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BANACH SPACES-BASED MIXED FINITE ELEMENT METHODS FOR THE COUPLED STOKES AND POISSON-NERNST-PLANCK EQUATIONS

por

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Tesis presentada a la Facultad de Ciencias Físicas y Matemáticas de la Universidad de Concepción para optar al título profesional de Ingeniero Civil Matemático

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2022,
Concepción, Chile.

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MÉTODOS DE ELEMENTOS FINITOS MIXTOS BASADOS EN ESPACIOS DE BANACH PARA LAS ECUACIONES ACOPLADAS DE STOKES Y DE POISSON-NERNST-PLANCK

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Fecha de Defensa: 20 de octubre 2022

Para mis padres Eduardo W. Correa y Yanina J. Barría

Acknowledgements

Quiero comenzar agradeciendo a mis padres, Yanina y Eduardo, por todo su apoyo brindado, porque me permitió poder estudiar de forma cómoda y segura, en condiciones mejores que en las que tuvieron ellos en su vida universitaria. Con su amor y sacrificio me simplificaron la vida, permitiéndome ahorrarme preocupaciones y dedicarme plenamente a los estudios. Siempre creyendo en mi potencial y confiando en mis decisiones. Estoy seguro de que sin ellos, este logro nunca hubiera sido posible y es por este motivo que este trabajo e hito va dedicados a ellos. Además quiero dar gracias a mi perrito Luke, su pura existencia me llena de felicidad todos los días. Por otra parte quiero conmemorar a mi abuelito Bernardo que en paz descanse.

Agradezco a mi profesor guía, Gabriel N. Gatica, por su orientación y consejo que me ha entregado en mi última etapa de formación. También de darme la oportunidad de trabajar con él, además de tutelar mi tesis. Su calidad de docente e investigador es únicamente superado por su calidad de persona. Aprovecho también de agradecer a su compadre, el Profesor Freddy Paiva, que desde mis inicios de formación me motivó a seguir adelante y aventurarme a este mundillo que son las EDPs. Ser su ayudante, me dio la madurez matemática que me permitió seguir en la rama de numérico. Dar gracias, igualmente, a mi otro profesor guía, Ricardo Ruiz-Baier, el cual me ayudó en la parte computacional de este trabajo, fue todo un honor colaborar con él. De igual forma, me gustaría agradecer a mi compañero y colega, Esteban Henríquez, por ayudarme a preparar mi presentación. Confío que en un futuro próximo estaremos colaborando, hasta que surja HDG in Banach-Spaces

Doy las gracias al CI^{2}MA por darme un espacio donde trabajar, aunque breve mi estadía, su hospitalidad ha sido insuperable. Destaco a Sra. Lorena, Sra. Paola y los estudiantes del doctorado Juan Paulo, Romel y Issac, los cuales me han hecho compañía en esta etapa. Agradezco su hospitalidad y apoyo.

Es necesario agradecer a mis compañeros y amigos que me han acompañado en esta etapa de mi vida y yo sé que seguirán siendo parte de ella: Bastian Ducumets, Carlos Valdes, Diana Ramirez, Javiera Arias, Maria Teresa, Nicolas Mellado, Nicol Echeverría, Miguel Cisterna, *Matias ahumada, Lucas Romero, Vicente Marchant, Yamira Roa, y muchos más. Ellos son los responsables de que yo pasara de alguien retraído y antisocial a una persona alegre y motivada. Los quiero a todos infinito. Deseo seguir conversando y jugando con ustedes eternamente. Aprovecho de agradecer a mis primos, Andrés Cánovas y Antonia Cánovas, los cuales me han aguantado y apoyado como roommate estos años.*

Agradecer también a la Universidad, que a parte de la gran formación que me entregó, mi transcurso en ella me ha permitido conocer gente única y fantástica, además de vivenciar momentos inolvidables que añoraré por siempre. Prometo trabajar desde aquí en adelante por el desarrollo libre del espíritu...

Este trabajo de tesis fue apoyado por ANID-Chile a través de los proyectos Centro de Modelamiento Matematico (FB210005), y Anillo of Computational Mathematics for Desalination Processes (ACT210087).

Abstract

This work is divided in two main parts. In the first part we provide sufficient conditions for perturbed saddle-point formulations in Banach spaces and their associated Galerkin schemes to be well-posed. Our approach, which extends a similar procedure employed with Hilbert spaces, proceeds in two slightly different ways depending on whether the kernel of the adjoint operator induced by one of the bilinear forms is trivial or not. The applicability of the continuous solvability is illustrated with a mixed formulation for the decoupled Nernst-Planck equation. This part yielded the following work already published:

> C.I. Correa and G.N. Gatica, *On the continuous and discrete well-posedness of perturbed saddle-point formulations in Banach spaces.* Comput. Math. Appl. 117 (2022), 14–23.

On the other hand, in the second part we employ a Banach spaces-based framework to introduce and analyze new mixed finite element methods for the numerical solution of the coupled Stokes and Poisson–Nernst–Planck equations, which is a nonlinear model describing the dynamics of electrically charged incompressible fluids. The pressure of the fluid is eliminated from the system (though computed afterwards via a postprocessing formula) thanks to the incompressibility condition and the incorporation of the fluid pseudostress as an auxiliary unknown. In turn, besides the electrostatic potential and the concentration of ionized particles, we use the electric field (rescaled gradient of the potential) and total ionic fluxes as new unknowns. The resulting fully mixed variational formulation in Banach spaces can be written as a coupled system. the well-posedness of the continuous formulation is a consequence of a fixed point strategy in combination with the Banach theorem, the Babuška–Brezzi theory, the solvability of abstract perturbed saddle point problem that will be developed in the first part of this thesis, and the Banach–Nečas–Babuška theorem. For this we also employ smallness assumptions on the data. An analogous approach, but using now both the Brouwer and Banach theorems, and invoking suitable stability conditions on arbitrary finite element subspaces, is employed to conclude the existence and uniqueness of solution for the associated Galerkin scheme. A priori

error estimates are derived, and examples of discrete spaces that fit the theory, include, e.g., Raviart–Thomas elements of order k along with piecewise polynomials of degree $\leqslant k$. Finally, rates of convergence are specified and several numerical experiments confirm the theoretical error bounds. These tests also illustrate the balance-preserving properties and applicability of the proposed family of methods. This part yielded the following work, presently submitted:

C.I. Correa, G.N. Gatica and R. Ruiz-Baier, *New mixed finite element methods for the coupled Stokes and Poisson-Nernst-Planck equations in Banach spaces.* Preprint 2022-26, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, (2022).

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On the continuous and discrete well-posedness of perturbed saddle-point formulations in Banach spaces

CHAPTER 1

Introduction

The purpose of this note in to analyze the solvability of the continuous and discrete schemes arising from perturbed saddle-point problems formulated in terms of Banach spaces. More precisely, given reflexive Banach spaces H and Q, bounded bilinear forms $a : \mathrm{H} \times \mathrm{H} \to \mathrm{R}$, $b : \mathrm{H} \times \mathrm{Q} \to \mathrm{R}$, and $c : \mathrm{Q} \times \mathrm{Q} \to \mathrm{R}$ and functionals $f \in \mathrm{H}'$ and $g \in \mathrm{Q}'$, the formulation of interest consists of seeking $(\sigma, u) \in \mathrm{H} \times \mathrm{Q}$ such that

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= f(\tau) \qquad \forall\, \tau \in \mathrm{H}\,, \\ b(\sigma,v) - c(u,v) &= g(v) \qquad \forall\, v \in \mathrm{Q}\,. \end{aligned} \tag{1.1}$$

In the particular case in which H and Q are Hilbert spaces, the well posedness of (1.1) and its associated Galerkin scheme is very well established nowadays. We refer to [10, Theorem 1.2, Section II.1.2] and [10, Proposition 2.11, Section II.2.4] for a through analysis of it, including the derivation of the corresponding Cea estimate. While several possible cases of the bilinear form c, which constitutes the so-called perturbation, are considered, the most frequent ones in applications are those in which, either the null space of the adjoint of the operator induced by b is trivial, or the bilinear form c is coercive on that kernel. Certainly, the non-perturbed formulation, that is when c vanishes, has already been fully studied, first in [9], then in [42] where two different bilinear forms b are assumed, and finally in [6] for the same abstract problem from [42], but within a Banach framework.

Going back to (1.1), we stress that an alternative setting is introduced in [8], where c is defined on a dense subspace Q_c of Q, and then multiplied by the square of a small parameter usually arising from the underlying physical model. For instance, in the case of the Reissner-Mindlin plate, which is used in [8] to illustrate the theory, the thickness of it defines that parameter. The approach in there assumes that Q_c is Hilbert with a c-dependent inner product, and then extends the classical results from [9] to the aforedescribed saddle point problem with penalty. Some of the tools employed in [8], particularly those regarding the handling of the inf-sup conditions involved, resemble the ones to be utilized below in Chapter 3 to prove our main theorems. On the other hand, Similar results to those in [10], though with slightly different proofs and providing further details, but still within a Hilbertian framework, are discussed in [7, Theorem 4.3.1, Sections 4.3.1] and [7, Theorem 5.5.1, Proposition 5.5.2, Section 5.5.1]. In turn, denoting by V and W the null spaces of the operator induced by b and its adjoint, respectively, we stress that a key result for the solvability analysis of (1.1) in the Hilbertian context is given by the identity (see, e.g. [7, eq. (4.3.18)])

$$\inf_{\substack{\tau\in\mathrm{V}^\perp\\ \tau\neq 0}}\sup_{\substack{v\in\mathrm{Q}\\ v\neq 0}}\frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}\,\|v\|_{\mathrm{Q}}} \;=\; \inf_{\substack{v\in\mathrm{W}^\perp\\ v\neq 0}}\sup_{\substack{\tau\in\mathrm{H}\\ \tau\neq 0}}\frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}\,\|v\|_{\mathrm{Q}}} \;>\; 0\,, \tag{1.2}$$

whose discrete version is also satisfied (see, e.g. [7, eq. (5.5.12)]).

According to the above discussion, and since the respective results do not seem to be available in the literature, the present work aims to extend the aforementioned theory to the Banach case. In this regard, we warn in advance that (1.2) is not going to hold for the continuous formulation nor for the discrete one, and hence the analysis and results to be presented below will take this fact into consideration, mainly when we deal with the Galerkin scheme of (1.1). Indeed, in this case the discrete inf-sup conditions arising from both sides of (1.2) require to be assumed separately with constants independent of the meshsizes. However, in the particular, though very frequent case in which W is the null subspace, we are able to apply a suitable characterization of closed range injective adjoint operators, so that for the solvability analysis it suffices to assume only the inf-sup condition arising from the right-hand side of (1.2). An analogous reasoning is valid if the discrete version of W, say W_h, is the null subspace as well.

The rest of the first part is organized as follows. In Chapter 2 we present some preliminary results on the spaces H and Q and the operators induced by the bilinear form b. In particular, we address here a key equivalence result between the inf-sup conditions involving b. Next, in Chapter 3 we establish the theorems providing the unique solvability of (1.1) and its associated Galerkin scheme. The presentation considers first a general situation in which nothing is said about W, and then the particular case in which it is assumed that $W = \{0\}$. We proceed analogously for the discrete solvability. Finally, an application of the continuous theory to mixed variational formulation of the decoupled Nernst-Planck equation for a single ionic species, is discussed in Chapter 4.

CHAPTER 2

Preliminary results

In this Chapter we present some previous results concerning the spaces and operators involved, which will be employed later on. To this end, we first let $\mathbf{B} : \mathrm{H} \to \mathrm{Q}'$ and $\mathbf{B}^{\mathrm{t}} : \mathrm{Q} \to \mathrm{H}'$ be the bounded linear operators induced by b, that is

$$\mathbf{B}(\tau)(v) := b(\tau, v) \quad \forall\, \tau \in \mathrm{H}\,,\ \forall\, v \in \mathrm{Q} \quad \text{and} \quad \mathbf{B}^{\mathrm{t}}(v)(\tau) := b(\tau, v) \quad \forall\, v \in \mathrm{Q}\,,\ \forall\, \tau \in \mathrm{H}\,, \tag{2.1}$$

and introduce the respective null spaces

$$\mathrm{V} := \mathrm{N}(\mathbf{B}) := \Big\{ \tau \in \mathrm{H} : \quad b(\tau, v) = 0 \quad \forall\, v \in \mathrm{Q} \Big\} \tag{2.2}$$

and

$$\mathrm{W} := \mathrm{N}(\mathbf{B}^{\mathrm{t}}) := \Big\{ v \in \mathrm{Q} : \quad b(\tau, v) = 0 \quad \forall\, \tau \in \mathrm{H} \Big\}\,. \tag{2.3}$$

Next, we assume that V and W admit topological complements, which means that there exist closed subspaces $\mathrm{V}^{\perp}$ and $\mathrm{W}^{\perp}$ of H and Q, respectively,such that

$$\mathrm{H} \,=\, \mathrm{V} \oplus \mathrm{V}^{\perp} \qquad \text{and} \qquad \mathrm{Q} \,=\, \mathrm{W} \oplus \mathrm{W}^{\perp}\,, \tag{2.4}$$

and let $i : \mathrm{V}^{\perp} \to \mathrm{H}$ and $j : \mathrm{W}^{\perp} \to \mathrm{Q}$ be the respective injections. Notice that these complements are denoted using the symbol $^{\perp}$ just to keep the analogy with the orthogonal decomposition theorem in the Hilbert spaces case, but certainly we are aware of the fact that in the present discussion we have no inner products and hence no orthogonality concepts.

Furthermore, a direct application of the open mapping theorem implies the existence of positive constants C_{H} and C_{Q}, depending only on H and Q, respectively, such that

$$\|\tau_0\|_{\mathrm{H}} + \|\bar{\tau}\|_{\mathrm{H}} \,\leqslant\, C_{\mathrm{H}}\,\|\tau\|_{\mathrm{H}} \quad \text{and} \quad \|v_0\|_{\mathrm{Q}} + \|\bar{v}\|_{\mathrm{Q}} \,\leqslant\, C_{\mathrm{Q}}\,\|v\|_{\mathrm{Q}} \tag{2.5}$$

for all $\tau \,=\, \tau_0 + \bar{\tau} \in \mathrm{V} \oplus \mathrm{V}^{\perp}$, and for all $v \,=\, v_0 + \bar{v} \in \mathrm{W} \oplus \mathrm{W}^{\perp}$. As a consequence of these boundedness properties, we have the following result.

Lemma 2.1. *There hold*

$$\frac{1}{C_{\mathrm{H}}}\,\|\tau\|_{\mathrm{H}} \,\leqslant\, \mathrm{dist}(\tau, \mathrm{V}) \,\leqslant\, \|\tau\|_{\mathrm{H}} \qquad \forall\, \tau \in \mathrm{V}^{\perp}\,, \tag{2.6}$$

and

$$\frac{1}{C_{\mathrm{Q}}}\,\|v\|_{\mathrm{Q}} \,\leqslant\, \mathrm{dist}(v, \mathrm{W}) \,\leqslant\, \|v\|_{\mathrm{H}} \qquad \forall\, v \in \mathrm{W}^{\perp}\,. \tag{2.7}$$

Proof. We begin by noticing that the upper bounds of (2.6) and (2.7) are straightforward, and that they are actually valid for all $(\tau, v) \in \mathrm{H} \times \mathrm{Q}$. In addition, being the respective lower bounds proved analogously, it suffices to provide the proof for one of them, say (2.6). To this end, we first recall that if X is a reflexive Banach space and T is a closed subspace of X', there holds

$$\mathrm{dist}(x, {}^{\circ}\mathrm{T}) \,=\, \sup_{\substack{\mathrm{F}\in\mathrm{T}\\ \mathrm{F}\neq 0}} \frac{|\mathrm{F}(x)|}{\|\mathrm{F}\|_{X'}} \qquad \forall\, x \in \mathrm{X}\,.$$

Thus, applying this identity to $\mathrm{X} = \mathrm{H}$ and $\mathrm{T} = \mathrm{V}^{\circ}$, and using that ${}^{\circ}\big(\mathrm{V}^{\circ}\big) = \mathrm{V}$, we deduce that

$$\text{dist}(\tau, \mathrm{V}) \;=\; \sup_{\substack{\mathrm{F}\in \mathrm{V}^\circ \\ \mathrm{F}\neq 0}} \frac{|\mathrm{F}(\tau)|}{\|\mathrm{F}\|_{\mathrm{H}'}} \qquad \forall\, \tau \in \mathrm{H}\,. \tag{2.8}$$

Next, we restrict to $\tau \in \mathrm{V}^\perp$. Then, given $\mathrm{G} \in \mathrm{H}'$, we define the functional $g : \mathrm{H} \to \mathrm{R}$ by $g(\zeta) := \mathrm{G}(\bar{\zeta})$ for all $\zeta = \zeta_0 + \bar{\zeta} \in \mathrm{H} = \mathrm{V} \oplus \mathrm{V}^\perp$. It follows that g is linear, $g|_{\mathrm{V}} \equiv 0$, and, using (2.5),

$$|g(\zeta)| \;=\; |\mathrm{G}(\bar{\zeta})| \;\leqslant\; \|\mathrm{G}\|_{\mathrm{H}'}\, \|\bar{\zeta}\|_{\mathrm{H}} \;\leqslant\; C_{\mathrm{H}}\, \|\mathrm{G}\|_{\mathrm{H}'}\, \|\zeta\|_{\mathrm{H}} \quad \forall\, \zeta \in \mathrm{H}\,,$$

which says that g is bounded, with $\|g\|_{\mathrm{H}'} \leqslant C_{\mathrm{H}}\, \|\mathrm{G}\|_{\mathrm{H}'}$, and hence $g \in \mathrm{V}^\circ$. In this way, according to (2.8), and noting that $g(\tau) = \mathrm{G}(\tau)$, we find that

$$\text{dist}(\tau, \mathrm{V}) \;\geqslant\; \frac{|g(\tau)|}{\|g\|_{\mathrm{H}'}} \;\geqslant\; \frac{|\mathrm{G}(\tau)|}{C_{\mathrm{H}}\, \|\mathrm{G}\|_{\mathrm{H}'}}$$

from which, taking supremum with respect to $\mathrm{G} \in \mathrm{H}'$, we conclude that

$$\text{dist}(\tau, \mathrm{V}) \;\geqslant\; \frac{1}{C_{\mathrm{H}}}\, \|\tau\|_{\mathrm{H}} \qquad \forall\, \tau \in \mathrm{V}^\perp\,,$$

thus finishing the proof of (2.6). □

Some equivalence properties connecting $\mathbf{B}$ and $\mathbf{B}^{\mathrm{t}}$ are established next.

Lemma 2.2. *The following statements are equivalent:*

i) $\mathbf{B}^{\mathrm{t}} \circ j : \mathrm{W}^\perp \to \mathrm{H}'$ *is injective and of closed range, that is there exists a constant* $\widetilde{\beta} > 0$ *such that*

$$\|\mathbf{B}^{\mathrm{t}}(v)\|_{\mathrm{H}'} \;:=\; \sup_{\substack{\tau\in \mathrm{H} \\ \tau\neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}} \;\geqslant\; \widetilde{\beta}\, \|v\|_{\mathrm{Q}} \qquad \forall\, v \in \mathrm{W}^\perp\,. \tag{2.9}$$

ii) $j' \circ \mathbf{B} : \mathrm{H} \to \left(\mathrm{W}^\perp\right)'$ *is surjective.*

iii) $\mathbf{B} \circ i : \mathrm{V}^\perp \to \mathrm{Q}'$ *is injective and of closed range, that is there exists a constant* $\widehat{\beta} > 0$ *such that*

that

$$\|\mathbf{B}(\tau)\|_{\mathrm{Q}'} := \sup_{\substack{v\in\mathrm{Q}\\ v\neq 0}} \frac{b(\tau,v)}{\|v\|_{\mathrm{Q}}} \geqslant \widehat{\beta}\,\|\tau\|_{\mathrm{H}} \qquad \forall\,\tau\in\mathrm{V}^{\perp}\,. \tag{2.10}$$

iv) $i'\circ\mathbf{B}^{\mathrm{t}}:\mathrm{Q}\to\big(\mathrm{V}^{\perp}\big)'$ *is surjective.*

Proof. Let $\mathcal{J}_{\mathrm{H}}:\mathrm{H}\to\mathrm{H}''$ and $\mathcal{J}_{\mathrm{Q}}:\mathrm{Q}\to\mathrm{Q}''$ be the isometric and bijective linear mappings given by

$$\mathcal{J}_{\mathrm{H}}(\tau)(\mathrm{F}) := \mathrm{F}(\tau) \quad \forall\,\tau\in\mathrm{H}\,, \quad \forall\,\mathrm{F}\in\mathrm{H}' \quad \text{and} \quad \mathcal{J}_{\mathrm{Q}}(v)(\mathrm{G}) := \mathrm{G}(v) \quad \forall\,v\in\mathrm{Q}\,, \quad \forall\,\mathrm{G}\in\mathrm{Q}'\,,$$

and observe, as suggested by the diagrams

$$\mathrm{H} \xrightarrow{\mathcal{J}_{\mathrm{H}}} \mathrm{H}'' \xrightarrow{(\mathbf{B}^{\mathrm{t}})'} \mathrm{Q}' \quad \text{and} \quad \mathrm{Q} \xrightarrow{\mathcal{J}_{\mathrm{Q}}} \mathrm{Q}'' \xrightarrow{\mathbf{B}'} \mathrm{H}'\,,$$

that there holds

$$\mathbf{B} = (\mathbf{B}^{\mathrm{t}})'\circ\mathcal{J}_{\mathrm{H}} \quad \text{and} \quad \mathbf{B}^{\mathrm{t}} = \mathbf{B}'\circ\mathcal{J}_{\mathrm{Q}}\,. \tag{2.11}$$

Indeed, given $\tau\in\mathrm{H}$ and $v\in\mathrm{Q}$, we obtain

$$\big((\mathbf{B}^{\mathrm{t}})'\circ\mathcal{J}_{\mathrm{H}}\big)(\tau)(v) = (\mathbf{B}^{\mathrm{t}})'\big(\mathcal{J}_{\mathrm{H}}(\tau)\big)(v) = \mathcal{J}_{\mathrm{H}}(\tau)\big(\mathbf{B}^{\mathrm{t}}(v)\big) = \mathbf{B}^{\mathrm{t}}(v)(\tau) = \mathbf{B}(\tau)(v)\,,$$

which proves the first identity of (2.11). The second one proceeds similarly or as a consequence of the first one after exchanging $\mathbf{B}$ with $\mathbf{B}^{\mathrm{t}}$ and the roles of the spaces H and Q. It follows from (2.11) that

$$j'\circ\mathbf{B} = \big(j'\circ(\mathbf{B}^{\mathrm{t}})'\big)\circ\mathcal{J}_{\mathrm{H}} = \big(\mathbf{B}^{\mathrm{t}}\circ j\big)'\circ\mathcal{J}_{\mathrm{H}}\,, \tag{2.12}$$

and hence, bearing in mind the bijectivity of $\mathcal{J}_{\mathrm{H}}$, we deduce that $j'\circ\mathbf{B}:\mathrm{H}\to\big(\mathrm{W}^{\perp}\big)'$ is surjective if and only if $\big(\mathbf{B}^{\mathrm{t}}\circ j\big)':\mathrm{H}''\to\big(\mathrm{W}^{\perp}\big)'$ is surjective as well, which, in turn, is equivalent to stating that $\mathbf{B}^{\mathrm{t}}\circ j:\mathrm{W}^{\perp}\to\mathrm{H}'$ is injective and of closed range. The above shows the equivalence between i) and ii). Analogously, employing from the second identity in (2.11) that

$$i'\circ\mathbf{B}^{\mathrm{t}} = \big(i'\circ\mathbf{B}'\big)\circ\mathcal{J}_{\mathrm{Q}} = \big(\mathbf{B}\circ i\big)'\circ\mathcal{J}_{\mathrm{Q}}\,, \tag{2.13}$$

we are able to prove that iii) and iv) are equivalent. In order to conclude the proof, it suffices to see, for instance, that i) and iii) share the same property, which is addressed in what follows. Indeed, let us assume now that i) holds. Then, knowing that $\mathbf{B}^{\rm t}\circ j$ has closed range, we have that $R(\mathbf{B}^{\rm t}\circ j) = {}^{\circ}\mathrm{N}\big((\mathbf{B}^{\rm t}\circ j)'\big)$, where, according to (2.12), $\mathrm{N}\big((\mathbf{B}^{\rm t}\circ j)'\big) = \mathcal{J}_{\mathrm{H}}\big(\mathrm{N}(j'\circ\mathbf{B})\big)$. A simple computation yields

$$\mathrm{N}(j'\circ\mathbf{B}) \,=\, \Big\{\tau\in\mathrm{H}: \quad j'\big(\mathbf{B}(\tau)\big)(v) = \mathbf{B}(\tau)(v) = 0 \quad \forall\, v\in\mathrm{W}^{\perp}\Big\} \,=\, \mathrm{V}\,,$$

and hence $R(\mathbf{B}^{\rm t}\circ j) = {}^{\circ}\mathcal{J}_{\mathrm{H}}(\mathrm{V}) = \mathrm{V}^{\circ}$. In this way, we conclude that $\mathbf{B}^{\rm t}\circ j:\mathrm{W}^{\perp}\to\mathrm{V}^{\circ}$ is bijective, and (2.9) implies that $\|(\mathbf{B}^{\rm t}\circ j)^{-1}\|\leqslant\frac{1}{\widetilde{\beta}}$. It follows that $(\mathbf{B}^{\rm t}\circ j)':(\mathrm{V}^{\circ})'\to(\mathrm{W}^{\perp})'$ is bijective as well, and

$$\big\|\big((\mathbf{B}^{\rm t}\circ j)'\big)^{-1}\big\| \,=\, \big\|\big((\mathbf{B}^{\rm t}\circ j)^{-1}\big)'\big\| \,=\, \|(\mathbf{B}^{\rm t}\circ j)^{-1}\| \,\leqslant\, \frac{1}{\widetilde{\beta}}\,,$$

which says, equivalently, that

$$\|(\mathbf{B}^{\rm t}\circ j)'(\mathcal{G})\|_{(\mathrm{W}^{\perp})'} \,\geqslant\, \widetilde{\beta}\,\|\mathcal{G}\|_{(\mathrm{V}^{\circ})'} \qquad \forall\,\mathcal{G}\in(\mathrm{V}^{\circ})'\,. \tag{2.14}$$

In particular, taking $\mathcal{G} = \mathcal{J}_{\mathrm{H}}(\tau)|_{\mathrm{V}^{\circ}}$, with $\tau\in H$, we obtain

$$\|(\mathbf{B}^{\rm t}\circ j)'(\mathcal{G})\|_{(\mathrm{W}^{\perp})'} \,=\, \sup_{\substack{v\in\mathrm{W}^{\perp}\\ v\neq 0}}\frac{(\mathbf{B}^{\rm t}\circ j)'(\mathcal{J}_{\mathrm{H}}(\tau))(v)}{\|v\|_{\mathrm{Q}}} \,=\, \sup_{\substack{v\in\mathrm{W}^{\perp}\\ v\neq 0}}\frac{\mathcal{J}_{\mathrm{H}}(\tau)\big(\mathbf{B}^{\rm t}(v)\big)}{\|v\|_{\mathrm{Q}}} \,=\, \sup_{\substack{v\in\mathrm{W}^{\perp}\\ v\neq 0}}\frac{b(\tau,v)}{\|v\|_{\mathrm{Q}}}\,, \tag{2.15}$$

whereas, making use of (2.8) in the last equality below, we find that

$$\|\mathcal{J}_{\mathrm{H}}(\tau)|_{\mathrm{V}^{\circ}}\|_{(\mathrm{V}^{\circ})'} \,:=\, \sup_{\substack{\mathrm{F}\in\mathrm{V}^{\circ}\\ \mathrm{F}\neq 0}}\frac{\mathcal{J}_{\mathrm{H}}(\tau)(\mathrm{F})}{\|\mathrm{F}\|_{\mathrm{H}'}} \,=\, \sup_{\substack{\mathrm{F}\in\mathrm{V}^{\circ}\\ \mathrm{F}\neq 0}}\frac{\mathrm{F}(\tau)}{\|\mathrm{F}\|_{\mathrm{H}'}} \,=\, \mathrm{dist}(\tau,\mathrm{V})\,. \tag{2.16}$$

In this way, replacing (2.15) and (2.16) back into (2.14), we conclude that

$$\|\mathbf{B}(\tau)\|_{\mathrm{Q}'} \,\geqslant\, \sup_{\substack{v\in\mathrm{W}^{\perp}\\ v\neq 0}}\frac{b(\tau,v)}{\|v\|_{\mathrm{Q}}} \,\geqslant\, \widetilde{\beta}\,\mathrm{dist}(\tau,\mathrm{V}) \qquad \forall\,\tau\in\mathrm{H}\,, \tag{2.17}$$

which, together with the lower bound of (2.6), yields iii) (cf. (2.10)) with $\widehat{\beta} := \frac{\widetilde{\beta}}{C_{\mathrm{H}}}$. Conversely, let us assume that iii) holds. Then, in order to prove i), we proceed analogously to the opposite implication. In particular, using now (2.13) one deduces that $R(\mathbf{B} \circ i) = \mathrm{W}^{\circ}$, so that $\mathbf{B} \circ i : \mathrm{V}^{\perp} \longrightarrow \mathrm{W}^{\circ}$ and $(\mathbf{B} \circ i)' : \big(\mathrm{W}^{\circ}\big)' \longrightarrow \big(\mathrm{V}^{\perp}\big)'$ are bijective with $\|\big((\mathbf{B} \circ i)'\big)^{-1}\| = \|\big(\mathbf{B} \circ i\big)^{-1}\| \leqslant \frac{1}{\widehat{\beta}}$. In this way, we get the analogue of (2.17), that is

$$\|\mathbf{B}^{\mathrm{t}}(v)\|_{\mathrm{H}'} \geqslant \sup_{\substack{\tau \in \mathrm{V}^{\perp} \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}} \geqslant \widehat{\beta}\, \mathrm{dist}(v, \mathrm{W}) \qquad \forall\, v \in \mathrm{Q}\,, \tag{2.18}$$

from which, along with (2.7), we arrive at (2.9) with $\widetilde{\beta} := \frac{\widehat{\beta}}{C_{\mathrm{Q}}}$. Further details are omitted. □

We find it important to emphasize here, as announced in Chapter 1, that the equivalence between the inf-sup conditions (2.9) (cf. i)) and (2.10) (cf. iii)) holds with different constants $\widetilde{\beta}$ and $\widehat{\beta}$. Indeed, from the proof of Lemma 2.2 we notice that, starting from i), we first derive the inequality (2.17) with the same constant $\widetilde{\beta}$, thus yielding the partial implication summarized as

$$\left\{ \sup_{\substack{\tau \in \mathrm{H} \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}} \geqslant \widetilde{\beta}\, \|v\|_{\mathrm{Q}} \quad \forall\, v \in \mathrm{W}^{\perp} \right\} \quad \Rightarrow \quad \left\{ \sup_{\substack{v \in \mathrm{Q} \\ v \neq 0}} \frac{b(\tau, v)}{\|v\|_{\mathrm{Q}}} \geqslant \widetilde{\beta}\, \mathrm{dist}(\tau, \mathrm{V}) \quad \forall\, \tau \in \mathrm{H} \right\}. \tag{2.19}$$

Similarly, starting from iii), we obtain (2.18) with the same constant $\widehat{\beta}$, which gives rise to the partial implication

$$\left\{ \sup_{\substack{v \in \mathrm{Q} \\ v \neq 0}} \frac{b(\tau, v)}{\|v\|_{\mathrm{Q}}} \geqslant \widehat{\beta}\, \|\tau\|_{\mathrm{H}} \quad \forall\, \tau \in \mathrm{V}^{\perp} \right\} \quad \Rightarrow \quad \left\{ \sup_{\substack{\tau \in \mathrm{H} \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}} \geqslant \widehat{\beta}\, \mathrm{dist}(v, \mathrm{W}) \quad \forall\, v \in \mathrm{Q} \right\}. \tag{2.20}$$

However, as observed in the aforementioned proof, the expressions given by $\widetilde{\beta}\, \mathrm{dist}(\tau, \mathrm{V})$ in (2.19) and $\widehat{\beta}\, \mathrm{dist}(v, \mathrm{W})$ in (2.20) are then bounded below, respectively, by $\widehat{\beta}\, \|\tau\|_{\mathrm{H}}$ for each $\tau \in \mathrm{V}^{\perp}$, with $\widehat{\beta} = \frac{\widetilde{\beta}}{C_{\mathrm{H}}}$, and by $\widetilde{\beta}\, \|v\|_{\mathrm{Q}}$ for each $v \in \mathrm{W}^{\perp}$, with $\widetilde{\beta} = \frac{\widehat{\beta}}{C_{\mathrm{Q}}}$. These estimates explain the above use of the concept "partial", which refers to the fact that, in order to obtain the same constant in both sides of each implication, the latter must be stated up to as indicated in (2.19) and (2.20). Differently from this case, when H and Q are Hilbert spaces, full implications are achieved in the sense that there hold $\mathrm{dist}(\tau, \mathrm{V}) = ||\tau||_{\mathrm{H}}$ for each $\tau \in V^{\perp}$, and $\mathrm{dist}(v, W) = ||v||_{\mathrm{Q}}$ for each $v \in W^{\perp}$, so that now the equivalence between i) and iii) does hold with the same constant $\widetilde{\beta} = \widehat{\beta}$, as it has already been established in the available bibliography (see, e.g. [7, eq. (4.3.18), Theorem 4.3.1], [26, Lemma 2.1], and [10, Proposition 1.2 and eqs. (1.15) and (1.16), Chapter II]). Moreover, this fact can be written, equivalently, as

$$\inf_{\substack{\tau\in \mathrm{V}^{\perp}\\ \tau\neq 0}} \sup_{\substack{v\in \mathrm{Q}\\ v\neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}\,\|v\|_{\mathrm{Q}}} \;=\; \inf_{\substack{v\in \mathrm{W}^{\perp}\\ v\neq 0}} \sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}\,\|v\|_{\mathrm{Q}}} \;=\; \widetilde{\beta} \;>\; 0\,,$$

which is exactly what what was highlighted in (1.2) (cf. Chapter 1).

CHAPTER 3

The main results

In this chapter we address the well-posedness of (1.1) and its associated Galerkin scheme.

3.1 An equivalent setting

We begin by observing that the perturbed saddle-point formulation (1.1) can be re-stated, equivalently, as: Find $(\sigma, u) \in \mathrm{H} \times \mathrm{Q}$ such that

$$\mathbf{A}\big((\sigma, u), (\tau, v)\big) \;=\; \mathbf{F}(\tau, v) \qquad \forall\, (\tau, v) \in \mathrm{H} \times \mathrm{Q}\,, \tag{3.1}$$

where $\mathbf{A} : (\mathrm{H} \times \mathrm{Q}) \times (\mathrm{H} \times \mathrm{Q}) \to \mathrm{R}$ and $\mathbf{F} : \mathrm{H} \times \mathrm{Q} \to \mathrm{R}$ are the bounded bilinear form and linear functional, respectively, defined by

$$\mathbf{A}\big((\zeta, w), (\tau, v)\big) \;:=\; a(\zeta, \tau) + b(\tau, w) + b(\zeta, v) - c(w, v) \quad \forall\, (\zeta, w),\, (\tau, v) \in \mathrm{H} \times \mathrm{Q}\,, \tag{3.2}$$

and

$$\mathbf{F}(\tau, v) \,:=\, f(\tau) + g(v) \qquad \forall\, (\tau, v) \in \mathrm{H} \times \mathrm{Q}\,. \tag{3.3}$$

Throughout the rest of this part we consider the product norm

$$\|(\tau, v)\|_{\mathrm{H}\times\mathrm{Q}} \,:=\, \|\tau\|_{\mathrm{H}} + \|v\|_{\mathrm{Q}} \qquad \forall\, (\tau, v) \in \mathrm{H} \times \mathrm{Q}\,.$$

Thus, resorting to the Banach-Nečas-Babuška theorem (cf. [23, Theorem 2.6]), also known as the generalized Lax-Milgram lemma, we deduce that (1.1) (equivalently (3.1)) is well-posed if and only if the following hypotheses are satisfied:

1) there exists a constant $\alpha > 0$ such that

$$S(\zeta, w) := \sup_{\substack{(\tau,v)\in\mathrm{H}\times\mathrm{Q}\\(\tau,v)\neq\mathbf{0}}} \frac{\mathbf{A}\big((\zeta, w), (\tau, v)\big)}{\|(\tau, v)\|_{\mathrm{H}\times\mathrm{Q}}} \geqslant \alpha\, \|(\zeta, w)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\, (\zeta, w) \in \mathrm{H} \times \mathrm{Q}\,. \tag{3.4}$$

2) for each $(\tau, v) \in \mathrm{H} \times \mathrm{Q}$, $(\tau, v) \neq \mathbf{0}$:

$$\sup_{(\zeta,w)\in\mathrm{H}\times\mathrm{Q}} \mathbf{A}\big((\zeta, w), (\tau, v)\big) \,>\, 0\,. \tag{3.5}$$

Certainly, when $\mathbf{A}$ is symmetric, which is equivalent to assume that a and c are, 2) is redundant and hence it suffices to prove 1). In this regard, we stress that the supremum in (3.4) is equivalent to the expression $\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}$, where

$$\mathrm{F}_{(\zeta,w)}(\tau) \,:=\, \mathbf{A}\big((\zeta, w), (\tau, 0)\big) \qquad \forall\, \tau \in \mathrm{H}\,, \tag{3.6}$$

and

$$\mathrm{G}_{(\zeta,w)}(v) \,:=\, \mathbf{A}\big((\zeta, w), (0, v)\big) \qquad \forall\, v \in \mathrm{Q}\,. \tag{3.7}$$

More precisely, it is easy to see that

$$\frac{1}{2}\Big\{\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}\Big\} \,\leqslant\, S(\zeta, w) \,\leqslant\, \|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} \quad \forall\, (\zeta, w) \in \mathrm{H} \times \mathrm{Q}\,. \tag{3.8}$$

Consequently, a necessary and sufficient condition for 1) is given by the existence of a constant $\widetilde{C} > 0$ such that

$$\|(\zeta, w)\|_{\mathrm{H}\times\mathrm{Q}} \leqslant \widetilde{C}\left\{\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}\right\} \qquad \forall\,(\zeta, w) \in \mathrm{H}\times\mathrm{Q}\,. \tag{3.9}$$

The above is basically the same procedure that was utilized in the proof of [10, Theorem 1.2, Chapter II] for the Hilbert version of (1.1), as well as the one that, except for some necessary modifications, we adopt below in Chapter 3.2 for the proof of the main theorem.

From now on we denote by $\|a\|$, $\|b\|$, and $\|c\|$, the smallest positive constants such that

$$\begin{aligned} |a(\zeta,\tau)| &\leqslant \|a\|\,\|\zeta\|_{\mathrm{H}}\,\|\tau\|_{\mathrm{H}} && \forall\,(\zeta,\tau) \in \mathrm{H}\times\mathrm{H}\,, \\ |b(\tau,v)| &\leqslant \|b\|\,\|\tau\|_{\mathrm{H}}\,\|v\|_{\mathrm{Q}} && \forall\,(\tau,v) \in \mathrm{H}\times\mathrm{Q}\,, \\ |c(w,v)| &\leqslant \|c\|\,\|w\|_{\mathrm{Q}}\,\|v\|_{\mathrm{Q}} && \forall\,(w,v) \in \mathrm{Q}\times\mathrm{Q}\,. \end{aligned} \tag{3.10}$$

3.2 Continuous solvability

The main result providing sufficient conditions for the solvability of (1.1) is established now. While some of the definitions and hypotheses have already been introduced, for sake of clearness we include them again in its statement.

Theorem 3.1. *Let* H *and* Q *be reflexive Banach spaces, and let* $a : \mathrm{H}\times\mathrm{H} \to \mathrm{R}$*,* $b : \mathrm{H}\times\mathrm{Q} \to \mathrm{R}$*, and* $c : \mathrm{Q}\times\mathrm{Q} \to \mathrm{R}$ *be given bounded bilinear forms* (*cf.* (3.10))*. In addition, let* $\mathbf{B} : \mathrm{H} \to \mathrm{Q}'$ *and* $\mathbf{B}^{\mathrm{t}} : \mathrm{Q} \to \mathrm{H}'$ *be the bounded linear operators induced by* b (*cf.* (2.1))*, and let* $\mathrm{V} := \mathrm{N}(\mathbf{B})$ *and* $\mathrm{W} := \mathrm{N}(\mathbf{B}^{\mathrm{t}})$ *be the respective null spaces* (*cf.* (2.2), (2.3))*. Assume that:*

i) *there exist closed subspaces* $\mathrm{V}^{\perp}$ *and* $\mathrm{W}^{\perp}$ *of* H *and* Q*, respectively, such that* $\mathrm{H} = \mathrm{V} \oplus \mathrm{V}^{\perp}$ *and* $\mathrm{Q} = \mathrm{W} \oplus \mathrm{W}^{\perp}$*,*

ii) a *and* c *are symmetric and positive semi-definite, the latter meaning that*

$$a(\tau,\tau) \geqslant 0 \quad \forall\,\tau \in \mathrm{H} \quad \textit{and} \quad c(v,v) \geqslant 0 \quad \forall\,v \in \mathrm{Q}\,, \tag{3.11}$$

iii) *there exists a constant* $\widetilde{\alpha} > 0$ *such that*

$$\sup_{\substack{\tau\in V\\ \tau\neq 0}} \frac{a(\vartheta,\tau)}{\|\tau\|_H} \geqslant \widetilde{\alpha}\,\|\vartheta\|_H \qquad \forall\,\vartheta\in V\,, \tag{3.12}$$

iv) *there exists a constant* $\widetilde{\beta} > 0$ *such that (cf.* (2.9))

$$\sup_{\substack{\tau\in H\\ \tau\neq 0}} \frac{b(\tau,v)}{\|\tau\|_H} \geqslant \widetilde{\beta}\,\|v\|_Q \qquad \forall\,v\in W^\perp\,, \tag{3.13}$$

v) *and there exists a constant* $\widetilde{\gamma} > 0$ *such that*

$$\sup_{\substack{v\in W\\ v\neq 0}} \frac{c(z,v)}{\|v\|_Q} \geqslant \widetilde{\gamma}\,\|z\|_Q \qquad \forall\,z\in W\,. \tag{3.14}$$

Then, for each pair $(f,g)\in H'\times Q'$ *there exists a unique* $(\sigma,u)\in H\times Q$ *solution to* (1.1) (*equivalently* (3.1)). *Moreover, there exists a constant* $\widetilde{C} > 0$, *depending only on* $\|a\|$, $\|c\|$, $\widetilde{\alpha}$, $\widetilde{\beta}$, C_H (*cf.* (2.5)), *and* $\widetilde{\gamma}$, *such that*

$$\|(\sigma,u)\|_{H\times Q} \leqslant \widetilde{C}\left\{\|f\|_{H'} + \|g\|_{Q'}\right\}. \tag{3.15}$$

Proof. Because of the assumed symmetry of a and c (cf. **ii)**), and as previously remarked, the proof reduces to show (3.9). In turn, it is easy to see that the assumptions on c allow to show that

$$c(w,v) \leqslant c(w,w)^{1/2}c(v,v)^{1/2} \qquad \forall w,v\in Q, \tag{3.16}$$

which constitutes a kind of Cauchy-Schwarz inequality for c, and hence $|\cdot|_c := c(\cdot,\cdot)^{1/2}$ defines a semi-norm in Q. Now, given $(\zeta,w)\in H\times Q$, we first define the functionals $F_{(\zeta,w)}\in H'$ and $G_{(\zeta,w)}\in Q'$ according to (3.6) and (3.7), respectively, that is

$$F_{(\zeta,w)}(\tau) := a(\zeta,\tau) + b(\tau,w) \qquad \forall\,\tau\in H\,, \tag{3.17}$$

and

$$\mathrm{G}_{(\zeta,w)}(v) := b(\zeta, v) - c(w, v) \qquad \forall\, v \in \mathrm{Q}\,. \tag{3.18}$$

Now, according to **i)**, we decompose ζ and w as

$$\zeta = \zeta_0 + \bar{\zeta} \quad \text{and} \quad w = w_0 + \bar{w}\,, \tag{3.19}$$

with $\zeta_0 \in \mathrm{V}$, $\bar{\zeta} \in \mathrm{V}^\perp$, $w_0 \in \mathrm{W}$, and $\bar{w} \in \mathrm{W}^\perp$. Therefore, the rest of the proof consists of bounding each one of the four components specified in (3.19). We begin by observing from (3.17) that $\mathrm{F}_{(\zeta,w)}(\tau) = a(\zeta,\tau)$ for all $\tau \in \mathrm{V}$, so that applying (3.12) (cf. **iii)**) to $\vartheta = \zeta_0$, we get

$$\widetilde{\alpha}\,\|\zeta_0\|_{\mathrm{H}} \leqslant \sup_{\substack{\tau\in\mathrm{V}\\ \tau\neq 0}} \frac{a(\zeta_0,\tau)}{\|\tau\|_{\mathrm{H}}} = \sup_{\substack{\tau\in\mathrm{V}\\ \tau\neq 0}} \frac{\mathrm{F}_{(\zeta,w)}(\tau) - a(\bar{\zeta},\tau)}{\|\tau\|_{\mathrm{H}}}\,,$$

from which it readily follows that

$$\|\zeta_0\|_{\mathrm{H}} \leqslant \frac{1}{\widetilde{\alpha}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \frac{\|a\|}{\widetilde{\alpha}}\,\|\bar{\zeta}\|_{\mathrm{H}}\,. \tag{3.20}$$

In turn, in order to bound $\bar{\zeta}$, we employ the equivalence between i) and iii) of Lemma 2.2, thanks to which and (3.13) (cf. **iv)**), there holds (cf. (2.10))

$$\widehat{\beta}\,\|\tau\|_{\mathrm{H}} \leqslant \sup_{\substack{v\in\mathrm{Q}\\ v\neq 0}} \frac{b(\tau,v)}{\|v\|_{\mathrm{Q}}} \qquad \forall\,\tau \in \mathrm{V}^\perp\,,$$

with $\widehat{\beta} := \frac{\widetilde{\beta}}{C_{\mathrm{H}}}$. Thus, noting from (3.18) that $\mathrm{G}_{(\zeta,w)}(v) = b(\bar{\zeta},v) - c(w,v)$ for all $v \in \mathrm{Q}$, and applying the foregoing inequality to $\tau = \bar{\zeta} \in \mathrm{V}^\perp$, we find that

$$\widehat{\beta}\,\|\bar{\zeta}\|_{\mathrm{H}} \leqslant \sup_{\substack{v\in\mathrm{Q}\\ v\neq 0}} \frac{b(\bar{\zeta},v)}{\|v\|_{\mathrm{Q}}} = \sup_{\substack{v\in\mathrm{Q}\\ v\neq 0}} \frac{\mathrm{G}_{(\zeta,w)}(v) + c(w,v)}{\|v\|_{\mathrm{Q}}}\,,$$

from which, using thanks to (3.16) and the boundedness of c, that $c(w,v) \leqslant \|c\|^{1/2}\,|w|_c\,\|v\|_{\mathrm{Q}}$, we deduce that

$$\|\bar{\zeta}\|_{\mathrm{H}} \leqslant \frac{1}{\widehat{\beta}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} + \frac{\|c\|^{1/2}}{\widehat{\beta}}\,|w|_c\,. \tag{3.21}$$

Thus, as a direct consequence of (3.20) and (3.21), we have the following preliminary bound

$$\|\zeta\|_{\mathrm{H}} \leqslant \frac{1}{\widetilde{\alpha}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \left(1+\frac{\|a\|}{\widetilde{\alpha}}\right)\frac{1}{\widehat{\beta}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} + \left(1+\frac{\|a\|}{\widetilde{\alpha}}\right)\frac{\|c\|^{1/2}}{\widehat{\beta}}\,|w|_c\,. \tag{3.22}$$

Certainly, it remains to bound $|w|_c$ in terms of $\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'}$ and $\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}$, which will be done later on. Meanwhile, we address the estimate of $\|w\|_{\mathrm{Q}}$. In fact, from the definition of $\mathrm{G}_{(\zeta,w)}$ (cf. (3.18)) we have $\mathrm{G}_{(\zeta,w)}(v) = -c(w,v) = -c(w_0,v) - c(\bar{w},v)$ for all $v \in \mathrm{W}$, and hence, applying (3.14) (cf. **v)**) to $z = w_0 \in \mathrm{W}$, we get

$$\widetilde{\gamma}\,\|w_0\|_{\mathrm{Q}} \leqslant \sup_{\substack{v\in \mathrm{W}\\ v\neq 0}} \frac{c(w_0,v)}{\|v\|_{\mathrm{Q}}} = \sup_{\substack{v\in \mathrm{W}\\ v\neq 0}} \frac{-\mathrm{G}_{(\zeta,w)}(v) - c(\bar{w},v)}{\|v\|_{\mathrm{Q}}}\,, \tag{3.23}$$

which yields

$$\|w_0\|_{\mathrm{Q}} \leqslant \frac{1}{\widetilde{\gamma}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} + \frac{\|c\|}{\widetilde{\gamma}}\,\|\bar{w}\|_{\mathrm{Q}}\,. \tag{3.24}$$

Furthermore, it is clear from (3.17) that $\mathrm{F}_{(\zeta,w)}(\tau) = a(\zeta,\tau) + b(\tau,\bar{w})$ for all $\tau \in \mathrm{H}$, so that making use of (3.13) (cf. **iv)**) with $v = \bar{w} \in \mathrm{W}^{\perp}$, we arrive at

$$\widetilde{\beta}\,\|\bar{w}\|_{\mathrm{Q}} \leqslant \sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}} \frac{b(\tau,\bar{w})}{\|\tau\|_{\mathrm{H}}} = \sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}} \frac{\mathrm{F}_{(\zeta,w)}(\tau) - a(\zeta,\tau)}{\|\tau\|_{\mathrm{H}}}\,, \tag{3.25}$$

which implies that

$$\|\bar{w}\|_{\mathrm{Q}} \leqslant \frac{1}{\widetilde{\beta}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \frac{\|a\|}{\widetilde{\beta}}\,\|\zeta\|_{\mathrm{H}}\,. \tag{3.26}$$

In this way, (3.24) and (3.26) give

$$\|w\|_{\mathrm{Q}} \leqslant \left(1+\frac{\|c\|}{\widetilde{\gamma}}\right)\frac{1}{\widetilde{\beta}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \frac{1}{\widetilde{\gamma}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} + \left(1+\frac{\|c\|}{\widetilde{\gamma}}\right)\frac{\|a\|}{\widetilde{\beta}}\,\|\zeta\|_{\mathrm{H}}\,. \tag{3.27}$$

On the other hand, we now aim to bound $|w|_c^2 := c(w,w)$. Indeed, evaluating $\mathrm{F}_{(\zeta,w)}$ (cf. (3.17)) and $\mathrm{G}_{(\zeta,w)}$ (cf. (3.18)) in ζ and w, respectively, and subtracting the resulting expressions, we obtain

$$a(\zeta,\zeta) + c(w,w) = \mathrm{F}_{(\zeta,w)}(\zeta) - \mathrm{G}_{(\zeta,w)}(w)\,,$$

from which, according to the positive semi-definiteness of a (cf. **ii)**), it follows that

$$|w|_c^2 \;\leqslant\; \|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'}\, \|\zeta\|_{\mathrm{H}} \;+\; \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}\, \|w\|_{\mathrm{Q}}\,. \tag{3.28}$$

Moreover, employing the bounds for $\|\zeta\|_{\mathrm{H}}$ and $\|w\|_{\mathrm{Q}}$ provided by (3.22) and (3.27), using Young's inequality conveniently, and performing several algebraic manipulations, we deduce from (3.28) that

$$|w|_c^2 \;\leqslant\; C_1\, \|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'}^2 \;+\; C_2\, \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}^2 \;+\; \frac{1}{2}\, |w|_c^2\,, \tag{3.29}$$

where C_1 and C_2, positive constants depending on $\|a\|$, $\|c\|$, $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widehat{\beta}$, and $\widetilde{\gamma}$, are given explicitly as

$$C_1 \;:=\; \left(1+\frac{\|a\|}{\widetilde{\alpha}}\right)\left\{\left(1+\frac{\|a\|}{\widetilde{\alpha}}\right)\frac{\|c\|}{\widehat{\beta}^2} \,+\, \left(1+\frac{\|c\|}{\widetilde{\gamma}}\right)\frac{1}{2\widetilde{\beta}} \,+\, \frac{1}{2\widehat{\beta}}\right\} \,+\, \frac{1}{\widetilde{\alpha}} \tag{3.30}$$

and

$$\begin{aligned} C_2 \;:=\; & \left(1+\frac{\|a\|}{\widetilde{\alpha}}\right)\left(1+\frac{\|c\|}{\widetilde{\gamma}}\right)\left\{\left(1+\frac{\|a\|}{\widetilde{\alpha}}\right)\left(1+\frac{\|c\|}{\widetilde{\gamma}}\right)\frac{\|a\|^2\|c\|}{\widetilde{\beta}^2\widehat{\beta}^2} \,+\, \frac{\|a\|}{\widetilde{\beta}\widehat{\beta}} \,+\, \frac{1}{2\widetilde{\beta}}\right\} \\ & \quad +\quad \left(1+\frac{\|a\|}{\widetilde{\alpha}}\right)\frac{1}{2\widehat{\beta}} \,+\, \frac{1}{\widetilde{\gamma}}\,. \end{aligned} \tag{3.31}$$

Finally, it is easy to see from (3.29) that

$$|w|_c \;\leqslant\; \Big(2\,\max\big\{C_1, C_2\big\}\Big)^{1/2} \Big\{\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \;+\; \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}\Big\}\,, \tag{3.32}$$

which, replaced back into (3.22), completes the upper bound of $\|\zeta\|_{\mathrm{H}}$. In turn, employing the latter in (3.27) leads to the respective estimate for $\|w\|_{\mathrm{Q}}$, and the proof is concluded. □

Bearing in mind the equivalence (3.8), we notice here that the proof of the previous theorem establishes, equivalently, that the global inf-sup condition for $\mathbf{A}$ holds, namely

$$\sup_{\substack{(\tau,v)\in \mathrm{H}\times\mathrm{Q}\\ (\tau,v)\neq \mathbf{0}}} \frac{\mathbf{A}\big((\zeta,w),(\tau,v)\big)}{\|(\tau,v)\|_{\mathrm{H}\times\mathrm{Q}}} \;\geqslant\; \frac{1}{2\,\widetilde{C}}\,\|(\zeta,w)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\zeta,w)\in \mathrm{H}\times\mathrm{Q}\,. \tag{3.33}$$

On the other hand, and related to a previous remark (right after the proof of Lemma 2.2) on the constants $\widetilde{\beta}$ and $\widehat{\beta}$ that appear in the inf-sup conditions (2.9) and (2.10), respectively, we stress here that the fact that they do not coincide does not yield any difficulty in the solvability result provided by Theorem 3.1. The reason is certainly because the difference between them is determined only by the reciprocals of the constants C_{H} and C_{Q}, which depend on the continuous spaces H and Q, which are fixed. However, this issue becomes a delicate point for the associated Galerkin scheme, to be addressed next, since the finite element subspaces employed are varying, and hence, the respective constants could vary as well with them, particularly with their dimensions. According to it, in this case we can not employ the equivalence between i) and iii) from Lemma 2.2 as such, but rather assume (which means proving when dealing with specific subspaces) that both discrete inf-sup conditions are satisfied with constants independent of those dimensions.

3.3 Discrete solvability

We now let $\{\mathrm{H}_h\}_{h>0}$ and $\{\mathrm{Q}_h\}_{h>0}$ be families of finite dimensional subspaces of H and Q, respectively, and introduce the Galerkin scheme associated with (1.1): Find $(\sigma_h, u_h) \in \mathrm{H} \times \mathrm{Q}_h$ such that

$$\begin{aligned} a(\sigma_h, \tau_h) + b(\tau_h, u_h) &= f(\tau_h) \qquad \forall\, \tau_h \in \mathrm{H}_h\,, \\ b(\sigma_h, v_h) - c(u_h, v_h) &= g(v_h) \qquad \forall\, v_h \in \mathrm{Q}_h\,. \end{aligned} \tag{3.34}$$

Then, we let $\mathbf{B}_h : \mathrm{H}_h \to \mathrm{Q}_h'$ and $\mathbf{B}_h^{\mathrm{t}} : \mathrm{Q}_h \to \mathrm{H}_h'$ be the discrete versions of the bounded linear operators induced by b (cf. (2.1)), and define the respective discrete null spaces

$$\mathrm{V}_h \,:=\, \mathrm{N}(\mathbf{B}_h) \,:=\, \Big\{ \tau_h \in \mathrm{H}_h : \quad b(\tau_h, v_h) = 0 \quad \forall\, v_h \in \mathrm{Q}_h \Big\} \tag{3.35}$$

and

$$\mathrm{W}_h \,:=\, \mathrm{N}(\mathbf{B}_h^{\mathrm{t}}) \,:=\, \Big\{ v_h \in \mathrm{Q}_h : \quad b(\tau_h, v_h) = 0 \quad \forall\, \tau_h \in \mathrm{H}_h \Big\}\,. \tag{3.36}$$

In this case, the existence of closed subspaces $\mathrm{V}_h^{\perp}$ and $\mathrm{W}_h^{\perp}$ of H_h and Q_h, respectively, satisfying the decompositions $\mathrm{H}_h = \mathrm{V}_h \oplus \mathrm{V}_h^{\perp}$ and $\mathrm{Q}_h = \mathrm{W}_h \oplus \mathrm{W}_h^{\perp}$, is guaranteed by the fact that both

H_h and Q_h are finite dimensional. As a consequence, the solvability result for (3.34), which is stated next, does not need to incorporate the aforementioned existence as an assumption (see hypothesis **i)** in Theorem 3.1) but rather as a fact. In this way, the discrete version of that theorem reads as follows. Hereafter, the expression "independent of h" means independent of the finite element subspaces H_h and Q_h.

Theorem 3.2. *In addition to the previous notations and definitions, assume that:*

i) *a and c are symmetric and positive semi-definite (cf. (3.11)),*

ii) *there exists a constant $\widetilde{\alpha}_d > 0$, independent of h, such that*

$$\sup_{\substack{\tau_h \in V_h \\ \tau_h \neq 0}} \frac{a(\vartheta_h, \tau_h)}{\|\tau_h\|_H} \geqslant \widetilde{\alpha}_d \|\vartheta_h\|_H \qquad \forall\, \vartheta_h \in V_h\,, \tag{3.37}$$

iii) *there exists a constant $\widetilde{\beta}_d > 0$, independent of h, such that*

$$\sup_{\substack{\tau_h \in H_h \\ \tau_h \neq 0}} \frac{b(\tau_h, v_h)}{\|\tau_h\|_H} \geqslant \widetilde{\beta}_d \|v_h\|_Q \qquad \forall\, v_h \in W_h^{\perp}\,, \tag{3.38}$$

iv) *there exists a constant $\widehat{\beta}_d > 0$, independent of h, such that*

$$\sup_{\substack{v_h \in Q_h \\ v_h \neq 0}} \frac{b(\tau_h, v_h)}{\|v_h\|_Q} \geqslant \widehat{\beta}_d \|\tau_h\|_H \qquad \forall\, \tau_h \in V_h^{\perp}\,, \tag{3.39}$$

v) *and there exists a constant $\widetilde{\gamma}_d > 0$, independent of h, such that*

$$\sup_{\substack{v_h \in W_h \\ v_h \neq 0}} \frac{c(z_h, v_h)}{\|v_h\|_Q} \geqslant \widetilde{\gamma}_d \|z_h\|_Q \qquad \forall\, z_h \in W_h\,. \tag{3.40}$$

Then, for each pair $(f, g) \in H' \times Q'$ there exists a unique $(\sigma_h, u_h) \in H_h \times Q_h$ solution to (3.34). Moreover, there exists a constant $\widetilde{C}_d > 0$, depending only on $\|a\|$, $\|c\|$, $\widetilde{\alpha}_d$, $\widetilde{\beta}_d$, $\widehat{\beta}_d$, and $\widetilde{\gamma}_d$, such that

$$\|\sigma_h\|_H + \|u_h\|_Q \leqslant \widetilde{C}_d \left\{ \|f\|_{H'} + \|g\|_{Q'} \right\}. \tag{3.41}$$

Proof. It follows analogously to the proof of Theorem 3.1, except for the fact that, instead of considering **iv)** as a consequence of **iii)**, the former is assumed here independently. Alternatively, this proof follows from a direct application of a slight modification of Theorem 3.1 in which a continuous version of the present hypothesis **iv)** is added. □

Similarly as noticed right after the proof of Theorem 3.1, we stress here that the previous theorem provides, equivalently, the global discrete inf-sup condition for $\mathbf{A}$, that is

$$\sup_{\substack{(\tau_h,v_h)\in \mathrm{H}_h\times \mathrm{Q}_h\\ (\tau_h,v_h)\neq \mathbf{0}}} \frac{\mathbf{A}\big((\zeta_h,w_h),(\tau_h,v_h)\big)}{\|(\tau_h,v_h)\|_{\mathrm{H}\times\mathrm{Q}}} \geqslant \frac{1}{2\,\widetilde{C}_{\mathrm{d}}}\,\|(\zeta_h,w_h)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\zeta_h,w_h)\in \mathrm{H}_h\times\mathrm{Q}_h\,. \tag{3.42}$$

Having established the well-posedness of the continuous and discrete formulations of interest, we now prove the respective Cea estimate. In what follows, given a subspace X_h of a generic Banach space $(X, \|\cdot\|_X)$, we set $\operatorname{dist}(x, X_h) := \inf_{x_h\in X_h} \|x-x_h\|_X$ for each $x\in X$.

Theorem 3.3. *Assume the hypotheses of Theorems* 3.1 *and* 3.2, *and let* $(\sigma,u)\in \mathrm{H}\times\mathrm{Q}$ *and* $(\sigma_h,u_h)\in \mathrm{H}_h\times\mathrm{Q}_h$ *be the unique solutions of* (1.1) *and* (3.34), *respectively. Then, there exists a constant* $\widehat{C}_{\mathrm{d}}>0$, *depending only on* $\|a\|$, $\|b\|$, $\|c\|$, $\widetilde{\alpha}_{\mathrm{d}}$, $\widetilde{\beta}_{\mathrm{d}}$, $\widehat{\beta}_{\mathrm{d}}$, *and* $\widetilde{\gamma}_{\mathrm{d}}$, *such that*

$$\|\sigma-\sigma_h\|_{\mathrm{H}} + \|u-u_h\|_{\mathrm{Q}} \leqslant \widehat{C}_{\mathrm{d}}\Big\{\operatorname{dist}(\sigma,\mathrm{H}_h) + \operatorname{dist}(u,\mathrm{Q}_h)\Big\}. \tag{3.43}$$

Proof. Due to the equivalence between (1.1) and (3.1), it is clear that (3.34) can be, equivalently, rewritten as: Find $(\sigma_h,u_h)\in \mathrm{H}_h\times\mathrm{Q}_h$ such that

$$\mathbf{A}\big((\sigma_h,u_h),(\tau_h,v_h)\big) = \mathbf{F}(\tau_h,v_h) \qquad \forall\,(\tau_h,v_h)\in \mathrm{H}_h\times\mathrm{Q}_h\,, \tag{3.44}$$

and hence, the derivation of (3.43) proceeds in the usual way for formulations of this kind. More precisely, we first apply the triangle inequality to obtain

$$\|(\sigma,u)-(\sigma_h,u_h)\|_{\mathrm{H}\times\mathrm{Q}} \leqslant \|(\sigma,u)-(\zeta_h,w_h)\|_{\mathrm{H}\times\mathrm{Q}} + \|(\sigma_h,u_h)-(\zeta_h,w_h)\|_{\mathrm{H}\times\mathrm{Q}}\,,$$

for each $(\zeta_h,w_h)\in \mathrm{H}_h\times\mathrm{Q}_h$, then we employ the global discrete inf-sup condition (3.42), which gives

$$\|(\sigma_h, u_h) - (\zeta_h, w_h)\|_{\mathrm{H}\times\mathrm{Q}} \leqslant 2\,\widetilde{C}_{\mathrm{d}} \sup_{\substack{(\tau_h,v_h)\in\mathrm{H}_h\times\mathrm{Q}_h\\(\tau_h,v_h)\neq\mathbf{0}}} \frac{\mathbf{A}\big((\sigma_h, u_h) - (\zeta_h, w_h), (\tau_h, v_h)\big)}{\|(\tau_h, v_h)\|_{\mathrm{H}\times\mathrm{Q}}}\,,$$

and finally we use that $\mathbf{A}\big((\sigma_h, u_h), (\tau_h, v_h)\big) = \mathbf{A}\big((\sigma, u), (\tau_h, v_h)\big)$ for each $(\tau_h, v_h) \in \mathrm{H}_h \times \mathrm{Q}_h$, along with the boundedness of $\mathbf{A}$. In this way, we readily arrive at (3.43) with $\widehat{C}_{\mathrm{d}} := 1 + 2\,\widetilde{C}_{\mathrm{d}}\,\|\mathbf{A}\|$. Alternatively, we can also derive (3.43) by proceeding similarly to [26, Theorem 2.5], that is by employing the corresponding Galerkin projection. □

3.4 Continuous solvability when $\mathrm{W} = \{0\}$

We now assume the particular case $\mathrm{W} = \{0\}$, equivalently $\mathrm{W}^{\perp} = \mathrm{Q}$, which means that the hypothesis **iv)** of Theorem 3.1 reduces to the existence of a constant $\widetilde{\beta} > 0$ such that

$$\|\mathbf{B}^{\mathrm{t}}(v)\|_{\mathrm{H}'} := \sup_{\substack{\tau\in\mathrm{H}\\ \tau\neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}} \geqslant \widetilde{\beta}\,\|v\|_{\mathrm{Q}} \qquad \forall\, v \in \mathrm{Q}\,. \tag{3.45}$$

Moreover, recalling from (2.11) that $\mathbf{B}^{\mathrm{t}} = \mathbf{B}' \circ \mathcal{J}_{\mathrm{Q}}$, and using the reflexivity of Q and the fact that $\mathcal{J}_{\mathrm{Q}}$ is an isometry, we observe that (3.45) can be rewritten, equivalently, as

$$\|\mathbf{B}'(\mathcal{G})\|_{\mathrm{H}'} \geqslant \widetilde{\beta}\,\|\mathcal{G}\|_{\mathrm{Q}''} \qquad \forall\, \mathcal{G} \in \mathrm{Q}''\,. \tag{3.46}$$

Note that the above establishes that $\mathbf{B}' : \mathrm{Q}'' \to \mathrm{H}'$ is injective and of closed range, which is equivalent to saying that $\mathbf{B} : \mathrm{H} \to \mathrm{Q}'$ is surjective. Thus, applying the converse implication of the characterization provided in [23, Lemma A.42], which is originally proved in [3], we deduce from (3.46) that for each $G \in \mathrm{Q}'$ there exists $\vartheta \in \mathrm{H}$ such that

$$\mathbf{B}(\vartheta) = G \qquad \text{and} \qquad \|\vartheta\|_{\mathrm{H}} \leqslant \frac{1}{\widetilde{\beta}}\,\|G\|_{\mathrm{Q}'}\,. \tag{3.47}$$

In this way, having the above result to our disposal in the present case, we can improve the statement of Theorem 3.1 as follows, highlighting in advance that no topological complement of V nor a continuous inf-sup condition for c are needed now.

Theorem 3.4. *Let* H *and* Q *be reflexive Banach spaces, and let* $a : \mathrm{H} \times \mathrm{H} \to \mathrm{R}$, $b : \mathrm{H} \times \mathrm{Q} \to \mathrm{R}$, *and* $c : \mathrm{Q} \times \mathrm{Q} \to \mathrm{R}$ *be given bounded bilinear forms (cf. (3.10)). In addition, let* $\mathbf{B} : \mathrm{H} \to \mathrm{Q}'$ *be one of the bounded linear operators induced by* b *(cf. (2.1)), and let* $\mathrm{V} := \mathrm{N}(\mathbf{B})$ *be the respective null space (cf. (2.2)). Assume that:*

i) *a and c are symmetric and positive semi-definite, the latter meaning that*

$$a(\tau, \tau) \geqslant 0 \quad \forall\, \tau \in \mathrm{H} \quad \textit{and} \quad c(v, v) \geqslant 0 \quad \forall\, v \in \mathrm{Q}\,, \tag{3.48}$$

ii) *there exists a constant* $\widetilde{\alpha} > 0$ *such that*

$$\sup_{\substack{\tau \in \mathrm{V} \\ \tau \neq 0}} \frac{a(\vartheta, \tau)}{\|\tau\|_{\mathrm{H}}} \geqslant \widetilde{\alpha}\, \|\vartheta\|_{\mathrm{H}} \qquad \forall\, \vartheta \in \mathrm{V}\,, \tag{3.49}$$

iii) *and there exists a constant* $\widetilde{\beta} > 0$ *such that*

$$\sup_{\substack{\tau \in \mathrm{H} \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}} \geqslant \widetilde{\beta}\, \|v\|_{\mathrm{Q}} \qquad \forall\, v \in \mathrm{Q}\,, \tag{3.50}$$

Then, for each pair $(f, g) \in \mathrm{H}' \times \mathrm{Q}'$ *there exists a unique* $(\sigma, u) \in \mathrm{H} \times \mathrm{Q}$ *solution to (1.1) (equivalently (3.1)). Moreover, there exists a constant* $\widetilde{C} > 0$*, depending only on* $\|a\|$, $\|c\|$, $\widetilde{\alpha}$, *and* $\widetilde{\beta}$*, such that*

$$\|(\sigma, u)\|_{\mathrm{H} \times \mathrm{Q}} \leqslant \widetilde{C} \left\{ \|f\|_{\mathrm{H}'} + \|g\|_{\mathrm{Q}'} \right\}. \tag{3.51}$$

Proof. We proceed analogously to the proof of Theorem 3.1, though with a key difference in the decomposition to be introduced below. Indeed, given $(\zeta, w) \in \mathrm{H} \times \mathrm{Q}$, we first define the functionals $\mathrm{F}_{(\zeta,w)} \in \mathrm{H}'$ and $\mathrm{G}_{(\zeta,w)} \in \mathrm{Q}'$ as we did in (3.17) and (3.18), respectively, and aim to establish the inequality (3.9). To this end, and bearing in mind **iii)**, we apply (3.47) to $G := \mathbf{B}(\zeta) \in \mathrm{Q}'$, thus yielding the existence of $\bar{\zeta} \in \mathrm{H}$ such that

$$\mathbf{B}(\bar{\zeta}) = \mathbf{B}(\zeta) \qquad \text{and} \qquad \|\bar{\zeta}\|_{\mathrm{H}} \leqslant \frac{1}{\widetilde{\beta}}\, \|\mathbf{B}(\zeta)\|_{\mathrm{Q}'} = \frac{1}{\widetilde{\beta}}\, \|\mathbf{B}(\bar{\zeta})\|_{\mathrm{Q}'}\,. \tag{3.52}$$

As a consequence, ζ can be decomposed as

$$\zeta \,=\, \zeta_0 + \bar{\zeta}\,, \tag{3.53}$$

with $\zeta_0 := \zeta - \bar{\zeta} \in \mathrm{V}$. As previously announced, we stress here that there is no need to identify a topological complement to which $\bar{\zeta}$ belongs, but rather to be able to bound $\|\bar{\zeta}\|_{\mathrm{H}}$, which is indeed guaranteed by the inequality from (3.52). Then, observing from (3.17) that $\mathrm{F}_{(\zeta,w)}(\tau) \,=\, a(\zeta,\tau)$ for all $\tau \in \mathrm{V}$, and applying (3.49) (cf. **ii)**) to $\vartheta = \zeta_0$, we deduce, exactly as for the derivation of (3.20), that

$$\|\zeta_0\|_{\mathrm{H}} \,\leqslant\, \frac{1}{\widetilde{\alpha}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \frac{\|a\|}{\widetilde{\alpha}}\,\|\bar{\zeta}\|_{\mathrm{H}}\,. \tag{3.54}$$

Next, noting from (3.18) that $b(\bar{\zeta},v) \,=\, \mathrm{G}_{(\zeta,w)}(v) + c(w,v)$ for all $v \in \mathrm{Q}$, it follows from the inequality in (3.52) that

$$\|\bar{\zeta}\|_{\mathrm{H}} \,\leqslant\, \frac{1}{\widetilde{\beta}}\,\|\mathbf{B}(\bar{\zeta})\|_{\mathrm{Q}'} \,=\, \frac{1}{\widetilde{\beta}}\,\sup_{\substack{v\in\mathrm{Q}\\ v\neq 0}} \frac{b(\bar{\zeta},v)}{\|v\|_{\mathrm{Q}}} \,=\, \frac{1}{\widetilde{\beta}}\,\sup_{\substack{v\in\mathrm{Q}\\ v\neq 0}} \frac{\mathrm{G}_{(\zeta,w)}(v) + c(w,v)}{\|v\|_{\mathrm{Q}}}\,,$$

from which, similarly to the derivation of (3.21), we arrive at

$$\|\bar{\zeta}\|_{\mathrm{H}} \,\leqslant\, \frac{1}{\widetilde{\beta}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} \,+\, \frac{\|c\|^{1/2}}{\widetilde{\beta}}\,|w|_c\,, \tag{3.55}$$

and hence, thanks to (3.54) and (3.55), the analogue of (3.22) becomes

$$\|\zeta\|_{\mathrm{H}} \,\leqslant\, \frac{1}{\widetilde{\alpha}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \Big(1 + \frac{\|a\|}{\widetilde{\alpha}}\Big)\,\frac{1}{\widetilde{\beta}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} \,+\, \Big(1 + \frac{\|a\|}{\widetilde{\alpha}}\Big)\,\frac{\|c\|^{1/2}}{\widetilde{\beta}}\,|w|_c\,. \tag{3.56}$$

Furthermore, we know from (3.17) that $b(\tau,w) \,=\, \mathrm{F}_{(\zeta,w)}(\tau) - a(\zeta,\tau)$ for all $\tau \in \mathrm{H}$, so that applying (3.50) (cf. **iii)**) with $v = w \in \mathrm{Q}$, we readily deduce that

$$\|w\|_{\mathrm{Q}} \,\leqslant\, \frac{1}{\widetilde{\beta}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \frac{\|a\|}{\widetilde{\beta}}\,\|\zeta\|_{\mathrm{H}}\,. \tag{3.57}$$

The rest of the proof proceeds exactly as the one of Theorem 3.1. In particular, we obtain (cf.

(3.28))

$$|w|_c^2 \leqslant \|F_{(\zeta,w)}\|_{H'} \|\zeta\|_H + \|G_{(\zeta,w)}\|_{Q'} \|w\|_Q , \tag{3.58}$$

and then, employing the bounds for $\|\zeta\|_H$ and $\|w\|_Q$ provided by (3.56) and (3.57), and applying Young's inequality conveniently, we arrive at

$$|w|_c \leqslant \Big(2 \max\big\{\widetilde{C}_1, \widetilde{C}_2\big\}\Big)^{1/2} \Big\{\|F_{(\zeta,w)}\|_{H'} + \|G_{(\zeta,w)}\|_{Q'}\Big\} , \tag{3.59}$$

where

$$\widetilde{C}_1 := \frac{1}{\widetilde{\alpha}} + \frac{1}{\widetilde{\beta}}\Big(1 + \frac{\|a\|}{\widetilde{\alpha}}\Big) + \frac{\|c\|}{\widetilde{\beta}^2}\Big(1 + \frac{\|a\|}{\widetilde{\alpha}}\Big)^2 \tag{3.60}$$

and

$$\widetilde{C}_2 := \frac{1}{\widetilde{\beta}}\Big(1 + \frac{\|a\|}{\widetilde{\alpha}}\Big) \left\{1 + \frac{\|a\|}{\widetilde{\beta}} + \frac{\|a\|^2 \|c\|}{\widetilde{\beta}^3}\Big(1 + \frac{\|a\|}{\widetilde{\alpha}}\Big)\right\} . \tag{3.61}$$

Finally, (3.59), (3.56), and (3.57) complete the proof. □

3.5 Discrete solvability when $W_h = \{0\}$

In what follows we consider the same notations and definitions given at the beginning of Chapter 3.3. Then, similarly to the analysis in Chapter 3.3, we now assume that $W_h = \{0\}$, which means that the hypothesis **iii)** of Theorem 3.2 reduces to the existence of a constant $\widetilde{\beta}_d > 0$, independent of h, such that

$$\|\mathbf{B}_h^t(v_h)\|_{H_h'} := \sup_{\substack{\tau_h \in H_h \\ \tau_h \neq 0}} \frac{b(\tau_h, v_h)}{\|\tau_h\|_H} \geqslant \widetilde{\beta}_d \|v_h\|_Q \qquad \forall\, v_h \in Q_h . \tag{3.62}$$

Therefore, noting that the discrete version of the respective identity in (2.11) becomes $\mathbf{B}_h^t = \mathbf{B}_h' \circ \mathcal{J}_{Q_h}$, we realize that (3.62) is equivalent to stating

$$\|\mathbf{B}_h'(\mathcal{G}_h)\|_{H_h'} \geqslant \widetilde{\beta}_d \|\mathcal{G}_h\|_{Q_h''} \qquad \forall\, \mathcal{G}_h \in Q_h'' , \tag{3.63}$$

so that applying again the converse implication of [23, Lemma A.42], we conclude that for each $G_h \in Q_h'$ there exists $\vartheta_h \in H_h$ such that

$$\mathbf{B}_h(\vartheta_h) \,=\, G_h \qquad \text{and} \qquad \|\vartheta_h\|_{\mathrm{H}} \,\leqslant\, \frac{1}{\widetilde{\beta}_{\mathrm{d}}}\, \|G_h\|_{\mathrm{Q}_h'} \,. \tag{3.64}$$

Consequently, we are now in position to present the discrete version of Theorem 3.4.

Theorem 3.5. *Assume that:*

i) *a and c are symmetric and positive semi-definite* (*cf.* (3.11)),

ii) *there exists a constant $\widetilde{\alpha}_{\mathrm{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\tau_h \in \mathrm{V}_h \\ \tau_h \neq 0}} \frac{a(\vartheta_h, \tau_h)}{\|\tau_h\|_{\mathrm{H}}} \,\geqslant\, \widetilde{\alpha}_{\mathrm{d}}\, \|\vartheta_h\|_{\mathrm{H}} \qquad \forall\, \vartheta_h \in \mathrm{V}_h \,, \tag{3.65}$$

iii) *and there exists a constant $\widetilde{\beta}_{\mathrm{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\tau_h \in \mathrm{H}_h \\ \tau_h \neq 0}} \frac{b(\tau_h, v_h)}{\|\tau_h\|_{\mathrm{H}}} \,\geqslant\, \widetilde{\beta}_{\mathrm{d}}\, \|v_h\|_{\mathrm{Q}} \qquad \forall\, v_h \in \mathrm{Q}_h \,. \tag{3.66}$$

Then, for each pair $(f, g) \in \mathrm{H}' \times \mathrm{Q}'$ there exists a unique $(\sigma_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ solution to (3.34)*. Moreover, there exists a constant $\widetilde{C}_{\mathrm{d}} > 0$, depending only on $\|a\|$, $\|c\|$, $\widetilde{\alpha}_{\mathrm{d}}$, and $\widetilde{\beta}_{\mathrm{d}}$, such that*

$$\|\sigma_h\|_{\mathrm{H}} \,+\, \|u_h\|_{\mathrm{Q}} \,\leqslant\, \widetilde{C}_{\mathrm{d}} \left\{ \|f\|_{\mathrm{H}'} \,+\, \|g\|_{\mathrm{Q}'} \right\}. \tag{3.67}$$

Proof. It proceeds analogously to the proof of Theorem 3.4, bearing in mind that, instead of (3.47), we now apply (3.64). In this way, given $(\zeta_h, w_h) \in \mathrm{H}_h \times \mathrm{Q}_h$, we deduce the existence of $\bar{\zeta}_h \in \mathrm{H}_h$ such that

$$\mathbf{B}_h(\bar{\zeta}_h) \,=\, \mathbf{B}_h(\zeta_h) \qquad \text{and} \qquad \|\bar{\zeta}_h\|_{\mathrm{H}} \,\leqslant\, \frac{1}{\widetilde{\beta}_{\mathrm{d}}}\, \|\mathbf{B}_h(\zeta_h)\|_{\mathrm{Q}'} \,=\, \frac{1}{\widetilde{\beta}_{\mathrm{d}}}\, \|\mathbf{B}_h(\bar{\zeta}_h)\|_{\mathrm{Q}'} \,, \tag{3.68}$$

so that ζ_h can be decomposed as

$$\zeta_h \,=\, \zeta_{0,h} + \bar{\zeta}_h \,, \tag{3.69}$$

with $\zeta_{0,h} := \zeta_h - \bar{\zeta}_h \in \mathrm{V}_h$. The rest of the proof is as the one of Theorem 3.4. Further details are omitted. □

Needless to say, we remark that the global inf-sup conditions stated in (3.33) and (3.42) are also consequence of the proofs of Theorems 3.4 and 3.5, respectively. We end this chapter with the corresponding Cea estimate, whose proof is exactly as that of Theorem 3.3.

Theorem 3.6. *Assume the hypotheses of Theorems* 3.4 *and* 3.5*, and let* $(\sigma, u) \in \mathrm{H} \times \mathrm{Q}$ *and* $(\sigma_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ *be the unique solutions of* (1.1) *and* (3.34)*, respectively. Then, there exists a constant* $\widehat{C}_{\mathrm{d}} > 0$*, depending only on* $\|a\|$, $\|b\|$, $\|c\|$, $\widetilde{\alpha}_{\mathrm{d}}$*, and* $\widetilde{\beta}_{\mathrm{d}}$*, such that*

$$\|\sigma - \sigma_h\|_{\mathrm{H}} + \|u - u_h\|_{\mathrm{Q}} \leqslant \widehat{C}_{\mathrm{d}} \Big\{ \mathrm{dist}(\sigma, \mathrm{H}_h) + \mathrm{dist}(u, \mathrm{Q}_h) \Big\}. \tag{3.70}$$

We end this chapter by emphasizing the main aspects of the present analysis that differ from those of the Hilbertian case. First of all, and regarding the proof of Theorem 3.1, we stress that instead of the identity (1.2), which is not valid in the Banach case, here we have to use Lemma 2.1 to be able to apply the equivalence (though with different constants $\widetilde{\beta}$ and $\widehat{\beta}$) between the statements i) and iii) of Lemma 2.2. A second difference is determined by the need of having to assume the existence of topological complements for the closed subspaces V and W, which, on the contrary, is for granted in the Hilbert case thanks to the orthogonal decomposition theorem. In addition, we do not make use of any ellipticity properties of a nor of c, but only of the respective inf-sup conditions. Furthermore, and concerning the solvability result provided by Theorem 3.2 for the associated Galerkin scheme, we notice that the lack of the finite dimensional version of (1.2) as well in the Banach case, forces us to assume two independent discrete inf-sup conditions instead of just one, as in the Hilbert case. On the other hand, when $\mathrm{W} = \mathrm{W}_h = 0$, in which case nor the identity (1.2) or its discrete version are valid either, we observe that the corresponding Theorems 3.4 and 3.5 make use of the characterization of surjective operators provided in [23, Lemma A.42], and more specifically of its consequences given by (3.47) and (3.64), respectively. In this way, and differently from the Hilbert case and the continuous and discrete Banach cases analyzed in Chapter 3.2 and 3.3, we do not need to identify any topological complements of V nor V_h in the proofs of the aforementioned theorems.

CHAPTER 4

Application to the decoupled Nernst Planck equation

The coupling of the Stokes and Poisson-Nernst-Planck equations is an electrohydrodinamic model describing the stationary flow of a Newtonian and incompressible fluid occupying a domain $\Omega \subseteq \mathrm{R}^n$, $n \in \{2,3\}$, with polygonal (resp. polyhedral) boundary Γ in R^2 (resp R^3) (see, e.g. [31], [32]). The dynamics of it is determined by the concentration of ionized particles ξ_1 and ξ_2, the electric current field $\boldsymbol{\varphi}$, and the velocity $\mathbf{u}$ and pressure p of the fluid. In particular, knowing the vector fields $\boldsymbol{\varphi}$ and $\mathbf{u}$, a simplified version of the decoupled Nernst-Planck equation for a single ionic species, whose concentration is denoted ξ, and for which the diffusion and dielectric coefficients are assumed to be equal to 1, is expressed in mixed form as

$$\begin{aligned} \boldsymbol{\sigma} &= \nabla\xi + \xi\,(\boldsymbol{\varphi} - \mathbf{u}) \quad \text{in} \quad \Omega\,, \\ \xi - \operatorname{div}(\boldsymbol{\sigma}) &= f \quad \text{in} \quad \Omega\,, \quad \xi = g \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{4.1}$$

where ∇ and div are the usual gradient and divergence operators acting on scalar and vector fields, respectively, and f and g are given data belonging to suitable function spaces. On purpose of this, in what follows we adopt standard notation for Lebesgue spaces $\mathrm{L}^t(\Omega)$, with $t \in (1, +\infty)$, and Sobolev spaces $\mathrm{H}^m(\Omega)$ and $\mathrm{H}_0^m(\Omega)$, with integer $m \geqslant 0$, whose corresponding norms and seminorms (in the case of the latter), either for the scalar or vectorial case, are denoted by $\|\cdot\|_{0,t;\Omega}$, $\|\cdot\|_{m,\Omega}$, and $|\cdot|_{m,\Omega}$, respectively. Furthermore, as usual we let $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$ be the space of traces of $\mathrm{H}^1(\Omega)$ and its dual, with norms $\|\cdot\|_{1/2,\Gamma}$ and $\|\cdot\|_{-1/2,\Gamma}$, respectively, and denote by $\langle\cdot,\cdot\rangle_\Gamma$ the corresponding duality pairing. On the other hand, given any generic scalar functional space S, we let $\mathbf{S}$ be its vector counterpart.

Now, in order to derive the variational formulation of (4.1), we stress that the right spaces where the unknowns are going to be sought is mainly determined by the term depending on $\boldsymbol{\varphi}$ and $\mathbf{u}$. Indeed, using the Cauchy-Schwarz and Hölder inequalities, we observe that

$$\left|\int_\Omega \xi\,(\boldsymbol{\varphi} - \mathbf{u})\cdot\boldsymbol{\tau}\right| \leqslant \|\xi\|_{0,\rho;\Omega}\left(\|\boldsymbol{\varphi}\|_{0,r;\Omega} + \|\mathbf{u}\|_{0,r;\Omega}\right)\|\boldsymbol{\tau}\|_{0,\Omega} \tag{4.2}$$

for all $\xi \in \mathrm{L}^\rho(\Omega)$, for all $\boldsymbol{\varphi}$, $\mathbf{u} \in \mathbf{L}^r(\Omega)$, and for all $\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)$, where $\rho = 2\ell$ and $r = 2j$, with ℓ, $j \in (1, +\infty)$ conjugate to each other, that is such that $\frac{1}{\ell} + \frac{1}{j} = 1$. Next, we let $\varrho \in (1, +\infty)$ be the conjugate of ρ, introduce the Banach space

$$\mathbf{H}(\mathrm{div}_\varrho;\Omega) := \left\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^\varrho(\Omega)\right\}, \tag{4.3}$$

which is endowed with the norm

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_\varrho;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,\varrho;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)\,,$$

and recall from [17, Section 3.1] (see also [12, Section 4.1] or [29, eq. (2.11)]) that for $\varrho \geqslant \frac{2n}{n+2}$ there holds

$$\langle\boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle_\Gamma = \int_\Omega\left\{\boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau})\right\} \qquad \forall\,(\boldsymbol{\tau}, v) \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)\times\mathrm{H}^1(\Omega)\,, \tag{4.4}$$

where $\boldsymbol{\nu}$ stands for the unit outward normal on Γ. Note that the integration by parts formula

(4.4) states implicitly that $\boldsymbol{\tau}\cdot\boldsymbol{\nu} \in \mathrm{H}^{-1/2}(\Gamma)$ for each $\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$. In addition, being $\rho = 2\ell > 2$, it follows that $\varrho \in (1,2)$, and hence the feasible range for ϱ becomes $\left(\frac{2n}{n+2}, 2\right)$. Thus, testing the first equation of (4.1) against $\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$, and then applying (4.4) with $v = \xi$, which requires to assume that, originally $\xi \in \mathrm{H}^1(\Omega)$, and that $g \in \mathrm{H}^{1/2}(\Gamma)$, we obtain

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \int_\Omega \xi\,\mathrm{div}(\boldsymbol{\tau}) - \int_\Omega \xi\,(\boldsymbol{\varphi} - \mathbf{u})\cdot\boldsymbol{\tau} = \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle_\Gamma\,. \tag{4.5}$$

In turn, assuming that $f \in \mathrm{L}^\varrho(\Omega)$, and testing the second equation of (4.1) against $\eta \in \mathrm{L}^\rho(\Omega)$, we get

$$\int_\Omega \eta\,\mathrm{div}(\boldsymbol{\sigma}) - \int_\Omega \xi\,\eta = -\int_\Omega f\,\eta\,. \tag{4.6}$$

In this way, placing together (4.5) and (4.6), we arrive at the following mixed variational formulation for (4.1): Find $(\boldsymbol{\sigma},\xi) \in \mathrm{H}\times\mathrm{Q}$ such that

$$\begin{aligned} a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\xi) - \int_\Omega \xi\,(\boldsymbol{\varphi} - \mathbf{u})\cdot\boldsymbol{\tau} &= \mathrm{F}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in \mathrm{H}\,,\\ b(\boldsymbol{\sigma},\eta) - c(\xi,\eta) &= \mathrm{G}(\eta) \qquad \forall\,\eta \in \mathrm{Q}\,, \end{aligned} \tag{4.7}$$

where

$$\mathrm{H} := \mathbf{H}(\mathbf{div}_\varrho;\Omega)\,, \qquad \mathrm{Q} := \mathrm{L}^\rho(\Omega)\,, \tag{4.8}$$

and the bilinear forms $a : \mathrm{H}\times\mathrm{H} \to \mathrm{R}$, $b : \mathrm{H}\times\mathrm{Q} \to \mathrm{R}$, and $c : \mathrm{Q}\times\mathrm{Q} \to \mathrm{R}$, and the functionals $\mathrm{F} : \mathrm{H} \longrightarrow \mathrm{R}$ and $\mathrm{G} : \mathrm{Q} \longrightarrow \mathrm{R}$, are defined, respectively, as

$$a(\boldsymbol{\zeta},\boldsymbol{\tau}) := \int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau} \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau}) \in \mathrm{H}\times\mathrm{H}\,, \tag{4.9}$$

$$b(\boldsymbol{\tau},\eta) := \int_\Omega \eta\,\mathrm{div}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau},\eta) \in \mathrm{H}\times\mathrm{Q}\,, \tag{4.10}$$

$$c(\lambda,\eta) := \int_\Omega \lambda\,\eta \qquad \forall\,(\lambda,\eta) \in \mathrm{Q}\times\mathrm{Q}\,, \tag{4.11}$$

$$\mathrm{F}(\boldsymbol{\tau}) := \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle \qquad \forall\,\boldsymbol{\tau} \in \mathrm{H}\,, \tag{4.12}$$

and

$$G(\eta) \; := \; - \int_{\Omega} f \, \eta \qquad \forall \, \eta \in Q \, . \tag{4.13}$$

Equivalently, introducing the bilinear forms $\mathbf{A}$, $\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}} : (H \times Q) \times (H \times Q) \to R$ given by

$$\mathbf{A}\big((\boldsymbol{\zeta}, \lambda), (\boldsymbol{\tau}, \eta)\big) \; := \; a(\boldsymbol{\zeta}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \lambda) + b(\boldsymbol{\zeta}, \eta) - c(\lambda, \eta) \tag{4.14}$$

and

$$\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}\big((\boldsymbol{\zeta}, \lambda), (\boldsymbol{\tau}, \eta)\big) \; := \; \mathbf{A}\big((\boldsymbol{\zeta}, \lambda), (\boldsymbol{\tau}, \eta)\big) \; - \; \int_{\Omega} \lambda \, (\boldsymbol{\varphi} \; - \; \mathbf{u}) \cdot \boldsymbol{\tau} \tag{4.15}$$

for all $(\boldsymbol{\zeta}, \lambda)$, $(\boldsymbol{\tau}, \eta) \in H \times Q$, we deduce that (4.7) can be re-stated as: Find $(\boldsymbol{\sigma}, \xi) \in H \times Q$ such that

$$\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}\big((\boldsymbol{\sigma}, \xi), (\boldsymbol{\tau}, \eta)\big) \; = \; F(\boldsymbol{\tau}) + G(\eta) \qquad \forall \, (\boldsymbol{\tau}, \eta) \in H \times Q \, . \tag{4.16}$$

According to the above, in what follows we show first that the bilinear forms forming part of $\mathbf{A}$ satisfy the assumptions of Theorem 3.4. Later on, we combine this fact with the effect of the extra term completing the definition of $\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}$ to conclude the solvability of (4.7) (or (4.16)).

We begin by observing that the reflexivity of $\mathbf{L}^2(\Omega)$, $L^{\varrho}(\Omega)$, and $L^{\rho}(\Omega)$, imply that H and Q are both reflexive Banach spaces. In addition, straightforward applications of the Cauchy-Schwarz and Hölder inequalities show that a, b, and c, are all bounded with $\|a\| \leqslant 1$, $\|b\| \leqslant 1$, and $\|c\| \leqslant |\Omega|^{(\rho-2)/\rho}$. Also, it is clear from (4.9) and (4.11) that a and b are symmetric and positive semi-definite (assumption **i)** of Theorem 3.4). Next, bearing in mind the definitions of b (cf. (4.10)) and the null space V of the operator $\mathbf{B}$ induced by b (cf. (2.2)), we find that

$$V \; = \; \Big\{ \tau \in \mathbf{H}(\mathrm{div}_{\varrho}; \Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) = 0 \Big\} \, , \tag{4.17}$$

and thus

$$a(\boldsymbol{\tau}, \boldsymbol{\tau}) \; = \; \|\boldsymbol{\tau}\|_{0,\Omega}^2 \; = \; \|\boldsymbol{\tau}\|_{\mathrm{div}_{\varrho};\Omega}^2 \qquad \forall \, \boldsymbol{\tau} \in V \, ,$$

from which it readily follows that a satisfies the continuous inf-sup condition (3.49) with constant $\widetilde{\alpha} = 1$ (assumption **ii)** of Theorem 3.4). It remains to show that b satisfies the continuous inf-sup condition (3.50) (assumption **iii)** of Theorem 3.4). While the corresponding proof is actually available in the literature (see, e.g. [29, Lemma 2.9] and the references mentioned there), we provide it again below for sake of completeness of the presentation.

Lemma 4.1. *There exists a constant* $\widetilde{\beta} > 0$*, depending only on* Ω *and* ρ*, such that*

$$\sup_{\substack{\boldsymbol{\tau}\in \mathrm{H}\\ \boldsymbol{\tau}\neq 0}} \frac{b(\boldsymbol{\tau},\eta)}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} \geqslant \widetilde{\beta}\,\|\eta\|_{\mathrm{Q}} \qquad \forall\,\eta\in \mathrm{Q}\,. \tag{4.18}$$

Proof. Given $\eta \in \mathrm{Q} := \mathrm{L}^\rho(\Omega)$, we first define $\eta_\varrho := |\eta|^{\rho-2}\,\eta$ and observe, thanks to simple algebraic computations, that $\eta_\varrho \in \mathrm{L}^\varrho(\Omega)$ and

$$\int_\Omega \eta\,\eta_\varrho \;=\; \|\eta\|_{0,\rho;\Omega}\,\|\eta_\varrho\|_{0,\varrho;\Omega}\,. \tag{4.19}$$

Then, we let $\widetilde{\boldsymbol{\tau}} := \nabla z \in \mathbf{L}^2(\Omega)$, where $z \in \mathrm{H}^1_0(\Omega)$ is the unique solution of the variational problem

$$\int_\Omega \nabla z\cdot\nabla w \;=\; -\int_\Omega \eta_\varrho\, w \qquad \forall\, w\in \mathrm{H}^1_0(\Omega)\,. \tag{4.20}$$

Indeed, Hölder's inequality and the continuous injection i_ρ of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^\rho(\Omega)$ guarantee that the right hand side of (4.20) constitutes a functional in $\mathrm{H}^1_0(\Omega)'$, so that the classical Lax-Milgram Lemma confirms the unique solvability of this problem. In turn, it follows from (4.20) that

$$\mathrm{div}(\widetilde{\boldsymbol{\tau}}) \;=\; \eta_\varrho \quad \text{in} \quad \Omega\,, \tag{4.21}$$

which yields $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$. Moreover, according to the continuous dependence result for (4.20) and the resulting bound for the norm of the aforementioned functional, we deduce the existence of a constant $c_\rho > 0$, depending on $\|i_\rho\|$, such that $\|z\|_{1,\Omega} \leqslant c_\rho\,\|\eta_\varrho\|_{0,\varrho;\Omega}$, and hence

$$\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_\varrho;\Omega} \;=\; |z|_{1,\Omega} \,+\, \|\eta_\varrho\|_{0,\varrho;\Omega} \;\leqslant\; \big(1+c_\rho\big)\,\|\eta_\varrho\|_{0,\varrho;\Omega}\,. \tag{4.22}$$

Finally, according to the definition of b (cf. (4.10)), and employing (4.21), (4.19), and (4.22), we obtain

$$\sup_{\substack{\boldsymbol{\tau}\in \mathrm{H}\\ \boldsymbol{\tau}\neq 0}} \frac{b(\boldsymbol{\tau},\eta)}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} \geqslant \frac{b(\widetilde{\boldsymbol{\tau}},\eta)}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}}} = \frac{\|\eta\|_{0,\rho;\Omega}\,\|\eta_\varrho\|_{0,\varrho;\Omega}}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}}} \geqslant \widetilde{\beta}\,\|\eta\|_{0,\rho;\Omega}\,,$$

with $\widetilde{\beta} := \big(1 + c_\rho\big)^{-1}$, thus proving the required continuous inf-sup condition (4.18). □

Having proved that a, b, and c verify the hypotheses of Theorem 3.4, we deduce that the global inf-sup condition (3.33) also holds for the present bilinear form $\mathbf{A}$ (cf. (4.14)), which means in this case that there exists a constant $\widehat{c} > 0$, depending only on $\|a\|$ ($\leqslant 1$), $\|c\|$ ($\leqslant |\Omega|^{(\rho-2)/\rho}$), $\widetilde{\alpha} = 1$, and $\widetilde{\beta} = \big(1 + c_\rho\big)^{-1}$, such that

$$\sup_{\substack{(\boldsymbol{\tau},\eta)\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau},\eta)\neq \mathbf{0}}} \frac{\mathbf{A}\big((\boldsymbol{\zeta},\lambda),(\boldsymbol{\tau},\eta)\big)}{\|(\boldsymbol{\tau},\eta)\|_{\mathrm{H}\times\mathrm{Q}}} \geqslant \widehat{c}\,\|(\boldsymbol{\zeta},\lambda)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\boldsymbol{\zeta},\lambda)\in \mathrm{H}\times\mathrm{Q}\,. \tag{4.23}$$

Thus, it readily follows from (4.15), (4.2), and (4.23) that

$$\sup_{\substack{(\boldsymbol{\tau},\eta)\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau},\eta)\neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}\big((\boldsymbol{\zeta},\lambda),(\boldsymbol{\tau},\eta)\big)}{\|(\boldsymbol{\tau},\eta)\|_{\mathrm{H}\times\mathrm{Q}}} \geqslant \Big\{\widehat{c} - \big(\|\boldsymbol{\varphi}\|_{0,r;\Omega} + \|\mathbf{u}\|_{0,r;\Omega}\big)\Big\}\,\|(\boldsymbol{\zeta},\lambda)\|_{\mathrm{H}\times\mathrm{Q}} \quad \forall\,(\boldsymbol{\zeta},\lambda)\in \mathrm{H}\times\mathrm{Q}\,, \tag{4.24}$$

from which, under the assumption that, say $\|\boldsymbol{\varphi}\|_{0,r;\Omega} + \|\mathbf{u}\|_{0,r;\Omega} \leqslant \frac{\widehat{c}}{2}$, we conclude that

$$\sup_{\substack{(\boldsymbol{\tau},\eta)\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau},\eta)\neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}\big((\boldsymbol{\zeta},\lambda),(\boldsymbol{\tau},\eta)\big)}{\|(\boldsymbol{\tau},\eta)\|_{\mathrm{H}\times\mathrm{Q}}} \geqslant \frac{\widehat{c}}{2}\,\|(\boldsymbol{\zeta},\lambda)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\boldsymbol{\zeta},\lambda)\in \mathrm{H}\times\mathrm{Q}\,. \tag{4.25}$$

Similarly, using the symmetry of $\mathbf{A}$ and (4.23), and under the same hypothesis on $\boldsymbol{\varphi}$ and $\mathbf{u}$, we find that

$$\sup_{\substack{(\boldsymbol{\zeta},\lambda)\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\zeta},\lambda)\neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}\big((\boldsymbol{\zeta},\lambda),(\boldsymbol{\tau},\eta)\big)}{\|(\boldsymbol{\zeta},\lambda)\|_{\mathrm{H}\times\mathrm{Q}}} \geqslant \frac{\widehat{c}}{2}\,\|(\boldsymbol{\tau},\eta)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\boldsymbol{\tau},\eta)\in \mathrm{H}\times\mathrm{Q}\,. \tag{4.26}$$

On the other hand, recalling from the proof of Lemma 4.1 that i_ρ is the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^\rho(\Omega)$, it is easy to see from (4.4) that there exists a constant $C_\rho > 0$, depending on $\|i_\rho\|$, such that $\|\boldsymbol{\tau}\cdot\boldsymbol{\nu}\|_{-1/2,\Gamma} \leqslant C_\rho\,\|\boldsymbol{\tau}\|_{\mathrm{div}_\varrho;\Omega}$ for all $\boldsymbol{\tau}\in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$, and hence we deduce from

(4.12) that $\mathrm{F} \in \mathrm{H}'$ with $\|\mathrm{F}\|_{\mathrm{H}'} \leqslant C_\rho \|g\|_{-1/2,\Gamma}$. In turn, (4.13) and Hölder's inequality yield $\mathrm{G} \in \mathrm{Q}'$ with $\|\mathrm{G}\|_{\mathrm{Q}} \leqslant \|f\|_{0,\varrho;\Omega}$.

In this way, we are now in position of establishing the well-posedness of (4.7) (equivalently (4.16)).

Theorem 4.2. *Let* $\boldsymbol{\varphi}$, $\mathbf{u} \in \mathbf{L}^r(\Omega)$ *such that* $\|\boldsymbol{\varphi}\|_{0,r;\Omega} + \|\mathbf{u}\|_{0,r;\Omega} \leqslant \frac{\widehat{c}}{2}$. *Then, there exists a unique* $(\boldsymbol{\sigma}, \xi) \in \mathrm{H} \times \mathrm{Q}$ *solution to* (4.7), *and there holds*

$$\|\boldsymbol{\sigma}\|_{\mathrm{div}_\varrho;\Omega} + \|\xi\|_{0,\rho;\Omega} \leqslant \frac{2}{\widehat{c}} \max\left\{1, C_\rho\right\} \left\{\|g\|_{-1/2,\Gamma} + \|f\|_{0,\varrho;\Omega}\right\}.$$

Proof. Thanks to (4.25), (4.26), and the boundedness of F and G, it follows from a straightforward application of the Banach-Nečas-Babuška Theorem (also known as generalized Lax-Milgram Lemma) (cf. [23, Theorem 2.6]). □

We end this part of the thesis by remarking that the continuous and discrete analyses of the full Poisson-Nernst-Planck and Stokes coupled model, which certainly contain those of (4.7), will be provided next in the second part.

Part II

New mixed finite element methods for the coupled Stokes and Poisson-Nernst-Planck equations in Banach spaces

CHAPTER 5

Introduction

Fluid mixtures with electrically charged ions are critical for many industrial processes and natural phenomena. Notable examples of current interest are efficient energy storage and electrodialysis cells, design of nanopore sensors, electro-osmotic water purification techniques, and even drug delivery in biological tissues [45]. One of the most well-known models for liquid electrolytes is the Poisson–Nernst–Planck / Stokes system. It describes the isothermal dynamics of the molar concentration of a number of charged species within a solvent. This classical model is valid for the regime of relatively small Reynolds numbers and it is written in terms of the concentrations, the barycentric velocity of the mixture, the pressure of the mixture, and the electrostatic potential. The system is strongly coupled and the set of equations consist of the transport equations for each dilute component of the electrolyte, a diffusion equation for the electrostatic equilibrium, the momentum balance for the mixture (including a force exerted by the electric field), and mass conservation.

Solving these systems lends itself difficult due to coupling nonlinearities of different nature. Numerical methods for incompressible flow equations coupled with Poisson–Nernst–Planck equations that are based on finite element schemes in primal formulation (also including stabilized and goal-adaptive methods) can be found in [4, 24, 36, 39, 41, 44], finite differences in e.g. [38], finite volume schemes in [43], spectral elements in [40], and also for virtual element methods in [21]. Regarding formulations using mixed methods, the first works addressing Stokes/PNP systems are relatively recent [31,32]. Mixed variational formulations are particularly interesting when direct discrete approximations of further variables of physical relevance are required. A recent approach to mixed methods consists in defining the corresponding variational settings in terms of Banach spaces instead of the usual Hilbertian framework, and without augmentation. As a consequence, the unknowns belong now to the natural spaces that are originated after carrying out the respective testing and integration by parts procedures, simpler and closer to the original physical model formulations arise, momentum conservative schemes can be obtained, and even other unknowns can be computed by postprocessing formulae. As a non-exhaustive list of contributions taking advantage of the use of Banach frameworks for solving the aforementioned kind of problems, we refer to [5,11,13–15,17,18,29,30,33], and among the different models considered there, we find Poisson, Brinkman–Forchheimer, Darcy–Forchheimer, Navier–Stokes, chemotaxis/Navier–Stokes, Boussinesq, coupled flow-transport, and fluidized beds. Nevertheless, and up to our knowledge, no mixed methods with the described advantages seem to have been developed so far for the coupled Stokes and Poisson–Nernst–Planck equations.

As motivated by the previous discussion, the goal of this part is to develop a Banach spaces-based formulation yielding new mixed finite element methods for, precisely, the coupled Stokes and Poisson–Nernst–Planck equations. The rest of the manuscript is organized as follows. Required notations and basic definitions are collected at the end of this introductory chapter. In Chapter 6 we describe the model of interest and introduce the additional variables to be employed. The mixed variational formulation is deduced in Chapter 7. After some preliminaries, the respective analysis is split according to the three equations forming the whole system. In particular, the right spaces to which the trial and test functions must belong are derived in each case by applying suitable integration by parts formulae jointly with the Cauchy–Schwarz and Hölder inequalities. In Chapter 8 we utilize a fixed-point approach to study the solvability of the continuous formulation. The Babuška–Brezzi theory and recent results on perturbed saddle-point problems, both in Banach spaces, along with the Banach–Nečas–Babuška theorem, are utilized to prove that the corresponding uncoupled problems are well-posed. The classical Banach fixed-point theorem is then applied to conclude the existence of a unique solution. In 9 we proceed analogously to Chapter 8 and, under suitable stability assumptions on the discrete spaces employed, show existence and then uniqueness of solution for the Galerkin scheme by applying the Brouwer and Banach theorems, respectively. A priori error estimates are also derived here. Next, in Chapter 10 we define explicit finite element subspaces satisfying those conditions, and provide the associated rates of convergence. Finally, several numerical examples confirming the latter and illustrating the good performance of the method, are reported in Chapter 11.

Preliminary notations

Throughout the second part, Ω is a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, which is star shaped with respect to a ball, and whose outward normal at $\Gamma := \partial\Omega$ is denoted by $\boldsymbol{\nu}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$ and Sobolev spaces $\mathrm{W}^{l,t}(\Omega)$ and $\mathrm{W}^{l,t}_0(\Omega)$, with $l \geqslant 0$ and $t \in [1,+\infty)$, whose corresponding norms, either for the scalar and vectorial case, are denoted by $\|\cdot\|_{0,t;\Omega}$ and $\|\cdot\|_{l,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and if $t = 2$ we write $\mathrm{H}^l(\Omega)$ instead of $\mathrm{W}^{l,2}(\Omega)$, with the corresponding norm and seminorm denoted by $\|\cdot\|_{l,\Omega}$ and $|\cdot|_{l,\Omega}$, respectively. In addition, letting t, $t' \in (1,+\infty)$ conjugate to each other, that is such that $1/t + 1/t' = 1$, we denote by $\mathrm{W}^{1/t',t}(\Gamma)$ the trace space of $\mathrm{W}^{1,t}(\Omega)$, and let $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the dual of $\mathrm{W}^{1/t',t}(\Gamma)$ endowed with the norms $\|\cdot\|_{-1/t',t';\Gamma}$ and $\|\cdot\|_{1/t',t;\Gamma}$, respectively. On the other hand, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$ will be employed for the norm of any element or operator whenever there is no confusion about the spaces to which they belong. Furthermore, as usual, $\mathbb{I}$ stands for the identity tensor in $\mathbb{R} := \mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbf{R} := \mathrm{R}^n$. Also, for any vector field $\mathbf{v} = (v_i)_{i=1,n}$ we set the gradient and divergence operators, respectively, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n} \quad \text{and} \quad \operatorname{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}\,.$$

Additionally, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product operators, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} = (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{n} \boldsymbol{\tau}_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}\,, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I}\,.$$

CHAPTER 6

The model problem

We consider the nonlinear system given by the coupled Stokes and Poisson–Nernst–Planck equations, which constitute an electrohydrodinamic model describing the stationary flow of a Newtonian and incompressible fluid occupying the domain $\Omega \subseteq \mathrm{R}^n$, $n \in \{2, 3\}$, with polygonal (resp. polyhedral) boundary Γ in R^2 (resp. R^3). Under the assumption of isothermal properties, equal molar volumes and molar masses for each species, the behavior of the system is determined by the concentrations ξ_1 and ξ_2 of ionized particles, and by the electric current field $\boldsymbol{\varphi}$. Mathematically speaking, and firstly regarding the fluid, we look for the barycentric velocity $\mathbf{u}$ and the pressure p of the mixture, such that $(\mathbf{u}, p)$ is solution to the Stokes equations

$$\begin{gathered}
-\mu\Delta\mathbf{u} + \nabla p \;=\; -(\xi_1 - \xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi} + \mathbf{f} \quad \text{in} \quad \Omega\,, \\
\operatorname{div}(\mathbf{u}) \;=\; 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} \;=\; \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega p = 0\,,
\end{gathered} \tag{6.1}$$

where μ is the constant viscosity, ε is the dielectric coefficient, also known as the electric conductivity coefficient, $\mathbf{f}$ is a source term, $\mathbf{g}$ is the Dirichlet datum for $\mathbf{u}$ on Γ, and the null mean value of p has been incorporated as a uniqueness condition for this unknown. In addition, $\boldsymbol{\varphi}$, ξ_1 and ξ_2 solve the Poisson–Nernst–Planck equations, which depend on the velocity $\mathbf{u}$ and are given by

$$\begin{gathered}\boldsymbol{\varphi} \,=\, \varepsilon\,\nabla\chi \quad \text{in} \quad \Omega\,, \qquad -\operatorname{div}(\boldsymbol{\varphi}) \,=\, (\xi_1-\xi_2) \,+\, f \quad \text{in} \quad \Omega\,, \\ \chi \,=\, g \quad \text{on} \quad \Gamma\,,\end{gathered} \tag{6.2}$$

where χ is the electrostatic potential, and for each $i \in \{1,2\}$

$$\begin{gathered}\xi_i - \operatorname{div}\big(\kappa_i(\nabla\xi_i \,+\, q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi}) \,-\, \xi_i\,\mathbf{u}\big) \,=\, f_i \quad \text{in} \quad \Omega\,, \\ \xi_i \,=\, g_i \quad \text{on} \quad \Gamma\,,\end{gathered} \tag{6.3}$$

where κ_1 and κ_2 are the diffusion coefficients, $q_i := \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = 2 \end{cases}$, f, f_1, and f_2 are external source/sink terms, and g, g_1 and g_2 are Dirichlet data for χ, ξ_1 and ξ_2, respectively, on Γ. The systems (6.2) and (6.3) correspond to the Poisson and Nernst–Planck equations, respectively. We end the description of the model by remarking that ε, κ_1, and κ_2 are all assumed to be bounded above and below, which means that there exist positive constants ε_0, ε_1, $\underline{\kappa}$, and $\bar{\kappa}$, such that

$$\varepsilon_0 \,\leqslant\, \varepsilon(\mathbf{x}) \,\leqslant\, \varepsilon_1 \quad \text{and} \quad \underline{\kappa} \,\leqslant\, \kappa_i(\mathbf{x}) \,\leqslant\, \bar{\kappa} \quad \text{for almost all } \mathbf{x} \in \Omega\,, \quad \forall\, i \in \{1,2\}\,. \tag{6.4}$$

We stress that in order to solve (6.3), $\mathbf{u}$ and $\boldsymbol{\varphi}$ are needed. In turn, (6.1) requires ξ_1, ξ_2 and $\boldsymbol{\varphi}$, whereas (6.2) makes use of ξ_1 and ξ_2. This multiple coupling is illustrated through the graph provided in Figure 6.1, where the vertexes represent the aforementioned equations and the arrows, properly labeled with the unknowns involved, show the respective dependence relationships.

Furthermore, since we are interested in employing a fully mixed variational formulation for the coupled model (6.1) – (6.3), we introduce the auxiliary variables of pseudostress

$$\boldsymbol{\sigma} \,:=\, \mu\,\nabla\mathbf{u} \,-\, p\,\mathbb{I} \quad \text{in} \quad \Omega\,, \tag{6.5}$$

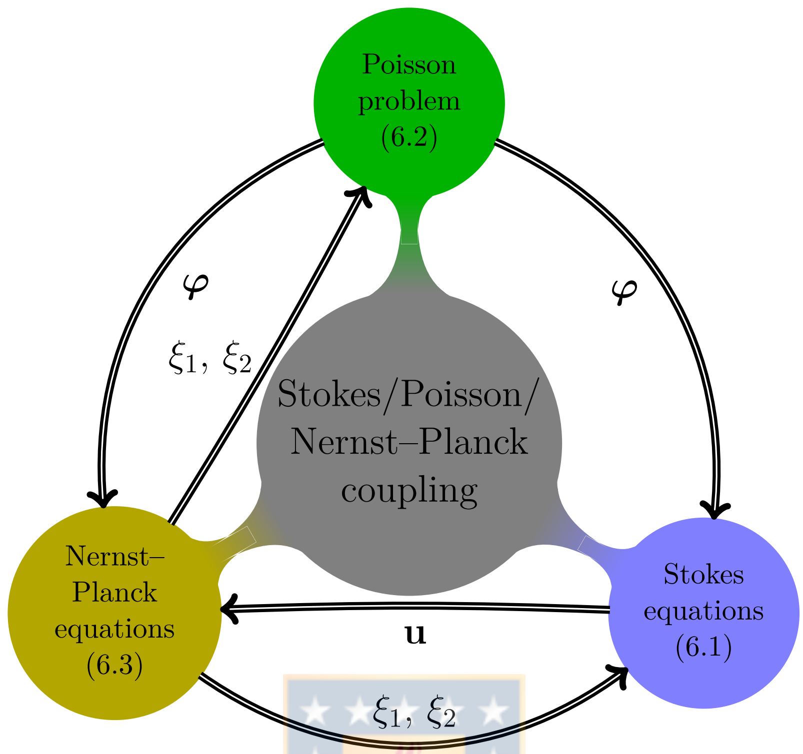

Figure 6.1: Illustrative graph of the coupling mechanisms connecting the three sub-problems (6.1), (6.2) and (6.3).

and, for each $i \in \{1,2\}$, the total (diffusive, cross-diffusive, and advective) ionic fluxes

$$\boldsymbol{\sigma}_i \;:=\; \kappa_i\,(\nabla\xi_i \;+\; q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi}) \;-\; \xi_i\,\mathbf{u} \quad \text{in} \quad \Omega\,. \tag{6.6}$$

Thus, applying the matrix trace in (6.5) and using the incompressibility condition, we deduce that

$$p \;=\; -\frac{1}{n}\,\mathrm{tr}(\boldsymbol{\sigma})\,, \tag{6.7}$$

so that, incorporating the latter expression into (6.5), p is eliminated from the system (6.1) - (6.3), which can then be rewritten in terms of the unknowns $\boldsymbol{\sigma}$, $\mathbf{u}$, $\boldsymbol{\varphi}$, χ, $\boldsymbol{\sigma}_i$ and ξ_i, $i \in \{1,2\}$, as

$$
\begin{gathered}
\frac{1}{\mu}\boldsymbol{\sigma}^{\mathrm{d}} \,=\, \nabla\mathbf{u} \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\boldsymbol{\sigma}) \,=\, (\xi_1 - \xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi} - \mathbf{f} \quad \text{in} \quad \Omega\,, \\
\mathbf{u} \,=\, \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) \,=\, 0\,, \\
\frac{1}{\varepsilon}\boldsymbol{\varphi} \,=\, \nabla\chi \quad \text{in} \quad \Omega\,, \quad -\operatorname{div}(\boldsymbol{\varphi}) \,=\, (\xi_1 - \xi_2) + f \quad \text{in} \quad \Omega\,, \\
\chi \,=\, g \quad \text{on} \quad \Gamma\,, \\
\frac{1}{\kappa_i}\boldsymbol{\sigma}_i \,:=\, \nabla\xi_i \,+\, q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi} \,-\, \kappa_i^{-1}\,\xi_i\,\mathbf{u} \quad \text{in} \quad \Omega\,, \\
\xi_i \,-\, \operatorname{div}(\boldsymbol{\sigma}_i) \,=\, f_i \quad \text{in} \quad \Omega\,, \quad \xi_i \,=\, g_i \quad \text{on} \quad \Gamma\,, \quad i \in \big\{1, 2\big\}\,.
\end{gathered}
\tag{6.8}
$$

We notice here that the uniqueness condition for p has been rewritten equivalently as the null mean value constraint for $\operatorname{tr}(\boldsymbol{\sigma})$.

CHAPTER 7

The fully mixed formulation

In this chapter we derive a suitable Banach spaces-based variational formulation for (6.8) by splitting the analysis in four chapters. The first one collects some preliminary discussions and known results, and the remaining three deal with each one of the pairs of equations forming the whole nonlinear coupled system (6.8), namely Stokes, Poisson, and Nernst-Planck.

7.1 Preliminaries

We begin by noticing that there are three key expressions in (6.8) that need to be looked at carefully before determining the right spaces where the unknowns must be sought, namely $(\xi_1 - \xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi}$, $q_i\,\xi_i\,\varepsilon^{-1}\boldsymbol{\varphi}$ and $\kappa_i^{-1}\,\xi_i\,\mathbf{u}$ in the first and fifth rows of (6.8). More precisely, ignoring the bounded above and below functions ε^{-1} and κ_i^{-1}, as well as the constant q_i, and given test functions $\mathbf{v}$ and $\boldsymbol{\tau}_i$ associated with $\mathbf{u}$ and $\boldsymbol{\sigma}_i$, respectively, straightforward applications of the

Cauchy–Schwarz and Hölder inequalities yield

$$\left|\int_\Omega (\xi_1 - \xi_2)\,\boldsymbol{\varphi}\cdot\mathbf{v}\right| \,\leqslant\, \|\xi_1 - \xi_2\|_{0,2\ell;\Omega}\,\|\boldsymbol{\varphi}\|_{0,2j;\Omega}\,\|\mathbf{v}\|_{0,\Omega}\,, \tag{7.1a}$$

$$\left|\int_\Omega \xi_i\,\boldsymbol{\varphi}\cdot\boldsymbol{\tau}_i\right| \,\leqslant\, \|\xi_i\|_{0,2\ell;\Omega}\,\|\boldsymbol{\varphi}\|_{0,2j;\Omega}\,\|\boldsymbol{\tau}_i\|_{0,\Omega}\,, \tag{7.1b}$$

and similarly

$$\left|\int_\Omega \xi_i\,\mathbf{u}\cdot\boldsymbol{\tau}_i\right| \,\leqslant\, \|\xi_i\|_{0,2\ell;\Omega}\,\|\mathbf{u}\|_{0,2j;\Omega}\,\|\boldsymbol{\tau}_i\|_{0,\Omega}\,, \tag{7.1c}$$

where ℓ, $j \in (1,+\infty)$ are conjugate to each other. In this way, denoting

$$\rho \,:=\, 2\ell\,, \quad \varrho \,:=\, \frac{2\ell}{2\ell-1}\ (\text{conjugate of } \rho)\,, \quad r \,:=\, 2j\,, \quad \text{and} \quad s \,:=\, \frac{2j}{2j-1}\ (\text{conjugate of } r)\,, \tag{7.2}$$

it follows that the above expressions make sense for $\xi_i \in \mathrm{L}^\rho(\Omega)$, $\boldsymbol{\varphi}$, $\mathbf{u} \in \mathbf{L}^r(\Omega)$, and $\mathbf{v}$, $\boldsymbol{\tau}_i \in \mathbf{L}^2(\Omega)$. The specific choice of ℓ, and hence of j, ρ, r and the respective conjugates ϱ and s, will be addressed later on, so that meanwhile we consider generic values for the indexes defined in (7.2).

Having set the above preliminary choice for the space to which $\boldsymbol{\varphi}$ belongs, we deduce from the first equation in the third row of (6.8) that χ should be initially sought in $\mathrm{W}^{1,r}(\Omega)$. In turn, using that $\mathrm{H}^1(\Omega)$ is embedded in $\mathrm{L}^t(\Omega)$ for $t \in [1,+\infty)$ in R^2 (resp. $t \in [1,6]$ in R^3), and for reasons that will become clear below, the unknowns ξ_i, $i \in \{1,2\}$, and $\mathbf{u}$ are initially sought in $\mathrm{H}^1(\Omega)$ and $\mathbf{H}^1(\Omega)$, respectively, certainly assuming that ρ and r verify the indicated ranges, namely ρ, $r \in (2,+\infty)$ in R^2, and ρ, $r \in (2,6]$ in R^3. Note that in terms of ℓ the latter constraint becomes $\ell \in [\frac{3}{2},3]$, which yields $\rho \in [3,6]$. Equivalently, $j \in [\frac{3}{2},3]$ and $r \in [3,6]$, though going through the respective intervals in the opposite direction to ℓ and ρ, respectively.

In turn, in order to derive the variational formulation of (6.8), we need to invoke a couple of integration by parts formulae, for which, given $t \in (1,+\infty)$, we first introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) \,:=\, \Big\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\Big\}\,, \tag{7.3a}$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) \,:=\, \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\Big\}\,, \tag{7.3b}$$

$$\mathbf{H}^t(\mathrm{div}_t;\Omega) := \left\{\boldsymbol{\tau} \in \mathbf{L}^t(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\right\}, \tag{7.3c}$$

which are endowed with the natural norms defined, respectively, by

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_t;\Omega)\,, \tag{7.4a}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,, \tag{7.4b}$$

$$\|\boldsymbol{\tau}\|_{t,\mathrm{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_t;\Omega)\,. \tag{7.4c}$$

Then, proceeding as in [26, eq. (1.43), Section 1.3.4] (see also [12, Section 4.1] and [17, Section 3.1]), it is easy to show that for each $t \geqslant \frac{2n}{n+2}$ there holds

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle = \int_\Omega \left\{\boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau})\right\} \qquad \forall\,(\boldsymbol{\tau}, v) \in \mathbf{H}(\mathrm{div}_t;\Omega)\times \mathrm{H}^1(\Omega)\,, \tag{7.5}$$

and analogously

$$\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{v}\rangle = \int_\Omega \left\{\boldsymbol{\tau} : \nabla\mathbf{v} + \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\right\} \qquad \forall\,(\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega)\times \mathbf{H}^1(\Omega)\,, \tag{7.6}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, as well as between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. Furthermore, given t, $t' \in (1,+\infty)$ conjugate to each other, there also holds (cf. [23, Corollary B. 57])

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle_\Gamma = \int_\Omega \left\{\boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau})\right\} \qquad \forall\,(\boldsymbol{\tau}, v) \in \mathbf{H}^t(\mathrm{div}_t;\Omega)\times \mathrm{W}^{1,t'}(\Omega)\,, \tag{7.7}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $\mathrm{W}^{-1/t,t}(\Gamma)$ and $\mathrm{W}^{1/t,t'}(\Gamma)$.

7.2 The Stokes equations

Let us first notice that, applying (7.6) with $t = s$ to $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_s;\Omega)$ and $\mathbf{u} \in \mathbf{H}^1(\Omega)$, and using the Dirichlet boundary condition on $\mathbf{u}$, for which we assume from now on that $\mathbf{g} \in \mathbf{H}^{1/2}(\Omega)$, we obtain

$$\int_\Omega \boldsymbol{\tau} : \nabla \mathbf{u} \;=\; -\int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \;+\; \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g} \rangle \,,$$

and thus, the testing of the first equation in the first row of (6.8) against $\boldsymbol{\tau}$ yields

$$\frac{1}{\mu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \;+\; \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \;=\; \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g} \rangle \,. \tag{7.8}$$

Note from the second term on the left-hand side of (7.8) that, knowing that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^s(\Omega)$, it actually suffices to look for $\mathbf{u}$ in $\mathbf{L}^r(\Omega)$, which is coherent with a previous discussion on the space to which this unknown should belong. In addition, testing the second equation in the first row of (6.8) against $\mathbf{v} \in \mathbf{L}^r(\Omega)$, for which we require that $\mathbf{f} \in \mathbf{L}^s(\Omega)$, we get

$$\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) \;=\; \int_\Omega (\xi_1 - \xi_2)\, \varepsilon^{-1} \boldsymbol{\varphi} \cdot \mathbf{v} - \int_\Omega \mathbf{f} \cdot \mathbf{v} \,, \tag{7.9}$$

which makes sense for $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^s(\Omega)$. Hence, due to the last equation in the second row of (6.8), it follows that we should look for $\boldsymbol{\sigma}$ in $\mathbb{H}_0(\mathbf{div}_s;\Omega)$, where

$$\mathbb{H}_0(\mathbf{div}_s;\Omega) \;:=\; \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_s;\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0 \right\}.$$

Moreover, it is easily seen that there holds the decomposition

$$\mathbb{H}(\mathbf{div}_s;\Omega) \;=\; \mathbb{H}_0(\mathbf{div}_s;\Omega) \oplus \mathrm{R}\mathbb{I} \,, \tag{7.10}$$

and that the incompressibility of the fluid forces the compatibility condition on $\mathbf{g}$ given by

$$\int_\Gamma \mathbf{g} \cdot \boldsymbol{\nu} \;=\; 0 \,.$$

As a consequence of the above, we realize that imposing (7.8) for each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_s;\Omega)$ is equivalent to doing it for each $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_s;\Omega)$. Furthermore, since $r > 2$ it follows that $\mathbf{L}^r(\Omega)$ is embedded in $\mathbf{L}^2(\Omega)$, which, along with the estimate (7.1a), confirms that the first term on the right-hand side of (7.9) is also well-defined. In this way, denoting from now on $\boldsymbol{\xi} := (\xi_1, \xi_2)$, and joining (7.8) and (7.9), we arrive at the following mixed variational formulation for the

Stokes equations (given by the first two rows of (6.8)): Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}\,, \\ \mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) &= \mathbf{G}_{\boldsymbol{\xi},\varphi}(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{Q}\,, \end{aligned} \tag{7.11}$$

where

$$\mathbf{H} \,:=\, \mathbb{H}_0(\mathbf{div}_s;\Omega)\,, \qquad \mathbf{Q} \,:=\, \mathbf{L}^r(\Omega)\,, \tag{7.12}$$

and the bilinear forms $\mathbf{a} : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$ and $\mathbf{b} : \mathbf{H} \times \mathbf{Q} \to \mathrm{R}$, and the functional $\mathbf{F} : \mathbf{H} \longrightarrow \mathrm{R}$, are defined, respectively, as

$$\mathbf{a}(\boldsymbol{\zeta}, \boldsymbol{\tau}) \,:=\, \frac{1}{\mu} \int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in \mathbf{H}\,, \tag{7.13a}$$

$$\mathbf{b}(\boldsymbol{\tau}, \mathbf{v}) \,:=\, \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H} \times \mathbf{Q}\,, \tag{7.13b}$$

$$\mathbf{F}(\boldsymbol{\tau}) \,:=\, \langle \boldsymbol{\tau}\, \boldsymbol{\nu}, \mathbf{g} \rangle \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}\,, \tag{7.13c}$$

whereas, given $\boldsymbol{\eta} := (\eta_1, \eta_2) \in \mathbf{L}^\rho(\Omega)$ and $\boldsymbol{\phi} \in \mathbf{L}^r(\Omega)$, the functional $\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} : \mathbf{Q} \longrightarrow \mathrm{R}$ is set as

$$\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}(\mathbf{v}) \,:=\, \int_\Omega (\eta_1 - \eta_2)\, \varepsilon^{-1} \boldsymbol{\phi} \cdot \mathbf{v} - \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{Q}\,. \tag{7.13d}$$

It is readily seen that, endowing $\mathbf{H}$ with the corresponding norm from (7.4b), that is

$$\|\boldsymbol{\tau}\|_{\mathbf{H}} \,:=\, \|\boldsymbol{\tau}\|_{\mathbf{div}_s;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbf{H}\,, \tag{7.14}$$

and recalling that $\|\cdot\|_{0,r;\Omega}$ is that of $\mathbf{Q}$, the bilinear forms $\mathbf{a}$ and $\mathbf{b}$, and the linear functionals $\mathbf{F}$ and $\mathrm{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}$, are all bounded. Indeed, applying the Cauchy–Schwarz and Hölder inequalities, noting that $\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} \leqslant \|\boldsymbol{\tau}\|_{0,\Omega}$ for all $\boldsymbol{\tau} \in \mathbf{H}$, invoking the identity (7.6) along with the continuous injection $\mathbf{i}_r : \mathbf{H}^1(\Omega) \to \mathbf{L}^r(\Omega)$, using (7.1a) together with the fact that $\|\cdot\|_{0,\Omega} \leqslant |\Omega|^{(r-2)/2r}\, \|\cdot\|_{0,r;\Omega}$, and bounding ε^{-1} according to (6.4), we deduce the existence of positive constants, denoted and given as

$$\begin{aligned} \|\mathbf{a}\| &:= \frac{1}{\mu}\,, \qquad \|\mathbf{b}\| := 1\,, \qquad \|\mathbf{F}\| := \big(1 + \|\mathbf{i}_r\|\big)\,\|\mathbf{g}\|_{1/2,\Gamma}\,, \\ &\text{and} \quad \|\mathbf{G}\| := \max\left\{\varepsilon_0^{-1}\,|\Omega|^{(r-2)/2r}, 1\right\}, \end{aligned} \tag{7.15}$$

such that

$$\begin{aligned} |\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau})| &\leqslant \|\mathbf{a}\|\,\|\boldsymbol{\zeta}\|_{\mathbf{H}}\,\|\boldsymbol{\tau}\|_{\mathbf{H}} \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\in\mathbf{H}\,, \\ |\mathbf{b}(\boldsymbol{\tau},\mathbf{v})| &\leqslant \|\mathbf{b}\|\,\|\boldsymbol{\tau}\|_{\mathbf{H}}\,\|\mathbf{v}\|_{\mathbf{Q}} \qquad \forall\,(\boldsymbol{\tau},\mathbf{v})\in\mathbf{H}\times\mathbf{Q}\,, \\ |\mathbf{F}(\boldsymbol{\tau})| &\leqslant \|\mathbf{F}\|\,\|\boldsymbol{\tau}\|_{\mathbf{H}} \qquad \forall\,\boldsymbol{\tau}\in\mathbf{H}\,, \quad \text{and} \\ |\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}(\mathbf{v})| &\leqslant \|\mathbf{G}\|\left\{\|\eta_1-\eta_2\|_{0,\rho;\Omega}\,\|\boldsymbol{\phi}\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,s;\Omega}\right\}\|\mathbf{v}\|_{\mathbf{Q}} \qquad \forall\,\mathbf{v}\in\mathbf{Q}\,. \end{aligned} \tag{7.16}$$

7.3 The electrostatic potential equations

We begin the derivation of the mixed formulation for the Poisson equation by testing the first equation in the third row of (6.8) against $\boldsymbol{\psi}\in\mathbf{H}^s(\mathrm{div}_s;\Omega)$. In this way, applying (7.7) with $t = s$ and $t' = r$ to the given $\boldsymbol{\psi}$ and $\chi\in\mathrm{W}^{1,r}(\Omega)$, and employing the Dirichlet boundary condition on χ, for which we assume that $g\in\mathrm{W}^{1/s,r}(\Gamma)$, we get

$$\int_\Omega \frac{1}{\varepsilon}\,\boldsymbol{\varphi}\cdot\boldsymbol{\psi} + \int_\Omega \chi\,\mathrm{div}(\boldsymbol{\psi}) = \langle\boldsymbol{\psi}\cdot\boldsymbol{\nu}, g\rangle_\Gamma\,. \tag{7.17}$$

In turn, testing the second equation in the third row of (6.8) against $\lambda\in\mathrm{L}^s(\Omega)$, which requires to assume that $f\in\mathrm{L}^r(\Omega)$, we obtain

$$\int_\Omega \lambda\,\mathrm{div}(\boldsymbol{\varphi}) = -\int_\Omega \lambda\,(\xi_1-\xi_2) - \int_\Omega f\,\lambda\,, \tag{7.18}$$

which certainly makes sense for $\mathrm{div}(\boldsymbol{\varphi})\in\mathrm{L}^r(\Omega)$. Thus, recalling from (7.1a) and (7.1b) that $\boldsymbol{\varphi}$ must belong to $\mathbf{L}^r(\Omega)$, it follows from the above that this unknown should be sought then in $\mathbf{H}^r(\mathrm{div}_r;\Omega)$. Furthermore, bearing in mind from (7.1a) - (7.1c) that ξ_1 and ξ_2 must belong to $\mathrm{L}^\rho(\Omega)$, we notice that in order for the first term on the right-hand side of (7.18) to make sense, we require that $\rho \geqslant r$, which is assumed from now on. Therefore, placing together (7.17) and (7.18), we obtain the following mixed variational formulation for the electrostatic potential equations (given by the third and fourth rows of (6.8)): Find $(\boldsymbol{\varphi}, \chi) \in \mathrm{X}_2 \times \mathrm{M}_1$ such that

$$\begin{aligned} a(\boldsymbol{\varphi}, \boldsymbol{\psi}) + b_1(\boldsymbol{\psi}, \chi) &= \mathrm{F}(\boldsymbol{\psi}) \qquad \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \\ b_2(\boldsymbol{\varphi}, \lambda) &= \mathrm{G}_{\boldsymbol{\xi}}(\lambda) \qquad \forall\, \lambda \in \mathrm{M}_2\,, \end{aligned} \tag{7.19}$$

where

$$\mathrm{X}_2 := \mathbf{H}^r(\mathrm{div}_r;\Omega)\,, \quad \mathrm{M}_1 := \mathrm{L}^r(\Omega)\,, \quad \mathrm{X}_1 := \mathbf{H}^s(\mathrm{div}_s;\Omega)\,, \quad \mathrm{M}_2 := \mathrm{L}^s(\Omega)\,, \tag{7.20}$$

and the bilinear forms $a : \mathrm{X}_2 \times \mathrm{X}_1 \to \mathrm{R}$ and $b_i : \mathrm{X}_i \times \mathrm{M}_i \to \mathrm{R}$, $i \in \{1, 2\}$, and the functional $\mathrm{F} : \mathrm{X}_1 \to \mathrm{R}$, are defined, respectively, as

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) := \int_\Omega \frac{1}{\varepsilon}\, \boldsymbol{\phi} \cdot \boldsymbol{\psi} \qquad \forall\, (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathrm{X}_2 \times \mathrm{X}_1\,, \tag{7.21a}$$

$$b_i(\boldsymbol{\psi}, \lambda) := \int_\Omega \lambda\, \mathrm{div}(\boldsymbol{\psi}) \qquad \forall\, (\boldsymbol{\psi}, \lambda) \in \mathrm{X}_i \times \mathrm{M}_i\,, \tag{7.21b}$$

$$\mathrm{F}(\boldsymbol{\psi}) := \langle \boldsymbol{\psi} \cdot \boldsymbol{\nu}, g \rangle_\Gamma \qquad \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \tag{7.21c}$$

whereas, given $\boldsymbol{\eta} := (\eta_1, \eta_2) \in \mathbf{L}^\rho(\Omega)$, the functional $\mathrm{G}_{\boldsymbol{\eta}} : \mathrm{M}_2 \to \mathrm{R}$ is defined by

$$\mathrm{G}_{\boldsymbol{\eta}}(\lambda) := -\int_\Omega \lambda\, (\eta_1 - \eta_2) - \int_\Omega f\, \lambda \qquad \forall\, \lambda \in \mathrm{M}_2\,. \tag{7.21d}$$

We end this chapter by establishing the boundedness of a, b_i, $i \in \{1, 2\}$, F, and $\mathrm{G}_{\boldsymbol{\eta}}$, for which we recall that the norms of X_1 and X_2 are defined by (7.4c) with $t = s$ and $t = r$, respectively, whereas those of M_1 and M_2 are certainly given by $\|\cdot\|_{0,r;\Omega}$ and $\|\cdot\|_{0,s;\Omega}$, respectively. Then, employing again the Cauchy–Schwarz and Hölder inequalities, bounding ε^{-1} according to (6.4), and using that $\|\cdot\|_{0,r;\Omega} \leqslant |\Omega|^{(\rho-r)/\rho r}\, \|\cdot\|_{0,\rho;\Omega}$, which follows from the fact that $\rho \geqslant r$, we find that there exist positive constants

$$\|a\| \,:=\, \frac{1}{\varepsilon_0}\,, \qquad \|b_1\| \,=\, \|b_2\| \,:=\, 1\,, \quad \text{and} \quad \|\mathrm{G}\| \,:=\, \max\left\{1, |\Omega|^{(\rho-r)/\rho r}\right\}, \tag{7.22}$$

such that

$$\begin{aligned} |a(\boldsymbol{\phi},\boldsymbol{\psi})| \,&\leqslant\, \|a\|\,\|\boldsymbol{\phi}\|_{\mathrm{X}_2}\,\|\boldsymbol{\psi}\|_{\mathrm{X}_1} \qquad \forall\,(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathrm{X}_2 \times \mathrm{X}_1\,,\\ |b_i(\boldsymbol{\psi},\lambda)| \,&\leqslant\, \|b_i\|\,\|\boldsymbol{\psi}\|_{\mathrm{X}_i}\,\|\lambda\|_{\mathrm{M}_i} \qquad \forall\,(\boldsymbol{\psi},\lambda) \in \mathrm{X}_i \times \mathrm{M}_1\,, \quad \forall\, i \in \left\{1,2\right\}, \quad \text{and}\\ |\mathrm{G}_{\boldsymbol{\eta}}(\lambda)| \,&\leqslant\, \|\mathrm{G}\|\left\{\|\eta_1-\eta_2\|_{0,\rho;\Omega} + \|f\|_{0,r;\Omega}\right\}\|\lambda\|_{0,s;\Omega} \qquad \forall\,\lambda \in \mathrm{M}_2\,. \end{aligned} \tag{7.23}$$

Regarding the boundedness of F, we need to apply [23, Lemma A.36], which, along with the surjectivity of the trace operator mapping $\mathrm{W}^{1,r}(\Omega)$ onto $\mathrm{W}^{1/s,r}(\Gamma)$, yields the existence of a fixed positive constant C_r, such that for the given $g \in \mathrm{W}^{1/s,r}(\Gamma)$, there exists $v_g \in \mathrm{W}^{1,r}(\Omega)$ satisfying $v_g|_\Gamma \,=\, g$ and

$$\|v_g\|_{1,r;\Omega} \,\leqslant\, C_r\,\|g\|_{1/s,r;\Gamma}\,.$$

Hence, employing (7.7) with $(t,t') \,=\, (s,r)$ and $(\boldsymbol{\tau},v) \,=\, (\boldsymbol{\psi},v_g)$, and then using Hölder's inequality, we arrive at

$$|\mathrm{F}(\boldsymbol{\psi})| \,\leqslant\, \|\mathrm{F}\|\,\|\boldsymbol{\psi}\|_{\mathrm{X}_1} \qquad \forall\,\boldsymbol{\psi} \in \mathrm{X}_1\,, \tag{7.24}$$

with

$$\|\mathrm{F}\| \,:=\, C_r\,\|g\|_{1/s,r;\Gamma}\,. \tag{7.25}$$

7.4 The ionized particles concentration equations

We now deal with the Nernst-Planck equations, that is the fifth and sixth rows of (6.8), for which we proceed analogously as we did for the Stokes equations in Chapter 7.2. More precisely, applying (7.5) with $t = \varrho$ to $\boldsymbol{\tau}_i \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$ and $\xi_i \in \mathrm{H}^1(\Omega)$, and using the Dirichlet boundary condition on ξ_i, for which we assume from now on that $g_i \in \mathrm{H}^{1/2}(\Gamma)$, we obtain

$$\int_\Omega \nabla\xi_i \cdot \boldsymbol{\tau}_i \,=\, -\int_\Omega \xi_i \operatorname{div}(\boldsymbol{\tau}_i) \,+\, \langle \boldsymbol{\tau}_i \cdot \boldsymbol{\nu}, g_i \rangle\,,$$

so that the testing of the equation in the fifth row of (6.8) against $\boldsymbol{\tau}_i$, yields

$$\int_\Omega \frac{1}{\kappa_i}\boldsymbol{\sigma}_i \cdot \boldsymbol{\tau}_i \,+\, \int_\Omega \xi_i \operatorname{div}(\boldsymbol{\tau}_i) \,-\, \int_\Omega \Big\{ q_i\, \xi_i\, \varepsilon^{-1} \boldsymbol{\varphi} \,-\, \kappa_i^{-1}\, \xi_i\, \mathbf{u} \Big\} \cdot \boldsymbol{\tau}_i \,=\, \langle \boldsymbol{\tau}_i \cdot \boldsymbol{\nu}, g_i \rangle\,. \tag{7.26}$$

Since $\operatorname{div}(\boldsymbol{\tau}_i) \in \mathrm{L}^{\varrho}(\Omega)$, we notice from the second term on the left-hand side of (7.26) that it suffices to look for ξ_i in $\mathrm{L}^{\rho}(\Omega)$, which, similarly as for Stokes, is coherent with a previous discussion on where to seek this unknown. In fact, as already commented, the corresponding estimates (7.1b) and (7.1c) confirm that the third term on the left-hand side of (7.26) is well-defined as well. We end this derivation by testing the first equation of the sixth row of (6.8) against a function in the same space to which ξ_i belongs, that is $\eta_i \in \mathrm{L}^{\rho}(\Omega)$, which gives

$$\int_\Omega \eta_i \operatorname{div}(\boldsymbol{\sigma}_i) \,-\, \int_\Omega \xi_i\, \eta_i \,=\, -\int_\Omega f_i\, \eta_i\,. \tag{7.27}$$

We remark that the above requires to assume that both f_i and $\operatorname{div}(\boldsymbol{\sigma}_i)$ belong to $\mathrm{L}^{\varrho}(\Omega)$, which is coherent with the fact that ξ_i is sought in $\mathrm{L}^{\rho}(\Omega)$ since, being $\rho > 2$, it follows that $\rho > \varrho$, and hence $\mathrm{L}^{\rho}(\Omega) \subseteq \mathrm{L}^{\varrho}(\Omega)$. Consequently, we arrive at the following mixed variational formulation for the ionized particles concentration equations: Find $(\boldsymbol{\sigma}_i, \xi_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ such that

$$\begin{aligned} a_i(\boldsymbol{\sigma}_i, \boldsymbol{\tau}_i) + c_i(\boldsymbol{\tau}_i, \xi_i) \,-\, c_{\boldsymbol{\varphi},\mathbf{u}}(\boldsymbol{\tau}_i, \xi_i) \quad &= \quad \mathrm{F}_i(\boldsymbol{\tau}_i) \qquad \forall\, \boldsymbol{\tau}_i \in \mathrm{H}_i\,, \\ c_i(\boldsymbol{\sigma}_i, \eta_i) \,-\, d_i(\xi_i, \eta_i) \quad &= \quad \mathrm{G}_i(\eta_i) \qquad \forall\, \eta_i \in \mathrm{Q}_i\,, \end{aligned} \tag{7.28}$$

where

$$\mathrm{H}_i \,:=\, \mathbf{H}(\operatorname{div}_{\varrho};\Omega)\,, \qquad \mathrm{Q}_i \,:=\, \mathrm{L}^{\rho}(\Omega)\,, \tag{7.29}$$

and the bilinear forms $a_i : \mathrm{H}_i \times \mathrm{H}_i \to \mathrm{R}$, $c_i : \mathrm{H}_i \times \mathrm{Q}_i \to \mathrm{R}$, and $d_i : \mathrm{Q}_i \times \mathrm{Q}_i \to \mathrm{R}$, and the functionals $\mathrm{F}_i : \mathrm{H}_i \longrightarrow \mathrm{R}$ and $\mathrm{G}_i : \mathrm{Q}_i \longrightarrow \mathrm{R}$, are defined, respectively, as

$$a_i(\boldsymbol{\zeta}_i, \boldsymbol{\tau}_i) := \int_\Omega \frac{1}{\kappa_i} \boldsymbol{\zeta}_i \cdot \boldsymbol{\tau}_i \qquad \forall\, (\boldsymbol{\zeta}_i, \boldsymbol{\tau}_i) \in \mathrm{H}_i \times \mathrm{H}_i\,, \tag{7.30a}$$

$$c_i(\boldsymbol{\tau}_i, \eta_i) := \int_\Omega \eta_i \,\mathrm{div}(\boldsymbol{\tau}_i) \qquad \forall\, (\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,, \tag{7.30b}$$

$$d_i(\vartheta_i, \eta_i) := \int_\Omega \vartheta_i\, \eta_i \qquad \forall\, (\vartheta_i, \eta_i) \in \mathrm{Q}_i \times \mathrm{Q}_i\,, \tag{7.30c}$$

$$\mathrm{F}_i(\boldsymbol{\tau}_i) := \langle \boldsymbol{\tau}_i \cdot \boldsymbol{\nu}, g_i \rangle \qquad \forall\, \boldsymbol{\tau}_i \in \mathrm{H}_i\,, \tag{7.30d}$$

$$\mathrm{G}_i(\eta_i) := -\int_\Omega f_i\, \eta_i \qquad \forall\, \eta_i \in \mathrm{Q}_i\,, \tag{7.30e}$$

whereas, given $(\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{Q} = \mathbf{H}^r(\mathrm{div}_r;\Omega) \times \mathbf{L}^r(\Omega)$, the bilinear form $c_{\boldsymbol{\phi},\mathbf{v}} : \mathrm{H}_i \times \mathrm{Q}_i \to \mathrm{R}$ is set as

$$c_{\boldsymbol{\phi},\mathbf{v}}(\boldsymbol{\tau}_i, \eta_i) := \int_\Omega \left\{ q_i\, \eta_i\, \varepsilon^{-1} \boldsymbol{\phi} - \kappa_i^{-1}\, \eta_i\, \mathbf{v} \right\} \cdot \boldsymbol{\tau}_i \qquad \forall\, (\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,. \tag{7.30f}$$

Similarly to the analysis at the end of Chapter 7.2 (cf. (7.15) and (7.16)), we conclude here that a_i, c_i, d_i, F_i, G_i, and $c_{\boldsymbol{\phi},\mathbf{v}}$ are all bounded with the norm defined by (7.4a) with $t = \varrho$ for H_i, and certainly the norm $\|\cdot\|_{0,\rho;\Omega}$ for Q_i. Indeed, applying the Cauchy–Schwarz and Hölder inequalities, bounding both ε^{-1} and κ_i^{-1} according to (6.4), noting that $\|\cdot\|_{0,\Omega} \leqslant |\Omega|^{(\rho-2)/2\rho}\, \|\cdot\|_{0,\rho;\Omega}$, invoking the identity (7.5) and the continuous injection $\mathrm{i}_\rho : \mathrm{H}^1(\Omega) \to \mathrm{L}^\rho(\Omega)$, and utilizing (7.1b) and (7.1c), we find that there exist positive constants

$$\begin{gathered} \|a_i\| := \frac{1}{\underline{\kappa}}\,, \quad \|c_i\| := 1\,, \quad \|d_i\| := |\Omega|^{(\rho-2)/\rho}\,, \quad \|\mathrm{F}_i\| := \big(1 + \|\mathrm{i}_\rho\|\big)\, \|g_i\|_{1/2,\Gamma}\,, \\ \|\mathrm{G}_i\| := \|f_i\|_{0,\varrho;\Omega}\,, \qquad \text{and} \quad \|c\| := \max\left\{\varepsilon_0^{-1}, \underline{\kappa}^{-1}\right\}, \end{gathered} \tag{7.31}$$

such that

$$
\begin{aligned}
&|a_i(\boldsymbol{\zeta}_i,\boldsymbol{\tau}_i)| \,\leqslant\, \|a_i\|\,\|\boldsymbol{\zeta}_i\|_{\mathrm{H}_i}\,\|\boldsymbol{\tau}_i\|_{\mathrm{H}_i} \qquad \forall\,(\boldsymbol{\zeta}_i,\boldsymbol{\tau}_i)\in \mathrm{H}_i\times\mathrm{H}_i\,,\\
&|c_i(\boldsymbol{\tau}_i,\eta_i)| \,\leqslant\, \|c_i\|\,\|\boldsymbol{\tau}_i\|_{\mathrm{H}_i}\,\|\eta_i\|_{\mathrm{Q}_i} \qquad \forall\,(\boldsymbol{\tau}_i,\eta_i)\in \mathrm{H}_i\times\mathrm{Q}_i\,,\\
&|d_i(\vartheta_i,\eta_i)| \,\leqslant\, \|d_i\|\,\|\vartheta_i\|_{\mathrm{Q}_i}\,\|\eta_i\|_{\mathrm{Q}_i} \qquad \forall\,(\vartheta_i,\eta_i)\in \mathrm{Q}_i\times\mathrm{Q}_i\,,\\
&|\mathrm{F}_i(\boldsymbol{\tau}_i)| \,\leqslant\, \|\mathrm{F}_i\|\,\|\boldsymbol{\tau}_i\|_{\mathrm{H}_i} \qquad \forall\,\boldsymbol{\tau}_i\in\mathrm{H}_i\,,\\
&|\mathrm{G}_i(\eta_i)| \,\leqslant\, \|\mathrm{G}_i\|\,\|\eta_i\|_{\mathrm{Q}_i} \qquad \forall\,\eta_i\in\mathrm{Q}_i\,,\quad\text{and}\\
&|c_{\boldsymbol{\phi},\mathbf{v}}(\boldsymbol{\tau}_i,\eta_i)| \,\leqslant\, \|c\|\,\Big\{\|\boldsymbol{\phi}\|_{0,r;\Omega}+\|\mathbf{v}\|_{0,r;\Omega}\Big\}\,\|\eta_i\|_{0,\rho;\Omega}\,\|\boldsymbol{\tau}_i\|_{0,\Omega} \qquad \forall\,(\boldsymbol{\tau}_i,\eta_i)\in\mathrm{H}_i\times\mathrm{Q}_i\,.
\end{aligned}
\tag{7.32}
$$

Throughout the rest of the part we will use indistinctly either $\|\boldsymbol{\eta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}$ or $\|\boldsymbol{\eta}\|_{0,\rho;\Omega}$, where

$$
\|\boldsymbol{\eta}\|_{0,\rho;\Omega} \,:=\, \|\eta_1\|_{0,\rho;\Omega} \,+\, \|\eta_2\|_{0,\rho;\Omega} \qquad \forall\,\boldsymbol{\eta}:=(\eta_1,\eta_2)\in\mathrm{Q}_1\times\mathrm{Q}_2\,.
$$

Summarizing, and putting together (7.11), (7.19), and (7.28), we find that, under the assumptions that $\mathbf{f}\in\mathbf{L}^s(\Omega)$, $\mathbf{g}\in\mathbf{H}^{1/2}(\Gamma)$, $f\in\mathrm{L}^r(\Omega)$, $g\in\mathrm{W}^{1/s,r}(\Gamma)$, $f_i\in\mathrm{L}^\varrho(\Omega)$, $g_i\in\mathrm{H}^{1/2}(\Gamma)$, and $\rho\geqslant r$, the mixed variational formulation of (6.8) reduces to: Find $(\boldsymbol{\sigma},\mathbf{u})\in\mathbf{H}\times\mathbf{Q}$, $(\boldsymbol{\varphi},\chi)\in\mathrm{X}_2\times\mathrm{M}_1$, and $(\boldsymbol{\sigma}_i,\xi_i)\in\mathrm{H}_i\times\mathrm{Q}_i$, $i\in\{1,2\}$, such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau})+\mathbf{b}(\boldsymbol{\tau},\mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) && \forall\,\boldsymbol{\tau}\in\mathbf{H}\,,\\
\mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) &= \mathbf{G}_{\boldsymbol{\xi},\boldsymbol{\varphi}}(\mathbf{v}) && \forall\,\mathbf{v}\in\mathbf{Q}\,,\\
a(\boldsymbol{\varphi},\boldsymbol{\psi})+b_1(\boldsymbol{\psi},\chi) &= \mathrm{F}(\boldsymbol{\psi}) && \forall\,\boldsymbol{\psi}\in\mathrm{X}_1\,,\\
b_2(\boldsymbol{\varphi},\lambda) &= \mathrm{G}_{\boldsymbol{\xi}}(\lambda) && \forall\,\lambda\in\mathrm{M}_2\,,\\
a_i(\boldsymbol{\sigma}_i,\boldsymbol{\tau}_i)+c_i(\boldsymbol{\tau}_i,\xi_i)\,-\,c_{\boldsymbol{\varphi},\mathbf{u}}(\boldsymbol{\tau}_i,\xi_i) &= \mathrm{F}_i(\boldsymbol{\tau}_i) && \forall\,\boldsymbol{\tau}_i\in\mathrm{H}_i\,,\\
c_i(\boldsymbol{\sigma}_i,\eta_i)\,-\,d_i(\xi_i,\eta_i) &= \mathrm{G}_i(\eta_i) && \forall\,\eta_i\in\mathrm{Q}_i\,.
\end{aligned}
\tag{7.33}
$$

CHAPTER 8

The continuous solvability analysis

In this chapter we proceed as in several related previous contributions (see, e.g. [13] and the references therein), and employ a fixed-point strategy to address the solvability of (7.33).

8.1 The fixed-point strategy

In order to rewrite (7.33) as an equivalent fixed point equation, we introduce suitable operators associated with each one of the three problems forming the whole nonlinear coupled system. Indeed, we first let $\widehat{\mathrm{T}} : (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2 \to \mathbf{Q}$ be the operator defined by

$$\widehat{\mathrm{T}}(\boldsymbol{\eta}, \boldsymbol{\phi}) := \hat{\mathbf{u}} \qquad \forall\, (\boldsymbol{\eta}, \boldsymbol{\phi}) \in (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2\,,$$

where $(\widehat{\boldsymbol{\sigma}}, \widehat{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution (to be confirmed below) of problem (7.11) (equivalently, the first two rows of (7.33)) with $(\boldsymbol{\eta}, \boldsymbol{\phi})$ instead of $(\boldsymbol{\xi}, \boldsymbol{\varphi})$, that is

$$\begin{aligned} \mathbf{a}(\widehat{\boldsymbol{\sigma}}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \widehat{\mathbf{u}}) &= \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}\,, \\ \mathbf{b}(\widehat{\boldsymbol{\sigma}}, \mathbf{v}) &= \mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{Q}\,. \end{aligned} \tag{8.1}$$

In turn, we let $\bar{\mathrm{T}} : \mathrm{Q}_1 \times \mathrm{Q}_2 \to \mathrm{X}_2$ be the operator given by

$$\bar{\mathrm{T}}(\boldsymbol{\eta}) := \bar{\boldsymbol{\varphi}} \qquad \forall\, \boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2\,,$$

where $(\bar{\boldsymbol{\varphi}}, \bar{\chi}) \in \mathrm{X}_2 \times \mathrm{M}_1$ is the unique solution (to be confirmed below) of problem (7.19) (equivalently, the third and fourth rows of (7.33)) with $\boldsymbol{\eta}$ instead of $\boldsymbol{\xi}$, that is

$$\begin{aligned} a(\bar{\boldsymbol{\varphi}}, \boldsymbol{\psi}) + b_1(\boldsymbol{\psi}, \bar{\chi}) &= \mathrm{F}(\boldsymbol{\psi}) \qquad \forall\, \boldsymbol{\psi} \in \mathrm{X}_1\,, \\ b_2(\bar{\boldsymbol{\varphi}}, \lambda) &= \mathrm{G}_{\boldsymbol{\eta}}(\lambda) \qquad \forall\, \lambda \in \mathrm{M}_2\,. \end{aligned} \tag{8.2}$$

Similarly, for each $i \in \{1, 2\}$, we let $\widetilde{\mathrm{T}}_i : \mathrm{X}_2 \times \mathbf{Q} \to \mathrm{Q}_i$ be the operator defined by

$$\widetilde{\mathrm{T}}_i(\boldsymbol{\phi}, \mathbf{v}) := \widetilde{\xi}_i \qquad \forall\, (\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{Q}\,,$$

where $(\widetilde{\boldsymbol{\sigma}}_i, \widetilde{\xi}_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ is the unique solution (to be confirmed below) of problem (7.28) (equivalently, the fifth and sixth rows of (7.33)) with $(\boldsymbol{\phi}, \mathbf{v})$ instead of $(\boldsymbol{\varphi}, \mathbf{u})$, that is

$$\begin{aligned} a_i(\widetilde{\boldsymbol{\sigma}}_i, \boldsymbol{\tau}_i) + c_i(\boldsymbol{\tau}_i, \widetilde{\xi}_i) - c_{\boldsymbol{\phi},\mathbf{v}}(\boldsymbol{\tau}_i, \widetilde{\xi}_i) &= \mathrm{F}_i(\boldsymbol{\tau}_i) \qquad \forall\, \boldsymbol{\tau}_i \in \mathrm{H}_i\,, \\ c_i(\widetilde{\boldsymbol{\sigma}}_i, \eta_i) - d_i(\widetilde{\xi}_i, \eta_i) &= \mathrm{G}_i(\eta_i) \qquad \forall\, \eta_i \in \mathrm{Q}_i\,, \end{aligned} \tag{8.3}$$

so that we can define the operator $\widetilde{\mathrm{T}} : \mathrm{X}_2 \times \mathbf{Q} \to (\mathrm{Q}_1 \times \mathrm{Q}_2)$ as:

$$\widetilde{\mathrm{T}}(\boldsymbol{\phi}, \mathbf{v}) := \big(\widetilde{\mathrm{T}}_1(\boldsymbol{\phi}, \mathbf{v}), \widetilde{\mathrm{T}}_2(\boldsymbol{\phi}, \mathbf{v})\big) = (\xi_1, \xi_2) =: \widetilde{\boldsymbol{\xi}} \qquad \forall\, (\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{Q}\,. \tag{8.4}$$

Finally, defining the operator $\mathbf{T} : (\mathrm{Q}_1 \times \mathrm{Q}_2) \to (\mathrm{Q}_1 \times \mathrm{Q}_2)$ as

$$\mathbf{T}(\boldsymbol{\eta}) := \widetilde{\mathrm{T}}\big(\bar{\mathrm{T}}(\boldsymbol{\eta}), \widehat{\mathrm{T}}\big(\boldsymbol{\eta}, \bar{\mathrm{T}}(\boldsymbol{\eta})\big)\big) \qquad \forall\, \boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2\,, \tag{8.5}$$

we observe that solving (7.33) is equivalent to seeking a fixed point of $\mathbf{T}$, that is: Find $\boldsymbol{\xi} \in \mathrm{Q}_1 \times \mathrm{Q}_2$ such that

$$\mathbf{T}(\boldsymbol{\xi}) = \boldsymbol{\xi}\,. \tag{8.6}$$

8.2 Well-posedness of the uncoupled problems

In this chapter we establish the well-posedness of the problems (8.1), (8.2), and (8.3), defining the operators $\widehat{\mathrm{T}}$, $\bar{\mathrm{T}}$, and $\widetilde{\mathrm{T}}_i$, respectively. To this end, we apply the Babuška–Brezzi theory in Banach spaces for the general case (cf. [6, Theorem 2.1, Corollary 2.1, Section 2.1]), and for a particular one [23, Theorem 2.34], as well as a recently established result for perturbed saddle point formulations in Banach spaces (cf. Theorem 3.4, [19, Theorem 3.4]) along with the Banach–Nečas–Babuška Theorem (also known as the generalized Lax–Milgram Lemma) (cf. [23, Theorem 2.6]).

8.2.1 Well-definedness of the operator $\widehat{\mathrm{T}}$

Here we apply [23, Theorem 2.34] to show that, given an arbitrary $(\boldsymbol{\eta}, \boldsymbol{\phi}) \in \big(\mathrm{Q}_1 \times \mathrm{Q}_2\big) \times \mathrm{X}_2$, (8.1) is well-posed, equivalently that $\widehat{\mathrm{T}}$ is well-defined. We remark that $(\boldsymbol{\eta}, \boldsymbol{\phi})$ only influences the functional $\mathbf{G}_{\eta,\phi}$, and that the boundedness of all the bilinear forms and linear functionals defining (8.1), has already been established in (7.15) and (7.16). Hence, the discussion below just refers to the remaining hypotheses to be satisfied by $\mathbf{a}$ and $\mathbf{b}$. We begin by letting $\mathbb{V}$ be the kernel of the operator induced by $\mathbf{b}$, that is

$$\mathbb{V} := \Big\{ \boldsymbol{\tau} \in \mathbf{H} : \qquad \mathbf{b}(\boldsymbol{\tau}, \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in \mathbf{Q} \Big\}\,,$$

which, according to the definitions of $\mathbf{H}$, $\mathbf{Q}$, and $\mathbf{b}$ (cf. (7.12), (7.13b)), along with the fact that $\mathbf{L}^s(\Omega)$ is isomorphic to the dual of $\mathbf{L}^r(\Omega)$, yields

$$\mathbb{V} := \Big\{ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_s;\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0} \Big\}. \tag{8.7}$$

Next, we recall that a slight modification of the proof of [26, Lemma 2.3] allows to prove that for each $t \geqslant \frac{2n}{n+2}$ (see, e.g., [11, Lemma 3.1] for the case $t = 4/3$, which is extensible almost verbatim for any t in the indicated range) there exists a constant C_t, depending only on Ω, such that

$$C_t \, \|\boldsymbol{\tau}\|_{0,\Omega}^2 \,\leqslant\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega}^2 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_t;\Omega)\,. \tag{8.8}$$

Then, assuming that $s \geqslant \frac{2n}{n+2}$, and using (8.8), we deduce from the definition of $\mathbf{a}$ (cf. (7.13a)), and similarly to [11, Lemma 3.2], that

$$\mathbf{a}(\boldsymbol{\tau},\boldsymbol{\tau}) \,\geqslant\, \boldsymbol{\alpha}\, \|\boldsymbol{\tau}\|_{\mathbf{div}_s;\Omega}^2 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{V}\,, \tag{8.9}$$

with $\boldsymbol{\alpha} := C_s/\mu$. Hence, thanks to (8.9), it is straightforward to see that $\mathbf{a}$ satisfies the hypotheses specified in [23, Theorem 2.34, eq. (2.28)] with the foregoing constant $\boldsymbol{\alpha}$. In order to fulfill all the hypotheses of the latter theorem, and knowing from (7.15) and (7.16) that the boundedness of the corresponding bilinear forms and linear functionals has already been established, it only remains to show the continuous inf-sup condition for $\mathbf{b}$. Moreover, being this result already proved for the particular case $s = 4/3$ (cf. [11, Lemma 3.3] and [30, Lemma 3.5] for a closely related one), and arising no significant differences for an arbitrary $s \geqslant \frac{2n}{n+2}$, we provide below, and for sake of completeness, only the main aspects of its proof.

Indeed, given $\mathbf{v} \in \mathbf{Q} := \mathbf{L}^r(\Omega)$, we first recall from (7.2) that $r > 2$, and set $\mathbf{v}_s := |\mathbf{v}|^{r-2}\,\mathbf{v}$, which is easily seen to satisfy

$$\mathbf{v}_s \,\in\, \mathbf{L}^s(\Omega) \quad \text{and} \quad \int_\Omega \mathbf{v}\cdot\mathbf{v}_s \,=\, \|\mathbf{v}\|_{0,r;\Omega}\,\|\mathbf{v}_s\|_{0,s;\Omega}\,.$$

In what follows, we make use of both, the Poincaré inequality, which refers to the existence of a positive constant c_{P}, depending on Ω, such that $c_{\mathrm{P}}\,\|\mathbf{z}\|_{1,\Omega}^2 \leqslant |\mathbf{z}|_{1,\Omega}^2 \quad \forall\, \mathbf{z} \in \mathbf{H}_0^1(\Omega)$, and the continuous injection $\mathbf{i}_r : \mathbf{H}^1(\Omega) \to \mathbf{L}^r(\Omega)$ for the indicated range of s. Then, we let $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ be the unique solution of: $\int_\Omega \nabla\mathbf{w}\cdot\nabla\mathbf{z} \,=\, -\int_\Omega \mathbf{v}_s\cdot\mathbf{z}$ for all $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$, which is guaranteed by the classical Lax–Milgram Lemma, and notice, thanks to the corresponding continuous dependence estimate, that $\|\mathbf{w}\|_{1,\Omega} \,\leqslant\, \frac{\|\mathbf{i}_r\|}{c_{\mathrm{P}}}\,\|\mathbf{v}_s\|_{0,s;\Omega}$. Hence, defining $\boldsymbol{\zeta} \,:=\, \nabla\mathbf{w} \,\in\, \mathbb{L}^2(\Omega)$, we deduce that $\mathbf{div}(\boldsymbol{\zeta}) = \mathbf{v}_s$ in Ω, so that $\boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div}_s;\Omega)$, and $\|\boldsymbol{\zeta}\|_{\mathbf{div}_s;\Omega} \,\leqslant\, \big(1 + \frac{\|\mathbf{i}_r\|}{c_{\mathrm{P}}}\big)\,\|\mathbf{v}_s\|_{0,s;\Omega}$. Finally, letting $\boldsymbol{\zeta}_0$ be the $\mathbb{H}_0(\mathbf{div}_s;\Omega)$-component of $\boldsymbol{\zeta}$, it is clear that $\mathbf{div}(\boldsymbol{\zeta}_0) = \mathbf{v}_s$ and that $\|\boldsymbol{\zeta}_0\|_{\mathbf{div}_s;\Omega} \,\leqslant\, \|\boldsymbol{\zeta}\|_{\mathbf{div}_s;\Omega}$, whence bounding by below with $\boldsymbol{\tau} := \boldsymbol{\zeta}_0 \in \mathbf{H}$, and using the definition of $\mathbf{b}$ (cf. (7.13b)) along with the above identities and estimates, we conclude that

$$\sup_{\substack{\boldsymbol{\tau}\in\mathbf{H}\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\mathbf{b}(\boldsymbol{\tau},\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbf{H}}} \,\geqslant\, \boldsymbol{\beta}\,\|\mathbf{v}\|_{\mathbf{Q}} \qquad \forall\,\mathbf{v}\in\mathbf{Q}\,, \tag{8.10}$$

with $\boldsymbol{\beta} \,:=\, \big(1 + \frac{\|\mathbf{i}_r\|}{c_{\mathrm{P}}}\big)^{-1}$. The foregoing inequality (8.10) proves [23, Theorem 2.34, eq. (2.29)] and completes the hypotheses of this theorem.

Consequently, the well-definedness of the operator $\widehat{\mathrm{T}}$ is stated as follows.

Theorem 8.1. *For each $(\boldsymbol{\eta},\boldsymbol{\phi}) \in (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2$ there exists a unique $(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{u}}) \in \mathbf{H}\times\mathbf{Q}$ solution to (8.1), and hence one can define $\widehat{\mathrm{T}}(\boldsymbol{\eta},\boldsymbol{\phi}) \,:=\, \widehat{\mathbf{u}} \in \mathbf{Q}$. Moreover, there exists a positive constant $C_{\widehat{\mathrm{T}}}$, depending only on μ, $\|\mathbf{i}_r\|$, ε_0, $|\Omega|$, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$, and hence independent of $(\boldsymbol{\eta},\boldsymbol{\phi})$, such that*

$$\|\widehat{\mathrm{T}}(\boldsymbol{\eta},\boldsymbol{\phi})\|_{\mathbf{Q}} \,=\, \|\widehat{\mathbf{u}}\|_{\mathbf{Q}} \,\leqslant\, C_{\widehat{\mathrm{T}}}\Big\{\|\mathbf{g}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,s,\Omega} \,+\, \|\boldsymbol{\eta}\|_{0,\rho;\Omega}\,\|\boldsymbol{\phi}\|_{0,r;\Omega}\Big\}\,. \tag{8.11}$$

Proof. Given $(\boldsymbol{\eta},\boldsymbol{\phi}) \in (\mathrm{Q}_1 \times \mathrm{Q}_2) \times \mathrm{X}_2$, a direct application of [23, Theorem 2.34] guarantees the existence of a unique $(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{u}}) \in \mathbf{H}\times\mathbf{Q}$ solution to (8.1). Then, the corresponding a priori estimate in [23, Theorem 2.34, eq. (2.30)] gives

$$\|\widehat{\mathbf{u}}\|_{\mathbf{Q}} \,\leqslant\, \frac{1}{\boldsymbol{\beta}}\left(1 + \frac{\|\mathbf{a}\|}{\boldsymbol{\alpha}}\right)\|\mathbf{F}\|_{\mathbf{H}'} \,+\, \frac{\|\mathbf{a}\|}{\boldsymbol{\beta}^2}\left(1 + \frac{\|\mathbf{a}\|}{\boldsymbol{\alpha}}\right)\|\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}\|_{\mathbf{Q}'}\,, \tag{8.12}$$

which, according to the identities and estimates given by (7.15) and (7.16), along with some algebraic manipulations, yields (8.11) and finishes the proof. □

Regarding the a priori bound for the component $\widehat{\boldsymbol{\sigma}}$ of the unique solution to (8.1), it follows from [23, Theorem 2.34, eq. (2.30)] that

$$\|\widehat{\boldsymbol{\sigma}}\|_{\mathbf{H}} \leqslant \frac{1}{\boldsymbol{\alpha}} \|\mathbf{F}\|_{\mathbf{H}'} + \frac{1}{\boldsymbol{\beta}} \left(1 + \frac{\|\mathbf{a}\|}{\boldsymbol{\alpha}}\right) \|\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}\|_{\mathbf{Q}'} ,$$

which yields the same inequality as (8.11), but with a different constant. Hence, choosing the largest of the respective constants, and still denoting it by $C_{\widehat{\mathrm{T}}}$, we can summarize the a priori estimates for $\widehat{\mathbf{u}}$ and $\widehat{\boldsymbol{\sigma}}$ by saying that both are given by the right-hand side of (8.11).

8.2.2 Well-definedness of the operator $\bar{\mathrm{T}}$

We now employ [6, Theorem 2.1, Section 2.1] to prove that, given an arbitrary $\boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2$, (8.2) is well-posed, equivalently that $\bar{\mathrm{T}}$ is well-defined. Similarly as for Chapter 8.2.1, we first stress that $\boldsymbol{\eta}$ is utilized only to define the functional $\mathrm{G}_{\boldsymbol{\eta}}$, and that the boundedness of all the bilinear forms and functionals defining (8.2), was already established by (7.22) and (7.23). In this way, it only remains to show that a, b_1, and b_2 satisfy the corresponding hypotheses from [6, Theorem 2.1, Section 2.1]. To this end, and because of the evident similarities, we follow very closely the analysis in [13, Section 3.2.3], which, in turn, suitably adopts the approach from [29, Section 2.4.2]. Indeed, we begin by letting K_i be the kernel of the operator induced by the bilinear form b_i, for each $i \in \{1, 2\}$, that is

$$\mathrm{K}_i := \Big\{ \boldsymbol{\psi} \in \mathrm{X}_i : \quad b_i(\boldsymbol{\psi}, \lambda) = 0 \quad \forall\, \lambda \in \mathrm{M}_i \Big\} , \tag{8.13}$$

which, according to the definitions of X_i and M_i (cf. (7.20)), and b_i (cf. (7.21b)), along again with the fact that $\mathbf{L}^r(\Omega)$ and $\mathbf{L}^s(\Omega)$ can be isomorphically identified with $\big(\mathbf{L}^s(\Omega)\big)'$ and $\big(\mathbf{L}^r(\Omega)\big)'$, respectively, gives

$$\mathrm{K}_1 := \Big\{ \boldsymbol{\psi} \in \mathbf{H}^s(\mathrm{div}_s; \Omega) : \quad \mathrm{div}(\boldsymbol{\psi}) = 0 \quad \text{in} \quad \Omega \Big\} , \tag{8.14}$$

and

$$\mathrm{K}_2 := \Big\{ \boldsymbol{\psi} \in \mathbf{H}^r(\mathrm{div}_r; \Omega) : \quad \mathrm{div}(\boldsymbol{\psi}) = 0 \quad \text{in} \quad \Omega \Big\} . \tag{8.15}$$

Next, in order to establish the inf-sup conditions required for the bilinear form a (cf. [6, eqs. (2.8) and (2.9)]), we resort to [13, Lemma 3.3], which is recalled below.

Lemma 8.2. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let $t, t' \in (1,+\infty)$ conjugate to each other with t (and hence t') lying in* $\begin{cases} [4/3,4] & \text{if } n=2 \\ [3/2,3] & \text{if } n=3 \end{cases}$. *Then, there exists a linear and bounded operator $D_t : \mathbf{L}^t(\Omega) \to \mathbf{L}^t(\Omega)$ such that*

$$\mathbf{div}(D_t(\mathbf{w})) \,=\, 0 \quad in \quad \Omega \qquad \forall\, \mathbf{w} \in \mathbf{L}^t(\Omega)\,. \tag{8.16}$$

In addition, for each $\mathbf{z} \in \mathbf{L}^{t'}(\Omega)$ such that $\mathbf{div}(\mathbf{z}) = 0$ in Ω, there holds

$$\int_\Omega \mathbf{z} \cdot D_t(\mathbf{w}) \,=\, \int_\Omega \mathbf{z} \cdot \mathbf{w} \qquad \forall\, \mathbf{w} \in \mathbf{L}^t(\Omega)\,. \tag{8.17}$$

Proof. It reduces to a minor modification of the proof of [29, Lemma 2.3], for which one needs to apply the well-posedness in $\mathrm{W}^{1,t}(\Omega)$ of a Poisson problem with homogeneous Dirichlet boundary conditions (see [27, Theorem 3.2] or [34, Theorems 1.1 and 1.3] for the vector version of it). The specified ranges for t and t' are precisely forced by the latter result. We omit further details and refer to the proof of [13, Lemma 3.3]. □

We are now in position to prove the required hypotheses on a.

Lemma 8.3. *Assume that s (and hence r) satisfy the ranges specified in Lemma 8.2. Then, there exists a positive constant $\bar{\alpha}$ such that*

$$\sup_{\substack{\boldsymbol{\psi}\in \mathrm{K}_1 \\ \boldsymbol{\psi}\neq \mathbf{0}}} \frac{a(\boldsymbol{\phi},\boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_{\mathrm{X}_1}} \,\geqslant\, \bar{\alpha}\, \|\boldsymbol{\phi}\|_{\mathrm{X}_2} \qquad \forall\, \boldsymbol{\phi} \in \mathrm{K}_2\,. \tag{8.18}$$

In addition, there holds

$$\sup_{\boldsymbol{\phi}\in \mathrm{K}_2} a(\boldsymbol{\phi},\boldsymbol{\psi}) \,>\, 0 \qquad \forall\, \boldsymbol{\psi} \in \mathrm{K}_1, \quad \boldsymbol{\psi} \neq \mathbf{0}\,. \tag{8.19}$$

Proof. Being almost verbatim to that of [13, Lemma 3.4], we just proceed to sketch it. Indeed, given $\boldsymbol{\phi} \in \mathrm{K}_2$, we recall from (7.2) that $r > 2$ and set $\boldsymbol{\phi}_s \,:=\, |\boldsymbol{\phi}|^{r-2}\,\boldsymbol{\phi}$, which belongs to $\mathbf{L}^s(\Omega)$

and satisfies

$$\int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{\phi}_s \;=\; \|\boldsymbol{\phi}\|_{0,r,\Omega}\, \|\boldsymbol{\phi}_s\|_{0,s,\Omega}\,. \tag{8.20}$$

In this way, bounding by below with $\boldsymbol{\psi} := D_s(\boldsymbol{\phi}_s)$, which, according to Lemma 8.2, belongs to K_1, and then using (8.17), (8.20), the boundedness of D_s, and the upper bound of ε (cf. (6.4)), we arrive at (8.18) with $\bar{\alpha} := \big(\|D_s\|\,\varepsilon_1\big)^{-1}$. On the other hand, given now $\boldsymbol{\psi} \in \mathrm{K}_1$, $\boldsymbol{\psi} \neq \mathbf{0}$, we define $\boldsymbol{\psi}_r := \begin{cases} |\boldsymbol{\psi}|^{s-2}\,\boldsymbol{\psi} & \text{if } \boldsymbol{\psi} \neq \mathbf{0} \\ \mathbf{0} & \text{if } \boldsymbol{\psi} = \mathbf{0} \end{cases}$, which lies in $\mathbf{L}^r(\Omega)$ and satisfies $\displaystyle\int_\Omega \boldsymbol{\psi}\cdot\boldsymbol{\psi}_r \;=\; \|\boldsymbol{\psi}\|^s_{0,s;\Omega} > 0$. Thus, bounding by below with $\boldsymbol{\phi} := D_r(\boldsymbol{\psi}_r) \in \mathrm{K}_2$, and proceeding similarly as for (8.18), we deduce (8.19) and conclude the proof. $\square$

Before continuing with the continuous inf-sup conditions for the bilinear forms b_i, $i \in \{1,2\}$, we now check the feasibility of the indexes employed so far, according to the different constraints that have arisen along the analysis. In fact, from the preliminary discussion provided in Chapter 7.1, we have the following initial ranges

$$\begin{cases} l,\, j \in (1,+\infty) \quad \text{and} \quad \rho,\, r \in (2,+\infty) \quad \text{if } n = 2\,, \\ l,\, j \in [3/2, 3] \quad \text{and} \quad \rho,\, r \in [3,6] \quad \text{if } n = 3\,, \end{cases} \tag{8.21}$$

which, being added the request $\rho \geqslant r$, equivalently $l \geqslant j$, becomes

$$\begin{cases} l \in [2,+\infty)\,, \quad j \in (1,2]\,, \quad \rho \in [4,+\infty)\,, \quad r \in (2,4] \quad \text{if } n = 2\,, \\ l \in [2,3]\,, \quad j \in [3/2,2]\,, \quad \rho \in [4,6]\,, \quad r \in [3,4] \quad \text{if } n = 3\,. \end{cases} \tag{8.22}$$

Finally, imposing to r (and hence to s) the ranges required by Lemma 8.2, and guaranteeing that $s \geqslant \frac{2n}{n+2}$, we arrive at the final feasible choices

$$\begin{cases} l \in [2,+\infty)\,,\ j \in (1,2]\,, \quad \rho \in [4,+\infty)\,,\ \varrho \in (1,4/3]\,, \quad r \in (2,4]\,,\ s \in [4/3,2) \quad \text{if } n = 2\,, \\ l = 3\,,\ j = 3/2\,, \quad \rho = 6\,,\ \varrho = 6/5\,, \quad r = 3\,,\ s = 3/2 \quad \text{if } n = 3\,. \end{cases} \tag{8.23}$$

Note that in (8.23) we have included the consequent ranges for $\varrho := \frac{\rho}{\rho-1}$ and $s := \frac{r}{r-1}$ as well.

However, we remark that the above indexes are not chosen independently, but once l (or its conjugate j) is chosen, then all the remaining ones are fixed.

We now go back to the well-definedness of $\bar{\mathrm{T}}$ by establishing the continuous inf-sup conditions for the bilinear forms b_i, $i \in \{1,2\}$. While the corresponding proofs are similar to those of [29, Lemma 2.7] and [13, Lemma 3.6], and very close to that of [28, Lemma 3.5], for sake of completeness we provide below the main details of them.

Lemma 8.4. *For each $i \in \{1,2\}$ there exists a positive constant $\bar{\beta}_i$ such that*

$$\sup_{\substack{\boldsymbol{\psi}\in \mathrm{X}_i \\ \boldsymbol{\psi}\neq \mathbf{0}}} \frac{b_i(\boldsymbol{\psi},\lambda)}{\|\boldsymbol{\psi}\|_{\mathrm{X}_i}} \geqslant \bar{\beta}_i\,\|\lambda\|_{\mathrm{M}_i} \qquad \forall\,\lambda\in \mathrm{M}_i\,. \tag{8.24}$$

Proof. We begin by noticing that the values of r and s specified in (8.23) are compatible with the range $[\frac{2n}{n+1},\frac{2n}{n-1}]$ required by [28, Theorem 3.2], an existence result to be applied below. According to it, and since the pairs $(\mathrm{X}_1,\mathrm{M}_1)$ and $(\mathrm{X}_2,\mathrm{M}_2)$ result from each other exchanging r and s, it suffices to prove (8.24) either for $i = 1$ or for $i = 2$. In what follows we consider $i = 1$, so that, given $\lambda \in \mathrm{M}_1 := \mathrm{L}^r(\Omega)$, we set $\lambda_s := |\lambda|^{r-2}\,\lambda$, which belongs to $\mathrm{L}^s(\Omega)$ and satisfies $\int_\Omega \lambda\,\lambda_s = \|\lambda\|_{0,r;\Omega}\,\|\lambda_s\|_{0,s;\Omega}$. Thus, a straightforward application of the scalar version of [28, Theorem 3.2] yields the existence of a unique $z \in \mathrm{W}^{1,s}_0(\Omega)$ such that $\Delta z = \lambda_s$ in Ω, $z = 0$ on Γ. Moreover, the corresponding continuous dependence result reads $\|z\|_{1,s;\Omega} \leqslant \bar{C}_s\,\|\lambda_s\|_{0,s;\Omega}$, where $\bar{C}_s$ is a positive constant depending on s. Next, defining $\boldsymbol{\phi} := \nabla z \in \mathbf{L}^s(\Omega)$, it follows that $\mathrm{div}(\boldsymbol{\phi}) = \lambda_s$ in Ω, whence $\boldsymbol{\phi} \in \mathbf{H}^s(\mathrm{div}_s;\Omega) =: \mathrm{X}_1$, and there holds $\|\boldsymbol{\phi}\|_{\mathrm{X}_1} = \|\boldsymbol{\phi}\|_{s,\mathrm{div}_s;\Omega} \leqslant \big(1+\bar{C}_s\big)\,\|\lambda_s\|_{0,s;\Omega}$. In this way, bounding by below with $\boldsymbol{\psi} := \boldsymbol{\phi} \in \mathrm{X}_1$, and bearing in mind the definition of b_1 (cf. (7.21b)) along with the foregoing identities and estimates, we arrive at (8.24) for $i = 1$ with $\beta_1 := \big(1+\bar{C}_s\big)^{-1}$. The proof for $i = 2$ proceeds analogously, except for the fact that, given $\lambda \in \mathrm{M}_2 := \mathrm{L}^s(\Omega)$, and since $s < 2$, one needs to define $\lambda_r := \begin{cases} |\lambda|^{s-2}\,\lambda & \text{if } \lambda \neq 0\,, \\ 0 & \text{if } \lambda = 0\,. \end{cases}$ Further details are omitted. □

As a consequence of Lemmas 8.3 and 8.4, and the boundedness properties given by (7.22), (7.23), (7.24), and (7.25), we are able to conclude now that the operator $\bar{\mathrm{T}}$ is well-defined.

Theorem 8.5. *For each* $\boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2$ *there exists a unique* $(\bar{\boldsymbol{\varphi}}, \bar{\chi}) \in \mathrm{X}_2 \times \mathrm{M}_1$ *solution to* (8.2), *and hence one can define* $\bar{\mathrm{T}}(\boldsymbol{\eta}) := \bar{\boldsymbol{\varphi}} \in \mathrm{X}_2$. *Moreover, there exists a positive constant* $C_{\bar{\mathrm{T}}}$, *depending only on* ε_0, C_r, $|\Omega|$, $\bar{\alpha}$, *and* $\bar{\beta}_2$, *such that*

$$\|\bar{\mathrm{T}}(\boldsymbol{\eta})\|_{\mathrm{X}_2} \,=\, \|\bar{\boldsymbol{\varphi}}\|_{\mathrm{X}_2} \,\leqslant\, C_{\bar{\mathrm{T}}} \left\{ \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\eta}\|_{0,\rho;\Omega} \right\}. \tag{8.25}$$

Proof. Given $\boldsymbol{\eta} \in \mathrm{Q}_1 \times \mathrm{Q}_2$, a straightforward application of [6, Theorem 2.1, Section 2.1] implies the existence of a unique $(\bar{\boldsymbol{\varphi}}, \bar{\chi}) \in \mathrm{X}_2 \times \mathrm{M}_1$ solution to (8.2). In turn, the a priori estimate provided in [6, Corollary 2.1, Section 2.1, eq. (2.15)] establishes

$$\|\bar{\boldsymbol{\varphi}}\|_{\mathrm{X}_2} \,\leqslant\, \frac{1}{\bar{\alpha}} \|\mathrm{F}\|_{\mathrm{X}_1'} \,+\, \frac{1}{\bar{\beta}_2} \left(1 + \frac{\|a\|}{\bar{\alpha}}\right) \|\mathrm{G}_{\boldsymbol{\eta}}\|_{\mathrm{M}_2'}\,, \tag{8.26}$$

which, along with the aforementioned boundedness properties, yields (8.25) and ends the proof. □

Similarly as for $\widehat{\mathrm{T}}$, and employing now [6, Corollary 2.1, Section 2.1, eq. (2.16)], we observe that the a priori bound for the $\bar{\chi}$ component of the unique solution to (8.2) reduces to

$$\|\bar{\chi}\|_{\mathrm{M}_1} \,\leqslant\, \frac{1}{\bar{\beta}_1} \left(1 + \frac{\|a\|}{\bar{\alpha}}\right) \|\mathrm{F}\|_{\mathrm{X}_1'} \,+\, \frac{\|a\|}{\bar{\beta}_1\,\bar{\beta}_2} \left(1 + \frac{\|a\|}{\bar{\alpha}}\right) \|\mathrm{G}_{\boldsymbol{\eta}}\|_{\mathrm{M}_2'}\,,$$

which yields the same inequality as (8.25), but with a different constant, in particular depending additionally on $\bar{\beta}_1$. Therefore, as before, we still denote the largest of them by $C_{\bar{\mathrm{T}}}$, and simply say that the right hand-side of (8.25) constitutes the a priori estimate for both $\bar{\boldsymbol{\varphi}}$ and $\bar{\chi}$.

8.2.3 Well-definedness of the operator $\widetilde{\mathrm{T}}$

In this section we employ the solvability result for perturbed saddle point formulations in Banach spaces provided by [19, Theorem 3.4], along with the Banach–Nečas–Babuška Theorem (cf. [23, Theorem 2.6]), to show that, given an arbitrary $(\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{Q}$, (8.3) is well-posed for each $i \in \{1, 2\}$, equivalently that T_i is well-defined. Since this result was already derived in [19, Theorem 4.2] as an application of the abstract theory developed there, and more specifically of [19, Theorem 3.4], we just discuss in what follows the main aspects of its proof.

To begin with, we introduce the bilinear forms $\mathbf{A}$, $\mathbf{A}_{\phi,\mathbf{v}} : (\mathrm{H}_i \times \mathrm{Q}_i) \times (\mathrm{H}_i \times \mathrm{Q}_i) \to \mathrm{R}$ given by

$$\mathbf{A}\big((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\big) \;:=\; a_i(\boldsymbol{\zeta}_i, \boldsymbol{\tau}_i) \;+\; c_i(\boldsymbol{\tau}_i, \vartheta_i) \;+\; c_i(\boldsymbol{\zeta}_i, \eta_i) \;-\; d_i(\vartheta_i, \eta_i)\,, \tag{8.27}$$

and

$$\mathbf{A}_{\phi,\mathbf{v}}\big((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\big) \;:=\; \mathbf{A}\big((\boldsymbol{\zeta}_i, \vartheta_i), (\boldsymbol{\tau}_i, \eta_i)\big) \;-\; c_{\phi,\mathbf{v}}(\boldsymbol{\tau}_i, \vartheta_i)\,, \tag{8.28}$$

for all $(\boldsymbol{\zeta}_i, \vartheta_i)$, $(\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i$, and realize that (8.3) can be re-stated as: Find $(\widetilde{\boldsymbol{\sigma}}_i, \widetilde{\xi}_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ such that

$$\mathbf{A}_{\phi,\mathbf{v}}((\widetilde{\boldsymbol{\sigma}}_i, \widetilde{\xi}_i), (\boldsymbol{\tau}_i, \eta_i)) \;=\; \mathrm{F}_i(\boldsymbol{\tau}_i) + \mathrm{G}_i(\eta_i) \qquad \forall\, (\boldsymbol{\tau}_i, \eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,. \tag{8.29}$$

In this way, the proof reduces to show first that the bilinear forms forming part of $\mathbf{A}$ satisfy the hypotheses of [19, Theorem 3.4], and then to combine the consequence of this result with the effect of the extra term given by $c_{\phi,\mathbf{v}}(\cdot,\cdot)$, to conclude that $\mathbf{A}_{\phi,\mathbf{v}}$ satisfies a global inf-sup condition.

Indeed, it is clear from (7.30a), (7.30c), and the upper bound of κ_i (cf. (6.4)) that a_i and d_i are symmetric and positive semi-definite, which proves the assumption i) of [19, Theorem 3.4]. Next, bearing in mind the definitions of c_i (cf. (7.30b)) and the spaces H_i and Q_i (cf. (7.29)), and using again that $\mathrm{L}^\rho(\Omega)$ is isomorphic to the dual of $\mathrm{L}^\varrho(\Omega)$, we readily find that the null space V_i of the operator induced by c_i becomes

$$\mathrm{V}_i \;:=\; \Big\{\boldsymbol{\tau}_i \in \mathrm{H}_i : \quad \mathrm{div}(\boldsymbol{\tau}_i) \;=\; 0\Big\}\,, \tag{8.30}$$

and thus

$$a_i(\boldsymbol{\tau}_i, \boldsymbol{\tau}_i) \;\geqslant\; \frac{1}{\bar{\kappa}}\,\|\boldsymbol{\tau}_i\|^2_{0,\Omega} \;=\; \frac{1}{\bar{\kappa}}\,\|\boldsymbol{\tau}_i\|^2_{\mathrm{div}_\varrho;\Omega} \qquad \forall\,\boldsymbol{\tau}_i \in \mathrm{V}_i\,, \tag{8.31}$$

from which the assumption ii) of [19, Theorem 3.4], namely the continuous inf-sup condition for a_i, is clearly satisfied with constant $\widetilde{\alpha} \;:=\; \bar{\kappa}^{-1}$.

In turn, while the continuous inf-sup condition for $\widetilde{c}_i$ was already established in [29, Lemma 2.9] (see also [19, Lemma 4.1]), for sake of clearness we provide below the main steps of its proof, which follows similarly to the one yielding the continuous inf-sup condition for $\mathbf{b}$ in the present

Chapter 8.2.1. More precisely, given $\eta_i \in \mathrm{Q}_i := \mathrm{L}^\rho(\Omega)$, we set $\eta_{i,\varrho} := |\eta_i|^{\rho-2}\,\eta_i$, which uses from (8.23) that $\rho \geqslant 2$, and notice that there hold $\eta_{i,\varrho} \in \mathrm{L}^\varrho(\Omega)$ and $\int_\Omega \eta_i\,\eta_{i,\varrho} = \|\eta_i\|_{0,\rho;\Omega}\,\|\eta_{i,\varrho}\|_{0,\varrho;\Omega}$. Then, we let $\boldsymbol{\zeta}_i := \nabla z \in \mathbf{L}^2(\Omega)$, where $z \in \mathrm{H}^1_0(\Omega)$ is the unique solution of the variational formulation: $\int_\Omega \nabla z \cdot \nabla w = -\int_\Omega \eta_{i,\varrho}\, w$ for all $w \in \mathrm{H}^1_0(\Omega)$, and deduce from the latter that $\mathrm{div}(\boldsymbol{\zeta}_i) = \eta_{i,\varrho}$ in Ω, which yields $\boldsymbol{\zeta}_i \in \mathrm{H}_i := \mathbf{H}(\mathrm{div}_\varrho;\Omega)$. In turn, denoting by c_{P} the positive constant guaranteeing the Poincaré inequality: $c_{\mathrm{P}}\,\|w\|^2_{1,\Omega} \leqslant |w|^2_{1,\Omega}$ $\forall\, w \in \mathrm{H}^1_0(\Omega$, and letting again $\mathrm{i}_\rho : \mathrm{H}^1(\Omega) \to \mathrm{L}^\rho(\Omega)$ be the continuous injection, we find that $\|z\|_{1,\Omega} \leqslant \frac{\|\mathrm{i}_\rho\|}{c_{\mathrm{P}}}\,\|\eta_{i,\varrho}\|_{0,\varrho;\Omega}$, and hence $\|\boldsymbol{\zeta}_i\|_{\mathrm{H}_i} \leqslant \big(1 + \frac{\|\mathrm{i}_\rho\|}{c_{\mathrm{P}}}\big)\,\|\eta_{i,\varrho}\|_{0,\varrho;\Omega}$. In this way, bounding by below with $\boldsymbol{\tau}_i := \boldsymbol{\zeta}_i \in \mathrm{H}_i$, recalling the definition of c_i (cf. (7.30b)), and employing the foregoing identities and estimates, we arrive at

$$\sup_{\substack{\boldsymbol{\tau}_i\in \mathrm{H}_i\\ \boldsymbol{\tau}_i\neq \mathbf{0}}} \frac{c_i(\boldsymbol{\tau}_i,\eta_i)}{\|\boldsymbol{\tau}_i\|_{\mathrm{H}_i}} \geqslant \widetilde{\beta}\,\|\eta_i\|_{\mathrm{Q}_i} \qquad \forall\, \eta_i \in \mathrm{Q}_i\,, \tag{8.32}$$

with $\widetilde{\beta} := \big(1 + \frac{\|\mathrm{i}_\rho\|}{c_{\mathrm{P}}}\big)^{-1}$, thus confirming the verification of assumption iii) of [19, Theorem 3.4].

Consequently, having shown that a_i, c_i, and d_i verify all the hypotheses of [19, Theorem 3.4], we conclude that $\mathbf{A}$ satisfies the global inf-sup condition, which means that there exists a positive constant $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}$, depending only on $\|a_i\|$, $\|c_i\|$, $\widetilde{\alpha}$, and $\widetilde{\beta}$, such that

$$\sup_{\substack{(\boldsymbol{\tau}_i,\eta_i)\in \mathrm{H}_i\times \mathrm{Q}_i\\ (\boldsymbol{\tau}_i,\eta_i)\neq \mathbf{0}}} \frac{\mathbf{A}\big((\boldsymbol{\zeta}_i,\vartheta_i),(\boldsymbol{\tau}_i,\eta_i)\big)}{\|(\boldsymbol{\tau}_i,\eta_i)\|_{\mathrm{H}_i\times \mathrm{Q}_i}} \geqslant \widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}\,\|(\boldsymbol{\zeta}_i,\vartheta_i)\|_{\mathrm{H}_i\times \mathrm{Q}_i} \qquad \forall\,(\boldsymbol{\zeta}_i,\vartheta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,. \tag{8.33}$$

Moreover, invoking the upper bound of $c_{\boldsymbol{\phi},\mathbf{v}}$ (cf. (7.31), (7.32)), it follows from (8.28) and (8.33) that

$$\sup_{\substack{(\boldsymbol{\tau}_i,\eta_i)\in \mathrm{H}_i\times \mathrm{Q}_i\\ (\boldsymbol{\tau}_i,\eta_i)\neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\boldsymbol{\zeta}_i,\vartheta_i),(\boldsymbol{\tau}_i,\eta_i)\big)}{\|(\boldsymbol{\tau}_i,\eta_i)\|_{\mathrm{H}_i\times \mathrm{Q}_i}} \geqslant \Big\{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}} - \|c\|\,\Big(\|\boldsymbol{\phi}\|_{0,r,\Omega} + \|\mathbf{v}\|_{0,r,\Omega}\Big)\Big\}\,\|(\boldsymbol{\zeta}_i,\vartheta_i)\|_{\mathrm{H}_i\times \mathrm{Q}_i} \tag{8.34}$$

for all $(\boldsymbol{\zeta}_i,\vartheta_i) \in \mathrm{H}_i \times \mathrm{Q}_i$, from which, under the assumption that, say

$$\|\boldsymbol{\phi}\|_{0,r,\Omega} + \|\mathbf{v}\|_{0,r,\Omega} \leqslant \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2\,\|c\|}\,, \tag{8.35}$$

we conclude that

$$\sup_{\substack{(\boldsymbol{\tau}_i,\eta_i)\in \mathrm{H}_i\times \mathrm{Q}_i\\ (\boldsymbol{\tau}_i,\eta_i)\neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\boldsymbol{\zeta}_i,\vartheta_i),(\boldsymbol{\tau}_i,\eta_i)\big)}{\|(\boldsymbol{\tau}_i,\eta_i)\|_{\mathrm{H}_i\times \mathrm{Q}_i}} \,\geqslant\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2}\,\|(\boldsymbol{\zeta}_i,\vartheta_i)\|_{\mathrm{H}_i\times \mathrm{Q}_i} \qquad \forall\,(\boldsymbol{\zeta}_i,\vartheta_i)\in \mathrm{H}_i\times \mathrm{Q}_i\,. \tag{8.36}$$

Similarly, using the symmetry of $\mathbf{A}$ and (8.33), and assuming again (8.35), we find that

$$\sup_{\substack{(\boldsymbol{\zeta}_i,\vartheta_i)\in \mathrm{H}_i\times \mathrm{Q}_i\\ (\boldsymbol{\zeta}_i,\vartheta_i)\neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\boldsymbol{\zeta}_i,\vartheta_i),(\boldsymbol{\tau}_i,\eta_i)\big)}{\|(\boldsymbol{\zeta}_i,\vartheta_i)\|_{\mathrm{H}_i\times \mathrm{Q}_i}} \,\geqslant\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2}\,\|(\boldsymbol{\tau}_i,\eta_i)\|_{\mathrm{H}_i\times \mathrm{Q}_i} \qquad \forall\,(\boldsymbol{\tau}_i,\eta_i)\in \mathrm{H}_i\times \mathrm{Q}_i\,. \tag{8.37}$$

In this way, we are now in position of establishing that, for each $i\in\{1,2\}$, (8.3) is well-posed, which means, equivalently, that $\widetilde{\mathrm{T}}_i$ is well-defined.

Theorem 8.6. *For each $i\in\{1,2\}$, and for each $(\boldsymbol{\phi},\mathbf{v})\in \mathrm{X}_2\times\mathbf{Q}$ such that* (8.35) *holds, there exists a unique $(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)\in \mathrm{H}_i\times \mathrm{Q}_i$ solution to* (8.3)*, and hence one can define $\widetilde{\mathrm{T}}_i(\boldsymbol{\phi},\mathbf{v}) := \widetilde{\xi}_i\in \mathrm{Q}_i$. Moreover, there exists a positive constant $C_{\widetilde{\mathrm{T}}}$, depending only on $\|\mathrm{i}_\rho\|$ and $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}$, such that*

$$\|\widetilde{\mathrm{T}}_i(\boldsymbol{\phi},\mathbf{v})\|_{\mathrm{Q}_i} \,=\, \|\widetilde{\xi}_i\|_{\mathrm{Q}_i} \,\leqslant\, \|(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)\|_{\mathrm{H}_i\times \mathrm{Q}_i} \,\leqslant\, C_{\widetilde{\mathrm{T}}}\Big\{\|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big\}\,. \tag{8.38}$$

Proof. Thanks to (8.36), (8.37), and the boundedness of F_i and G_i (cf. (7.31), (7.32)), the unique solvability of (8.3) follows from a straightforward application of [23, Theorem 2.6]. In turn, the a priori estimate given by [23, Theorem 2.6, eq. (2.5)] reads

$$\|(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)\|_{\mathrm{H}_i\times \mathrm{Q}_i} \,\leqslant\, \frac{2}{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}\Big\{\|\mathrm{F}_i\|_{\mathrm{H}_i'} \,+\, \|\mathrm{G}_i\|_{\mathrm{Q}_i'}\Big\}\,,$$

which, along with the upper bounds for $\|\mathrm{F}_i\|_{\mathrm{H}_i'}$ and $\|\mathrm{G}_i\|_{\mathrm{Q}_i'}$ derived from (7.31) and (7.32), yields (8.38) with $C_{\widetilde{\mathrm{T}}} := \dfrac{2}{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}\big(1+\|\mathrm{i}_\rho\|\big)$. □

We end this chapter by observing from the definition of $\widetilde{\mathrm{T}}$ (cf. (8.4)) and the priori estimates given by (8.38) for each $i\in\{1,2\}$, that

$$\|\widetilde{\mathrm{T}}(\boldsymbol{\phi},\mathbf{v})\|_{\mathrm{Q}_1\times \mathrm{Q}_2} \,:=\, \sum_{i=1}^{2}\|\widetilde{\mathrm{T}}_i(\boldsymbol{\phi},\mathbf{v})\|_{\mathrm{Q}_i} \,\leqslant\, C_{\widetilde{\mathrm{T}}}\sum_{i=1}^{2}\Big\{\|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\Big\} \tag{8.39}$$

for each $(\boldsymbol{\phi}, \mathbf{v}) \in \mathrm{X}_2 \times \mathbf{Q}$ satisfying (8.35).

8.3 Solvability analysis of the fixed-point scheme

Knowing that the operators $\widehat{\mathrm{T}}$, $\bar{\mathrm{T}}$, $\widetilde{\mathrm{T}}$, and hence $\mathbf{T}$ as well, are well defined, we now address the solvability of the fixed-point equation (8.5). For this purpose, and in order to finally apply the Banach Theorem, we first derive sufficient conditions under which $\mathbf{T}$ maps a closed ball of $\mathrm{Q}_1 \times \mathrm{Q}_2$ into itself. Thus, letting δ be an arbitrary radius to be properly chosen later on, we define

$$\mathrm{W}(\delta) := \Big\{ \boldsymbol{\eta} := (\eta_1, \eta_2) \in \mathrm{Q}_1 \times \mathrm{Q}_2 : \quad \|\boldsymbol{\eta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \leqslant \delta \Big\} . \tag{8.40}$$

Then, given $\boldsymbol{\eta} \in \mathrm{W}(\delta)$, we have from the definition of $\mathbf{T}$ (cf. (8.5)) and the a priori estimate for $\widetilde{\mathrm{T}}$ (cf. (8.39)) that, under the assumption (cf. (8.35))

$$\mathcal{S}(\boldsymbol{\eta}) := \|\bar{\mathrm{T}}(\boldsymbol{\eta})\|_{0,r,\Omega} + \|\widehat{\mathrm{T}}\big(\boldsymbol{\eta}, \bar{\mathrm{T}}(\boldsymbol{\eta})\big)\|_{0,r,\Omega} \leqslant \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2\,\|c\|} , \tag{8.41}$$

there holds

$$\|\mathbf{T}(\boldsymbol{\eta})\|_{\mathrm{Q}_1\times\mathrm{Q}_2} = \|\widetilde{\mathrm{T}}\big(\bar{\mathrm{T}}(\boldsymbol{\eta}), \widehat{\mathrm{T}}\big(\boldsymbol{\eta}, \bar{\mathrm{T}}(\boldsymbol{\eta})\big)\big)\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \leqslant C_{\widetilde{\mathrm{T}}} \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega} \Big\} . \tag{8.42}$$

In turn, applying the a priori estimates for $\widehat{\mathrm{T}}$ (cf. (8.11)) and $\bar{\mathrm{T}}$ (cf. (8.25)), we find that

$$\begin{aligned}
\mathcal{S}(\boldsymbol{\eta}) &\leqslant \big(1 + C_{\widehat{\mathrm{T}}}\,\|\boldsymbol{\eta}\|\big)\,\|\bar{\mathrm{T}}(\boldsymbol{\eta})\| + C_{\widehat{\mathrm{T}}} \Big\{ \|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega} \Big\} \\
&\leqslant C_0\big(1 + \|\boldsymbol{\eta}\|\big)\|\boldsymbol{\eta}\| + C_0\big(1 + \|\boldsymbol{\eta}\|\big)\Big\{ \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} \Big\} + C_{\widehat{\mathrm{T}}}\Big\{ \|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega} \Big\} ,
\end{aligned}$$

with $C_0 := \max\big\{1, C_{\widehat{\mathrm{T}}}\big\}\, C_{\bar{\mathrm{T}}}$, so that, bounding $\|\boldsymbol{\eta}\|$ by δ, we deduce that a sufficient condition for (8.41) reduces to

$$C_0\big(1 + \delta\big)\delta + C_0\big(1 + \delta\big)\Big\{ \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} \Big\} + C_{\widehat{\mathrm{T}}}\Big\{ \|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega} \Big\} \leqslant \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{2\,\|c\|} . \tag{8.43}$$

For instance, defining

$$\delta \,:=\, \min\left\{1, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{8C_0\|c\|}\right\}, \tag{8.44}$$

letting $C_1 \,:=\, 2C_0$, and imposing

$$C_1\left\{\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\right\} + C_{\widehat{\mathrm{T}}}\left\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega}\right\} \,\leqslant\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}{4\,\|c\|}\,, \tag{8.45}$$

it is easily seen that (8.43) holds. We have therefore proved the following result.

Lemma 8.7. *Assume that δ and the data are sufficiently small so that there hold* (8.43) *and*

$$C_{\widetilde{\mathrm{T}}}\sum_{i=1}^{2}\left\{\|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega}\right\} \,\leqslant\, \delta\,. \tag{8.46}$$

Then, $\mathbf{T}\big(\mathrm{W}(\delta)\big) \,\subseteq\, \mathrm{W}(\delta)$*. In particular, with the definition* (8.44) *of δ, and under the assumptions* (8.45) *and* (8.46)*, the same conclusion is attained.*

We now address the continuity properties of $\widehat{\mathrm{T}}$, $\bar{\mathrm{T}}$, $\widetilde{\mathrm{T}}$, and hence of $\mathbf{T}$. We begin with that of $\widehat{\mathrm{T}}$.

Lemma 8.8. *There exists a positive constant $\mathrm{L}_{\widehat{\mathrm{T}}}$, depending only on ε_0, $|\Omega|$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\|\mathbf{a}\|$, such that*

$$\|\widehat{\mathrm{T}}(\boldsymbol{\eta},\boldsymbol{\phi}) - \widehat{\mathrm{T}}(\boldsymbol{\vartheta},\boldsymbol{\psi})\|_{\mathbf{Q}} \,\leqslant\, \mathrm{L}_{\widehat{\mathrm{T}}}\left\{\|\boldsymbol{\eta}\|_{0,\rho,\Omega}\,\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{0,r,\Omega} \,+\, \|\boldsymbol{\eta}-\boldsymbol{\vartheta}\|_{0,\rho,\Omega}\,\|\boldsymbol{\psi}\|_{0,r,\Omega}\right\} \tag{8.47}$$

for all $(\boldsymbol{\eta},\boldsymbol{\phi})$, $(\boldsymbol{\vartheta},\boldsymbol{\psi}) \in \big(\mathrm{Q}_1 \times \mathrm{Q}_2\big) \times \mathrm{X}_2$.

Proof. Given $(\boldsymbol{\eta},\boldsymbol{\phi})$, $(\boldsymbol{\vartheta},\boldsymbol{\psi}) \in \big(\mathrm{Q}_1 \times \mathrm{Q}_2\big) \times \mathrm{X}_2$, we let $\widehat{\mathrm{T}}(\boldsymbol{\eta},\boldsymbol{\phi}) \,:=\, \widehat{\mathbf{u}}$ and $\widehat{\mathrm{T}}(\boldsymbol{\vartheta},\boldsymbol{\psi}) \,:=\, \widehat{\mathbf{w}}$, where $(\widehat{\boldsymbol{\sigma}},\widehat{\mathbf{u}}) \in \mathbf{H}\times\mathbf{Q}$ and $(\widehat{\boldsymbol{\zeta}},\widehat{\mathbf{w}}) \in \mathbf{H}\times\mathbf{Q}$ are the corresponding unique solutions of (8.1). Then, subtracting both systems, we obtain

$$\begin{aligned} \mathbf{a}(\widehat{\boldsymbol{\sigma}}-\widehat{\boldsymbol{\zeta}},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},\widehat{\mathbf{u}}-\widehat{\mathbf{w}}) \;&=\; 0 \qquad && \forall\,\boldsymbol{\tau}\in\mathbf{H}\,,\\ \mathbf{b}(\widehat{\boldsymbol{\sigma}}-\widehat{\boldsymbol{\zeta}},\mathbf{v}) \;&=\; \big(\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} - \mathbf{G}_{\boldsymbol{\vartheta},\boldsymbol{\psi}}\big)(\mathbf{v}) \qquad && \forall\,\mathbf{v}\in\mathbf{Q}\,, \end{aligned} \tag{8.48}$$

which says that $(\widehat{\boldsymbol{\sigma}} - \widehat{\boldsymbol{\zeta}}, \widehat{\mathbf{u}} - \widehat{\mathbf{w}}) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution of a system like (8.1), but with $\mathbf{F} = \mathbf{0}$ and $\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} - \mathbf{G}_{\boldsymbol{\vartheta},\boldsymbol{\psi}}$ instead of just $\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}$. Hence, similarly as for the derivation of (8.11), that is employing [23, Theorem 2.34, eq. (2.30)] (see also (8.12)), we deduce that

$$\|\widehat{\mathrm{T}}(\boldsymbol{\eta},\boldsymbol{\phi}) - \widehat{\mathrm{T}}(\boldsymbol{\vartheta},\boldsymbol{\psi})\|_{\mathbf{Q}} \,=\, \|\widehat{\mathbf{u}} - \widehat{\mathbf{w}}\|_{\mathbf{Q}} \,\leqslant\, \frac{\|\mathbf{a}\|}{\boldsymbol{\beta}^2}\left(1 + \frac{\|\mathbf{a}\|}{\boldsymbol{\alpha}}\right)\|\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} - \mathbf{G}_{\boldsymbol{\vartheta},\boldsymbol{\psi}}\|_{\mathbf{Q}'}\,. \tag{8.49}$$

In turn, it is clear from (7.13d), and then subtracting and adding $\boldsymbol{\psi}$ to the factor $\boldsymbol{\phi}$ in the first term, that for each $\mathbf{v} \in \mathbf{Q}$ there holds

$$\begin{aligned}(\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} - \mathbf{G}_{\boldsymbol{\vartheta},\boldsymbol{\psi}})(\mathbf{v}) \,&=\, \int_\Omega \varepsilon^{-1}\Big\{(\eta_1 - \eta_2)\,\boldsymbol{\phi} - (\vartheta_1 - \vartheta_2)\,\boldsymbol{\psi}\Big\}\cdot\mathbf{v}\\ &=\, \int_\Omega \varepsilon^{-1}\Big\{(\eta_1 - \eta_2)\,(\boldsymbol{\phi} - \boldsymbol{\psi}) + \big((\eta_1 - \vartheta_1) - (\eta_2 - \vartheta_2)\big)\,\boldsymbol{\psi}\Big\}\cdot\mathbf{v}\,,\end{aligned}$$

from which, proceeding as for the boundedness of $\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}}$ (cf. (7.15), (7.16)), that is employing the lower bound of ε (cf. (6.4)), (7.1a), and the fact that $\|\cdot\|_{0,\Omega} \leqslant |\Omega|^{(r-2)/2r}\|\cdot\|_{0,r;\Omega}$, we conclude that

$$\|\mathbf{G}_{\boldsymbol{\eta},\boldsymbol{\phi}} - \mathbf{G}_{\boldsymbol{\vartheta},\boldsymbol{\psi}}\|_{\mathbf{Q}'} \,\leqslant\, \varepsilon_0^{-1}\,|\Omega|^{(r-2)/2r}\Big\{\|\boldsymbol{\eta}\|_{0,\rho,\Omega}\,\|\boldsymbol{\phi} - \boldsymbol{\psi}\|_{0,r,\Omega} \,+\, \|\boldsymbol{\eta} - \boldsymbol{\vartheta}\|_{0,\rho,\Omega}\,\|\boldsymbol{\psi}\|_{0,r,\Omega}\Big\}\,. \tag{8.50}$$

In this way, replacing (8.50) back into (8.49), we arrive at (8.47) and finish the proof. □

The next result establishes the continuity of $\bar{\mathrm{T}}$, whose proof follows similarly to that of Lemma 8.8.

Lemma 8.9. *There exists a positive constant* $\mathrm{L}_{\bar{\mathrm{T}}}$*, depending only on* $|\Omega|$*,* $\bar{\alpha}$*,* $\bar{\beta}_2$*, and* $\|a\|$*, such that*

$$\|\bar{\mathrm{T}}(\boldsymbol{\eta}) - \bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2} \,\leqslant\, \mathrm{L}_{\bar{\mathrm{T}}}\,\|\boldsymbol{\eta} - \boldsymbol{\vartheta}\|_{0,\rho;\Omega} \qquad \forall\,\boldsymbol{\eta},\,\boldsymbol{\vartheta} \in \mathrm{Q}_1 \times \mathrm{Q}_2\,. \tag{8.51}$$

Proof. Given $\boldsymbol{\eta}$, $\boldsymbol{\vartheta} \in \mathrm{Q}_1 \times \mathrm{Q}_2$, we let $\bar{\mathrm{T}}(\boldsymbol{\eta}) := \bar{\boldsymbol{\varphi}}$ and $\bar{\mathrm{T}}(\boldsymbol{\vartheta}) := \bar{\boldsymbol{\phi}}$, where $(\bar{\boldsymbol{\varphi}}, \bar{\chi}) \in \mathrm{X}_2 \times \mathrm{M}_1$ and $(\bar{\boldsymbol{\phi}}, \bar{\omega}) \in \mathrm{X}_2 \times \mathrm{M}_1$ are the corresponding unique solutions of (8.2). Then, subtracting both systems, we get

$$\begin{aligned} a(\bar{\varphi} - \bar{\phi}, \psi) + b_1(\psi, \bar{\chi} - \bar{\omega}) &= 0 && \forall\, \psi \in \mathrm{X}_1\,, \\ b_2(\bar{\varphi} - \bar{\phi}, \lambda) &= \big(\mathrm{G}_{\eta} - \mathrm{G}_{\vartheta}\big)(\lambda) && \forall\, \lambda \in \mathrm{M}_2\,, \end{aligned} \tag{8.52}$$

which states that $(\bar{\varphi} - \bar{\phi}, \bar{\chi} - \bar{\omega}) \in \mathrm{X}_2 \times \mathrm{M}_1$ is the unique solution of a problem like (8.2) with $\mathrm{G} = \mathbf{0}$ and $\mathrm{G}_{\eta} - \mathrm{G}_{\vartheta}$ instead of G_{η}. In this way, proceeding as for the derivation of (8.25), which means applying the a priori estimate given by [6, Corollary 2.1, Section 2.1, eq. (2.15)] (see also (8.26)), we find that

$$\|\bar{\mathrm{T}}(\eta) - \bar{\mathrm{T}}(\vartheta)\|_{\mathrm{X}_2} \;=\; \|\bar{\varphi} - \bar{\phi}\|_{\mathrm{X}_2} \;\leqslant\; \frac{1}{\bar{\beta}_2}\left(1 + \frac{\|a\|}{\bar{\alpha}}\right) \|\mathrm{G}_{\eta} - \mathrm{G}_{\vartheta}\|_{\mathrm{M}_2'}\,. \tag{8.53}$$

Now, it is clear from (7.21d) that for each $\lambda \in \mathrm{M}_2$ there holds

$$\big(\mathrm{G}_{\eta} - \mathrm{G}_{\vartheta}\big)(\lambda) \;=\; \mathrm{G}_{\eta - \vartheta}(\lambda) \;=\; -\int_{\Omega} \lambda \big\{(\eta_1 - \vartheta_1) - (\eta_2 - \vartheta_2)\big\}\,,$$

from which, applying Hölder's inequality, as we did for the boundedness of G_{η} (cf. (7.22), (7.23)), and using that $\|\cdot\|_{0,r;\Omega} \leqslant |\Omega|^{(\rho-r)/\rho r}\|\cdot\|_{0,\rho;\Omega}$, we deduce that

$$\|\mathrm{G}_{\eta} - \mathrm{G}_{\vartheta}\|_{\mathrm{M}_2'} \;\leqslant\; |\Omega|^{(\rho-r)/\rho r}\,\|\eta - \vartheta\|_{0,\rho;\Omega}\,. \tag{8.54}$$

Finally, employing (8.54) in (8.53), we obtain (8.51) and conclude the proof. □

It remains to prove the continuity of $\widetilde{\mathrm{T}}$, which is provided by the following lemma.

Lemma 8.10. *There exists a positive constant* $\mathrm{L}_{\widetilde{\mathrm{T}}}$*, depending only on* ε_0*,* $\underline{\kappa}$*,* $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}$*, and* $C_{\widetilde{\mathrm{T}}}$*, such that*

$$\|\widetilde{\mathrm{T}}(\phi, \mathbf{v}) - \widetilde{\mathrm{T}}(\psi, \mathbf{w})\|_{\mathrm{Q}_1 \times \mathrm{Q}_2} \;\leqslant\; \mathrm{L}_{\widetilde{\mathrm{T}}} \sum_{i=1}^{2} \Big\{\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega}\Big\}\, \|(\phi, \mathbf{v}) - (\psi, \mathbf{w})\|_{\mathrm{X}_2 \times \mathbf{Q}} \tag{8.55}$$

for all $(\phi, \mathbf{v}), (\psi, \mathbf{w}) \in \mathrm{X}_2 \times \mathbf{Q}$ *satisfying* (8.35).

Proof. Given $(\phi, \mathbf{v})$ and $(\psi, \mathbf{w})$ as indicated, we let, for each $i \in \{1, 2\}$, $\widetilde{\mathrm{T}}_i(\phi, \mathbf{v}) := \widetilde{\xi}_i \in \mathrm{Q}_i$

and $\widetilde{\mathrm{T}}_i(\boldsymbol{\psi},\mathbf{w}) := \widetilde{\vartheta}_i \in \mathrm{Q}_i$, where $(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ and $(\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i) \in \mathrm{H}_i \times \mathrm{Q}_i$ are the corresponding unique solutions of (8.3), equivalently (cf. (8.29))

$$\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i),(\boldsymbol{\tau}_i,\eta_i)\big) \;=\; \mathrm{F}_i(\boldsymbol{\tau}_i) + \mathrm{G}_i(\eta_i) \qquad \forall\,(\boldsymbol{\tau}_i,\eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,, \tag{8.56}$$

and

$$\mathbf{A}_{\boldsymbol{\psi},\mathbf{w}}\big((\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i),(\boldsymbol{\tau}_i,\eta_i)\big) \;=\; \mathrm{F}_i(\boldsymbol{\tau}_i) + \mathrm{G}_i(\eta_i) \qquad \forall\,(\boldsymbol{\tau}_i,\eta_i) \in \mathrm{H}_i \times \mathrm{Q}_i\,. \tag{8.57}$$

It follows from (8.56) and (8.57), along with the definitions of the bilinear forms $\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}$ (cf. (8.28)) and $c_{\boldsymbol{\phi},\mathbf{v}}$ (cf. (7.30f)), that

$$\begin{aligned}\mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)-(\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i),(\boldsymbol{\tau}_i,\eta_i)\big) \;&=\; \mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i),(\boldsymbol{\tau}_i,\eta_i)\big) - \mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i),(\boldsymbol{\tau}_i,\eta_i)\big)\\ &=\; \mathbf{A}_{\boldsymbol{\psi},\mathbf{w}}\big((\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i),(\boldsymbol{\tau}_i,\eta_i)\big) - \mathbf{A}_{\boldsymbol{\phi},\mathbf{v}}\big((\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i),(\boldsymbol{\tau}_i,\eta_i)\big) \;=\; c_{\boldsymbol{\phi}-\boldsymbol{\psi},\mathbf{v}-\mathbf{w}}(\boldsymbol{\tau}_i,\widetilde{\vartheta}_i)\,,\end{aligned} \tag{8.58}$$

so that applying the global inf-sup condition (8.36) to $(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)-(\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i)$, and then using (8.58) and the boundedness of $c_{\boldsymbol{\phi},\mathbf{v}}$ (cf. (7.31), (7.32)), we conclude that

$$\|\widetilde{\xi}_i - \widetilde{\vartheta}_i\|_{\mathrm{Q}_i} \;\leqslant\; \|(\widetilde{\boldsymbol{\sigma}}_i,\widetilde{\xi}_i)-(\widetilde{\boldsymbol{\zeta}}_i,\widetilde{\vartheta}_i)\|_{\mathrm{H}_i\times\mathrm{Q}_i} \;\leqslant\; \frac{2\,\|c\|}{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}\Big\{\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{0,r;\Omega} + \|\mathbf{v}-\mathbf{w}\|_{0,r;\Omega}\Big\}\,\|\widetilde{\vartheta}_i\|_{\mathrm{Q}_i}\,.$$

Next, invoking the a priori bound (8.38) for $\|\widetilde{\vartheta}_i\|_{\mathrm{Q}_i}$, the foregoing inequality yields

$$\|\widetilde{\mathrm{T}}_i(\boldsymbol{\phi},\mathbf{v}) - \widetilde{\mathrm{T}}_i(\boldsymbol{\psi},\mathbf{w})\|_{\mathrm{Q}_i} \;\leqslant\; \frac{2\,\|c\|\,C_{\widetilde{\mathrm{T}}}}{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A}}}\Big\{\|g_i\|_{1/2,\Gamma} \;+\; \|f_i\|_{0,\varrho;\Omega}\Big\}\,\|(\boldsymbol{\phi},\mathbf{v})-(\boldsymbol{\psi},\mathbf{w})\|_{\mathrm{X}_2\times\mathbf{Q}}\,,$$

from which, summing over $i \in \big\{1,2\big\}$, we arrive at (8.55) and end the proof. □

Having proved Lemmas 8.8, 8.9, and 8.10, we now aim to derive the continuity property of the fixed point operator $\mathbf{T}$. To this end, given $\boldsymbol{\eta}$, $\boldsymbol{\vartheta} \in \mathrm{W}(\delta)$ (cf. (8.40)), we first recall from the definition of $\mathbf{T}$ (cf. (8.5)) and Theorem 8.6 that, in order to define $\mathbf{T}(\boldsymbol{\eta})$ and $\mathbf{T}(\boldsymbol{\vartheta})$, we need that the pairs $\big(\bar{\mathrm{T}}(\boldsymbol{\eta}),\widehat{\mathrm{T}}(\boldsymbol{\eta},\bar{\mathrm{T}}(\boldsymbol{\eta}))\big)$ and $\big(\bar{\mathrm{T}}(\boldsymbol{\vartheta}),\widehat{\mathrm{T}}(\boldsymbol{\vartheta},\bar{\mathrm{T}}(\boldsymbol{\vartheta}))\big)$ satisfy (8.35). Then, according to the discussion at the beginning of the present chapter, we know that a sufficient condition for the latter is given by (8.43), which we assume in what follows. Alternatively, and as indicated there as well, (8.44) and (8.45) are in turn sufficient for (8.43).

Thus, under the aforementioned assumption on δ and the data, a direct application of (8.55) (cf. Lemma 8.10) yields

$$\begin{aligned}\|\mathbf{T}(\boldsymbol{\eta}) - \mathbf{T}(\boldsymbol{\vartheta})\|_{\mathrm{Q}_1\times\mathrm{Q}_2} &= \|\widetilde{\mathrm{T}}\big(\bar{\mathrm{T}}(\boldsymbol{\eta}), \widehat{\mathrm{T}}(\boldsymbol{\eta}, \bar{\mathrm{T}}(\boldsymbol{\eta}))\big) - \widetilde{\mathrm{T}}\big(\bar{\mathrm{T}}(\boldsymbol{\vartheta}), \widehat{\mathrm{T}}(\boldsymbol{\vartheta}, \bar{\mathrm{T}}(\boldsymbol{\vartheta}))\big)\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \\ &\leqslant \mathrm{L}_{\widetilde{\mathrm{T}}} \sum_{i=1}^{2} \Big\{\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega}\Big\} \Big\{\|\bar{\mathrm{T}}(\boldsymbol{\eta}) - \bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2} + \|\widehat{\mathrm{T}}(\boldsymbol{\eta}, \bar{\mathrm{T}}(\boldsymbol{\eta})) - \widehat{\mathrm{T}}(\boldsymbol{\vartheta}, \bar{\mathrm{T}}(\boldsymbol{\vartheta}))\|_{\mathbf{Q}}\Big\}.\end{aligned} \tag{8.59}$$

In addition, employing now (8.51) (cf. Lemma 8.9) and (8.47) (cf. Lemma 8.8), we obtain

$$\|\bar{\mathrm{T}}(\boldsymbol{\eta}) - \bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2} \leqslant \mathrm{L}_{\bar{\mathrm{T}}}\, \|\boldsymbol{\eta} - \boldsymbol{\vartheta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}\,, \tag{8.60}$$

and

$$\begin{aligned}&\|\widehat{\mathrm{T}}(\boldsymbol{\eta}, \bar{\mathrm{T}}(\boldsymbol{\eta})) - \widehat{\mathrm{T}}(\boldsymbol{\vartheta}, \bar{\mathrm{T}}(\boldsymbol{\vartheta}))\|_{\mathbf{Q}} \\ &\quad\leqslant \mathrm{L}_{\widehat{\mathrm{T}}} \Big\{\|\boldsymbol{\eta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}\, \|\bar{\mathrm{T}}(\boldsymbol{\eta}) - \bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2} + \|\boldsymbol{\eta} - \boldsymbol{\vartheta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}\, \|\bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2}\Big\},\end{aligned} \tag{8.61}$$

respectively, whereas the a priori estimate for $\bar{\mathrm{T}}(\boldsymbol{\vartheta})$ (cf. (8.25), Theorem 8.5) states

$$\|\bar{\mathrm{T}}(\boldsymbol{\vartheta})\|_{\mathrm{X}_2} \leqslant C_{\bar{\mathrm{T}}} \Big\{\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\vartheta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}\Big\}. \tag{8.62}$$

In this way, using (8.60) in both (8.59) and (8.61), and then replacing the resulting (8.61) along with (8.62) in (8.59), as well as recalling that $\|\boldsymbol{\eta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}$ and $\|\boldsymbol{\vartheta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}$ are bounded by δ, we deduce the existence of a positive constant $\mathrm{L}_{\mathbf{T}}$, depending only on $\mathrm{L}_{\widetilde{\mathrm{T}}}$, $\mathrm{L}_{\bar{\mathrm{T}}}$, $\mathrm{L}_{\widehat{\mathrm{T}}}$, and $C_{\bar{\mathrm{T}}}$, such that

$$\begin{aligned}&\|\mathbf{T}(\boldsymbol{\eta}) - \mathbf{T}(\boldsymbol{\vartheta})\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \\ &\quad\leqslant \mathrm{L}_{\mathbf{T}} \Big(1 + \delta + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\Big) \sum_{i=1}^{2} \Big\{\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega}\Big\}\, \|\boldsymbol{\eta} - \boldsymbol{\vartheta}\|_{\mathrm{Q}_1\times\mathrm{Q}_2}\,,\end{aligned} \tag{8.63}$$

for all $\boldsymbol{\eta}$, $\boldsymbol{\vartheta} \in \mathrm{W}(\delta)$. We are thus in position to establish the main result of this chapter.

Theorem 8.11. *In addition to the hypotheses of Lemma* 8.7*, that is* (8.43) *and* (8.46)*, or* *alternatively* (8.44), (8.45), *and* (8.46), *assume that*

$$\mathrm{L}_{\mathbf{T}}\left(1+\delta+\|g\|_{1/s,r;\Gamma}+\|f\|_{0,r;\Omega}\right)\sum_{i=1}^{2}\left\{\|g_i\|_{1/2,\Gamma}+\|f_i\|_{0,\varrho,\Omega}\right\} \,<\, 1\,. \tag{8.64}$$

Then, the operator $\mathbf{T}$ *has a unique fixed point* $\boldsymbol{\xi} \in \mathrm{W}(\delta)$. *Equivalently, the coupled problem* (7.33) *has a unique solution* $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbf{H}\times\mathbf{Q}$, $(\boldsymbol{\varphi},\chi) \in \mathrm{X}_2\times\mathrm{M}_1$, *and* $(\boldsymbol{\sigma}_i,\xi_i) \in \mathrm{H}_i\times\mathrm{Q}_i$, $i \in \{1,2\}$, *with* $\boldsymbol{\xi} := (\xi_1,\xi_2) \in \mathrm{W}(\delta)$. *Moreover, there hold the following a priori estimates*

$$\begin{aligned}
\|(\boldsymbol{\sigma},\mathbf{u})\|_{\mathbf{H}\times\mathbf{Q}} \,&\leqslant\, C_{\hat{\mathrm{T}}}\left\{\|\mathbf{g}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,s,\Omega} \,+\, \|\boldsymbol{\xi}\|_{0,\rho;\Omega}\,\|\boldsymbol{\varphi}\|_{0,r;\Omega}\right\},\\
\|(\boldsymbol{\varphi},\chi)\|_{\mathrm{X}_2\times\mathrm{M}_1} \,&\leqslant\, C_{\bar{\mathrm{T}}}\left\{\|g\|_{1/s,r;\Gamma}+\|f\|_{0,r;\Omega}+\|\boldsymbol{\xi}\|_{0,\rho;\Omega}\right\}, \quad \textit{and}\\
\|(\sigma_i,\xi_i)\|_{\mathrm{H}_i\times\mathrm{Q}_i} \,&\leqslant\, C_{\tilde{\mathrm{T}}}\left\{\|g_i\|_{1/2,\Gamma}+\|f_i\|_{0,\varrho,\Omega}\right\} \quad i\in\{1,2\}\,.
\end{aligned} \tag{8.65}$$

Proof. We first recall that the assumptions of Lemma 8.7 guarantee that $\mathbf{T}$ maps $\mathrm{W}(\delta)$ into itself. Then, bearing in mind the Lipschitz-continuity of $\mathbf{T}:\mathrm{W}(\delta)\to\mathrm{W}(\delta)$ (cf. (8.63)) and the assumption (8.64), a straightforward application of the classical Banach theorem yields the existence of a unique fixed point $\boldsymbol{\xi} \in \mathrm{W}(\delta)$ of this operator, and hence a unique solution of (7.33). Finally, it is easy to see that the a priori estimates provided by (8.11) (cf. Theorems 8.1), (8.25) (cf. Theorem 8.5), and (8.38) (cf. Theorem 8.6) yield (8.65) and finish the proof. □

CHAPTER 9

The Galerkin scheme

We now introduce the Galerkin scheme of the fully mixed variational formulation (7.33), analyze its solvability by applying a discrete version of the fixed point approach adopted in Chapter 8.1, and derive the corresponding a priori error estimate.

9.1 Preliminaries

We first let $\mathbf{H}_h$, $\mathbf{Q}_h$, $\mathrm{X}_{i,h}$, $\mathrm{M}_{i,h}$, $\mathrm{H}_{i,h}$,and $\mathrm{Q}_{i,h}$, $i \in \{1,2\}$, be arbitrary finite element subspaces of the spaces $\mathbf{H}$, $\mathbf{Q}$, X_i, M_i, H_i, and Q_i, $i \in \{1,2\}$, respectively. Hereafter, h denotes both the sub-index of each subspace and the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, so that $h := \max\{h_K : K \in \mathcal{T}_h\}$. Explicit finite element subspaces satisfying the stability hypotheses to be introduced throughout the forthcoming analysis, will be defined later on in Chapter 10. Then, the Galerkin scheme associated with (7.33) reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, $(\boldsymbol{\varphi}_h, \chi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$, and

$(\boldsymbol{\sigma}_{i,h}, \xi_{i,h}) \in \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h},\ i \in \{1,2\}$, such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau} \in \mathbf{H}_h\,,\\
\mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= \mathbf{G}_{\boldsymbol{\xi}_h,\boldsymbol{\varphi}_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,,\\
a(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) + b_1(\boldsymbol{\psi}_h, \chi_h) &= \mathrm{F}(\boldsymbol{\psi}_h) && \forall\, \boldsymbol{\psi}_h \in \mathrm{X}_{1,h}\,,\\
b_2(\boldsymbol{\varphi}_h, \lambda_h) &= \mathrm{G}_{\boldsymbol{\xi}_h}(\lambda_h) && \forall\, \lambda_h \in \mathrm{M}_{2,h}\,,\\
a_i(\boldsymbol{\sigma}_{i,h}, \boldsymbol{\tau}_{i,h}) + c_i(\boldsymbol{\tau}_{i,h}, \xi_{i,h}) - c_{\boldsymbol{\varphi}_h,\mathbf{u}_h}(\boldsymbol{\tau}_{i,h}, \xi_{i,h}) &= \mathrm{F}_i(\boldsymbol{\tau}_{i,h}) && \forall\, \boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h}\,,\\
c_i(\boldsymbol{\sigma}_{i,h}, \eta_{i,h}) - d_i(\xi_{i,h}, \eta_{i,h}) &= \mathrm{G}_i(\eta_{i,h}) && \forall\, \eta_{i,h} \in \mathrm{Q}_{i,h}\,.
\end{aligned}
\tag{9.1}
$$

In what follows, we adopt the discrete version of the strategy employed in Chapter 8.1 to analyse the solvability of (9.1). We now let $\widehat{\mathrm{T}}_h : (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \times \mathrm{X}_{2,h} \to \mathbf{Q}_h$ be the operator defined by

$$\widehat{\mathrm{T}}_h(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h) := \widehat{\mathbf{u}}_h \qquad \forall\, (\boldsymbol{\eta}_h, \boldsymbol{\phi}_h) \in (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \times \mathrm{X}_{2,h}\,,$$

where $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\mathbf{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the unique solution (to be confirmed below) of the first two rows of (9.1) with $(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h)$ instead of $(\boldsymbol{\xi}_h, \boldsymbol{\varphi}_h)$, that is

$$
\begin{aligned}
\mathbf{a}(\widehat{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \widehat{\mathbf{u}}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbf{H}_h\,,\\
\mathbf{b}(\widehat{\boldsymbol{\sigma}}_h, \mathbf{v}_h) &= \mathbf{G}_{\boldsymbol{\eta}_h,\boldsymbol{\phi}_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,.
\end{aligned}
\tag{9.2}
$$

In turn, we let $\overline{\mathrm{T}}_h : \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h} \to \mathrm{X}_{2,h}$ be the operator given by

$$\overline{\mathrm{T}}_h(\boldsymbol{\eta}_h) := \bar{\boldsymbol{\varphi}}_h \qquad \forall\, \boldsymbol{\eta}_h \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}\,,$$

where $(\bar{\boldsymbol{\varphi}}_h, \bar{\chi}_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ is the unique solution (to be confirmed below) of the third and fourth rows of (9.1) with $\boldsymbol{\eta}_h$ instead of $\boldsymbol{\xi}_h$, that is

$$\begin{aligned} a(\bar{\boldsymbol{\varphi}}_h, \boldsymbol{\psi}_h) + b_1(\boldsymbol{\psi}_h, \bar{\chi}_h) &= \mathrm{F}(\boldsymbol{\psi}_h) \qquad \forall\, \boldsymbol{\psi}_h \in \mathrm{X}_{1,h}\,, \\ b_2(\bar{\boldsymbol{\varphi}}_h, \lambda_h) &= \mathrm{G}_{\boldsymbol{\eta}_h}(\lambda_h) \qquad \forall\, \lambda_h \in \mathrm{M}_{2,h}\,. \end{aligned} \tag{9.3}$$

Similarly, for each $i \in \{1,2\}$, we let $\widetilde{\mathrm{T}}_{i,h} : \mathrm{X}_{2,h} \times \mathbf{Q}_h \to \mathrm{Q}_{i,h}$ be the operator defined by

$$\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_h, \mathbf{v}_h) := \widetilde{\xi}_{i,h} \qquad \forall\, (\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{Q}_h\,,$$

where $(\widetilde{\boldsymbol{\sigma}}_{i,h}, \widetilde{\xi}_{i,h}) \in \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}$ is the unique solution (to be confirmed below) of the fifth and sixth rows of (9.1) with $(\boldsymbol{\phi}_h, \mathbf{v}_h)$ instead of $(\boldsymbol{\varphi}_h, \mathbf{u}_h)$, that is

$$\begin{aligned} a_i(\widetilde{\boldsymbol{\sigma}}_{i,h}, \boldsymbol{\tau}_{i,h}) + c_i(\boldsymbol{\tau}_{i,h}, \widetilde{\xi}_{i,h}) - c_{\boldsymbol{\phi}_h, \mathbf{v}_h}(\boldsymbol{\tau}_{i,h}, \widetilde{\xi}_{i,h}) &= \mathrm{F}_i(\boldsymbol{\tau}_{i,h}) \qquad \forall\, \boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h}\,, \\ c_i(\widetilde{\boldsymbol{\sigma}}_{i,h}, \eta_{i,h}) - d_i(\widetilde{\xi}_{i,h}, \eta_{i,h}) &= \mathrm{G}_i(\eta_{i,h}) \qquad \forall\, \eta_{i,h} \in \mathrm{Q}_{i,h}\,, \end{aligned} \tag{9.4}$$

so that we can define the operator $\widetilde{\mathrm{T}}_h : \mathrm{X}_{2,h} \times \mathbf{Q}_h \to (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h})$ as:

$$\widetilde{\mathrm{T}}_h(\boldsymbol{\phi}_h, \mathbf{v}_h) := \big(\widetilde{\mathrm{T}}_{1,h}(\boldsymbol{\phi}_h, \mathbf{v}_h), \widetilde{\mathrm{T}}_{2,h}(\boldsymbol{\phi}_h, \mathbf{v}_h)\big) = (\widetilde{\xi}_{1,h}, \widetilde{\xi}_{2,h}) =: \widetilde{\boldsymbol{\xi}}_h \qquad \forall\, (\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{Q}_h\,. \tag{9.5}$$

Finally, defining the operator $\mathbf{T}_h : (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \to (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h})$ as

$$\mathbf{T}(\boldsymbol{\eta}_h) := \widetilde{\mathrm{T}}_h\big(\bar{\mathrm{T}}_h(\boldsymbol{\eta}_h), \widehat{\mathrm{T}}_h\big(\boldsymbol{\eta}_h, \bar{\mathrm{T}}_h(\boldsymbol{\eta}_h)\big)\big) \qquad \forall\, \boldsymbol{\eta}_h \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}\,, \tag{9.6}$$

we observe that solving (9.1) is equivalent to seeking a fixed point of $\mathbf{T}_h$, that is: Find $\boldsymbol{\xi}_h \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}$ such that

$$\mathbf{T}_h(\boldsymbol{\xi}_h) = \boldsymbol{\xi}_h\,. \tag{9.7}$$

9.2 Discrete solvability analysis

In this chapter we proceed analogously to Chapters 8.2 and 8.3 and establish the well-posedness of the discrete system (9.1) by means of the solvability study of the equivalent fixed point equation (9.7). In this regard, we emphasize in advance that, being the respective analysis very similar to that developed in the aforementioned chapters, here we simply collect the main results and provide selected details of the corresponding proofs.

According to the above, we first aim to prove that the discrete operators $\widehat{\mathrm{T}}_h$, $\bar{\mathrm{T}}_h$, and $\widetilde{\mathrm{T}}_{i,h}$, $i \in \{1,2\}$, and hence $\widetilde{\mathrm{T}}_h$ and $\mathbf{T}_h$, are all well-defined, which reduces, equivalently, to show that the problems (9.2), (9.3), and (9.4) are well-posed. To this end, we now apply the discrete versions of [23, Theorem 2.34], [6, Theorem 2.1, Section 2.1], and [19, Theorem 3.4], which are given by [23, Proposition 2.42], [6, Corollary 2.2, Section 2.2], and [19, Theorem 3.5], respectively. More precisely, following similar approaches from related works (see, e.g. [13, Section 4.2]), our analysis throughout the rest of this chapter is based on suitable hypotheses that need to be satisfied by the finite element subspaces utilized in (9.1), which are split according to the requirements of the associated decoupled problems. Explicit examples of discrete spaces verifying these assumptions will be specified later on in Chapter 10.

We begin by addressing the well-definedness of $\widehat{\mathrm{T}}_h$, for which we let $\mathbb{V}_h$ be the discrete kernel of $\mathbf{b}$, that is

$$\mathbb{V}_h := \Big\{ \boldsymbol{\tau}_h \in \mathbf{H}_h : \quad \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{Q}_h \Big\}\,, \tag{9.8}$$

and assume that

(**H.1**) there holds $\mathbf{div}(\mathbf{H}_h) \subseteq \mathbf{Q}_h$, and

(**H.2**) there exists a positive constant $\boldsymbol{\beta}_{\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{H}}} \geq \boldsymbol{\beta}_{\mathrm{d}}\, \|\mathbf{v}_h\|_{\mathbf{Q}} \qquad \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,. \tag{9.9}$$

Then, according to the definition of $\mathbf{b}$ (cf. (7.13b)), it follows from (9.8) and (**H.1**) that

$$\mathbb{V}_h := \Big\{ \boldsymbol{\tau}_h \in \mathbf{H}_h : \quad \mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0} \Big\}, \tag{9.10}$$

which says that $\mathbb{V}_h$ is contained in the continuous kernel $\mathbb{V}$ (cf. (8.7)), and hence the discrete version of (8.9) is automatically satisfied, that is

$$\mathbf{a}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geqslant \boldsymbol{\alpha}_{\mathrm{d}} \, \|\boldsymbol{\tau}\|^2_{\mathbf{div}_s;\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{V}_h\,, \tag{9.11}$$

with $\boldsymbol{\alpha}_{\mathrm{d}} = \boldsymbol{\alpha} := C_s/\mu$. Recall here that C_s is the constant provided by inequality (8.8) with $t = s$. In this way, it is clear from (9.11) that $\mathbf{a}$ satisfies the hypotheses given by [23, Proposition 2.42, eq. (2.35)] with the constant $\boldsymbol{\alpha}_{\mathrm{d}}$, whereas (**H.2**) states that $\mathbf{b}$ fulfills [23, Proposition 2.42, eq. (2.36)] with the constant $\boldsymbol{\beta}_{\mathrm{d}}$. We are thus in position to establish next the following result.

Theorem 9.1. *For each $(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h) \in (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \times \mathrm{X}_{2,h}$ there exists a unique $(\widehat{\boldsymbol{\sigma}}_h, \widehat{\mathbf{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ solution to (9.2), and hence one can define $\widehat{\mathrm{T}}_h(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h) := \widehat{\mathbf{u}}_h \in \mathbf{Q}_h$. Moreover, there exists a positive constant $C_{\widehat{\mathrm{T}},\mathrm{d}}$, depending only on μ, $\|\mathbf{i}_r\|$, ε_0, $|\Omega|$, $\boldsymbol{\alpha}_{\mathrm{d}}$, and $\boldsymbol{\beta}_{\mathrm{d}}$, and hence independent of $(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h)$, such that*

$$\|\widehat{\mathrm{T}}_h(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h)\|_{\mathbf{Q}} = \|\widehat{\mathbf{u}}_h\|_{\mathbf{Q}} \leqslant C_{\widehat{\mathrm{T}},\mathrm{d}} \Big\{ \|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega} + \|\boldsymbol{\eta}_h\|_{0,\rho;\Omega} \, \|\boldsymbol{\phi}_h\|_{0,r;\Omega} \Big\}. \tag{9.12}$$

Proof. Given $(\boldsymbol{\eta}_h, \boldsymbol{\phi}_h) \in (\mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}) \times \mathrm{X}_{2,h}$, the existence of a unique solution to (9.2) follows from a straightforward application of [23, Proposition 2.42]. In turn, the corresponding a priori bound from [23, Theorem 2.34, eq. (2.30)] and the boundedness properties of $\mathbf{F}$ and $\mathbf{G}_{\boldsymbol{\eta}_h,\boldsymbol{\phi}_h}$ imply (9.12). □

Similarly as observed for the continuous operator $\widehat{\mathrm{T}}$, we remark here that the right-hand side of (9.12) can also be assumed as the respective a priori estimate for $\widehat{\boldsymbol{\sigma}}_h$.

Furthermore, for the well-definedness of $\bar{\mathrm{T}}_h$, we need to introduce the discrete kernels of b_1 and b_2, namely

$$\mathrm{K}_{1,h} := \Big\{ \boldsymbol{\psi}_h \in \mathrm{X}_{1,h} : \quad b_1(\boldsymbol{\psi}_h, \lambda_h) = 0 \quad \forall\, \lambda_h \in \mathrm{M}_{1,h} \Big\}, \tag{9.13}$$

and

$$\mathrm{K}_{2,h} \;:=\; \Big\{ \boldsymbol{\psi}_h \in \mathrm{X}_{2,h} : \quad b_2(\boldsymbol{\psi}_h, \lambda_h) = 0 \quad \forall\, \lambda_h \in \mathrm{M}_{2,h} \Big\}\,, \tag{9.14}$$

respectively, and consider the following assumptions

(**H.3**) there exists a positive constant $\bar{\alpha}_{\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\psi}_h \in \mathrm{K}_{1,h} \\ \boldsymbol{\psi}_h \neq \mathbf{0}}} \frac{a(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_{\mathrm{X}_1}} \;\geqslant\; \bar{\alpha}_{\mathrm{d}}\, \|\boldsymbol{\phi}_h\|_{\mathrm{X}_2} \qquad \forall\, \boldsymbol{\phi}_h \in \mathrm{K}_{2,h}\,, \quad \text{and} \tag{9.15a}$$

$$\sup_{\boldsymbol{\phi}_h \in \mathrm{K}_{2,h}} a(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) \;>\; 0 \qquad \forall\, \boldsymbol{\psi}_h \in \mathrm{K}_{1,h}, \quad \boldsymbol{\psi}_h \neq \mathbf{0}\,. \tag{9.15b}$$

(**H.4**) for each $i \in \{1, 2\}$ there exists a positive constant $\bar{\beta}_{i,\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\psi}_h \in \mathrm{X}_{i,h} \\ \boldsymbol{\psi}_h \neq \mathbf{0}}} \frac{b_i(\boldsymbol{\psi}_h, \lambda_h)}{\|\boldsymbol{\psi}_h\|_{\mathrm{X}_i}} \;\geqslant\; \bar{\beta}_{i,\mathrm{d}}\, \|\lambda_h\|_{\mathrm{M}_i} \qquad \forall\, \lambda_h \in \mathrm{M}_{i,h}\,. \tag{9.16}$$

As a consequence of (**H.3**) and (**H.4**) we provide next the discrete version of Theorem 8.5.

Theorem 9.2. *For each $\boldsymbol{\eta}_h \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}$ there exists a unique $(\bar{\boldsymbol{\varphi}}_h, \bar{\chi}_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$ solution to* (9.3)*, and hence one can define $\bar{\mathrm{T}}_h(\boldsymbol{\eta}_h) := \bar{\boldsymbol{\varphi}}_h \in \mathrm{X}_{2,h}$. Moreover, there exists a positive constant $C_{\bar{\mathrm{T}},\mathrm{d}}$, depending only on ε_0, C_r, $|\Omega|$, $\bar{\alpha}_{\mathrm{d}}$, and $\bar{\beta}_{2,\mathrm{d}}$, such that*

$$\|\bar{\mathrm{T}}_h(\boldsymbol{\eta}_h)\|_{\mathrm{X}_2} \;=\; \|\bar{\boldsymbol{\varphi}}_h\|_{\mathrm{X}_2} \;\leqslant\; C_{\bar{\mathrm{T}},\mathrm{d}} \Big\{ \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\eta}_h\|_{0,\rho;\Omega} \Big\}\,. \tag{9.17}$$

Proof. Given $\boldsymbol{\eta}_h \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h}$, a direct application of [6, Corollary 2.2, Section 2.2] implies the existence of a unique solution to (9.3), whereas the a priori estimate provided in [6, Corollary 2.2, eq. (2.24)] and the boundedness properties of F and $\mathrm{G}_{\boldsymbol{\eta}_h}$ yield (9.17). □

Analogously as explained for the continuous operator $\bar{\mathrm{T}}$, here we can also assume that, except for a constant $C_{\bar{\mathrm{T}},\mathrm{d}}$ depending additionally on $\bar{\beta}_{1,\mathrm{d}}$, the a priori estimate for $\bar{\chi}_h$, which follows now from [6, Corollary 2.2, eq. (2.25)], is also given by the right-hand side of (9.17).

It remains to prove the well-definedness of $\widetilde{\mathrm{T}}_h := (\widetilde{\mathrm{T}}_{1,h}, \widetilde{\mathrm{T}}_{2,h})$, for which we first observe that, being a_i and c_i symmetric and positive semi-definite in the whole spaces H_i and Q_i, they certainly keep these properties in $\mathrm{H}_{i,h}$ and $\mathrm{Q}_{i,h}$, respectively, so that the assumption i) of [19, Theorem 3.5] is clearly satisfied. Next, given $i \in \{1,2\}$, we let $\mathrm{V}_{i,h}$ be the discrete kernel of c_i, that is

$$\mathrm{V}_{i,h} := \Big\{ \boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h} : \quad c_i(\boldsymbol{\tau}_{i,h}, \eta_{i,h}) = 0 \quad \forall\, \eta_{i,h} \in \mathrm{Q}_{i,h} \Big\}, \tag{9.18}$$

and consider the hypotheses

(**H.5**) for each $i \in \{1,2\}$ there holds $\mathrm{div}\big(\mathrm{H}_{i,h}\big) \subseteq \mathrm{Q}_{i,h}$, and

(**H.6**) there exists a positive constant $\widetilde{\beta}_{\mathrm{d}} > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h} \\ \boldsymbol{\tau}_{i,h} \neq \mathbf{0}}} \frac{c_i(\boldsymbol{\tau}_{i,h}, \eta_{i,h})}{\|\boldsymbol{\tau}_{i,h}\|_{\mathrm{H}_i}} \geqslant \widetilde{\beta}_{\mathrm{d}}\, \|\eta_{i,h}\|_{\mathrm{Q}_i} \qquad \forall\, \eta_{i,h} \in \mathrm{Q}_{i,h}\,. \tag{9.19}$$

It follows from (9.18), the definition of c_i (cf. (7.30b)), and (**H.5**) that

$$\mathrm{V}_{i,h} := \Big\{ \boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h} : \quad \mathrm{div}(\boldsymbol{\tau}_{i,h}) = 0 \Big\}, \tag{9.20}$$

whence, similarly to the case of $\widehat{\mathrm{T}}_h$, $\mathrm{V}_{i,h}$ is contained in the continuous kernel V_i (cf. (8.30)) of c_i, thus yielding the discrete analogue of (8.31), that is

$$a_i(\boldsymbol{\tau}_{i,h}, \boldsymbol{\tau}_{i,h}) \geqslant \frac{1}{\bar{\kappa}}\, \|\boldsymbol{\tau}_{i,h}\|^2_{\mathrm{div}_{\varrho};\Omega} \qquad \forall\, \boldsymbol{\tau}_{i,h} \in V_{i,h}\,. \tag{9.21}$$

In this way, it is clear from (9.21) that a_i satisfies the hypothesis ii) of [19, Theorem 3.5] with the constant $\widetilde{\alpha}_{\mathrm{d}} := \bar{\kappa}^{-1}$, whereas (**H.6**) constitutes itself the corresponding assumption iii). Consequently, a straightforward application of [19, Theorem 3.5] implies the discrete global inf-sup condition for $\mathbf{A}$ (cf. (8.27)) with a positive constant $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathrm{d}}$ depending only on $\|a_i\|$, $\|c_i\|$, $\widetilde{\alpha}_{\mathrm{d}}$, and $\widetilde{\beta}_{\mathrm{d}}$, and thus the same property is shared by $\mathbf{A}_{\boldsymbol{\phi}_h,\mathbf{v}_h}$ for each $(\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{Q}_h$ satisfying the discrete version of (8.35), that is

$$\|\boldsymbol{\phi}_h\|_{0,r,\Omega} + \|\mathbf{v}_h\|_{0,r,\Omega} \leqslant \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathrm{d}}}{2\,\|c\|}\,. \tag{9.22}$$

We are now in position of establishing the well-definedness of $\widetilde{\mathrm{T}}_{i,h}$ for each $i \in \{1,2\}$.

Theorem 9.3. *Given* $i \in \{1,2\}$ *and* $(\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{Q}_h$ *such that* (9.22) *holds, there exists a unique* $(\widetilde{\boldsymbol{\sigma}}_{i,h}, \widetilde{\xi}_{i,h}) \in \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}$ *solution to* (9.4)*, and hence one can define* $\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_h, \mathbf{v}_h) := \widetilde{\xi}_{i,h} \in \mathrm{Q}_{i,h}$. *Moreover, there exists a positive constant* $C_{\widetilde{\mathrm{T}},\mathrm{d}}$*, depending only on* $\|\mathrm{i}_\rho\|$ *and* $\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathrm{d}}$*, such that*

$$\|\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_h, \mathbf{v}_h)\|_{\mathrm{Q}_i} = \|\widetilde{\xi}_{i,h}\|_{\mathrm{Q}_i} \leqslant \|(\widetilde{\boldsymbol{\sigma}}_{i,h}, \widetilde{\xi}_{i,h})\|_{\mathrm{H}_i \times \mathrm{Q}_i} \leqslant C_{\widetilde{\mathrm{T}},\mathrm{d}} \left\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega} \right\}. \tag{9.23}$$

Proof. It reduces to a direct application of [23, Theorem 2.22], whose corresponding a priori estimate, yielding (9.23), makes use of the boundedness of F_i and G_i (cf. (7.31) and (7.32)). □

Analogously to the continuous case, it follows from the definition of $\widetilde{\mathrm{T}}_h$ (cf. (9.5)) and the a priori estimates given by (9.23) for each $i \in \{1,2\}$, that

$$\|\widetilde{\mathrm{T}}_h(\boldsymbol{\phi}_h, \mathbf{v}_h)\|_{\mathrm{Q}_1 \times \mathrm{Q}_2} := \sum_{i=1}^{2} \|\widetilde{\mathrm{T}}_{i,h}(\boldsymbol{\phi}_h, \mathbf{v}_h)\|_{\mathrm{Q}_i} \leqslant C_{\widetilde{\mathrm{T}},\mathrm{d}} \sum_{i=1}^{2} \left\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho;\Omega} \right\} \tag{9.24}$$

for each $(\boldsymbol{\phi}_h, \mathbf{v}_h) \in \mathrm{X}_{2,h} \times \mathbf{Q}_h$ satisfying (9.22).

Having established that the discrete operators $\widehat{\mathrm{T}}_h$, $\bar{\mathrm{T}}_h$, $\widetilde{\mathrm{T}}_h$, and hence $\mathbf{T}_h$ (under the constraint imposed by (9.22)), are all well defined, we now proceed as in Chapter 8.3 to address the solvability of the corresponding fixed-point equation (9.7). Then, letting δ_d be an arbitrary radius, we set

$$\mathrm{W}(\delta_\mathrm{d}) := \left\{ \boldsymbol{\eta}_h := (\eta_{1,h}, \eta_{2,h}) \in \mathrm{Q}_{1,h} \times \mathrm{Q}_{2,h} : \quad \|\boldsymbol{\eta}_h\|_{\mathrm{Q}_1 \times \mathrm{Q}_2} \leqslant \delta_\mathrm{d} \right\}, \tag{9.25}$$

and, reasoning analogously to the derivation of Lemma 8.7 (cf. beginning of Chapter 8.3), we deduce that $\mathbf{T}_h$ maps $\mathrm{W}(\delta_\mathrm{d})$ into itself under the discrete versions of (8.43) and (8.46), which, denoting $C_{0,\mathrm{d}} := \max\{1, C_{\widehat{\mathrm{T}},\mathrm{d}}\} C_{\bar{\mathrm{T}},\mathrm{d}}$, are given, respectively, by

$$C_{0,\mathrm{d}}(1+\delta_\mathrm{d})\delta_\mathrm{d} + C_{0,\mathrm{d}}(1+\delta_\mathrm{d})\left\{\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\right\} + C_{\widehat{\mathrm{T}},\mathrm{d}}\left\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega}\right\} \leqslant \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathrm{d}}}{2\,\|c\|} \tag{9.26}$$

and

$$C_{\widetilde{\mathrm{T}},\mathrm{d}} \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} \,+\, \|f_i\|_{0,\varrho;\Omega} \Big\} \,\leqslant\, \delta_{\mathrm{d}}\,. \tag{9.27}$$

Alternatively, the same conclusion is attained if, instead of (9.26), we define

$$\delta_{\mathrm{d}} \,:=\, \min\left\{ 1, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathrm{d}}}{8 C_{0,\mathrm{d}} \|c\|} \right\}, \tag{9.28}$$

and, letting $C_{1,\mathrm{d}} \,:=\, 2 C_{0,\mathrm{d}}$, impose

$$C_{1,\mathrm{d}} \Big\{ \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} \Big\} + C_{\widehat{\mathrm{T}},\mathrm{d}} \Big\{ \|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega} \Big\} \,\leqslant\, \frac{\widetilde{\boldsymbol{\alpha}}_{\mathbf{A},\mathrm{d}}}{4\,\|c\|}\,. \tag{9.29}$$

Note, however, that only (9.26) is required for $\mathbf{T}_h$ to be well-defined. Furthermore, employing analogue arguments to those utilized in the proofs of Lemmas 8.8, 8.9, and 8.10, we are able to show the continuity properties of $\widehat{\mathrm{T}}_h$, $\bar{\mathrm{T}}_h$, and $\widetilde{\mathrm{T}}_h$, that is the discrete versions of (8.47), (8.51), and (8.55), which are exactly as the latter, but with corresponding constants denoted $\mathrm{L}_{\widehat{\mathrm{T}},\mathrm{d}}$, $\mathrm{L}_{\bar{\mathrm{T}},\mathrm{d}}$, and $\mathrm{L}_{\widetilde{\mathrm{T}},\mathrm{d}}$. Therefore, following an analogue procedure to the one that yielded (8.63), we deduce that, under the assumption (9.26), there exists a positive constant $\mathrm{L}_{\mathbf{T},\mathrm{d}}$, depending only on $\mathrm{L}_{\widetilde{\mathrm{T}},\mathrm{d}}$, $\mathrm{L}_{\bar{\mathrm{T}},\mathrm{d}}$, $\mathrm{L}_{\widehat{\mathrm{T}},\mathrm{d}}$, and $C_{\bar{\mathrm{T}},\mathrm{d}}$, such that

$$\begin{aligned} & \|\mathbf{T}_h(\boldsymbol{\eta}_h) - \mathbf{T}_h(\boldsymbol{\vartheta}_h)\|_{\mathrm{Q}_1\times\mathrm{Q}_2} \\ & \quad \leqslant\, \mathrm{L}_{\mathbf{T},\mathrm{d}} \Big(1 + \delta_{\mathrm{d}} + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} \Big) \sum_{i=1}^{2} \Big\{ \|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega} \Big\} \|\boldsymbol{\eta}_h - \boldsymbol{\vartheta}_h\|_{\mathrm{Q}_1\times\mathrm{Q}_2}\,, \end{aligned} \tag{9.30}$$

for all $\boldsymbol{\eta}_h$, $\boldsymbol{\vartheta}_h \in \mathrm{W}(\delta_{\mathrm{d}})$.

Consequently, we can establish next the main result of this chapter.

Theorem 9.4. *Assume that δ_{d} and the data are sufficiently small so that* (9.26) *and* (9.27) *are satisfied, or alternatively that there holds* (9.28), (9.29), *and* (9.27). *Then, the operator $\mathbf{T}_h$ has a fixed point $\boldsymbol{\xi}_h \in \mathrm{W}(\delta_{\mathrm{d}})$. Equivalently, the coupled problem* (9.1) *has a solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, $(\boldsymbol{\varphi}_h, \chi_h) \in \mathrm{X}_{2,h} \times \mathrm{M}_{1,h}$, and $(\boldsymbol{\sigma}_{i,h}, \xi_{i,h}) \in \mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}$, $i \in \{1, 2\}$, with $\boldsymbol{\xi}_h := (\xi_{1,h}, \xi_{2,h}) \in \mathrm{W}(\delta_{\mathrm{d}})$.*

Moreover, there hold the following a priori estimates

$$\begin{aligned}
\|(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} &\leqslant C_{\widehat{\mathrm{T}},\mathrm{d}}\Big\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega} + \|\boldsymbol{\xi}_h\|_{0,\rho;\Omega}\,\|\boldsymbol{\varphi}_h\|_{0,r;\Omega}\Big\}\,,\\
\|(\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} &\leqslant C_{\bar{\mathrm{T}},\mathrm{d}}\Big\{\|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \|\boldsymbol{\xi}_h\|_{0,\rho;\Omega}\Big\}\,, \quad \textit{and} \qquad (9.31)\\
\|(\sigma_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i} &\leqslant C_{\widetilde{\mathrm{T}},\mathrm{d}}\Big\{\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega}\Big\} \quad i\in\{1,2\}\,.
\end{aligned}$$

In addition, under the extra assumption

$$\mathrm{L}_{\mathbf{T},\mathrm{d}}\Big(1+\delta_{\mathrm{d}} + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega}\Big)\sum_{i=1}^{2}\Big\{\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega}\Big\} \,<\, 1\,, \qquad (9.32)$$

the aforementioned solutions of (9.7) *and* (9.1) *are unique.*

Proof. As previously observed, the fact that $\mathbf{T}_h$ maps $\mathrm{W}(\delta_{\mathrm{d}})$ into itself is consequence of (9.26) and (9.27), or alternatively of (9.28), (9.29), and (9.27). Then, the continuity of $\mathbf{T}_h$ (cf. (9.30)) and Brouwer's theorem (cf. [16, Theorem 9.9-2]) imply the existence of solution of (9.7), and hence of (9.1). In turn, under the additional hypothesis (9.32), the Banach fixed point theorem guarantees the uniqueness of solution. In either case, (8.11), (8.25), and (8.38) yield the a priori estimates (9.31) and conclude the proof. □

9.3 A priori error analysis

In this chapter we consider arbitrary finite element subspaces satisfying the assumptions specified in Chapter 9.2, and establish the Céa estimate for the Galerkin error

$$\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\boldsymbol{\varphi},\chi) - (\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} + \sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i,\xi_i) - (\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\,, \qquad (9.33)$$

where $\big((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\varphi},\chi),(\boldsymbol{\sigma}_i,\xi_i)\big) \in \big(\mathbf{H}\times\mathbf{Q}\big)\times\big(\mathrm{X}_2\times\mathrm{M}_1\big)\times\big(\mathrm{H}_i\times\mathrm{Q}_i\big)$, $i\in\{1,2\}$, is the unique solution of (7.33), and $\big((\boldsymbol{\sigma}_h,\mathbf{u}_h),(\boldsymbol{\varphi}_h,\chi_h),(\boldsymbol{\sigma}_{i,h},\xi_{i,h})\big) \in \big(\mathbf{H}_h\times\mathbf{Q}_h\big)\times\big(\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}\big)\times\big(\mathrm{H}_{i,h}\times\mathrm{Q}_{i,h}\big)$, $i\in\{1,2\}$, is a solution of (9.1). We proceed as in previous related works (see, e.g. [13]) by applying suitable Strang-type estimates to the pairs of associated continuous and discrete schemes arising from (7.33) and (9.1) after splitting them according to the three decoupled equations. Throughout the rest of this chapter, given a subspace Z_h of an arbitrary Banach space $\big(Z, \|\cdot\|_Z\big)$, we set

$$\operatorname{dist}\big(z, Z_h\big) \;:=\; \inf_{z_h \in Z_h} \|z - z_h\|_Z \qquad \forall\, z \in Z\,.$$

We begin the analysis by considering the first two rows of (7.33) and (9.1), so that, employing the estimates provided by [6, Proposition 2.1, Corollary 2.3, Theorem 2.3], we deduce the existence of a positive constant $\widehat{c}$, independent of h, such that

$$\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} \;\leqslant\; \widehat{c}\Big\{\operatorname{dist}(\boldsymbol{\sigma},\mathbf{H}_h) \,+\, \operatorname{dist}(\mathbf{u},\mathbf{Q}_h) \,+\, \|\mathbf{G}_{\boldsymbol{\xi},\boldsymbol{\varphi}} - \mathbf{G}_{\boldsymbol{\xi}_h,\boldsymbol{\varphi}_h}\|_{\mathbf{Q}_h'}\Big\}\,. \tag{9.34}$$

Thus, proceeding analogously to the derivation of (8.50), we readily obtain

$$\|\mathbf{G}_{\boldsymbol{\xi},\boldsymbol{\varphi}} - \mathbf{G}_{\boldsymbol{\xi}_h,\boldsymbol{\varphi}_h}\|_{\mathbf{Q}_h'} \;\leqslant\; \varepsilon_0^{-1}\,|\Omega|^{(r-2)/2r}\Big\{\|\boldsymbol{\xi}\|_{0,\rho,\Omega}\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,r,\Omega} \,+\, \|\boldsymbol{\varphi}_h\|_{0,r,\Omega}\,\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{0,\rho,\Omega}\Big\}\,, \tag{9.35}$$

which, substituted back in (9.34), yields

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} \;&\leqslant\; c_{\widehat{\mathrm{T}}}\Big\{\operatorname{dist}(\boldsymbol{\sigma},\mathbf{H}_h) \,+\, \operatorname{dist}(\mathbf{u},\mathbf{Q}_h)\\ &+ \|\boldsymbol{\xi}\|_{0,\rho,\Omega}\,\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,r,\Omega} \,+\, \|\boldsymbol{\varphi}_h\|_{0,r,\Omega}\,\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{0,\rho,\Omega}\Big\}\,,\end{aligned} \tag{9.36}$$

with $c_{\widehat{\mathrm{T}}} \,:=\, \widehat{c}\,\max\big\{1, \varepsilon_0^{-1}\,|\Omega|^{(r-2)/2r}\big\}$.

Next, employing the same estimates from [6, Proposition 2.1, Corollary 2.3, Theorem 2.3] to the context given by the third and fourth rows of (7.33) and (9.1), we find that there exists a positive constant $\bar{c}$, independent of h, such that

$$\|(\boldsymbol{\varphi},\chi) - (\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} \;\leqslant\; \bar{c}\Big\{\operatorname{dist}(\boldsymbol{\varphi},\mathrm{X}_{2,h}) \,+\, \operatorname{dist}(\chi,\mathrm{M}_{1,h}) \,+\, \|\mathrm{G}_{\boldsymbol{\xi}} - \mathrm{G}_{\boldsymbol{\xi}_h}\|_{\mathrm{M}_{2,h}'}\Big\}\,. \tag{9.37}$$

In turn, proceeding as for the deduction of (8.54), we obtain

$$\|\mathrm{G}_{\boldsymbol{\xi}} - \mathrm{G}_{\boldsymbol{\xi}_h}\|_{\mathrm{M}_{2,h}'} \;\leqslant\; |\Omega|^{(\rho-r)/\rho r}\,\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{0,\rho;\Omega}\,, \tag{9.38}$$

which, along with (9.37), gives

$$\|(\boldsymbol{\varphi},\chi)-(\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} \leqslant c_{\bar{\mathrm{T}}}\Big\{\mathrm{dist}(\boldsymbol{\varphi},\mathrm{X}_{2,h}) + \mathrm{dist}(\chi,\mathrm{M}_{1,h}) + \|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{0,\rho;\Omega}\Big\}, \tag{9.39}$$

with $c_{\bar{\mathrm{T}}} := \bar{c}\max\big\{1,|\Omega|^{(\rho-r)/\rho r}\big\}$.

Furthermore, we now focus on the last two rows of (7.33) and (9.1), with the terms $c_{\boldsymbol{\varphi},\mathbf{u}}(\boldsymbol{\tau}_i,\xi_i)$ and $c_{\boldsymbol{\varphi}_h,\mathbf{u}_h}(\boldsymbol{\tau}_{i,h},\xi_{i,h})$ being considered as part of the respective functionals on the right-hand side. In this way, applying the estimate from [23, Lemma 2.27], we conclude that there exists a positive constant $\widetilde{c}$, independent of h, such that

$$\begin{aligned}&\|(\boldsymbol{\sigma}_i,\xi_i)-(\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\\ &\quad\leqslant \widetilde{c}\Big\{\mathrm{dist}(\boldsymbol{\sigma}_i,\mathrm{H}_{i,h}) + \mathrm{dist}(\xi_i,\mathrm{Q}_{i,h}) + \|c_{\boldsymbol{\varphi},\mathbf{u}}(\cdot,\xi_i)-c_{\boldsymbol{\varphi}_h,\mathbf{u}_h}(\cdot,\xi_{i,h})\|_{\mathrm{H}'_{i,h}}\Big\}.\end{aligned} \tag{9.40}$$

Then, subtracting and adding $\xi_{i,h}$ to the second component of $c_{\boldsymbol{\varphi},\mathbf{u}}(\cdot,\xi_i)$, making use of the triangle inequality, bearing in mind the definition of $c_{\boldsymbol{\phi},\mathbf{v}}$ (cf. (7.30f)), and employing its boundedness property (cf. (7.31), (7.32)), we get

$$\begin{aligned}&\|c_{\boldsymbol{\varphi},\mathbf{u}}(\cdot,\xi_i)-c_{\boldsymbol{\varphi}_h,\mathbf{u}_h}(\cdot,\xi_{i,h})\|_{\mathrm{H}'_{i,h}} \leqslant \|c_{\boldsymbol{\varphi},\mathbf{u}}(\cdot,\xi_i-\xi_{i,h})\|_{\mathrm{H}'_{i,h}} + \|c_{\boldsymbol{\varphi}-\boldsymbol{\varphi}_h,\mathbf{u}-\mathbf{u}_h}(\cdot,\xi_{i,h})\|_{\mathrm{H}'_{i,h}}\\ &\leqslant \|c\|\Big\{\big(\|\boldsymbol{\varphi}\|_{0,r;\Omega}+\|\mathbf{u}\|_{0,r;\Omega}\big)\|\xi_i-\xi_{i,h}\|_{0,\rho;\Omega} + \|\xi_{i,h}\|_{0,\rho;\Omega}\big(\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,r;\Omega}+\|\mathbf{u}-\mathbf{u}_h\|_{0,r;\Omega}\big)\Big\},\end{aligned}$$

which, jointly with (9.40), and summing over $i\in\{1,2\}$, imply

$$\begin{aligned}\sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i,\xi_i)-(\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i} &\leqslant c_{\widetilde{\mathrm{T}}}\Big\{\sum_{i=1}^{2}\big(\mathrm{dist}(\boldsymbol{\sigma}_i,\mathrm{H}_{i,h}) + \mathrm{dist}(\xi_i,\mathrm{Q}_{i,h})\big)\\ &\quad+\big(\|\boldsymbol{\varphi}\|_{0,r;\Omega}+\|\mathbf{u}\|_{0,r;\Omega}\big)\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_{0,\rho;\Omega}\\ &\quad+\|\boldsymbol{\xi}_h\|_{0,\rho;\Omega}\big(\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,r;\Omega}+\|\mathbf{u}-\mathbf{u}_h\|_{0,r;\Omega}\big)\Big\},\end{aligned} \tag{9.41}$$

with $c_{\widetilde{\mathrm{T}}} := \widetilde{c}\max\big\{1,\|c\|\big\}$.

For the rest of the analysis we introduce the partial error

$$\mathtt{E} \,:=\, \|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \sum_{i=1}^{2} \|(\boldsymbol{\sigma}_i,\xi_i) - (\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\,,$$

and suitably combine the estimates (9.36), (9.39), and (9.41). More precisely, employing the right-hand side of (9.39) to bound $\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,r;\Omega}$ in (9.36) and (9.41), adding the resulting inequalities, performing some algebraic manipulations, and then utilizing the a priori bounds for $\|\boldsymbol{\varphi}\|_{0,r;\Omega}$, $\|\boldsymbol{\varphi}_h\|_{0,r;\Omega}$, $\|\boldsymbol{\xi}\|_{0,\rho;\Omega}$, $\|\boldsymbol{\xi}_h\|_{0,\rho;\Omega}$, and $\|\mathbf{u}\|_{0,r;\Omega}$ provided by Theorems 8.11 and 9.4, we find that there exists a positive constant C_e, depending on $c_{\widehat{\mathrm{T}}}$, $c_{\bar{\mathrm{T}}}$, $c_{\widetilde{\mathrm{T}}}$, δ, δ_{d}, $C_{\widehat{\mathrm{T}}}$, $C_{\bar{\mathrm{T}}}$, $C_{\widetilde{\mathrm{T}}}$, $C_{\bar{\mathrm{T}},\mathrm{d}}$, and $C_{\widetilde{\mathrm{T}},\mathrm{d}}$, and hence independent of h, such that

$$\begin{aligned}\mathtt{E} \,\leqslant\, C_e\Big\{&\mathrm{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbf{H}_h\times\mathbf{Q}_h\big) + \mathrm{dist}\big((\boldsymbol{\varphi},\chi),\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}\big) + \sum_{i=1}^{2}\mathrm{dist}\big((\boldsymbol{\sigma}_i,\xi_i),\mathrm{H}_{i,h}\times\mathrm{Q}_{i,h}\big)\Big\}\\ &+ C_e\Big\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega} + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \sum_{i=1}^{2}\big(\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega}\big)\Big\}\,\mathtt{E}\,. \end{aligned} \tag{9.42}$$

Consequently, we are in position to establish the announced Céa estimate.

Theorem 9.5. *In addition to the hypotheses of Theorems* 8.11 *and* 9.4*, assume that*

$$C_e\Big\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,s,\Omega} + \|g\|_{1/s,r;\Gamma} + \|f\|_{0,r;\Omega} + \sum_{i=1}^{2}\big(\|g_i\|_{1/2,\Gamma} + \|f_i\|_{0,\varrho,\Omega}\big)\Big\} \leqslant \frac{1}{2}\,. \tag{9.43}$$

Then, there exists a positive constant C, independent of h, such that

$$\begin{aligned}&\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\boldsymbol{\varphi},\chi) - (\boldsymbol{\varphi}_h,\chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} + \sum_{i=1}^{2}\|(\boldsymbol{\sigma}_i,\xi_i) - (\boldsymbol{\sigma}_{i,h},\xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\\ &\leqslant C\Big\{\mathrm{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbf{H}_h\times\mathbf{Q}_h\big) + \mathrm{dist}\big((\boldsymbol{\varphi},\chi),\mathrm{X}_{2,h}\times\mathrm{M}_{1,h}\big) + \sum_{i=1}^{2}\mathrm{dist}\big((\boldsymbol{\sigma}_i,\xi_i),\mathrm{H}_{i,h}\times\mathrm{Q}_{i,h}\big)\Big\}. \end{aligned} \tag{9.44}$$

Proof. Under the assumption (9.43), the a priori estimate for $\mathtt{E}$ follows from (9.42), which, along with (9.39), yield (9.44) and ends the proof. □

We end this chapter by remarking that (6.7) suggests the following postprocessed approximation for the pressure p

$$p_h \;=\; -\frac{1}{n}\,\mathrm{tr}(\boldsymbol{\sigma}_h)\,, \tag{9.45}$$

for which it is easy to show that

$$\|p - p_h\|_{0,\Omega} \;\leqslant\; \frac{1}{\sqrt{n}}\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}\,. \tag{9.46}$$

CHAPTER 10

Specific finite element subspaces

In this chapter we define explicit finite element subspaces satisfying the hypotheses (**H.1**) - (**H.6**) that were introduced in Chapter 9.2 for the well posedness of the Galerkin scheme (9.1), and provide the corresponding rates of convergence.

10.1 Preliminaries

In what follows we make use of the notations introduced at the beginning of Chapter 9.1. Thus, given an integer $k \geqslant 0$, for each $K \in \mathcal{T}_h$ we let $\mathrm{P}_k(K)$ and $\mathbf{P}_k(K)$ be the spaces of polynomials of degree $\leqslant k$ defined on K and its vector version, respectively. Similarly, letting $\mathbf{x}$ be a generic vector in R^n, $\mathbf{RT}_k(K) := \mathbf{P}_k(K) + \mathrm{P}_k(K)\mathbf{x}$ and $\mathbb{RT}_k(K)$ stand for the local Raviart-Thomas space of order k defined on K and its associated tensor counterpart. In addition, we let $\mathrm{P}_k(\mathcal{T}_h)$, $\mathbf{P}_k(\mathcal{T}_h)$, $\mathbf{RT}_k(\mathcal{T}_h)$ and $\mathbb{RT}_k(\mathcal{T}_h)$ be the corresponding global versions of $\mathrm{P}_k(K)$, $\mathbf{P}_k(K)$, $\mathbf{RT}_k(K)$

and $\mathbb{RT}_k(K)$, respectively, that is

$$\mathrm{P}_k(\mathcal{T}_h) := \left\{ v_h \in \mathrm{L}^2(\Omega) : \quad v_h|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \right\},$$

$$\mathbf{P}_k(\mathcal{T}_h) := \left\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \right\},$$

$$\mathbf{RT}_k(\mathcal{T}_h) := \left\{ \boldsymbol{\tau}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \right\},$$

and

$$\mathbb{RT}_k(\mathcal{T}_h) := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \right\}.$$

We notice here that for each $t \in (1,+\infty)$ there hold the inclusions $\mathrm{P}_k(\mathcal{T}_h) \subseteq \mathrm{L}^t(\Omega)$, $\mathbf{P}_k(\mathcal{T}_h) \subseteq \mathbf{L}^t(\Omega)$, $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}(\mathrm{div}_t;\Omega)$, $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}^t(\mathrm{div}_t;\Omega)$, and $\mathbb{RT}_k(\mathcal{T}_h) \subseteq \mathbb{H}(\mathbf{div}_t;\Omega)$, which are employed below to introduce our specific finite element subspaces. Indeed, we now set

$$\mathbf{H}_h := \mathbb{RT}_k(\mathcal{T}_h) \cap \mathbb{H}_0(\mathbf{div}_s;\Omega), \quad \mathbf{Q}_h := \mathbf{P}_k(\mathcal{T}_h), \quad \mathrm{H}_{i,h} := \mathbf{RT}_k(\mathcal{T}_h), \quad \mathrm{Q}_{i,h} := \mathrm{P}_k(\mathcal{T}_h),$$

$$\mathrm{X}_{2,h} := \mathbf{RT}_k(\mathcal{T}_h), \quad \mathrm{M}_{1,h} := \mathrm{P}_k(\mathcal{T}_h), \quad \mathrm{X}_{1,h} := \mathbf{RT}_k(\mathcal{T}_h), \quad \text{and} \quad \mathrm{M}_{2,h} := \mathrm{P}_k(\mathcal{T}_h). \tag{10.1}$$

10.2 Verification of the hypotheses (H.1) - (H.6)

We begin by observing from (10.1) that (**H.1**) is trivially satisfied, whereas (**H.2**) was proved in [17, Lemma 5.5] (see, also, [11, Lemma 4.3]) for the particular case given by $r = 4$ and $s = 4/3$. In turn, a vector version of (**H.2**) was established in [29, Lemma 4.5] for $s \in (1,2)$ in 2D (with local notation there given by ϱ instead of s). In both cases, the preliminary result provided by [17, Lemma 5.4] plays a key role in the respective proofs. While we could simply say, at least in 2D, that (**H.2**) follows basically from a direct extension of [29, Lemma 4.5], we provide its explicit proof below for sake of completeness. To this end, following [29, Section 4.1], we now introduce for each $t \in (1,+\infty)$ the space

$$\mathbf{H}_t := \left\{ \boldsymbol{\tau} \in \mathbf{H}^t(\mathrm{div}_t;\Omega) \cup \mathbf{H}(\mathrm{div}_t;\Omega) : \quad \boldsymbol{\tau}|_K \in \mathbf{W}^{1,t}(K) \quad \forall\, K \in \mathcal{T}_h \right\},$$

and let $\Pi_h^k : \mathbf{H}_t \to \mathbf{RT}_k(\mathcal{T}_h)$ be the global Raviart-Thomas interpolator (cf. [7, Section 2.5]). Then, we recall from [7, Proposition 2.5.2 and eq. (2.5.27)] the commuting diagram property

$$\operatorname{div}\big(\Pi_h^k(\boldsymbol{\tau})\big) \;=\; \mathcal{P}_h^k\big(\operatorname{div}(\boldsymbol{\tau})\big) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}_t\,, \tag{10.2}$$

where $\mathcal{P}_h^k : \mathrm{L}^1(\Omega) \to \mathrm{P}_k(\mathcal{T}_h)$ is the projector defined, for each $v \in \mathrm{L}^1(\Omega)$, as the unique element $\mathcal{P}_h^k(v) \in \mathrm{P}_k(\mathcal{T}_h)$ such that

$$\int_\Omega \mathcal{P}_h^k(v)\, q_h \;=\; \int_\Omega v\, q_h \qquad \forall\, q_h \in \mathrm{P}_k(\mathcal{T}_h)\,. \tag{10.3}$$

In turn, it follows from [23, Proposition 1.135] (see, also, [13, eq. (A.5)]) that there exists a positive constant $C_{\mathcal{P}}$, independent of h, such that for each $t \in (1, +\infty)$ there holds

$$\|\mathcal{P}_h^k(v)\|_{0,t;\Omega} \;\leqslant\; C_{\mathcal{P}}\, \|v\|_{0,t;\Omega} \qquad \forall\, v \in \mathrm{L}^t(\Omega)\,. \tag{10.4}$$

On the other hand, while here we could use again [17, Lemma 5.4], we prefer to resort to the slightly more general result provided by [13, Lemma A.2], thus giving a greater visibility to it, which establishes that, given an integer l such that $1 \leqslant l \leqslant k+1$, and given t, $p \in (1,+\infty)$, such that $p \leqslant t \leqslant \frac{np}{n-p}$ if $p < n$, or $p \leqslant t < +\infty$ if $p = n$, there exists a positive constant C, independent of h, such that

$$\|\boldsymbol{\tau} \,-\, \Pi_h^k(\boldsymbol{\tau})\|_{0,t;\Omega} \;\leqslant\; C\, h^{l+\frac{n}{t}-\frac{n}{p}}\, |\boldsymbol{\tau}|_{l,p;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{l,p}(\Omega)\,. \tag{10.5}$$

Note that for the first set of constraints on t and p, there holds $\frac{n}{t} - \frac{n}{p} \geqslant -1$, which yields $l + \frac{n}{t} - \frac{n}{p} \geqslant 0$, whereas for the second one, there holds $l + \frac{n}{t} - \frac{n}{p} = l - 1 + \frac{n}{t} \geqslant \frac{n}{t}$, thus proving that in any case the power of h in (10.5) is non-negative. In this way, it follows from (10.5) that, for $l = 1$, and under the specified ranges of t and p, there exists a positive constant C_Π, independent of h, such that (cf. [13, Lemma A.3])

$$\|\Pi_h^k(\boldsymbol{\tau})\|_{0,t;\Omega} \;\leqslant\; C_\Pi\, \|\boldsymbol{\tau}\|_{1,p;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{1,p}(\Omega)\,. \tag{10.6}$$

In particular, for $p < n$ and $t = 2$, the inequality $t \leqslant \frac{np}{n-p}$ becomes $p \geqslant \frac{2n}{n+2}$, so that for the resulting range of p, that is $p \in \big[\frac{2n}{n+2}, 2\big)$ in 2D, and $p \in \big[\frac{2n}{n+2}, 2\big]$ in 3D, we obtain

$$\|\Pi_h^k(\boldsymbol{\tau})\|_{0,\Omega} \leqslant C_\Pi \|\boldsymbol{\tau}\|_{1,p;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{1,p}(\Omega)\,. \tag{10.7}$$

Analogue identities and inequalities to those stated above are valid with the tensor and vector versions of Π_h^k and $\mathcal{P}_h^k$, which are denoted by $\boldsymbol{\Pi}_h^k$ and $\boldsymbol{\mathcal{P}}_h^k$, respectively.

We are now in position to prove that (**H.2**) holds.

Lemma 10.1. *Under the ranges for r and s specified by* (8.23)*, there exists a positive constant $\boldsymbol{\beta}_{\mathrm{d}}$, independent of h, such that*

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{H}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_s;\Omega}} \geqslant \boldsymbol{\beta}_{\mathrm{d}} \|\mathbf{v}_h\|_{0,r;\Omega} \qquad \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,, \tag{10.8}$$

Proof. Given $\mathbf{v}_h \in \mathbf{Q}_h$, $\mathbf{v}_h \neq \mathbf{0}$, we set $\mathbf{v}_{h,s} := |\mathbf{v}_h|^{r-2}\,\mathbf{v}_h$, which belongs to $\mathbf{L}^s(\Omega)$, and notice that

$$\int_\Omega \mathbf{v}_h \cdot \mathbf{v}_{h,s} = \|\mathbf{v}_h\|_{0,r;\Omega}\,\|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,. \tag{10.9}$$

Next, we let $\mathcal{O}$ be a bounded convex polygonal domain that contains $\bar{\Omega}$, and define

$$\mathbf{g} := \begin{cases} \mathbf{v}_{h,s} & \text{in} \quad \Omega\,, \\ \mathbf{0} & \text{in} \quad \mathcal{O}\backslash\bar{\Omega}\,, \end{cases}.$$

It is readily seen that $\mathbf{g} \in \mathbf{L}^s(\mathcal{O})$ and $\|\mathbf{g}\|_{0,s;\mathcal{O}} = \|\mathbf{v}_{h,s}\|_{0,s;\Omega}$. Then, applying the elliptic regularity result provided by [25, Corollary 1], we deduce that there exists a unique $\mathbf{z} \in \mathbf{W}^{2,s}(\mathcal{O}) \cap \mathbf{W}_0^{1,s}(\mathcal{O})$ such that: $\Delta\mathbf{z} = \mathbf{g}$ in $\mathcal{O}$, $\mathbf{z} = \mathbf{0}$ on $\partial\mathcal{O}$. Moreover, there exists a positive constant C_{reg}, depending only on $\mathcal{O}$, such that

$$\|\mathbf{z}\|_{2,s;\mathcal{O}} \leqslant C_{\mathrm{reg}}\,\|\mathbf{g}\|_{0,s;\mathcal{O}} = C_{\mathrm{reg}}\,\|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,. \tag{10.10}$$

Hence, defining $\boldsymbol{\zeta} := \nabla\mathbf{z}|_\Omega \in \mathbb{W}^{1,s}(\Omega)$, it follows that $\mathbf{div}(\boldsymbol{\zeta}) = \mathbf{v}_{h,s}$ in Ω, and, according to

(10.10),

$$\|\boldsymbol{\zeta}\|_{1,s;\Omega} \leqslant \|\mathbf{z}\|_{2,s;\mathcal{O}} \leqslant C_{\mathrm{reg}}\,\|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,. \tag{10.11}$$

Now, since the identity tensor $\mathbb{I}$ clearly belongs to $\mathbb{RT}_k(\mathcal{T}_h)$, we can let $\boldsymbol{\zeta}_h$ be the $\mathbb{H}_0(\mathbf{div}_s;\Omega)$-component (cf. (7.10)) of $\boldsymbol{\Pi}_h^k(\boldsymbol{\zeta})$, so that $\boldsymbol{\zeta}_h \in \mathbf{H}_h$. In this way, employing the analogue of (10.2), we find that

$$\mathbf{div}(\boldsymbol{\zeta}_h) = \mathbf{div}\big(\boldsymbol{\Pi}_h^k(\boldsymbol{\zeta})\big) = \boldsymbol{\mathcal{P}}_h^k\big(\mathbf{div}(\boldsymbol{\zeta})\big) = \boldsymbol{\mathcal{P}}_h^k\big(\mathbf{v}_{h,s}\big)\,, \tag{10.12}$$

which, along with the analogue of (10.4) for $t = s$, give

$$\|\mathbf{div}(\boldsymbol{\zeta}_h)\|_{0,s;\Omega} \leqslant C_{\boldsymbol{\mathcal{P}}}\,\|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,. \tag{10.13}$$

In turn, noting that the range for s (cf. (8.23)) fits into the one for p in (10.7), we can apply this inequality (with $p = s$) and the regularity estimate (10.11), to arrive at

$$\|\boldsymbol{\zeta}_h\|_{0,\Omega} \leqslant \|\boldsymbol{\Pi}_h^k(\boldsymbol{\zeta})\|_{0,\Omega} \leqslant C_{\boldsymbol{\Pi}}\,\|\boldsymbol{\zeta}\|_{1,s;\Omega} \leqslant C_{\boldsymbol{\Pi}}\,C_{\mathrm{reg}}\,\|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,, \tag{10.14}$$

which, combined with (10.13), implies

$$\|\boldsymbol{\zeta}_h\|_{\mathbf{div}_s;\Omega} \leqslant \big(C_{\boldsymbol{\mathcal{P}}} + C_{\boldsymbol{\Pi}}\,C_{\mathrm{reg}}\big)\,\|\mathbf{v}_{h,s}\|_{0,s;\Omega}\,. \tag{10.15}$$

Consequently, bounding below the supremum in (10.8) with $\boldsymbol{\zeta}_h$, and making use of (10.12), the analogue of (10.3), (10.9), and (10.15), we conclude the required discrete inf-sup condition with the constant $\boldsymbol{\beta}_{\mathrm{d}} := \big(C_{\boldsymbol{\mathcal{P}}} + C_{\boldsymbol{\Pi}}\,C_{\mathrm{reg}}\big)^{-1}$. □

Furthermore, for the hypotheses (**H.3**) and (**H.4**), we first stress that (**H.3**) corresponds exactly to [13, (**H.5**)], and hence we omit most details and refer to [13, Section 5.2, Lemma 5.2]. We just make a few remarks here. First of all, we observe that the discrete kernels of the bilinear forms b_1 and b_2 coincide algebraically, which reduces to

$$\mathrm{K}_h^k := \Big\{\boldsymbol{\psi}_h \in \mathbf{RT}_k(\mathcal{T}_h) : \quad \mathrm{div}(\boldsymbol{\psi}_h) = 0 \quad \text{in} \quad \Omega\Big\}\,.$$

Then, we let $\Theta_h^k : \mathbf{L}^1(\Omega) \to \mathrm{K}_h^k$ be the projector defined similarly to (10.3), that is, given $\boldsymbol{\phi} \in \mathbf{L}^1(\Omega)$, $\Theta_h^k(\boldsymbol{\phi})$ is the unique element in K_h^k such that

$$\int_\Omega \Theta_h^k(\boldsymbol{\phi}) \cdot \boldsymbol{\psi}_h = \int_\Omega \boldsymbol{\phi} \cdot \boldsymbol{\psi}_h \qquad \forall\, \boldsymbol{\psi}_h \in \mathrm{K}_h^k\,.$$

In this way, a quasi-uniform boundedness property of Θ_h^k in 2D (cf. [13, eq.(5.8)]), along with the properties of the operators D_t (cf. Lemma 8.2), play a key role in the proof of (**H.3**). Whether the aforementioned boundedness is satisfied or not in 3D is still an open problem, and hence, similarly to [13], the assumption (**H.3**) is the only aspect of the analysis in this chapter that does not hold in 3D. All the other conditions are valid in both 2D and 3D. Regarding (**H.4**), we remark that the discrete inf-sup conditions for b_1 and b_2, which adapt the continuous analysis from Lemma 8.4 to the present discrete setting, follow from slight modifications of the proofs of [29, Lemma 4.5] and [13, Lemma 5.3]. Further details are omitted here.

Finally, it is clear from (10.1) that (**H.5**) is trivially satisfied, whereas (**H.6**) was proved precisely by [29, Lemma 4.5]. Alternatively, for the discrete inf-sup condition for c_i we can proceed analogously to the proof of Lemma 10.1 by observing that the range of ϱ (cf. (8.23), recall that $\mathrm{H}_i := \mathbf{H}(\mathrm{div}_\varrho;\Omega)$) also fits into the one for p in (10.7), whence this inequality can be applied to $p = \varrho$ as well.

10.3 The rates of convergence

Here we provide the rates of convergence of the Galerkin scheme (9.1) with the specific finite element subspaces introduced in Chapter 10.1, for which we previously collect the respective approximation properties. In fact, thanks to [23, Proposition 1.135] and its corresponding vector version, along with interpolation estimates of Sobolev spaces, those of $\mathbf{Q}_h$, $\mathrm{Q}_{i,h}$, and $\mathrm{M}_{1,h}$, are given as follows

($\mathbf{AP}_h^{\mathbf{u}}$) there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{v}, \mathbf{Q}_h\big) \ := \inf_{\mathbf{v}_h \in \mathbf{Q}_h} \|\mathbf{v} \,-\, \mathbf{v}_h\|_{0,r;\Omega} \ \leqslant \ C\, h^l \, \|\mathbf{v}\|_{l,r;\Omega} \,,$$

$\Big(\mathbf{AP}_h^{\xi_i}\Big)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\eta_i \in \mathrm{W}^{l,\rho}(\Omega)$, there holds

$$\operatorname{dist}\big(\eta_i, \mathrm{Q}_{i,h}\big) \ := \inf_{\eta_{i,h} \in \mathrm{Q}_{i,h}} \|\eta_i \,-\, \eta_{i,h}\|_{0,\rho;\Omega} \ \leqslant \ C\, h^l \, \|\eta_i\|_{l,\rho;\Omega} \,,$$

$(\mathbf{AP}_h^{\chi})$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\lambda \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}\big(\lambda, \mathrm{M}_{1,h}\big) \ := \inf_{\lambda_h \in \mathrm{M}_{1,h}} \|\lambda \,-\, \lambda_h\|_{0,r;\Omega} \ \leqslant \ C\, h^l \, \|\lambda\|_{l,r;\Omega} \,.$$

Furthermore, from [29, eq. (4.6), Section 4.1] and its tensor version, which, as the foregoing ones, are derived in the classical way by using the Deny–Lions Lemma and the corresponding scaling estimates (cf. [23, Lemmas B.67 and 1.101]), we state next the approximation properties of $\mathbf{H}_h$ and $\mathrm{H}_{i,h}$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_s; \Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,s}(\Omega)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\tau}, \mathbf{H}_h\big) \ := \inf_{\boldsymbol{\tau}_h \in \mathbf{H}_h} \|\boldsymbol{\tau} \,-\, \boldsymbol{\tau}_h\|_{\mathbf{div}_s;\Omega} \ \leqslant \ C\, h^l \left\{ \|\boldsymbol{\tau}\|_{l,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{l,s;\Omega} \right\},$$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}_i})$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\tau}_i \in \mathbb{H}^l(\Omega)$ with $\operatorname{div}(\boldsymbol{\tau}_i) \in \mathbf{W}^{l,\varrho}(\Omega)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\tau}_i, \mathrm{H}_{i,h}\big) \ := \inf_{\boldsymbol{\tau}_{i,h} \in \mathrm{H}_{i,h}} \|\boldsymbol{\tau}_i \,-\, \boldsymbol{\tau}_{i,h}\|_{\operatorname{div}_\varrho;\Omega} \ \leqslant \ C\, h^l \left\{ \|\boldsymbol{\tau}_i\|_{l,\Omega} \,+\, \|\operatorname{div}(\boldsymbol{\tau}_i)\|_{l,\varrho;\Omega} \right\}.$$

Finally, that of $X_{2,h}$, which we recall from [29, Section 4.5, $(\mathbf{AP}_h^{\mathbf{u}})$], becomes

$(\mathbf{AP}^{\boldsymbol{\varphi}}_{\mathbf{h}})$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\phi} \in \mathbf{W}^{l,r}(\Omega)$ with $\mathrm{div}(\boldsymbol{\phi}) \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\mathrm{dist}\big(\boldsymbol{\phi}, X_{2,h}\big) \,:=\, \inf_{\boldsymbol{\phi}_h \in X_{2,h}} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{r,\mathrm{div}_r;\Omega} \,\leqslant\, C\, h^l \Big\{ \|\boldsymbol{\phi}\|_{l,r;\Omega} \,+\, \|\mathrm{div}(\boldsymbol{\phi})\|_{l,r;\Omega} \Big\}.$$

The rates of convergence of (9.1) are now provided by the following theorem.

Theorem 10.2. *Let* $\big((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\varphi}, \chi), (\boldsymbol{\sigma}_i, \xi_i)\big) \in \big(\mathbf{H} \times \mathbf{Q}\big) \times \big(\mathrm{X}_2 \times \mathrm{M}_1\big) \times \big(\mathrm{H}_i \times \mathrm{Q}_i\big)$, $i \in \{1,2\}$ *be the unique solution of* (7.33) *with* $\boldsymbol{\xi} := (\xi_1, \xi_2) \in \mathrm{W}(\delta)$, *and let* $\big((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\varphi}_h, \chi_h), (\boldsymbol{\sigma}_{i,h}, \xi_{i,h})\big) \in \big(\mathbf{H}_h \times \mathbf{Q}_h\big) \times \big(\mathrm{X}_{2,h} \times \mathrm{M}_{1,h}\big) \times \big(\mathrm{H}_{i,h} \times \mathrm{Q}_{i,h}\big)$, $i \in \{1,2\}$ *be a solution of* (9.1) *with* $\boldsymbol{\xi}_h := (\xi_{1,h}, \xi_{2,h}) \in \mathrm{W}(\delta_\mathrm{d})$, *which is guaranteed by Theorems* 8.11 *and* 9.4, *respectively. In turn, let* p *and* p_h *given by* (6.7) *and* (9.45), *respectively. Assume the hypotheses of Theorem* 9.5, *and that there exists* $l \in [1, k+1]$ *such that* $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_s; \Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,s}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{l,r}(\Omega)$, $\boldsymbol{\varphi} \in \mathbf{W}^{l,r}(\Omega)$, $\mathrm{div}(\boldsymbol{\varphi}) \in \mathrm{W}^{l,r}(\Omega)$, $\chi \in \mathrm{W}^{l,r}(\Omega)$, $\boldsymbol{\sigma}_i \in \mathbb{H}^l(\Omega)$, $\mathrm{div}(\boldsymbol{\sigma}_i) \in \mathbf{W}^{l,\varrho}(\Omega)$, *and* $\xi_i \in \mathrm{W}^{l,\rho}(\Omega)$, $i \in \{1,2\}$. *Then, there exists a positive constant* C, *independent of* h, *such that*

$$\begin{aligned}
&\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|p - p_h\|_{0,\Omega} + \|(\boldsymbol{\varphi}, \chi) - (\boldsymbol{\varphi}_h, \chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1} + \sum_{i=1}^{2} \|(\boldsymbol{\sigma}_i, \xi_i) - (\boldsymbol{\sigma}_{i,h}, \xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i} \\
&\quad \leqslant\, C\, h^l \Big\{ \|\boldsymbol{\sigma}\|_{l,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,s;\Omega} \,+\, \|\mathbf{u}\|_{l,r;\Omega} \,+\, \|\boldsymbol{\varphi}\|_{l,r;\Omega} \,+\, \|\mathrm{div}(\boldsymbol{\varphi})\|_{l,r;\Omega} \,+\, \|\chi\|_{l,r;\Omega} \\
&\quad +\, \sum_{i=1}^{2} \big(\|\boldsymbol{\sigma}_i\|_{l,\Omega} \,+\, \|\mathrm{div}(\boldsymbol{\sigma}_i)\|_{l,\varrho;\Omega} \,+\, \|\xi_i\|_{l,\rho;\Omega} \big) \Big\}.
\end{aligned}$$

Proof. It follows straightforwardly from Theorem 9.5, (9.46), and the above approximation properties. □

CHAPTER 11

Computational results

We turn now to the numerical verification of the rates of convergence anticipated by Theorem 10.2. The following examples in 2D and 3D have been realized with the finite element library FEniCS [1]. The linearization of the nonlinear algebraic equations that arise after discretization is done using either a fixed-point Picard algorithm or an exact Newton–Raphson method (with the zero vector as initial guess and iterations are stopped once the absolute or relative residual drops below 10^{-8}) and the linear systems are solved with the multifrontal massively parallel sparse direct method MUMPS [2].

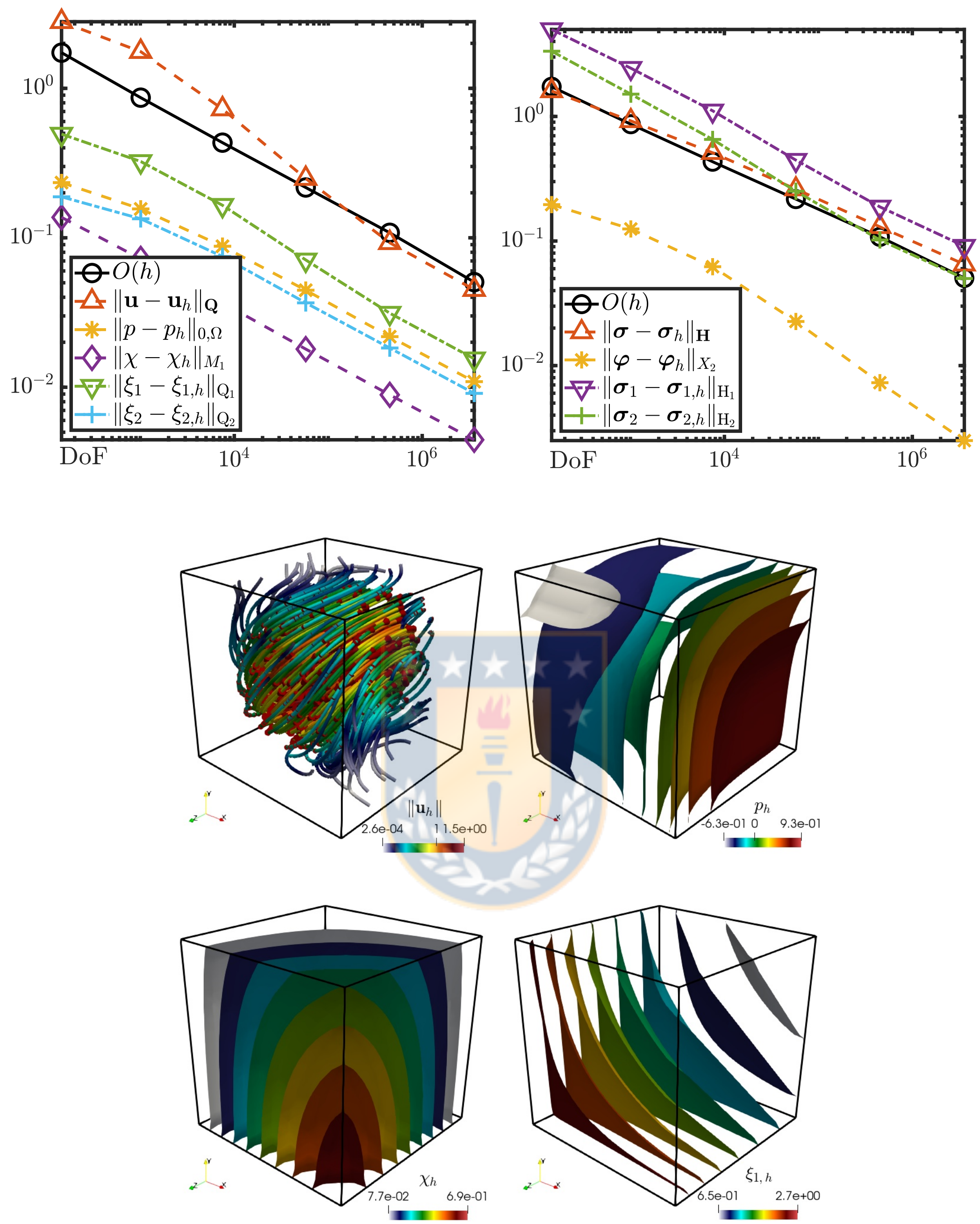

Figure 11.1: Example 1. Error history associated with the finite element family (10.1) with $k = 0$ in 3D for primal variables (top left) and mixed variables (top right), and samples of approximate primal variables (velocity streamlines $\mathbf{u}_h$, iso-surfaces of postprocessed pressure p_h, electrostatic potential χ_h, and positive ion concentration $\xi_{1,h}$; bottom plots). In all mesh refinements the number of Newton–Raphson iterations was 4.

Example 1. Considering first the spatial domain $\Omega = (0,1)^3$ along with the arbitrarily chosen parameters

$$\mu = 10^{-3}, \quad \varepsilon = 0.1, \quad \kappa_1 = 0.25, \quad \kappa_2 = 0.5,$$

we define the following manufactured exact solutions to (6.8)

$$\mathbf{u} = \begin{pmatrix} \sin^2(\pi x)\sin(\pi y)\sin(2\pi z) \\ \sin(\pi x)\sin^2(\pi y)\sin(2\pi z) \\ -[\sin(2\pi x)\sin(\pi y) + \sin(\pi x)\sin(2\pi y)]\sin^2(\pi z) \end{pmatrix},$$

$$p = x^4 - \frac{1}{2}(y^4 + z^4), \quad \xi_1 = \exp(-xy + z),$$

$$\xi_2 = \cos^2(xyz), \quad \chi = \sin(x)\cos(y)\sin(z), \quad \boldsymbol{\sigma} = \mu\nabla\mathbf{u} - p\mathbb{I},$$

$$\boldsymbol{\sigma}_i = \kappa_i(\nabla\xi_i + q_i\xi_i\varepsilon^{-1}\boldsymbol{\varphi}) - \xi_i\mathbf{u}, \quad \boldsymbol{\varphi} = \varepsilon\nabla\chi,$$

and construct forcing/source terms and non-homogeneous Dirichlet boundary conditions $\mathbf{f}, \mathbf{g}, f_i, g_i$ from these closed-form solutions. Using the lowest-order version of the finite element spaces defined in (10.1) (with polynomial degree $k = 0$), we solve problem (9.1) on a sequence of six succesively refined regular meshes. The zero-mean pressure condition is enforced using a real Lagrange multiplier approach. At each refinement level we compute errors between approximate and smooth exact solutions using the norms in (9.33) and Theorem 10.2 (but we split their contribution coming from the error on each individual field variable). For this 3D accuracy test we consider the Banach spaces indexes specified in (8.23)

$$r = 3, \quad s = 3/2, \quad \rho = 6, \quad \varrho = 6/5.$$

The results of this convergence study are collected in Figure 11.1 (top panels), where we plot in log-log scale the error decay as the number of degrees of freedom increases. Apart from the electric field $\boldsymbol{\varphi}$ which converges with rate of approximately 1.5, all other variables exhibit an optimal rate of convergence. In the bottom panel of the figure we show approximate solutions for some of the field variables, which indicate well resolved profiles.

In addition, the balance-preserving property of the proposed mixed formulation is assessed

DoF	h	e	r	momentum$_h$	potential$_h$	transport$_{1,h}$	transport$_{2,h}$
145	1.732	1.40e+1	$\star$	2.37e-07	7.29e-17	1.83e-15	8.64e-16
1009	0.866	7.44e+0	0.91	8.61e-08	2.45e-16	4.14e-15	1.81e-15
7489	0.433	3.43e+0	1.12	6.07e-10	4.53e-16	5.10e-15	4.85e-15
57601	0.217	1.40e+0	1.29	1.27e-11	6.76e-16	1.45e-14	8.77e-15
451585	0.108	6.00e-01	1.22	1.04e-11	6.29e-15	1.47e-14	2.48e-11
3575809	0.051	2.97e-01	1.13	5.88e-11	4.20e-15	2.38e-15	2.95e-15

Table 11.1: Example 1. Total error, experimental rates of convergence, and ℓ^∞-norm of the projected residual of the momentum, potential, and ionic transport equations.

by computing the quantities

$$\begin{aligned}
\texttt{momentum}_h &:= \|\boldsymbol{\mathcal{P}}_h^k(\mathbf{div}(\boldsymbol{\sigma}_h) - (\xi_{1,h} - \xi_{2,h})\,\varepsilon^{-1}\boldsymbol{\varphi}_h + \mathbf{f})\|_{\ell^\infty},\\
\texttt{potential}_h &:= \|\mathcal{P}_h^k(\mathrm{div}(\boldsymbol{\varphi}_h) + (\xi_{1,h} - \xi_{2,h}) + f)\|_{\ell^\infty},\\
\texttt{transport}_{i,h} &:= \|\mathcal{P}_h^k(\xi_{i,h} - \mathrm{div}(\boldsymbol{\sigma}_{i,h}) - f_i)\|_{\ell^\infty}.
\end{aligned}$$

These values, for each refinement level, are collected in Table 11.1. We tabulate the total error

$$\begin{aligned}
\mathtt{e} := \|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \|p - p_h\|_{0,\Omega} \,+\, \|(\boldsymbol{\varphi}, \chi) - (\boldsymbol{\varphi}_h, \chi_h)\|_{\mathrm{X}_2\times\mathrm{M}_1}\\
+ \quad \sum_{i=1}^{2} \|(\boldsymbol{\sigma}_i, \xi_i) - (\boldsymbol{\sigma}_{i,h}, \xi_{i,h})\|_{\mathrm{H}_i\times\mathrm{Q}_i}\,,
\end{aligned}$$

(as indicated by Theorem 10.2) as well as the rates of convergence computed as

$$\mathtt{r} = \log(\mathtt{e}/\widehat{\mathtt{e}})[\log(h/\widehat{h})]^{-1}\,,$$

where $\mathtt{e}$ and $\widehat{\mathtt{e}}$ denote errors produced on two consecutive meshes associated with mesh sizes h and $\widehat{h}$, respectively. From the last columns we see that the potential and transport balance equations are satisfied to machine precision while the error for the momentum balance is higher. This may be explained by the presence of the term $\boldsymbol{\varphi}_h$ on the right-hand side (which has a H(div)-component).

Example 2. In addition, and in order to illustrate the implementation of fixed-point solvers, we have realized numerically Picard versions of the linearization of (9.1). In case A we follow the fixed-point structure used in the analysis of Chapter 9.1, that is, solving sequentially problems

$$(9.2) \to (9.3) \to (9.4),$$

and iterating until the ℓ^2-norm of the vector containing the residual of the Picard iterates reaches 10^{-8}. Next, in case B we choose a different fixed-point splitting where we apply two modifications with respect to case A. First, in (9.4) instead of the linear functional for the second discrete electrostatic potential equation (discrete version of (7.21d)) we consider $G(\lambda_h) := -\int_\Omega f\lambda_h$ and the coupling term appears as a bilinear form contribution (and no longer as part of the linear functional), say

$$\widehat{g}(\lambda_h, (\xi_{1,h}, \xi_{2,h})) := \int_\Omega \lambda_h(\xi_{1,h} - \xi_{2,h}).$$

Secondly, with regards to the constitutive equation in the ionized particle equations, we swap the bilinearity in the flux definition (discrete version of (7.30f)) from $\xi_{i,h}$ to the pair $(\boldsymbol{\phi}_h, \mathbf{u}_h)$, that is, we consider

$$\widehat{c}_{\hat{\xi}_{i,h}}(\boldsymbol{\tau}_{i,h}, (\boldsymbol{\phi}_h, \mathbf{u}_h)) := \int_\Omega \left\{ q_i\, \hat{\xi}_i\, \varepsilon^{-1} \boldsymbol{\phi}_h \,-\, \kappa_i^{-1}\, \hat{\xi}_i\, \mathbf{u}_h \right\} \cdot \boldsymbol{\tau}_i.$$

For both fixed-point cases we have taken as initial guess solution the zero vector. Moreover, we consider a 2D problem with manufactured solutions defined on $\Omega = (0,1)^2$

$$\mathbf{u} = \begin{pmatrix} \cos(\pi x)\sin(\pi y) \\ -\sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad p = x^4 - y^4, \quad \chi = \sin(x)\cos(y), \quad \xi_1 = \exp(-xy), \quad \xi_2 = \cos^2(xy),$$

and take the same model constants as before. In 2D, and according to (8.23) we now choose

$$r = 4, \quad s = 4/3, \quad \rho = 4, \quad \varrho = 4/3.$$

We focus on the number of Picard iterations required in each case, displaying the obtained results in Table 11.2. While we confirm that all methods give exactly the same errors (and

		case A			case B			case C		
DoF	h	e	r	iter	e	r	iter	e	r	iter
$k = 0$										
221	0.500	6.64e+0	$\star$	80	6.64e+0	$\star$	9	6.64e+0	$\star$	5
841	0.250	2.36e+0	1.49	83	2.36e+0	1.49	8	2.36e+0	1.49	4
3281	0.125	8.34e-01	1.50	72	8.34e-01	1.50	8	8.34e-01	1.50	4
12961	0.062	3.32e-01	1.33	70	3.32e-01	1.33	9	3.32e-01	1.33	4
51521	0.031	1.51e-01	1.14	68	1.51e-01	1.14	9	1.51e-01	1.14	4
$k = 1$										
681	0.500	6.87e-01	$\star$	68	6.87e-01	$\star$	9	6.87e-01	$\star$	4
2641	0.250	1.20e-01	2.51	68	1.20e-01	2.51	9	1.20e-01	2.51	3
10401	0.125	2.57e-02	2.23	68	2.57e-02	2.23	9	2.57e-02	2.23	4
41281	0.062	6.11e-03	2.08	68	6.11e-03	2.08	9	6.11e-03	2.08	4
164481	0.031	1.51e-03	2.01	77	1.50e-03	2.02	9	1.50e-03	2.02	4

Table 11.2: Example 2. Total error, experimental rates of convergence, and number of iterations required for two types of fixed-point methods as well as for Newton–Raphson linearization.

consequently also the same convergence rates, which are optimal in view of the theoretical bounds), from the number of fixed-point iterations we readily note that case B performs much better than case A, for the two polynomial degrees we tested $k = 0$, $k = 1$. This behaviour could be explained by the stability of different linearizations of advective nonlinearities and by the strength of the coupling for this particular choice of model parameters. We stress that the analysis of case B is, however, not at all straightforward since the decoupled linear electrostatic potential problem resulting from the first modification is no longer symmetric. For sake of reference we also tabulate total errors and number of nonlinear iterates obtained with the method we use also in Examples 1 and 3: an exact Newton–Raphson linearization (labelled here as case C). Needless to say, the latter is actually the one that one would employ in practical computations. Samples of the approximate solutions (only the mixed variables) computed with the method in case A are portrayed in Figure 11.2.

Example 3. We conclude this chapter with an application problem where we demonstrate the use of the mixed finite element scheme in simulating the transport process in an electrokinetic system with an ion-selective interface, where the development of an electroosmotic instability is expected. The problem configuration is adopted from [21, 22]. This system corresponds to a transient counterpart of (6.8) in the absence of external forces and sources ($\mathbf{f} = \mathbf{0}$, $f = f_i = 0$),

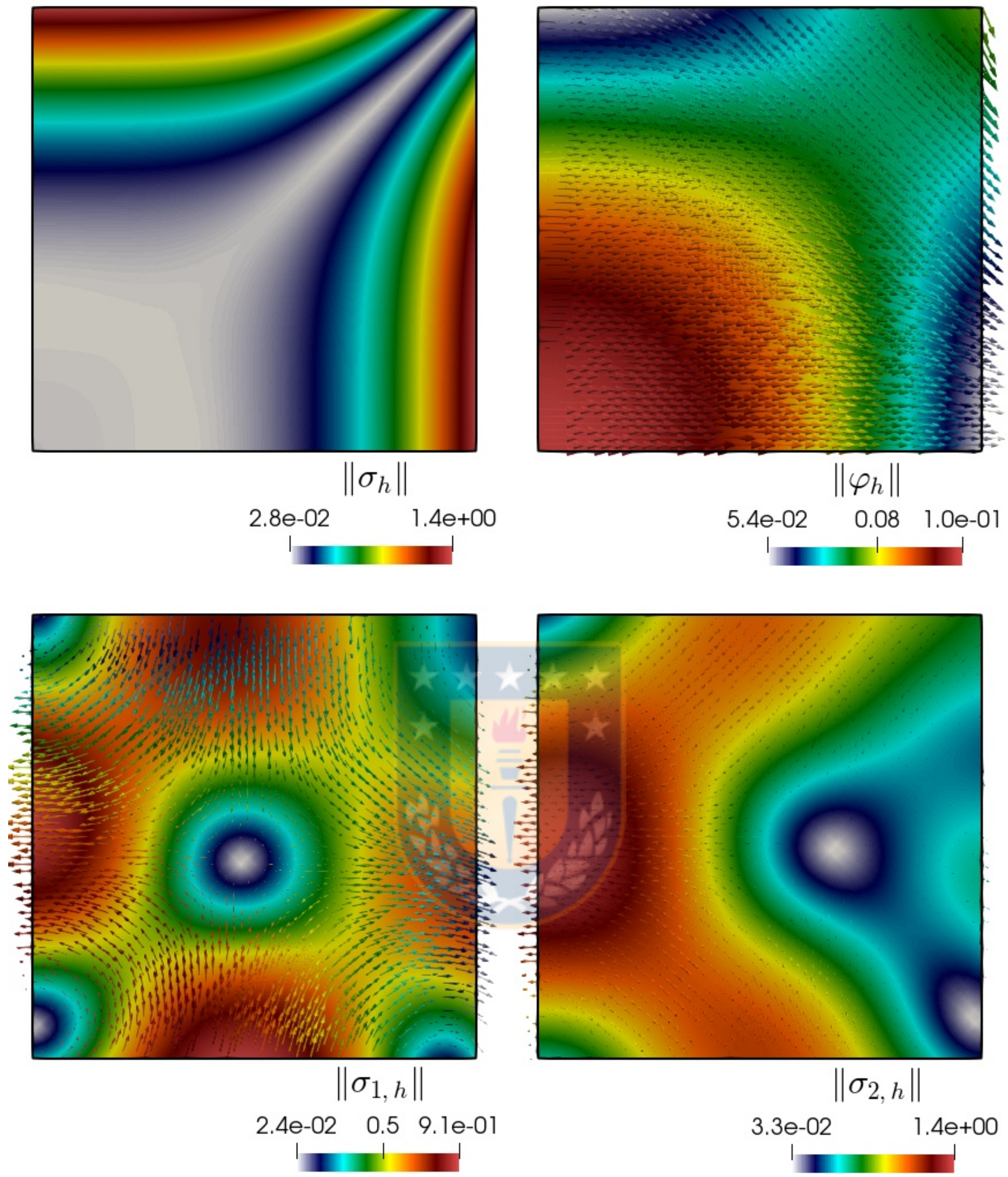

Figure 11.2: Example 2. Samples of approximate mixed variables (stress magnitude, electric field magnitude and arrows, and ionic fluxes) obtained with the fixed-point algorithm labelled case A, and for $k = 1$.

where the following additional terms appear in the momentum and concentration equations (note also the different scaling of ε on the right-hand side of the momentum balance, required to match the adimensionalization in [22])

$$-\frac{1}{\mathrm{Sc}}\partial_t \mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) \;=\; (\xi_1 - \xi_2)\,\frac{1}{2\varepsilon^2}\boldsymbol{\varphi}, \quad -\partial_t \xi_i + \mathrm{div}(\boldsymbol{\sigma}_i) = 0.$$

The time derivatives are discretized using backward Euler's method. In the problem setup a boundary layer is present in the vicinity of the solid boundary (the bottom edge of the rectangular domain), and therefore we employ a graded mesh with a higher refinement close to the layer. For this problem we select the second-order family of finite element subspaces (setting $k = 1$ in Chapter 10.1), which gives for the chosen mesh 865201 degrees of freedom.

The physical properties of the system are as follows. The cation species is Na^+ having diffusivity $\kappa_1 = 1$ and the anion species is Cl^- with the same diffusivity $\kappa_2 = 1$. The dynamic viscosity of the mixture is $\mu = 1$. Initial conditions are given by $\mathbf{u} = \mathbf{0}$, and a 2% random perturbation on a linearly varying initial ionic concentrations $\xi_1 = \zeta(2-y)$, $\xi_2 = \zeta x$, where ζ is a uniform random variable between 0.98 and 1. On the top boundary we set $\xi_1 = \xi_2 = 1$, $\mathbf{u} = \mathbf{0}$, and an applied voltage of $\chi = 120$. On the bottom boundary we impose $\chi = 0$, $\xi_1 = 2$, $\boldsymbol{\sigma}_2 \cdot \boldsymbol{\nu} = 0$, and $\mathbf{u} = \mathbf{0}$. On the vertical walls we prescribe periodic boundary conditions. The other model parameters take the values $\varepsilon = 8 \cdot 10^{-6}$, $\mathrm{Sc} = 10^3$, and we use a timestep $\Delta t = 10^{-6}$. We plot snapshots of the anion concentration $\xi_{2,h}$ in Figure 11.3 at times $t = 10^{-4}, 10^{-3}$. We observe similar ionic patterns to those produced also in [35, 37].

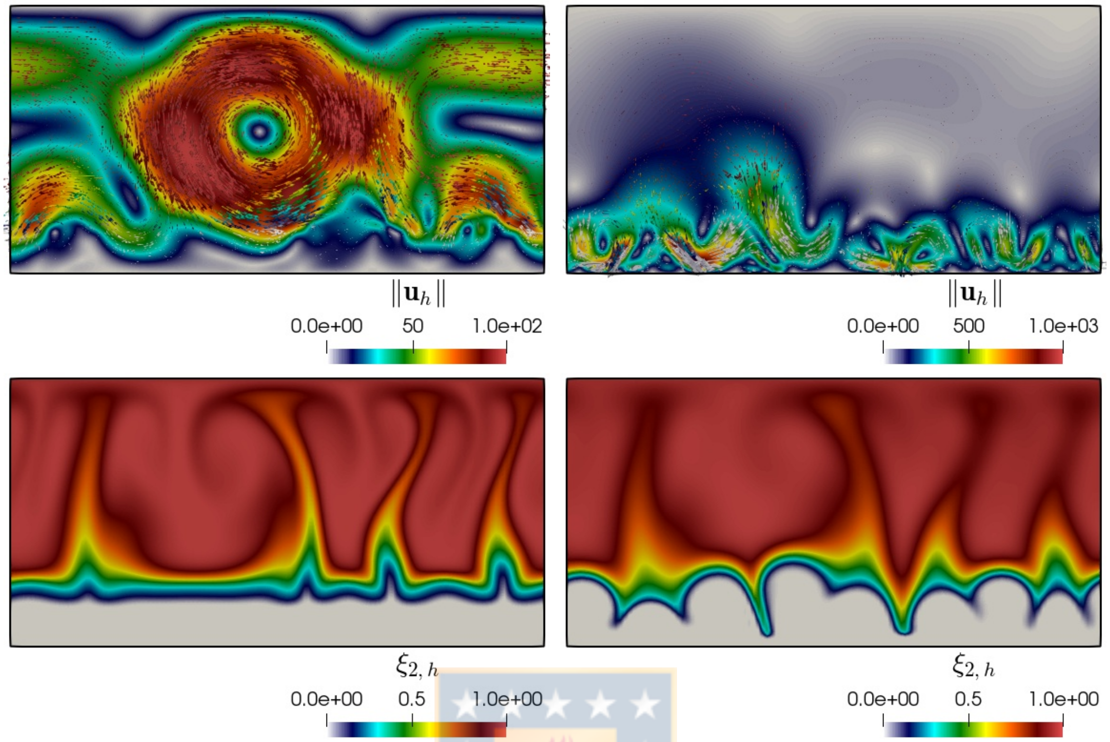

Figure 11.3: Example 3. Samples of approximate velocity (top) and anion concentration (bottom) at times $t = 10^{-4}$ and 10^{-3} (left and right, respectively), produced with the mixed method and using $k = 1$.

Concluding Remarks

In the first part of this thesis we developed a new theory to continuous and discrete schemes of perturbed problems in Banach spaces to be well-posed. The main results of this part are:

- We provided sufficient conditions for the well-posedness of perturbed saddle-point formulations in Banach spaces and their associated Galerkin schemes in the case in which the kernel of the adjoint operator induced by one of the bilinear forms is not trivial.

- In the case in which the kernel of the adjoint operator induced by one of the bilinear forms is trivial, we employ a suitable characterization of a closed range injective adjoint operator, to lighten the sufficient conditions for the solvability.

In the second part of this thesis we developed mixed finite element methods for a partial differential equation of physical interest in Biology and Nanotechnology, namely, the coupled Stokes and Poisson-Nernst-Planck equations. We proved the solvability of the continuous and discrete formulations as well as their convergence results, and we also provided corresponding numerical examples and simulations. The main results of this part are:

- We develop a new mixed formulation in Banach spaces for the coupled problem given by the Stokes and Poisson–Nernst-Planck equations.

- The well-posedness of the continuous formulation was proved using a fixed point strategy in combination with the Banach theorem.

- An analogous approach is employed to conclude the existence and uniqueness of a solution for the associated Galerkin scheme. In addition, a priori error estimates are derived.

- Finally we use Raviart-Thomas elements of order k with their appropriate convergence rates, followed by several numerical experiments confirming the theoretical error bounds.

- We also showed the applicability of the theory presented in the first part.

Future work

The methods developed and the results obtained in this thesis have motivated several on-going and future projects. Some of them are described below:

- We are interested in extending the applicability of the theory developed in the first part of this thesis to others problems.
- We are interested in extending the analysis and results to the Navier-Stokes case, that is to the coupled problem given by

$$-\mu\,\Delta\mathbf{u} + \lambda\,(\nabla\mathbf{u})\,\mathbf{u} + \nabla p = -(\xi_1-\xi_2)\,\varepsilon^{-1}\boldsymbol{\varphi} + \mathbf{f} \quad \text{in} \quad \Omega\,,$$

$$\mathrm{div}(\mathbf{u}) = 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega p = 0\,.$$

- We are interested in developing a posteriori error analysis for the method studied in part II.
- We are also interested in extending the analysis and results to time dependent case.

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