84	2.37e-08	3.97	1.29e-07	4.12	8.38e-09	4.31	4.21e-09	4.29
	70030	8.05e-09	3.77	1.43e-09	4.14	8.29e-09	4.04	4.44e-10	4.32	2.24e-10	4.31

Following our assumption that $h_2 > h_1$, we added the hanging nodes on the top mesh as one can observe in Figure 4.5. For the case $\nu = 1$, shown in Table 4.13, we observe the loss of half a power of h for the approximation of the velocity gradient, the velocity trace and the postprocessed velocity, while the rest of the variables achieve optimal convergence as before. Nonetheless, this is not the case in Tables 4.14 and 4.15, where small enough ν allows us to recover the optimality of the method. This suggests that the ratio between ν and the meshsize may dominate the error estimate for the previously sub-optimal quantities.

Table 4.14: History of convergence of the HDG scheme for $\nu = 10^{-3}$, $\delta = \mathcal{O}(h^2)$ and hanging nodes present on the discrete interfaces.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{u^\star}$	e.o.c.
1	76	3.91e+01	—	2.67e+01	—	4.64e-02	—	5.77e+00	—	2.47e+00	—
	290	1.43e+01	1.50	8.97e+00	1.63	1.54e-02	1.64	7.92e-01	2.97	2.92e-01	3.19
	1150	3.88e+00	1.90	2.29e+00	1.98	3.95e-03	1.98	1.00e-01	3.00	4.00e-02	2.88
	4492	1.02e+00	1.96	6.02e-01	1.96	1.03e-03	1.97	1.38e-02	2.91	5.11e-03	3.02
	17986	2.62e-01	1.97	1.51e-01	1.99	2.60e-04	1.99	1.76e-03	2.98	6.70e-04	2.93
	70030	6.48e-02	2.05	3.70e-02	2.07	6.40e-05	2.06	2.14e-04	3.10	8.46e-05	3.04
2	76	1.32e+01	—	5.42e+00	—	2.11e-02	—	2.87e+00	—	1.37e+00	—
	290	1.36e+00	3.40	8.02e-01	2.85	1.69e-03	3.77	1.30e-01	4.63	6.12e-02	4.64
	1150	1.94e-01	2.83	1.16e-01	2.81	2.13e-04	3.01	6.03e-03	4.45	2.68e-03	4.54
	4492	2.46e-02	3.03	1.53e-02	2.97	2.73e-05	3.02	3.65e-04	4.12	1.59e-04	4.14
	17986	3.26e-03	2.92	1.96e-03	2.97	3.55e-06	2.94	2.17e-05	4.07	9.04e-06	4.14
	70030	4.10e-04	3.05	2.40e-04	3.09	4.45e-07	3.06	1.34e-06	4.10	5.52e-07	4.11
3	76	4.63e+00	—	8.09e-01	—	7.19e-03	—	1.10e+00	—	4.89e-01	—
	290	1.25e-01	5.40	7.17e-02	3.62	1.51e-04	5.77	9.76e-03	7.06	4.31e-03	7.07
	1150	8.40e-03	3.92	4.87e-03	3.90	9.65e-06	3.99	2.08e-04	5.58	8.27e-05	5.74
	4492	5.73e-04	3.94	3.34e-04	3.93	6.50e-07	3.96	5.81e-06	5.26	1.82e-06	5.60
	17986	3.78e-05	3.92	2.14e-05	3.96	4.22e-08	3.94	1.81e-07	5.00	5.31e-08	5.09
	70030	2.32e-06	4.10	1.30e-06	4.13	2.59e-09	4.11	5.54e-09	5.13	1.68e-09	5.08

Table 4.15: History of convergence of the HDG scheme for $\nu = 10^{-6}$, $\delta = \mathcal{O}(h^2)$ and hanging nodes present on the discrete interfaces.

k	N	$e_{\mathbb{L}}$	e.o.c.	$e_{\boldsymbol{u}}$	e.o.c.	e_p	e.o.c.	$e_{\hat{\boldsymbol{u}}}$	e.o.c.	$e_{u^\star}$	e.o.c.
1	76	3.91e+04	—	2.67e+04	—	4.64e-02	—	5.78e+03	—	2.48e+03	—
	290	1.43e+04	1.50	8.97e+03	1.63	1.54e-02	1.64	7.92e+02	2.97	2.92e+02	3.19
	1150	3.88e+03	1.90	2.29e+03	1.98	3.95e-03	1.98	1.00e+02	3.00	4.00e+01	2.88
	4492	1.02e+03	1.96	6.02e+02	1.96	1.03e-03	1.97	1.38e+01	2.91	5.11e+00	3.02
	17986	2.62e+02	1.97	1.51e+02	1.99	2.60e-04	1.99	1.76e+00	2.98	6.70e-01	2.93
	70030	6.48e+01	2.05	3.70e+01	2.07	6.40e-05	2.06	2.14e-01	3.10	8.46e-02	3.04
2	76	1.32e+04	—	5.42e+03	—	2.11e-02	—	2.86e+03	—	1.37e+03	—
	290	1.36e+03	3.40	8.02e+02	2.85	1.69e-03	3.77	1.29e+02	4.63	6.11e+01	4.64
	1150	1.94e+02	2.83	1.16e+02	2.81	2.13e-04	3.00	6.03e+00	4.45	2.67e+00	4.54
	4492	2.46e+01	3.03	1.53e+01	2.97	2.73e-05	3.02	3.64e-01	4.12	1.59e-01	4.14
	17986	3.26e+00	2.92	1.96e+00	2.97	3.55e-06	2.94	2.17e-02	4.07	9.03e-03	4.14
	70030	4.10e-01	3.05	2.40e-01	3.09	4.45e-07	3.06	1.34e-03	4.10	5.52e-04	4.11
3	76	4.61e+03	—	8.07e+02	—	7.13e-03	—	1.10e+03	—	4.87e+02	—
	290	1.24e+02	5.39	7.17e+01	3.62	1.50e-04	5.77	9.71e+00	7.06	4.28e+00	7.07
	1150	8.39e+00	3.91	4.87e+00	3.90	9.64e-06	3.99	2.06e-01	5.59	8.12e-02	5.76
	4492	5.73e-01	3.94	3.34e-01	3.93	6.50e-07	3.96	5.75e-03	5.25	1.77e-03	5.61
	17986	3.78e-02	3.92	2.14e-02	3.96	4.22e-08	3.94	1.79e-04	5.00	5.13e-05	5.11
	70030	2.32e-03	4.10	1.30e-03	4.13	2.59e-09	4.11	5.43e-06	5.15	1.58e-06	5.12

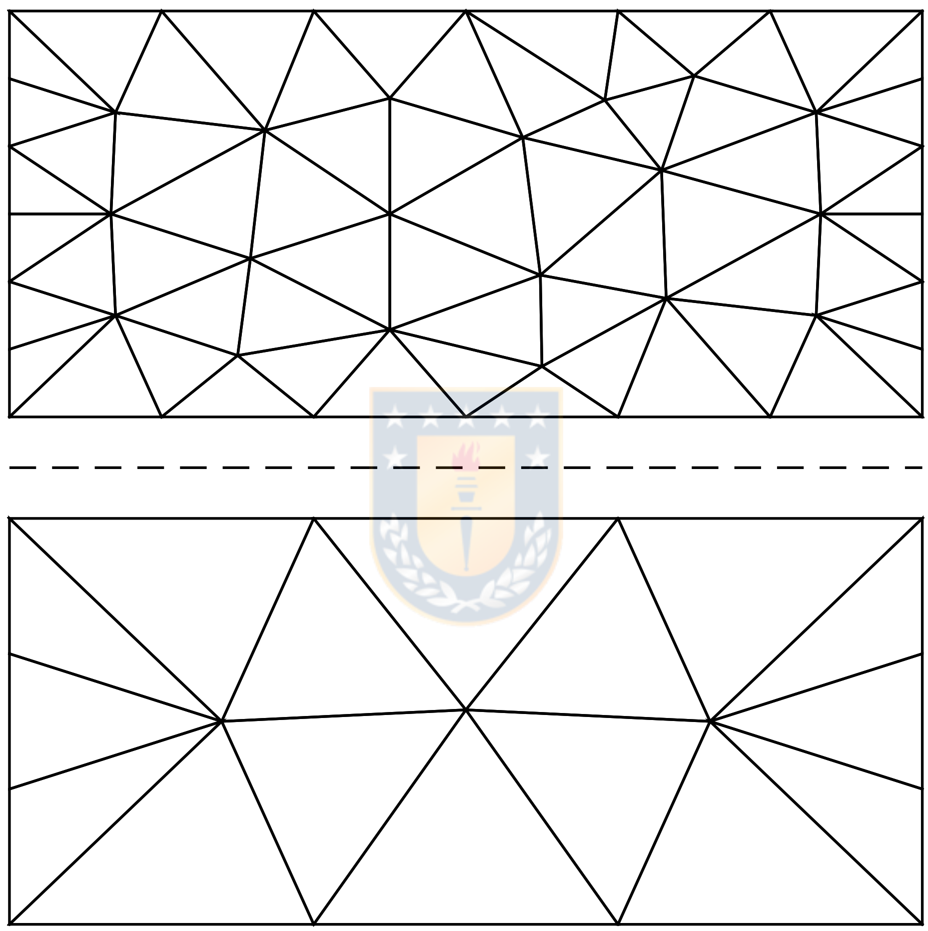

Figure 4.5: Dissimilar meshes with hanging nodes between the discrete interfaces. In this case, the flat interface (dashed line) is far away from the meshes.

CHAPTER 5

Conclusions and discussions

In this work we developed an HDG method for the Stokes equations of an incompressible fluid whose domain is discretized by two independent polygonal subdomains with different meshsizes. In order to obtain a stability estimate, we employed an energy argument, a duality argument to bound the norm of the discrete velocity and an inf-sup condition for the norm of the discrete pressure. In particular, the proposed scheme is stable under certain hypothesis related to the size of the gap δ in comparison to the meshsize h.

To obtain the previous estimates, a transferring technique, originally designed for non-polygonal domains, was successfully adapted to our context. On the other hand, to deduce the error estimates we used the stability bounds and the properties of the HDG projection. This allows to conclude that our method is optimal under the assumptions in section 3.2 and the numerical experiments presented validate these results, showing $k+1$ convergence for all the variables and $k+2$ convergence for the divergence-free postprocess of the discrete velocity. Furthermore, experiments that do not exactly fit under our assumptions were presented with positive results, suggesting the robustness of the method in broader contexts.

5.1 On the swapped transmission conditions

While neither an explicit analysis or numerical experiments for the swapped tranmission conditions (2.2.4) were given in this work, the symmetries present in terms such as those in Lemmas 2 and 6 would allow us to carry out almost identical arguments should we have used the swapped conditions and thus obtain the exact same error estimates. Moreover, the numerical results presented in [42] strongly suggest that this would be case as the choice of transmission conditions did not affect the optimality of the method for the diffusion equation.

5.2 On the presence of hanging nodes

While the analysis presented in this work assumed the absence of hanging nodes between the discrete interfaces, the numerical experiments show that the method works even when these are present. Comparing to the results obtained in [42], where the discrete gradient also loses half a power of h when hanging nodes are present, from the analysis carried out for the diffusion equation we see that the loss of optimality is due to the fact that an extrapolated polynomial from one interface might not be a polynomial in the other, but rather a piece-wise polynomial, which would force us to use $\boldsymbol{L}^2$ projectors leaving extra terms in the estimates. In fact, the similarities between both analyses suggests that it would not be difficult to adapt the results to the case with hanging nodes to corroborate the numerical behaviour shown in this work. Nonetheless, as discussed in Chapter 4, the errors show dependence on ν as small enough viscosities compared to the meshsize allow us to recover the optimality of the method and so this is an important result to establish explicitly in future work.

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