Universidad de Concepción
Dirección de Postgrado
Facultad de Ciencias Físicas y Matemáticas - Programa Doctorado en Matemática

On Some Isoperimetric Inequalities Involving Nonlocal Operators

Algunas desigualdades isoperimétricas que involucran operadores no locales

Tesis para optar al grado de Doctor en Matemática

Franco Olivares-Contador

CONCEPCIÓN-CHILE

2022

Profesor Guía: Rajesh Mahadevan
Departamento de Matemática, Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción

Universidad de Concepción
Dirección de Postgrado
Facultad de Ciencias Físicas y Matemáticas - Programa Doctorado en Matemática

On Some Isoperimetric Inequalities Involving Nonlocal Operators

Algunas desigualdades isoperimétricas que involucran operadores no locales

Tesis para optar al grado de Doctor en Matemática

FRANCO OLIVARES CONTADOR
CONCEPCIÓN-CHILE
2022

Comisión evaluadora:
Mark Ashbaugh (University of Missouri, United States)
Rafael Benguria(Pontificia Universidad Catolica de Chile, Chile)
Almut Burchard (University of Toronto, Canada)
Dominique Spehner (UdeC, Chile)
Rajesh Mahadevan (UdeC, Chile)

Dedicatorias

A la memoria de mi abuelo Ciro Olivares y mi abuela Oriela Azolas muertos en mis años de permanencia en el doctorado.

Agradecimientos

Agradezco a CONICYT por otorgarme la beca para estudios de Doctorado nacional CONICYT-PCHA/Doctorado Nacional/2020-21161103. Agradezco a todos los que han colaborado en la realización de esta tesis, en especial, a Rajesh Mahadevan, por su invaluable apoyo en el desarrollo de esta. Por darme la libertad de investigar lo que mi curiosidad y sentido estético me indicara; también le agradezco que me ayudara a salir de mi estancos en el desarrollo de investigación. A Mark Ashbaugh, le agradezco sus ayuda brindada a pulir el texto y corregir inconsistencias del artículo [41]. Le agradezco Almut Burchard, por su cálida acogida en mi pasantía en la Universidad de Toronto, fue muy estimulante las reuniones en que discutimos sobre operadores no locales. Agradecimento, especial a mi familia, en especial a mi abuela Amanda Araya por su cariño y sus conversaciones, que me han hecho año a año más humano, gracias por regalarme tantos libros en la infancia, gracias ha esto me he convertido en una persona curiosa, una persona que se asombra ante la infinitud del conocimiento humano. Un especial agradecimiento a Daniela Narbona, mi compañera de vida, espero que sigamos avanzando en la armonización del realismo y idealismo en nuestras vidas.

Publications

The contents of Chapter 3 appears in the work (Mahadevan and Olivares, [41]) published in 2021. The contents of Chapter 4 form part of a manuscript which is to be submitted for publication.

Contents

Chapter 1

Introduction

1.1 Description of the problems along with some historical background

The classical isoperimetric inequality, says the following: "among all regions with a given area, the circle has the smallest perimeter," or its equivalent form: "among all regions with a given perimeter, the circle has the largest area."
Isoperimetric inequalities restricted to classes of polygons apparently date from Zenodorus (2nd century BC). The isoperimetric inequality in the class of n-sided polygons says the following: "A regular n-gon has greater area than all other n-gons with the same perimeter" (see [6]).

Although these kinds of questions have been of interest since antiquity and intuitive answers to some of these questions were proposed, one can say for sure that rigorous answers to these questions were forthcoming only since the development of modern calculus. The pioneering contributions of the Bernoulli brothers, Euler and Lagrange during the 18th century, followed by those of Weierstrass during the 19th century gave rise to the development of the area of Calculus of Variations which addresses such questions. In the 20th century it reached greater heights with the developments of more efficient tools of analysis with notable contributions by Hilbert, Noether, Tonelli, de Giorgi, Almgren among many others. It is often the case that, to solve a particular shape optimization problem, one has to go beyond shapes and use a mathematical framework where classical shapes form a part. The eventual answer in the relaxed setup may not be a shape but in certain problems it may turn out that a classical shape is a (or the) solution.

Lord Rayleigh, in his treatise, “The Theory of Sound” (see [47], 1894, pp. 339-340) colorfully phrased the following conjecture “If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle.” The conjecture was proved independently by Faber [22] and Krahn [34], [35] in the 1920s. The result in any dimension says that the ball minimizes the first eigenvalue of the Dirichlet Laplacian and is now referred to as the Faber-Krahn inequality (which we refer to as FKI for short). Since then many minimization or maximization problems for the principal eigenvalue of standard elliptic operators with respect to the shape of the domain (with constraints on the volume, perimeter etc.) have been studied. These results, usually, go under the name of Faber-Krahn inequalities or in general, isoperimetric inequalities. This thesis is dedicated to the study of a few such problems. In order to put the results of the thesis in context, we mention some preceding works on such inequalities. The discussion below is by no means exhaustive.

Pólya and Szegő ([45], 1951) conjectured that among all N-gons of fixed area, the regular N-gon of the same area minimizes the first eigenvalue of the Dirichlet Laplacian. Even up until now this conjecture has only been established among the classes of triangles and quadrilaterals (see [45], [28]). For other N-gons ($N \geq 5$) this still remains a challenging open problem.

Apart from the Laplacian, FKIs have been obtained for other elliptic operators, for both locally and non-locally defined operators. Among locally defined elliptic operators, the FKI for the $p-$Laplacian which was first studied by Bhattacharya ([5], 1999) has been revisited in recent years in a few papers [40, 18]. The sharp lower and upper bounds for the Dirichlet Laplacian for a triangle has been studied by Siudeja [55]. Besides this, Laugesen and Siudeja have establised in [36] that, among all triangles of given diameter, the equilateral triangle minimizes the sum of the first n eigenvalues of the Dirichlet Laplacian. Whereas, in the non-local setting, the FKI for the fractional p-Laplacian has been studied by Brasco et al ([10], 2014) in the class of bounded open sets and by Olivares-Contador ([43], 2017) for the class of triangles and quadrilaterals for Dirichlet boundary conditions (in this work, symmetrization method has been used, as in this thesis). At the same time, FKIs for the Riesz potential operator have also been studied by Rozenblum et al ([50], 2016) and by Kalmenov et al ([30], 2017) in the class of open bounded domains in Euclidean space. Analogous questions in other geometric settings have also been studied by Ruzhansky et al ([53], 2016). Another functional of interest in geometric problems is the Cheeger constant of a domain which appears in Cheeger's pioneering work ([17], 1970). The Cheeger constant for a domain is the infimum of the quotient of the perimeter of a subset divided by its area, among all subsets of the domain and it also can be interpreted as the first Dirichlet eigenvalue of the $1-$Laplacian (see [32], 2008). Isoperimetric estimates for the Cheeger constant were studied in Kawohl et al ([31], 2003). In the work of D. Bucur et al ([11], 2016) the following FKI was proved: "the regular $N-$gon minimizes the Cheeger constant among all N-gons with a given area". In the non-local setting, the fractional Cheeger problem has been studied by Brasco et al in ([10], 2014) where, among other results, a fractional analogue of the corresponding result established in [[31] was proved.

In this thesis we obtain some isoperimetric inequalities for some non-local shape functionals in the class of triangular and quadrilateral domains. The main tool that we use to get these isoperimetric inequalities is Steiner symmetrization.

1.2 Synopsis of the chapters

Synopsis of Chapter 2

This chapter contains the technical preliminaries required for studying the problems stated above and for their proof. In particular, we include the definition and properties of symmetric decreasing rearrangement and Steiner symmetrization of sets and functions (Section 2.1). In that section we will especially recall the Riesz's inequality for both symmetric decreasing rearrangement and Steiner symmetrization. In Section 2.3, we give, the necessary background on the Riesz potential operator and the eigenvalue functionals which will be studied in Chapter 3. Section 2.2 provides an introduction to the functionals which will be studied in Chapter 4. In Section 2.4, an outline of the proofs of some previously known Faber- Krahn inequalities, both local and non-local, will be given based on Riesz's inequality. These outlines will allow us to visualize the technique that will be used later in proving the main results. Section 2.5 gives a brief treatment of Hausdorff convergence for sets including some convergence results used later in Chapters 3 and 4.

Synopsis of Chapter 3

Chapter 3 contains inequalities of Faber-Krahn type for the first (largest) eigenvalue of the Riesz potential operator. It is shown that the principal eigenvalue of the Riesz potential operator in the class of triangular domains of given area is maximum when the triangle is equilateral. Similarly, it is proved that the maximum value of this functional in the class of quadrilateral domains of given area is attained when the quadrilateral is a square. It is also established that the equilateral triangle and the square are, respectively, the unique maximizers.

The precise mathematical statements are to be found below. Given $d > 1$, $\Omega \subseteq \mathbb{R}^d$ a bounded domain (simply connected and open) and $0 < \alpha < d$, the Riesz potential operator is a compact self-adjoint operator defined on $L^2(\Omega)$ by

$$(I_\alpha u)(x) = C(d, \alpha) \int_\Omega \frac{u(y)}{|x-y|^{d-\alpha}} dy \tag{1.2.1}$$

where $C(d, \alpha) = \pi^{-\frac{d}{2}} 2^{-\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}$. The principal (largest) eigenvalue of I_α admits the characterization

$$\lambda_1(\Omega) = \max \left\{ \frac{\int_\Omega \int_\Omega C(d, \alpha) \frac{u(x)u(y)}{|x-y|^{d-\alpha}} dx dy}{\int_\Omega u^2(x)\, dx} : u \in L^2(\Omega) \setminus \{0\} \right\}. \tag{1.2.2}$$

The main theorem of Chapter 3 is the following.

Theorem 1.2.1. *The maximum of $\lambda_1(\Omega)$ among all triangles of given area is obtained when Ω is an equilateral triangle and only when Ω is an equilateral triangle. Similarly, the maximum of $\lambda_1(\Omega)$ among all quadrilaterals of given area is obtained when Ω is a square and only when Ω is a square.*

We also briefly treat the corresponding isoperimetric inequalities for the Schatten p-norms of the Riesz potential operator for integer values of p with $p > \max\left(\frac{d}{\alpha}, 2\right)$. Indeed, in Section 2.3, it is shown, under this hypothesis, that the singular values of the operator form a p-summable sequence so that, by Definition 2.3.8, the operator belongs to the Schatten p-class. Note that the Riesz operator, for $p > \frac{d}{\alpha}$, although not in the trace class, could be Hilbert-Schmidt if $p = 2 > \frac{d}{\alpha}$. Moreover, being a positive compact self-adjoint operator the maximal eigenvalue is the operator norm as an operator on L^2 whereas the Schatten norms are other norms of the operator containing different information on the operator.

Theorem 1.2.2. *Let $p > \max\left(\frac{2}{\alpha}, 2\right)$ be a natural number. Then, the maximum of $\|I_{\alpha,\Omega}\|_p$ among all triangles of given area is obtained when Ω is an equilateral triangle. Similarly, the maximum of $\|I_{\alpha,\Omega}\|_p$ among all quadrilaterals of given area is obtained when Ω is a square.*

An introduction to the Riesz potential operator, which includes a discussion of eigenvalue functionals of interest in the thesis, is provided in Section 2.3. At the beginning of Section 3.1 we give two preliminary results which are used in the proof of the main result and are of interest in themselves. These are; firstly, the continuity of the first eigenvalue of the Riesz operator and of the Schatten norm with respect to the convergence in the Hausdorff complementary metric of a sequence of uniformly bounded convex open sets (namely, Proposition 3.1.4 and Corollary 4.2.3) and; secondly, a discussion of the equality case in the Riesz's inequality for Steiner symmetrization (see Proposition 2.1.6). A proof of the latter result in dimension one appears in Lieb [39]. The equality case for this inequality with respect to the symmetric decreasing rearrangement, in any dimension, is treated in Lieb [38]. However, a discussion of the analogous result with respect to Steiner symmetrization, for any dimension, is not easy to find.

Synopsis of Chapter 4

In Chapter 4, the functionals under consideration are the non-local s-perimeter of a domain, and the corresponding Cheeger constant. It is shown that the minimum of these functionals in the class of triangular domains of given area is attained for equilateral triangles and, in the case of quadrilateral domains of given area, is attained for a square. One is able to show that equality in the isoperimetric inequality for the non$-$local $s-$perimeter in the class of triangles holds only for an equilateral triangle and likewise, is a square, in the case of quadrilaterals but similar results could not be established in the case of the non-local Cheeger constant.

More precisely, for $0 < s < 1$, for any Borel set $E \subset \mathbb{R}^d$ the nonlocal s-perimeter $P_s(E)$ is, by definition,

$$P_s(E) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{d+s}} dxdy. \tag{1.2.3}$$

The following scale-invariant nonlocal isoperimetric inequality

$$\frac{P_s(E)}{|E|^{1-\frac{s}{d}}} \geq \frac{P_s(B)}{|B|^{1-\frac{s}{d}}} \tag{1.2.4}$$

where B is any $d-$dimensional ball, holds with equality occurring if only if E is a ball (see Frank et al [23] or Brasco et al [10]).

For any $\Omega \subseteq \mathbb{R}^d$, a bounded open set, the nonlocal s-Cheeger constant is defined by

$$h_s(\Omega) = \inf_{E \subset \Omega} \frac{P_s(E)}{|E|} \tag{1.2.5}$$

over all measurable subsets E of Ω. From (1.2.4) the following Faber-Krahn inequality for the Cheeger constant can be deduced (see Brasco et al [10])

$$|\Omega|^{\frac{s}{d}} h_s(\Omega) \geq |B|^{\frac{s}{d}} h_s(B) \, . \tag{1.2.6}$$

We obtain the following analogues of the isoperimetric inequalities (1.2.4) and (1.2.6) while restricting the class of domains to be triangles or quadrilaterals.

Theorem 1.2.3. *(a nonlocal isoperimetric inequality in the class of triangles and quadrilaterals) Let $0 < s < 1$. The minimum of $P_s(\Omega)$ among all triangles (open) of given area is obtained when Ω is an equilateral triangle and only when Ω is an equilateral triangle. Similarly, the minimum of $P_s(\Omega)$ among all quadrilaterals (open) of given area is obtained when Ω is a square and only when Ω is a square.*

Theorem 1.2.4. *(isoperimetric inequality for the nonlocal s-Cheeger constant for triangles and quadrilaterals) Let $0 < s < 1$. The minimum of $h_s(\Omega)$ among all triangles (open) of given area is obtained when Ω is an equilateral triangle. Similarly, the minimum of $h_s(\Omega)$ among all quadrilaterals (open) of given area is obtained when Ω is a square.*

The Theorems 1.2.3 and 1.2.4 are proved in a similar way to the Theorem 1.2.1 but this is done using, instead of the Riesz's inequality for Steiner symmetrizations, the nonlocal Pólya-Szegő inequality.

Chapter 2

Preliminaries

2.1 Symmetric decreasing rearrangement and Steiner symmetrization of sets and functions

In this subsection we recall, mainly, the notion of Steiner symmetrization and some of its properties. The definitions given below follow Lieb and Loss [39] or Brascamp, Lieb and Luttinger [9]. We also refer to the same texts and Gruber [26] for the main properties (see [12, 3, 21, 28] for complementary information).

Proposition 2.1.1. *(layer-cake representation) Let ν be a measure on the Borel sets of $\mathbb{R}^+$ such that $\phi(t) := \nu([0,t))$ is finite for every $t > 0$. Let (Ω, Σ, μ) be a sigma-finite measure space and f any nonnegative measurable function on Ω. Let (Ω, Σ, μ) be a sigma-finite measure space and f any nonnegative measurable function on Ω. Then*

$$\int_\Omega \phi(f(x))\mu(dx) = \int_0^\infty \mu(\{x \in \Omega : f(x) > t\})\nu(dt). \tag{2.1.1}$$

In particular, if $\phi(t) = t^p$ where $p > 0$, then we have

$$\|f\|^p_{L^p(X,d\mu)} = p\int_0^\infty t^{p-1}\mu(\{x \in \Omega : f(x) > t\})dt, \tag{2.1.2}$$

and if μ is the Dirac measure at $x \in \mathbb{R}^n$ and $\phi(t) = t$, (2.1.1) takes the following form

$$f(x) = \int_0^\infty \chi_{\{y\in\Omega: f(y)>t\}}(x)dt = \int_0^\infty \chi_{\{y\in\Omega: f(y)\geq t\}}(x)dt\,. \tag{2.1.3}$$

Proof: See Theorem 1.13 of [39]. ∎

The definition of the symmetric decreasing rearrangement used here is based on the layer cake representation of a function $f(\cdot)$ in terms of its "slices" $\{x \in \Omega : f(x) > t\}$.

Definition 2.1.2. (Symmetric decreasing rearrangement) *For any Borel measurable subset $\Omega \subset \mathbb{R}^d$ with finite Lebesgue measure we denote by Ω^* the open ball with centre at 0 having the same measure as Ω. For any nonnegative Borel measurable function f on $\mathbb{R}^d$ vanishing at infinity (in the sense that the level sets $\{f > t\}$ all have finite measures for any $t > 0$) we define the symmetric-decreasing rearrangement f^* of f using the layer cake representation, by symmetrizing its level sets, that is*

$$f^*(x) = \int_0^\infty \chi_{\{y \in \mathbb{R}: f(y) > t\}^*}(x)dt. \tag{2.1.4}$$

f^* so defined is a Borel measurable function.

Steiner symmetrization: By definition, the Steiner symmetrization of a function, in one dimension, is the same as the symmetric decreasing rearrangement given above. The Steiner symmetrization of a function defined on a domain in $\mathbb{R}^d (d > 1)$ with respect to a hyperplane is given in the following definition.

Definition 2.1.3. *Let f be a nonnegative, Borel measurable function on $\mathbb{R}^d$ which vanishes at infinity, and let H be any hyperplane ($(d-1)$-dimensional plane) through the origin of $\mathbb{R}^d$. We set up an orthogonal coordinate system on $\mathbb{R}^d$ in such a way that, if $(x', x_d) = (x_1, x_2, \ldots, x_{d-1}, x_d)$ stand for the coordinates of a generic point x, then H is the plane $x_d = 0$.*

A nonnegative, Borel measurable function $\mathcal{S}f$ on $\mathbb{R}^d$ is called the Steiner symmetrization with respect to H of f, if $\mathcal{S}f(x_1, x_2, \ldots, x_{d-1}, \cdot)$ is the symmetric decreasing rearrangement of $f(x_1, x_2, \ldots, x_{d-1}, \cdot)$ with respect to the x_d variable, for each fixed $x_1, \ldots, x_{d-1}$.

It can be seen that this naturally leads to the following definition for the Steiner symmetrization of a bounded measurable set Ω with respect to the hyperplane H.

Definition 2.1.4. (Steiner symmetrization of a set) *For any bounded Borel measurable set $\Omega \subset \mathbb{R}^d$ the Steiner symmetrization of Ω with respect to H, to be denoted by $\mathcal{S}\Omega$,* *is given by*

$$\mathcal{S}\Omega = \bigcup_{\substack{b\in H\\ \Omega\cap L_b\neq\emptyset}} \{b + t\, e_d : |t| \leq \frac{1}{2}|\Omega\cap L_b|\} \tag{2.1.5}$$

where $|\Omega\cap L_b|$ *is the one-dimensional Lebesgue measure of* $\Omega\cap L_b$ *with* L_b *being the line with direction* e_d *passing through the point* b *for any* $b\in H$.

We recall the following properties of the Steiner symmetrization of functions.

Proposition 2.1.5. *1. The definitions of* $\mathcal{S}A$ *and* $\mathcal{S}f$ *are consistent, that is,*

$$\chi_{\mathcal{S}A} = \mathcal{S}\chi_A \text{ and } \mathcal{S}\{x : f(x) > t\} = \{x : \mathcal{S}f(x) > t\}.$$

for all Borel measurable sets A *with finite Lebesgue measure and for all non-negative Borel measurable functions* f *which vanish at infinity.*

2. The super-level sets are equimeasurable, that is,

$$|\{x : f(x) > t\}| = |\mathcal{S}\{x : f(x) > t\}|.$$

In particular, for any measurable set $A\subset\mathbb{R}^d$ *with finite Lebesgue measure we have* $V(A) = V(\mathcal{S}A)$.

3. Let f *be a nonnegative Borel measurable function with* $f\in L^2(\mathbb{R}^d)$. *Then,*

$$\|f\|_2 = \|\mathcal{S}f\|_2.$$

4. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ *be a non-negative Borel measurable function which vanishes at infinity and let* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *be continuous and monotonically increasing with* $\Phi(0) = 0$. *Then,*

$$\int_{\mathbb{R}^d}\Phi(f(x))dx = \int_{\mathbb{R}^d}\Phi(\mathcal{S}f(x))dx.$$

Proof: For (1) see pages 183-184 of [3]. By the part (1) of the Proposition 2.1.5

$$|\{x : f(x) > t\})| = |\{x : \mathcal{S}f(x) > t\}| = |\mathcal{S}\{x : f(x) > t\}|.$$

The equality $V(A) = V(\mathcal{S}A)$ can be seen by decomposing the volume integral into one dimensional sections perpendicular to the hyperplane H and observing that the sections have the same measure (length) before and after the rearrangement. As to (4), it is easy to see that the hypotheses of Φ guarantee the existence of a measure ν such that $\Phi(t) = \nu([0, t))$ for all $t > 0$. Then, by Proposition 2.1.1, we get

$$\int_{\mathbb{R}^d} \Phi(f(x)dx = \int_0^\infty |\{x : f(x) > t\})|\nu(dt)$$

and

$$\int_{\mathbb{R}^d} \Phi(\mathcal{S}f(x))dx = \int_0^\infty |\{x : \mathcal{S}f(x) > t\}|\nu(dt).$$

Hence

$$\int_{\mathbb{R}^d} \Phi(f(x))dx = \int_{\mathbb{R}^d} \Phi(\mathcal{S}f(x))dx.$$

Then, (3) is just a special case of (4) for the choice $\Phi(t) = t^2$. ■

We now recall Riesz's inequality for the Steiner symmetrization in one dimension. This inequality for symmetric-decreasing rearrangement goes back to F. Riesz (1930,[48]). The equality case in above inequality was studied by A. Burchard [14]. We will give the proof given in Lieb and Loss [39, Lemma 3.6] (but we give some extra details) which is originally from Rogers [49] and Brascamp-Lieb-Luttinger [9].

Proposition 2.1.6. (Riesz's inequality in one-dimension) *Let f, g and h be non-negative Borel measurable functions that vanish at infinity on $\mathbb{R}$, and let f^*, g^* and h^* be their respective symmetric-decreasing rearrangement.*
Then, for $I(f, g, h) := \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(x-y)h(y)\ dxdy$, we have

$$I(f, g, h) \leq I(f^*, g^*, h^*). \tag{2.1.6}$$

Proof: STEP 1: Using Fubini's theorem and the layer-cake decomposition on f, g and h, we have

$$\begin{aligned} I(f, g, h) &= \int_0^\infty\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{\{f>s\}}(x)\chi_{\{g>r\}}(x-y)\chi_{\{h>t\}}(y)dxdydsdrdt \\ &= \int_0^\infty\int_0^\infty\int_0^\infty I(\chi_{\{f>s\}}, \chi_{\{g>r\}}, \chi_{\{h>t\}})dsdrdt. \end{aligned} \tag{2.1.7}$$

By the same argument, we obtain

$$I(f^*,g^*,h^*) = \int_0^\infty \int_0^\infty \int_0^\infty I(\chi_{\mathcal{S}\{f>s\}}, \chi_{\mathcal{S}\{g>r\}}, \chi_{\mathcal{S}\{h>t\}}) ds dr dt. \tag{2.1.8}$$

Since f, g and h are nonnegative functions that vanish at infinity, their respective level set have finite measure. Thus, from (2.1.16) and (2.1.8), it can be seen that it is enough to prove (2.1.6) for characteristic functions of measurable sets of finite measure.

STEP 2: Let O^1, O^2 and O^3 be sets of finite measure. By outer regularity of the Lebesgue measure, for each $k = 1,2,3$, there exists a sequence of open sets $\{O_r^k\}_{r\in\mathbb{N}}$ (having finite measure for every $r \in \mathbb{N}$) such that $O^k \subset O_r^k \subset O_{r-1}^k$for all $r \in \mathbb{N}$ and $\lim_{r\to\infty}|O_r^k| = |O^k|$. Obviously, then we also have $\lim_{r\to\infty}|(O_r^k)^*| = |(O^k)^*|$. This means that $\chi_{O_r^k} \to \chi_{O^k}$ and $\chi_{(O_r^k)^*} \to \chi_{(O^k)^*}$ in $L^1(\mathbb{R})$ and so, for a subsequence which we denote by the same index, $\chi_{O_r^k} \to \chi_{O^k}$ and $\chi_{(O_r^k)^*} \to \chi_{(O^k)^*}$ almost everywhere. The dominated convergence theorem shows that

$$\begin{aligned} \lim_{r\to\infty} I(\chi_{O_r^1}, \chi_{O_r^2}, \chi_{O_r^3}) &= I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) \quad \text{and} \\ \lim_{r\to\infty} I(\chi_{(O_r^1)^*}, \chi_{(O_r^2)^*}, \chi_{(O_r^3)^*}) &= I(\chi_{(O^1)^*}, (\chi_{(O^2)^*}, \chi_{(O^3)^*}) . \end{aligned}$$

So, we can further deduce that it is enough to prove (2.1.6) for characterisitic functions of open sets O^1, O^2, O^3 having finite measure.

STEP 3: Now, we recall that every open subset of real line is the disjoint union of countably many open intervals. For $k \in \{1,2,3\}$, denote these intervals by $I_1^k, I_2^k, \dots$ such that $O^k = \cup_{n=1}^\infty I_n^k$, arranged in such a way that $|I_{n+1}^k| \le |I_n^k|$. If we set

$$F_m^k = \bigcup_{n=1}^m I_n^k \quad \text{con} \quad k \in \{1,2,3\} \tag{2.1.9}$$

we get

$$\lim_{m\to\infty}|F_m^k| = \textstyle\sum_{l=1}^\infty |I_l^k| = |O^j| \text{ for every } k \in \{1,2,3\}.$$

Using the monotone convergence theorem, we see that

$$\begin{aligned} \lim_{m\to\infty} I(\chi_{F_m^1}, \chi_{F_m^2}, \chi_{F_m^3}) &= I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) \quad \text{and} \\ \lim_{m\to\infty} I(\chi_{(F_m^1)^*}, \chi_{(F_m^2)^*}, \chi_{(F_m^3)^*}) &= I(\chi_{(O^1)^*}, \chi_{(O^2)^*}, \chi_{(O^3)^*}). \end{aligned}$$

The above, further reduces the problem of showing (2.1.6) to that of showing (2.1.6) for characteristic functions of sets O^1, O^2, O^3 each of which is a finite union of disjoint open intervals of finite length.

STEP 4: Thus, we can write

$$I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) = \sum_{j=1}^{J} \sum_{m=1}^{M} \sum_{n=1}^{N} \int\int f_j(x - a_j) g_m(x - y - b_m) h_n(y - c_n) dxdy$$

where f_j, g_m and h_n are characteristic functions of intervals centered at the origin and the a_j, b_m and c_n are real numbers. We set

$$I_{jmn}(t) = \int\int f_j(x - ta_j) g_m(x - y - tb_m) h_n(y - tc_n) dxdy, \tag{2.1.10}$$

and

$$I_t(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) = \sum_{j=1}^{J} \sum_{m=1}^{M} \sum_{n=1}^{N} I_{jmn}(t)\,,$$

so that $I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) = I_1(\chi_{O^1}, \chi_{O^2}, \chi_{O^3})$. We observe that $I_{jmn}(t)$ is symmetric and decreasing with respect to t. Indeed, by introducing the change of variables, $w = x - ta_j$, $z = y - ta_j + tb_m$, we obtain

$$\begin{aligned} I_{jmn}(t) &= \int\int f_j(w) g_m(w - z) h_n(z + t(a_j - b_m - c_n)) dwdz \\ &= \int u_{jm}(z) h_n(z + t(a_j - b_m - c_n)) dz\,. \end{aligned}$$

Then, $u_{jm}(z) = \int f_j(w) g_m(w - z) dw$ is of the form $|(-r, r) \cap (-s + z, s + z)|$ since f_j and g_m are characteristic functions of symmetric intervals about the origin and so, u_{jm} is a symmetric non-increasing function in z. Now, $I_{jmn}(t) = \int u_{jm}(z) h_n(z + t(a_j - b_m - c_n)) dz$ is the integral of a symmetric non-increasing function u_{jm} on shifts of a symmetric interval and therefore, is seen to be symmetric and decreasing with respecto to t.

Now, going back to (2.1.10), notice that, as $t \to 0$, the supporting intervals of the functions $f_j(x - ta_j)$, $g_m(x - y - tb_m)$ and $h_n(y - tc_n)$ move toward the origin and so when the supporting intervals of any two of the f_js or g_ms or h_ns touch each other, the couple may be replaced by the characteristic function of a unified interval. At this stage we reinitialise the sets O^1, O^2 and O^3 by using the current position of the intervals and observe that $I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3})$ for the reinitiailised sets O^1, O^2 and O^3 is larger than it was at the beginning of Step 4. We repeat the process of Step 4 with the current values of O^1, O^2 and O^3.

STEP 5: After a finite number of repetitions of Step 4, we obtain the desired inequality

$$I(\chi_{O^1}, \chi_{O^2}, \chi_{O^3}) \leq I(\chi_{(O^1)^*}, \chi_{(O^2)^*}, \chi_{(O^3)^*}). \qquad \blacksquare$$

Corollary 2.1.7. (Riesz's inequality for Steiner symmetrization in higher dimensions) *Let f, h and g be nonnegative measurable functions vanishing at infinity on $\mathbb{R}^d$ and V a plane passing through the origin of $\mathbb{R}^d$. Then, taking for the definition, $I(f,g,h) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)g(x-y)h(y)\ dxdy$, we have*

$$I(f,g,h) \leq I(\mathcal{S}f, \mathcal{S}g, \mathcal{S}h). \tag{2.1.11}$$

where $\mathcal{S}f$ is the Steiner symmetrization of f with respect to V (see Definition 2.1.3).

Proof: For any fixed $z' = z_1, \ldots, z_{d-1}$, $f^*(z', \cdot)$, $g^*(z', \cdot)$ and $h^*(z', \cdot)$ are, respectively, the one dimensional symmetric decreasing rearrangement of $f(z', \cdot)$, $g(z', \cdot)$ and $h(z', \cdot)$. So, for any x', y' in $\mathbb{R}^{d-1}$, we apply Riesz's inequality for one dimensional symmetric decreasing rearrangement for the functions $f(x', \cdot)$, $g(x'-y', \cdot)$ and $h(y', \cdot)$ to obtain

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} f(x', x_d) g(x'-y', x_d - y_d) h(y', y_d) dx_d dy_d \\ &\quad \leq \int_{\mathbb{R}} \int_{\mathbb{R}} f^*(x', x_d) g^*(x'-y', x_d - y_d) h^*(y', y_d) dx_d dy_d\,. \end{aligned}$$

Then, integrating the inequality with respect to x', y', we obtain the desired inequality.

$\blacksquare$

Note: An analogous result for the symmetric decreasing rearrangement of a function in any dimension is the following Riesz's inequality

$$I(f,g,h) \leq I(f^*, g^*, h^*). \tag{2.1.12}$$

where f^* is the symmetric decreasing rearrangement of f (see Definition 2.1.2). For a detailed treatment of this we refer to Burchard [13] and Lieb and Loss [39].

A more general rearrrangement inequality published in the work of Brascamp, Lieb and Luttinger [9] asserts the following.

Theorem 2.1.8. *Let $f_1, f_2, \dots, f_m$ be non-negative functions on $\mathbb{R}^d$ which vanish at infinity. Let $k \leq m$ and let $B = ((b_{ij}))$ be a $k \times m$ matrix. If we set*

$$I(f_1, f_2, \dots, f_m) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^{m} f_j \left(\sum_{i=1}^{k} b_{ij} x_i \right) dx_1 dx_2 \dots dx_k$$

then we have

$$I(f_1, f_2, \dots, f_m) \leq I(f_1^*, f_2^*, \dots, f_m^*) \,. \tag{2.1.13}$$

The following passage is dedicated to a discussion of the equality case of Riesz's inequality in one dimension. A proof is sketched in Lieb and Loss [39] leaving some of the work to be done by the reader. This proof is detailed in my master's thesis [44](in Spanish). We present here a different proof given in Carlen and Maggi [15] which is based on the Brunn-Minkowski inequality. Before studying this proof, let us recall the Brunn-Minkowski inequality and the definition of a Lebesgue point of a measurable set and some relevant facts about the latter.

Definition 2.1.9. *We shall say that a point x is a Lebesgue point of a measurable set A in $\mathbb{R}$, if*

$$\lim_{\epsilon \to 0^+} \frac{|A \cap B(x, \epsilon)|}{|B(x, \epsilon)|} = 1.$$

We will denote the set of Lebesgue points of A as D_A. Clearly, by its definition, the interior of a set A is contained in D_A and D_A is contained in the closure of A. Also note that if x is a Lebesgue point of A, then $x + y$ is a Lebesgue point of $A + y$ and $-x$ is a a Lebesgue point of $-A$. Finally, it is true that x is a Lebesgue point of $C \cap D$ if x is a Lebesgue point of C and x is a Lebesgue point of D.

Theorem 2.1.10. *(Lebesgue's density theorem) Let $A \subseteq \mathbb{R}$ such that $m(A) > 0$. Almost every point of A is a Lebesgue point of A.*

Proof: See apendix D in van Rooij and Schikhof [57]. ■

The following lemma has its inspiration in Lemma 1 in A. Burchard [13].

Lemma 2.1.11. *Let A and B be measurable sets in $\mathbb{R}$, χ_A and χ_B their characteristic functions. Together with the above assumptions, suppose that A and B have no isolated points. Then,*

$$\overline{A+B} = \operatorname{supp}(\chi_A * \chi_B)$$

where $(\chi_A * \chi_B)(x) = \displaystyle\int_{\mathbb{R}} \chi_A(x-y)\chi_B(y)dy.$

Proof: It is clear that

$$(\chi_A * \chi_B)(x) = |(-A+x) \cap B|$$

and that, if $\chi_A * \chi_B(x) > 0$ then $x \in A+B$. Therefore,

$$\operatorname{supp}(\chi_A * \chi_B) = \overline{\{x : \chi_A * \chi_B(x) > 0\}} \subset \overline{A+B}\,. \tag{2.1.14}$$

Conversely, if $x \in D_A + D_B$ then $x = a + b$ with $a \in D_A$ and $b \in D_B$. This means that $b = x - a$ is a Lebesgue point of $B \cap (x - A)$. Therefore, $|(x-A) \cap B| > 0$ which means $x \in \operatorname{supp}(\chi_A * \chi_B)$. We have shown that

$$D_A + D_B \subset \operatorname{supp}(\chi_A * \chi_B)\,.$$

By the assumption that A and B have no isolated points and since D_A and A differ only by a set of measure zero and similarly, for D_B and B, it follows that that D_A and A have the same closures and so do D_B and B. Therefore, since the support of a function is a closed set, this implies that

$$\overline{D_A + D_B} = \overline{A+B} \subset \operatorname{supp}(\chi_A * \chi_B)\,. \tag{2.1.15}$$

So, we obtain the desired conclusion from (2.1.14) and (2.1.15). ■

The Brunn-Minkowski inequality in one dimension (see [27]) states that

$$|A+B| \geq |A| + |B|,$$

with A and B two nonempty measurable sets in $\mathbb{R}$. There is equality if and only if either

A consists of a single point or B consists of a single point, or A and B are of the form $A = \overline{I} \setminus N_A$ and $B = \overline{J} \setminus N_B$ where N_A and N_B are sets of measure zero, and $\overline{I}$ and $\overline{J}$ are closed intervals.

Proposition 2.1.12. *(Lieb's theorem on cases of equality in the Riesz rearrangement inequality). Let f, g and h be non-negative and non-zero integrable functions that vanish at infinity on $\mathbb{R}$ where g is symmetric and strictly decreasing. Then there is equality in* (2.1.6) *if and only if $f(x) = f^*(x-a)$ and $h(x) = h^*(x-a)$ for some $a \in \mathbb{R}$.*

Proof: Let $F_r = \{f > r\}$, $G_s = \{g \geq s\}$ and $H_t = \{h > t\}$. Since we may write

$$I(f, g, h) \quad = \quad \int_0^\infty \int_0^\infty \int_0^\infty I(\chi_{F_r}, \chi_{G_s}, \chi_{H_t}) dsdrdt$$

and

$$I(f^*, g^*, h^*) \quad = \quad \int_0^\infty \int_0^\infty \int_0^\infty I(\chi_{F_r^*}, \chi_{G_s^*}, \chi_{H_t^*}) dsdrdt \,,$$

under the hypothesis that $g = g^*$, equality in the Riesz's inequality implies that

$$I(\chi_{F_r}, \chi_{G_s}, \chi_{H_t}) \quad = \quad I(\chi_{F_r^*}, \chi_{G_s}, \chi_{H_t^*}) \ \text{a.e.} \ r, s, t. \tag{2.1.16}$$

Our aim is to show that, for almost all r, F_r is an interval and they are all symmetric with respect to some a and similarly, for almost all t, H_t is an interval and they are all symmetric intervals about the same a.

Since g is symmetric and strictly decreasing about intervals for every l with $0 \leq l$ there exists s such that $[-l, l] = \{g \geq s\}$. For example s may taken to be $s = \sup\{h : \{g \geq h\} \subset [-l, l]\}$. So, for any given r and t, assuming without loss of generality that $|F_r| \leq |H_t|$ we can find s such that

$$(F_r)^* + G_s = (H_t)^* \,. \tag{2.1.17}$$

That is, by Lemma 2.1.11, $(H_t)^*$ and $\text{supp}(\chi_{F_r^*} \star \chi_{G_s}(x))$ coincide except for a null set and so,

$$\begin{aligned} I(\chi_{F_r^*}, \chi_{G_s}, \chi_{H_t^*}) \quad &= \quad \int_{\mathbb{R}} (\chi_{F_r^*} \star \chi_{G_s})(x) \ \chi_{H_t^*}(x) dx \\ &= \quad \int_{\mathbb{R}} (\chi_{F_r^*} \star \chi_{G_s}(x) \ dx \end{aligned}$$

$$= \quad |(F_r)^*||G_s| = |F_r||G_s| \tag{2.1.18}$$

If it happens that F_r is not an interval and since G_s is an interval, then by the equality condition for Brunn Minkowski inequality, necesarrily one must have

$$|F_r + G_s| > |F_r| + |G_s| = |(F_r)^*| + |G_s| = |(F_r)^* + G_s| = |(H_t)^*| = |H_t|\,. \tag{2.1.19}$$

Therefore, $\{x \notin H_t, x \in \mathrm{supp}(\chi_{F_r} \star \chi_{G_s})(x)\}$ is of positive measure and so,

$$\begin{aligned} 0 < \int_{\mathbb{R}} (\chi_{F_r} \star \chi_{G_s})(x)\ \chi_{(H_t)^c}(x)dx &= \int_{\mathbb{R}} \chi_{F_r} \star \chi_{G_s}(x)\ dx - \int_{\mathbb{R}} (\chi_{F_r^*} \star \chi_{G_s})(x)\ \chi_{H_t}(x)dx \\ &= |F_r||G_s| - \int_{\mathbb{R}} (\chi_{F_r} \star \chi_{G_s})(x)\ \chi_{H_t(}x)dx\,. \end{aligned} \tag{2.1.20}$$

So, by using (2.1.18) in (2.1.20) we get

$$I(\chi_{F_r}, \chi_{G_s}, \chi_{H_t}) < I(\chi_{F_r^*}, \chi_{G_s}, \chi_{H_t^*})$$

and by the continuity of the integrals on either side with respecto to r, s and t, the inequality has to hold on an open set of values for r, s and t contradicting (2.1.16). So, F_r and H_t are intervals (up to sets of measure zero). We shall now prove that, $F_r := A$ and $H_t := B$ are centered at the same point. For this purpose, let us write (2.1.16) as

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_A(x)\chi_{(-l,l)}(x-y)\chi_B(y)dxdy = \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{A^*}(x)\chi_{(-l,l)}(x-y)\chi_{B^*}(y)dxdy, \tag{2.1.21}$$

for $l > 0$. Let's consider that A has center a and B has center b. Suppose that $a \neq b$. Without loss of generality, we assume that $b > a$. Using the above, the equation (2.1.21) is equivalent to

$$\int_{B^*} |A^* \cap [(-l,l) + y]|dy = \int_{B^*} |A^* \cap [(-l,l) + y + b - a]|dy. \tag{2.1.22}$$

From step 4 of the proof of the Proposition 2.1.6 we know that $|A^* \cap [(-l,l) + y]|$ is a non-increasing function of y (this function is also a continuous function), in particular this function is strictly decreasing in $y \in \left(\frac{|A|}{2} - l, \frac{|A|}{2} + l\right)$ and hence injective. Using the above and (2.1.22) we have

$$|A^* \cap [(-l,l)+y]| = |A^* \cap [(-l,l)+y+b-a]|. \tag{2.1.23}$$

Therefore, by the injectivity in $\left(\frac{|A|}{2}-l, \frac{|A|}{2}+l\right)$ we get $a=b$.
Fixing r_0 and varying t(and vice versa) shows that the center will not depend on r or t. So almost all intervals $\{h>t\}$ have the same center (the center of $\{f>r_0\}$). Then $\{h>t\}$ and $\{h^*>t\}$ are translations of each other by an independent level t. This means $h=h^*$ except for a translation.
In the same way, fixing t and varying r we conclude as before that $f=f^*$ except for a translation. ■

We end this subsection by mentioning the following properties of the Steiner symmetrization of sets. Let us first fix some notions. A convex body is a compact convex set with non-empty interior. By $\mathcal{K}^d$ we denote the set of all convex bodies in $\mathbb{R}^d$. For a convex body A in $\mathbb{R}^d$, the inradius $r(A)$ is the maximum of the radii of balls contained in A and the circumradius $R(A)$ is the minimum of the radii of balls containing A.

Proposition 2.1.13. *Let $A,B\in\mathcal{K}^d$. Then,*

1. *$\mathcal{S}A\subseteq\mathcal{S}B$ for $A\subseteq B$.*

2. *$r(A)\le r(\mathcal{S}A)$.*

3. *$R(\mathcal{S}A)\le R(A)$.*

4. *$S(\mathcal{S}A)\le S(A)$ where $S(A)$ denotes the surface area (perimeter) of A.*

Proof: We refer to Gruber [26, Proposition 9.1] for a proof. ■

2.2 The fractional perimeter, the fractional Cheeger constant and fractional order spaces

Our aim here is to provide the definitions and some main properties of the non-local perimeter and non-local Cheeger constant of sets.

Perimeter functionals: We first recall that a Borel subset Ω of $\mathbb{R}^d$ is said to be of finite perimeter if

$$P(\Omega) := \sup\left\{\int_\Omega \mathrm{div} v\, dx : v \in C_c^1(\Omega;\mathbb{R}^d), \|v\|_\infty \leq 1\right\} < \infty.$$

For $0 < s < 1$, for any Borel set $E \subset \mathbb{R}^d$ the nonlocal s-perimeter $P_s(E)$ is, by definition,

$$P_s(E) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\chi_E(x) - \chi_E(y)|}{|x-y|^{d+s}} dxdy\,. \tag{2.2.1}$$

The right hand side is just the Gagliardo semi-norm $[\chi_E]_{W^{s,1}}$ of χ_E. In general, for $1 \leq p < \infty$ the fractional order space $\widetilde{W}_0^{s,p}(\Omega)$ is taken to be the closure of $C_0^\infty(\Omega)$ in the fractional order Sobolev space $W^{s,p}(\mathbb{R}^d)$ equipped with the norm

$$\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}} dxdy + \int_{\mathbb{R}^d} |u|^p(x)\, dx\right)^{1/p}.$$

Example 2.2.1. *For $0 < s < 1$, a direct computation gives the non-local perimeter of a interval $(-a, a)$ with $a > 0$*

$$\begin{aligned} P_s((-a,a)) &= 2\int_{-a}^{a}\left(\int_{-\infty}^{-a} \frac{dx}{|x-y|^{1+s}} + \int_a^{+\infty} \frac{dx}{|x-y|^{1+s}}\right) dy \\ &= 4\int_{-a}^{a}\int_a^{+\infty} \frac{1}{|x-y|^{1+s}} dxdy \\ &= \frac{4(2a)^{1-s}}{s(1-s)}. \end{aligned}$$

Example 2.2.2. *For $0 < s$, using spherical coordinates one can explicitly calculate the non-local perimeter of a ball $B(0, R)$,*

$$P_s(B(0,R)) = P_s(B(x,R)) = 2\int_{B(x,R)}\int_{B(x,R)^C} \frac{1}{|x-y|^{d+s}} dxdy = \frac{2\omega_{d-1}\omega_d R^{d-s}}{s}.$$

We refer to Mazya et al [42] and Bourgain et al [8], respectively, for the following asymptotic behavior as $s \to 0^+$ and $s \to 1$ of the fractional perimeter:

$$\lim_{x\to 0^+} sP_s(\Omega) = C(d)|\Omega| \quad \text{and} \quad \lim_{x\to 1}(1-s)P_s(\Omega) = C'(d)P(\Omega), \tag{2.2.2}$$

where $C(d)$ and $C'(d)$ are constants.

So, the fractional perimeter is intermediate between the perimeter and the Lebesgue measure. An interpolation result between the s-perimeter $P_s(\cdot)$ and the classical perimeter $P(\cdot)$ is given below.

Proposition 2.2.3. *Let $s \in (0,1)$, for every finite perimeter set $E \subset \mathbb{R}^d$ we have*

$$P_s(E) \leq \frac{2^{1-s}}{(1-s)s} P(E)^s |E|^{1-s} \tag{2.2.3}$$

Proof: This a special case of Proposition 4.2 [10] given as Corollary 4.4 therein. ■

Note: The above Proposition 2.2.3 says, in particular that, if a set E has finite perimeter $P(E)$ then its fractional perimeter $P_s(E)$ is also finite.

Cheeger constants: The Cheeger constant of a set $\Omega \subseteq \mathbb{R}^d$ has the following definition

$$h_1(\Omega) = \inf_{A \subseteq \Omega} \frac{P(A)}{|A|}. \tag{2.2.4}$$

The Cheeger constant, which appears in a well-known work of Cheeger [17], is an important quantity and appears in many different contexts. For a recent overview of this theme we refer to Leonardi [37]. Here, we are interested in some questions involving a non-local version of the Cheeger constant.

For any $\Omega \subseteq \mathbb{R}^d$, a bounded open set, the nonlocal s-Cheeger constant is defined by

$$h_s(\Omega) = \inf_{E \subset \Omega} \frac{P_s(E)}{|E|}, \tag{2.2.5}$$

over all measurable subsets E of Ω. The existence of a measurable set $E_\Omega \subset \Omega$ achieving the infimum in (2.2.5) has been shown in Brasco et al [10, Proposition 5.3]. A minimizer for $h_s(\Omega)$ is said to be an $s-$Cheeger set of Ω.

The following properties are easy to prove starting from the definition.

Proposition 2.2.4. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set.*

1. *(homothety law) $P_s(k\Omega) = k^{d-s} P_s(\Omega)$ and $h_s(k\Omega) = k^{-s} h_s(\Omega)$ for $k > 0$.*

2. *(translation invariance) We have $P_s(\Omega + x) = P_s(\Omega)$ and $h_s(\Omega + x) = h_s(\Omega)$ for any $x \in \mathbb{R}^d$.*

3. *(rotation invariance) We have* $P_s(R\Omega) = P_s(\Omega)$ *and* $h_s(R\Omega) = h_s(\Omega)$ *for any isometry* R.

4. *(domain monotonicity)* $h_s(A) \geq h_s(B)$ *for bounded open sets* A, B *such that* $A \subseteq B$.

Note: Note that the perimeter functional has, in general, no monotonicity properties with respect to domain inclusions.

Definition 2.2.5. *It is said that* Ω *is* $s-$*calibrable if it is an* $s-$*Cheeger set of itself, that is,*

$$h_s(\Omega) = \frac{P_s(\Omega)}{|\Omega|}\,.$$

Example 2.2.6. *Every ball* $B := B(0, R)$ *of* $\mathbb{R}^d$ *is* $s-$*calibrable. This is a direct consequence of the isoperimetric inequality* (1.2.4) *which is proved in Brasco et al [10], which gives for every* $E \subset B$

$$\frac{P_s(E)}{|E|} \geq \frac{P_s(B)}{|B|}\left(\frac{|B|}{|E|}\right)^{\frac{s}{d}} \geq \frac{P_s(B)}{|B|},$$

and so

$$h_s(B) = \frac{P_s(B)}{|B|}\,.$$

Using the Example 2.2.2 we have $h_s(B) = \frac{2\omega_{d-1}R^{-s}}{s}$.

As compared to the local Cheeger constant, several issues around the non-locality of s-Cheeger constants are not yet well understood. For example, while in the local setting for any convex set the Cheeger set is known to be unique [16], it is not known if this is true in the non-local setting. These are not however the questions that we will be interested in, in this thesis, but we are rather interested in some isoperimetric inequalities involving these quantities.

2.3 Riesz operators

In this subsection we give a brief introduction to the Riesz potential operator and discuss some spectral functionals which are of our interest. The main references for this section are Stein [56], Vladimirov [58] and articles by Ruzhansky et al [50, 52, 53].

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain (simply connected, open) and $d > 1$. Let $0 < \alpha < d$.

The Riesz potential operator on $L^2(\Omega)$ is defined by

$$(I_{\alpha,\Omega}u)(x) = C(d,\alpha)\int_{\Omega}\frac{u(y)}{|x-y|^{d-\alpha}}dy \tag{2.3.1}$$

with $C(d,\alpha) = \pi^{-\frac{d}{2}}2^{-\alpha}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}$. If $2\alpha < d$, for $u \in L^2(\Omega)$, the Riesz potential $I_\alpha(u)$ is well defined almost everywhere.

Note: On $\mathbb{R}^d$, the Riesz potential operator $I_{\alpha,\mathbb{R}^d}$ is the inverse of fractional Laplacian operator $(-\Delta)^s$, with $\alpha = 2s$, which may be defined for smooth functions as follows:

$$(-\Delta)^s u(x) := a(d,s)\lim_{\delta\to 0^+}\int_{\{y\in\mathbb{R}^n:\delta\le|x-y|\}}\frac{(u(x)-u(y))}{|x-y|^{n+2s}}dy$$

where $a(d,s)$ is a suitably defined constant.

Proposition 2.3.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, $d > 1$. For $0 < \alpha < d$, the Riesz potential operator $I_{\alpha,\Omega}$ is a bounded linear map from $L^2(\Omega)$ to $L^2(\Omega)$.*

Proof: The proof given here is extracted from Section 15.4 of Vladimirov [58]. Let R be large enough such that $\Omega \subseteq \overline{B}(0,R)$. Then, using Cauchy–Schwarz's inequality followed by Tonelli's theorem, we obtain the following estimate

$$\begin{aligned}
\|I_{\alpha,\Omega}u\|^2_{L^2(\Omega)} &= \int_\Omega |I_{\alpha,\Omega}u(x)|^2 dx \\
&= \int_\Omega \left|\int_\Omega \frac{C(d,\alpha)u(y)}{|x-y|^{d-\alpha}}dy\right|^2 dx \\
&= (C(d,\alpha))^2\int_\Omega\left|\int_\Omega\left(\frac{1}{|x-y|^{(d-\alpha)/2}}\right)\left(\frac{u(y)}{|x-y|^{(d-\alpha)/2}}\right)dy\right|^2 dx \\
&\le (C(d,\alpha))^2\int_\Omega\left(\int_\Omega\frac{1}{|x-y'|^{d-\alpha}}dy'\right)\left(\int_\Omega\frac{|u(y)|^2}{|x-y|^{d-\alpha}}dy\right)dx \\
&\le (C(d,\alpha))^2\max_{x\in\overline{B}(0,R)}\left(\int_{\overline{B}(0,R)}\frac{1}{|x-y'|^{d-\alpha}}dy'\right)\max_{y\in\overline{B}(0,R)}\left(\int_{\overline{B}(0,R)}\frac{1}{|x-y|^{d-\alpha}}dx\right)\int_\Omega|u(y)|^2dy \\
&= (C(d,\alpha))^2N^2\int_\Omega|u(y)|^2dy < \infty
\end{aligned}$$

where it is enough to take $N = \int_{\overline{B}(0,2R)}\frac{1}{|x|^{d-\alpha}}dx$ (being finite if $\alpha > 0$). This shows that $I_{\alpha,\Omega}$ is a bounded operator on $L^2(\Omega)$. ■

Proposition 2.3.2. *For $0 < \alpha, \beta$ and $\alpha + \beta < d$, the Riesz potential operator satisfies the semigroup property*

$$I_\alpha I_\beta = I_{\alpha+\beta}\,, \tag{2.3.2}$$

with $I_\alpha := I_{\alpha,\mathbb{R}^d}$.

Proof: See Stein [56, Chapter 5, Section 1]. ■

Proposition 2.3.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain where $d > 1$. For $0 < \alpha < d$, the Riesz potential operator $I_{\alpha,\Omega}$ is a nonnegative operator.*

Proof: Since the kernel of the operator I_α is symmetric, being a bounded symmetric operator it is self-adjoint.

The non-negaitivity can be deduced using the semigroup property (2.3.2) argument from Proposition 2.1 of [50]. Consider $T : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ defined by

$$T(u)(x) = C(d, \alpha/2) \int_{\mathbb{R}^d} \frac{\chi_\Omega(y)u(y)}{|x-y|^{d-\frac{\alpha}{2}}}dy, \tag{2.3.3}$$

and its adjoint is given by

$$T^*(u)(y) = \chi_\Omega(y)\; C(d, \alpha/2) \int_{\mathbb{R}^d} \frac{u(x)}{|x-y|^{d-\frac{\alpha}{2}}}dx. \tag{2.3.4}$$

Then, we note that $T^*T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is given by

$$C(d, \alpha)\chi_\Omega(x) \int_{\mathbb{R}^d} \frac{\chi_\Omega(y)u(y)}{|x-y|^{d-\alpha}}dy,$$

by using the semi-group property (2.3.2). We shall denote T^*T by $\widetilde{I}_{\alpha,d}$ which is, clearly, a non-negative operator. Finally, we note that under the direct sum decomposition $L^2(\mathbb{R}^d) = L^2(\Omega) \bigoplus L^2(\mathbb{R}^d \setminus \Omega)$, it's clear that $\widetilde{I}_{\alpha,d} \cong I_{\alpha,\Omega} \bigoplus \mathbf{0}$. Therefore, the component $I_{\alpha,\Omega}$ is a non-negative operator. ■

We recall a general result on the compactness of an integral operator on L^2 space which will give us the compactness of the Riesz operator. We refer the reader to Cwikel [19] for details of this result while we limit ourselves to the description of the result in what follows. For this we first recall the definition of the weak L^p space.

Definition 2.3.4. *Let (X, μ) be a σ-finite measure space. For $0 < p < \infty$, the space of weak L^p functions on X, denoted by $L^{p,\infty}(X, \mu)$ (or weak-$L^p(X, \mu)$) is the set of equivalence classes of μ-almost everywhere equal functions for which*

$$\|f\|_{L^{p,\infty}(X)} := \sup(\{\alpha^p \mu_f(\alpha) : \alpha > 0\})^{\frac{1}{p}}$$

is finite, where $\mu_f(\alpha) = \mu(\{|f| \geq \alpha\})$ is the μ measure of the super-level set $\{|f| \geq \alpha\}$.

Note: For any $0 < p < \infty$ we have the continuous inclusion $L^p(X, \mu) \subset L^{p,\infty}(X, \mu)$ which follows from Chebyshev's inequality

$$\alpha^p \mu_u(\alpha) = \int_{\mathbb{R}^d} \alpha^p \chi_{\{|u|>\alpha\}} d\mu \leq \int_{\mathbb{R}^d} |u(x)|^p \chi_{\{|u|>\alpha\}} d\mu \leq \int_{\mathbb{R}^d} |u(x)|^p d\mu \,.$$

Example 2.3.5. *For $0 < p < \infty$, it is easy to check that $u(x) := \frac{1}{|x|^{\frac{d}{p}}}$ belong to $L^{p,\infty}(\mathbb{R}^d)$ but not does not belong to $L^p(\mathbb{R}^d)$. This shows that the inclusion $L^p(X, \mu) \subset L^{p,\infty}(X, \mu)$ is, in general, strict.*

Proposition 2.3.6. *Let $2 < p < \infty$ and let p' be the conjugate exponent satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. Given $g \in L^p(\mathbb{R}^d)$ and $f \in$ weak-$L^{p'}(\mathbb{R}^d)$ let us define, for any $u \in L^2(\mathbb{R}^d)$,*

$$B_{f,g}u(x) := \int_{\mathbb{R}^d} f(x - y)g(y)u(y)dy \,.$$

Then, $B_{f,g}$ is a compact operator on $L^2(\mathbb{R}^d)$. In particular its singular values s_k (the eigenvalues of $((B_{f,g})^* B_{f,g})^{1/2}$) satisfy:

$$\sup_{k\geq 1} k^{\frac{1}{p}} s_k(B_{f,g}) \leq C_p \|f\|_{L^{p'},\infty} \|g\|_{L^p}. \tag{2.3.5}$$

Proof: See the estimate (1) of Cwikel [19]. ■

Proposition 2.3.7. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, $d > 1$. For $0 < \alpha < d$, the Riesz potential operator $I_{\alpha,\Omega}$ is a compact operator. The eigenvalues of $I_{\alpha,\Omega}$ satisfy the estimates*

$$\lambda_j(\Omega) \leq C j^{-\alpha/d} |\Omega|^{\frac{\alpha}{2d}} \,. \tag{2.3.6}$$

Proof: We follow [50, Proposition 2.1] here. We apply Proposition 2.3.6, while choosing $f = \frac{1}{|\cdot|^{d-\frac{\alpha}{2}}}$ which is in weak $L^{\frac{2d}{2d-\alpha}}(\mathbb{R}^d)$ and $g = \chi_\Omega$ which is in $L^{\frac{2d}{\alpha}}(\mathbb{R}^d)$, to deduce that T

defined by (2.3.3) is a compact operator. Therefore the operator T^*T is compact, but this is the operator $\widetilde{I}_{\alpha,d}$ as we have seen during the proof of Proposition 2.3.7 and consequently, the summand $I_{\alpha,\Omega}$ is a compact operator.

The estimate (2.3.5) then yields

$$j^{\frac{\alpha}{2d}}\, s_j(T) \le C|\Omega|^{\frac{\alpha}{2d}} \ . \tag{2.3.7}$$

Since the eigenvalues of $I_{\alpha,\Omega}$ equal the squares of the singular numbers of T, we get

$$\lambda_j(I_{\alpha,\Omega}) = s_j(T)^2 \le K j^{-\frac{\alpha}{d}} |\Omega|^{\frac{\alpha}{d}} \,. \tag{2.3.8}$$

■

We recall the following definition of operators of Schatten class from [52] and refer the reader to the details therein for more information.

Definition 2.3.8. *Let $1 \le p < \infty$. A compact operator $T : H \to H$ on a Hilbert space H belongs to the Schatten class $S_p(H)$ if*

$$\|T\|_p = \left(\sum_{k=1}^{\infty} s_k^p\right)^{\frac{1}{p}} < \infty$$

where s_k are the singular number of T. $\|T\|_p$ is called the Schatten-von Neumann norm of T. When T is self-adjoint and belongs to $S_p(H)$ for $p = 2$, it is called a Hilbert-Schmidt operator and when true for $p = 1$ it is said to be of trace-class. If T is a self-adjoint operator in $S_p(H)$ and p is a positive integer, notice that $\|T\|_p^p$ is just $\operatorname{Tr} T^p$.

Corollary 2.3.9. *The operator $I_{\alpha,\Omega}$ belongs to the Schatten class $S_p(L^2(\Omega))$ for $p > \frac{d}{\alpha}$.*

Proof: This follows immediately from the estimate (2.3.6) and the definition of $S_p(L^2(\Omega))$. ■

It will also be useful to keep in mind the following result (see Goffeng [25, Theorem 2.4]) which allows to give an alternate description of the Schatten p-norm for integer values of $p(> 2)$. Given, $1 \le p, q < \infty$, take $L^{p,q}$ to be defined as consisting of the class of functions $k(x, y)$ on $\Omega \times \Omega$ such that

$$\|k\|_{L^{p,q}} = \left(\int_\Omega \left(\int_\Omega |k(x,y)|^p dx\right)^{\frac{q}{p}} dy\right)^{\frac{1}{q}} < \infty.$$

The Hermitian conjugate of the function k is defined by $k^*(x,y) := \overline{k(y,x)}$.

Theorem 2.3.10. *Suppose that $K_j : L^2(\Omega) \to L^2(\Omega)$ are operators with integral kernels k_j for $j = 1, \cdots, m$ such that $\|k_j\|_{L^{p'},p}, \|k_j^*\|_{L^{p'},p} < \infty$ for certain $p > 2$. Then, for $m \geq p$, the operator $K_1K_2 \dots K_m$ is a trace class operator, and we have the trace formula*

$$Tr(K_1K_2 \dots K_m) = \int_{\Omega^m} \prod_{j=1}^{m} k_j(x_j, x_{j+1}) dx_1 dx_2 \dots dx_m$$

where we identify x_{m+1} with x_1.

As an immediate corollary we have.

Corollary 2.3.11. *Let $p > \max\left(\frac{d}{\alpha}, 2\right)$ be a natural number. Then, we have*

$$\|I_{\alpha,\Omega}\|_p^p = \sum_{k=1}^{\infty} s_k^p = Tr(I_{\alpha,\Omega}^p) = (C(d,\alpha))^p \int_{\Omega^p} \prod_{k=1}^{p} |x_k - x_{k+1}|^{\alpha-d} dx_1 \dots dx_p, \qquad (2.3.9)$$

where x_{p+1} is identified with x_1.

Proof: The hypothesis on p guarantees that $|\cdot|^{-(d-\alpha)}$ is locally in $L^{p'}$ allowing us to apply Theorem 2.3.10 with $m = p$ and obtain the desired conclusion. ■

The spectral functionals of interest in the thesis are, the first eigenvalue $\lambda_1(\Omega)$ of the Riesz operator and the Schatten norm $\|I_{\alpha,\Omega}\|_p$ of the Riesz operator (for p in $\mathbb{N}$). We gather below of some of their relevant properties which will be used in Chapters 3 and 4. The first (largest) eigenvalue of the Riesz operator is characterized as follows:

$$\lambda_1(\Omega) = \max\left\{ \frac{\int_\Omega \int_\Omega C(d,\alpha) \frac{u(x)u(y)}{|x-y|^{d-\alpha}} dxdy}{\int_\Omega u^2(x)\, dx} : u \in L^2(\Omega) \setminus \{0\} \right\}. \qquad (2.3.10)$$

It can be shown that the maximizer u_1 for (1.2.2) exists, and is, in fact, continuous and satisfies the following Euler-Lagrange equation in the weak form

$$\int_\Omega\int_\Omega C(d,\alpha) \frac{u_1(x)\phi(y)}{|x-y|^{d-\alpha}} dxdy = \lambda_1(\Omega) \int_\Omega u_1(x)\phi(x) dx \qquad \text{for all } \phi \in L^2(\Omega)\,. \qquad (2.3.11)$$

The strong formulation being

$$C(d,\alpha) \int_\Omega \frac{u_1(y)}{|x-y|^{d-\alpha}} dy = \lambda_1(\Omega)\, u_1(x) \qquad \text{in } L^2(\Omega)\,. \qquad (2.3.12)$$

Moreover, the first eigenvalue is simple and the eigenfunction is of constant sign as stated in Lemma 3.1 of [53] and referred to as Jentzsch's theorem.

Proposition 2.3.12. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, $d > 1$. For $0 < \alpha < d$, the first eigenvalue λ_1 (the largest) of the Riesz potential operator $I_{\alpha,\Omega}$ is positive and simple; the corresponding eigenfunction u_1 can be chosen positive.*

Proof: We reproduce the following proof from Lemma 3.1 of [53]. It will use the fact that $I^2_{\alpha,\Omega}$ being the square of $I_{\alpha,\Omega}$ is also a compact, self-adjoint and positive linear operator, and by the variational principle we have that

$$\lambda_1^2 = \sup_{f \in L^2(\Omega), f \neq 0} \frac{\langle I^2_{\alpha,\Omega} f, f \rangle}{\|f\|^2} . \tag{2.3.13}$$

Let λ_1 be the first eigenvalue of $I_{\alpha,\Omega}$ and u_1 an eigenfunction corresponding to λ_1, that is, $I_{\alpha,\Omega} u_1 = \lambda_1 u_1$. We keep in mind that $I^2_{\alpha,\Omega} u_1 = \lambda_1^2 u_1$. We assume that without loss of generality that the L^2 norm of u_1 is 1.
First, we shall prove that u_1 cannot change sign in the domain Ω, which is equivalent to,

$$|u_1(x)u_1(y)| = u_1(x)u_1(y) \qquad \text{for all } x \in \Omega \text{ and } y \in \Omega . \tag{2.3.14}$$

We can proceed by assuming that (2.3.14) is false, so there is x_0 and $y_0 \in \Omega$ such that

$$|u_1(x_0)u_1(y_0)| > u_1(x_0)u_1(y_0). \tag{2.3.15}$$

From the continuity of u_1, there are open neighborhoods $U(x_0, r) \subset \Omega$ and $U(y_0, r) \subset \Omega$ such that

$$|u_1(x)u_1(y)| > u_1(x)u_1(y) \qquad \text{for all } x \in U(x_0, r) \text{ and } y \in U(y_0, r) .$$

Therefore,

$$\begin{aligned} \langle I^2_{\alpha,\Omega}|u_1|, |u_1|\rangle = \langle I_{\alpha,\Omega}|u_1|, I_{\alpha,\Omega}|u_1|\rangle &= C(d,\alpha)^2 \int_\Omega \int_\Omega \int_\Omega \frac{|u_1|(y)}{|y-z|^{d-\alpha}} \frac{|u_1|(x)}{|x-z|^{d-\alpha}} dydxdz \\ &> C(d,\alpha)^2 \int_\Omega \int_\Omega \int_\Omega \frac{u_1(y)}{|y-z|^{d-\alpha}} \frac{u_1(x)}{|x-z|^{d-\alpha}} dydxdz \\ &= \langle I_{\alpha,\Omega} u_1, I_{\alpha,\Omega} u_1 \rangle = \lambda_1^2 . \end{aligned} \tag{2.3.16}$$

which contradicts the variational principle (2.3.13). So, u_1 is of the same sign in Ω and moreover, there cannot be any x_0 in Ω such that $u(x_0) = 0$. Indeed, it follows from the relation

$$u_1(x_0) = \frac{1}{\lambda_1}\pi^{-\frac{d}{2}}2^{-\alpha}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}\int_\Omega \frac{u(y)}{|x_0-y|^{d-\alpha}}dy$$

that if $u_1(x_0) = 0$ then it is zero identically in the whole of Ω which is not possible.

Finally, we show that λ_1 is simple. Let's suppose that u_1 and f_1 are two eigenfunctions corresponding to λ_1 which are linearly independent, then $u_1 + cf_1$ is also an eigenfunction for λ_1 for an arbitrary real number c. By what has been proved, without loss of generality $u_1 + cf_1 > 0$ in Ω, that is, $c > -\frac{u_1}{f_1}$. This contradicts that c is an arbitrary real number. ■

Proposition 2.3.13. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set. Then, the domain functionals, the first eigenvalue of the Riesz operator $\lambda_1(\cdot)$ and the Schatten norm for the Riesz operator $\|I_{\alpha,\cdot}\|_p$ satisfy the following properties*

1. *(translation invariance) $F(\Omega) = F(\Omega + x)$ for all $x \in \mathbb{R}^d$.*
2. *(invariance under orthonormal transformations) $F(\Omega) = F(T(\Omega))$ for every orthonormal transformation T.*
3. *(homothety law) $F(k\Omega) = k^\alpha F(\Omega)$ for any $k > 0$.*
4. *(domain monotony) If $A \subset B$ are open sets, then $F(A) \leq F(B)$.*

Proof: The properties for λ_1 are easily established starting from the variational formulation of λ_1. In the case of the Schatten norm it is more convenient to start from the expression (2.3.9). We sketch the proof of properties 1 and 3 for the functional $\lambda_1(\cdot)$. Let $z \in \mathbb{R}^d$, for this note that $u \mapsto w$, given by $w(x) := u(x+z)$, is an isomorphism from $L^2(\Omega)$ to $L^2(\Omega + z)$ and under this isomorphism, we have

$$\begin{aligned}
\frac{\int_{\Omega+z}\int_{\Omega+z}\frac{w(x)w(y)}{|x-y|^{d-\alpha}}dxdy}{\int_{\Omega+z}|w(x)|^2dx} &= \frac{\int_{\Omega+z}\int_{\Omega+z}\frac{u(x+z)u(y+z)}{|x-y|^{d-\alpha}}dxdy}{\int_{\Omega+z}|u(x+z)|^2dx} \\
&= \frac{\int_\Omega\int_\Omega\frac{u(t)u(r)}{|t-r|^{d-\alpha}}dtdr}{\int_\Omega|u(t)|^2dt}
\end{aligned}$$

from which property 1 follows by taking the maximum. Now we show property 3, for this note that $u \mapsto w$, given by $w(x) := u\left(\frac{x}{k}\right)$, is an isomorphism from $L^2(\Omega)$ to $L^2(k\Omega)$ and under this isomorphism, we have

$$\begin{aligned}\frac{\int_{k\Omega}\int_{k\Omega}\frac{w(x)w(y)}{|x-y|^{d-\alpha}}dxdy}{\int_{k\Omega}|w|^2(x)dx} &= \frac{\int_{k\Omega}\int_{k\Omega}\frac{u\left(\frac{x}{k}\right)u\left(\frac{y}{k}\right)}{|x-y|^{d-\alpha}}dxdy}{\int_{k\Omega}\left|u\left(\frac{x}{k}\right)\right|^2dx}\\ &= k^{\alpha}\frac{\int_{\Omega}\int_{\Omega}\frac{u(z)u(r)}{|z-r|^{d-\alpha}}dzdr}{\int_{\Omega}|u(z)|^2dz}\end{aligned}$$

from which property 3 follows by taking the maximum. The properties 2 and 4 can be proved analogously. ■

2.4 Isoperimetric inequalities

In this section we recall some isoperimetric inequalities, both local and non-local, already known in the literature along with short sketches of their proofs. A few of the most basic isoperimetric inequalities are the following:

$P(A^*) \leq P(A)$ and $P(\mathcal{S}A) \leq P(A)$ for any Borel measurable set in $\mathbb{R}^d$ of finite perimeter.

Closely related to these isoperimetric inequalities are the Pólya-Szegőand Riesz inequalities.

Theorem 2.4.1. (local Pólya-Szegőinequality) *Let u a non-negative function belonging to $H_0^1(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n}|\nabla u^*(x)|^2dx \leq \int_{\mathbb{R}^n}|\nabla u(x)|^2dx\,.$$

Proof: The following proof is from Lieb and Loss [39].

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla u(x)|^2dx = \int_{\mathbb{R}^n}|\widehat{\nabla} u(x)|^2dx &= \int_{\mathbb{R}^n}4\pi^2|\xi||\hat{u}(\xi)|^2d\xi\\ &= \lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^n}(1-e^{-4\pi^2|\xi|^2t})|\hat{u}(\xi)|^2d\xi\\ &= \lim_{t\to 0}\left(\frac{1}{t}\int_{\mathbb{R}^n}|\hat{u}|^2-\frac{1}{t}\int_{\mathbb{R}^n}\hat{u}(\xi)(G_t*u)\hat{}(\xi)d\xi\right)\\ &= \lim_{t\to 0}\left(\frac{1}{t}\int_{\mathbb{R}^n}|u|^2-\frac{1}{t}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}G_t(x-y)u(x)u(y)dxdy\right)\\ &\geq \lim_{t\to 0}\left(\frac{1}{t}\int_{\mathbb{R}^n}|u^*|^2-\frac{1}{t}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}u^*(x)G_t(x-y)u^*(y)dxdy\right)\end{aligned}$$

where $G_t(x-y) = \frac{1}{(4\pi t)^{n/2}}e^{\frac{-|x-y|^2}{4t}}$ is the heat kernel. The inequality in the last line is a consequence of the Riesz's inequality (2.1.12). ■

Isoperimetric inequality for the fractional perimeter: The following scale-invariant nonlocal isoperimetric inequality

$$\frac{P_s(E)}{|E|^{1-\frac{s}{d}}} \geq \frac{P_s(B)}{|B|^{1-\frac{s}{d}}} \tag{2.4.1}$$

where B is any $d-$dimensional ball, holds with equality ocurring if only if E is a ball. This result follows directly from the fractional Sobolev inequality Theorem 4.1 proved in Frank and Seiringer [24] which they proved using a fractional Pólya-Szegő inequality on L^p. If we restrict ourselves to sets of finite perimeter in $\mathbb{R}^d$, then the above result also can be obtained following Proposition 4.2 and Corollary 4.4 of Brasco et al [10]. The non-local isoperimetric inequality may be viewed upon as saying that the symmetric decreasing rearrangement diminishes the fractional perimeter. As regards the Steiner symmetrization we may state the following and it, basically, follows directly from Corollary 4.1.2 (which is proved in Chapter 4 of this thesis) for the choice of the function χ_E in $W^{s,1}(\mathbb{R}^d)$.

$$P_s(\mathcal{S}E) \leq P_s(E) \text{ for every Borel set } E \text{ of finite perimeter} \tag{2.4.2}$$

where $\mathcal{S}E$ is the Steiner symmetrization with respect a hyperplane.

Theorem 2.4.2. (non-local Pólya-Szegőinequality under symmetric decreasing rearrangement) *Let $d \geq 1$, $0 < s < 1$, $2s < d$ and $u \in \widetilde{W}_0^{s,2}(\Omega)$. Then,*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u^*(x) - u^*(y)|^2}{|x-y|^{d+2s}} dxdy \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dxdy\,.$$

The equality holds iff u is proportional to a translate of a symmetric decreasing function.

Proof: See [1] and [24] (where also the equality case is discussed). ■

Remark 2.4.3. *In [43] the nonlocal Pólya-Szegő inequality for Steiner symmetrization is stated without proof. A proof of this can be found in my master's thesis [44](in Spanish). In Chapter 4, we give an English translation of the same for the sake of completeness.*

(Rayleigh-Faber-Krahn inequalities): We briefly recall the following Faber-Krahn inequalities.

Theorem 2.4.4. (FKI for Dirichlet Laplacian) *Let Ω be an open set of finite volume in $\mathbb{R}^d$ and let*

$$\lambda_1(\Omega) = \inf \left\{ \int_\Omega |\nabla u(x)|^2 dx : u \in H_0^1(\Omega), \|u\|_{L^2(\Omega)} = 1 \right\}, \tag{2.4.3}$$

be the first eigenvalue of the Laplacian eigenvalue problem on Ω with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = \lambda u & in\ \Omega \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{2.4.4}$$

Then, we have

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$$

where Ω^* is the unit ball having the same volume as Ω.

Proof: Let ϕ_1 be a minimizer for $\lambda_1(\Omega)$ in (2.4.3) so that

$$\lambda_1(\Omega) = \int_{\mathbb{R}^n} |\nabla \phi_1(x)|^2 dx\,. \tag{2.4.5}$$

Applying Pólya-Szegő inequality (Theorem 2.4.1) together with the observation that the L_2-norm is unchanged under symmetric decreasing rearrangement, that is, it holds that $\|\phi_1\|_2 = \|\phi_1^*\|_2 = 1$, we get

$$\lambda_1(\Omega) = \int_\Omega |\nabla \phi_1(x)|^2 dx \geq \int_{\Omega^*} |\nabla \phi_1^*(x)|^2 dx \geq \lambda_1(\Omega^*)\,. \tag{2.4.6}$$

■

Theorem 2.4.5. (FKI for the Riesz potential operator) *Let Ω be an open set of finite volume in $\mathbb{R}^d$. Abusing notation, let $\lambda_1(\Omega)$ be the first eigenvalue of the Riesz potential operator given as in* (1.2.2). *Then, we have*

$$\lambda_1(\Omega^*) \geq \lambda_1(\Omega).$$

Where Ω^* is the symmetric rearrangement of Ω.

Proof: This is given following Lemma 3.2 of [50]. Let ϕ_1 be a maximizer of $\lambda_1(\Omega)$.

Then, using Riesz's inequality for the symmetric decreasing rearrangement (2.1.12) together with the observation that the L_2-norm is unchanged under symmetric decreasing rearrangement, we have

$$\lambda_1(\Omega) = C(d,\alpha)\int_\Omega\int_\Omega \frac{\phi_1(x)\phi_1(y)}{|x-y|^{d-\alpha}}dxdy \leq C(d,\alpha)\int_{\Omega^*}\int_{\Omega^*}\frac{\phi_1^*(x)\phi_1^*(y)}{|x-y|^{d-\alpha}}dxdy \leq \lambda_1(\Omega^*)\,. \tag{2.4.7}$$

■

Theorem 2.4.6. *For $\Omega \subset \mathbb{R}^d$ which is open and bounded, let $\lambda_1^s(\Omega)$ be the principal Dirichlet eigenvalue for the fractional Laplacian defined as below*

$$\lambda_1^s(\Omega) = \inf_{u\in \widetilde{W}_0^{s,2}(\Omega)}\left\{[u]^2_{W^{s,2}(\Omega)} := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy : \|u\|_{L^2(\Omega)} = 1\right\} \tag{2.4.8}$$

over the fractional order Sobolev space $\widetilde{W}_0^{s,2}(\Omega)$. Then, we have

$$\lambda_1^s(\Omega) \geq \lambda_1^s(\Omega^*). \tag{2.4.9}$$

Moreover, if equality holds in (2.4.9) *then Ω is a ball.*

Proof: The inequality and the equality condition, can be obtained using Theorem 2.4.2. ■

The following is an isoperimetric inequality for the Schatten norms of Riesz potential proved in Rozenblum et al [50].

Theorem 2.4.7. *For any integer p with $p_0 := \frac{d}{\alpha} < p < \infty$, we get*

$$\|I_{\alpha,\Omega}\|_p \leq \|I_{\alpha,\Omega^*}\|_p\,.$$

Proof: By the trace formula for the Schatten norm of $I_{\alpha,\Omega}$ given in Corollary 2.3.11, we get

$$\sum_{j=1}^\infty \lambda_j(\Omega)^p = Tr(I^p_{\alpha,\Omega}) = (C(d,\alpha))^p\int_{\Omega^p}\prod_{k=1}^p|x_k - x_{k+1}|^{\alpha-d}dx_1\ldots dx_p,\;\; x_{p+1}\equiv x_1 \tag{2.4.10}$$

By Brascamp-Lieb-Luttinger inequality (2.1.13) for symmetric-decreasing rearrangement, we get

$$\int_{\Omega} \cdots \int_{\Omega} |y_1 - y_2|^{\alpha-d} |y_2 - y_3|^{\alpha-d} \ldots |y_p - y_1|^{\alpha-d} dy_1 \ldots dy_p$$
$$\leq \int_{\Omega^*} \cdots \int_{\Omega^*} |y_1 - y_2|^{\alpha-d} |y_2 - y_3|^{\alpha-d} \ldots |y_p - y_1|^{\alpha-d} dy_1 \ldots dy_p.$$

By the above inequality and (2.4.10), we get

$$\sum_{j=1}^{\infty} \lambda_j^p(\Omega) \leq \sum_{j=1}^{\infty} \lambda_j^p(\Omega^*) .$$

■

2.5 Hausdorff convergence of sets

Definition 2.5.1. *The Minkowski addition of two sets $X, Y \subset \mathbb{R}^d$ is defined by*

$$X \oplus Y := \bigcup_{y \in Y} (X + y). \tag{2.5.1}$$

The Minkowski difference of two sets $X, Y \subset \mathbb{R}^d$ is defined by

$$X \ominus Y := \bigcap_{y \in Y} (X - y). \tag{2.5.2}$$

If $Y = -Y$, then

$$X \ominus Y = \bigcap_{y \in Y} (X + y). \tag{2.5.3}$$

Note: If K is a convex body the set $K \ominus B(0, \epsilon)$ is called the inner parallel body of K at distance ϵ (see page 93 and 148 of [54]).

Definition 2.5.2. *Let K and C be two non-empty compact sets in $\mathbb{R}^d$. Then their Hausdorff distance is defined as*

$$d^H(K, C) = \inf\{\epsilon \geq 0 ;\ K \subseteq C \oplus B(0, \epsilon) \text{ and } C \subseteq K \oplus B(0, \epsilon)\} .$$

Let O_1, O_2 be two open subsets of a compact set B. Then the so called complementary

Hausdorff distance is defined by:

$$d_H(O_1, O_2) = d^H(B \backslash O_1, B \backslash O_2). \tag{2.5.4}$$

Remark 2.5.3. *$d^H(K_n, K) \longrightarrow 0$ when $n \to \infty$ if and only if every $\epsilon > 0$ there is n_ϵ such that $K_n \subseteq K \oplus B(0, \epsilon)$ and $K \subseteq K_n \oplus B(0, \epsilon)$ for every $n \geq n_\epsilon$.*

The following compactness result with respect to Hausdorff convergence shall be useful.

Theorem 2.5.4. *Let B be a fixed compact set in $\mathbb{R}^d$ and $\{\Omega_n\}$ a sequence of open sets in B. Then, there is an open set $\Omega \subset B$ and a subsequence $\{\Omega_{n_k}\}$ of $\{\Omega_n\}$ which converges with respect to the Hausdorff distance to Ω.*

Proof: See Theorem 2.3.15 of Henrot [28]. ■

Remark 2.5.5. *Let $\{P_k\}$ a sequence of polygons of n sides, contained in a closed ball B, then by Theorem 2.5.4 we have there is a set P such that a subsequence of $\{P_k\}$ converges with respect to the Hausdorff distance to P. By the Heine-Borel theorem, upto a subsequence, the vertices of P_k converge to some number m of distinct points. So, in fact, these point should be the vertices of P, a polygon of m sides with $m \leq n$. In particular, it follows that if $\{\triangle_n\}$ is a sequence of triangular domains with fixed area say A that converges with respect to the Hausdorff distance to a domain $\triangle$, then $\lim_{n \longrightarrow \infty} Area(\triangle_n) = Area(\triangle) = A$. If $\triangle$ is a interval, we $Area(\triangle) = 0 < A$. Therefore $\triangle$ is a triangle.* ■

Finally, we end this section by stating a lower semicontinuity result for the perimeter with respect to the convergence, in the complementary Hausdorff distance for which we refer to Henrot and Pierre [29, Chapter 2.3] for the details. Given a family of convex open sets $\{\Omega_n\}$, all contained in a fixed ball, that converges with respect to the complementary Hausdorff distance to an open convex set Ω we have

$$P(\Omega) \leq \varliminf_{n \to \infty} P(\Omega_n)\,. \tag{2.5.5}$$

Chapter 3

Isoperimetric inequalities for the Riesz potential operator in the class of triangles and quadrilaterals

The main results of this chapter are the isoperimetric inequalities for the first eigenvalue and the Schatten norm of the Riesz potential operator while restricting the class of domains to be triangles or quadrilaterals. Some of the existing results has been reviewed briefly in the Introduction and recalled in Chapter 2.4. We recall that, given $d > 1$, $0 < \alpha < d$ and a bounded open domain $\Omega \subseteq \mathbb{R}^d$, the Riesz operator $I_{\alpha,\Omega} : L^2(\Omega) \to L^2(\Omega)$ has been defined in Chapter 2.3 by

$$(I_\alpha u)(x) = C(d,\alpha) \int_\Omega \frac{u(y)}{|x-y|^{d-\alpha}} dy$$

where $C(d,\alpha) = \pi^{-\frac{d}{2}} 2^{-\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}$. This was seen to be a compact, self-adjoint and positive operator whose maximal eigenvalue is given by

$$\lambda_1(\Omega) = \max\left\{ \int_\Omega \int_\Omega C(d,\alpha) \frac{u(x)u(y)}{|x-y|^{d-\alpha}} dxdy : u \in L^2(\Omega), \|u\|_{L^2(\Omega)} = 1 \right\}.$$

The maximizer in the above exists, is an eigenfunction which satisfies the following Euler-Lagrange equation

$$C(d,\alpha) \int_\Omega \frac{u(y)}{|x-y|^{d-\alpha}} dy = \lambda_1(\Omega) u(x) \text{ in } L^2(\Omega) \tag{3.0.1}$$

having the weak formulation

$$\int_\Omega\int_\Omega C(d,\alpha)\frac{u(x)\phi(y)}{|x-y|^{d-\alpha}}dxdy=\lambda_1(\Omega)\int_\Omega u(x)\phi(x)dx \text{ for all } \phi\in L^2(\Omega)\,. \tag{3.0.2}$$

Moreover, the first eigenvalue is simple and the eigenfunction is of constant sign. The main aims of this chapter are to prove Theorems 1.2.1 and 1.2.2. We also prove two secondary results which are used in the proof of the main results. The first is the continuity of the functionals $\lambda_1(\Omega)$ and $\|I_{\alpha,\Omega}\|_p$ of the Riesz operator with respect to the convergence in the Hausdorff complementary metric of a family of uniformly bounded convex open sets (Proposition 3.1.4 and Corollary 4.2.3). The second, is a discussion of the equality case in the Riesz's inequality for the Steiner symmetrization in higher dimensions, a result which is not thoroughly treated in the literature.

3.1 Continuity of the shape functionals for the Riesz potential operator

Before we can prove the continuity results Proposition 3.1.4 and Corollary 4.2.3 for the spectral functionals for the Riesz operator we need the following observations on convex bodies.

The first proposition shows formally that the inner parallel body of a set X at distance ϵ is contained in the complement, within any bigger set Z, of the fattening of $Z\setminus X$ by ϵ.

Proposition 3.1.1. *Let $X,Z\subset\mathbb{R}^d$ with $X\subset Z$ and let $\epsilon>0$. Then*

$$X\ominus B(0,\epsilon)\subseteq Z\backslash((Z\backslash X)\oplus B(0,\epsilon))\,. \tag{3.1.1}$$

Proof: From (2.5.2) it follows that $(X\ominus B(0,\epsilon))^c=X^c\oplus B(0,\epsilon)$ from which

$$Z\cap(X\ominus B(0,\epsilon))^c=Z\cap(X^c\oplus B(0,\epsilon)). \tag{3.1.2}$$

On the other hand, we have trivially

$$Z\cap((Z\cap X^c)\oplus B(0,\epsilon))\subseteq Z\cap(X^c\oplus B(0,\epsilon)).$$

So, after taking the complement with respect to Z in the anterior and then using (3.1.2), we get

$$Z \backslash (Z \cap (X \ominus B(0,\epsilon))^c) \subseteq Z \backslash (Z \cap ((Z \cap X^c) \oplus B(0,\epsilon))).$$

From this, in view of the hypothesis that $X \subseteq Z$, we get

$$X \ominus B(0,\epsilon) \subseteq Z \backslash ((Z \cap X^c) \oplus B(0,\epsilon)).$$

■

Proposition 3.1.2. *Let X be an open convex set of $\mathbb{R}^d$. Then, the following holds:*

$$X \ominus B(0,\epsilon) = \overline{X} \ominus B(0,\epsilon)\,.$$

Proof: On the one hand, it is clear that $X \ominus B(0,\epsilon) \subseteq \overline{X} \ominus B(0,\epsilon)$.
On the other hand, for any $x \in \overline{X} \ominus B(0,\epsilon)$, it follows from the definition (2.5.2) that $B(x,\epsilon) \subseteq \overline{X}$. Since for an open convex set it is true that $\dot{\overline{X}} = X$ (see Theorem 2.28 of [46]) we get $B(x,\epsilon) \subseteq X$, and so, $x \in X \ominus B(0,\epsilon)$. This proves the inclusion which is less obvious. ■

The main ingredient in the proof of the Proposition 3.1.4 is the following Lemma.

Lemma 3.1.3. *Let K be a convex body in $\mathbb{R}^d$, with $B(0,r) \subset K \subset B(0,R)$ for some numbers $r > 0$ and $R > 0$. If $0 < \epsilon < \frac{r^2}{4R}$, then*

$$\left(1 - 4\frac{R\epsilon}{r^2}\right) K \subset K \ominus B(0,\epsilon) \subset K\,. \tag{3.1.3}$$

Proof: See Lemma 2.3.6, page 93 of [54]. ■

The following abstract continuity property for a shape functional can be deduced, essentially following Proposition 1.3 [43] or Proposition 2.9 [41].

Proposition 3.1.4. *Let F be a functional over the classes of nonempty convex open subsets in $\mathbb{R}^d$ which satisfies the following conditions*

1. *(invariance under translations) $F(B+x) = F(B)$ for any $x \in \mathbb{R}^d$ and B open convex subset of $\mathbb{R}^n$.*

2. *(domain monotonicity)* $F(B) \leq F(A)$ *whenever* $A \subseteq B$ *(or,* $F(A) \leq F(B)$ *whenever* $A \subseteq B$*).*

3. *(homothety law) There exist* $\alpha \in \mathbb{R}$*, such that* $F(kA) = k^{\alpha} F(A)$ *for all* $k > 0$ *and for every bounded open set* A*.*

Let B *be a fixed compact set in* $\mathbb{R}^d$ *and* Ω_n *be a family of nonempty convex open subsets of* B *which converges, for the complementary Hausdorff distance, to a nonempty convex open set* Ω*. Then,* $F(\Omega) = \lim_{n\to\infty} F(\Omega_n)$*.*

Proof: Since F is invariant under translation we can assume that $0 \in \Omega$. Since Ω is an open set, there is an open ball such that $B(0, r) \subseteq \Omega$. We assume, without loss of generality, that B is the closure of the ball $B(0, R)$ for some R large enough. STEP 1: Since, $d_H(\Omega_n, \Omega) \to 0$, by the definition of the complementary Hausdorff distance, for any $\epsilon > 0$ there exist n_ϵ such that

$$B\backslash\Omega \subset (B\backslash\Omega_n) \oplus B(0, \epsilon) \text{ for all } n \geq n_\epsilon \tag{3.1.4}$$

and

$$B\backslash\Omega_n \subset (B\backslash\Omega) \oplus B(0, \epsilon) \text{ for all } n \geq n_\epsilon\,. \tag{3.1.5}$$

Further, by taking the relative complement in (3.1.5) with respect to B and thereafter applying Proposition 3.1.1 with the choices $X = \Omega$ and $Z = B$, we obtain

$$\Omega \ominus B(0, \epsilon) \subseteq \Omega_n \text{ for all } n \geq n_\epsilon. \tag{3.1.6}$$

Therefore, by Proposition 3.1.2, we also have

$$\overline{\Omega} \ominus B(0, \epsilon) \subseteq \Omega_n \text{ for all } n \geq n_\epsilon \tag{3.1.7}$$

From the above, by choosing $0 < \epsilon < r$, we get

$$B(0, r - \epsilon) = B(0, r) \ominus B(0, \epsilon) \subseteq \Omega \ominus B(0, \epsilon) \subseteq \Omega_n \text{ for all } n \geq n_\epsilon.$$

So, if $0 < \epsilon < \frac{r}{2}$, then we shall also have

$$B(0, r/2) \subseteq \Omega_n \text{ for all } n \geq n_\epsilon \,. \tag{3.1.8}$$

Step 2: Let us now fix $0 < \epsilon < \frac{r^2}{16R}$. For this choice, we also have $0 < \epsilon < \frac{r^2}{4R}$ and so, applying Lemma 3.1.3 with the compact set $\overline{\Omega}$ in mind, we get

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)\Omega \subset \left(1 - 16\frac{R\epsilon}{r^2}\right)\ \overline{\Omega} \subset \overline{\Omega} \ominus B(0,\epsilon)\,. \tag{3.1.9}$$

So, using (3.1.7), it follows that

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)\Omega \subset \Omega_n \text{ for all } n \geq n_\epsilon \,. \tag{3.1.10}$$

Then, by using the domain monotonicity and homothety properties of F, we can obtain the inequality

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)^\alpha F(\Omega) \leq F(\Omega_n)\,.$$

After taking the liminf, as $n \to \infty$, we obtain

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)^\alpha F(\Omega) \leq \varliminf_{n\to\infty} F(\Omega_n). \tag{3.1.11}$$

If we now take the limit as $\epsilon \to 0$ in (3.1.11) we get

$$F(\Omega) \leq \varliminf_{n\to\infty} F(\Omega_n). \tag{3.1.12}$$

Step 3: Arguing similarly as in Step 1, but starting from (3.1.4), we can also obtain the inclusion

$$\Omega_n \ominus B(0,\epsilon) \subseteq \Omega \text{ for all } n \geq n_\epsilon \,.$$

In view of (4.2.2) and since we have chosen $0 < \epsilon < \frac{r^2}{16R}$, by applying Lemma 3.1.3 with the compact set $\overline{\Omega}_n$ in mind we obtain

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)\Omega_n \subset \left(1 - 16\frac{R\epsilon}{r^2}\right)\ \overline{\Omega}_n \subset \overline{\Omega}_n \ominus B(0,\epsilon)\,.$$

So, continuing similarly as in Step 2, we conclude that

$$\overline{\lim_{n\to\infty}} F(\Omega_n) \leq F(\Omega)\,. \tag{3.1.13}$$

The desired result follows from (3.1.12) and (3.1.13). ∎

Corollary 3.1.5. *The first eigenvalue $\lambda_1(\cdot)$ and the Schatten norm $\|I_{\alpha,\cdot}\|_p$ of the Riesz potential operator satisfy the hypotheses of Proposition 3.1.4 and, so we have that λ_1 and $\|I_{\alpha,\cdot}\|_p$ are continuous with respect to the convergence, in the complementary Hausdorff distance, of a family of uniformly bounded non-empty convex open sets.*

3.2 Case of equality in the Riesz's inequality for Steiner symmetrization

We requiere the following lemma.

Lemma 3.2.1. *Let $A \subseteq \mathbb{R}$ be a Lebesgue measurable set with $|A| > 0$. If $A = A + x$ for some $x \in \mathbb{R} - \{0\}$, then $|A| = \infty$.*

Proof: Since $|A| > 0$, necessarily there exists an $n \in \mathbb{Z}$ for which $B := A \cap [n, n+1]$ has positive measure. Notice that $B + x \subseteq A + x = A$ and then, using induction, we also obtain $B + mx \subseteq A$ for every $m \in \mathbb{Z}$. Now, we assume, without loss of generality, that $x > 0$ and then choose $M \in \mathbb{N}$ such that $Mx > 1$. Then it follows that the intervals $[sMx + n, sMx + n + 1]$ are disjoint for distinct $s \in \mathbb{Z}$. Since, $B + sMx \subset [sMx + n, sMx + n + 1]$, we obtain that the sets $B + sMx$ are disjoint for distinct $s \in \mathbb{Z}$. Therefore, necessarily it follows that $|A| = \infty$, since A contains infinitely many disjoint copies of B. ∎

Proposition 3.2.2. *In addition to the hypotheses of Corollary 2.1.7, suppose that g is symmetric with respect to the hyperplane H and strictly decreasing in the orthogonal direction (moving away) and that both f and h are non-zero measurable functions, then there is equality in (2.1.11) if and only if there exists $w \in \mathbb{R}^d$ of the form $(0, 0, \ldots, 0, k)$ for some $k \in \mathbb{R}$, such that $f(x) = \mathcal{S}f(x - w)$ and $h(x) = \mathcal{S}h(x - w)$ for almost all $x \in \mathbb{R}^d$.*

Proof: The equality case in the d-dimensional case, for $d \geq 2$, is discussed below. The proof, unlike the proof of the equality case in Riesz's inequality under the symmetric decreasing rearrangement or Schwarz symmetrization sketched in Theorem 3.9 of Lieb and Loss [39], does not require induction on the dimension. We exploit, directly, the one-dimensional result. Since we have chosen an orthogonal coordinate system wherein H is the plane $\{(x',0) : x' = (x_1, x_2, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}\}$, the hypotheses on g gives us $g(z',\cdot) = \mathcal{S}g(z',\cdot)$ and so the equality $I(f,g,h) = I(\mathcal{S}f, g, \mathcal{S}h)$ may be written as

$$\int_{\mathbb{R}^{d-1}\times\mathbb{R}^{d-1}} \int_{\mathbb{R}\times\mathbb{R}} f(x',x_d)g(x'-y',x_d-y_d)h(y',y_d)dx_d dy_d dx' dy'$$
$$= \int_{\mathbb{R}^{d-1}\times\mathbb{R}^{d-1}} \int_{\mathbb{R}\times\mathbb{R}} \mathcal{S}f(x',x_d)g(x'-y',x_d-y_d)\mathcal{S}h(y',y_d)dx_d dy_d dx' dy',$$

with $(x',y') \in \mathbb{R}^{d-1}\times\mathbb{R}^{d-1}$. For any fixed $(x',y') \in \mathbb{R}^{d-1}\times\mathbb{R}^{d-1}$, by definition, $\mathcal{S}f(x',\cdot)$ and $\mathcal{S}h(y',\cdot)$ are the one dimensional symmetric decreasing rearrangements of $f(x',\cdot)$ and $h(y',\cdot)$ respectively. So, for any $x', y' \in \mathbb{R}^{d-1}$, Riesz's inequality applied to the functions $f(x',\cdot), g(x'-y',\cdot)$ and $h(y',\cdot)$ viewed as functions of the final variable gives us

$$\int_{\mathbb{R}\times\mathbb{R}} f(x',x_d)g(x'-y',x_d-y_d)h(y',y_d)dx_d dy_d$$
$$\leq \int_{\mathbb{R}\times\mathbb{R}} \mathcal{S}f(x',x_d)g(x'-y',x_d-y_d)\mathcal{S}h(y',y_d)dx_d dy_d\,.$$

From the above it follows that

$$\int_{\mathbb{R}\times\mathbb{R}} f(x',x_d)g(x'-y',x_d-y_d)h(y',y_d)dx_d dy_d$$
$$= \int_{\mathbb{R}\times\mathbb{R}} \mathcal{S}f(x',x_d)g(x'-y',x_d-y_d)\mathcal{S}h(y',y_d)dx_d dy_d \quad \text{a.e.} \quad x',y'\,. \tag{3.2.1}$$

Let S be the set of (x',y') for which equality holds in (3.2.1), so that $(\mathbb{R}^{d-1}\times\mathbb{R}^{d-1})\setminus S$ has measure 0. Also, let

$$M = \{x' \in \mathbb{R}^{d-1} : f(x',\cdot) \text{ is non-zero}\}\,, \quad N = \{y' \in \mathbb{R}^{d-1} : h(y',\cdot) \text{ is non-zero}\}\,.$$

Note that, by our hypothesis, both M and N are of positive measure. Also, for any $(x',y') \in M \times N$, by the definition of M and N, both $f(x',\cdot)$ and $h(y',\cdot)$ are non-zero functions. Therefore, for any $(x',y') \in S\bigcap(M\times N)$, since we have equality for the Riesz's inequality in one-dimension, we deduce that there exists $k \in \mathbb{R}$ which, *a priori*, is a function $k(x', y')$ of (x', y') such that

$$\begin{aligned} f(x', x_d) &= \mathcal{S}f(x', x_d - k) \;\; a.e. \;\; x_d \quad \text{and} &(3.2.2)\\ h(y', y_d) &= \mathcal{S}h(y', y_d - k) \;\; a.e. \;\; y_d\,. &(3.2.3)\end{aligned}$$

Now, $S \bigcap (M \times N)$ is of full measure in $(M \times N)$ and so, for almost all $y' \in N$ the section $(S \bigcap (M \times N))_{y'}$ which is the section of $S \bigcap (M \times N)$ at y' is of full measure in M. Consider any y'_0 in N for which $(S \bigcap (M \times N))_{y'_0}$ has the same measure as M. Now, for any $x', z' \in (S \bigcap (M \times N))_{y'_0}$, we observe that

$$\begin{aligned} f(x', x_d) = \mathcal{S}f(x', x_d - k_{x',y'_0}) \;\; a.e. \;\; x_d\,, \;\; h(y'_0, y_d) = \mathcal{S}h(y'_0, y_d - k_{x',y'_0}) \;\; a.e. \;\; y_d \\ f(z', x_d) = \mathcal{S}f(z', x_d - k_{z',y'_0}) \;\; a.e. \;\; x_d\,, \;\; h(y'_0, y_d) = \mathcal{S}h(y'_0, y_d - k_{z',y'_0}) \;\; a.e. \;\; y_d \end{aligned}$$

from which we obtain

$$h(y'_0, y_d) = h(y'_0, y_d + k_{z',y'_0} - k_{x',y'_0}) \; a.e. \; y_d \tag{3.2.4}$$

Since y'_0 belongs to N we see that $\{y_d : h(y'_0, y_d) \neq 0\}$ is of positive measure. Also, since $h(y'_0, \cdot)$ vanishes at infinity, it is possible to fix a t with $0 < t < \infty$ such that the measure of $A = \{y_d : h(y'_0, y_d) > t\}$ is finite and positive. Also notice that, by (3.2.4), we have $A = A + k_{x',y'_0} - k_{z',y'_0}$. Therefore, using Lemma 3.2.1, we have $k_{z',y'_0} = k_{x',y'_0}$. Let us denote this common value by $k(y'_0)$. This implies that $f(x', \cdot)$ is symmetric decreasing about $x_d = k(y'_0)$ independently of x' in $(S \bigcap (M \times N))_{y'_0}$. That is, $f(x', \cdot)$ is symmetric decreasing about $x_d = k(y'_0)$ for almost every x' in M. Since, $f(x', \cdot)$ is independent of y'_0 we can also conclude that $k(y'_0)$ does not really depend on y'_0 since a non-zero function vanishing at infinity cannot be symmetric-decreasing with respect to two distinct points. Therefore, for almost every x' in M, $f(x', \cdot)$ is symmetric-decreasing about $x_d = k$ where k now is independent of x', y'.

Similarly, we could start with any $x'_0 \in M$ such that the section $(S \bigcap (M \times N))_{x'_0}$ which is the section of $S \bigcap (M \times N)$ at x'_0 is of full measure in N. Then, arguing as above, after reversing the roles of f and h and using the point x'_0 in M, we also conclude that, for almost all $w' \in M$, $h(w', \cdot)$ is Steiner symmetric about $y_d = k$ also, using (3.2.2)-(3.2.3), since k turns out to be independent of x' and y'. ■

3.3 Proof of the main theorems

The maximization problem of Theorem 1.2.1 for maximizing the Riesz eigenvalue $\lambda_1(\cdot)$ in the class of triangles or quadrilaterals of given area may be formulated as

$$\max\{\lambda_1(P) : P \in P_N, \quad \text{area}(P) = a\} \tag{3.3.1}$$

where P_N is the class of non-empty open convex polygons with N edges. It is possible to show using the compactness of the class of uniformly bounded open convex sets for the complementary Hausdorff distance (see Theorem 2.3.15 of [28]) and Corollary 3.1.5 the existence of an open set Ω for which the maximum is attained in (3.3.1). As mentioned in the introduction, even for the eigenvalues of the Laplacian with Dirichlet boundary condition the corresponding FKI is still open for polygons with 5 or more sides. Our aim is to show the existence of solutions of (3.3.1) in the class of triangles and quadrilaterals and to characterize them, that is, prove a FKI.

Proof of Theorem 1.2.1 Let $\triangle_1$ be an arbitrary triangle of area a. We successively define triangles $\triangle_{n+1}$ by taking the Steiner symmetrization of $\triangle_n$ with respect to the perpendicular bisector of a side with respect to which there is no symmetry. Then, by part 2 of the Proposition of 2.1.5, each of the triangles has area a and also the triangles are uniformly bounded since, by part 3 of the proposition by Proposition 2.1.13, the circumradius decreases after successive Steiner symmetrizations. Then, it can be shown that the sequence $\triangle_n$ converges with respect to the complementary Hausdorff distance to an equilateral triangle $\triangle$ (see page 158 of [45], 20.7 Theorem, pages 153-154 of [4] or Theorem 3.3.3 [28]).

Let f_n be an eigenfunction for $\lambda_1(\triangle_n)$, that is, a function for which the maximum is attained in (1.2.2) which we can take to be continuous, non-negative and having unit L^2 norm. By property 1 of Proposition 2.3.13 the eigenvalue $\lambda_1(\triangle_n)$ is invariant under translations, and so in this expression we may always assume that the coordinate system has its origin at the circumcenter of the triangle. Furthermore, for the ease of considering Steiner symmetrization of functions with respect to the chosen line of Steiner symmterization of Δ_n, by the rotation invariance of the eigenvalue, we may that this line is oriented along the x_1-axis. Let $\mathcal{S}f_n$ be the corresponding Steiner symmetrization of $\tilde{f}_n$ (the extension of f_n by zero outside $\triangle_n$) which is a Borel measurable function vanishing at infinity. We

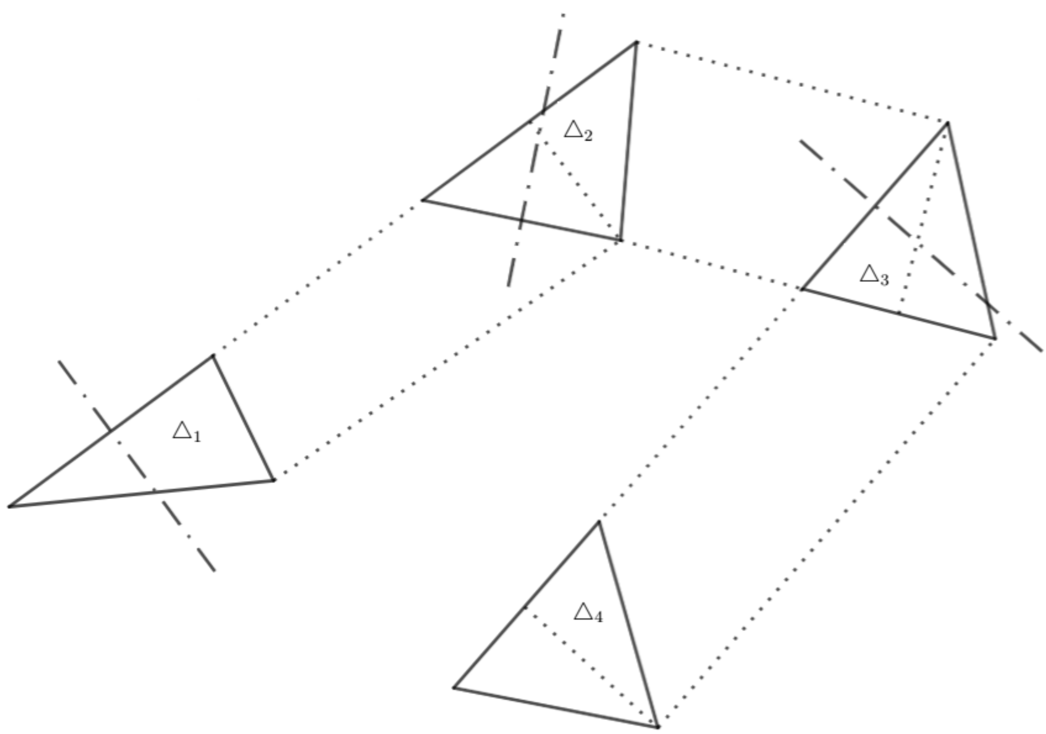

Figure 3.1: Sequence of triangles

note that $\mathcal{S}f_n$ has to be supported on the closure of $\triangle_{n+1}$. By property 2 in Proposition 2.1.5, we note that $\mathcal{S}f_n$ also has norm 1 in the L^2 norm. Notice that the Riesz potential $|x|^{-(2-\alpha)}$, is Steiner symmetric with respect to the x_1 axis. Since $\alpha < 2$, it is also a strictly decreasing function away from the x_1-axis in the x_2-direction. Therefore, we can apply Corollary 2.1.7 to the function $\tilde{f}_n$ (taken twice) and with the function in the middle taken as the Riesz potential to obtain

$$\begin{aligned}
\lambda_1(\triangle_n) &= \int_{\triangle_n}\int_{\triangle_n} C(2,\alpha)\frac{f_n(x)f_n(y)}{|x-y|^{2-\alpha}}dxdy \\
&\leq \int_{\triangle_{n+1}}\int_{\triangle_{n+1}} C(2,\alpha)\frac{\mathcal{S}f_n(x)\mathcal{S}f_n(y)}{|x-y|^{2-\alpha}}dxdy \\
&\leq \max_{w\in L^2(\triangle_{n+1})}\int_{\triangle_{n+1}}\int_{\triangle_{n+1}} C(2,\alpha)\frac{w(x)w(y)}{|x-y|^{2-\alpha}}dxdy \\
&= \lambda_1(\triangle_{n+1}) \qquad (3.3.2)
\end{aligned}$$

for each n. Therefore,

$$\lambda_1(\triangle_1) \leq \lambda_1(\triangle_n) \text{ for all } n\,.$$

We then use the continuity property of the Riesz eigenvalue for the complementary Hausdorff convergence (Corollary 4.2.3) and get

$$\lambda_1(\triangle_1) \leq \lim_{n\to\infty} \lambda_1(\triangle_n) = \lambda_1(\triangle).$$

This shows that λ_1 has its maximum value for an equilateral triangle of given area.

The proof, in the case of quadrilaterals, that the maximum is attained for a square uses a similar argument as in the case of triangles. To start with, apply Steiner symmetrization with respect to an axis l_1 which is perpendicular to a diagonal of the quadrilateral (not necessarily convex, see one of the figures below for the non-convex case), for which the other two vertices aren't on the same side of this diagonal. The resulting object is a *convex* quadrilateral which is symmetric with respect to this axis. Next, we Steiner symmetrize with respect to a perpendicular axis l_2 and thereby get a rhombus. This is to be followed by a Steiner symmetrization with respect to an axis l_3 perpendicular to one of the sides to produce a rectangle. The rectangle is then Steiner symmetrized with respect to an axis perpendicular to a diagonal to get, again, a rhombus. By repeating the procedures for the rhombus and rectangle we end up with an infinite sequence of rhombi and rectangles which converge, ultimately, in the complementary Hausdorff distance, to a square (refer to pages 158-159 of [45] or 20.8 Theorem, pages 154-155 of [4]). Now we will address the proof of the uniqueness. We consider the case of triangular domains. Suppose that Δ is any triangle for which the maximum is attained in (3.3.1). If we suppose that Δ is not an equilateral triangle, then there is at least one side m of Δ such that Δ is not symmetric with respect to the perpendicular bisector to m. Let $\mathcal{S}\Delta$ the Steiner symmetrization of Δ respect to the perpendicular bisector of m. Let f be the first normalized eigenfunction associated to $\lambda_1(\Delta)$ and $\mathcal{S}f$ its Steiner symmetrization respect to the perpendicular bisector of m. Using the property 3 of the Proposition 2.1.5 and Corollary 2.1.7 and we obtain

$$\begin{aligned}
\lambda_1(\mathcal{S}\Delta) &\geq \int_{\mathcal{S}\Delta}\int_{\mathcal{S}\Delta} C(2,\alpha)\frac{\mathcal{S}f(x)\mathcal{S}f(y)}{|x-y|^{2-\alpha}}dxdy \\
&\geq \int_{\Delta}\int_{\Delta} C(2,\alpha)\frac{f(x)f(y)}{|x-y|^{2-\alpha}}dxdy \\
&= \lambda_1(\Delta)\,. \qquad (3.3.3)
\end{aligned}$$

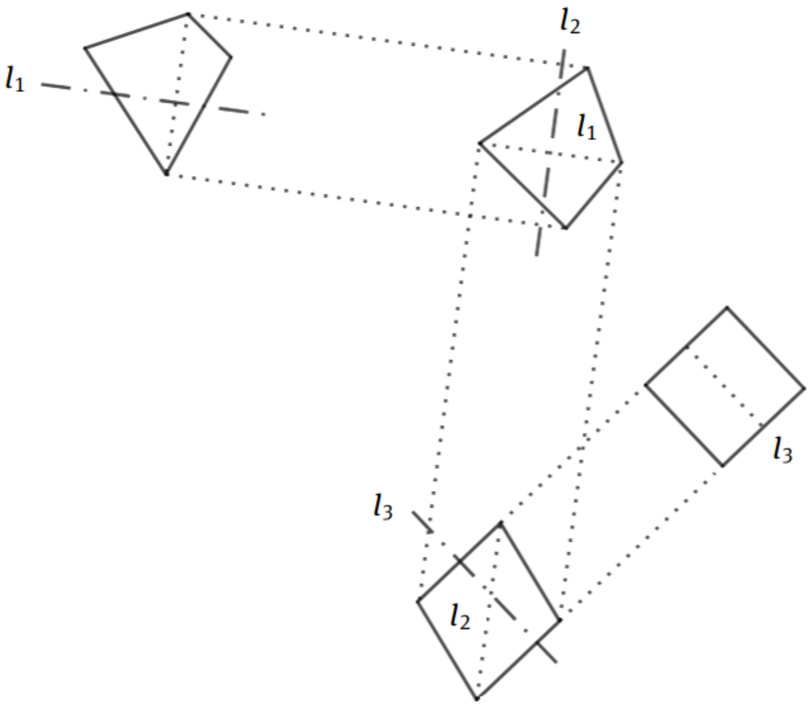

Figure 3.2: Sequence of quadrilaterals

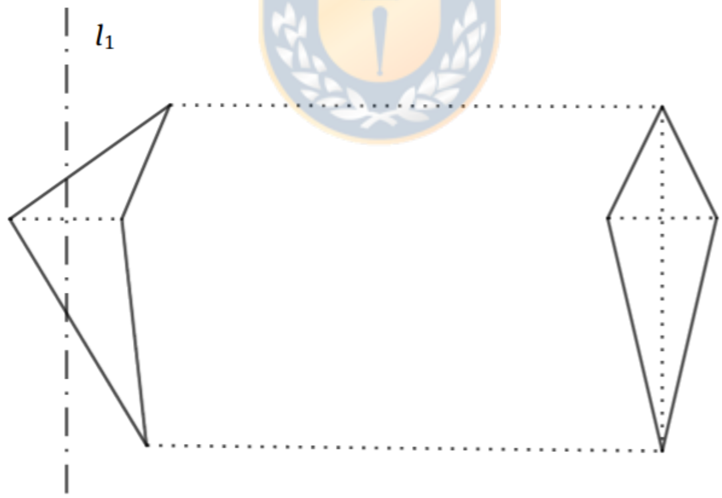

Figure 3.3: Symmetrization of a nonconvex quadrilateral

Now, the fact that Δ maximizes λ_1 leads to the observation that $\lambda_1(\mathcal{S}\Delta) = \lambda_1(\Delta)$ and so, we get the equality case in Riesz's inequality. Then, by Proposition 3.2.2 it follows that f is a translate of $\mathcal{S}f$. Furthermore, $\mathcal{S}f$ is a maximizer for $\lambda_1(\mathcal{S}\Delta)$. As mentioned in the Proposition 2.3.12, the first eigenfunction for the Riesz operator $f, \mathcal{S}f$ are strictly positive on Δ and $\mathcal{S}\Delta$ respectively, and so

$$\begin{aligned} \Delta = \{x : f(x) > 0\} &= \{x : \mathcal{S}f(x-y) > 0\} \\ &= \mathcal{S}\Delta + y. \end{aligned}$$

Thus, Δ is Steiner symmetric with respect to the perpendicular bisector of m contrary to our supposition. So, we conclude that the equilateral triangle is the only minimizer. The uniqueness in the quadrilateral case is similarly proved by contradiction. ■

Remark 3.3.1. *Given an arbitrary pentagon, it is not possible to guarantee the existence of a line with respect to which the Steiner symmetrization results in a pentagon; in general, the number of sides will be increased. For example, the pentagon in the following figure becomes a hexagon. Note that the interval* $\overline{\text{AB}}$ *gives two sides after applying a Steiner symmetrization with respect to* l*. This situation has been an obstacle in making progress on such isoperimetric problems for* $n > 4$*.*

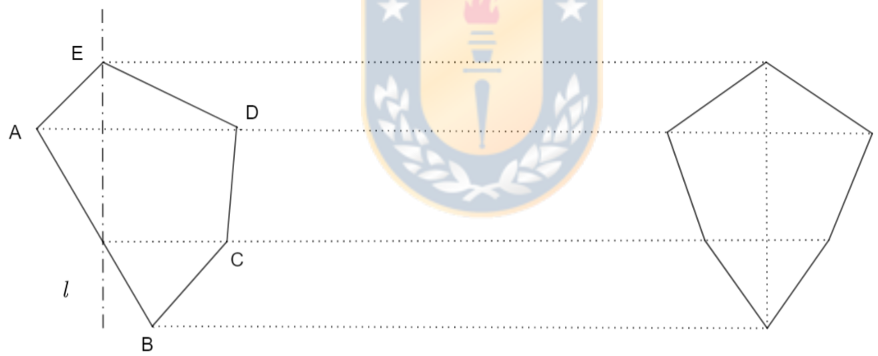

Figure 3.4: The Steiner symmetrization of a pentagon has, in general, six edges

We now briefly consider possible 3−dimensional analogs of our planar results for triangles and quadrilaterals. These results apply to tetrahedra and certain prisms and are outlined in the following Theorem. We begin with tetrahedra, the 3−dimensional analogue of triangles. A reference to Steiner symmetrization in 3 dimension is p. 5 of [45].

Definition 3.3.2. *Let* E_1 *and* E_2 *be two parallel planes (visualize the first one being under the second),* R *a polygonal region in* E_1*, and* L *a line that intersects* E_1 *and* E_2*, but not* R*. For each point* P *of* R*, let* $\overline{\text{PP}'}$ *be a segment parallel to* L *and joining the point* P *with*

its other extreme P' in E_2. The union of all segments $\overline{\mathrm{PP'}}$ is called a prism.
The polygonal region R is called the lower base, or simply the base of the prism. The part of the prism that is at E_2 is called the top base. The edges of the faces of the prism that do not lie in the planes E_1 and E_2 are called the lateral edges of the prism. If L is perpendicular to E_1 and E_2, then the prism is called a right prism; otherwise it is an oblique prism.

Theorem 3.3.3. *The maximum of $\lambda_1(\Omega)$ among all tetrahedra of given volume V is obtained when Ω is a regular tetrahedron and only when Ω is a regular tetrahedron. Similarly, the maximum of $\lambda_1(\Omega)$ among all prisms(right or oblique) of given volume V and a quadrilateral base is obtained when Ω is a cube and only when Ω is a cube.*

Proof: Let T be an arbitrary tetrahedron of a given volume V. First we Steiner symmetrize T with respect to a plane perpendicular to one of its edges. Next we Steiner symmetrize our new tetrahedron with respect to a plane perpendicular to the edge that lies in the former plane of symmetrization (this is the one edge of the tetrahedron that isn't the edge with which we started and that doesn't meet it). The tetrahedron so obtained has two planes of symmetry perpendicular to each other. By repeating these above two steps alternatively we end up with an infinite sequence of tetrahedron which converge, ultimately, in the complementary Hausdorff distance, to a regular tetrahedron T_R with the given volume V(see [45] p. 159). By the same arguments given in the previous proof we have that

$$\lambda_1(T_R) \geq \lambda_1(T). \tag{3.3.4}$$

Now we will address the proof of the uniqueness of the case of tetrahedron domains. Suppose that T_0 is any tetrahedron for which the maximum is attained in

$$\max\{\lambda_1(T) : T \quad \text{is a tetrahedron}, \quad \text{volume}(P) = a\} \tag{3.3.5}$$

If we suppose that T_0 is not a regular tetrahedron, then there is at least one edge m of T_0 such that T_0 is not symmetric with respect to the perpendicular bisector plane to m. Let $\mathcal{S}T_0$ the Steiner symmetrization of T_0 respect to the perpendicular bisector plane of m. Let f be the first normalized eigenfunction associated to $\lambda_1(T_0)$ and $\mathcal{S}f$ its Steiner symmetrization respect to the perpendicular bisector plane of m. Using the property 3 of the Proposition 2.1.5 and Corollary 2.1.7 and we obtain

$$\begin{aligned}
\lambda_1(\mathcal{S}T_0) &\geq \int_{\mathcal{S}T_0}\int_{\mathcal{S}T_0} C(3,\alpha)\frac{\mathcal{S}f(x)\mathcal{S}f(y)}{|x-y|^{2-\alpha}}dxdy \\
&\geq \int_{T_0}\int_{T_0} C(3,\alpha)\frac{f(x)f(y)}{|x-y|^{2-\alpha}}dxdy \\
&= \lambda_1(T_0)\,. \qquad (3.3.6)
\end{aligned}$$

Now, the fact that T_0 maximizes λ_1 leads to the observation that $\lambda_1(\mathcal{S}T_0) = \lambda_1(T_0)$ and so, we get the equality case in Riesz's inequality. Then, by Proposition 3.2.2 it follows that f is a translate of $\mathcal{S}f$. Furthermore, $\mathcal{S}f$ is a maximizer for $\lambda_1(\mathcal{S}T_0)$. As mentioned in the Proposition 2.3.12, the first eigenfunction for the Riesz operator $f, \mathcal{S}f$ are strictly positive on T_0 and $\mathcal{S}T_0$ respectively, and so

$$\begin{aligned}
T_0 = \{x : f(x) > 0\} &= \{x : \mathcal{S}f(x-y) > 0\} \\
&= \mathcal{S}T_0 + y.
\end{aligned}$$

Thus, T_0 is Steiner symmetric with respect to the perpendicular bisector plane of m contrary to our supposition. So, we conclude that the regular tetrahedron is the only minimizer.

The proof, in the case of prisms, uses a similar argument as in the case of tetrahedron. Let P be an arbitrary prism of a given volume V. First we Steiner symmetrize P with respect to a plane perpendicular to one of its lateral edges and change it into a right prism. We then take a sequence of Steiner symmetrizations with respect to planes orthogonal to the base of the right prism so that the bases converge to a square (it is enough to choose the same sequence of lines as in the successive Steiner symmetrizations of the quadrilateral and the perpendicular to the base to form these planes). This leads to a a right prism with square base (see [45] p. 159). The above process is repeated taking one of the lateral faces now as the base to get another right prism and so on until the right prisms converge to a cube.

The uniqueness in the case of prisms is proved similarly by contradiction.

Proof of Theorem 1.2.2: We now briefly sketch the proof of corresponding isoperimetric inequalities for Schatten norms of Riesz potential. The proof goes along similar lines as in that of Theorem 1.2.1, in the case of triangles we start with the same construction of a sequence of triangles Δ_n starting from an arbitrary triangle. By Brascamp-Lieb-Luttinger inequality for Steiner symmetrization (Theorem 2.1.8), we get

$$\sum_{j=1}^{\infty} \lambda_j^p(\triangle_n) = (C(2,\alpha))^p \int_{\triangle_n} \dots \int_{\triangle_n} |y_1 - y_2|^{\alpha-2} |y_2 - y_3|^{\alpha-2} \dots |y_p - y_1|^{\alpha-2} dy_1 \dots dy_p$$
$$\leq (C(2,\alpha))^p \int_{\triangle_{n+1}} \dots \int_{\triangle_{n+1}} |y_1 - y_2|^{\alpha-2} |y_2 - y_3|^{\alpha-2} \dots |y_p - y_1|^{\alpha-2} dy_1 \dots dy_p = \sum_{j=1}^{\infty} \lambda_j^p(\triangle_{n+1}).$$

We then use the continuity property of Schatten norms of Riesz potential for the complementary Hausdorff convergence and get

$$\sum_{j=1}^{\infty} \lambda_j^p(\triangle_1) \leq \lim_{n\to\infty} \sum_{j=1}^{\infty} \lambda_j^p(\triangle_n) = \sum_{j=1}^{\infty} \lambda_j^p(\triangle).$$

The quadrilateral case is similarly proved.

Now we will address the proof of the uniqueness. We consider the case of triangular domains. Suppose that Δ is any triangle for which the maximum is attained in $\|I_{\alpha,\Omega}\|_p$ where Ω has area fixed. If we suppose that Δ is not an equilateral triangle, then there is at least one side m of Δ such that Δ is not symmetric with respect to the perpendicular bisector to m. Let $\mathcal{S}\Delta$ the Steiner symmetrization of Δ respect to the perpendicular bisector of m. We have,

$$\sum_{j=1}^{\infty} \lambda_j^p(\Delta) \leq \sum_{j=1}^{\infty} \lambda_j^p(\mathcal{S}\Delta).$$

Now, the fact that Δ maximizes $\|I_{\alpha,\Omega}\|_p$ leads to the observation that $\|I_{\alpha,\Delta}\|_p = \|I_{\alpha,\mathcal{S}\Delta}\|_p$ and so, we get

$$\begin{aligned} &\int_{\Delta} \dots \int_{\Delta} |y_1 - y_2|^{\alpha-2} |y_2 - y_3|^{\alpha-2} \dots |y_p - y_1|^{\alpha-2} dy_1 \dots dy_p \\ &= \int_{\mathcal{S}\Delta} \dots \int_{\mathcal{S}\Delta} |y_1 - y_2|^{\alpha-2} |y_2 - y_3|^{\alpha-2} \dots |y_p - y_1|^{\alpha-2} dy_1 \dots dy_p \end{aligned} \tag{3.3.7}$$

On the other hand, for almost all y_{p-2} and y_1, there are no translates of $|y_{p-1} - y_{p-2}|^{\alpha-2}$ and $|y_p - y_1|^{\alpha-2}$ which are both Steiner symmetric. Therefore, by Theorem 3.2.2, we conclude that

$$\int_{\Delta}\int_{\Delta} |y_{p-2} - y_{p-1}|^{\alpha-2} |y_{p-1} - y_p|^{\alpha-2} |y_p - y_1|^{\alpha-2} dy_{p-1} dy_p$$

$$< \int_{S\Delta}\int_{S\Delta} |y_{p-2}-y_{p-1}|^{\alpha-2}|y_{p-1}-y_p|^{\alpha-2}|y_p-y_1|^{\alpha-2}dy_{p-1}dy_p,$$

for almost every $(y_1, y_2, \ldots, y_{p-2})$.

Integrating into the remaining variables we get

$$\begin{aligned}\int_{\Delta} \cdots \int_{\Delta} & |y_1-y_2|^{\alpha-2}|y_2-y_3|^{\alpha-2}\ldots|y_p-y_1|^{\alpha-2}dy_1\ldots dy_p \\ < & \int_{S\Delta}\cdots\int_{S\Delta} |y_1-y_2|^{\alpha-2}|y_2-y_3|^{\alpha-2}\ldots|y_p-y_1|^{\alpha-2}dy_1\ldots dy_p\end{aligned}$$

The above is contrary to (3.3.7). So, we conclude that the equilateral triangle is the only minimizer. The uniqueness in the quadrilateral case is similarly proved by contradiction.

■

Theorem 3.3.4. *Let $p > \frac{d}{\alpha}$. The maximum of $\|I_{\alpha,\Omega}\|_p$ among all tetrahedra of given volume is obtained when Ω is a regular tetrahedron. Similarly, the maximum of $\|I_{\alpha,\Omega}\|_p$ among all prisms(right or oblique) of given volume and a quadrilateral base is obtained when Ω is a cube.*

Proof: The proof of this theorem is analogous to the proof of Theorem 4.3.3. ■

Theorem 3.3.5. *Let $p > \max\left(\frac{d}{\alpha}, 2\right)$. The maximum of $\|I_{\alpha,\Omega}\|_p$ among all tetrahedra of given volume is obtained when Ω is a regular tetrahedron. Similarly, the maximum of $\|I_{\alpha,\Omega}\|_p$ among all prisms(right or oblique) of given volume and a quadrilateral base is obtained when Ω is a cube.*

Proof: The proof of this theorem is analogous to the proof of Theorem 4.3.3

Chapter 4

An isoperimetric inequality for the fractional Cheeger constant and nonlocal perimeter for triangles and quadrilaterals

In this chapter we will prove Theorem 1.2.3 and Theorem 1.2.4. The proof uses a non-local Pólya-Szegő inequality under Steiner symmetrization and continuity results for the respective functionals, for the convergence of a family of uniformly bounded non-empty convex open sets, in the complementary Hausdorff distance. This inequality was also used in a previous work on isoperimetric inequalities for the first eigenvalue of the fractional p-Laplacian with Dirichlet boundary conditions [43]. Since, in the literature, it is difficult to find anything beyond a statement of the non-local Pólya-Szegő under Steiner symmetrization its proof was detailed in the master's thesis Olivares [43] in Spanish. Here, we reproduce the proof for the sake of completeness (only for bounded functions). The proof is an adaptation of the proof of Lemma A.2 of [24] to the case of Steiner symmetrization. The preliminary results include the continuity of the fractional perimeter and the continuity of the fractional Cheeger constant with respect to the convergence, in the complementary Hausdorff distance, of a family of uniformly bounded non-empty convex open sets.

4.1 Non-local Pólya-Szegő inequality

Following the same ideas given in Lemma $A.2.$ of [24], for Steiner symmetrization instead of symmetric decreasing rearrangement, we get the following lemma:

For J a nonnegative, convex function on $\mathbb{R}^d$ with $J(0) = 0$ and k a nonnegative measurable function on $\mathbb{R}^d$, we let

$$E[u] := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(u(x) - u(y))k(x - y)dxdy.$$

Lemma 4.1.1. *Let J be a nonnegative, convex function on $\mathbb{R}$, $J(0) = 0$ and $k \in L_1(\mathbb{R}^d)$ be a nonnegative strictly decreasing function. Then for a bounded function u with $E[u] < \infty$ and $|\{u > \tau\}| < \infty$ finite for all $\tau > 0$ we have*

$$E[u] \geq E[\mathcal{S}u],$$

where $\mathcal{S}u$ is the Steiner symmetrization of u with respect a hyperplane H. If $J(t) = |t|$, then equality holds if and only if the level sets $\{u > \tau\}$ are is symmetric respect to H (up to a translation) for a.e. $\tau > 0$.

Proof: Let's write first $J = J_+ + J_-$ with

$$J_+(t) = \begin{cases} J(t) & \text{si } \ t \geq 0 \\ 0 & \text{si } \quad t < 0, \end{cases}$$

and $J_-(t) = J_+(-t)$. We can define via the above

$$E_+[u] := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J_+(u(x) - u(y))k(x - y)dxdy. \tag{4.1.1}$$

Similarly , we define E_-. Following the above definition we can write $E[u] = E_+[u]+E_-[u]$. By the above, it is enough to prove the theorem for the case E_+, because the statement for E_- (and hence for the original E) follows by $J_-(u(x) - u(y)) = J_+(u(y) - u(x))$ and the symmetry of k with respect to the center.

Since J is convex so is J_+. From the above J_+ it is locally Lipschitz, so J_+ is differentiable almost everywhere on $[0, \infty)$ and $J_+(t) = \displaystyle\int_0^t J'_+(\tau)d\tau$, with J'_+ the right derivative of J_+.

This allows us to write

$$\begin{aligned}
\int_0^\infty J_+'(u(x)-\tau)\chi_{\{\tau:u(y)\le\tau\}}(\tau)d\tau &= \int_{u(y)}^{u(x)} J_+'(u(x)-\tau)\chi_{\{\tau:u(y)\le\tau\}}(\tau)d\tau \\
&= \int_{u(y)}^{u(x)} J_+'(u(x)-\tau)d\tau \\
&= -\int_{u(x)-u(y)}^{0} J_+'(w)dw \\
&= J_+(u(x)-u(y)). \qquad (4.1.2)
\end{aligned}$$

Let

$$e_\tau^+[u] := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J_+'(u(x)-\tau)k(x-y)\chi_{\{\tau:u(y)\le\tau\}}(\tau)dxdy.$$

Using Tonelli's Theorem, we get

$$\begin{aligned}
\int_0^\infty e_\tau^+[u]d\tau &= \int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J_+'(u(x)-\tau)k(x-y)\chi_{\{\tau:u(y)\le\tau\}}(\tau)dxdyd\tau \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x-y)\int_0^\infty J_+'(u(x)-\tau)\chi_{\{\tau:u(y)\le\tau\}}(\tau)d\tau dxdy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J_+(u(x)-u(y))k(x-y)dxdy \\
&= E_+[u]. \qquad (4.1.3)
\end{aligned}$$

We claim that $\int_{\mathbb{R}^d} J_+'(u(x)-\tau)dx < \infty$ for every $\tau\in\mathbb{R}^+$. Indeed, since u is bounded there is $M\in\mathbb{R}^+$ such that $u(x)\le M$ for every $x\in\mathbb{R}^d$. From the above, together with the fact that J_+' is an increasing function and u is a function vanishing at infinity, we can get

$$\begin{aligned}
\int_{\mathbb{R}^d} J_+'(u(x)-\tau)dx &= \int_{\{z:u(z)\ge\tau\}} J_+'(u(x)-\tau)dx \\
&\le J_+'(M-\tau)\int_{\{z:u(z)\ge\tau\}} dx < \infty.
\end{aligned}$$

Writing $\chi_{\{u\le\tau\}} = 1-\chi_{\{u>\tau\}}$ and using that the Lebesgue integral is invariant under translation, we obtain that

$$e_\tau^+[u] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J_+'(u(x)-\tau)k(x-y)\chi_{\{u\le\tau\}}(y)dxdy$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J'_+(u(x)-\tau)k(x-y)(1-\chi_{\{u>\tau\}})(y)dxdy \\
&= \|k\|_1\int_{\mathbb{R}^d} J'_+(u(x)-\tau)dx - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J'_+(u(x)-\tau)k(x-y)\chi_{\{u>\tau\}}dxdy. \quad (4.1.4)
\end{aligned}
$$

By part 3 of Proposition 2.1.5 we have

$$
\int_{\mathbb{R}^d} J'_+(u(x)-\tau)dx = \int_{\mathbb{R}^d} J'_+(\mathcal{S}u(x)-\tau)dx. \tag{4.1.5}
$$

Applying Corollary 2.1.7 to the second term of (4.1.4) together with the relation (4.1.5) we get

$$
e^+_\tau[\mathcal{S}u] \leq e^+_\tau[u]. \tag{4.1.6}
$$

By (4.1.6) and the above we have that

$$
E_+[\mathcal{S}u] \leq E^+[u],
$$

now we will consider the case of equality. Now, let k be strictly decreasing and suppose that $E_+[\mathcal{S}u] = E_+[u]$ for a bounded function u. It follows from the representation (4.1.3) that we have $e^+_\tau[\mathcal{S}u] = e^+_\tau[u]$ for $\tau > 0$ a.e. For equality to hold in (4.1.6), for any $\tau > 0$ a.e. by Proposition 3.2.2 necessarily there exists an $a_\tau \in \mathbb{R}^n$ such that

$$
\chi_{\{u>\tau\}}(x', x_d) = \chi_{\{u>\tau\}}(x', x_d - a_\tau) \text{ and } \; J'_+(u(x', x_d)-\tau) = J'_+(\mathcal{S}u(x', x_d - a_\tau)-\tau).
$$

for almost everywhere (x', x_d). From this we conclude that $\mathcal{S}\{u > \tau\} + (0, a_\tau) = \{u > \tau\}$ for a.e. $\tau > 0$. ■

The nonlocal Pólya-Szegő inequality (Theorem 2.4.2) can be rewritten by as follows

$$
\int_{\mathbb{R}^d} (-\triangle)^s u^*(x)u^*(x)dx \leq \int_{\mathbb{R}^d} (-\triangle)^s u(x)u(x)dx \tag{4.1.7}
$$

Recall the definition of fractional Laplacian and its relation to the Riesz potential operator

$$
(-\Delta)^s u(x) = 2\int_{\mathbb{R}^d} \frac{(u(x)-u(y))}{|x-y|^{n+2s}}dy := I_{-2s}u(x). \tag{4.1.8}
$$

Using above we can rewrite the inequality (4.1.9)

$$\int_{\mathbb{R}^d} I_{-2s}u^*(x)u^*(x)dx \leq \int_{\mathbb{R}^d} I_{-2s}u(x)u(x)dx \tag{4.1.9}$$

The following nonlocal Pólya-Szegő inequality under the Steiner symmetrization is a consequence of Lemma 4.1.1. This inequality can be interpreted as the energy generated by the fractional $p-$Laplacian operator for $p = 1$ (See, [10]).
The proof of Proposition 4.1.2 given in Theorem A.1 of [24] for the symmetric decreasing rearrangement. We sketch the proof of the adaptation to the case of Steiner symmetrization given in [44] for the sake of completeness.

Proposition 4.1.2. *(nonlocal Pólya-Szegő inequality). Let $d \geq 1$, $0 < s < 1$ and $u \in \widetilde{W}_0^{s,1}(\Omega)$ which is bounded. Then,*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\mathcal{S}u(x) - \mathcal{S}u(y)|}{|x-y|^{d+s}}dxdy \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x) - u(y)|}{|x-y|^{d+s}}dxdy, \tag{4.1.10}$$

where $\mathcal{S}u$ is the Steiner symmetrization of u with respect to a given hyperplane H. The equality holds iff the level sets $\{u > \tau\}$ are symmetric respect to H (up to a translation) for a.e. $\tau > 0$.

Proof: Since $\mathcal{S}u(x)$ is nonnegative and $||u(x)| - |u(y)|| \leq |u(x) - u(y)|$, it suffices to prove the theorem for nonnegative functions. Using the definition of the Gamma function and a change of variables we obtain

$$\frac{1}{\Gamma(\frac{n+ps}{2})}\int_0^\infty \alpha^{\frac{n+ps}{2}-1}\mathrm{e}^{-\alpha|x-y|^2}d\alpha = \frac{1}{|x-y|^{n+ps}} \tag{4.1.11}$$

Using (4.1.11) and Tonelli's theorem for nonnegative integrands we have

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^{n+s}}dxdy &= C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^\infty \alpha^{\frac{n+s}{2}-1}\mathrm{e}^{-\alpha|x-y|^2}\ |u(x)-u(y)|d\alpha dxdy \\ &= C\int_0^\infty I_\alpha[u]\alpha^{\frac{n+s}{2}-1}d\alpha,\end{aligned}$$

with

$$I_\alpha[u] := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u(x)-u(y)|\mathrm{e}^{-\alpha|x-y|^2}dxdy \qquad \text{amd} \qquad C = \frac{1}{\Gamma(\frac{n+s}{2})}.$$

The function $J(t) = |t|$ is convex and nonnegative with $J(0) = 0$. The function $k(x) = e^{-\alpha|x|^2}$ is symmetric respect to H (up to a translation) which belongs to $L_1(\mathbb{R}^n)$. Therefore, applying Lemma 4.1.1 to the functional I_α we obtain the desired result. ■

4.2 (Semi-)Continuity results for the shape functionals

We prove the continuity of the fractional perimeter with respect to the convergence in the Hausdorff complementary metric of a sequence of uniformly bounded convex open sets(for open sets in general we also prove a lower semicontinuity property) and the continuity of the fractional Cheeger constant with respect to the convergence in this metric.

Proposition 4.2.1. *(lower semicontinuity of P_s) Let B be a fixed compact set in $\mathbb{R}^d$ and Ω_n be a family of nonempty open subsets of B which converges, for the complementary Hausdorff distance, to a nonempty open set Ω. Furthermore, if Ω_n are sets of finite perimeter (classic perimeter) in $\mathbb{R}^d$, with*

$$\sup_{n\in\mathbb{N}} P(\Omega_n) < \infty,$$

then

$$\varliminf_{n\to\infty} P_s(\Omega_n) \geq P_s(\Omega). \tag{4.2.1}$$

Proof: By compactness theorem for a family of sets with a uniformly bounded perimeter, there exists a subsequence $\{\Omega_{n_k}\}$ such $\chi_{\Omega_{n_k}} \longrightarrow \chi_\Omega$ in $L^1(B)$ as $k \longrightarrow \infty$. So, we can extract a further subsequence such that $\chi_{\Omega_{n_k}} \longrightarrow \chi_\Omega$ almost everywhere and, consequently, $\chi_{\Omega^c_{n_k}} \longrightarrow \chi_{\Omega^c}$ almost everywhere. By Fatou's lemma in the expression (2.2.1) for the fractional perimeter, we get the lower semicontinuity of P_s

$$\begin{aligned} P_s(\Omega) &= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\chi_\Omega(x)\chi_{\Omega^C}(y)}{|x-y|^{d+s}}dxdy \\ &= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\varliminf_{n_k\to\infty}(\chi_{\Omega_{n_k}}(x)\chi_{\Omega^C_{n_k}}(y))}{|x-y|^{d+s}}dxdy \\ &\leq \varliminf_{n_k\to\infty} P_s(\Omega_{n_k}) \end{aligned}$$ ■

Proposition 4.2.2. *Let B be a fixed compact set in $\mathbb{R}^d$ and Ω_n be a family of non-empty convex open subsets of B which converges, for the complementary Hausdorff distance, to a non-empty convex open Ω. Then, $P_s(\Omega) = \lim_{n\to+\infty} P_s(\Omega_n)$.*

Proof: Arguing similarly as in Step 1 of the proof of Proposition 3.1.4 shows that there a open ball $B(0, r/2)$ such that

$$B(0, r/2) \subseteq \Omega_n \text{ for all } n \geq n_\epsilon\,. \tag{4.2.2}$$

On the other hand,

$$P_s(\Omega_n) \quad = \quad 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\chi_{\Omega_n^C}(x)\chi_{\Omega_n}(y)}{|x-y|^{d+s}}dxdy \tag{4.2.3}$$

It is not difficult to prove that $\chi_{\Omega_n^C}(x)\chi_{\Omega_n}(y) \leq \chi_{B(0,r/2)^C}(x)\chi_B(y)$ and

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\chi_B(x)\chi_{B(0,r/2)^C}(y)}{|x-y|^{d+s}}dxdy < \infty. \tag{4.2.4}$$

For other hand, using the monotonicity of perimeters for convex bodies, we have $P(\Omega_n) \leq P(\Omega)$ for all $n \in \mathbb{N}$ (see [7], §7).
By compactness theorem for a family of sets with a uniformly bounded perimeter (see Theorem 6.3., Chapter 5 of [20]), there exists a subsequence $\{\Omega_{n_k}\}$ such $\chi_{\Omega_{n_k}} \longrightarrow \chi_\Omega$ in $L^1(B)$ as $k \longrightarrow \infty$. So, we can extract a further subsequence such that $\chi_{\Omega_{n_k}} \longrightarrow \chi_\Omega$ almost everywhere and, consequently, $\chi_{\Omega_{n_k}^c} \longrightarrow \chi_{\Omega^c}$ almost everywhere. Since the above and using the Dominated convergence theorem, we get $P_s(\Omega) = \lim_{n\to+\infty} P_s(\Omega_n)$. ∎

Corollary 4.2.3. *Then functional $h_s(\cdot)$ satisfies the hypotheses of Proposition 3.1.4, so we have that h_s is continuous with respect to the convergence, in the complementary Hausdorff distance, of a family of uniformly bounded non-empty convex open sets.*

Theorem 4.2.4. *Among all non-empty convex open sets in a ball $B \subseteq \mathbb{R}^d$, with the same non-local perimter P_s then exists at least one with maximum volume*

Proof: Let C be the class of all non-empty convex open sets in a ball $B \subseteq \mathbb{R}^d$ with perimter P_s equals to l. Now let Ω_n be a maximizing sequence for $V := \sup_{\Omega\in C} |\Omega| < \infty$. Then, there exists an convex open $\Omega \subset B$ and a subsequence in B that we name again

Ω_n such that $d^H(\Omega_n, \Omega) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\lim_{n\to\infty} |\Omega_n| = V$. From Propostion 4.2.2 we get $P_s(\Omega) = \lim_{n\to+\infty} P_s(\Omega_n) = l$. From the continuity property of the volume with respect to the convergence, in the complementary Hausdorff distance, we have $\lim_{n\to\infty} |\Omega_n| = V$. Therefore $V = |\Omega|$. ■

4.3 Proofs of the main theorems

Proof of Theorem 1.2.3 Let $\triangle_1$ be an arbitrary open triangle of area a. We, consider the sequence of open triangles $\triangle_n$ defined by successive Steiner symmetrizations in the proof of Theorem 1.2.1. By Proposition 2.2.4(3), $P_s(\triangle_n)$ is invariant under translations, and so in this expression we may always assume that the coordinate system has its origin at the circumcenter of the triangle. Furthermore, for the ease of considering Steiner symmetrization of functions with respect to the chosen line of Steiner symmetrization of $\triangle_n$, by the rotation invariance of the P_s, we may assume that this line is oriented along the x_1-axis. Since $\triangle_n$ has finite perimeter $P(\triangle_n)$, we can use the Corollary 4.4. of [10] to conclude that $P_s(\triangle_n)$ is finite, so $\chi_{\triangle_n} \in W_0^{s,1}(\triangle_n))$. We can apply the Proposition 4.1.2 to the function $\chi_{\triangle_n}$, we have

$$\begin{aligned} P_s(\triangle_n) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\chi_{\triangle_n}(x) - \chi_{\triangle_n}(y)|}{|x-y|^{2+s}} dxdy \\ &\geq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\chi_{S\triangle_n}(x) - \chi_{S\triangle_n}(y)|}{|x-y|^{2+s}} dxdy \\ &= P_s(\triangle_{n+1}) \end{aligned}$$

for each n. Therefore,

$$P_s(\triangle_1) \geq P_s(\triangle_n) \text{ for all } n\,.$$

Using Propositions 4.2.2 or 4.2.1, we have

$$P_s(\triangle_1) \geq \varliminf_{n\to\infty} P_s(\triangle_n) \geq P_s(\triangle).$$

This shows that P_s attains its minimum value for an equilateral triangle. The proof, in the case of quadrilaterals, that the minimum is attained for a square uses a similar argument as in the case of triangles. The construction of the sequence of quadrilaterals is the same as in the proof of Theorem 1.2.1.

We now turn to the question of uniqueness in the case of triangular domains. Suppose that Δ is an open triangle of given area for which the minimum is attained. If Δ is not already an equilateral triangle, then there is at least one axis m (perpendicular to one of its sides) such that Δ is not Steiner symmetric with respect to m. Let $\mathcal{S}\Delta$ be the Steiner symmetrization of Δ respect to m. Without loss of generality, m passes through the origin.

$$\begin{aligned} P_s(\triangle) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\chi_\triangle(x)-\chi_\triangle(y)|}{|x-y|^{d+s}}dxdy \\ &\geq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\chi_{\mathcal{S}\triangle}(x)-\chi_{\mathcal{S}\triangle}(y)|}{|x-y|^{d+s}}dxdy \\ &= P_s(\mathcal{S}\triangle) \end{aligned}$$

Now, since we assumed that Δ minimizes P_s this leads to $P_s(\mathcal{S}\Delta) = P_s(\Delta)$ and so, we get the equality case in nonlocal Pólya-Szegő inequality (Proposition 4.1.2). Then, by Proposition 4.1.2 it follows that

$$\mathcal{S}\{\chi_\triangle > \tau\} + a_\tau = \{\chi_\triangle > \tau\}, \tag{4.3.1}$$

for every $\tau > 0$ a.e. Equation (4.3.1) is satisfied in particular for $0 < \tau < 1$ a.e. Taking one of those τ, let's name it by τ', we have so

$$\mathcal{S}\triangle + a_{\tau'} = \triangle. \tag{4.3.2}$$

Thus, Δ is Steiner symmetric with respect to m contrary to our supposition. So, we conclude that the equilateral triangle is the only minimizer.
The quadrilateral case for P_s is proved similarly to the triangular case. ■

Theorem 4.3.1. *The maximum of $P_s(\Omega)$ among all tetrahedra of given volume is obtained when Ω is a regular tetrahedron and only when Ω is a regular tetrahedron. Similarly, the maximum of $P_s(\Omega)$ among all prisms(right or oblique) of given volume and a quadrilateral base is obtained when Ω is a cube and only when Ω is a cube.*

Proof: The proof of this theorem is analogous to the proof of Theorem 4.3.3 ■

Proof of Theorem 1.2.4 Let $E_{\triangle_n}$ $s-$Cheeger set of $\triangle_n$, that is, $h_s(\triangle_n) = \frac{P_s(E_{\triangle_n})}{|E_{\triangle_n}|}$, we have

$$\begin{aligned} h_s(\triangle_n) &= \frac{\displaystyle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\chi_{E_{\triangle_n}}(x) - \chi_{E_{\triangle_n}}(y)|}{|x-y|^{2+s}} dxdy}{|E_{\triangle_n}|} \\ &\geq \frac{\displaystyle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\chi_{\mathcal{S}E_{\triangle_n}}(x) - \chi_{\mathcal{S}E_{\triangle_n}}(y)|}{|x-y|^{2+s}} dxdy}{|\mathcal{S}E_{\triangle_n}|} \\ &= \frac{P_s(\mathcal{S}E_{\triangle_n})}{|\mathcal{S}E_{\triangle_n}|} \\ &\geq h_s(\mathcal{S}E_{\triangle_n}) = h_s(\triangle_{n+1}) \end{aligned} \tag{4.3.3}$$

for each n. Therefore,

$$h_s(\triangle_1) \geq h_s(\triangle_n) \text{ for all } n\,.$$

We then use the continuity property of h_s for the complementary Hausdorff convergence (Corollary 4.2.3) and get

$$h_s(\triangle_1) \geq \lim_{n\to\infty} h_s(\triangle_n) = h_s(\triangle). \tag{4.3.4}$$

This shows that h_s attains its minimum value for an equilateral triangle. The proof of the quadrilateral case is obtained in the same way as in the previous theorem. ■

Remark 4.3.2. *The case of equality for the case of triangles and quadrilaterals the techniques given above, are not conclusive. Suppose that Δ is an open triangle of given area for which the minimum is attained. If Δ is not already an equilateral triangle, then there is at least one axis m (perpendicular to one of its sides) such that Δ is not Steiner symmetric with respect to m. Let $\mathcal{S}\Delta$ the Steiner symmetrization of Δ respect to the perpendicular bisector of m. Let E_Δ be $s-$Cheeger sets of Δ. Following the same steps as the proof of* (4.3.3), *we get*

$$\begin{aligned} h_s(\Delta) = \frac{P_s(E_\triangle)}{|E_\triangle|} &= \frac{\displaystyle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\chi_{E_\Delta}(x) - \chi_{E_\Delta}(y)|}{|x-y|^{2+s}} dxdy}{|E_\Delta|} \\ &\geq \frac{\displaystyle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\chi_{\mathcal{S}E_\Delta}(x) - \chi_{\mathcal{S}E_\Delta}(y)|}{|x-y|^{2+s}} dxdy}{|\mathcal{S}E_\triangle|} \end{aligned}$$

$$
\begin{aligned}
&= \frac{P_s(E_{\mathcal{S}\triangle})}{|E_{\mathcal{S}\triangle}|} \\
&\geq h_s(\mathcal{S}\triangle) \qquad (4.3.5)
\end{aligned}
$$

Now, the fact that Δ *minimizes* h_s *leads to the observation that* $h_s(\mathcal{S}\Delta) = h_s(\Delta)$ *and so*

$$\frac{P_s(E_\Delta)}{|E_\Delta|} = \frac{P_s(\mathcal{S}E_\triangle)}{|\mathcal{S}E_\triangle|}. \qquad (4.3.6)$$

Considering $|E_\triangle| = |\mathcal{S}E_\triangle|$, *and so, we get the equality case in nonlocal Pólya-Szegőinequality. Therefore*

$$E_\triangle = \mathcal{S}E_\triangle + a_{\tau'} \qquad (4.3.7)$$

Though we cannot prove it, we suspect that $\triangle = \mathcal{S}\triangle + a_{\tau'}$.

Theorem 4.3.3. *The maximum of* $h_s(\Omega)$ *among all tetrahedra of given volume is obtained when* Ω *is a regular tetrahedron. Similarly, the maximum of* $h_s(\Omega)$ *among all prisms(right or oblique) of given volume and a quadrilateral base is obtained when* Ω *is a cube.*

Proof: The proof of this theorem is analogous to the proof of Theorem 4.3.3 ■

Bibliography

[1] F. Almgren and E. H. Lieb, Symmetric decreasing rearrangement is sometimes continuous, J. Amer. Math. Soc., 2 (1989), 683-773.

[2] L. Ambrosio, N. Fusco and D. Pallara, Functions of bounded variation and free discontinuity problems, Clarendon/Oxford, 2000.

[3] A. Baernstein II, D. Drasin and R. S. Laugesen, Symmetrization in Analysis, Cambridge University Press, 2019

[4] R. V. Benson, Euclidean Geometry and Convexity, McGraw-Hill, New York, 1966.

[5] T. Bhattacharya, A proof of the Faber-Krahn inequality for the first eigenvalue of the p-Laplacian, Ann. Mat. Pura. Appl. 177 (1999), 225-240.

[6] V. Blåsjö, The evolution of the isoperimetric problem, Amer. Math. Monthly 112 (2005), 526-566.

[7] T. Bonnesen and W. Fenchel, Theory of convex bodies, BCS Associates, Moscow, ID, 1987. Translated from the German and edited by L. Boron, C. Christenson and B. Smith.

[8] J. Bourgain, H. Brezis, and P. Mironescu, Another look at Sobolev spaces, In: Optimal Control and Partial Differential Equations (J. L. Menaldi, E. Rofman and A. Sulem, eds.). A volume in honor of A. Bensoussans's 60th birthday, Amsterdam: IOS Press; Tokyo: Ohmsha, 2001.

[9] H. Brascamp, E. H. Lieb and J. M. Luttinger, A general rearrangement inequality for multiple integrals, J. Funct. Anal. 17 (1974), 227-237.

[10] L. Brasco, E. Lindgren and E. Parini, The fractional Cheeger problem, Interfaces Free Boundaries, 16 (2014), 419-458.

[11] D. Bucur and I. Fragalà, A Faber-Krahn Inequality for the Cheeger constant of $N-$gons, J. Geom. Anal. 26 (2016), 88-117.

[12] A. Burchard, A Short Course on Rearrangement Inequalities (2009), http://www.math.toronto.edu/almut/rearrange.pdf.

[13] A. Burchard, Cases of equality in the Riesz rearrangement inequality, supervised by Michael Loss, Ph.D. thesis, Georgia Institute of Technology, 1994, http://www.math.utoronto.ca/almut/theses/Almut-thesis.pdf

[14] A. Burchard, Cases of equality in the Riesz rearrangement inequality, Ann. Math. 143 (1996), 499-527.

[15] E. Carlen and F. Maggi, Stability for the Brunn-Minkowski and Riesz Rearrangement Inequalities, with Applications to Gaussian Concentration and Finite Range Non-local Isoperimetry, Canad. J. Math. 16 (2017), 1036-1063.

[16] V. Caselles, A. Chambolle and M. Novaga, Uniqueness of the Cheeger set of a convex body, Pacific J. Math. 232 (2007), 77-90.

[17] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian, In: Gunning R (ed.) Problems in Analysis: A Symposium in Honor of Salomon Bochner, Princeton Univ. Press, (1970), 195-199.

[18] A. M. H. Chorwadwala, R. Mahadevan and F. Toledo, On the Faber-Krahn inequality for the Dirichlet p-Laplacian, ESAIM-Control Optim. Calc. Var. 21 (2015), 60-72.

[19] M. Cwikel, Weak type estimates for singular values and the number of bound states of Schrödinger operators, Ann. of Math. 106 (1977), 93-100.

[20] M. C. Delfour and J. P. Zolésio, Shapes and geometries: metrics, analysis, differential calculus, and optimization, 2nd edition, SIAM series on Advances in Design and Control, Society for Industrial and Applied Mathematics, Philadelphia, 2011.

[21] L.C. Evans and R.F. Gariepy. Measure Theory and Fine Properties of Functions. Boca Raton, CRC Press, 1992.

[22] G. Faber, Beweis dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt, Sitzungberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München Jahrgang, (1923), 169-172.

[23] R. L. Frank, E.H. Lieb and R. Seiringer, Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators, J. Amer. Math. Soc. 21(2008), 925-950.

[24] R. L. Frank and R. Seiringer, Non-linear ground state representations and sharp Hardy inequalities, J. Funct. Anal. 255 (2008), 3407-3430.

[25] M. Goffeng, Analytic formulas for the topological degree of non-smooth mappings: The odd-dimensional case, Adv. Math. 23(2012), 357-377.

[26] P. Gruber, Convex and Discrete Geometry, Berlin, Springer-Verlag, 2007.

[27] H. Hadwiger and D. Ohmann, Brunn-Minkowskischer Satz und Isoperimetrie, Math. Z., 66(1956/57), 1-8.

[28] A. Henrot, Extremum Problems for Eigenvalues of Elliptic Operators, Basel, Birkhäuser Verlag, 2006.

[29] A. Henrot and M. Pierre, Shape Variation and Optimization. A Geometrical Analysis, vol. 28. (European Mathematical Society (EMS), Zürich, 2018).

[30] T. Sh. Kal'menov and D. Suragan, Inequalities for the eigenvalues of the Riesz potential, Mat. Zametki, 6 (2017), 844-850.

[31] B. Kawohl and V. Fridman, Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant, Comment. Math. Univ. Carolin, 44 (2003), 659-667.

[32] B. Kawohl and M. Novaga, The $p-$Laplace eigenvalue problem as $p \to 1$ and Cheeger sets in a Finsler metric, J. Convex Anal. 15 (2008), 623-634.

[33] A. Kassymov, M. Ruzhansky and B.T. Torebek, Rayleigh-Faber-Krahn, Lyapunov and Hartmann-Wintner inequalities for fractional elliptic problems, arXiv:2006.16672 [math.AP], preprint (2020)

[34] E. Krahn, Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises, Math. Ann., 94 (1925), 97-100.

[35] E. Krahn, Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen, Acta Comm. Univ. Tartu (Dorpat), A9 (1926), 1-44.

[36] R. S. Laugesen and B. A. Siudeja, Dirichlet eigenvalue sums on triangles are minimal for equilaterals, Commun. Anal. Geom., 19(2011), 855-885,

[37] G. P. Leonardi, An overview on the Cheeger problem. New trends in shape optimization, 117-139, Birkhäuser/Springer, 2015.

[38] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Studies in Appl. Math. 57 (1977), 93-105.

[39] E. H. Lieb and M. Loss, Analysis, 2nd ed., Graduate Studies in Mathematics 14, American Mathematical Society, Providence, RI, 2001.

[40] I. Ly, The eigenvalues for the p-Laplacian operator, JIPAM. J. Inequal. Pure Appl. Math., 6 (2005).

[41] R. Mahadevan and F. Olivares, A Faber-Krahn inequality for the Riesz potential operator for triangles and quadrilaterals, J. Spectr. Theory, 4 (2021), 1935-1951.

[42] V. Maz'ya and T. Shaposhnikova, On the Bourgain, Brezis and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces, J. Functional Anal. 195 (2002), 230-238.

[43] F. Olivares-Contador, The Faber-Krahn inequality for the first eigenvalue of the fractional Dirichlet p-Laplacian for triangles and quadrilaterals, Pac. J. Math., 288 (2017), 425-434.

[44] F. Olivares-Contador, La desigualdad de Faber-Krahn en Polígonos para el $p-$Laplaciano fraccionario-Dirichlet.
$http://152.74.17.92/bitstream/11594/1965/3/Tesis\_La\_Desigualdad\_de\_Faber\_Krahn.Image.Marked.pdf$

[45] G. Pólya and G. Szegő, Isoperimetric Inequalities in Mathematical Physics, Princeton University Press, Princeton, 1951.

[46] W. Prenowitz and J. Jantosciak, Join Geometries a Theory of Convex Sets and Linear Geometry, New York, Heidelberg, Berlin, Springer-Verlag,1979.

[47] J. W. S. Rayleigh, The Theory of Sound, vol. 1, 2nd ed., McMillan, London, 1894. reprinted Dover, New York, 1945.

[48] F. Riesz, Sur une inégalité intégrale, J. London Math. Soc., 5 (1930), 162-168.

[49] C. A. Rogers, A single integral inequality, J. London Math. Soc. 32 (1957), 102-108.

[50] G. Rozenblum, M. Ruzhansky and D. Suragan, Isoperimetric inequalities for Schatten norms of Riesz potentials, J. Funct. Anal., 271 (2016), 224-239 .

[51] B. Russo, On the Hausdorff-Young theorem for integral operators, Pacific J. Math., 68(1977), 241-253.

[52] M. Ruzhansky, M. Sadybekov and D. Suragan, Spectral Geometry of Partial Differential Operators, CRC Press, 2020.

[53] M. Ruzhansky and D. Suragan, On first and second eigenvalues of Riesz transforms in spherical and hyperbolic geometries, Bull. Math. Sci., 6 (2016), 325-334 .

[54] R. Schneider, Convex Bodies: the Brunn-Minkowski theory, 2nd ed., Encyclopedia of Mathematics and its Applications 151, Cambridge University Press, 2014.

[55] B. A. Siudeja, Sharp bounds for eigenvalues of triangles, Michigan Math. J. 55 (2007), no. 2, 243-254.

[56] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, 1970.

[57] A. C. M. van Rooij and W.H. Schikhof, A Second Course on Real Functions, Cambridge, 1982.

[58] V. S. Vladimirov, Equations of Mathematical Physics, translated by Audrey Littlewood, Marcel Dekker, New York, 1971.

[59] W. P. Ziemer, Weakly differentiable functions. Sobolev spaces and functions of bounded variation, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989.