An agent-based model for teaching–learning processes

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A B S T R A C T

This paper presents an agent-based model for describing the increase the knowledge by accumulating the information needed to complete a learning task or objectives, based on phenomena studied by behavioral and learning scientists. From the simulations, the average increase rate in knowledge, the skewness and kurtosis of knowledge distributions, and grade distributions are determined. These tools make it possible to evaluate the efficiency of teaching strategies and the performance of learning in the classroom. The present model significantly reproduces the phenomenology obtained in Bordogna and Albano (2001), showing first and second-order phase transitions and the temporal dynamics of knowledge. Furthermore, the results of our model allow us to built a gas model analogy. Some of the study cases show characteristics of systems far from the state of thermodynamic equilibrium. This allows us to use the known techniques from gas models to interpret the dynamics of the simulated learning process. The presented model does not describe the teaching–learning process in all its complexity. We use a simple behavioral characteristic of the persons, namely, the inattentional private experience. However, it allows us to prove learning strategies to optimize the learning process. Also, this model is a starting point to propose new models with more elements from neuroscience and sociophysics to study in greater depth the dynamics of the classroom.

1. Introduction

The study of the learning process has been developed by psychologists from the 60's [1–3]. Research in educational psychology suggests that the teacher plays a fundamental and active role in the teaching process since he has the responsibility of the rhythm, content and sequence of the lecture [4]. However, the learning process occurs while people interact in groups [5,6] and depends on the individual ability to integrate the information of the environment with their internal representations [7]. According to Cameron and Giuntoli [8], 46% of students pay attention at any given time during the lecture, suggesting that in the classroom students focus less than half of the attention on the content of class. This happens for *inattentional private experience*, such as *attentional lapses*, *daydreaming* and *mindwandering*, which are highly relevant in the educational context [9]. These phenomena are fundamental to generate the exchange between internal and external representation that ensures learning [7,10]. It is estimated that students daydream 30% to 50% of the time [11], as well as mindwandering, which depending on the task to be performed and the concentration required, also varies between 30% and 50% [7]. On the other hand, both individual and collective work associated with bibliographic material and student study habits are of great importance in the academic performance of students.

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According to Gettinger and Seibert [12], study skills are academic enablers; they function as critical tools for learning, and that enhance the effectiveness and efficiency of students' learning. Furthermore, study skills, study habits, study attitudes, and study motivation exhibit particularly strong and robust relationships with academic performance in college, as noted by Credé and Kuncel [13].

From another point of view, in the last decades, different studies have been carried out that seek to describe the society in the framework of complex systems [14]. However, research on the learning process from the multidisciplinary perspective of complex systems is at a relatively early stage [15–17]. Particularly from sociophysics [18–20], the development of a theory to describe the teaching–learning process in different contexts has approached through applications of the Monte Carlo method to a generalized Ising Model [21], for example, learning via the Internet [22], the teaching process in the classroom [23] and social learning applications [24]. Also, modeling based on the kinetic theory of active particles [25] provides a more general theoretical framework for studying the collective learning process.

The goal of this paper is to model the increase the knowledge by accumulating the information needed to complete a learning task or objectives, based on phenomena studied by behavioral and learning scientists. The basic assumptions to modeling the acquisition of information from the agents are that this process is affected by (1) students' cognitive abilities and proficiencies, and (2) the learning environment. The basic idealizations of the model, concerning each issue, are that: (1) students' cognitive abilities only depend on the loss of attention due to the inattentional private experiences; and (2) the agents obtain information from (a) the direct instruction for a teacher, (b) social interaction in workgroup sessions, (c) student's study occurring in an environment that is conducive to studying. To describe this process, we using one of the most used modeling approaches to study complex systems, the so-called *bottom up* [26]. This approach is characterized by the use of agent-based models (ABMs) to reproduce artificial worlds or virtual societies [27,28]. These models have been widely used to describe social dynamics such as segregation [29], cultural diffusion [30], civil violence [31,32], as well as in econophysics [33,34].

The order in which the contents are presented will be as follows. In Section 2, we will briefly describe the proposed model. Then, in Section 3, we present the comparative study with other developed models, and present new case studies analyzed and discuss based on the theoretical elements already proposed. Finally, in Section 4, we establish the conclusions.

2. The model

Our model consists of a system with N agents that increase their knowledge by accumulating the amount of information A_I needed to complete a learning task or objectives during a training time t. Let C_i be a vector of length A_I with components σ_{if}. If agent i acquire a unit of information f then $\sigma_{if} = 1$, otherwise $\sigma_{if} = 0$. In this way, the knowledge of each agent is a sum of information that varies over time.

This process is determined by two parameters. The local parameter α_i measures the loss of attention of each agent of the system during the learning process. The loss of attention is produced by the inattentional private experience as quoted in [7]. We assume that this parameter refers to the intrinsic capacity of each agent to acquire information to increase their knowledge. On the other hand, the global parameter β is the amount of noise in the environment during the process of accumulating information and it taking values between [0, 1] uniformly distributed. Note that the β parameter corresponds to the reciprocal of the noise parameter in [21].

We consider three different ways that agents may acquire the information needed to complete a learning task or objectives:

1. Student–Teacher Interaction: In this process, each unit of time is equivalent to a lecture was given by a teacher. In this lecture, the agents can acquire a unit of information if $R \geq \alpha_i + \beta$, where R is a random number between [0, 1] uniformly distributed. So the knowledge vector component takes the value one ($\sigma_{if} = 1$).

2. Student–Student Interaction: In this process, the acquire the information occurs in rounds of group work. These take place after a given number of lectures. Here agents can increase their knowledge by exchanging a unit of information between them. So, subgroups G of N_G agents are formed, where they can exchange a randomly a unit of information according to the following steps:

    (a) An agent i of a subgroup G will randomly choose another agent j of the same subgroup.
    (b) With R a random number, If $R \geq \alpha_i$, agent i will randomly select a unit of information k that it does not have and copy it from agent j, such that $\sigma_{ik} = \sigma_{jk}$.
    (c) This process will be repeated for each agent i in the G subgroups.

3. Student–Bibliography Interaction: In this process, the increase of knowledge occurs when each agent independently studies the bibliography or learning material provided by the teacher. This process will occur after a certain amount of time N_{ST} and an agent i acquire a unit of information k from the study material B when $R \geq \alpha_i$, with R a selected random number. Such that $\sigma_{ik} = \sigma_{Bk}$.

In this work, we only considered noise in the Student–Teacher interaction. In all other interactions, we assumed that work in suitable conditions so that the noise does not impede to acquire information during interactions. For example, in the case of group work, participatory methodologies are used, tasks that require different levels of concentration, where the noise not be an impediment. On the other hand, study habits are characterized by appropriate studying routines (e.g., reviews of material) occurring in an environment that is conducive to studying [13].

Although the similarity of these interactions with those already defined by Bordogna and Albano [21] is evident, they differ in the way agents interact. In this case, the acquisition of learning objectives in the Student–Student and Student–Bibliography interaction was inspired by the exchange of cultural features of the Axelrod model [30]. These forms of interaction are also present in the models described by kinetic theory [25], money exchange [33,34] and opinion formation [35,36].

Finally, We carried out a number of evaluations, N_{EV}, to quantify the amount of information A_I acquired by each agent during the training time. The grade calculated as follows

$$grade(i) = \left(\frac{\sum\limits_{j}^{A_I} \sigma_{ij}}{A_I/N_{EV}} \right) * 6 + 1. \tag{1}$$

The ratio between A_I and N_{EV} sets a similar amount of information for each evaluation. This formula has set to obtain a scale of notes between 1 and 7.

3. Study cases

Initially, our model compared with the results reported by Bordogna and Albano [21] through performing simulations that consider the same general conditions. That is, the same instructional time, the number of agents and the number of students that randomly make up the work groups. This simulations have two kind of classrooms. One classroom has students who only attend lectures (Case I), and in other classroom has students attend lectures and work groups (Case II). However, considering the differences in the simulation approach, some parameters are different to generate similar conditions.

For Bordogna and Albano [21], knowledge increases over time and is assumed to be discrete. Thus they called $\Delta\sigma$ as a "quantum" of knowledge. With this, the minimum knowledge is 0.1, and the maximum is 1. Our model, based on the exchange of information to complete a learning task or objectives maintains the discrete nature of knowledge but, varies in quantitative value due to the simulation approach. Thus, in our case the minimum of knowledge is zero, and the maximum is the total of information to be acquired in the instructional period, $A_I = 100$.

The Student–Student interaction in [21] occurs throughout the whole simulation period, which in our model is equivalent to considering group work at each step. Therefore, the number of group work is equal to the number of lectures. The α_i parameter takes values between 0 and 1 evenly distributed, considering that for Bordogna and Albano [21], there is a possibility of unsuccessful interactions due to persuasion and affinity between those who interact. In our model, the increase in knowledge is only conditioned by periods of loss of attention of agents during the learning process.

Fig. 1(a) shows the variation of the average knowledge of the system in the time $\sigma(t)$. Here the regimes described by Bordogna and Albano in their work [21] (Fig. 1, case AA, page 118701-2) can be observed. For short time regimes ($t < 5$) there is no significant difference in obtaining information to increase their knowledge. But in intermediate time regimes ($5 < t < 100$) the difference between Case I and Case II is observed. Note that this difference is due to students of Case I only attending lectures, while students in Case II attend lectures and group sessions. The advantage of Case II is finally expressed in prolonged time regimes ($t > 100$), where the difference in accumulated information is remarkable.

Fig. 1(b) shows the noise effects during the learning process in the classroom. Here it is observed that the noise considerably reduces the acquire of information as it increases. Because the noise does not take effect during the Student–Student interaction, the lessons given during the lecture are reinforced through cooperation in group work.

In addition, this figure shows for both cases a phase transition around the point $\beta_c \approx 0.99$, as described in [23,24]. As quoted by Bordogna y Albano, the phase transition for Cases I and II are characterized as second order and first order transition respectively. The inset of Fig. 1(b) shows how significant is the difference between the variation of the maximum knowledge of both cases under the effects of noise. As the noise increases, group work becomes more relevant to achieve meaningful learning in the classroom.

These results shows that our model significantly reproduces the phenomenology obtained by Bordogna and Albano [21]. The differences observed in Fig. 1(b) originate in the assumption that in this work we do not consider that the Student–Student interaction has noise. Thus, as noise increases in the classroom, the possibility of acquiring information of the Student–Teacher interaction decreases. In the critical point β_c, students stop acquiring information. So in the Student–Student interaction, they have no information to exchange. It is important to note that the results of our model in Case II Fig. 1(b) remarkably shows the characteristic form of the first-order phase transition.
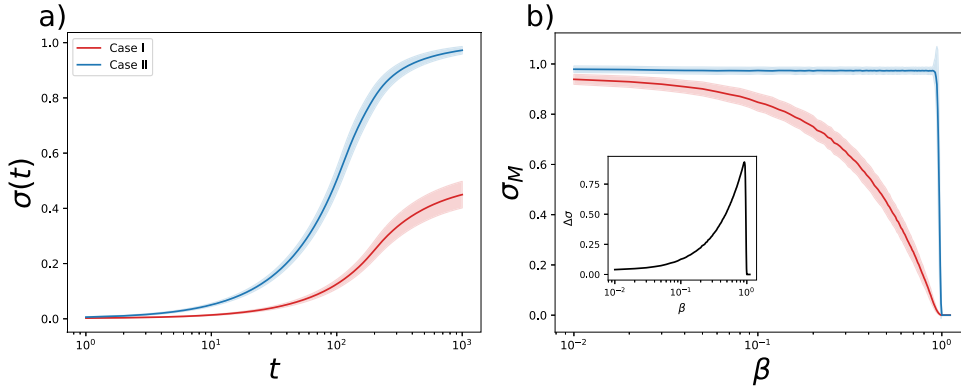
Fig. 1. Results obtained with our model under equivalent conditions to the ones by Bordogna and Albano [21]. The red line corresponds to the case in which the students only attend lectures, and the blue line corresponds to the case in which the students attend the class and have group work sessions. Panel (a) shows the variation of the average knowledge of the system $\sigma(t)$ over time. Panel (b) shows the variation of the maximum knowledge in the system versus the noise in a classroom. The inset shows the difference between the variation of the maximum knowledge of both cases. For these results averaged over 10^3 different experiments, $N = 96$ agents were considered, $A_I = 100$, $N_G = 3$, $\alpha \sim U[0, 1]$, $\beta = 0.5$ and a number of lectures equal to the simulation time $t = 10^3$.

Next, we study cases with small training times to analyze situations comparable to learning in a university classroom. We considered a training period of 36 lectures, with a maximum of information of 36 (one per lecture) and 12 group work sessions. In this way, we simulate a training period of 3 months with 3 h of teaching and 1 group work session per week. The working groups are formed by $N_G = 3$ students chosen at random, and $N_{EV} = 2$ evaluations are carried out during the training time. Considering the evidence developed from studies in educational psychology [7,11], the α_i parameter was distributed taking values between [0.30, 0.50].

3.1. Homogeneous classroom

We have considered four kinds of classrooms with all students in the same conditions. In case I, the students only attend to lectures. In case II, the students attend lectures and work groups. In the case III, the students attend lectures and study individually reviewing the bibliography after each class. Finally, in the case IV, the students attend lectures, work groups and study individually.

Fig. 2(a) shows the evolution of the average knowledge in the time $\sigma(t)$ for each case. In times before the first group work session, $t < 4$, it is not possible to find large differences between those who attend the group work sessions and those who only attend lectures. Nonetheless, the advantage of those who additionally study class-by-class from the beginning is significant, resulting in an initial knowledge ($t = 1$) greater than for the other cases. Note that the small breaks in the blue line and the green line from $t \geq 4$, is due to the beginning of the group work sessions. Here the students increase their knowledge as a result of the exchange of information. Then, between the first group work session and the first evaluation, $4 \leq t \leq 18$, the increase in knowledge is progressive and determines the dynamics. In this time interval, the advantage of those who attend the group work sessions over those who only attend lectures is observable. Also, the notorious advantage of the students in Case III and Case IV is due to the independent study of the first class, where the advantage of those who attend the group work sessions is observable again.

For times after the first evaluation, $t > 18$, the difference in knowledge between the different cases analyzed is maintained. Note that students who attend all activities and study class-by-class reach a steady state, obtaining maximum knowledge in the instructional time. Students who only attend lectures reach an average knowledge of 0.6, while for those who attend lectures and attend in group work sessions they get a value close to 0.8.

In general, the increase in knowledge is fast in all cases showing slight differences. However, for intermediate times regimes, the differences in student's knowledge increase due to the start of group work sessions. Thus, after the first evaluation, the dynamics of the system is determined by the rest of the instructional time.

Fig. 2(b) shows the temporal evolution of the average knowledge of all cases on a logarithmic scale. A linear fit was to the data between $4 \leq t \leq 19$ using the least-squares method. Before $t = 4$, workgroups have not yet begun, so the dynamics for the cases are not comparable. From $t = 19$ onward, the difference in knowledge between the cases analyzed maintaining during the instruction time. Here, the slopes can be interpreted as a rate of acquiring information and bring us a measure of performance of the students to complete a learning task or objectives during training time. This measure allows us to rank studied cases from the most to the least efficient, and verify that study skills are related to academic performance and improve the effectiveness and efficiency of student learning [12,13].

Fig. 2(c) shows the noise effects during the learning process in the homogeneous classroom. Here it is observed that the acquisition of information reduces as it increases the noise.
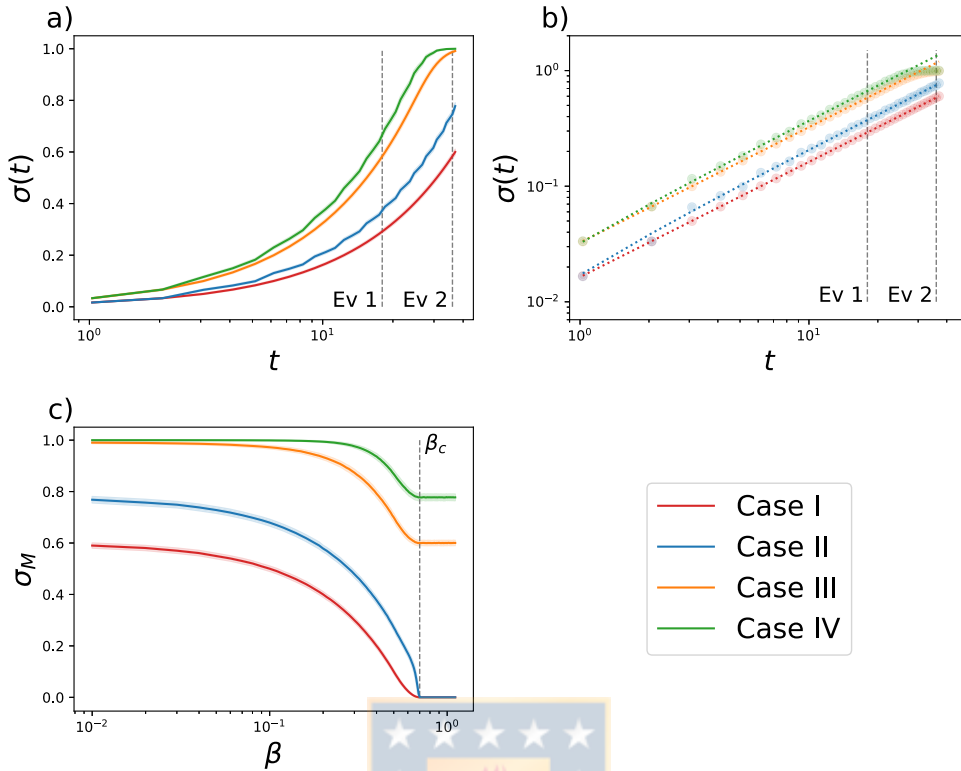
Fig. 2. Results obtained for a homogeneous classroom. The colors red, blue, yellow, and green corresponds to the case I, II, III, and IV respectively. The segmented vertical lines indicate the time at which the evaluations were performed. Panel (a) shows the variation of the average knowledge of the system $\sigma(t)$ over time. Panel (b) shows the variation of the average knowledge of the system on a logarithmic scale. The dotted lines correspond to a linear fit made using least squares, and the colors used correspond to the respective adjustment of each case. Also, the slope of each of the lines with their respective color is shown. Panel (c) shows the variation of the maximum knowledge in the system versus the noise in a classroom. The segmented vertical lines in $\beta_c \approx 0.70$ indicate the critical noise point. For these results averaged over 10^3 different experiments, $N = 96$ agents were considered, $A_I = 36$, $N_{ST} = 36$, $N_G = 3$, $N_{EV} = 2$, $\alpha \sim U[0.3, 0.5]$, and a number of lectures equal to the simulation time $t = 36$. In panel (a) and panel (b) $\beta = 0.0$. In panel (c) $\beta \in [0, 1]$.

The Case I and Case II in the homogeneous classroom decrease continuously, similarly to Case I observed in Fig. 1(b). This behavior is characteristic of a second-order phase transition, as described in [23,24]. Note that the behavior of Case II in the homogeneous classroom is different from Case II observed in Fig. 1(b) because there is a difference in the number of workgroup sessions. For Case II observed in Fig. 1 b, the workgroup sessions are after each class. Instead, for Case II in the homogeneous classroom, the group sessions are every three classes. Finally, in Case III and Case IV, the acquire of information decreases continuously but does not reach zero because the workgroup session and individual study allows reinforces the information given during the lecture through cooperation.

Fig. 3(a) shows the temporal variation of the distribution of knowledge for each case study. Until before the first evaluation, similar behavior observed for all cases. The mean of the knowledge distribution increases progressively over time, as noted in Fig. 3(b). The standard deviation increases progressively for Case I and II. But in Case III y Case IV decreases for $t > 20$ and $t > 25$ respectively, because the distribution of knowledge for Case III and Case IV condenses at the maximum value when reaching the end of instruction time.

Note that the dispersion variation in knowledge distribution shows that the increase in knowledge is different for each agent. The origin in the knowledge distribution dispersion is given by of variety attention capacities of the agents parameterized by α_i. So the students in Case III and Case IV reach the maximum of knowledge due to the individual study and the collaboration in the workgroup sessions.

On the other hand, detail the change in the asymmetry of the knowledge distributions is observed in Fig. 3(b). For Case I, there is a slight increase in negative asymmetry where the mean is less than the median. Case II has a negative asymmetry in the beginning but between $8 < t \leq 32$, the knowledge distribution has positive asymmetry where the mean is greater than the median. Finally, when $t > 32$ arrives, the knowledge distribution has a negative asymmetry again. For Case III, the asymmetry is negative throughout the training time, with a significant increase in value from $t > 24$. For case IV, the asymmetry change on time, reaching values significantly high due to the condensation of the distribution on the maximum of knowledge.
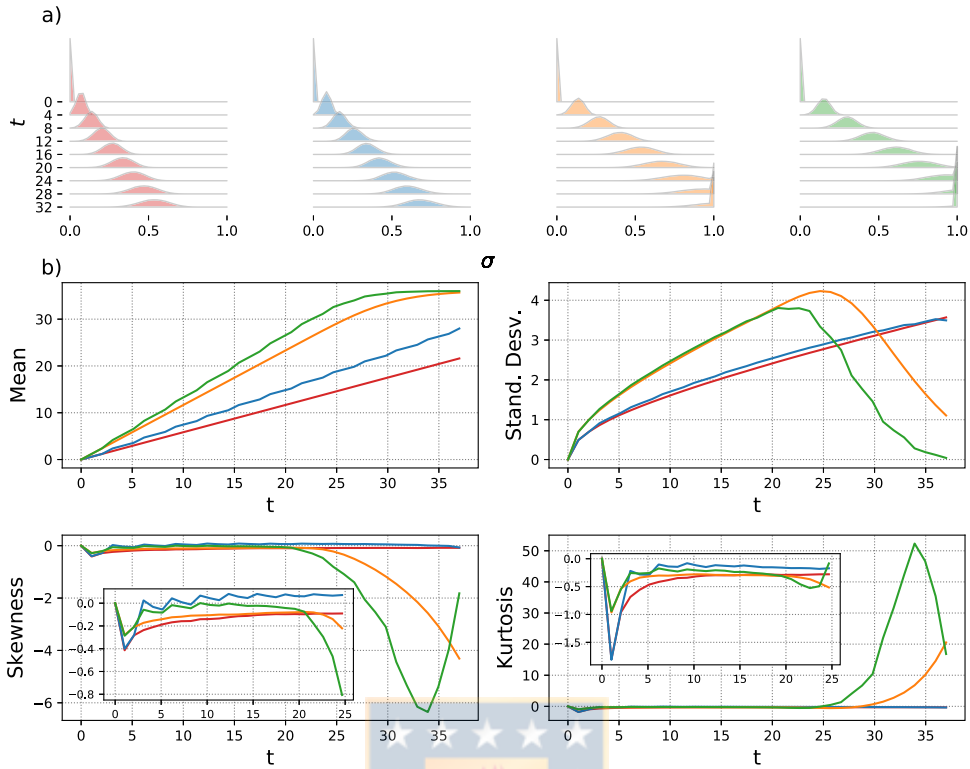
Fig. 3. Results obtained for a homogeneous classroom. The colors red, blue, yellow, and green corresponds to the case I, II, III, and IV respectively. Panel (a) shows the variation in time of the distribution of knowledge σ_i for each of the case studies. Panel (b) shows the variation in time of the mean, standard deviation, skewness, and kurtosis for each of the cases analyzed. The inset figures show greater detail of skewness and kurtosis in the region of $t \in [0, 25]$. For these results averaged over 10^3 different experiments, $N = 96$ agents were considered, $A_I = 36$, $N_{ST} = 36$, $N_G = 3$, $N_{EV} = 2$, $\alpha \sim U[0.3, 0.5]$, $\beta = 0.0$, and a number of lectures equal to the simulation time $t = 36$.

The change of the knowledge distribution form with the variation of the kurtosis on time showed in Fig. 3(b). For $t = 0$, the knowledge for all agents is null. Then, it show that the knowledge distributions take platykurtic form for Cases I and II throughout the instructional time. Case III and case IV show a transition from the Platykurtic to a Leptokurtic from $t \leq 30$ and $t \leq 25$, respectively.

Finally, Fig. 4 shows the distribution of the grades obtained by the students for the first and the second evaluation, and the average grade at the end of the course. Note that for Cases I and II, the forms of the distributions of grades in the first evaluation show a similar dispersion but for different average values. This difference indicates that Case II is more efficient. In the second evaluation, Case II is more efficient than Case I but the dispersion in both distributions is smaller than the previous one. This difference generates homogenization in the students. In Case III and Case IV, the distribution of grades condenses at the maximum value, showing that they reached the maximum information needed to complete a learning task or objectives. Table 1 summarizes these results.

3.2. Heterogeneous classroom

In this subsection we analyze eight cases with different regimes of class attendance, group work sessions and individual study. The attendance can be obligatory or random distributed uniformly. The characteristics of each case are described in Table 2. As previously, we consider a training period of 36 lectures, with a maximum of information of 36 (one per lecture) and 12 group work sessions.

Fig. 5(a) shows the time variation of the average knowledge $\sigma(t)$ for each Case. It is appreciated that the Cases OOO and ORO (blue and purple lines, respectively) reach maximum knowledge. Then, Cases OOR and ROO (the orange line and the green dotted line, respectively) are those that reach an average value closer to the maximum values obtained in Cases OOO and ORO. Cases ORR and RRO (the dotted brown line and the pink line, respectively) have the same final result. Finally, Cases ROR and RRR (red and gray lines, respectively) obtain the lowest maximum values, in which the maximum knowledge obtained in Case ROR is greater than in Case RRR. As expected, the acquisition of information is more favorable when the agent's attendance is obligatory to each of the three learning instances. Likewise, the acquisition of information is poor when the attendance at each of the learning instances is random.
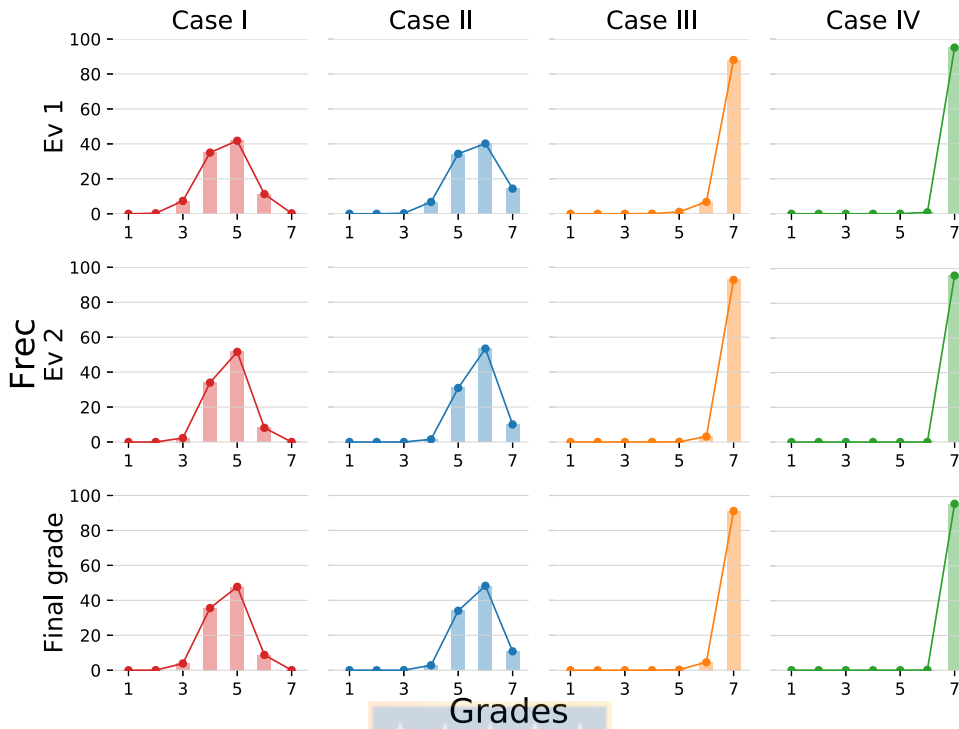
Fig. 4. Results obtained for a homogeneous classroom. The rows correspond to the distribution of grades obtained in the first evaluation, the second evaluation and the final average grade. The colors red, blue, yellow, and green corresponds to the Case I, II, III, and IV respectively. For these results averaged over 10^3 different experiments, $N = 96$ agents were considered, $A_I = 36$, $N_{ST} = 36$, $N_G = 3$, $N_{EV} = 2$, $\alpha \sim U[0.3, 0.5]$, $\beta = 0.0$, and a number of lectures equal to the simulation time $t = 36$.

Table 1
This table shows the mean, standard deviation, median, skewness, and kurtosis for the distribution of grades obtained in the first evaluation, the second evaluation and the final average grade. These results averaged over 10^3 different experiments.

Cases		Ev 1	Ev 2	Final
I	Mean	4.598	4.600	4.599
	Stand. Desv.	0.054	0.040	0.045
	Median	4.601	4.601	4.599
	Skewness	−0.063	−0.083	−0.087
	Kurtosis	−0.041	−0.009	−0.054
II	Mean	5.639	5.667	5.653
	Stand. Desv.	0.052	0.041	0.043
	Median	5.639	5.667	5.651
	Skewness	0.027	0.143	0.124
	Kurtosis	−0.252	−0.154	−0.264
III	Mean	6.897	6.948	6.923
	Stand. Desv.	0.071	0.052	0.054
	Median	6.899	6.950	6.924
	Skewness	−0.289	−0.346	−0.197
	Kurtosis	0.199	0.077	−0.080
IV	Mean	6.988	6.999	6.994
	Stand. Desv.	0.058	0.016	0.031
	Median	6.990	7.000	6.995
	Skewness	−1.267	−3.411	−1.297
	Kurtosis	2.106	14.155	2.210

It is worth noting that attendance at all lectures as well as individual study class-by-class is equally important. Because in both Cases, acquiring a information in the instructional period can occur the same number of times. Therefore, agents who attend all lectures and study class-by-class have more advantage over agents who only attend lectures or study individually. Note that the group work sessions are less periodic, due to them occurring every three lectures. However, they are an educational instance that guarantees the increase of knowledge through peer interaction, which allows OOR

Table 2
This table shows the different Cases studied according to the attendance assigned for lectures, group work sessions and individual study.

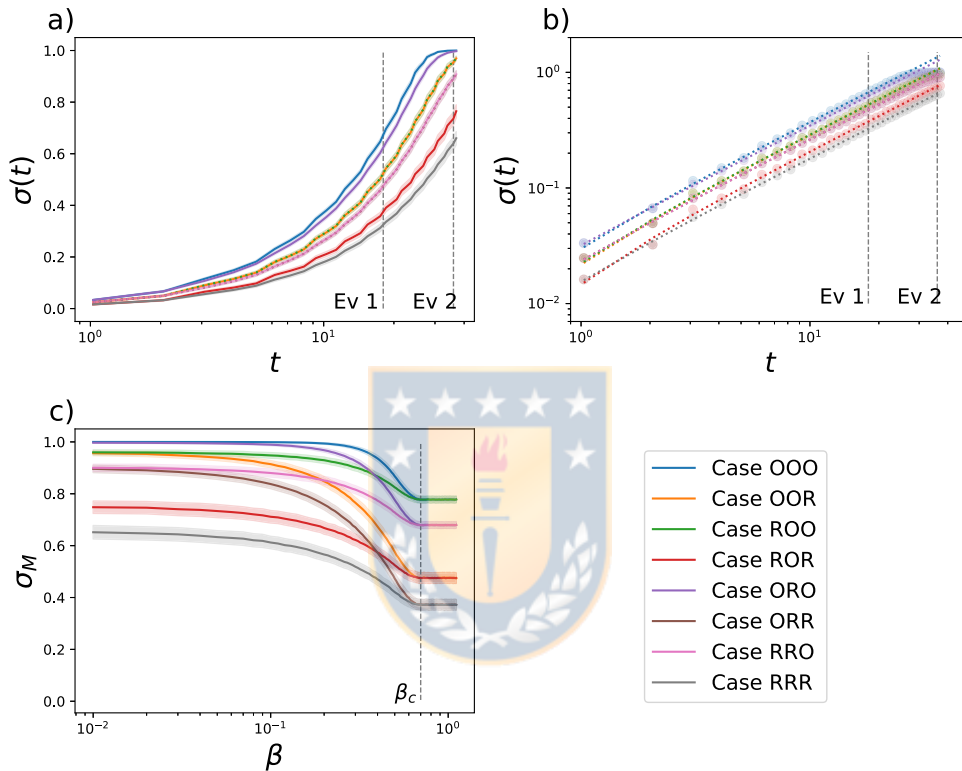| Cases	Class attendance	Group work attendance	Individual study
OOO	Obligatory	Obligatory	Obligatory
OOR	Obligatory	Obligatory	Random
ROO	Random	Obligatory	Obligatory
ROR	Random	Obligatory	Random
ORO	Obligatory	Random	Obligatory
ORR	Obligatory	Random	Random
RRO	Random	Random	Obligatory
RRR	Random	Random	Random

Fig. 5. Results obtained for the heterogeneous classroom of the Cases described in Table 2. The segmented vertical lines indicate the time at which the evaluations were carried out. Panel (a) shows the variation of the average knowledge of the system $\sigma(t)$ in time. Panel (b) shows the variation of the system average knowledge on a logarithmic scale. The dotted lines correspond to a linear fit made using least squares, and the colors used correspond to the respective adjustment of each Case. Also, the slope of each of the lines with their respective color is shown. Panel (c) shows the variation of the maximum knowledge in the system versus the noise in a classroom. The segmented vertical lines in $\beta_c \approx 0.70$ indicate the critical noise point. For these results averaged over 10^3 different experiments, $N = 96$ agents were considered, $A_I = 36$, $N_G = 3$, $N_{EV} = 2$, $\alpha \sim U[0.3, 0.5]$, and a number of lectures equal to the simulation time $t = 36$. In panel (a) and panel (b) $\beta = 0.0$. In panel (c) $\beta \in [0, 1]$.

and ROO Cases and ORR and RRO Cases to be efficient. Finally, Cases ROR and RRR have the lowest average performance. In these Cases class attendance and the individual study are random. Thus, they do not guarantee a constant source of information. However, the differences in efficiency observed between Cases ROR and RRR shows that attendance at group work sessions creates an increase in the information achieved.

As in the Cases described in Fig. 2, it is possible to notice small breaks in the lines of average knowledge due to the beginning of the group work sessions. However, these breaks are more significant for Cases with full attendance at group work sessions. Also, it is again possible to note that after the first evaluation, the difference in knowledge acquired in the different Cases analyzed becomes a permanent trend.

Fig. 5(b) shows the temporal evolution of the average knowledge in the classroom on a logarithmic scale. A linear fit was made to the data between $4 \leq t \leq 19$ using the least-squares method. Because before $t = 4$ the rounds of study have not yet begun, so the dynamics for the three Cases are not comparable. On the other hand, from $t = 19$ onward, the difference in knowledge between the Cases analyzed is maintained, so the dynamics do not change, and some students

Table 3
This table shows all the study Cases with noise null ranked from the most efficient to the least efficient according to the value of the slope.

Cases	OOO	ORO	OOR	ROO	RRO	ORR	ROR	RRR
Slope	0.0372	0.0346	0.0291	0.0290	0.0264	0.0263	0.0209	0.0181
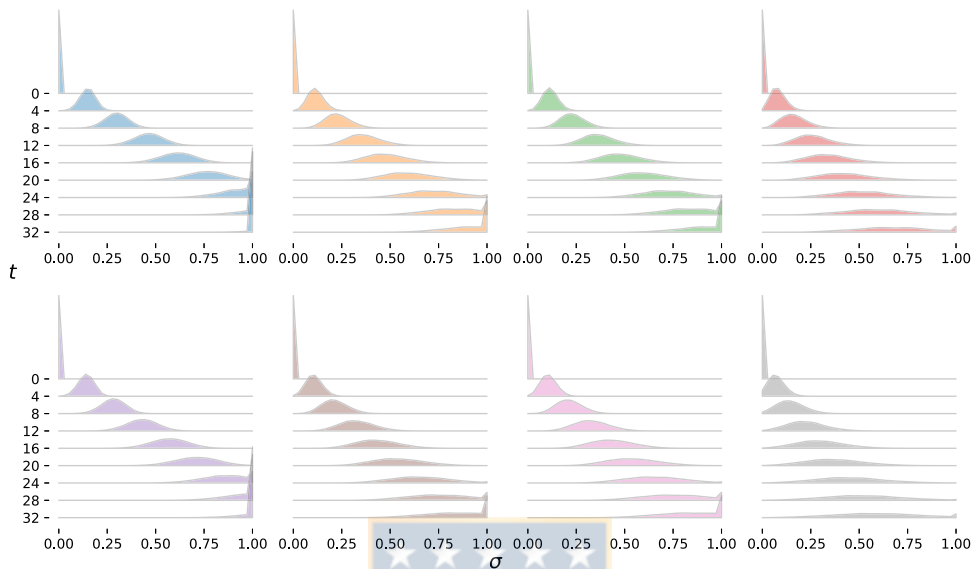
Fig. 6. Results obtained for a heterogeneous classroom. This figure shows the variation in time of the distribution of knowledge σ_i for each of the Case studies. For these results averaged over 10^3 different experiments, $N = 96$ agents were considered, $A_I = 36$, $N_G = 3$, $N_{EV} = 2$, $\alpha \sim U[0.3, 0.5]$, $\beta = 0.0$, and a number of lectures equal to the simulation time $t = 36$.

reach maximum knowledge. The slopes obtained in each Case (see Table 3) allows us to rank studied Cases from the most to the least efficient as follows: OOO, ORO, OOR, ROO, RRO, ORR, ROR, RRR.

Fig. 5(c) shows the noise effects during the learning process in the heterogeneous classroom. Here it is observed that the acquisition of information reduces as it increases the noise. Again, the behavior obtained is characteristic of a second-order phase transition, as in the Cases described in Fig. 2, and quoted in [23,24].

Note the change in the grouping of cases in the noiseless and the noisier scenario. When the noise is null, we observe similarities in the cases OOR and ROO, ORR and RRO, and ROR and RRR. However, when the noise increments, the maximum knowledge change, and new grouping cases are OOO and ROO, ORO and RRO, OOR and ROR, and ORR and RRR. Here emerge a new rank of efficiency because the individual study and the workgroup take more relevance when the noisier scenarios do not allow to acquire information in the lecture.

Fig. 6 shows the time variation of the distribution of knowledge for each Case study. The temporal increase of the mean and the standard deviation of the distributions observed in Fig. 7(a) and (b), allow to classify the different studied Cases. Thus, we can establish similarity in the form of the distributions in Cases OOO and ORO, OOR and ROO, RRO and ORR, and in ROR and RRR. Note that, in all Cases, this increase accounts for the progressive increase in student knowledge. The increase in dispersion indicates that each student increased their knowledge differently.

Also, Fig. 7(c) and (d) shows changes in asymmetry as well as in the forms of the distributions in each Case. In Cases OOO and ORO there is a slight increase in the negative asymmetry, where the mean is less than the median. Also, a transition from a platykurtic to a leptokurtic form is observed at $t = 25$ for Case OOO and $t = 27$ for Case ORO. For Cases, OOR and ROO, asymmetry and kurtosis are very similar. The distributions show positive asymmetry, and at $t = 26$ the asymmetry becomes negative. At $t = 33$, for both Cases, a change from the platykurtic to a leptokurtic form is observed. For Cases ORR and RRO, we can observe a similar behavior described Cases OOR and ROO. The asymmetry change from positive to a negative at $t = 27$ and the change from platykurtic to leptokurtic form occurs at $t = 36$ and $t = 35$ for Cases ORR and RRO respectively. Finally, for Cases, ROR and RRR a change from positive to negative asymmetry are observed at $t = 32$ and $t = 33$, respectively. However, the distribution maintains a platykurtic form throughout the instructional time. These changes in the distributions of knowledge account for the existence of the dynamics within the classroom, which is observed in Fig. 6.

Fig. 8 shows the distribution of grades of the Case studies in a heterogeneous classroom. The rows correspond to the distributions of grades obtained in the first evaluation, the second evaluation and the final average grade. The columns correspond to Cases from OOO to RRR. Again the similarities described above can be observed for Cases OOO and ORO, OOR and ROO, RRO and ORR, and in ROR and RRR. These similarities can be confirmed by looking at Table 4.
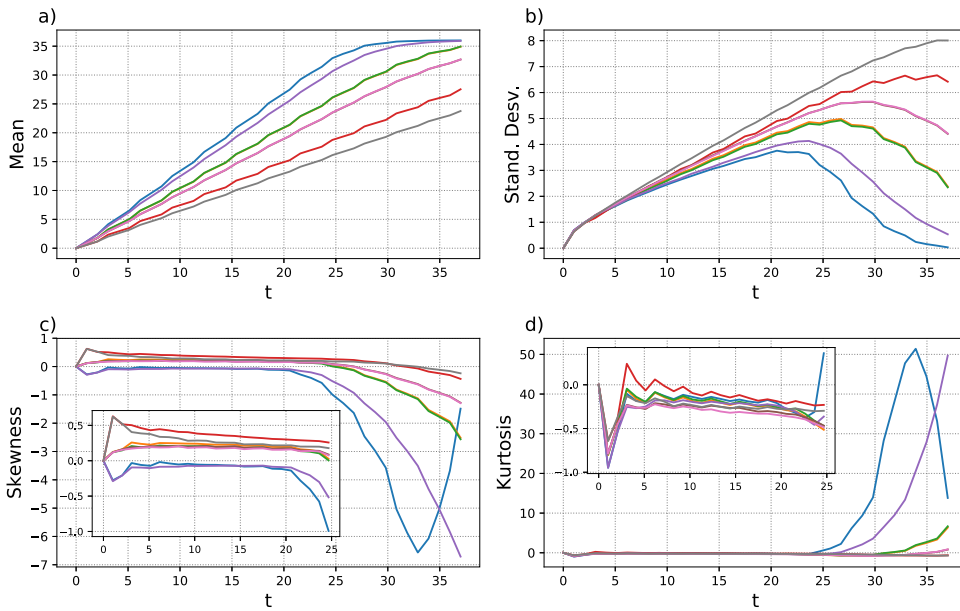
Fig. 7. Results obtained for a heterogeneous classroom. This figure shows the variation in time of the mean, standard deviation, skewness, and kurtosis for each of the Cases analyzed. The inset figures show greater detail of skewness and kurtosis in the region of $t \in [0, 25]$. For these results averaged over 10^3 different experiments, $N = 96$ agents were considered, $A_I = 36$, $N_G = 3$, $N_{EV} = 2$, $\alpha \sim U[0.3, 0.5]$, $\beta = 0.0$, and a number of lectures equal to the simulation time $t = 36$.

Fig. 8. Results obtained for a heterogeneous classroom. The rows correspond to the distribution of grades obtained in the first evaluation, the second evaluation and the final average grade. The columns correspond to the Cases from OOO to RRR. For these results averaged over 10^3 different experiments, $N = 96$ agents were considered, $A_I = 36$, $N_G = 3$, $N_{EV} = 2$, $\alpha \sim U[0.3, 0.5]$, $\beta = 0.0$, and a number of lectures equal to the simulation time $t = 36$.

3.3. An analogy with a gas model

To get an idea of the detailed evolution of the distribution of knowledge in a group of agents, we first analyze the simplest possible case in this model. All agents have the same measures the loss of attention α_i. Unlike the cases already described, we studied its asymptotic evolution considering a number of lectures equal to the instruction time $t = 144$ (see Fig. 9).

This result of the evolution of the system where the agents obtain information from the learning environment with equal probability, without interacting with each other, reassembles the normal distribution. It is possible to think that the evolution of the system, from its initial condition, where all the agents are without information, to a state characterized by a normal distribution, corresponds to a gas in a thermal bath that expands and tends towards thermodynamic equilibrium. This evolution where all agents have the same conditions for the acquisition of information, but the differences are induced by the random occurrence of episodes of loss of attention. If the agents' loss of attention probability is distributed as in

Table 4

This table shows the mean, standard deviation, median, skewness and kurtosis for the distribution of grades obtained by students for the first, second and final grades in the heterogeneous classroom with noise null. These results averaged over 10^3 different experiments.

| Cases		Ev 1	Ev 2	Final	Cases		Ev 1	Ev 2	Final
OOO	Mean	6.989	6.999	6.994	ORO	Mean	6.952	6.984	6.968
	Stand. Desv.	0.054	0.018	0.029		Stand. Desv.	0.070	0.048	0.050
	Median	6.990	7.000	6.995		Median	6.955	6.986	6.969
	Skewness	−1.254	−3.678	−1.391		Skewness	−0.607	−0.925	−0.724
	Kurtosis	−2.073	15.729	2.736		Kurtosis	0.426	1.076	0.694
OOR	Mean	6.645	6.764	6.704	ROO	Mean	6.647	6.763	6.705
	Stand. Desv.	0.073	0.064	0.063		Stand. Desv.	0.074	0.064	0.064
	Median	6.646	6.767	6.706		Median	6.649	6.766	6.709
	Skewness	−0.041	0.072	0.011		Skewness	−0.324	−0.387	−0.381
	Kurtosis	−0.140	−0.134	−0.175		Kurtosis	−0.015	0.160	0.028
ORR	Mean	6.326	6.415	6.371	RRO	Mean	6.332	6.415	6.373
	Stand. Desv.	0.076	0.066	0.068		Stand. Desv.	0.075	0.064	0.067
	Median	6.990	7.000	6.995		Median	6.335	6.417	6.376
	Skewness	−0.119	−0.076	−0.077		Skewness	−0.105	−0.066	−0.089
	Kurtosis	0.087	0.277	0.224		Kurtosis	−0.007	0.009	0.017
ROR	Mean	5.450	5.522	5.486	RRR	Mean	4.897	4.934	4.916
	Stand. Desv.	0.073	0.068	0.069		Stand. Desv.	0.083	0.080	0.081
	Median	5.438	5.516	5.475		Median	4.894	4.936	4.916
	Skewness	0.122	0.092	0.125		Skewness	0.014	−0.016	−0.006
	Kurtosis	−0.084	0.014	−0.017		Kurtosis	−0.191	−0.170	−0.188
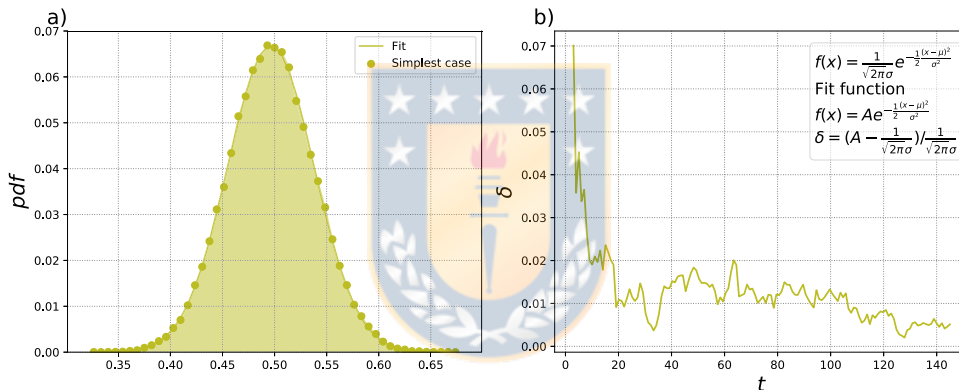
Fig. 9. In panel (a) the distribution observed in $t = 144$ is shown with points in olive. An olive line shows the fit of a normal distribution. Panel (b) shows the relative error of the adjustment function as the time elapses. For these results averaged over 10^3 different experiments, $N = 96$ agents were considered, $A_I = 144$, $N_G = 3$, $N_{EV} = 2$, $\alpha = 0.5$, $\beta = 0.0$, and a number of lectures equal to the instruction time $t = 144$.

the range already described, since the agents do not interact with each other, the total distribution will evolve from the initial condition to a superposition of normal distributions.

The interaction between the agents gives rise to distributions that deviate from the normal distribution. As can be seen in Fig. 10, the cases shown in the upper and lower panels are not equilibrium distributions, but rather have deviations being wider than the normal distribution (see upper panels Fig. 10) or presenting two modes as in the distributions of the lower panels. The latter is generated by the association of agents in the study process based on their previous performance. This is a form of Maxwell's demon, where by homophilia the agents are grouped to study in triples according to their previous performance. It is clear that this is the reason for the appearance of two modes in the distributions in the lower panels of Fig. 10.

4. Conclusions

We developed an agent-based model based on phenomena studied by behavioral and learning scientists, to study the increase the knowledge by accumulating the information needed to complete a learning task or objectives.

Our model has basic idealizations in common with the model reported by Bordogna and Albano [21], as the interactions with a teacher, with other agents, and the bibliography. However, they differ in the way of operationalizing some basic assumptions, generating differences in the possibles phenomena to simulate. For example, Bordogna and Albano consider the possibility of unsuccessful interactions due to persuasion and affinity between those who interact. In our model, the increase in knowledge is only conditioned by periods of loss of attention of agents during the learning process and is
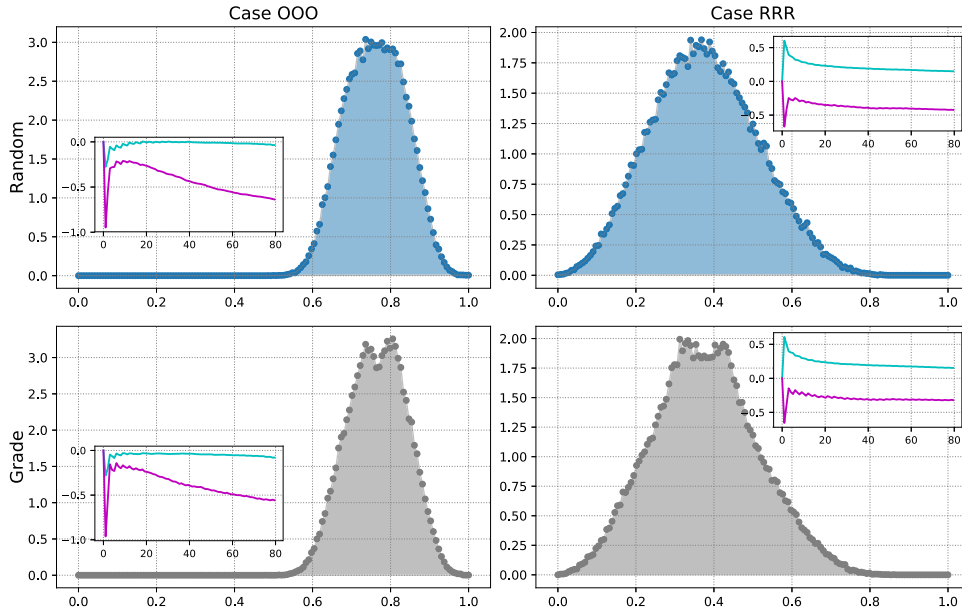
Fig. 10. Distributions of Cases in a heterogeneous classroom observed in $t = 80$. The columns correspond to the Cases OOO and RRR. The rows corresponding to agents are grouped random and to academic performance (amount of information). The inset figures show greater detail of skewness (cyan line) and kurtosis (magenta line) in the region of $t \in [0, 80]$. For these results averaged over 10^3 different experiments, $N = 96$ agents were considered, $A_I = 144$, $N_G = 3$, $N_{EV} = 2$, $\alpha \sim U[0.3, 0.5]$, $\beta = 0.0$, and a number of lectures equal to the instruction time $t = 144$.

incremental. These characteristics could lead to the model of Bordogna and Albano shows some phenomena (like clusters of fixed opinions) than our model cannot.

Our model can be considered a equivalent version of Bordogna and Albano's model but has the advantage of reproducing the temporal dynamics of knowledge and show the characteristic behavior of first and second-order phase transitions, as observed in Fig. 1(b). Furthermore, the effects of noise introduced during the learning process in the homogeneous classroom generate changes in the nature of phase transitions. Thus, our model, unlike the Bordognas and Albano model, shows a change in the type of phase transition, from first order to second order due to the limitation of the range of the loss of attention, as shown in Fig. 2(c). In addition, in the heterogeneous classroom there are changes in the grouping of the case studies presented, showing efficient cases in scenarios without noise, as well as efficient cases in noisier scenarios, which are different from those without noise. This change produced by the inclusion of noise, can be interpreted as an adaptation property of the system that allows it to reach the maximum knowledge only from the intrinsic parameters of agents and the rules of interactions.

It is important to note that the slopes showed in Figs. 2(b) and 5(b) can be interpreted as a rate of acquiring information and bring us a measure of performance of the students to complete a learning task during training time. The slopes allow us to rank the study cases and differentiate all of them when the attendance to any learning activity can be random. Furthermore, this fact allows us to verify that the study skills, study habits, study attitudes, and study motivation exhibit particularly strong and robust relationships with academic performance in college, as noted by Cred and Kuncel [13].

The distributions of grades from evaluation one and two contain information on the dynamics of learning. Thus, grades distributions can be used as a tool to measure the efficiency of teaching strategies. It also makes it possible to create and propose new schemes for the assessment process based on learning achievements.

Our model allows to analyze the temporal variation of the distribution of knowledge in the classroom, as well as the kinetic theory of active particles developed by Burini et al. [25]. Our results allow us to built a representation of our model as a gas model, showing characteristics of systems far from the state of thermodynamic equilibrium. It allows us to use the known techniques of the gas models to interpret the dynamics of the simulated learning process. For example, an analogy can be made between the increase of the kinetic temperature and that of the dispersion observed in the distributions of acquisition of information. It is also possible an analogy between the heat flux with the skewness of the distributions of the classroom dynamics, which due to the workgroup sessions.

The basic assumptions and idealizations used in our model allow us to identify some elements of the learning process, to obtain a first modeling approach based on phenomena as loss of attention. Besides, our model shows phase transitions, and other characteristics of ABMs such as Heterogeneity, Autonomy of agents, and Local Interactions, as noted by Epstein [26]. Nonetheless, we do not pay particular attention to how agents construct knowledge in pair interaction. Thus, our model lacks some key features of complex systems. For example, self-organization, sensitivity to initial conditions,

or nonlinearities, as quoted by Jacobson [17]. This is a disadvantage because our model does not describe the teaching–learning process in all its complexity. However, our model can be improved and include, for example, pre-knowledge in agents, non-linear feedback effects, as shown in Koponen [37]. Also this model can be improved to consider the influence of groups formed by affinity with a topological basis in complex networks, the influence of work and teaching methodologies on classroom learning or phenomena that arise from the coexistence of people in the classroom such as homophilia or the formation of groups or communities.

CRediT authorship contribution statement

Ignacio Ormazábal: Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing - original draft, Writing - review & editing. **Félix A. Borotto:** Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing - original draft , Writing - review & editing. **Hernán F. Astudillo:** Conceptualization, Methodology, Software, Formal analysis, Investigation, Writing - original draft, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Phase diagram in a one-dimensional civil disorder model

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Epstein's model for a civil disorder is an agent-based model that simulates a social protest process where the central authority uses the police force to dissuade it. The interactions of police officers and citizens produce dynamics that do not yet have any analysis from the sociophysics approach. We present numerical simulations to characterize the properties of the one-dimensional civil disorder model on stationary state. To do this, we consider interactions on a Moore neighborhood and a random neighborhood with two different visions. We introduce a Potts-like energy function and construct the phase diagram using the agent state concentration. We find order-disorder phases and reveal the principle of minimum grievance as the underlying principle of the model's dynamics. Besides, we identify when the system can reach stable or an instability conditions based on the agents' interactions. Finally, we identified the most relevant role of the police based on their capacity to dissuade a protest and their effect on facilitating a stable scenario.

I. INTRODUCTION

In recent years, the sociophysics or statistical physics of social dynamics has described different social phenomena as collective effects of the interaction between individuals [1–3]. In particular, the study of opinion dynamics has generated various models describing consensus, agreement, or uniformity using tools from statistical physics [4–7]. Recent efforts aim at describing these models, considering a diversity of individual traits of a population and size of group discussion [8] or their multistate variations, such as, for example, the majority voting model [9], the multistate voter model [10], multichoice opinion dynamics models [11], or the multistate noisy q-voter model [12].

In parallel, social scientists have used agent-based models to reproduce emerging social phenomena [13,14], such as the Schelling model of urban segregation [15] and the Axelrod model of cultural dissemination [16]. These models have attracted the attention of physicists, who have described the Schelling model as interacting physical particles [17] and as an Ising-like model [18]. Furthermore, they have characterized the static and dynamic properties in one and two dimensions [19] and their different behaviors using a phase diagram [20]. They have recently used a similar energy function to characterize the Schelling and Sakoda models [21–23]. On the other hand, physicists described the Axelrod model in two dimensions showing order-disorder phase transitions [24]. Then, they described the one-dimensional Axelrod model as a starting point for its description in more complex topologies [25]. In addition, they described the role of dimensionality on the order-disorder phase transitions [26] and the stability model using Lyapunov functions [27,28]. In this context,

Epstein presented an agent-based model to describe the social dynamics of protests and rebellions through recognizable macroscopic phenomena [29]. This model simulates a social protest process where the central authority uses police force to dissuade a protest.

For a generalized rebellion, Epstein presents five study cases and reports statistical regularities observed in the punctuated equilibrium dynamics, opening new questions about how the civil disorder dynamics work. Different scientists have modified this model to describe other social conflicts such as workers' protests [30], the spread of criminal activity [31], or civil war cases between ethnic groups [32]. In addition, some variations include legitimacy with endogenous feedback [33] or the influence of the distribution of money on the dynamics [34]. Despite these modifications, nobody characterized this agent-based model using concepts and tools of statistical physics in its original form.

This paper aims to characterize the one-dimensional civil disorder model on a stationary state as a first approach to studying this model in other dimensions or topologies. To do this, we perform numerical simulations of the Epstein model with and without police officers and use two visions to define interactions in a Moore neighborhood and a random neighborhood. We introduced two macroscopic quantities and built the phase diagram to identify different behaviors. On the one side, we define a Potts-like energy function to deduce a guiding principle to understand civil disorder dynamics. For the Schelling and Sakoda model [22,23], this energy-like function allows us to identify a minimization principle to understand the spatial segregation patterns as efficient or inefficient. In the case of the dynamics of a factory workers' protest [35], a function similar to the Ising model's free energy allows us to identify the steady state of the system and to describe two phases based on the *principle of minimum dissatisfaction*. On the other side, we used the concentration of agents to identify

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and characterize phase changes. The concentration allows us to describe phase transitions in the q-voter model with two types of stochasticity [36] and the multistate noisy q-voter model [12]. Besides, this macroscopic quantity is helpful in building phase diagrams. In the Schelling and Axelrod model [20,24], this strategy allows us to identify domain boundaries of the different qualitative behaviors. Hence, in this work, the agents' concentration and the phase diagram are crucial elements to characterize the original Epstein model and allow us to describe order-disorder transitions not reported in other works.

When considering the system without police officers, we identify transitions from two orders with a majority phase, a disordered phase, and a consensus phase. Furthermore, we identify when the system can reach stability or an instability conditions based on the agent's interactions. Besides, we reveal the *principle of minimum grievance*, the underlying principle of the model's dynamics. On the other hand, in the system with police officers, we study the effects of police officers' concentrations in different scenarios generated by the kind of neighborhoods and vision. We find the same order-disorder transition, but now we observe six ordered phases with a majority, one disordered phase, and the consensus phase. With the global quantities that we introduced, we can determine the role of police officers to dissuade a social protest. Finally, we identify stability and instability conditions of the system dynamics, and we show the energetic cost of using the police force to facilitate a stable scenario. These results from the perspective of sociophysics yield new qualitative elements and contribute to the future to study the dynamics of this model in other dimensions and topologies to approach the complexity of the dynamics of social protest.

The paper is organized as follows: In Sec. II, we introduce Epstein's model for a civil disorder and the global quantities used to describe the model's behavior to reach the stationary state. The simulations for the model without and with police officers and their respective phase diagrams are presented in Sec. III. The discussion of our results and concluding remarks are in Sec. IV.

II. THE MODEL AND GLOBAL QUANTITIES

A. The Epstein model

The civil disorder model has two agents: citizens and police officers. Citizens can be active when they participate in social protest, passive when they do not participate, or jailed when the police officers catch them. Citizens can switch from one state to another depending on their neighborhood, local parameters, and the global parameters of the system. On the other hand, a police officer agent represents the central authority's force. They are responsible for deterring a protest by capturing the active agents in their neighborhood. The neighborhood for all agents can be a von Neumann neighborhood used by Epstein [29] or a Moore neighborhood as in other works [31,34].

The system's dynamics emerge by relating the legitimacy of the authority and the grievance of the population, i.e., it depends on the relationship between the global parameters

and the agents' parameters. The global parameters are the same for all agents: legitimacy L, a state change threshold T, the maximum jail term J_{max}, and the vision v. The original model's vision determines the neighborhood's size, similarly to the rule radius in cellular automata [37] and range in other opinion models [38,39]. On the other hand, the local agent parameters are hardship H and risk aversion R. Both parameters are random values between zero and one uniformly distributed among all agents.

The rules that determine the agents' actions are as follows:

(1) State change rule. Each agent will decide whether to join the protest, evaluating the equation $G - N_R > T$, where $G = H(1 - L)$ symbolizes the grievance and $N_R = RP$ the net risk. The arrest probability equation $P = 1 - \exp[-k(C/A)_v]$ depends on the active agents and police officers ratio in the neighborhood defined by the vision. Hence when any agents evaluate if they switch their state, consider all active agents and police officers in their neighborhood. Therefore, for a fixed number of police officers, the agent's arrest probability falls the more active agents there are. Notice that A is always at least one because the agent always counts himself as active when computing P. The value of k is 2.3 for ensuring plausible P values, as reported by Epstein [29]. In the complete form of the state switch equation,

$$H(1 - L) - R(1 - \exp[-k(C/A)_v]) > T, \qquad (1)$$

we notice that the first element on the left depends on a combination of local and global values, and the other depends on the neighborhood conditions. In this way, when the difference of the agents' state variables exceeds the threshold, they switch from passive to active; otherwise, they remain passive agents.

(2) Capture rule. Police officers randomly capture an active agent from their neighborhood. If there are no active agents, they do nothing. A jailed agent stops participating in the dynamics according to the jail parameter assigned value randomly, with values between zero and the maximum determined at the beginning of the simulation. We used 30 time steps as a maximum jail term, the same value used by Epstein [29]. When jailed agents finish their sentences, they return to the model dynamics as passive agents.

(3) Movement rule: Each agent will move to an empty space at random within their neighborhood.

After setting global and local parameters of the model, we placed all the agents in random positions in the lattice to start the simulations. At each time step, all agents evaluate the dynamics rules asynchronously [29]. We show a schematic visualization of the changes of the agents' state changes due to the interaction rules in Fig. 1.

B. Global quantities

In order to characterize the model, we labeled each agent with the variable α, which can take values between one and four to represent an active agent, passive, jailed, or police officer. Then, we have defined the following quantities.

1. Concentration of agents

To see the predominant state in the system and study its macroscopic behavior as a function of the global parameters,

(a) System without police

Agent state transition

$$H(1 - L) > T$$

Pasive   Active

Local and global parameters

(b) System with police

Agent state transition

$$G - N_R > T$$ Police

Pasive   Active   Jailed

$$J_{max}$$
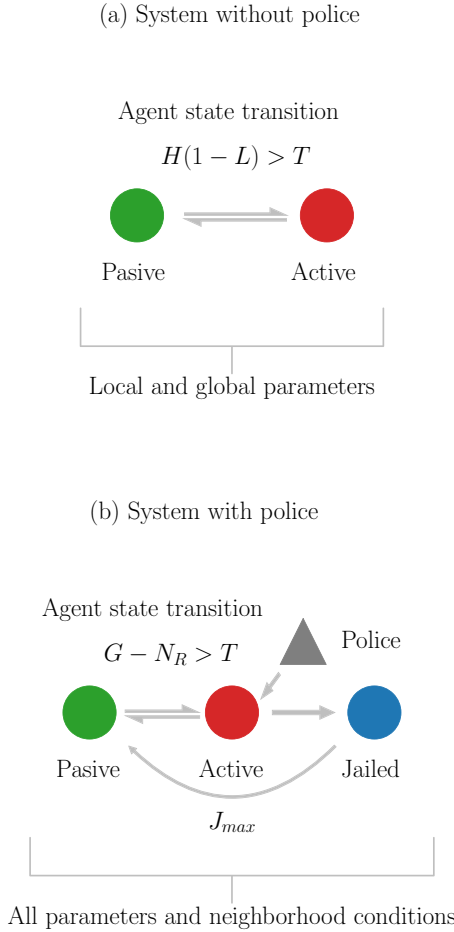
All parameters and neighborhood conditions

FIG. 1. Schematic visualization of agents state changes in the Epstein's model for a civil disorder. For the system without police officers, the citizen agents can switch between two states depending on their local and global parameters. For the system with police officers, the agents can change among three states depending on their local and global parameters and neighborhood conditions. The switch from the active to jailed state is a product of the interaction with the police officers' agents.

we define:

$$C_\alpha = \frac{N_\alpha}{N}, \qquad (2)$$

where N_α denotes the number of agents in the state α and N the number of agents in the system. As usually used in opinion dynamics models [12,36], $\sum C_\alpha = 1$ and we distinguish the following phases:

(i) The disordered phase, when all agent states are of a similar concentration in the system.

(ii) The ordered phase, when one agent state is majority over the others.

(iii) The consensus phase is when the system reaches a particular ordered phase where all agents have the same state.

It is essential to note what we define order from the opinion dynamics perspective to describe order-disorder transitions. Thus, by order, we refer to a macroscopic pattern in which we could find a majority opinion state. We do not refer to the common idea related to public order as the absence of criminal

or political violence in society. Moreover, these definitions are convenient because they allow us to identify a macroscopic state with the agents' state and the system's dynamics.

2. Energy

Now, we introduce a global quantity that allows us to analyze and interpret the system based on the macroscopic states that emerge from the agents' states of the system. Hence, we introduce a Potts-like energy function [40],

$$E[\alpha] = -\frac{1}{2vN} \sum_{i=1}^{N} \sum_{j \in V_i} J_{ij} \delta(\alpha_i, \alpha_j), \qquad (3)$$

where the symbol $\sum_{j \in V_i}$ means the sum over all neighbors j of the agent i with the same state α. Here J_{ij} is called the interaction strength. However, we will take $J_{ij} = 1$ (for all i and j), due to the characteristics original model. In other words, in this work, we do not consider different interaction strength values. $\delta(\alpha_i, \alpha_j)$ is a Kronecker delta, i.e., $\delta(\alpha_i, \alpha_j) = 1$ if $\alpha_i = \alpha_j$ and zero for all $\alpha_i \neq \alpha_j$.

Note that the energy functions introduced in the Schelling model [21,22] are like the Ising model because the models have two possible states. For the case of the Sakoda model [23] and this model, it is more natural to use the Potts energy because these models have more than two states.

Besides, we can observe that the energy definition shows the absolute minimum or ground state when all the agents in the system are in the same state. On the other hand, the energy may reach the maximum energy when the system takes a chessboard aspect. This behavior is convenient because it allows us to establish analogies or interpretations of the dynamics.

III. SIMULATIONS AND PHASE DIAGRAMS

This paper aims to characterize the Epstein's model for a civil disorder in a one-dimensional lattice with periodic boundary conditions as a first approximation to understand the dynamics of social protests. Thus, we perform simulations on a one-dimensional lattice with $N = 2^8$ and $N = 2^{10}$ sites, considering a system without and with police officers. The first one has agents with two possible states, active and passive. The second one has police officers then the agents can be active, passive, or jailed.

Furthermore, to study the effects of interactions in the system dynamics, we consider agents with visions one and seven interacting in Moore and random neighborhoods. Note that vision determines the neighborhood's size, so when we consider a one-dimensional lattice, the vision represents the number of pairs of agents to consider to evaluate an agent's state switch. For example, when the agent's vision is one, the Moore neighborhood of the agents consists of its nearest neighbors. Thus each agent considers two sites, one to their left and one to their right. When the agent's vision is seven, the Moore neighborhood of the agents counts seven sites on the left and seven sites on the right, with 14 agents in total.

On the other hand, a random neighborhood is when an agent can randomly select other agents to form their neighborhood. Then, when the agent's vision is one, it chooses
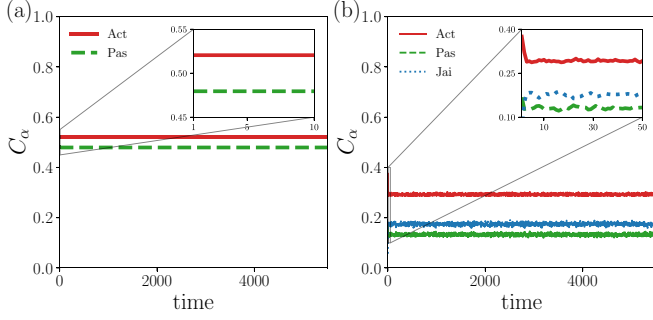
FIG. 2. Agents' concentration variation on time. Panel (a) corresponds to a system without police officers and (b) a system with police officers. In both cases, the system reaches a stationary state quickly.

two agents randomly. When its vision is seven, the agent can randomly select 14 agents from the lattice.

We have not used Epstein's motion rule from the original model in this work. So the agents occupying the whole of the one-dimensional lattice and the system can enter a stationary regime, as shown in Fig. 2. As we can see, the dynamics converge quickly around the same value for the agents' concentration in any state and remain constant on average after some time. For this reason, in this paper, we can study the asymptotic properties of the dynamics and characterize the model on the steady state. In this way, we use 20 realizations with 5500 time steps for all the study cases. In each realization, all agents have different initial positions and state variables. Then, we discard the first 500 time steps to obtain a steady state. Finally, we calculated the average quantities over 20 realizations to characterize the model.

A. System without police officers

When we study the system without police officers, the state switch equation (1) changes to

$$H(1-L) > T. \tag{4}$$

Hence, the agents' state only depends on its local parameters and is independent of their neighborhood. Therefore,

changes in the system's dynamics depend on the threshold and the initial simulation's conditions. To study the whole system, we run simulations for threshold and legitimacy values between 0.00 and 0.99 with a step of 0.01 for both variables.

To obtain a first idea of the model dynamics, we study the concentration and energy variations for different threshold fixed values. We observe the variation of the concentration of agents when the legitimacy increases in Fig. 3. With low values to legitimacy, the active agents are predominant. Then as legitimacy increases, the passive agents are predominant. When the threshold is $T = 0.10$, the system's dominant state changes, as shown in Fig. 3(a). Note that when the legitimacy is $L < 0.80$, the active agents predominate, when $L = 0.80$, the concentration for two states are similar, and when $L > 0.80$, the passive agents are dominant. For $L \geqslant 0.90$, all agents of the system are in the passive state. We can see a translation of the point of concentration similarity and the point when all agents of the system are passive states when the threshold increases in Figs. 3(a), 3(b), 3(c), and 3(d) figures. The translations of these points indicate transitions in the system. It is important to note that all showed results collapsed on the same curves for all visions, neighborhoods, and sites considered in our simulations. This behavior is because the state switch equation (4) is independent of the neighborhood and indicates the system's dynamic depending on the threshold and the initial simulation's conditions.

We present the global average energy versus legitimacy with different threshold values in Fig. 4. When the threshold is $T = 0.10$, the energy starts with a lower value of around $\langle E \rangle \approx -0.8$. As the legitimacy increases, the energy reaches a maximum value around $\langle E \rangle \approx -0.5$ when the legitimacy is $L = 0.80$. Next, the energy converges to the minimum value when the legitimacy is $L \geqslant 0.90$. The initial energy value is a local minimum and indicates when the active agents predominate. Then, when the agents' concentrations are similar, the energy reaches a maximum. Finally, the energy minimizes when the system has only passive agents and reaches the absolute minimum or the ground state.

On the other hand, when the system possesses higher thresholds values, the energy reaches a maximum and then converges to the minimum quickly. Besides, we can see a
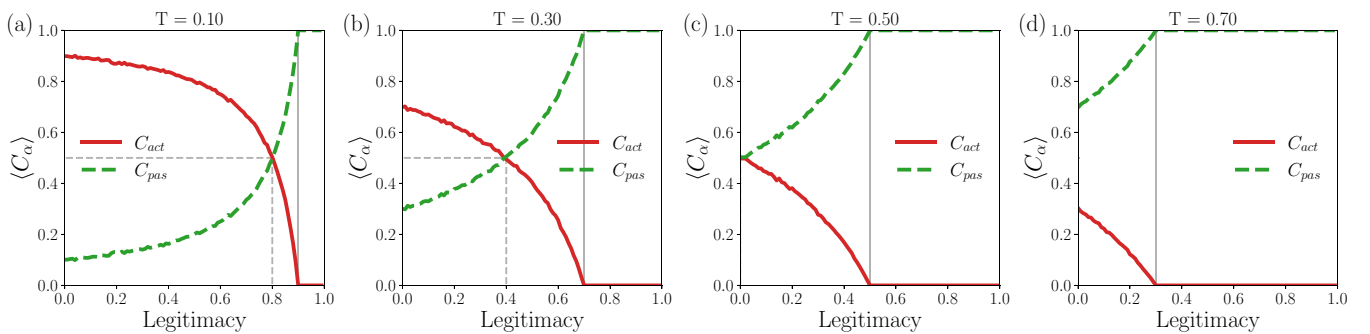

FIG. 3. Concentration variations for different threshold fixed values in a system without police officers. With low values to legitimacy, the active agents are predominant, but as legitimacy increases, the passive agents dominate. There are points when active and passive agents have a similar concentration (segmented lines) and when all the agents in the system are passives (solid lines). As the threshold increases, we can see a translation of these points suggesting a transition. This figure shows simulations results for a one-dimensional lattice with $N = 2^{10}$ sites, Moore neighborhood, and vision one. However, the results obtained for all visions, neighborhoods, and sites considered collapse on the same curve.
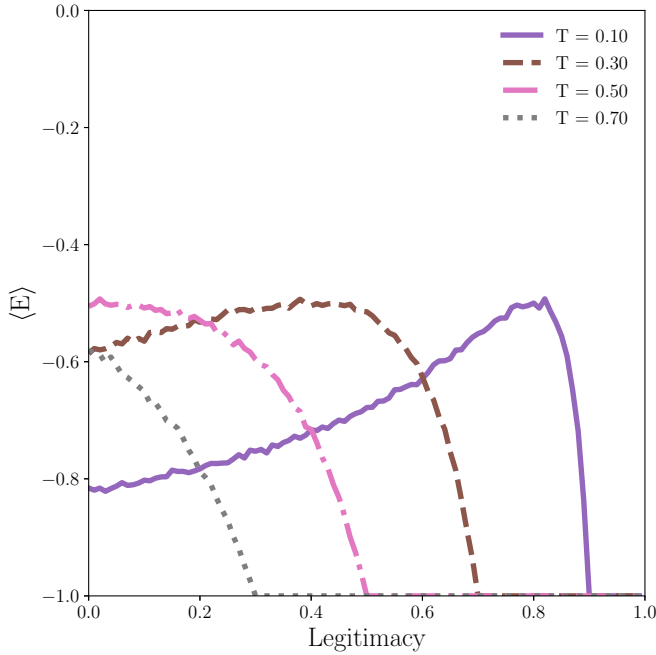
FIG. 4. The global average energy variations for different threshold fixed values in a system without police officers. The energy shows two minimum values. The first is a local minimum, where the active agents predominate in the system. The other one is a global minimum or the ground state, and all agents in the system are passive. On the other hand, the system reaches a maximum value when the active and passive agent concentrations are similar. This figure shows simulations results for a one-dimensional lattice with $N = 2^{10}$ sites, Moore neighborhood, and vision one. However, the results obtained for all visions, neighborhoods, and sites considered collapse on the same curve.
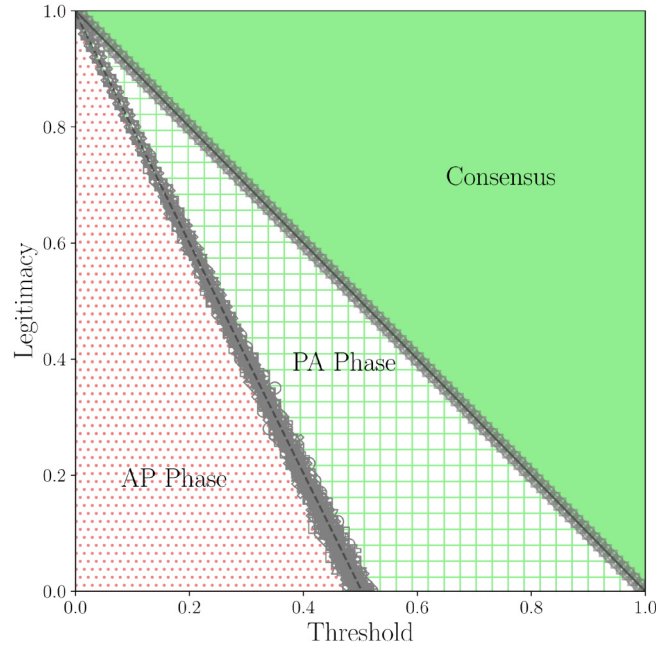

FIG. 5. Phase diagram for a system without police officers. Phases AP and PA are ordered phases with a majority agent state. Active agents are predominant in the AP phase and passive in phase PA. The dashed line between these two phases shows when the system has similar concentrations and is therefore disordered. The consensus phase is a particular ordered phase when all agents are passives. The solid line shows the transition from the order with a majority to consensus. Every point in this diagram corresponds to an L and T value when the concentrations of active and passive agents are similar or when the system reaches a consensus. The points for all visions, neighborhoods, and sites considered collapse on the same curve.

translation of these energy points as the threshold increases suggesting a transition.

To build the phase diagram, we search the coordinates (T, L) where the agents' concentrations are similar, and all agents are passive. As shown in Fig. 5, these points define the phase boundaries. Following the phases described in opinion models, the results show order-disorder transitions. Phase AP and PA are ordered phases with a majority agent state. Active agents are dominant in phase AP and passives in phase PA. As a result of crossing the dashed line between these two phases, we observe a disordered phase with similarly active and passive agents concentrations. Note that our numerical results are consistent with the assumption for an average agent. We can find the critical legitimacy to obtain equal concentrations of active and passive. As a result, we obtained this dashed line of critical legitimacy $L_c = 1 - 2T$. The consensus phase is a particular ordered phase when all agents are passive. The solid line shows the transition from majority order to consensus order.

To observe the system transition, we study the stationary probability density function of the agents' concentration. We show distributions for a system with Moore neighborhood with vision one and $T = 0.10$ in Fig. 6 because we observe the same behavior independent of the neighborhood, vision, and sites considered in our simulations. We can see the

system transition from the order with active agents majority in $L = 0.75$ to a disordered phase in $L = 0.80$ and then a change to order with passive agent majority in $L = 0.85$, in Figs. 6(a), 6(b), and 6(c). As for legitimacy increases, we can observe a transition to a consensus in $L = 0.90$ in Figs. 6(d), 6(e), and 6(f).

B. System with police officers

To study the model with police officers, we used legitimacy values and the concentration of police officers from 0.00 to 0.99 with a step of 0.01 for both variables. We vary the concentrations of police officers as an initial condition because its value determines the system's dynamics. The police officers' role is to dissuade a social protest, preventing citizen agents from becoming active agents and arrest active agents in the system producing jailed agents. Besides, the police officers' action depends on the vision, so we study the system separately with two different visions. On the other hand, the police officers' inclusion makes the change of the agents state depends on the state parameters and the neighborhood conditions, as we can see in Eq. (1). We used a fixed threshold value in $T = 0.10$ for these simulations because its role is to determine a limit value to the state switch equation. Furthermore, this value coincides with Epstein's reported value in the
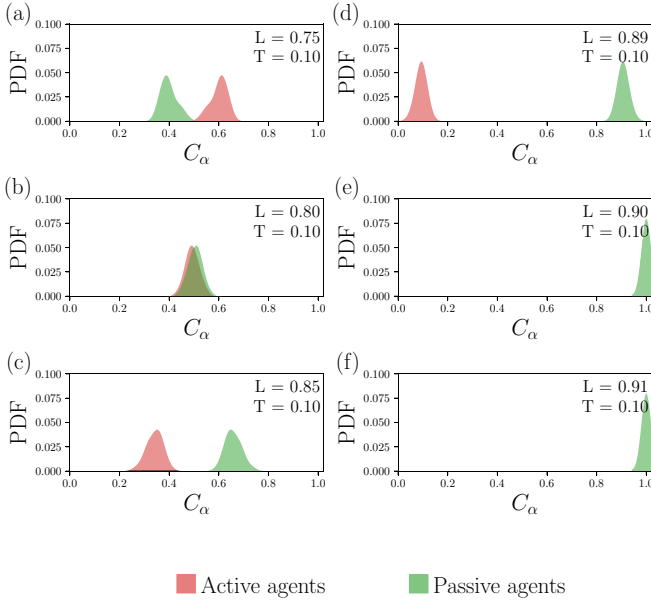
FIG. 6. Stationary probability density function of the agents' concentration for a system without police officers. We see two transitions when the system increases legitimacy and threshold fixed. A continuous transition from AP phase to PA phase across a disordered phase in $L = 0.80$, as seen in (a), (b), and (c). Then, a continuous transition from PA phase to consensus, as seen in (d), (e), and (f). This figure shows simulations results for a one-dimensional lattice with $N = 2^{10}$ sites, Moore neighborhood, and vision one. However, these behaviors are the same for all visions, neighborhoods, and sites considered in our simulations.

five study cases from the original model for a generalized rebellion. Thus, this value allows us to characterize this context with a more significant parameters set.

1. Results with vision one

We show the agents' concentration variations for different values of police officers' concentrations and vision one in Fig. 7. When the interactions occur in a Moore neighborhood, we observe a variation of the concentration of agents when the legitimacy increases in Figs. 7(a), 7(b), 7(c), and 7(d). With low values to legitimacy, the active agents are predominant. Then as legitimacy increases, the passive agents are dominant. The jailed agents' concentration depends on the police officers' concentration. Then their variations only occur as police officers' concentration increases and produce a change of active agents concentration. With interactions in the random neighborhood, the jailed agent concentrations have predominant values for low legitimacy, as we can see in Figs. 7(e), 7(f), 7(g), and 7(h). Then, passive agents are predominant as legitimacy increases. The active agent concentrations depend on the police officers' concentrations and decrease as the number of police officers in the system increases.

Now, we can observe the global average energy versus legitimacy for a Moore and random neighborhood in Figs. 8(a) and 8(b), respectively. On the one side, when the

interactions occur in the Moore neighborhood, and police officers' concentration equals $C_{\text{pol}} = 0.10$, the energy starts around $\langle E \rangle \approx -0.6$. Next, it has an increasing behavior to around $\langle E \rangle \approx -0.4$, and a legitimacy value is close to $L = 0.80$. Then, the energy decreases quickly to the lower value $\langle E \rangle \approx -0.8$. We observe the local energy minimum when active agents are predominant. Then, the energy maximum shows when the agents' states have similar concentrations. In particular, the active and passive agents concentration is approximately 0.4, and jailed agents and police officers concentrations are close to 0.10. Finally, we see the global energy minimum when all agents are passive. Note that this global minimum is not absolute because the police officers' concentration equals $C_{\text{pol}} = 0.10$. As the police officers' concentration increases, the energy maintains a constant value before reaching the minimum energy value. Note that this global energy minimum increases as a police officer's concentration. On the other side, for a random neighborhood, the energy started around $\langle E \rangle \approx -0.4$, with a police officer concentration equal to 0.10, 0.30, and 0.50. Next, the energy had an increasing behavior until it reached a maximum around $\langle E \rangle \approx -0.3$ and eventually converged rapidly to different energy minima. We observe similar behavior in both neighborhoods for police officer concentration is $C_{\text{pol}} = 0.70$. Furthermore, the global minimum of energy increases as a police officers' concentration increases for all cases.

As we noticed in the results for a system without police officers, in the systems with police officers, there are also points where the state concentrations are similar. Their positions move as the police officers' concentrations increase. These translations suggest a change in the state predominant in the system. Furthermore, there is a point when the three states of the systems are similar, configuring the order-disorder transitions like opinions models [12,36].

To verify this idea, we search for phase boundaries defined by the points in which the agents' concentrations are similar and when all agents reach a passive state. Every point corresponds to police officers' concentration and legitimacy value and depends on a pair of similar agent states. Thus, the C_{ap} coordinate (●) is when the concentrations of active and passive agents are similar. The C_{ja} coordinate (★) indicates similarity in the jailed and passive state, and the C_{ja} point (■) when jailed and actives agents are similar concentration. Each point formed a curve defining different regions on the phase diagram shown in Fig. 9. For both the Moore neighborhood in Fig. 9(a) and the random neighborhood in Fig. 9(b), we observe phases classified according to the transitions described for the system without police officers. There are six ordered phases with a majority state. Each one has a label indicating the order of the predominant state. For example, the PAJ phase has dominant passives agents, followed by active and the jailed agents, and so on for the other phases. The system reaches a consensus in the passive phase when all agents are passives and legitimacy equal 0.90. This value is determined by the threshold value selected. The black region indicates when there are only police officers in the system.

The system reaches a disordered phase labeled a triple point when the active, passive, and jailed states are in similar
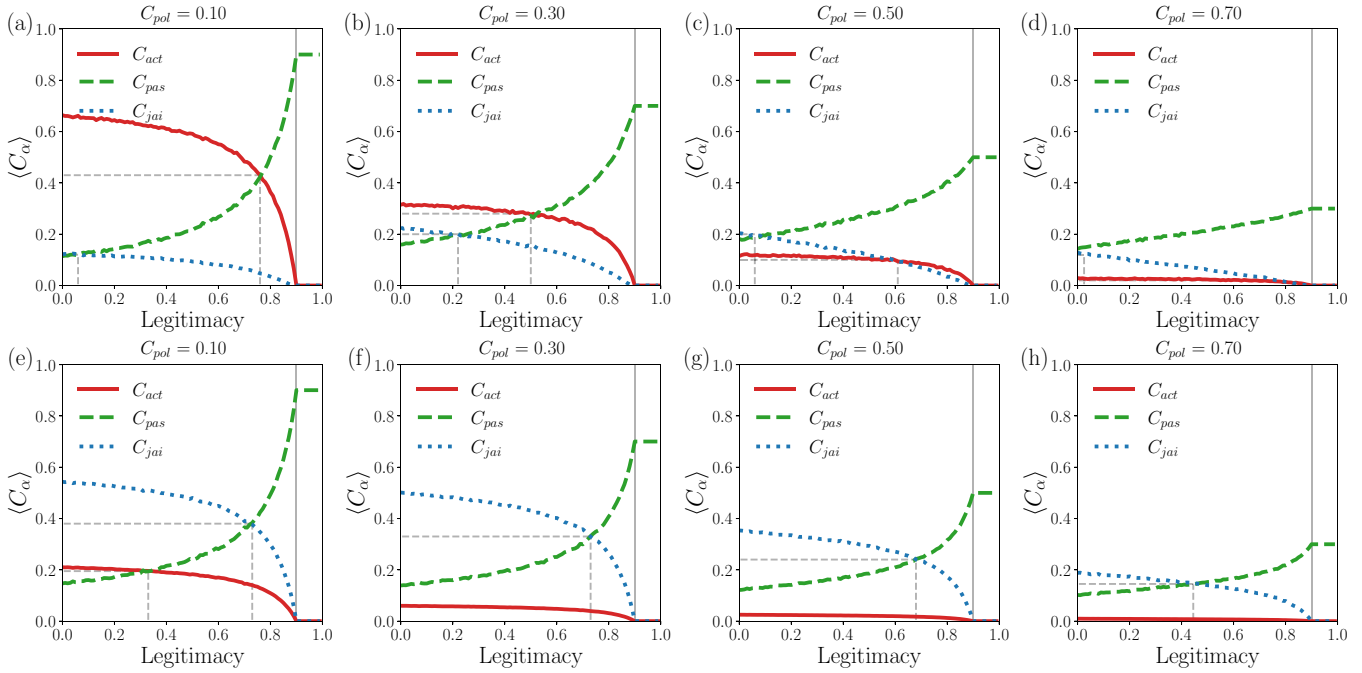
FIG. 7. Agents concentration variations in a system with police officers for different values of police officers concentrations, legitimacy, vision one, and threshold fixed. When the interactions occur in a Moore neighborhood, the predominance of active agents decreases when the police officers' concentrations increase because of increased jailed agents. As a result, we can see a translation of the concentration similarity points (segmented lines) in (a), (b), (c), and (d). These translations show changes of the predominant state and the existence of a point in which the three states of the system are similar, suggesting a phase change. In a random neighborhood, we can see the same dynamics of translation of the concentration similarity point in (e), (f), (g), and (h). However, police officers can capture more active agents because of the random selection of their neighborhoods. Thus, the predominance of jailed agents results until the system reaches a high legitimacy. The vertical solid line depends on the fixed threshold and indicates when the system reaches a consensus. This figure shows simulations results for a one-dimensional lattice with $N = 2^{10}$ sites, but with $N = 2^8$ sites, we observe the same result.
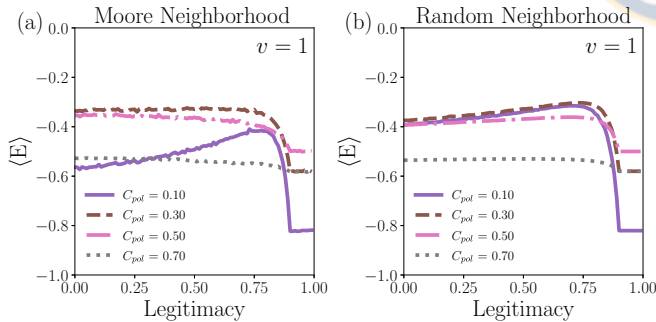

FIG. 8. The global average energy variations in a system with police officers for different values of police officers concentrations, legitimacy, vision one, and threshold fixed. In a Moore neighborhood, the energy reaches a minimum value for low police officers concentrations and increases as legitimacy increases, as shown in (a). When the system reaches a consensus, the energy shows a global minimum. As it increases the police officers' concentration, it is possible to observe that the energy maintains a constant value before reaching a consensus of passive agents. The maximum energy value indicates when the majority agent states have similar concentrations. In a random neighborhood, the energy has a constant value and increases as legitimacy increases, as shown in (b). In both cases, the global minimum of energy increase as a police officer's concentration increases. This figure shows simulations results for a one-dimensional lattice with $N = 2^{10}$ sites, but with $N = 2^8$ sites, we observe the same result.

concentrations. The position of this point and the regions of the phases depends on the neighborhood. Most phases have an observable area for the Moore neighborhood. The part where jailed agents predominate is minor because police officers can only capture active agents among their nearest neighbors. In contrast, police officers are more likely to catch an active agent in a random neighborhood. Thus, we can observe a translation of the triple point and an increase in the regions' size with predominant jailed agents and a decrease in the areas where active agents are dominant. It is important to note that now it is more difficult to see the order-disorder transition. Unlike the system without police officers, where we observe the transition for all parameters, in the system with police officers, the disordered phase is only a point in this diagram.

We can observe the system transition with the stationary probability density function of the agents' concentration in Fig. 10. As for legitimacy increases, the system changes from the disordered phase to an ordered phase with a passive state majority. It then reaches a consensus phase in a Moore neighborhood in Figs. 10(a), 10(b), and 10(c). For a random neighborhood, the transition from the ordered phase with the active state to a disordered phase, then the consensus phase, as we see in Figs. 10(d), 10(e), and 10(f). The final passive agents' concentration depends on the police officers's concentration fixed to observe the transition. So, in the Moore neighborhood, the final concentration is 0.6 and,
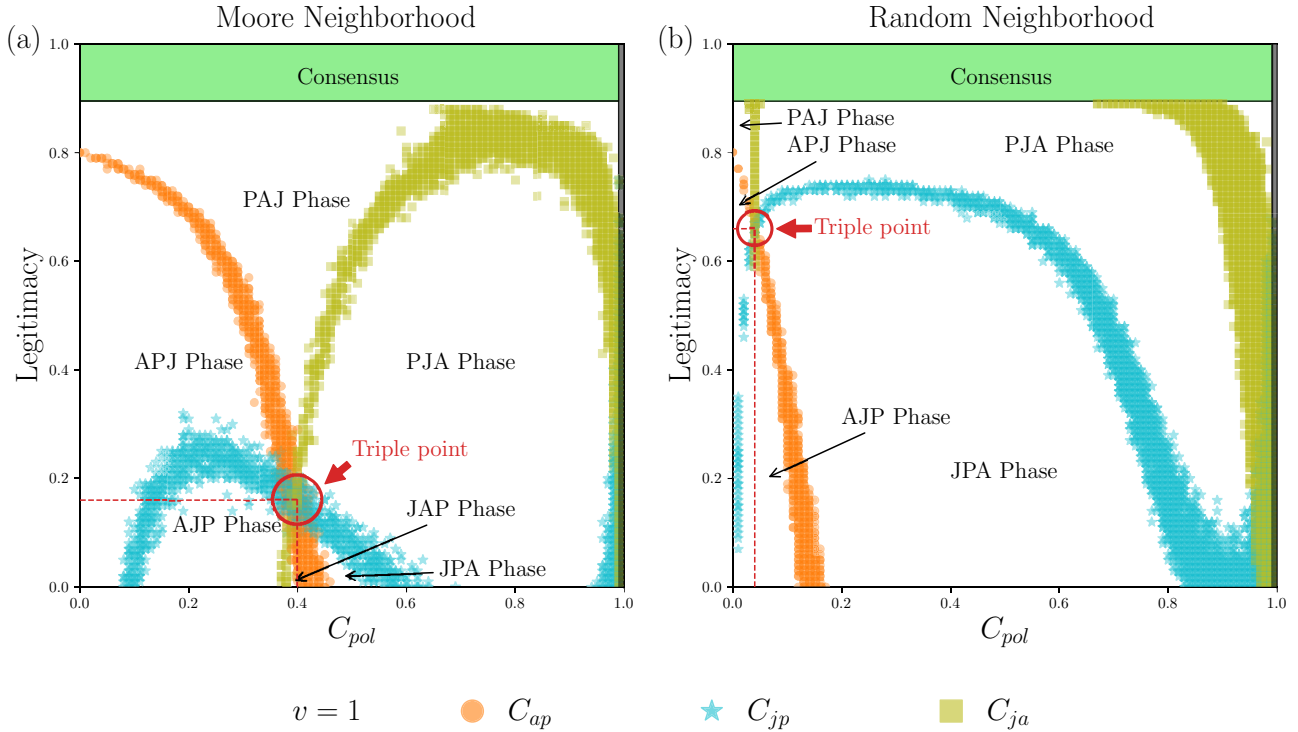
FIG. 9. Phase diagrams for a system with police officers and vision one. Every point in this diagram corresponds to police officers' concentration and legitimacy value when the different states are at similar concentrations. For example, the C_{ap} points (●) are the points at which the concentrations of active and passive agents are similar. These points determine different ordered phases with a majority, and each label indicates the order of the predominant state. For example, in the PAJ phase, passive agents are predominate, followed by active and jailed agents, and so on. The triple point in these curves' intersection indicates when the three states are in similar concentrations, so we observe a disorder in the system. The system reaches a consensus when legitimacy is greater than or equal to 0.90. The black region indicates when there are only police officers in the system. With Moore neighborhood show six differents order phases with a majority, one point where the system reaches a disorder, and a consensus phase. With random neighborhoods, order phases with majority change because police officers can capture more active agents and predominantly jailed agents. However, the system reaches a consensus phase in the same conditions because the fixed threshold determines this change. This figure shows simulations results for a one-dimensional lattice with $N = 2^{10}$ sites, but with $N = 2^8$ sites, we observe the same result.

in a random neighborhood, the passive agents' concentration is 0.9.

2. Results with vision seven

We show the agents' concentration variations for different police officers' concentrations and vision seven in Fig. 11. When the interactions occur in a Moore neighborhood, we can see a variation of the concentration of agents when the legitimacy increases in Figs. 11(a), 11(b), 11(c), and 11(d). With low legitimacy, active agents exist, but jailed agents are predominant. Then as legitimacy increases, the passive agents are dominant. As police officers' concentration increases, the active agents' concentration minimizes. Passive agents increase constantly, and jailed agents decrease.

With interactions in the random neighborhood, the jailed agent concentration has predominant values for low legitimacy, as we can see in Figs. 11(e), 11(f), 11(g), and 11(h). Furthermore, the active agent's concentration starts initially with a low value, and passive agents are predominant as legitimacy increases. As police officers' concentration increases, the concentration of active agents disappears, passive agents

increase constantly, and jailed agents decrease. This behavior indicates a change in the importance of the police officers' role to dissuade a protest. When the agents have vision seven, the relevant police officers' role is to prevent citizen agents from becoming active and prevent the emergence of a protest.

Now, we can observe the global average energy versus legitimacy for a Moore and random neighborhood in Figs. 12(a) and 12(b), respectively. On the one side, when the interactions occur in the Moore neighborhood, and police officers' concentration equals $C_{pol} = 0.10$, the energy starts close to $\langle E \rangle \approx -0.4$. Next, it maintains a constant behavior until the energy decreases quickly to the lower value of around $\langle E \rangle \approx -0.8$. On the other side, in the random neighborhood, and police officers' concentration equals $C_{pol} = 0.10$, the energy has a lower value, around $\langle E \rangle \approx -0.3$. Then increases to reach a maximum value close to $\langle E \rangle \approx -0.4$ and decreases quickly to the lower value around $\langle E \rangle \approx -0.8$. Note that, in both cases, we observe the initial energy value when jailed agents are predominant. Then, the energy maximum shows when the jailed and passive agents concentration is approximately 0.4, and police officers concentration is 0.10. Finally, we see the global energy minimum when all agents are passive. However, in both neighborhoods, as police officers' concentrations
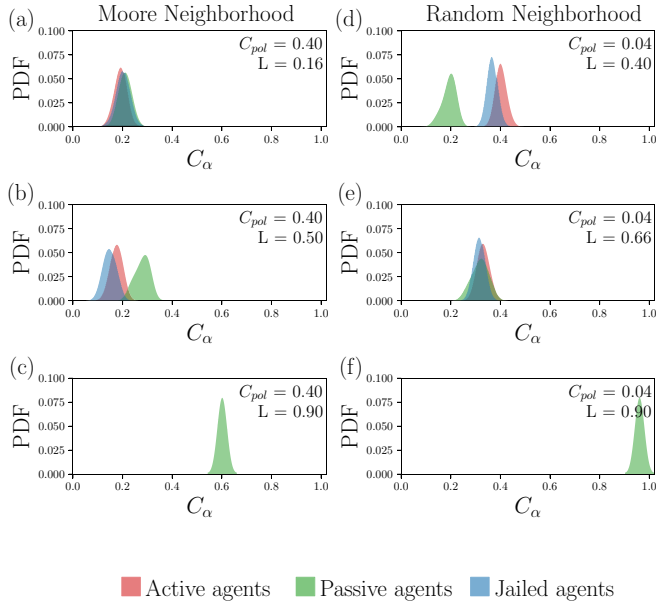
FIG. 10. Stationary probability density function of the concentration of agents for a system with police officers and vision one. We see two transitions when the system increases legitimacy and fixes police officers' concentration. In the Moore neighborhood, we can see a disordered phase, then order with passive agents majority, and, finally, a consensus phase in (a), (b), and (c). The same dynamics occur in the random neighborhood, with an order with active agents majority, a disordered phase, and then a consensus phase in (d), (e), and (f). This figure shows simulations results for a one-dimensional lattice with $N = 2^{10}$ sites, but with $N = 2^8$ sites, we observe the same result.

increase, the energy has a decreasing behavior until reaching the lower energy value. We note that the global energy minimum increases as a police officer's concentration for all cases.

For low police officers concentrations, points where the state concentrations are similar exist, suggesting changes in the predominant state in the system. To search for a point where the system shows an order-disorder transition, we built a phase diagram. Every point corresponds to police officers' concentration and legitimacy value and depends on a pair of similar agent states. Then, we can observe the phases formed for the system with police officers's vision seven in Fig. 13. We see the same six phases observed in the system for the Moore neighborhood with vision one in Fig. 13(a). The region's size for every phase changes notably because of the increase in agents' vision and the police officers' activity. As a result, we note that the regions with the predominance of passive agents are more significant than the others. Besides, we can observe a triple point where the disordered phase occurs, the area where the system only has police officers, and a consensus phase with only passive agents. For the random neighborhood, the effect of vision seven is more significant, as shown in Fig. 13(b). Although the size of the regions dominated by passive agents is similar to those of the Moore neighborhood, we can notice that the JAP phase and the point at which all concentrations are similar disappears. The JAP phase has most jailed agents followed by active and passive agents. As a result, we observe that there is no order-disorder transition. However, the system change between different order phases with a majority state depending on the police officers' concentration and legitimacy values. It reaches a consensus when all agents are in the passive state at the value of legitimacy is 0.90 because this depends on the threshold value fixed at the beginning of simulations.

We show the stationary probability density function of the agents' concentration in Fig. 14 to observe the system transition. For the Moore neighborhood, we selected the police officers' concentrations 0.06 and varied the legitimacy. We can see, as legitimacy increases, the system change from the disordered phase to an ordered phase with a passive state majority in Figs. 14(a) and 14(b). Then reaches a consensus phase in Fig. 14(c). We selected a lower value of police officers' concentrations for the random neighborhood to observe the possibility of finding similar concentrations for the three states as legitimacy increases. Nevertheless, only find order with majority phases, as shown in Figs. 14(d) and 14(e), and the consensus phase in Fig. 14(f).

IV. DISCUSSION AND CONCLUDING REMARKS

This paper studied the one-dimensional civil disorder model with the whole lattice occupied to characterize their evolution on the steady state. To do this, we performed extensive numerical simulations of the model with and without police officers, considering visions one and seven in Moore and random neighborhoods to study the effects of interactions on the system's dynamics. We used the agent state concentration and introduced a Potts-like energy function as global quantities to characterize the model.

In the system without police, the dynamics only depend on values assigned as initial conditions. One of them is hardship, a local parameter uniformly distributed between values zero and one for each agent. In the model, this parameter allows for a heterogeneous society of agents. The other two, legitimacy and threshold, are global parameters that we use as control parameters for our simulations. The threshold is a quantity defined as non-negative and determines a limit for an agent's state change.

Given that the product of hardship and legitimacy symbolizes the grievance in the state change equation, we interpret this threshold as a tolerance for grievance. The higher the threshold value, the more disposed agents are to tolerate grievance before rebelling against authority. This parameter takes different values between zero and one, which we interpret as an essential property of a community that depends on its culture or way of life. Thus, we find communities with a low threshold and quickly protest for a grievance and communities with a very high threshold that lives with a minimal grievance and does not rebel against authority. On the other hand, legitimacy is the community's perception of the regime or the system's authority. Thus, a low legitimacy produces more significant grievance, and high legitimacy favors the passivity of the system.

In our results for the variations in the concentration of agents as a function of legitimacy, we observe that active agents are predominant for low values of legitimacy. As legitimacy increases, we find a point where the concentrations of
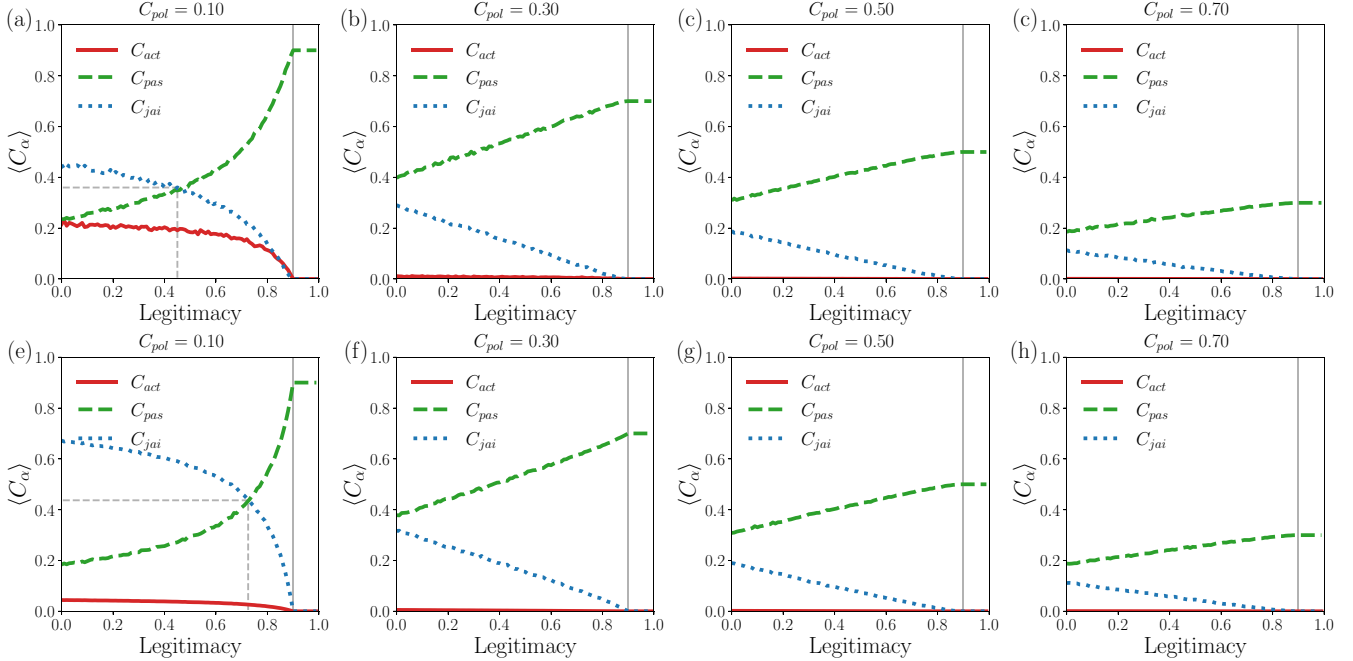
FIG. 11. Agents concentration variations in a system with police officers for different values of police officers concentrations, legitimacy, vision seven, and threshold fixed. The first and second-row figures show interactions in a Moore and random neighborhood respectably. We can see the predominance of jailed agents with low police officers concentrations in Moore and random neighborhoods in (a) and (e). Note that the concentration of jailed agents is more significant in the random neighborhood because police officers can capture more agents. However, as the police officers' concentrations increase, we can see the predominance of passive agents for both kinds of neighborhoods. Besides, we observe the same dynamics of translation of the concentration similarity (segmented lines) as in a system with vision one but only for low police officers' concentrations. The vertical solid line depends on the fixed threshold and indicates when the system reaches a consensus. This figure shows simulations results for a one-dimensional lattice with $N = 2^{10}$ sites, but with $N = 2^8$ sites, we observe the same result.

active and passive agents are similar, and then passive agents become predominant. According to the legitimacy variation, the predominance changes depend on the threshold values. We note that passive agents are always dominant for threshold
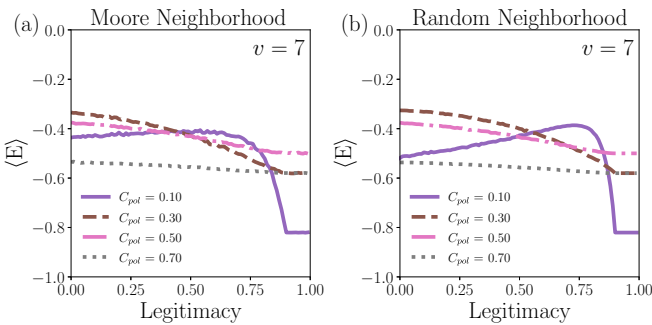

FIG. 12. The global average energy variations in a system with police officers for different values of police officers concentrations, legitimacy, vision seven, and threshold fixed. As for legitimacy increases, the energy reaches a global minimum in Moore and random neighborhoods. However, the energy only shows a maximum with police officers' concentration values lower or equal to 0.10. Then, as police officers' concentration increases, the energy constantly decreases until the system reaches a consensus. Besides, in both cases, the global minimum of energy increase as a police officer's concentration increases, as we can see in (a) and (b). This figure shows simulations results for a one-dimensional lattice with $N = 2^{10}$ sites, but with $N = 2^8$ sites, we observe the same result.

values greater than 0.50 and coexist with active agents. However, for high legitimacy values, we notice that all agents in the system become passive. These changes in the predominance of a state among the agents are indicators of phase changes. We build the phase diagram based on the concentration points of similar agents or when all agents are passive, and we observe order-disorder transitions. We identify the AP and PA phases as order phases with a majority state. Active agents are predominant in the first and passive agents in the second. We find a disordered phase in which the agents' concentrations are similar when crossing between these phases. When all the agents are passive, the system reaches the consensus phase. We study the transitions with the stationary probability density function of the concentration of the agents, and we observe a typical scenario of continuous transitions [36].

The energy shows one maximum and two minimum values for low threshold values. One of them is a local minimum, where the active agents predominate in the system. The other is a global minimum or ground state, where all agents in the system are passive. The system reaches the maximum value when the concentrations of active and passive agents are similar. In contrast, when the system has higher threshold values, the energy is maximum and quickly converges to a minimum. We can see the translation of these energy points as the threshold increases, allowing us to identify the phase changes. Thus, the local minimum indicates the ordered phase with an active agents majority, the maximum the disordered
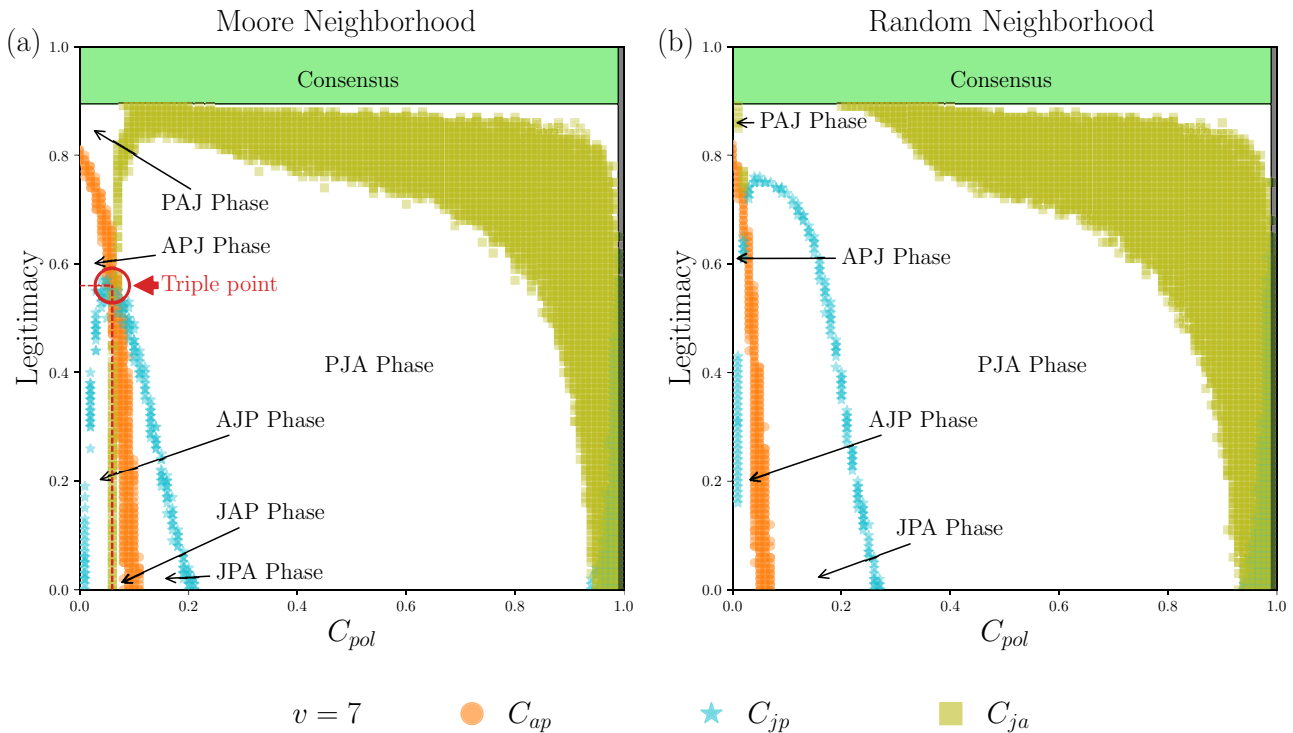
FIG. 13. Phase diagrams for a system with police officers and vision seven. Every point in this diagram corresponds to police officers' concentration and legitimacy value when the different states are at similar concentrations. We can see the same phases and transitions order-disorder from the system with vision one in the Moore neighborhood in (a). Nevertheless, the size of each phase change. With random neighborhood, the disordered phase and the JAP phase disappear due to increased police officers' action with vision seven, as shown in (b). In both cases, the system reaches a consensus when the legitimacy is greater than or equal to 0.90 because the fixed threshold determines this change. The black region indicates only police officers in the system. This figure shows simulations results for a one-dimensional lattice with $N = 2^{10}$ sites, but with $N = 2^8$ sites, we observe the same result.

phase, and the global minimum or ground state is the consensus phase.

The local energy minimum observed for threshold values greater than zero and less than 0.50 is a metastable point. The active agents' predominance generates energy and stops the system from reaching the ground state. In other words, the system is not entirely stable due to grievance. However, as legitimacy increases, the system can reach a consensus. On the other hand, when the threshold values are greater than 0.50, the local minimum disappears and becomes an energy maximum. The maximum is an unstable point for all threshold values because the system can fall to the local or global minimum depending on how legitimacy varies. Since the concentrations of active and passive agents are similar, the system increases energy due to grievance generating a scenario of instability comparable to a polarized society. In this context, we understand polarization as a situation of equal opinion searching for a consensus. It is important to note that the maximum energy that our results show is the maximum possible given the initial conditions of random positioning of the agents. However, it is possible to find an absolute maximum for the energy by positioning the agents deliberately to form the checkerboard appearance.

Finally, the global minimum is a stable point of the system since all the agents are in the same passive state in the consensus phase. When threshold values exceed 0.50, the system can reach the global minimum for low legitimacy values.

Nevertheless, for threshold values less than 0.50, we find that reaching consensus requires higher legitimacy values. Indeed, the greater the tolerance for grievance, the less legitimacy is required to reach consensus. This result reflects the existence of societies with high thresholds, in which reaching consensus requires low legitimacy values, unlike other societies in which reaching or maintaining consensus requires high legitimacy.

From these results, we can reveal that the principle underlying the dynamics of the model is a principle of minimum grievance, equivalent to that observed in the model of worker protest in a factory. This principle allows us to interpret that the system seeks minimal grievance or a consensus. Naturally, reaching and maintaining consensus requires high values of legitimacy or high values of tolerance, the latter being the one that determines the value of legitimacy necessary in a heterogeneous society. Thus, the emergence of protests is due to global conditions of legitimacy or threshold that generate a grievance, increasing the system's energy. This grievance generates energy variations that can lead the system to instability or metastability.

In the system with police, the agents can be active, passive, or jailed, and the dynamic depends on the vision and the neighborhood. For this reason, we study the system separately, considering two different views. We fix the threshold value and consider the legitimacy and concentration of police officers in the system as a control parameter because the police officer's role is to deter a protest. Besides, the agents' state
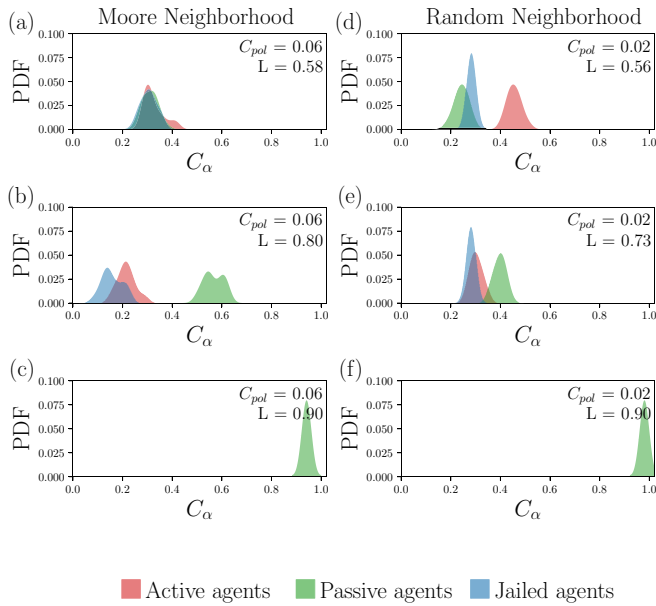
FIG. 14. Stationary probability density function of the concentration of agents for a system with police officers and vision seven. We can see two transitions in the Moore neighborhood when the system increased legitimacy and fixed police officers' concentration. A disordered phase, then orders with passive agents majority, and finally, a consensus phase in (a), (b), and (c). However, in the random neighborhood, only observe order-with majority phases and then a consensus phase in (d), (e), and (f). This figure shows simulations results for a one-dimensional lattice with $N = 2^{10}$ sites, but with $N = 2^8$ sites, we observe the same result.

switch depends on the number of active agents and police officers in their neighborhood.

The concentration of agents allows us to identify the predominance of states in the system. We find that active agents predominate in the low legitimacy regime only when the system has vision one and Moore neighborhood. For the rest of the cases, the jailed agents always predominate. As in the system without police, as legitimacy increases, the predominant state changes, and passive agents become the majority. In addition, we again note points where concentrations are similar. For legitimacy values greater than or equal to 0.90, all agents in the system are passive. It is essential to mention that the increased concentration of police officers in the system facilitates the predominance of passive agents. Therefore, in general, the concentration of passive agents increases as legitimacy increases. In particular, this increase is linear in the system with vision seven because the agents do not switch to the active state as police presence increases in a neighborhood. In addition, when the system reached the legitimacy value in which jailed and passive concentrations are similar, the number of jailed agents began to drop rapidly.

The phase diagrams allow us to identify the domains' limits based on the changes in the predominance of a state in the system. The diagrams show order-disorder transitions for the system with vision equal to one. We identify six ordered phases with a majority state, a disordered phase at a specific point in the diagram, and a consensus phase where all agents in the system are passive. The disorder point's position
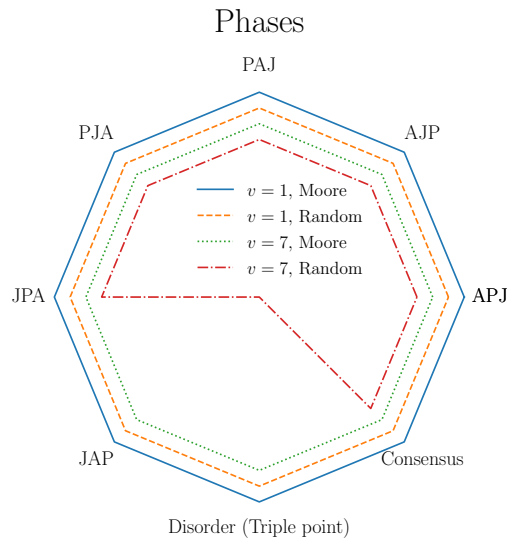

FIG. 15. Schematic visualization of different phases found at the one-dimensional civil disorder model with police officers on the steady state. We observe six ordered phases with a majority state, a disordered phase, and a consensus with vision one. With vision seven and Moore neighborhood, we observe the same phases. However, there is neither the disordered phase nor the ordered phase, with most jailed agents followed by active and passive agents (JAP phase) in a random neighborhood.

and the phases' sizes depend on the type of neighborhood. The random neighborhood makes it easier to capture active agents. So, the disorder point requires less concentration of police than in the Moore neighborhood, and the most extensive phases are those with a predominance of passive agents. When the system has vision seven, the differences depend on the neighborhood. In a Moore neighborhood, we observe order-disorder transitions with the same phases observed in the system with vision one. However, in a random neighborhood, we only observe 5 phases of order with a majority, and we do not observe the order-disorder transition. In this neighborhood, police officers significantly increase activity. So, they can capture many active agents in the low legitimacy regime, preventing the emergence of active agents with high legitimacy values. We show a summary of the different scenarios of the system with police officers in steady state in Fig. 15. Finally, the study of the stationary probability density functions of the concentration of the officers shows us that the phase changes are continuous at the same as in the system without police.

From the results we obtained for energy, we can note essential differences in behavior according to the concentration of police officers in the system. When the police concentration has values less than or equal to 0.10, we can see that the energy shows a local minimum, a maximum, and a global minimum. The local minimum shows an ordered state with a majority. We observe the most active agents in the case of vision one and a Moore neighborhood. In the other cases, the jailed agents always predominate. As for legitimacy increases, the energy reaches a maximum when at least two agent states are in similar concentrations. For example, for the system with vision one and a Moore neighborhood, this maximum

shows when the concentration of active and passive agents are similar. In the other cases, we observe that the jailed and the passive are in a similar concentration. Therefore, the maximum energy only shows changes between phases of order with a majority and cannot be identified with the state of disorder of the system. Finally, we observe the global energy minimum when all agents in the system are passive, indicating that the system is in the consensus phase.

When the concentration of police officers has values greater than 0.10 but less than 0.70, we observe different behavior of energy depending on the vision and the type of neighborhood. For vision one and a Moore neighborhood, the initial energy is maximum. Then, as legitimacy increases, energy remains constant until the legitimacy value indicates the phase with passive agents majority. Then, the energy decays quickly to the minimum in the consensus phase. The initial energy for vision one and a random neighborhood is a local minimum. We observe a minimal increase until it reaches a maximum and quickly decays to a global minimum. The maximum shows the transition from jailed agents' majority phase to the passive agents' majority phase and then reaches the consensus phase. For vision seven and both neighborhoods, we see that the energy starts at the maximum possible value, indicating a phase where the prisoners predominate. Then, energy steadily decays toward the global minimum in the consensus phase.

Finally, for regimes where the police concentration is greater than or equal to 0.70, we see for all cases that the initial energy has a closer value to the minimum. As legitimacy increases, the energy constantly decreases until it reaches the minimum associated with consensus. In general, we can note that the minimum energy for all cases is not an absolute minimum. Despite reaching the consensus phase and all the agents being passive, there are police officers in the system. Therefore, its value depends on the police officers' concentration and their random position on the lattice. This result leads us to conjecture that the energy value will be closer to the absolute minimum for an arbitrary initial configuration with two clusters of police officers and passive agents.

Regarding the observed stability points of energy, we see similarities for the system without police officers. The local energy minimum observed for low police concentration is a metastable point for the same reason as the system without police. For all visions and neighborhoods, this point shows the existence of grievance, either because there are active agents or prisoners. Therefore, the system can change to the global minimum seeking consensus as legitimacy increases. The point of maximum energy is generally unstable for low values of police concentration because it can fall to the local or global minimum with variations in legitimacy. For high police concentrations, the maximum remains constant at low legitimacy, and as legitimacy increases, it reaches the minimum rapidly. Finally, the global energy minimum is stable for all cases because the system reaches a consensus. However, the increase of police officers' concentration makes its value change. Based on the results obtained, we can still interpret the system based on the principle of minimum grievance because the system tends toward a global minimum as legitimacy increases. However, the energy is insufficient to identify

the system's prevailing state and the effects of police concentration on the dynamics.

Since the system now considers three possible states for agents and police officers, we must complement the analysis with the concentration of agents. With these two macroscopic quantities together, we can identify the most relevant function of the police based on the vision and the type of neighborhood. When considering vision one, the capture of active agents is the most relevant police function to dissuade a protest. In the Moore neighborhood with low police concentration, the system requires high legitimacy values for active officers are not predominant. As the concentration of police officers increases, we observe a significant decrease in the activity of active agents, so the system needs a lower value of legitimacy to change the active majority. In the random neighborhood, police officers increase the capture of active agents, allowing them not to be a dominant state in the system for low values of legitimacy. In addition, as the concentration of police officers increases, the activity of active agents rapidly decreases, increasing the concentration of jailed agents. Nevertheless, high values of legitimacy are still needed for passive agents to be predominant, and subsequently, the system reaches consensus.

When the system has vision seven, we observe that there are still active agents with a low concentration of police officers, but they are not predominant. Therefore, the system requires increasing legitimacy to reach the predominance of passive agents. As the concentration of police officers increases, the activity of active agents disappears. As for legitimacy increases, the passive agents increase, and the jailed agents decrease linearly. Hence, in these cases, the relevant role of police officers is to prevent the appearance of a protest by preventing the officers from becoming active.

Although the police activity considerably reduces the active agents' activity in both cases, the permanent presence of jailed agents indicates a grievance in the population and an increase in the system's energy. Hence, we can conclude that the system will not be utterly stable if there is an internal grievance. This affirmation indicates that dissuading a protest by capturing active agents is ineffective in reducing grievance. Nevertheless, our results show that as the concentration of police officers in the system increases, the system needs a lower value of legitimacy to reach consensus or minimum energy. Observing the minimum obtained, we note that it is not an absolute minimum and depends directly on the concentration of police officers in the system. Therefore, the greater the concentration of police officers, the value of the minimum energy increases. In other words, dissuading protests to facilitate or maintain consensus has an energy cost for the system proportional to the number of police officers or the amount of force used.

We do not observe any significant variation of the system's behavior without police officers in terms of the variations in lattice size, the vision of the agents, or the type of neighborhood. So, we confirm that the dynamics only depend on the initial random conditions, such as the positioning of agents in the lattice and the definition of local parameters. Therefore, we conjecture that the system's dynamics will be the same regardless of the topology or dimension of the grid. In addition, the use of the movement rule will not generate state changes in the agents because the defined rules do not depend

on the neighborhood in the system without police. In this context, the global quantities introduced and the phase diagram present new elements to analyze the model's dynamics and make new interpretations to understand the dynamics of social protests.

In the results obtained for the system with police officers, we did not notice significant variations of the dynamics with the variation of the size of the system. However, the dynamics dependent directly on the vision and the neighborhood product of the definition of the agents' rules. The police officers' activity increased significantly with the elections of a random neighborhood and increased vision. The global quantities introduced allowed us to identify the most relevant role and effect of the police officers in the system. It is essential to mention that the size of the system determines the maximum possible vision of the agents. Hence, for visions close to the maximum possible, we conjecture that the results will be similar to those shown for vision seven.

On the other hand, considering the movement of the agents and other topologies or lattice dimensions, we expect considerable changes in the system's dynamics with police officers. When Epstein presented this model in two dimensions, he reported very particular dynamics for specific parameters, such as the punctuated equilibrium phenomenon. Therefore, our results can serve future research as a first approximation to characterize the results reported by Epstein or even find new states or phenomena not yet reported in the current literature.

It is essential to note that we base our interpretations on the numerical results obtained from this simplified model. However, in the real world, the dynamics of social protests are more complex and involve many other factors. Nevertheless, studying this model from the perspective of sociophysics can yield new elements that allow us to address the complexities of the dynamics of social protest.

ACKNOWLEDGMENTS

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