Non-linear pseudo-differential evolution equation in fractional time

by

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Dedication

I dedicate my dissertation work to my children,
my husband
and my mother.

Acknowledgement

I want to thank God, for giving me health and strength during my doctoral studies. A special feeling of gratitude to my advisor, Vicente Vergara Aguilar, for his guidance and prompt feedback throughout this process. I would like to thank all the academics of the doctoral program for their wise teachings. I also appreciate all the support I received from those who made this thesis possible.

Introduction

Fractional calculus is the part of mathematical analysis that studies derivatives and integrals of any arbitrary order, real or complex. We usually refer to it when we want to describe evolution problems with *memory*. The first known historical record where a fractional derivative is mentioned, can be found in a letter written to Guillaume de l'Hôpital (1661−1704) by Gottfried Wilhelm Leibniz (1646−1716) in 1695. Long after, the 19th and 20th centuries would be the witnesses of just how important is this branch of mathematics.

Researchers from various areas has been motivated by the increasing use of fractional calculus in the mathematical modeling of processes in health sciences, natural sciences, economy and engineering (see, e.g. [11], [12], [42] and [48]). This calculus is more reliable in predicting the evolution of some phenomena or processes, such as particle motion, conservation of mass, propagation of acoustic waves and anomalous diffusion in complex media. In this thesis we are particularly interested in the later.

The non-local nature of a fractional derivative when its order is non-integer, introduces memory into the system. Unlike a derivative of integer order at a point, a derivative of non-integer order depends on all values of the function, even those far away from the point. Numerous experiments in some media have demonstrated that the mean squared displacement (MSD) of a particle is directly proportional to a power of time. The exponent of such power is the order of the derivative in time for the corresponding evolution equation. For instance, an order between zero and one may model an anomalous diffusion process (see, e.g. [18], [49]).

On the other hand, the diffusion term classically represented by the Laplacian operator can be replaced by a pseudo-differential operator, which is an extension of the concept of differential operator acting on the spatial variable. The Laplacian operator has been studied for long time due to the Gaussian laws governing some processes of heat conduction, but currently others singular integral operators can be used as a natural extension to non-Gaussian laws. The theory of pseudo-differential operators arose in the mid 1960s, with Lars Valter Hörmander (1931−2012) being the foremost contributor to it.

Following this path, in this thesis we have investigated the Cauchy problem

$$\begin{aligned} \partial_t^\alpha (u-u_0)(t,x) + \Psi_\beta(x,-i\nabla)u(t,x) &= f(t,x,u), \quad t>0,\ x\in\mathbb{R}^d, \\ u(t,x)|_{t=0} &= u_0(x), \quad x\in\mathbb{R}^d, \end{aligned} \tag{1}$$

as a generalization of non-Gaussian diffusions that include memory, considering a non-regular class of solutions u and a probabilistic interpretation of the operator $-\Psi_\beta(x,-i\nabla)$.

$\mathbb{R}^d$ is the d-dimensional Euclidean space, ∂_t^α denotes the *Riemann–Liouville fractional derivative* of order $\alpha \in (0,1)$ in the time variable t and $\Psi_\beta(x, -i\nabla)$ stands for a *singular integral operator* of constant order $\beta \in (0,2)$. The notation $\partial_t^\alpha(u - u_0)(t,x)$ is understood as $\partial_t^\alpha u(t,x) - \partial_t^\alpha(1) \times u_0(x)$ and the function u_0 stands for the initial data in a certain Lebesgue space. The function f is proportional to $|u(t,x)|^{\gamma-1}u(t,x)$, $\gamma > 1$.

Our time derivative is also called the *Caputo fractional derivative*, for instance, in the sense of [35, Section 2.4] with $0 < \alpha < 1$. However, other authors may require smoothness conditions on the function to define the Caputo derivative (see, e.g. [55, Sub-section 2.4.1]). The symbol associated with the operator $\Psi_\beta(x, -i\nabla)$ is of the type Lévy-Khintchine and the corresponding stochastic process is called a *localised Feller-Courrège process*. Therefore, it makes sense to use the notation $\Psi_\beta(x, -i\nabla)$ for the associated generator (see, e.g. [38, Chapter 6, Appendices C and D]).

Problems like (1) can describe, for instance, a *stable jump-diffusion process* which comes from stochastic control theory with coefficients depending on the position (see, e.g. [37, Section 3]). In this case, the value β is called the *index of stability* of the random process in the sense of a stable distribution (see, e.g. [62, Definition 1.1.1 and Theorem 1.1.2]). Stable distributions appear in the analysis of Markov processes, specially those with a high sensitivity dependence between random variables, such as telecommunications, finance, epidemiology or diffusion. In a suitable setting, the Green function of the Cauchy problem $\dfrac{\partial u}{\partial t} + \Psi_\beta(x, -i\nabla)u = 0$ is non-Gaussian and it is considered as the transition probability density of the corresponding *stable non-Gaussian process* [39, Chapter 7].

Similar to the case of Gaussian processes, which have been widely studied (see, e.g. [2], [21], [24], [27], [28]), the general objective of this thesis is to investigate the existence of solutions to (1), their qualitative properties, asymptotic behaviour and blow-up phenomena, developing methods of non-linear analysis in Banach spaces.

For results on the existence of solutions, we have studied the operator $\Psi_\beta(x, -i\nabla)$ with variable coefficients and with constant coefficients separately. This is why the case with variable coefficients only admits local solutions. In both cases, we apply the representation of solutions with sub-Markovian semigroups and Mittag-Leffler functions, as well as principles of fixed-point and strong solutions in the sense of Jan Prüss (1951−2018). Our methods also illustrate the importance of defining appropriate spaces of functions in evolutions of non-linear diffusions, beyond the classical sense.

In Chapter 1, we give the standard definitions and notations which are used in this thesis, as well as the necessary theory on fractional calculus, Mittag-Leffler functions, multipliers and Volterra equations. In the theory of Mittag-Leffler functions, we also prove a result on Gronwall's inequalities including singularities. This result is stated in Lemma 1.2.1.

In Chapter 2, we investigate solutions to (1) for the case when $\Psi_\beta(x, -i\nabla)$ has variable coefficients. In order to adapt the theory of sub-Markovian semigroups and Dirichlet operators as a tool for locally solving the problem (1), the analytic properties of the corresponding symbol associated are examined. A representation for the solutions is given by exploiting the Fourier analysis, strongly continuous semigroups and Volterra equations. The main results that summarize our method for variable coefficients are stated in Theorems 2.3.7 and 2.4.2.

Chapter 3 deals with the solvability of (1) when the operator $\Psi_\beta(x, -i\nabla)$ has constant coefficients. We start studying the two fundamental solutions associated with the evolutionary problem, whose estimates for the L_p-norm are found to obtain three main results concerning mild and global solutions. The existence and uniqueness of a mild solution is based on the conditions required in some parameters, one of which is the order of stability of the stochastic process. The existence and uniqueness of a global solution is found for the case of small initial conditions and another for non-negative initial conditions. In addition, the chapter includes the asymptotic behaviour of global solutions as a linear combination of the fundamental solutions with L_p-decay. The main results of this chapter are stated in Theorems 3.2.1, 3.3.1, 3.4.1 and 3.5.1.

In the last chapter, we show that the non-linearity of (1) leads to the blow-up of positive solutions in a finite time. For instance, when the operator $\Psi_\beta(x, -i\nabla)$ becomes the negative Laplacian $(-\Delta)$, all positive solutions to the Cauchy problem

$$\begin{aligned} \partial_t\, u(t,x) + (-\Delta)u(t,x) &= u(t,x)^\gamma, \quad t>0,\ x\in\mathbb{R}^d,\\ u(t,x)|_{t=0} &= u_0(x) \geq 0, \quad x\in\mathbb{R}^d, \end{aligned}$$

blow-up in a finite time under some considerations on the parameter γ. This fact was investigated by Fujita (1928−) in 1966 ([22]) and since then, many other researchers have explored blow-up phenomena (see, e.g. [36], [41], [51], [67]). Following the analysis of this phenomenon, Chapter 4 is devoted to the question of the existence of Fujita-type critical exponents. The main result of this chapter is Theorem 4.0.1.

Contents

Chapter 1

Preliminaries

In this chapter we give the basic notations and definitions which we need throughout this thesis. As usual, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers, respectively. The dimension of the euclidean space is $d \in \mathbb{N}$. For a real number a we denote by $[a]$ the maximal integer not exceeding it.

Let $\mathbb{N}_0$ be the set $\mathbb{N} \cup \{0\}$. A multi-index $\delta = (\delta_1, \cdots, \delta_d)$ is an element of $\mathbb{N}_0^d$ whose order is $|\delta| = \delta_1 + \cdots + \delta_d$. The differential operator of order δ is given by $\partial^\delta = \dfrac{\partial^{|\delta|}}{\partial x_1^{\delta_1} \cdots \partial x_d^{\delta_d}}$ and we understand $\partial^\delta f = f$ whenever $|\delta| = 0$. For a vector $x \in \mathbb{R}^d$, we define $x^\delta = x_1^{\delta_1} \cdots x_d^{\delta_d}$ and $|x|^\delta = |x_1|^{\delta_1} \cdots |x_d|^{\delta_d}$. The usual norms are $\|x\|_d = |x_1| + \cdots + |x_d|$ and the standard Euclidean norm $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$. We recall that all norms are equivalent in $\mathbb{R}^d$.

The positive half-line is given by $\mathbb{R}_+ = [0, \infty)$. The convolution with a scalar-valued function, on $\mathbb{R}_+$ or $\mathbb{R}$, is denoted by $*$ and the usual convolution with respect to the spatial variable is denoted by $\star$. The Euler Gamma function is $\Gamma(z) := \int_0^\infty s^{z-1} e^{-s} ds$ and the Beta function is defined by $B(z, y) := \int_0^1 s^{z-1} (1-s)^{y-1} ds$, $\mathrm{Re} z, \mathrm{Re} y > 0$. The relation between them is $B(z, y) = \frac{\Gamma(z)\Gamma(y)}{\Gamma(z+y)}$ ([6, Chapter 1]).

If (Ω, Σ, μ) is an arbitrary measure space and $1 \leq p \leq \infty$, we denote by $L_p(\Omega, \Sigma, \mu)$ the **Lebesgue space** (of equivalence classes) of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_p := \left(\int_\Omega |f(x)|^p \mu(dx) \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$

and

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| < \infty.$$

For any $1 \leq p \leq \infty$ the normed space $(L_p(\Omega, \Sigma, \mu), \|\cdot\|_p)$ is Banach ([60, Theorem 3.11]) and for $p = 2$ is Hilbert. On this last case, we denote by $\langle \cdot, \cdot \rangle_2$ the usual inner product.

If $(X, \|\cdot\|_X)$ is an arbitrary Banach space and $1 \leq p \leq \infty$, we denote by $L_p(\Omega, \Sigma, \mu; X)$ the **Bochner space** (of equivalence classes) of Bochner-measurable functions $f : \Omega \to X$ such that $\|f\|_X$ lies in $L_p(\Omega, \Sigma, \mu)$. These spaces of Bochner are Banach endowed with the norm

$$\|f\|_p := \left(\int_\Omega \|f(x)\|_X^p \mu(dx) \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

and for $L_\infty(\Omega, \Sigma, \mu; X)$ the norm is given by

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x \in \Omega} \|f(x)\|_X.$$

We say that f is integrable, in the sense of Lebesgue or Bochner, whenever $\|f\|_X$ belongs to $L_1(\Omega, \Sigma, \mu)$. When Σ is the Lebesgue σ-algebra and μ is the Lebesgue measure, we write $L_p(\Omega)$ or $L_p(\Omega; X)$ for the corresponding Lebesgue or Bochner space, respectively. For a locally compact space K, its Borel σ-algebra will be denoted by $\mathcal{B}(K)$.

If $J \subset \mathbb{R}$ is non-empty, $C(J; X)$ is the space of continuous functions $f : J \to X$ and $L_p(J; X)$ is the Bochner space, $1 \le p \le \infty$. For $f \in C([a, b]; X)$, its Riemann integral in the usual way coincides with the Bochner integral ([29, Lemma 2.3.24, Definition 2.3.25]).

By $C^\infty(\mathbb{R}^d)$ we denote the set of real-valued functions on $\mathbb{R}^d$ which are arbitrarily differentiable. As usual, $C_0^\infty(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$ is the set of the **test functions** and it is a dense subspace of $L_p(\mathbb{R}^d)$ for all $1 \le p < \infty$. The space of bounded continuous functions real-valued on $\mathbb{R}^d$ is denoted by $C_b(\mathbb{R}^d)$. A function $f : \mathbb{R}^d \to \mathbb{R}$ vanishes at infinity if for any $\epsilon > 0$ there exists a compact set $K \subset \mathbb{R}^d$ such that

$$|f(x)| < \epsilon, \quad x \in \mathbb{R}^d \setminus K.$$

The set of these functions is denoted by $C_\infty(\mathbb{R}^d)$ and $C_0^\infty(\mathbb{R}^d)$ is dense in it with the usual norm $\|\cdot\|_\infty$ ([60, Theorem 3.17], [59, Ejercicio 6.1]). Clearly, $C_\infty(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$.

We understand the above notions regarding non-negative measures, however, sometimes we will need to use *signed measures* on a measurable space (Ω, Σ), that is, a set function $\mu : \Sigma \to (-\infty, \infty]$ (or $[-\infty, \infty)$), such that $\mu(\emptyset) = 0$ and μ is countably additive. By Jordan's decomposition theorem it is known that $\mu = \mu_1 - \mu_2$, where μ_1 and μ_2 are *mutually singular*, non-negative and unique. In this case, we define $\|\mu\| := \mu_1(\Omega) + \mu_2(\Omega) = |\mu|(\Omega)$ as the *total variation* of μ. The set of signed measures on Ω is denoted by $M(\Omega)$ and it is a linear space with the usual operations. Whenever $\|\mu\| < \infty$ we say that μ is *bounded*.

If K is a locally compact space, we define $M_b(K)$ as the set of bounded signed measures on K. If $(\mu_n)_{n \in \mathbb{N}}$ is a sequence in $M_b(K)$ and $\mu \in M_b(K)$, we say that $(\mu_n)_{n \in \mathbb{N}}$ converges in norm to μ if and only if

$$\lim_{n \to \infty} \|\mu_n - \mu\| = 0$$

and we say that $(\mu_n)_{n \in \mathbb{N}}$ converges weakly to μ if for all function $f \in C_b(K)$ we have

$$\lim_{n \to \infty} \int_K f \mu_n = \int_K f \mu.$$

It is easy to see that norm convergence implies weak convergence. Besides, $(M_b(K); \|\cdot\|)$ is a normed linear space ([60, Sections 6.5 and 6.6]).

In what follows we use the notations $f \asymp g$ and $f \lesssim g$ in D, which means that there exists constants $C, C_1, C_2 > 0$ such that $C_1 g \leq f \leq C_2 g$ and $f \leq Cg$ in D, respectively. Such constants may change line by line. We say that $f(x)$ and $g(x)$ are asymptotically equivalent as $x \to \infty$, if the quotient $\dfrac{f(x)}{g(x)}$ tends to unity. In this case, our notation is $f(x) \sim g(x)$ $(x \to \infty)$ (see [14]).

As a linear operator A on a Banach space X we understand a linear mapping whose domain $D(A)$ is a linear subspace of X and its range $Ran(A)$ is contained in X. We say that $A : D(A) \to X$ is continuous (or bounded) if $\|A(v)\|_X \leq C\|v\|_X$ for all $v \in D(A)$, that is, the continuity of A is satisfied when $D(A)$ carries the topology induced by X. The family of continuous linear operators $T : X \to X$ is denoted by $\mathbf{B}(X)$ and $\|T\|_{\mathbf{B}(X)} = \sup\limits_{v\in X\setminus\{0\}} \frac{\|Tv\|_X}{\|v\|_X}$ is the usual operator norm, sometimes shortly denoted by $\|T\|$ if no ambiguity arises.

We say that A is invertible if there is a bounded operator A^{-1} such that $A^{-1} : X \to D(A)$, $AA^{-1} = I_X$ and $A^{-1}A = I_{D(A)}$, where I_X and $I_{D(A)}$ is the identity on X and $D(A)$, respectively. The *resolvent* of A is the set $\rho(A) = \{\lambda \in \mathbb{C} : (A - \lambda) \text{ is invertible}\}$. If $\lambda \in \rho(A)$, then the *resolvent operator* of A at λ is written as $R_A(\lambda) = (A - \lambda)^{-1}$. The spectrum of A, denoted by $\sigma(A)$, is the set of all points $\lambda \in \mathbb{C}$ for which $(A - \lambda)$ is not invertible (see, e.g. [26, Chapter 1]).

We complete this review, giving some notions related to sub-Markovian semigroups on $C_\infty(\mathbb{R}^d)$ and on $L_p(\mathbb{R}^d)$, $1 \leq p < \infty$, which are taken from [29, Chapter 4]. For a real or complex Banach space $(X, \|\cdot\|_X)$, a family of bounded linear operators $(T_t)_{t\geq 0}$ on X is a semigroup of operators, if $T_0 = I_X$ and $T_{s+t} = T_s \circ T_t$ holds for all $s, t \geq 0$. We say that the semigroup $(T_t)_{t\geq 0}$ is strongly continuous if $\lim_{t\to 0}\|T_t x - x\|_X = 0$ for all $x \in X$, and the semigroup is called a contraction semigroup if $\|T_t\| < 1$ for all $t \geq 0$.

If $(T_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on $\big(C_\infty(\mathbb{R}^d), \|\cdot\|_\infty\big)$ which is positivity preserving, i.e., $T_t f \geq 0$ whenever $f \geq 0$, then it is called a *Feller semigroup* ([29, Definition 4.1.4]). If $(T_t)_{t\geq 0}$ is defined on $\big(L_p(\mathbb{R}^d), \|\cdot\|_p\big)$, $1 \leq p < \infty$, such that $0 \leq T_t f \leq 1$ almost everywhere whenever $0 \leq f \leq 1$ almost everywhere, then it is called a *sub-Markovian semigroup* ([29, Definition 4.1.6]). The semigroup $(T_t)_{t\geq 0}$ is *symmetric* if for all f, g belonging to $C_\infty(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ or to $L_p(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, respectively, we have that $\langle T_t f, g\rangle_2 = \langle f, T_t g\rangle_2$.

1.1 Fractional integrals and derivatives

Depending on what we need to model and the conditions we have at our disposal, many types of fractional derivatives have emerged, including Riemann–Liouville, Caputo, Caputo-Fabrizio, Marchaud, Grünwald–Letnikov, among others. There are also several fractional integrals, such as Riemann–Liouville, Hadamard and Atangana–Baleanu. Unfortunately, in the literature there are different notations and definitions for some of them. In this thesis, we employ the Riemann-Liouville fractional integral and derivative. We refer to [61, Chapter 2] for the following notions.

Definition 1.1.1. *Let $f \in L_1([0,T])$. The Riemann–Liouville integral of order $\vartheta > 0$ is defined as*

$$J^{\vartheta} f(t) := \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} f(s) ds, \quad t > 0.$$

Since $f \in L_1([0,T])$, this integral exists almost everywhere and it has the semigroup property

$$J^{\vartheta_1} J^{\vartheta_2} = J^{\vartheta_1+\vartheta_2}, \quad \vartheta_1, \vartheta_2 > 0.$$

Further, we set $J^0 f := f$.

It is very common to use a class of *scalar kernels* given by

$$g_{\vartheta}(t) := \begin{cases} \frac{1}{\Gamma(\vartheta)} t^{\vartheta-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

to denote

$$J^{\vartheta} f(t) = (g_{\vartheta} * f)(t), \quad t > 0.$$

On $(0,\infty)$ and for $\alpha \in (0,1)$ we have that

$$g_{\alpha} * g_{1-\alpha} \equiv 1.$$

Definition 1.1.2. *Let $\alpha \in (0,1)$ and $f \in L_1([0,T])$, such that the first derivative of $J^{1-\alpha} f$ exists almost everywhere. The Riemann–Liouville derivative of order α is defined as*

$$\partial_t^{\alpha} f := \frac{d}{dt}(g_{1-\alpha} * f)(t) = \frac{d}{dt}(J^{1-\alpha} f)(t).$$

If f is absolutely continuous, this derivative exists almost everywhere ([61, Lemma 2.2]).

From these definitions and the semigroup property, it follows that

$$\partial^{\alpha} J^{\alpha} f = f$$

for all $f \in L_1([0,T])$ and $0 < \alpha < 1$. The equality $J^{\alpha} \partial^{\alpha} f = f$ requires stronger assumptions (see [61, Theorem 2.4]).

1.2 Mittag-Leffler functions

In this section we review some analytical properties of the Mittag-Leffler functions and their connection with the densities of stable laws (see e.g., [44] and [66]). These functions are so named from the Swedish mathematician Gösta Mittag-Leffler (1846-1927) who introduced them at the beginning of the century XX (1903, 1904, 1905).

In the present thesis we work with the Mittag-Leffler function of two real parameters $\alpha, \vartheta > 0$ ([7, Chapter 18]), given by

$$E_{\alpha,\vartheta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \vartheta)}, \quad z \in \mathbb{C}.$$

However, in the literature we can find a generalization of this function with three complex parameters, as well as its relation with the Mellin-Barnes integral and the H-Function (also called Fox's H-function). See e.g., [46, Definition 1.4].

It is also known that $E_{\alpha,\vartheta}$, $\alpha, \vartheta > 0$, is an entire function. Whenever $\vartheta = 1$,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in \mathbb{C},$$

is the standard Mittag-Leffler function and $E_\alpha(-x)$ is completely monotonic for $x \geq 0$ if $0 < \alpha < 1$. The latter is thanks to the work of the American mathematician Harry Pollard (1919-1985).

The importance of these functions is because they can be defined for any operator that generates a strongly continuous semigroup in a Banach space, using Zolotarev's formula (or Zolotarev-Pollard formula), in terms of Green functions or strongly continuous semigroups; see [40, Section 8.1]. This representation plays a fundamental role for obtaining estimates of the Green functions in evolution equations with fractional time, as we will see in Chapters 2 and 3. In particular, any bounded operator on a Banach space generates an uniformly (hence strongly) continuous semigroup ([52, Section 1.1 Theorem 1.1]).

On the other hand, the standard Mittag-Leffler function E_α has become a useful tool to obtain Gronwall type inequalities with singularities. This is particularly important in this thesis for analysing uniqueness of solutions. For this reason, we have derived the following inequality including a singularity.

Lemma 1.2.1. *Let $\alpha \in (0,1)$ and $\vartheta \geq 0$ such that $\alpha - \vartheta > 0$. Let $g(t)$ a non-negative function locally bounded on $t \in [0,T)$ with some $T > 0$. Suppose that $f(t)$ is non-negative and locally bounded on $[0,T)$ such that*

$$f(t) \leq g(t) + C \int_0^t (t-s)^{\alpha-1} s^{-\vartheta} f(s) ds$$

for all $t \in [0,T)$, with some positive constant C. Then

$$f(t) \leq g(t) + \int_0^t \left[\sum_{n=1}^{\infty} a_n (t-s)^{n\alpha-(n-1)\vartheta-1} s^{-\vartheta} g(s) \right] ds, \quad 0 \leq t < T,$$

where

$$a_n = C^n \prod_{k=1}^{n-1} \frac{\Gamma(\alpha)\Gamma(k(\alpha-\vartheta))}{\Gamma((k+1)\alpha - k\vartheta)}.$$

Proof. The case $\vartheta = 0$ is straightforward from [69, Theorem 1]. For the case $\vartheta > 0$ we require some adjustments in its proof. First, we define the operator B given by

$$B\phi(t) := C \int_0^t (t-s)^{\alpha-1} s^{-\vartheta} \phi(s) ds, \quad t \geq 0,$$

for locally bounded functions ϕ. By construction, the operator B is linear and $B\phi_1 \leq B\phi_2$ whenever $\phi_1 \leq \phi_2$. Therefore, we have that

$$f(t) \leq \sum_{k=0}^{n-1} B^k g(t) + B^n f(t), \quad n \geq 1. \tag{1.1}$$

Next, we prove that

$$B^n f(t) \le C^n \prod_{k=1}^{n-1} \frac{\Gamma(\alpha)\Gamma(k(\alpha-\vartheta))}{\Gamma((k+1)\alpha-k\vartheta)} \int_0^t (t-s)^{n\alpha-(n-1)\vartheta-1} s^{-\vartheta} f(s)ds \tag{1.2}$$

is true for all $n \in \mathbb{N}$ by induction. The case $n=1$ follows straightforwardly from the definition of B. Now, we suppose that (1.2) is true for $N \in \mathbb{N}$ and applying B we obtain

$$\begin{aligned}
&B\left(B^N f\right)(t) \\
&= C\int_0^t (t-s)^{\alpha-1} s^{-\vartheta} B^N f(s)ds \\
&\le C^{N+1} \prod_{k=1}^{N-1} \frac{\Gamma(\alpha)\Gamma(k(\alpha-\vartheta))}{\Gamma((k+1)\alpha-k\vartheta)} \int_0^t (t-s)^{\alpha-1} s^{-\vartheta} \left[\int_0^s (s-\tau)^{N\alpha-(N-1)\vartheta-1}\tau^{-\vartheta} f(\tau)d\tau\right] ds \\
&= C^{N+1} \prod_{k=1}^{N-1} \frac{\Gamma(\alpha)\Gamma(k(\alpha-\vartheta))}{\Gamma((k+1)\alpha-k\vartheta)} \int_0^t \left[\int_\tau^t (t-s)^{\alpha-1} s^{-\vartheta} (s-\tau)^{N\alpha-(N-1)\vartheta-1} ds\right] \tau^{-\vartheta} f(\tau)d\tau,
\end{aligned}$$

where the last line comes from the Fubini's theorem. Besides, the integral in the square brackets can be estimated with the substitution $s=\tau+z(t-\tau)$ as follows.

$$\begin{aligned}
&\int_\tau^t (t-s)^{\alpha-1} s^{-\vartheta} (s-\tau)^{N\alpha-(N-1)\vartheta-1} ds \\
&= \int_0^1 ((t-\tau)(1-z))^{\alpha-1} (\tau+z(t-\tau))^{-\vartheta} (z(t-\tau))^{N\alpha-(N-1)\vartheta-1}(t-\tau)dz \\
&\le \int_0^1 ((t-\tau)(1-z))^{\alpha-1} (z(t-\tau))^{-\vartheta} (z(t-\tau))^{N\alpha-(N-1)\vartheta-1}(t-\tau)dz \\
&= (t-\tau)^{(N+1)\alpha-N\vartheta-1} \int_0^1 (1-z)^{\alpha-1} z^{N(\alpha-\vartheta)-1} dz \\
&= (t-\tau)^{(N+1)\alpha-N\vartheta-1} \frac{\Gamma(\alpha)\Gamma(N(\alpha-\vartheta))}{\Gamma((N+1)\alpha-N\vartheta)}.
\end{aligned}$$

Consequently,

$$B\left(B^N f\right)(t) \le C^{N+1} \prod_{k=1}^{N} \frac{\Gamma(\alpha)\Gamma(k(\alpha-\vartheta))}{\Gamma((k+1)\alpha-k\vartheta)} \int_0^t (t-\tau)^{(N+1)\alpha-N\vartheta-1} \tau^{-\vartheta} f(\tau)d\tau$$

which proves the inductive step in (1.2).

Finally, since $\frac{\Gamma((k+1)(\alpha-\vartheta))}{\Gamma((k+1)\alpha-k\vartheta)} \le 1$ for k large enough, we have that

$$\lim_{n\to\infty} B^n f(t) = 0$$

and the expression (1.1) can be written as

$$f(t) \le \sum_{n=0}^{\infty} B^n g(t).$$

The proof is complete. □

1.3 Multipliers

We start this section defining the *Schwartz space* given by

$$S(\mathbb{R}^d) := \{v \in C^\infty(\mathbb{R}^d) : \|v\|_{(N,\delta)} < \infty \text{ for all } N \in \mathbb{N}_0,\ \delta \in \mathbb{N}_0^d\},$$

where

$$\|v\|_{(N,\delta)} := \sup_{x\in\mathbb{R}^d} |\partial^\delta v(x)|(1+\|x\|)^N.$$

On this space we define the Fourier transform as

$$\widehat{v}(\xi) = \mathcal{F}(v)(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} v(x)dx, \quad v \in S(\mathbb{R}^d),$$

and the inverse Fourier transform as

$$\mathcal{F}^{-1}(w)(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} w(\xi)d\xi, \quad w \in S(\mathbb{R}^d).$$

Sometimes we write $\mathcal{F}_{x\mapsto\xi}(v)(\xi)$ and $\mathcal{F}^{-1}_{\xi\mapsto x}(w)(x)$ for denoting the Fourier transform and its inverse, respectively.

The family of norms $\|\cdot\|_{(N,\delta)}$ defines a Fréchet topology on $S(\mathbb{R}^d)$ ([20, Proposition 8.2]) and $\mathcal{F}$ is an homeomorphism into itself with that topology, that is, $\mathcal{F} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ is a bijective mapping, where both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous ([29, Theorem 3.1.7]). We want to point out that in the literature there are some definitions of the Fourier transform which differ in the factor 2π, however this does not affect our results.

According to our definition and ([29, Remark 3.1.11]), we have that

$$\langle v, w\rangle_2 = \frac{1}{(2\pi)^d}\langle \widehat{v}, \widehat{w}\rangle_2$$

for all $v, w \in S(\mathbb{R}^d)$, which extends to all $L_2(\mathbb{R}^d)$ by density. In particular, by the Plancherel's theorem ([29, Theorem 3.2.18]),

$$\|v\|_2^2 = \frac{1}{(2\pi)^d}\|\widehat{v}\|_2^2, \quad v \in L_2(\mathbb{R}^d).$$

Having these properties in mind, we can introduce some function spaces such as the *Bessel potential* $H_2^m(\mathbb{R}^d)$, which is defined as the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|v\|_{H_2^m} = \frac{1}{(2\pi)^{\frac{d}{2}}}\left\|\left(1+\|\cdot\|^2\right)^{\frac{m}{2}}\widehat{v}\right\|_2.$$

If m is a non-negative integer number then the classical Sobolev space $W_2^m(\mathbb{R}^d)$ and $H_2^m(\mathbb{R}^d)$ are isomorphic ([29, Theorem 3.11.10]).

In a similar way, using a *continuous negative definite function* $\psi : \mathbb{R}^d \to \mathbb{C}$, we can define the ψ*-Bessel potential* spaces

$$H^{\psi,2}(\mathbb{R}^d) := \{v \in L_2(\mathbb{R}^d) : \|v\|_{H^{\psi,2}} < \infty\}$$

endowed with the norm

$$\|v\|_{H^{\psi,2}} := \frac{1}{(2\pi)^{\frac{d}{2}}} \left\| (1+|\psi(\cdot)|)\, \widehat{v} \right\|_2 .$$

In this thesis is particularly important the theory of the continuous negative definite functions, hence we have taken the following notions from [29, Sections 3.6 and 3.10].

Definition 1.3.1. *The function $\psi : \mathbb{R}^d \to \mathbb{C}$ is called a continuous negative definite function if ψ is continuous, $\psi(0) \geq 0$ and $\xi \mapsto e^{-t\psi(\xi)}$ is positive definite for all $t \geq 0$.*

A typical example of continuous negative definite functions is

$$\mathbb{R}^d \ni \xi \mapsto \|\xi\|^{2s} \in \mathbb{R}$$

for any $s \in (0,1]$ ([29, Example 3.9.17]).

Nevertheless, the following lemma is often useful when dealing with lesser known functions ([29, Lemma 3.6.8 and Theorem 3.6.11]).

Lemma 1.3.1. *The function $\psi : \mathbb{R}^d \to \mathbb{C}$ is negative definite if and only if*

$\psi(0) \geq 0$,

$\psi(\xi) = \overline{\psi(-\xi)}$ and

for any $k \in \mathbb{N}$, for any $\xi^1, \cdots, \xi^k \in \mathbb{R}^d$ and $z_1, \cdots, z_k \in \mathbb{C}$,

$$\sum_{j=1}^{k} z_j = 0 \text{ implies } \sum_{j,l=1}^{k} \psi(\xi^j - \xi^l) z_j \bar{z}_l \leq 0.$$

As we have seen in the definition of the space $H^{\psi,2}(\mathbb{R}^d)$, the function $|\psi|$ acts on $\widehat{v}$ as a *pointwise multiplier.*

In general terms, a function g is called a pointwise multiplier from a space S_1 to another space S_2, if for every function $v \in S_1$ the product $g\,v \in S_2$. Other definitions also require that the linear mapping $v \mapsto g\,v$ is bounded. Due to the algebraic properties of the pointwise multipliers, its use has expanded to the context of linear operators giving rise to the *multiplier operators.*

A multiplier operator is a linear operator defined on a functions space which changes the frequency spectrum of the function via the Fourier transform. In fact, for a continuous negative definite function ψ, we can define a linear operator Λ given by

$$v \mapsto \Lambda(v) := \mathcal{F}^{-1}\left(\psi(\cdot)\widehat{v}\right)$$

on a suitable domain of functions, for instance $C_0^\infty(\mathbb{R}^d) \subset S(\mathbb{R}^d)$. The natural next step is to study its possible extensions.

The role of these type of multipliers have been investigated widely as a part of Fourier Analysis (see e.g., [5] and [58]). Thereby, we arrive at the theory of pseudo-differential operators with negative definite symbols ([30]) and the theory of Sobolev multipliers ([47]). As we will see later, we find that in the literature the term multiplier often refers to the symbol associated with the corresponding linear operator that generates a strongly continuous contraction semigroup on some Banach space.

For instance, by the theorem of Hille-Yosida ([29, Theorem 4.1.33]) we know that a linear operator $A : D(A) \to X$, on a Banach space $(X, \|\cdot\|_X)$, is the generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$, if and only if $D(A)$ is dense in X, A is dissipative and $Ran(A - \lambda) = X$ for some $\lambda > 0$. Therefore, the properties of the pseudo-differential operator A, the semigroup $(T_t)_{t\geq 0}$ and the resolvent operator $R_A(\lambda)$ at λ, $\lambda \in \rho(A)$, are determined by the symbol associated with A (see, e.g. [31, Chapter 6]).

1.4 Volterra equations

Here, we present the necessary background on the Volterra integral equations in the sense of Prüss ([57]), of the *scalar type*

$$u(t) = f(t) + \int_0^t k(t-s)Au(s)ds, \quad t \in [0, T], \tag{1.3}$$

where A is a closed linear unbounded operator defined on a Banach space X, with dense domain $D(A) \subset X$. The scalar function $k \in L_{1,loc}(\mathbb{R}^+)$ is called the *kernel* of the convolution $k * Au$. The unknown function u and the forcing term f belong to the space $C([0, T]; X)$.

The notation $Au(s)$ is understood as $A(u(s))$ and thus Au can be considered as a function on $[0, T]$.

We recall that an operator $A : D(A) \to X$ is closed if its graph is closed in $X \times X$ and it is closable if it has a closed extension ([29, Definition 2.7.3, Lemma 2.7.12, Definition 2.7.13]). When A is closed, the domain of A equipped with the graph norm $\|\cdot\|_A := \|\cdot\|_X + \|A(\cdot)\|_X$ is a Banach space, which is usually denoted by X_A (see [57, Chapter 1]).

The term $\int_0^t k(t-s)Au(s)ds$ is understood in the sense of Bochner, that is, the integral is an element of X if and only if $\int_0^t k(t-s)\|Au(s)\|_X ds < \infty$ ([25, Theorem 3.7.4]). If $k = g_\vartheta$ with $\vartheta > 0$ as in Section 1.1, the integral exists almost everywhere whenever $\|Au(\cdot)\|_X$ is locally integrable on $\mathbb{R}^+$. In the case of $X = L_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, we have that $\left(\int_0^t g_\vartheta(t-s)Au(s)ds\right)(x) = \int_0^t g_\vartheta(t-s)Au(s)(x)ds$.

If $u(t) \in D(A)$ for all $t \in [0, T]$, u and Au belong to $C([0, T]; X)$, then it holds $\int_0^T u(t)dt \in D(A)$ and $A\left(\int_0^T u(t)dt\right) = \int_0^T Au(t)dt$ ([29, Lemma 2.3.24 C.]).

We say that a function $u \in C([0, T]; X)$ is a *strong solution* of (1.3) if $u \in C([0, T]; X_A)$ and it satisfies (1.3). We say that u is a *mild solution* of (1.3) if $k * u \in C([0, T]; X_A)$ and $u(t) = f(t) + A(k * u)(t)$ for all $t \in [0, T]$. Every strong solution of (1.3) is a mild solution ([57, Definition 1.1]).

On the other hand, it is known that the pair of kernels $(g_{1-\alpha}, g_\alpha)$, $0 < \alpha < 1$, are of type $(\mathcal{PC})$, i.e., they satisfy the following condition (see [56]):

> $g_{1-\alpha} \in L_{1,loc}(\mathbb{R}^+)$ is non-negative and non-increasing, the kernel $g_\alpha \in L_{1,loc}(\mathbb{R}^+)$ and $(g_{1-\alpha} * g_\alpha) = 1$ in $(0, \infty)$.

In fact, g_α is completely positive ([57, Definition 4.5]). The class of completely positive kernels and its properties are very useful for applying subordination principles and obtaining strong solution to (1.3). For instance, there exists a unique scalar *propagation function* $w(t,\tau)$, $t,\tau \geq 0$, associated with the completely positive function g_α. Besides, associated with w we have the so-called *relaxation function* $s(t,\tau)$, $t,\tau \geq 0$, which is the solution of the scalar Volterra equation, fixing τ,

$$s(t,\tau) + \tau\left(g_\alpha * s(t,\tau)\right)(t) = 1, \quad t \geq 0.$$

The precise relation between w and s and their properties, can be found in [57, Section 4.5] and [56, Sections 2 and 3].

The property $(\mathcal{PC})$, together with the previous abstract notions, allows that the problem (1) can be rewritten as a Volterra equation of the form (1.3), therefore one has to choose a suitable Banach space X for finding strong solutions in accordance with the particular structure of the evolution problem. This theory is illustrated on various sections of this thesis.

Chapter 2

The semi-linear problem with variable coefficients

This chapter deals with the solvability of the semi-linear Cauchy problem

$$\begin{aligned} \partial_t^\alpha (u-u_0)(t,x) + \Psi_\beta(x,-i\nabla)u(t,x) &= \lambda |u(t,x)|^{\gamma-1}u(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ u(t,x)|_{t=0} &= u_0(x), \quad x\in\mathbb{R}^d. \end{aligned} \tag{2.1}$$

Here, $\gamma > 1$ and $\lambda \in \mathbb{R}$ are parameters of the non-linear term. The symbol of the operator $\Psi_\beta(x,-i\nabla)$ is a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ given by

$$\psi(x,\xi) = \|\xi\|^\beta \omega_\mu\left(x, \frac{\xi}{\|\xi\|}\right) \tag{2.2}$$

such that

$$\Psi_\beta(x,-i\nabla)v(x) = \mathcal{F}^{-1}_{\xi\to x}[\psi(x,\xi)(\mathcal{F}v)(\xi)],\ v\in C_0^\infty(\mathbb{R}^d).$$

The function ω_μ is real-valued, defined on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ by

$$\omega_\mu(x,\theta) := \int_{\mathbb{S}^{d-1}} |\theta\cdot\eta|^\beta \mu(x,d\eta), \quad \theta\in\mathbb{S}^{d-1}, \tag{2.3}$$

where $\theta = \frac{\xi}{\|\xi\|}$ and $\mathbb{S}^{d-1}$ denotes the unit $(d-1)$-sphere contained in $\mathbb{R}^d$.

Since ω_μ is defined on $\mathbb{R}^d \times \mathbb{S}^{d-1}$, we need first to collect some elementary concepts that are used for the analysis of functions on $\mathbb{S}^{d-1}$. Besides, in Section 2.1 we recall the existing relationship between multivariate stable distributions and Borel measures defined on $\mathbb{S}^{d-1}$. Other properties of these measures are addressed in Sections 2.2 and 2.3, some of them very useful for obtaining solutions to (2.1) in the last section of this chapter.

In what follows, we consider $\mathbb{S}^{d-1}$ as a Hausdorff topological space with the subspace topology induced by $\mathbb{R}^d$. Thereby, $\mathbb{S}^{d-1}$ is a $(d-1)$-*manifold* and it is endowed with a C^∞ atlas, that is, a collection of homeomorphisms or real charts $\phi : U \to \mathbb{R}^{d-1}$ pairwise C^∞-compatible, whose domains are open sets in $\mathbb{S}^{d-1}$ that cover $\mathbb{S}^{d-1}$ (see [50, Chapter I Definition 1.22]). For simplicity we assume that the collection has two charts, ϕ_1 and ϕ_2, with domains $U_1 = \mathbb{S}^{d-1} \setminus (0,0,\cdots,0,1)$ and $U_2 = \mathbb{S}^{d-1} \setminus (0,0,\cdots,0,-1)$, respectively.

These charts have the form

$$U_1 \ni \theta \longmapsto \phi_1(\theta) = \left(\frac{\theta_1}{1-\theta_d}, \cdots, \frac{\theta_{d-1}}{1-\theta_d} \right) \in \mathbb{R}^{d-1},$$

$$U_2 \ni \theta \longmapsto \phi_2(\theta) = \left(\frac{\theta_1}{1+\theta_d}, \cdots, \frac{\theta_{d-1}}{1+\theta_d} \right) \in \mathbb{R}^{d-1}$$

and their inverses are, respectively,

$$\mathbb{R}^{d-1} \ni z \longmapsto \phi_1^{-1}(z) = \left(\frac{2z_1}{1+\|z\|^2}, \cdots, \frac{2z_{d-1}}{1+\|z\|^2}, -\frac{1-\|z\|^2}{1+\|z\|^2} \right) \in U_1,$$

$$\mathbb{R}^{d-1} \ni z \longmapsto \phi_2^{-1}(z) = \left(\frac{2z_1}{1+\|z\|^2}, \cdots, \frac{2z_{d-1}}{1+\|z\|^2}, \frac{1-\|z\|^2}{1+\|z\|^2} \right) \in U_2.$$

We see that $\{U_1, U_2\}$ is the standard open covering of $\mathbb{S}^{d-1}$ and the transition functions, $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$, are smooth on its domain of definition. By the compactness of $\mathbb{S}^{d-1}$, there exists a smooth partition of unity on $\mathbb{S}^{d-1}$ given by $\{\nu_1, \nu_2\}$, which is subordinate to the covering $\{U_1, U_2\}$ (see, e.g. [15, Lemma 7]). This means that $\nu_j \in C^\infty(\mathbb{S}^{d-1})$ and $\operatorname{supp} \nu_j \subset U_j$ is a compact set, $j = 1, 2$.

We say that a function h defined on $\mathbb{S}^{d-1}$ belongs to $L_2(\mathbb{S}^{d-1})$ if

$$(\nu_j h) \circ \phi_j^{-1} \in L_2(\mathbb{R}^{d-1})$$

for $j \in \{1, 2\}$. Besides, we equip $L_2(\mathbb{S}^{d-1})$ with the norm

$$\|h\|_{L_2(\mathbb{S}^{d-1})} := \left(\|(\nu_1 h) \circ \phi_1^{-1}\|^2_{L_2(\mathbb{R}^{d-1})} + \|(\nu_2 h) \circ \phi_2^{-1}\|^2_{L_2(\mathbb{R}^{d-1})} \right)^{\frac{1}{2}}.$$

If we change the collection $\{U_j, \phi_j, \nu_j\}$ we obtain an equivalent norm.

In this way, we suppose that $\mu(x, d\eta)$ is a *transition kernel* on $\mathbb{R}^d \times \mathcal{B}(\mathbb{S}^{d-1})$ in the sense of [29, Definition 2.3.19], that is, for every fixed $A \in \mathcal{B}(\mathbb{S}^{d-1})$ the function

$$\mathbb{R}^d \ni x \mapsto \mu(x, A) \in [0, \infty]$$

is measurable, and for every fixed $x \in \mathbb{R}^d$ the set function

$$\mathcal{B}(\mathbb{S}^{d-1}) \ni A \mapsto \mu(x, A) \in [0, \infty]$$

is a centrally symmetric finite (non-negative) Borel measure on $\mathbb{S}^{d-1}$. All these measures are called *spectral measures* (see, e.g. [40, Section 1.8], [37]). As we show below, from the properties of these spectral measures one can derive analogous properties for the function ω_μ given in (2.3).

Whenever we say that $\omega_\mu(x, \cdot)$ is a continuous function on $\mathbb{S}^{d-1}$ for fixed x, we understood the continuity (or C^∞) in the natural way with the structure of atlas introduced above (see [15, Definition 4]).

2.1 Spectral measures

A spectral measure on $\mathbb{S}^{d-1}$ arises from processes involving multivariate stable distributions, in our particular case β-stable because of the probabilistic interpretation of the operator $-\Psi_\beta(x,-i\nabla)$. We refer to [62, Chapter 2] and [54] for such notions and some examples.

The spectral measure is a data-structure containing the correlation structure of a stable distribution on $\mathbb{R}^d$. For a symmetric β-stable random vector in $\mathbb{R}^d$, existence and uniqueness of its *characteristic function* and the corresponding *symmetric spectral measure* is proved in [62, Theorem 2.4.3].

In order to study the properties of the spectral measures associated with the transition kernel μ, we consider the map, which we again denote by μ,

$$\mathbb{R}^d \ni x \longmapsto \mu(x) \in M(\mathbb{S}^{d-1})$$

including the usual topology of $\big(\mathbb{R}^d; \|\cdot\|\big)$ and $\big(M_b(\mathbb{S}^{d-1}); \|\cdot\|\big)$.

Next, we show two lemmata in order to set conditions on the transition kernel μ that can be inherited by ω_μ.

For the first lemma we introduce the notion of *density of the measure* $\mu(x)$, for every $x \in \mathbb{R}^d$, that is, the measure can be formulated as $\mu(x,d\eta) = \varrho(x,\eta)d\eta$, where $\varrho(x,\cdot)$ is a continuous function on $\mathbb{S}^{d-1}$. Additionally, it is worth clarifying that for the differentiation of a function on $\mathbb{S}^{d-1}$ we understand the order as $\delta \in \mathbb{N}_0^{d-1}$, since $\mathbb{S}^{d-1}$ is a $(d-1)$-dimensional manifold. For the derivatives of functions on $\mathbb{S}^{d-1}$, we think of them as *directional derivatives* along differentiable curves on $\mathbb{S}^{d-1}$.

Lemma 2.1.1. *Let $N \in \mathbb{N}$. Suppose that the map $\mathbb{R}^d \ni x \mapsto \mu(x) \in M(\mathbb{S}^{d-1})$ fulfills:*

(i) $\mu(x,d\eta) = \varrho(x,\eta)d\eta$ with $\varrho(x,\cdot)$ a continuous and non-negative function on $\mathbb{S}^{d-1}$, for all $x \in \mathbb{R}^d$, whose upper bound is uniform in x.

(ii) The functions

$$\mathbb{R}^{d-1} \ni z \longmapsto \int_{\mathbb{S}^{d-1}} |\phi_k^{-1}(z)\cdot\eta|^\beta (\phi_k^{-1}(z)\cdot\eta)^{-|\delta|} \prod_{n=1}^{|\delta|} \eta_{j_n}\ \mu(x,d\eta)$$

are continuous and bounded, for all $x \in \mathbb{R}^d$, $\delta \in \mathbb{N}_0^{d-1}$, $1 \le |\delta| \le N$, $k \in \{1,2\}$ and $j_n \in \{1,\cdots,d\}$, with bounds that are uniform in x.

Then the function ω_μ given by (2.3) is N-times continuously differentiable in the second variable and it satisfies the condition

$$\sup_{x\in\mathbb{R}^d} \{\|\partial^\delta \omega_\mu(x,\cdot)\|_{L_\infty(\mathbb{S}^{d-1})} : |\delta| \le N\} < \infty.$$

Proof. We fix $x \in \mathbb{R}^d$. The continuity of $\theta \mapsto \omega_\mu(x,\theta)$ on $\mathbb{S}^{d-1}$ is straightforward since this space is compact and $\varrho(x,\cdot)$ is continuous on it, besides the upper bound is uniform in x.

For the differentiability we use the structure of atlas that was previously introduced. By definition, $\omega_\mu(x,\cdot) \in C^m(\mathbb{S}^{d-1})$ at the point $\theta \in U_k$ if and only if the function

$$\omega_\mu(x,\cdot) \circ \phi_k^{-1}$$

is of class $C^m(\mathbb{R}^{d-1})$ at the point $z = \phi_k(\theta)$. Without loss of generality we put $k = 1$. The partial derivative of this function w.r.t. z_j, $j = 1, \cdots, d-1$, is given by

$$\begin{aligned}
&\frac{\partial}{\partial z_j}\left(\omega_\mu(x,\cdot)\circ\phi_1^{-1}\right)(z) \\
&= \int_{\mathbb{S}^{d-1}} \frac{\partial}{\partial z_j}|\phi_1^{-1}(z)\cdot\eta|^\beta \mu(x,d\eta) \\
&= \int_{\mathbb{S}^{d-1}} \beta|\phi_1^{-1}(z)\cdot\eta|^\beta(\phi_1^{-1}(z)\cdot\eta)^{-1}\frac{\partial}{\partial z_j}(\phi_1^{-1}(z)\cdot\eta)\mu(x,d\eta) \\
&= \int_{\mathbb{S}^{d-1}} \beta|\phi_1^{-1}(z)\cdot\eta|^\beta(\phi_1^{-1}(z)\cdot\eta)^{-1}\left[\frac{\partial}{\partial z_j}\left(\frac{2z_j\eta_j}{1+\|z\|^2}\right) - \frac{\partial}{\partial z_j}\left(\frac{2\eta_d}{1+\|z\|^2}\right)\right]\mu(x,d\eta) \\
&= \frac{\partial}{\partial z_j}\left(\frac{2z_j}{1+\|z\|^2}\right)\int_{\mathbb{S}^{d-1}} \beta|\phi_1^{-1}(z)\cdot\eta|^\beta(\phi_1^{-1}(z)\cdot\eta)^{-1}\eta_j\,\mu(x,d\eta) \\
&\quad - \frac{\partial}{\partial z_j}\left(\frac{2}{1+\|z\|^2}\right)\int_{\mathbb{S}^{d-1}} \beta|\phi_1^{-1}(z)\cdot\eta|^\beta(\phi_1^{-1}(z)\cdot\eta)^{-1}\eta_d\,\mu(x,d\eta).
\end{aligned}$$

The continuity of the integrals as functions of z, which comes from the hypothesis (ii), proves the continuity of this derivative. Finally, the uniform boundedness in x and similar arguments leads to the desired condition. □

Corollary 2.1.1. *Under the assumption of Lemma 2.1.1 part (i), the symbol ψ given in (2.2) is a negative definite function.*

Proof. According to [30, Definition 2.3.1], we only need to show that the function

$$\mathbb{R}^d \ni \xi \mapsto \psi(x,\xi) = \|\xi\|^\beta \omega_\mu\left(x, \frac{\xi}{\|\xi\|}\right) \in \mathbb{R}$$

is negative definite for each $x \in \mathbb{R}^d$. For this purpose, from (2.3) together with the representation given in [37, Formula (1.9)], we see that

$$\begin{aligned}
\omega_\mu\left(x,\frac{\xi}{\|\xi\|}\right) &= \int_{\mathbb{S}^{d-1}}\left|\frac{\xi}{\|\xi\|}\cdot\eta\right|^\beta \mu(x,d\eta) \\
&= \int_0^\infty\int_{\mathbb{S}^{d-1}}\left(1+\frac{i\langle\xi,y\rangle}{\|\xi\|(1+\|y\|^2)} - e^{i\langle\frac{\xi}{\|\xi\|},y\rangle}\right)\frac{d\|y\|}{\|y\|^{1+\beta}}\tilde{\mu}(x,d\eta),
\end{aligned}$$

with $\eta = \frac{y}{\|y\|}$ and the measure $\tilde{\mu}(x,\cdot)$ is proportional to $\mu(x,\cdot)$ ([37, Formula (1.4)]). Thus we obtain

$$\psi(x,\xi) = \int_0^\infty\int_{\mathbb{S}^{d-1}}\left(1+\frac{i\langle\xi,y\rangle}{1+\|y\|^2} - e^{i\langle\xi,y\rangle}\right)\frac{d\|y\|}{\|y\|^{1+\beta}}\tilde{\mu}(x,d\eta).$$

Let $N \in \mathbb{N}$, $z_1, \cdots, z_N \in \mathbb{C}$ and $\xi_1, \cdots, \xi_N \in \mathbb{R}^d$, such that

$$\sum_{j=1}^N z_j = 0.$$

It follows that

$$
\begin{aligned}
&\sum_{j,l=1}^{N} z_j \overline{z_l} \psi(x, \xi_j - \xi_l) \\
&= \int_0^\infty \int_{\mathbb{S}^{d-1}} \left(\sum_{j,l=1}^{N} z_j \overline{z_l} + i \sum_{j,l=1}^{N} \frac{z_j \overline{z_l} \langle \xi_j - \xi_l, y \rangle}{1 + \|y\|^2} - \sum_{j,l=1}^{N} z_j e^{i\langle \xi_j, y\rangle} \overline{z_l e^{i\langle \xi_l, y\rangle}} \right) \frac{d\|y\|}{\|y\|^{1+\beta}} \tilde{\mu}(x, d\eta) \\
&= \int_0^\infty \int_{\mathbb{S}^{d-1}} \left(0 - \sum_{j=1}^{N} z_j e^{i\langle \xi_j, y\rangle} \overline{\sum_{l=1}^{N} z_l e^{i\langle \xi_l, y\rangle}} \right) \frac{d\|y\|}{\|y\|^{1+\beta}} \tilde{\mu}(x, d\eta) \\
&= -\int_0^\infty \int_{\mathbb{S}^{d-1}} \left| \sum_{j=1}^{N} z_j e^{i\langle \xi_j, y\rangle} \right|^2 \tilde{\mu}(x, d\eta) \\
&\leq 0.
\end{aligned}
$$

The proof is completed by Lemma 1.3.1. □

In the sequel we suppose that the hypotheses of Lemma 2.1.1 part (i) hold throughout this thesis without further reminder.

Lemma 2.1.2. *Let $N \in \mathbb{N}$. Suppose that the map $\mathbb{R}^d \ni x \mapsto \mu(x) \in M(\mathbb{S}^{d-1})$ fulfills:*

(i) $\partial^\delta \mu(x)$ is a measure (not necessarily positive) on $\mathbb{S}^{d-1}$ for all $x \in \mathbb{R}^d$ and for all $\delta \in \mathbb{N}_0^d$, $|\delta| \leq N$, with the uniformly bounded total variation.

(ii) The maps $\mathbb{R}^d \ni x \mapsto \partial^\delta \mu(x) \in M(\mathbb{S}^{d-1})$ are continuous for all $|\delta| \leq N$.

Then the function ω_μ given by (2.3) is N-times continuously differentiable in the first variable and it satisfies the condition

$$\sup_{\theta \in \mathbb{S}^{d-1}} \{ \|\partial^\delta \omega_\mu(\cdot, \theta)\|_{L_\infty(\mathbb{R}^d)} : |\delta| \leq N \} < \infty.$$

Proof. We fix $\theta \in \mathbb{S}^{d-1}$. The continuity of $\omega_\mu(\cdot, \theta)$ follows from the continuity of the map μ and the properties of the space $M_b(\mathbb{S}^{d-1})$ (see Chapter 1). Indeed, for an arbitrary $(x_n)_{n \in \mathbb{N}} \to x$ in $\mathbb{R}^d$ we have that

$$
\begin{aligned}
\lim_{n\to\infty} \omega_\mu(x_n, \theta) &= \lim_{n\to\infty} \int_{\mathbb{S}^{d-1}} |\theta \cdot \eta|^\beta \mu(x_n, d\eta) \\
&= \int_{\mathbb{S}^{d-1}} |\theta \cdot \eta|^\beta \lim_{n\to\infty} \mu(x_n, d\eta) \\
&= \int_{\mathbb{S}^{d-1}} |\theta \cdot \eta|^\beta \mu(x, d\eta) \\
&= \omega_\mu(x, \theta).
\end{aligned}
$$

For the derivative w.r.t. x_j at the point x, $j\in\{1,2,\cdots,d\}$, we consider $(h_n)_{n\in\mathbb{N}}\to 0$ in $\mathbb{R}$ and $e_j=(0,\cdots,\underbrace{1}_{j^{th}\ position},\cdots,0)$. Therefore,

$$\begin{aligned}\frac{\partial\omega_\mu(\cdot,\theta)}{\partial x_j}(x)&=\lim_{n\to\infty}\frac{1}{h_n}[\omega_\mu(x+h_ne_j,\theta)-\omega_\mu(x,\theta)]\\&=\lim_{n\to\infty}\frac{1}{h_n}\int_{\mathbb{S}^{d-1}}|\theta\cdot\eta|^\beta[\mu(x+h_ne_j,d\eta)-\mu(x,d\eta)]\\&=\int_{\mathbb{S}^{d-1}}|\theta\cdot\eta|^\beta\lim_{n\to\infty}\frac{1}{h_n}[\mu(x+e_jh_n,d\eta)-\mu(x,d\eta)]\\&=\int_{\mathbb{S}^{d-1}}|\theta\cdot\eta|^\beta\frac{\partial\mu}{\partial x_j}(x,d\eta).\end{aligned}$$

The continuity of this derivative is proved by repeating the arguments in the case of ω_μ. Using the same reasoning we find the existence and continuity of the others derivatives up to order N. Finally, uniform boundedness of the total variation of each measure $\partial^\delta\mu(x)$, for all $|\delta|\le N$ and $x\in\mathbb{R}^d$, yields the desired condition. □

An open problem for future research could be the study of β-stable random diffusion processes, whose corresponding spectral measure guarantees the hypothesis of the previous lemmata.

2.2 Some properties of the operator $-\Psi_\beta(x,-i\nabla)$

In this section, we show some properties of the operator $-\Psi_\beta(x,-i\nabla)$ in terms of the transition kernel μ. For this purpose, we analyse the symbol $\psi(x,\xi)=\|\xi\|^\beta\omega_\mu\left(x,\frac{\xi}{\|\xi\|}\right)$ given by (2.2), as a product between the symbols $\|\cdot\|^\beta$ and ω_μ. Since $\beta\in(0,2)$, we know that $\|\cdot\|^\beta$ is a continuous negative definite function (see Section 1.3) and we can define the operator associated with $-\|\cdot\|^\beta$ in the form

$$\Upsilon v(x):=-\mathcal{F}^{-1}_{\xi\to x}[\|\xi\|^\beta(\mathcal{F}v)(\xi)],\ v\in C_0^\infty(\mathbb{R}^d).\tag{2.4}$$

From [29, Example 4.6.29] we have that $(\Upsilon,C_0^\infty(\mathbb{R}^d))$ maps from $C_0^\infty(\mathbb{R}^d)$ into $L_p(\mathbb{R}^d)$ for any $1<p<\infty$. By choosing $p=2$, together with Plancherel's theorem and (2.4), it follows that

$$\Upsilon: C_0^\infty(\mathbb{R}^d)\to L_2(\mathbb{R}^d)$$

is a linear operator well defined and

$$\mathcal{F}(\Upsilon v)=-\|\cdot\|^\beta\mathcal{F}(v).$$

Now, we define the operator associated with the symbol ω_μ given by

$$\Theta v(x):=\mathcal{F}^{-1}_{\xi\to x}\left[\omega_\mu\left(x,\frac{\xi}{\|\xi\|}\right)(\mathcal{F}v)(\xi)\right],\ v\in C_0^\infty(\mathbb{R}^d).\tag{2.5}$$

Our goal is to show that Θ is bounded under suitable conditions on ω_μ and therefore it can be extended to all of $L_2(\mathbb{R}^d)$. With this extension, which is again denoted by Θ, we obtain that

$$-\Psi_\beta(x,-i\nabla)=\Theta\circ\Upsilon$$

maps from $C_0^\infty(\mathbb{R}^d)$ into $L_2(\mathbb{R}^d)$. Finally, we want to show that $-\Psi_\beta(x,-i\nabla)$ is closable, again, under suitable conditions on ω_μ.

We prove the boundedness of $\Theta : C_0^\infty(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ by using a result stated in [47, Theorem 16.3.2.]. First, we define on $\mathbb{R}^{d-1}$ the function

$$\zeta(z) := (1+\|z\|^2)^l$$

with some integer number $l > \frac{d-1}{4}$, thus we get

$$\int_{\mathbb{R}^{d-1}} \frac{1}{\zeta^2(z)} dz < \infty. \tag{2.6}$$

For every fixed $x \in \mathbb{R}^d$ we need that $\omega_\mu(x,\cdot) \in L_2(\mathbb{S}^{d-1})$. Here, we observe that the continuity of $\omega_\mu(x,\cdot)$ on $\mathbb{S}^{d-1}$ and the nature of the collection $\{U_j, \phi_j, \nu_j\}$, $j=1,2$, as mentioned at the beginning of the chapter, yield

$$\|\omega_\mu(x,\cdot)\|_{L_2(\mathbb{S}^{d-1})} = \left(\|(\nu_1\omega_\mu(x,\cdot))\circ\phi_1^{-1}\|^2_{L_2(\mathbb{R}^{d-1})} + \|(\nu_2\omega_\mu(x,\cdot))\circ\phi_2^{-1}\|^2_{L_2(\mathbb{R}^{d-1})} \right)^{\frac{1}{2}} < \infty.$$

As in [47, Sub-section 16.3.1], we introduce the norm

$$\|h\|_{\mathcal{H}_\zeta} := \left(\int_{\mathbb{R}^{d-1}} |\zeta(z)\mathcal{F}(h)(z)|^2 dz \right)^{\frac{1}{2}}, \quad h \in C_0^\infty(\mathbb{R}^{d-1}),$$

and we define the space $\mathcal{H}_\zeta(\mathbb{R}^{d-1})$ as the completion of $C_0^\infty(\mathbb{R}^{d-1})$ in this norm.

Remark 2.2.1. *Due to our choice of the function ζ and conditions on l, the space $\mathcal{H}_\zeta(\mathbb{R}^{d-1})$ is that of Bessel potentials $H_2^{2l}(\mathbb{R}^{d-1})$ which is isomorphic to the Sobolev space $W_2^{2l}(\mathbb{R}^{d-1})$ (see Section 1.3).*

We also define the space of functions w on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ given by

$$H_\zeta(\mathbb{R}^d \times \mathbb{S}^{d-1}) := \left\{ w : \|w\|_{H_\zeta} < \infty \right\},$$

where

$$\|w\|_{H_\zeta} := \left(\int_{\mathbb{R}^d} \|w(x,\cdot);\mathbb{S}^{d-1}\|^2_{\mathcal{H}_\zeta} dx \right)^{\frac{1}{2}}$$

and

$$\|w(x,\cdot);\mathbb{S}^{d-1}\|_{\mathcal{H}_\zeta} := \left(\|(\nu_1 w(x,\cdot))\circ\phi_1^{-1}\|^2_{\mathcal{H}_\zeta} + \|(\nu_2 w(x,\cdot))\circ\phi_2^{-1}\|^2_{\mathcal{H}_\zeta} \right)^{\frac{1}{2}}.$$

Since (2.6) holds, it is known that the space $\mathcal{H}_\zeta(\mathbb{R}^{d-1})$ is an algebra in the sense that

$$\|h_1h_2\|_{\mathcal{H}_\zeta} \le c\|h_1\|_{\mathcal{H}_\zeta}\|h_2\|_{\mathcal{H}_\zeta}, \quad h_1,h_2 \in \mathcal{H}_\zeta(\mathbb{R}^{d-1}).$$

The proof of this statement is similar to that of [53, Lemma 1].

Now, we suppose that $\omega_\mu(x,\cdot) \in C^{2l}(\mathbb{S}^{d-1})$ for every $x \in \mathbb{R}^d$ and that

$$k := \sup_{x\in\mathbb{R}^d} \{ \|\partial^\delta \omega_\mu(x,\cdot)\|_{L_\infty(\mathbb{S}^{d-1})} : |\delta| \le 2l \} < \infty. \tag{2.7}$$

According to this assumption, it follows that

$$\left|\partial^\delta\left(\omega_\mu(x,\cdot)\circ\phi_j^{-1}\right)(z)\right|\le k$$

for all $x\in\mathbb{R}^d$, $z\in\mathbb{R}^{d-1}$ and $|\delta|\le 2l$, $j=1,2$.

Let $g_j\in C_0^\infty(U_j)$ such that $g_j\nu_j=\nu_j$, $j=1,2$. We note that the support of $g_j\omega_\mu(x,\cdot)\circ\phi_j^{-1}$ and its derivatives up to order $2l$, is a compact set in $\mathbb{R}^{d-1}$. Therefore,

$$\partial^\delta\left(g_j\omega_\mu(x,\cdot)\circ\phi_j^{-1}\right)\in L_2(\mathbb{R}^{d-1})$$

and

$$\sum_{|\delta|\le 2l}\left\|\partial^\delta\left(g_j\omega_\mu(x,\cdot)\circ\phi_j^{-1}\right)\right\|_{L_2(\mathbb{R}^{d-1})}<k_j,$$

with some constant $k_j>0$ which depends on k and an upper bound for all $\partial^\delta\left(g_j\circ\phi_j^{-1}\right)$, $|\delta|\le 2l$. This and Remark 2.2.1 show that $g_j\omega_\mu(x,\cdot)\circ\phi_j^{-1}\in\mathcal{H}_\varsigma(\mathbb{R}^{d-1})$ for every $x\in\mathbb{R}^d$ and that

$$\|g_j\omega_\mu(x,\cdot)\circ\phi_j^{-1}\|_{\mathcal{H}_\varsigma}\le c_j,\quad j=1,2. \tag{2.8}$$

We are in position now to prove that the map

$$H_\varsigma(\mathbb{R}^d\times\mathbb{S}^{d-1})\ni w\longmapsto\omega_\mu w\in H_\varsigma(\mathbb{R}^d\times\mathbb{S}^{d-1})$$

is bounded. Indeed, let $w\in H_\varsigma(\mathbb{R}^d\times\mathbb{S}^{d-1})$. Since $\mathcal{H}_\varsigma(\mathbb{R}^{d-1})$ is an algebra because of condition (2.6), it follows that

$$\begin{aligned}
\|\omega_\mu w\|^2_{H_\varsigma} &= \int_{\mathbb{R}^d}\|(\omega_\mu w)(x,\cdot);\mathbb{S}^{d-1}\|^2_{\mathcal{H}_\varsigma}dx\\
&=\sum_{j=1}^2\int_{\mathbb{R}^d}\|(\nu_j(\omega_\mu w)(x,\cdot))\circ\phi_j^{-1}\|^2_{\mathcal{H}_\varsigma}dx\\
&=\sum_{j=1}^2\int_{\mathbb{R}^d}\|((g_j\omega_\mu)(x,\cdot)(\nu_j w)(x,\cdot))\circ\phi_j^{-1}\|^2_{\mathcal{H}_\varsigma}dx\\
&=\sum_{j=1}^2\int_{\mathbb{R}^d}\|(g_j\omega_\mu)(x,\cdot)\circ\phi_j^{-1}\ (\nu_j w)(x,\cdot)\circ\phi_j^{-1}\|^2_{\mathcal{H}_\varsigma}dx\\
&\le c^2\sum_{j=1}^2\int_{\mathbb{R}^d}\|(g_j\omega_\mu)(x,\cdot)\circ\phi_j^{-1}\|^2_{\mathcal{H}_\varsigma}\|(\nu_j w)(x,\cdot)\circ\phi_j^{-1}\|^2_{\mathcal{H}_\varsigma}dx\\
&\le c^2\sum_{j=1}^2\int_{\mathbb{R}^d}c_j^2\|(\nu_j w)(x,\cdot)\circ\phi_j^{-1}\|^2_{\mathcal{H}_\varsigma}dx\\
&\le \underbrace{c^2\max(c_1^2,c_2^2)}_{K}\int_{\mathbb{R}^d}\sum_{j=1}^2\|(\nu_j w)(x,\cdot)\circ\phi_j^{-1}\|^2_{\mathcal{H}_\varsigma}dx\\
&= K\int_{\mathbb{R}^d}\|w(x,\cdot);\mathbb{S}^{d-1}\|^2_{\mathcal{H}_\varsigma}dx\\
&= K\|w\|^2_{H_\varsigma},
\end{aligned}$$

where c_1 and c_2 come from (2.8).

This implies that $\Theta : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$, given by (2.5), is bounded ([47, Theorem 16.3.2.]) and that the composition with the operator in (2.4), i.e.,

$$\begin{aligned} -\Psi_\beta(x,-i\nabla) : C_0^\infty(\mathbb{R}^d) &\longrightarrow L_2(\mathbb{R}^d) \\ v \quad &\longmapsto -\mathcal{F}^{-1}_{\xi\to x}[\psi(x,\xi)(\mathcal{F}v)(\xi)], \end{aligned} \tag{2.9}$$

is well defined.

Next, we show that $A := -\Psi_\beta(x,-i\nabla)$ is closable using [29, Lemma 2.7.12]. For this purpose, let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $C_0^\infty(\mathbb{R}^d)$ converging in $L_2(\mathbb{R}^d)$ to zero, such that $(Av_n)_{n\in\mathbb{N}}$ converges in $L_2(\mathbb{R}^d)$ to some $\varphi \in L_2(\mathbb{R}^d)$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be arbitrary. We have that

$$\begin{aligned} \langle -Av_n, \phi\rangle_2 &= \int_{\mathbb{R}^d} -Av_n(x)\phi(x)dx \\ &= \int_{\mathbb{R}^d}\left[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\psi(x,\xi)\widehat{v}_n(\xi)d\xi\right]\phi(x)dx \\ &= \int_{\mathbb{R}^d}\left[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\psi(x,\xi)\left[\int_{\mathbb{R}^d} e^{-iy\cdot\xi}v_n(y)dy\right]d\xi\right]\phi(x)dx \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}e^{-iy\cdot\xi}\psi(x,\xi)v_n(y)\phi(x)dyd\xi dx \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} v_n(y)\left\{\int_{\mathbb{R}^d} e^{-iy\cdot\xi}\underbrace{\int_{\mathbb{R}^d} e^{ix\cdot\xi}\psi(x,\xi)\phi(x)dx}_{=:g(\xi)}\,d\xi\right\}dy \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} v_n(y)\left\{\int_{\mathbb{R}^d} e^{-iy\cdot\xi}g(\xi)d\xi\right\}dy. \end{aligned}$$

Let $m \in \mathbb{N}$ such that $m > \frac{2\beta+d}{2}$. We suppose now that the symbol ω_μ and its derivatives of order up to m w.r.t. x are uniformly bounded on $\mathbb{S}^{d-1}$, that is,

$$k_e := \sup_{\theta\in\mathbb{S}^{d-1}}\{\|\partial^\delta\omega_\mu(\cdot,\theta)\|_{L_\infty(\mathbb{R}^d)} : |\delta| \le m\} < \infty. \tag{2.10}$$

Denoting by B_1 the ball in $\mathbb{R}^d$ of radius 1 centered at the origin, we observe that

$$\int_{\mathbb{R}^d}|g(\xi)|^2 d\xi = \int_{B_1}|g(\xi)|^2 d\xi + \int_{\mathbb{R}^d\setminus B_1}|g(\xi)|^2 d\xi =: I_1 + I_2$$

and that

$$\begin{aligned} I_1 &\le \int_{B_1}\left(\int_{\mathbb{R}^d}\psi(x,\xi)|\phi(x)|dx\right)^2 d\xi \\ &\le \int_{B_1}\left(\int_{\mathbb{R}^d} k_e\|\xi\|^\beta|\phi(x)|dx\right)^2 d\xi \\ &\le (k_e\|\phi\|_1)^2\int_{B_1}\|\xi\|^{2\beta}d\xi < \infty. \end{aligned}$$

Here, we may also use the constant k instead of k_e. For the integral I_2 we note that

$$\begin{aligned}
i\xi_j g(\xi) &= \int_{\mathbb{R}^d} e^{ix\cdot\xi} i\xi_j \psi(x,\xi)\phi(x)dx, \quad j=1,\cdots,d,\\
&= \int_{\mathbb{R}^d} \frac{\partial e^{ix\cdot\xi}}{\partial x_j}\psi(x,\xi)\phi(x)dx\\
&= -\int_{\mathbb{R}^d} e^{ix\cdot\xi}\frac{\partial}{\partial x_j}\left(\psi(x,\xi)\phi(x)\right)dx\\
&= -\int_{\mathbb{R}^d} e^{ix\cdot\xi}\left(\frac{\partial\psi(x,\xi)}{\partial x_j}\phi(x)+\psi(x,\xi)\frac{\partial\phi(x)}{\partial x_j}\right)dx
\end{aligned}$$

and thus we obtain

$$\begin{aligned}
|\xi_j||g(\xi)| &\le \int_{\mathbb{R}^d}\left(\left|\frac{\partial\psi(x,\xi)}{\partial x_j}\right||\phi(x)|+\psi(x,\xi)\left|\frac{\partial\phi(x)}{\partial x_j}\right|\right)dx\\
&\le k_e\|\xi\|^\beta\left(\|\phi\|_1+\left\|\frac{\partial\phi(x)}{\partial x_j}\right\|_1\right)\\
&\le k_e\|\xi\|^\beta\|\phi\|_{W_1^1}.
\end{aligned}$$

Similarly, it can be shown that

$$\|\xi\|^m|g(\xi)| \le K_e\|\xi\|^\beta\|\phi\|_{W_1^m}$$

where the constant K_e depends on m, d and k_e. Therefore,

$$\begin{aligned}
I_2 &\le \left(K_e\|\phi\|_{W_1^m}\right)^2\int_{\mathbb{R}^d\setminus B_1}\|\xi\|^{2\beta-2m}d\xi\\
&\lesssim \left(K_e\|\phi\|_{W_1^m}\right)^2\int_1^\infty\|\xi\|^{2\beta-2m+d-1}d\|\xi\|,
\end{aligned}$$

which is finite because we chose $m>\frac{2\beta+d}{2}$. This proves that $g\in L_2(\mathbb{R}^d)$ and it must be the Fourier transform of $\mathcal{F}^{-1}(g)$ from Plancherel's theorem. Consequently,

$$\langle -Av_n,\phi\rangle_2 = \int_{\mathbb{R}^d} v_n(y)\mathcal{F}^{-1}(g)(-y)dy$$

and Hölder's inequality yields

$$\begin{aligned}
|\langle Av_n,\phi\rangle_2| &\le \|v_n\|_2\|\mathcal{F}^{-1}(g)\|_2\\
&\lesssim \|v_n\|_2\|g\|_2.
\end{aligned}$$

Taking the limit $n\to\infty$ we obtain that

$$\langle\varphi,\phi\rangle_2 = 0$$

holds for all $\phi\in C_0^\infty(\mathbb{R}^d)$. The hypothesis $\varphi\in L_2(\mathbb{R}^d)$ and the density of $C_0^\infty(\mathbb{R}^d)$ in $L_2(\mathbb{R}^d)$ show that $\varphi\equiv 0$ and hence $A=-\Psi_\beta(x,-i\nabla)$ given by (2.9) is closable. From [29, Theorem 2.7.14] we define the domain of its closure as

$$D(\overline{A}) := \overline{C_0^\infty}^{\|\cdot\|_{A;L_2(\mathbb{R}^d)}}$$

with respect to the graph norm

$$\|\cdot\|_{A;L_2(\mathbb{R}^d)} := \|\cdot\|_2 + \|A(\cdot)\|_2.$$

Therefore, we have proved the following result.

Theorem 2.2.1. *Let $\beta \in (0,2)$, $l, m \in \mathbb{N}$ such that $l > \frac{d-1}{4}$ and $m > \frac{2\beta+d}{2}$. Suppose that the transition kernel μ satisfies the assumptions of Lemma 2.1.1 and Lemma 2.1.2 with the integer number $N = 2l$ and $N = m$, respectively. Then the conditions (2.7) and (2.10) hold, the linear operator $-\Psi_\beta(x, -i\nabla)$ defined by (2.9) maps from $C_0^\infty(\mathbb{R}^d)$ into $L_2(\mathbb{R}^d)$ and it is closable.*

Remark 2.2.2. *We want to point out the case where we fix $x_0 \in \mathbb{R}^d$, which leads to the symbol $\|\xi\|^\beta \omega_\mu(\theta) := \|\xi\|^\beta \omega_\mu(x_0, \theta)$ independent of x. The corresponding operator $-\Psi_\beta(x_0, -i\nabla)$, **freezing** the coefficients at x_0, will be denoted by $-\Psi_\beta(-i\nabla)$. This situation is particularly exploited in the next chapters.*

2.3 Sub-Markovian semigroup in $L_2(\mathbb{R}^d)$

In this section we study other conditions on the transition kernel μ, which defines the symbol $\psi(x,\xi) = \|\xi\|^\beta \omega_\mu \left(x, \frac{\xi}{\|\xi\|}\right)$ given in (2.2), such that (2.1) is solvable in the strong sense of Prüss. For instance, as in [30, Corollary 2.6.7], for a large enough $\tilde{\lambda} > 0$ it is known that the operator

$$A_{\tilde{\lambda}} := -\Psi_\beta(x, -i\nabla) - \tilde{\lambda}$$

is a Dirichlet operator and generates a sub-Markovian semigroup on the Banach space $L_2(\mathbb{R}^d)$, whose domain is the ψ-Bessel potential space

$$H^{\psi,2}(\mathbb{R}^d) := \{v \in L_2(\mathbb{R}^d) : \|v\|_{H^{\psi,2}} < \infty\}$$

introduced in Section 1.3 with the negative definite function $\mathbb{R}^d \ni \xi \mapsto \|\xi\|^\beta \in \mathbb{R}$. Thereby, the norm of this space is

$$\|v\|_{H^{\psi,2}} := \frac{1}{(2\pi)^{\frac{d}{2}}} \left\| \left(1 + \|\cdot\|^\beta\right) \widehat{v} \right\|_2.$$

Similarly, for all $s \geq 0$,

$$H^{\psi,s}(\mathbb{R}^d) := \{v \in L_2(\mathbb{R}^d) : \|v\|_{H^{\psi,s}} < \infty\}$$

and

$$\|v\|_{H^{\psi,s}} := \frac{1}{(2\pi)^{\frac{d}{2}}} \left\| \left(1 + \|\cdot\|^\beta\right)^{\frac{s}{2}} \widehat{v} \right\|_2.$$

It is clear that $H^{\psi,0}(\mathbb{R}^d) = L_2(\mathbb{R}^d)$ and that $\|\cdot\|_{H^{\psi,s}} \leq \|\cdot\|_{H^{\psi,s+r}}$, $s, r \geq 0$.

Remark 2.3.1. *The space $S(\mathbb{R}^d)$ is dense in $\left(H^{\psi,s}(\mathbb{R}^d), \|\cdot\|_{H^{\psi,s}}\right)$, $s \geq 0$. (see, e.g., [30, Proposition 3.3.14]).*

Remark 2.3.2. *We note that* $\left(H^{\psi,s}(\mathbb{R}^d), \|\cdot\|_{H^{\psi,s}}\right)$ *is a Hilbert space with the inner product* $\langle v, w\rangle_{H^{\psi,s}} := \dfrac{1}{(2\pi)^d}\displaystyle\int_{\mathbb{R}^d}(1+\|\xi\|^\beta)^s\,\widehat{v}(\xi)\overline{\widehat{w}(\xi)}d\xi$, *for all* $s \geq 0$.

In order to exploit the theory of sub-Markovian semigroups, together with the relation between the constants of our estimates, we rewrite

$$\psi(x,\xi) = \|\xi\|^\beta\omega_\mu(x_0,\theta) + \|\xi\|^\beta[\omega_\mu(x,\theta) - \omega_\mu(x_0,\theta)] =: \psi_1(\xi) + \psi_2(x,\xi),$$

with $\theta = \frac{\xi}{\|\xi\|}$ and a suitable fixed $x_0 \in \mathbb{R}^d$. We denote by P_j the operator associated with the symbol ψ_j, $j \in \{1,2\}$, and thus

$$\Psi_\beta(x,-i\nabla) = P_1 + P_2(x).$$

Now, we investigate the behaviour of the symbols ψ_1 and ψ_2 separately according to the following assumptions $(\mathcal{C}_1)$-$(\mathcal{C}_3)$.

$(\mathcal{C}_1)$ $\mu(x) - \mu(x_0) \geq 0$ for all $x \in \mathbb{R}^d$ and the density of $\mu(x_0)$ is strictly positive.

$(\mathcal{C}_2)$ The function $\mathbb{R}^d \ni x \mapsto \mu(x) \in M(\mathbb{S}^{d-1})$ belongs to $C^m(\mathbb{R}^d)$ and there exist functions $\varphi_\delta \in L_1(\mathbb{R}^d)$, independent of θ, such that

$$|\partial_x^\delta \omega_\mu(x,\theta)| \leq \varphi_\delta(x)$$

for all $\delta \in \mathbb{N}_0^d$, $|\delta| \leq m$ and $m = \left[\frac{d}{\beta}\right] + d + 4$.

To establish the last assumption, we use the fact that the norms $\|\cdot\|$ and $\|\cdot\|_d$ in $\mathbb{R}^d$ are equivalent. Besides, we define the number

$$c := \sup_{\eta\in\mathbb{R}^d\setminus\{0\}} \frac{\|\eta\|}{\|\eta\|_d}$$

and the maps

$$\mathbb{N}_0 \ni k \mapsto \begin{cases} \varsigma(k) := \frac{2^{\frac{k}{2}}\pi^{\frac{d}{2}}\Gamma\left(\frac{1}{2}\right)}{(2\pi)^d\Gamma\left(\frac{d+1}{2}\right)} \sum_{|\delta|\leq d+k+1} c^{|\delta|}\|\varphi_\delta\|_1 \\ \iota(k) := \varsigma(k+1) \end{cases}. \tag{2.11}$$

$(\mathcal{C}_3)$ We assume:

(i) $\varsigma\left(\left[\frac{d}{\beta}\right]+3\right) < c_0 < \tilde{\lambda}$,

(ii) $-\tilde{\lambda}^2 + 2\tilde{\lambda}c_0 \leq (c_0 - \varsigma(0))^2 \leq \tilde{\lambda}(c_0 - \varsigma(1))$.

Remark 2.3.3. *(a) Assumption* $(\mathcal{C}_1)$ *and definition of* ω_μ *in* (2.3) *imply that*

$$\omega_\mu(x,\theta) - \omega_\mu(x_0,\theta) \geq 0$$

for all $x \in \mathbb{R}^d, \theta \in \mathbb{S}^{d-1}$, *and that there exist constants* $c_0, c_1 > 0$ *such that*

$$c_0 \leq \omega_\mu(x_0,\theta) \leq c_1, \quad \textit{for all } \theta \in \mathbb{S}^{d-1}.$$

As we will see later, the strict positivity of c_0 is essential in this section.
(b) Assumption $(\mathcal{C}_2)$ yields

$$|\partial_x^\delta \psi_2(x,\xi)| \le \varphi_\delta(x)\|\xi\|^\beta, \quad |\delta| \le m.$$

In particular, the Fourier transform of $\psi_2(x,\xi)$ in the variable x, i.e.,

$$\widehat{\psi_2}(\eta,\xi) := \mathcal{F}_{x\to\eta}(\psi_2(\cdot,\xi))(\eta,\xi), \quad \eta \in \mathbb{R}^d,$$

exists.
(c) It is remarkable that a large enough $\tilde{\lambda}$ is sufficient to satisfy assumption $(\mathcal{C}_3)$ and thus the assumption can be reduced to $\varsigma\left(\left[\frac{d}{\beta}\right]+3\right) < c_0$. However, as we will see in one of the main results of this chapter (Theorem 2.4.2), as $\tilde{\lambda}$ increases, the existence time of the local solution could decrease.

Lemma 2.3.1. *Under assumptions $(\mathcal{C}_1)$-$(\mathcal{C}_2)$, $P_1 v \in L_2(\mathbb{R}^d)$ and $P_2 v \in L_2(\mathbb{R}^d)$ for all $v \in S(\mathbb{R}^d)$, respectively.*

Proof. Let $v \in S(\mathbb{R}^d)$. From Section 1.3 we know that $\widehat{v} \in S(\mathbb{R}^d)$ and Remark 2.3.3(a) implies that

$$\begin{aligned}\int_{\mathbb{R}^d} |\psi_1(\xi)\widehat{v}(\xi)|^2\, d\xi &= \int_{\mathbb{R}^d} \left|\|\xi\|^\beta \omega_\mu(x_0,\theta)\widehat{v}(\xi)\right|^2 d\xi \\ &\le c_1^2 \int_{\mathbb{R}^d} \|\xi\|^{2\beta}|\widehat{v}(\xi)|^2 d\xi < \infty.\end{aligned}$$

This shows that $P_1 v = \mathcal{F}^{-1}[\psi_1 \widehat{v}] \in L_2(\mathbb{R}^d)$. For the operator $P_2(x)$, by definition we have that

$$\begin{aligned}\widehat{P_2 v}(\eta) &= \int_{\mathbb{R}^d} e^{-ix\cdot\eta} P_2(x)v(x)dx \\ &= \int_{\mathbb{R}^d} e^{-ix\cdot\eta}\left[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\psi_2(x,\xi)\widehat{v}(\xi)d\xi\right] dx \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-ix\cdot(\eta-\xi)}\psi_2(x,\xi)\widehat{v}(\xi)dxd\xi \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{\psi_2}(\eta-\xi,\xi)\widehat{v}(\xi)d\xi.\end{aligned}$$

From assumption $(\mathcal{C}_2)$ it follows that

$$c^{|\delta|}|\eta|^\delta|\widehat{\psi_2}(\eta,\xi)| \le c^{|\delta|}\|\varphi_\delta\|_1\|\xi\|^\beta$$

and

$$(1+c\|\eta\|_d)^k\,|\widehat{\psi_2}(\eta,\xi)| \le \|\xi\|^\beta \sum_{|\delta|\le k} c^{|\delta|}\|\varphi_\delta\|_1$$

for any $k \le m$. Therefore,

$$|\widehat{\psi_2}(\eta,\xi)| \le \left(1+\|\eta\|^2\right)^{-\frac{k}{2}}\|\xi\|^\beta \sum_{|\delta|\le k} c^{|\delta|}\|\varphi_\delta\|_1.$$

This estimate and the Minkowski's integral inequality yield

$$\begin{aligned}
\|\widehat{P_2 v}\|_2 &\le \frac{1}{(2\pi)^d}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|\widehat{\psi_2}(\eta-\xi,\xi)|\,|\widehat{v}(\xi)|d\xi\right)^2 d\eta\right)^{\frac{1}{2}}\\
&\le \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|\widehat{\psi_2}(\eta-\xi,\xi)|^2|\widehat{v}(\xi)|^2 d\eta\right)^{\frac{1}{2}} d\xi\\
&\le \frac{\sum_{|\delta|\le d+1} c^{|\delta|}\|\varphi_\delta\|_1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(1+\|\eta-\xi\|^2\right)^{-(d+1)}d\eta\right)^{\frac{1}{2}}\|\xi\|^\beta|\widehat{v}(\xi)|d\xi\\
&\le \frac{\sum_{|\delta|\le d+1} c^{|\delta|}\|\varphi_\delta\|_1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(1+\|\eta-\xi\|^2\right)^{-\frac{d+1}{2}}d\eta\right)^{\frac{1}{2}}\|\xi\|^\beta|\widehat{v}(\xi)|d\xi\\
&= \frac{\sum_{|\delta|\le d+1} c^{|\delta|}\|\varphi_\delta\|_1}{(2\pi)^d}\sqrt{\frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}}\int_{\mathbb{R}^d}\|\xi\|^\beta|\widehat{v}(\xi)|d\xi.
\end{aligned}$$

But this integral is finite because $\widehat{v}\in S(\mathbb{R}^d)$. □

Lemma 2.3.2. *Let $s\ge 0$. Under the assumption $(\mathcal{C}_1)$,*

$$c_0\left(\|v\|^2_{H^{\psi,s+1}}-\|v\|^2_{H^{\psi,s}}\right)\le \langle P_1 v,v\rangle_{H^{\psi,s}}\le c_1\left(\|v\|^2_{H^{\psi,s+1}}-\|v\|^2_{H^{\psi,s}}\right)$$

and

$$c_0^2\left(\|v\|^2_{H^{\psi,s+2}}-2\|v\|^2_{H^{\psi,s+1}}+\|v\|^2_{H^{\psi,s}}\right)\le \|P_1 v\|^2_{H^{\psi,s}}\le c_1^2\left(\|v\|^2_{H^{\psi,s+2}}-2\|v\|^2_{H^{\psi,s+1}}+\|v\|^2_{H^{\psi,s}}\right)$$

hold for all $v,w\in S(\mathbb{R}^d)$.

Proof. By using Remark 2.3.2 the proof is almost identical to the proof of [30, Proposition 2.3.6]. □

Lemma 2.3.3. *Under the assumption $(\mathcal{C}_2)$ and the map ς in (2.11), the estimates*

$$|\langle P_2 v,w\rangle_{H^{\psi,k}}|\le \varsigma(k)\|w\|_{H^{\psi,k}}\left(\|v\|^2_{H^{\psi,k+2}}-2\|v\|^2_{H^{\psi,k+1}}+\|v\|^2_{H^{\psi,k}}\right)^{\frac{1}{2}}$$

and

$$\|P_2 v\|_{H^{\psi,k}}\le \varsigma(k)\left(\|v\|^2_{H^{\psi,k+2}}-2\|v\|^2_{H^{\psi,k+1}}+\|v\|^2_{H^{\psi,k}}\right)^{\frac{1}{2}}$$

for all $0\le k\le m-d-1$, $v,w\in S(\mathbb{R}^d)$ hold.

Proof. Let $v, w \in S(\mathbb{R}^d)$. From the proof of Lemma 2.3.1 we obtain

$$\begin{aligned}
&|\langle P_2 v, w\rangle_{H^{\psi,k}}| \\
&\leq \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(1+\|\eta\|^\beta)^k|\widehat{P_2 v}(\eta)|\,|\widehat{w}(\eta)|d\eta \\
&\leq \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{\psi_2}(\eta-\xi,\xi)|\,|\widehat{v}(\xi)|d\xi\right](1+\|\eta\|^\beta)^k|\widehat{w}(\eta)|d\eta \\
&\leq \frac{\sum_{|\delta|\leq d+k+1} c^{|\delta|}\|\varphi_\delta\|_1}{(2\pi)^{2d}} \\
&\quad\times\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\left(1+\|\eta-\xi\|^2\right)^{-\frac{d+k+1}{2}}\|\xi\|^\beta|\widehat{v}(\xi)|d\xi\right](1+\|\eta\|^\beta)^k|\widehat{w}(\eta)|d\eta \\
&= \frac{\sum_{|\delta|\leq d+k+1} c^{|\delta|}\|\varphi_\delta\|_1}{(2\pi)^{2d}} \\
&\quad\times\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\left(1+\|\eta-\xi\|^2\right)^{-\frac{d+1}{2}}\left(\frac{1+\|\eta\|^\beta}{(1+\|\eta-\xi\|^2)(1+\|\xi\|^\beta)}\right)^{\frac{k}{2}}(1+\|\eta\|^\beta)^{\frac{k}{2}}|\widehat{w}(\eta)|d\eta\right] \\
&\qquad\times(1+\|\xi\|^\beta)^{\frac{k}{2}}\|\xi\|^\beta|\widehat{v}(\xi)|d\xi.
\end{aligned}$$

We write

$$\left(1+\|\eta-\xi\|^2\right)^{-\frac{d+1}{2}} = \left(1+\|\eta-\xi\|^2\right)^{-\frac{d+1}{4}-\frac{d+1}{4}}$$

and we use the fact that

$$\frac{1+\|\eta\|^\beta}{(1+\|\xi-\eta\|^2)(1+\|\xi\|^\beta)}\leq 2, \quad \eta,\xi\in\mathbb{R}^d.$$

Hölder's inequality in the square brackets brings

$$\begin{aligned}
&|\langle P_2 v, w\rangle_{H^{\psi,k}}| \\
&\leq 2^{\frac{k}{2}}\frac{\sum_{|\delta|\leq d+k+1} c^{|\delta|}\|\varphi_\delta\|_1}{(2\pi)^{2d}}\sqrt{\frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}} \\
&\times\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(1+\|\eta-\xi\|^2\right)^{-\frac{d+1}{2}}(1+\|\eta\|^\beta)^k|\widehat{w}(\eta)|^2d\eta\right)^{\frac{1}{2}}(1+\|\xi\|^\beta)^{\frac{k}{2}}\|\xi\|^\beta|\widehat{v}(\xi)|d\xi.
\end{aligned}$$

Again Hölder and Young's inequality for convolutions produce

$$\begin{aligned}
&|\langle P_2 v, w\rangle_{H^{\psi,k}}| \\
&\leq \frac{2^{\frac{k}{2}}}{(2\pi)^d}\sum_{|\delta|\leq d+k+1} c^{|\delta|}\|\varphi_\delta\|_1\frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\|w\|_{H^{\psi,k}}\left(\|v\|^2_{H^{\psi,k+2}}-2\|v\|^2_{H^{\psi,k+1}}+\|v\|^2_{H^{\psi,k}}\right)^{\frac{1}{2}}.
\end{aligned}$$

This proves the first estimate.

For the second one, previous arguments show that

$$\begin{aligned}
&\|P_2 v\|_{H^{\psi,k}} \\
&\leq \frac{1}{(2\pi)^{\frac{3d}{2}}}\left(\int_{\mathbb{R}^d}(1+\|\eta\|^\beta)^k\left(\int_{\mathbb{R}^d}|\widehat{\psi_2}(\eta-\xi,\xi)|\,|\widehat{v}(\xi)|d\xi\right)^2 d\eta\right)^{\frac{1}{2}} \\
&\leq \frac{1}{(2\pi)^{\frac{3d}{2}}}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|\widehat{\psi_2}(\eta-\xi,\xi)|^2(1+\|\eta\|^\beta)^k|\widehat{v}(\xi)|^2 d\eta\right)^{\frac{1}{2}} d\xi \\
&\leq \frac{\sum_{|\delta|\leq d+k+1} c^{|\delta|}\|\varphi_\delta\|_1}{(2\pi)^{\frac{3d}{2}}}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(1+\|\eta-\xi\|^2\right)^{-(d+k+1)}(1+\|\eta\|^\beta)^k d\eta\right)^{\frac{1}{2}}\|\xi\|^\beta|\widehat{v}(\xi)|d\xi \\
&\leq \frac{\sum_{|\delta|\leq d+k+1} c^{|\delta|}\|\varphi_\delta\|_1}{(2\pi)^{\frac{3d}{2}}} \\
&\quad\times\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(1+\|\eta-\xi\|^2\right)^{-\frac{d+1}{2}}\left(\frac{1+\|\eta\|^\beta}{(1+\|\eta-\xi\|^2)(1+\|\xi\|^\beta)}\right)^k d\eta\right)^{\frac{1}{2}} \\
&\qquad\times(1+\|\xi\|^\beta)^{\frac{k}{2}}\|\xi\|^\beta|\widehat{v}(\xi)|d\xi \\
&\leq 2^{\frac{k}{2}}\frac{\sum_{|\delta|\leq d+k+1} c^{|\delta|}\|\varphi_\delta\|_1}{(2\pi)^{\frac{3d}{2}}}\sqrt{\frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}}\int_{\mathbb{R}^d}(1+\|\xi\|^\beta)^{\frac{k}{2}}\|\xi\|^\beta|\widehat{v}(\xi)|d\xi \\
&< \infty
\end{aligned}$$

whenever $v \in S(\mathbb{R}^d)$. Applying the density of this space in $H^{\psi,k}(\mathbb{R}^d)$ in the first estimate, we obtain the second one. □

Remark 2.3.4. *Lemma 2.3.2 and Lemma 2.3.3 show that $\|P_1 v\|_{H^{\psi,s}} \leq c_1\|v\|_{H^{\psi,s+2}}$ and that $\|P_2 v\|_{H^{\psi,k}} \leq \varsigma(k)\|v\|_{H^{\psi,k+2}}$ for all $v \in S(\mathbb{R}^d)$. Hence, from Remark 2.3.1, we can extend these results for all $v \in H^{\psi,s+2}(\mathbb{R}^d)$ and $v \in H^{\psi,k+2}(\mathbb{R}^d)$ with the corresponding values of s and k.*

Lemma 2.3.4. *Under the assumptions $(\mathcal{C}_1)$-$(\mathcal{C}_2)$ and the map ι in (2.11), the estimates,*

$$|\langle P_1 v, w\rangle_{H^{\psi,s}}| \leq c_1\|v\|_{H^{\psi,s+1}}\|w\|_{H^{\psi,s+1}}$$

for all $s \geq 0$, $v, w \in H^{\psi,s+1}(\mathbb{R}^d)$, and

$$|\langle P_2 v, w\rangle_{H^{\psi,k}}| \leq \iota(k)\|v\|_{H^{\psi,k+1}}\|w\|_{H^{\psi,k+1}}$$

for all $0 \leq k \leq m-d-2$, $v, w \in H^{\psi,k+1}(\mathbb{R}^d)$ hold.

Proof. The first estimate is obtained as in the proof of [30, Proposition 2.3.20], together with Remark 2.3.2, for all $v, w \in S(\mathbb{R}^d)$.

The density of $S(\mathbb{R}^d)$ in $\left(H^{\psi,s+1}(\mathbb{R}^d), \|\cdot\|_{H^{\psi,s+1}}\right)$ allows to extend the linear mappings

$$S(\mathbb{R}^d) \ni w \mapsto \langle P_1 v, w\rangle_{H^{\psi,s}} \in \mathbb{R}$$

to $H^{\psi,s+1}(\mathbb{R}^d)$, fixing $v \in S(\mathbb{R}^d)$, and

$$S(\mathbb{R}^d) \ni v \mapsto \langle P_1 v, w\rangle_{H^{\psi,s}} \in \mathbb{R}$$

to $H^{\psi,s+1}(\mathbb{R}^d)$, fixing $w \in H^{\psi,s+1}(\mathbb{R}^d)$.

For the second estimate we use previous arguments and we obtain

$$\begin{aligned}
&|\langle P_2 v, w\rangle_{H^{\psi,k}}| \\
&\le \frac{\sum_{|\delta|\le d+k+2} c^{|\delta|}\|\varphi_\delta\|_1}{(2\pi)^{2d}} \\
&\times \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\left(1+\|\eta-\xi\|^2\right)^{-\frac{d+k+2}{2}}(1+\|\eta\|^\beta)^k|\widehat{w}(\eta)|d\eta\right](1+\|\xi\|^\beta)|\widehat{v}(\xi)|d\xi \\
&= \frac{\sum_{|\delta|\le d+k+2} c^{|\delta|}\|\varphi_\delta\|_1}{(2\pi)^{2d}} \\
&\times \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\left(1+\|\eta-\xi\|^2\right)^{-\frac{d+1}{2}-\frac{k+1}{2}}\left(\frac{1+\|\xi\|^\beta}{1+\|\eta\|^\beta}\right)^{\frac{1}{2}}\left(\frac{1+\|\eta\|^\beta}{1+\|\xi\|^\beta}\right)^{\frac{k}{2}}(1+\|\eta\|^\beta)^{\frac{k+1}{2}}|\widehat{w}(\eta)|d\eta\right] \\
&\qquad \times (1+\|\xi\|^\beta)^{\frac{k+1}{2}}|\widehat{v}(\xi)|d\xi.
\end{aligned}$$

As before,

$$\frac{1+\|\xi\|^\beta}{(1+\|\xi-\eta\|^2)(1+\|\eta\|^\beta)} \le 2$$

and thus it follows that

$$|\langle P_2 v, w\rangle_{H^{\psi,k}}| \le \frac{\sqrt{2}\,2^{\frac{k}{2}}}{(2\pi)^d}\sum_{|\delta|\le d+k+2} c^{|\delta|}\|\varphi_\delta\|_1 \frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\|w\|_{H^{\psi,k+1}}\|v\|_{H^{\psi,k+1}}.$$

Again, the result can be extended to $\left(H^{\psi,k+1}(\mathbb{R}^d), \|\cdot\|_{H^{\psi,k+1}}\right)$ by a density argument. ☐

Theorem 2.3.1. *Under the assumptions $(\mathcal{C}_1)$-$(\mathcal{C}_2)$ the operator $\Psi_\beta(x,-i\nabla)$ satisfies*

$$\|\Psi_\beta(x,-i\nabla)v\|_{H^{\psi,k}} \ge \left(\frac{c_0}{\sqrt{2}} - \varsigma(k)\right)\|v\|_{H^{\psi,k+2}} - c_0\|v\|_{H^{\psi,k}}$$

for all $v \in H^{\psi,k}(\mathbb{R}^d)$ and $0 \le k \le m-d-1$.

Proof. Let $v \in H^{\psi,k}(\mathbb{R}^d)$. We find that

$$\begin{aligned}
\|P_1 v\|^2_{H^{\psi,k}} &\ge \frac{c_0^2}{(2\pi)^d}\int_{\mathbb{R}^d}(1+\|\xi\|^\beta)^k\|\xi\|^{2\beta}|\widehat{v}(\xi)|^2 d\xi \\
&= \frac{c_0^2}{(2\pi)^d}\int_{\mathbb{R}^d}(1+\|\xi\|^\beta)^k(1+\|\xi\|^{2\beta})|\widehat{v}(\xi)|^2 d\xi - \frac{c_0^2}{(2\pi)^d}\int_{\mathbb{R}^d}(1+\|\xi\|^\beta)^k|\widehat{v}(\xi)|^2 d\xi \\
&\ge \frac{c_0^2}{2(2\pi)^d}\int_{\mathbb{R}^d}(1+\|\xi\|^\beta)^{k+2}|\widehat{v}(\xi)|^2 d\xi - \frac{c_0^2}{(2\pi)^d}\int_{\mathbb{R}^d}(1+\|\xi\|^\beta)^k|\widehat{v}(\xi)|^2 d\xi \\
&= \frac{c_0^2}{2}\|v\|^2_{H^{\psi,k+2}} - c_0^2\|v\|^2_{H^{\psi,k}} \\
&\ge \left(\frac{c_0}{\sqrt{2}}\|v\|_{H^{\psi,k+2}} - c_0\|v\|_{H^{\psi,k}}\right)^2.
\end{aligned}$$

From here and Lemma 2.3.3 we have that

$$\begin{aligned}\|\Psi_\beta(x,-i\nabla)v\|_{H^{\psi,k}} &= \|P_1v+P_2v\|_{H^{\psi,k}}\\ &\geq \|P_1v\|_{H^{\psi,k}}-\|P_2v\|_{H^{\psi,k}}\\ &\geq \frac{c_0}{\sqrt{2}}\|v\|_{H^{\psi,k+2}}-c_0\|v\|_{H^{\psi,k}}-\varsigma(k)\|v\|_{H^{\psi,k+2}}.\end{aligned}$$

□

Theorem 2.3.2. *Under the assumptions* $(\mathcal{C}_1)$*-*$(\mathcal{C}_3)$ *there exists a constant* $c_6>0$ *such that the operator* $\Psi_\beta(x,-i\nabla)+\tilde{\lambda}$ *satisfies*

$$\|\Psi_\beta(x,-i\nabla)v+\tilde{\lambda}v\|_2\geq c_6\|v\|_{H^{\psi,2}}$$

for all $v\in H^{\psi,2}(\mathbb{R}^d)$*.*

Proof. Let $v\in H^{\psi,2}(\mathbb{R}^d)$. We have that

$$\begin{aligned}\|\Psi_\beta(x,-i\nabla)v+\tilde{\lambda}v\|_2^2 &= \langle\Psi_\beta(x,-i\nabla)v+\tilde{\lambda}v,\Psi_\beta(x,-i\nabla)v+\tilde{\lambda}v\rangle_2\\ &= \|\Psi_\beta(x,-i\nabla)v\|_2^2+2\tilde{\lambda}\langle\Psi_\beta(x,-i\nabla)v,v\rangle_2+\tilde{\lambda}^2\|v\|_2^2\\ &= \|(P_1+P_2)v\|_2^2+2\tilde{\lambda}\langle(P_1+P_2)v,v\rangle_2+\tilde{\lambda}^2\|v\|_2^2.\end{aligned}$$

The first term is estimated using Lemmata 2.3.2 and 2.3.3 with $k=s=0$, and $c_2:=\varsigma(0)$:

$$\begin{aligned}\|(P_1+P_2)v\|_2^2 &= \|P_1v\|_2^2+2\langle P_1v,P_2v\rangle_2+\|P_2v\|_2^2\\ &\geq \|P_1v\|_2^2-2|\langle P_2v,P_1v\rangle_2|+\|P_2v\|_2^2\\ &\geq \|P_1v\|_2^2-2\|P_1v\|_2\|P_2v\|_2+\|P_2v\|_2^2\\ &= (\|P_1v\|_2-\|P_2v\|_2)^2\\ &\geq \left(c_0\left(\|v\|_{H^{\psi,2}}^2-2\|v\|_{H^{\psi,1}}^2+\|v\|_2^2\right)^{\frac{1}{2}}-c_2\left(\|v\|_{H^{\psi,2}}^2-2\|v\|_{H^{\psi,1}}^2+\|v\|_2^2\right)^{\frac{1}{2}}\right)^2\\ &= (c_0-c_2)^2\left(\|v\|_{H^{\psi,2}}^2-2\|v\|_{H^{\psi,1}}^2+\|v\|_2^2\right)\end{aligned}$$

whenever $c_0\geq c_2$. In a similar way, Lemmata 2.3.2 and 2.3.4 with $k=s=0$, and $c_3:=\iota(0)$, yield

$$\begin{aligned}\langle(P_1+P_2)v,v\rangle_2 &= \langle P_1v,v\rangle_2+\langle P_2v,v\rangle_2\\ &\geq \langle P_1v,v\rangle_2-|\langle P_2v,v\rangle_2|\\ &\geq c_0\left(\|v\|_{H^{\psi,1}}^2-\|v\|_2^2\right)-c_3\|v\|_{H^{\psi,1}}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Psi_\beta(x,-i\nabla)v+\tilde{\lambda}v\|_2^2 &\geq (c_0-c_2)^2\|v\|_{H^{\psi,2}}^2\\ &\quad+2\left[\tilde{\lambda}(c_0-c_3)-(c_0-c_2)^2\right]\|v\|_{H^{\psi,1}}^2\\ &\quad+\left[(c_0-c_2)^2-2\tilde{\lambda}c_0+\tilde{\lambda}^2\right]\|v\|_2^2\\ &\geq (c_0-c_2)^2\|v\|_{H^{\psi,2}}^2\end{aligned}$$

under assumption $(\mathcal{C}_3)$. □

At the beginning of this section we defined the operator

$$A_{\tilde{\lambda}} = -\Psi_\beta(x, -i\nabla) - \tilde{\lambda}$$

on the Banach space $\left(H^{\psi,2}(\mathbb{R}^d), \|\cdot\|_{H^{\psi,2}}\right)$. Now, we are in a position to prove its dissipativity and other properties.

Theorem 2.3.3. *Under the assumptions* $(\mathcal{C}_1)$*-*$(\mathcal{C}_3)$ *the operator* $\left(A_{\tilde{\lambda}}, H^{\psi,2}(\mathbb{R}^d)\right)$ *is* L_2*-dissipative.*

Proof. As in the proof of Theorem 2.3.2 we find that

$$\begin{aligned}
\langle -A_{\tilde{\lambda}} v, v\rangle_2 &= \langle \left(\Psi_\beta(x, -i\nabla) + \tilde{\lambda}\right) v, v\rangle_2 \\
&= \langle (P_1 + P_2)v, v\rangle_2 + \tilde{\lambda}\|v\|_2^2 \\
&\geq c_0 \left(\|v\|_{H^{\psi,1}}^2 - \|v\|_2^2\right) - c_3\|v\|_{H^{\psi,1}}^2 + \tilde{\lambda}\|v\|_2^2 \\
&= (\tilde{\lambda} - c_0)\|v\|_2^2 + (c_0 - c_3)\|v\|_{H^{\psi,1}}^2 \\
&\geq 0
\end{aligned}$$

for all $v \in H^{\psi,2}(\mathbb{R}^d)$. Let $\tau > 0$. It follows that

$$\begin{aligned}
\|\tau v - A_{\tilde{\lambda}} v\|_2^2 &= \tau^2\|v\|_2^2 - 2\tau\langle A_{\tilde{\lambda}} v, v\rangle_2 + \|A_{\tilde{\lambda}} v\|_2^2 \\
&\geq \tau^2\|v\|_2^2.
\end{aligned}$$

□

Theorem 2.3.4. *Under the assumptions* $(\mathcal{C}_1)$*-*$(\mathcal{C}_3)$ *the operator*

$$A_{\tilde{\lambda}} : H^{\psi,2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$$

is bijective and closed.

Proof. From Lemma 2.3.4 and c_3 as in the proof of Theorem 2.3.2, we find that

$$\begin{aligned}
|\langle A_{\tilde{\lambda}} v, w\rangle_2| &\leq |\langle P_1 v, w\rangle_2| + |\langle P_2 v, w\rangle_2| + \tilde{\lambda}|\langle v, w\rangle_2| \\
&\leq c_1\|v\|_{H^{\psi,1}}\|w\|_{H^{\psi,1}} + c_3\|v\|_{H^{\psi,1}}\|w\|_{H^{\psi,1}} + \tilde{\lambda}\|v\|_{H^{\psi,1}}\|w\|_{H^{\psi,1}} \\
&= (c_1 + c_3 + \tilde{\lambda})\|v\|_{H^{\psi,1}}\|w\|_{H^{\psi,1}}
\end{aligned}$$

and from the proof of Theorem 2.3.3 we see that

$$\begin{aligned}
|\langle A_{\tilde{\lambda}} v, v\rangle_2| &\geq (\tilde{\lambda} - c_0)\|v\|_2^2 + (c_0 - c_3)\|v\|_{H^{\psi,1}}^2 \\
&\geq (c_0 - c_3)\|v\|_{H^{\psi,1}}^2
\end{aligned}$$

for all $v, w \in H^{\psi,1}(\mathbb{R}^d)$. Therefore, the existence of the unique $v \in H^{\psi,1}(\mathbb{R}^d)$ such that

$$A_{\tilde{\lambda}} v = f, \quad f \in L_2(\mathbb{R}^d),$$

comes from [29, Theorem 2.7.41] and the proof of [30, Theorem 2.3.27]. But this function v belongs to $H^{\psi,2}(\mathbb{R}^d)$ under the same arguments as in the proof of [30, Theorem 2.3.28], using Theorem 2.3.2 instead of [30, Theorem 2.3.13]. This proves the first statement.

Let $\tau > 0$. We note that

$$A_{\tilde{\lambda}} - \tau = -\Psi_\beta(x, -i\nabla) - (\tilde{\lambda} + \tau)$$

and assumption $\mathcal{C}_3$ is also true for $\tilde{\lambda}+\tau$ instead of $\tilde{\lambda}$. Hence, $(A_{\tilde{\lambda}} - \tau)$ is also bijective for all $\tau > 0$ and $\big(A_{\tilde{\lambda}}, H^{\psi,2}(\mathbb{R}^d)\big)$ is closed by [29, Lemma 4.1.26]. □

Theorem 2.3.5. *Under the assumptions* $(\mathcal{C}_1)$-$(\mathcal{C}_3)$ *the operator*

$$A_{\tilde{\lambda}} : H^{\psi,2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$$

generates a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ *on* $L_2(\mathbb{R}^d)$.

Proof. Follows from Theorems 2.3.3 and 2.3.4, together with Theorem of Hille and Yosida ([29, Theorem 4.1.33]). □

Theorem 2.3.6. *Under the assumptions* $(\mathcal{C}_1)$-$(\mathcal{C}_3)$ *the operator*

$$\begin{aligned} B : H^{\psi,\left[\frac{d}{\beta}\right]+4}(\mathbb{R}^d) &\to H^{\psi,\left[\frac{d}{\beta}\right]+2}(\mathbb{R}^d) \\ v &\longmapsto A_{\tilde{\lambda}} v \end{aligned}$$

is closable and its closure $\big(\bar{B}, D(\bar{B})\big)$ *generates a Feller semigroup* $(S_t)_{t\geq 0}$ *on* $C_\infty(\mathbb{R}^d)$.

Proof. We choose $s = k = \left[\frac{d}{\beta}\right] + 2$ in Lemmata 2.3.2 and 2.3.3. Thus we see that the operator $A_{\tilde{\lambda}}$ maps $H^{\psi,\left[\frac{d}{\beta}\right]+4}(\mathbb{R}^d)$ into $H^{\psi,\left[\frac{d}{\beta}\right]+2}(\mathbb{R}^d)$. Besides,

$$H^{\psi,\left[\frac{d}{\beta}\right]+4}(\mathbb{R}^d) \hookrightarrow H^{\psi,\left[\frac{d}{\beta}\right]+2}(\mathbb{R}^d) \hookrightarrow C_\infty(\mathbb{R}^d)$$

by the choice of s and k ([30, Formula (2.296)]). We want to point out that these embeddings are continuous with the corresponding norm on the spaces $H^{\psi,k}(\mathbb{R}^d)$.

Applying Lemmata 2.3.2 and 2.3.4, with $s = k = \left[\frac{d}{\beta}\right] + 2$ and $c_5 := \iota\left(\left[\frac{d}{\beta}\right] + 2\right)$, we find

$$\begin{aligned} \left|\langle A_{\tilde{\lambda}} v, w\rangle_{H^{\psi,\left[\frac{d}{\beta}\right]+2}}\right| &\leq \left|\langle P_1 v, w\rangle_{H^{\psi,\left[\frac{d}{\beta}\right]+2}}\right| + \left|\langle P_2 v, w\rangle_{H^{\psi,\left[\frac{d}{\beta}\right]+2}}\right| + \tilde{\lambda}\left|\langle v, w\rangle_{H^{\psi,\left[\frac{d}{\beta}\right]+2}}\right| \\ &\leq (c_1 + c_5 + \tilde{\lambda})\|v\|_{H^{\psi,\left[\frac{d}{\beta}\right]+3}}\|w\|_{H^{\psi,\left[\frac{d}{\beta}\right]+3}} \end{aligned}$$

and

$$\begin{aligned} \left|\langle A_{\tilde{\lambda}} v, v\rangle_{H^{\psi,\left[\frac{d}{\beta}\right]+2}}\right| &\geq \langle P_1 v, v\rangle_{H^{\psi,\left[\frac{d}{\beta}\right]+2}} - \left|\langle P_2 v, v\rangle_{H^{\psi,\left[\frac{d}{\beta}\right]+2}}\right| + \tilde{\lambda}\langle v, v\rangle_{H^{\psi,\left[\frac{d}{\beta}\right]+2}} \\ &\geq c_0\left(\|v\|^2_{H^{\psi,\left[\frac{d}{\beta}\right]+3}} - \|v\|^2_{H^{\psi,\left[\frac{d}{\beta}\right]+2}}\right) - c_5\|v\|^2_{H^{\psi,\left[\frac{d}{\beta}\right]+3}} + \tilde{\lambda}\|v\|^2_{H^{\psi,\left[\frac{d}{\beta}\right]+2}} \\ &= (c_0 - c_5)\|v\|^2_{H^{\psi,\left[\frac{d}{\beta}\right]+3}} + (\tilde{\lambda} - c_0)\|v\|^2_{H^{\psi,\left[\frac{d}{\beta}\right]+2}} \\ &\geq (c_0 - c_5)\|v\|^2_{H^{\psi,\left[\frac{d}{\beta}\right]+3}} \end{aligned}$$

for all $v, w \in H^{\psi,[\frac{d}{\beta}]+3}(\mathbb{R}^d)$. Once again the existence of the unique $v \in H^{\psi,[\frac{d}{\beta}]+3}(\mathbb{R}^d)$ such that

$$A_{\tilde{\lambda}}v = f, \quad f \in H^{\psi,[\frac{d}{\beta}]+2}(\mathbb{R}^d),$$

comes from [29, Theorem 2.7.41] and similar arguments of the proof of [30, Theorem 2.3.27].

But this function v belongs to $H^{\psi,[\frac{d}{\beta}]+4}(\mathbb{R}^d)$ under the same arguments as in the proof of [30, Theorem 2.3.19], using Theorem 2.3.1 instead of [30, Theorem 2.3.13]. Indeed, from Theorem 2.3.1 with $k = \left[\frac{d}{\beta}\right] + 2$ and $c_4 := \varsigma\left(\left[\frac{d}{\beta}\right] + 2\right)$, we see that

$$\|\Psi_\beta(x, -i\nabla)v\|_{H^{\psi,[\frac{d}{\beta}]+2}} \geq \frac{c_0 - \sqrt{2}c_4}{\sqrt{2}}\|v\|_{H^{\psi,[\frac{d}{\beta}]+4}} - c_0\|v\|_{H^{\psi,[\frac{d}{\beta}]+2}}$$

and we note that $c_5 \geq \sqrt{2}c_4$. Therefore $c_0 > \sqrt{2}c_4$ by assumption $(\mathcal{C}_3)$.

Taken together, these results show that

$$B : H^{\psi,[\frac{d}{\beta}]+4}(\mathbb{R}^d) \to H^{\psi,[\frac{d}{\beta}]+2}(\mathbb{R}^d)$$

is bijective.

Now, as in the proof of Theorem 2.3.4, this procedure also works for $\tilde{\lambda} + \tau$ with any $\tau > 0$. That is,

$$Ran(B - \tau) = H^{\psi,[\frac{d}{\beta}]+2}(\mathbb{R}^d)$$

for all $\tau > 0$.

On the other hand, from Corollary 2.1.1 and [29, Theorem 4.5.6] we know that the operator $\left(-\Psi_\beta(x, -i\nabla), C_0^\infty(\mathbb{R}^d)\right)$ satisfies the positive maximum principle. Since $C_0^\infty(\mathbb{R}^d)$ is a dense subspace of $H^{\psi,[\frac{d}{\beta}]+4}(\mathbb{R}^d)$ ([29, Remark 3.10.2 and Theorem 3.10.3]), we have that $C_0^\infty(\mathbb{R}^d)$ is an operator core in the sense of [30, Theorem 2.6.1]. Hence, the operator $\left(B, H^{\psi,[\frac{d}{\beta}]+4}(\mathbb{R}^d)\right)$ also satisfies the positive maximum principle and the assertion follows from [29, Theorem 4.5.3]. □

Theorem 2.3.7. *Under the assumptions* $(\mathcal{C}_1)$-$(\mathcal{C}_3)$,

$$A_{\tilde{\lambda}} : H^{\psi,2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$$

is a Dirichlet operator and generates an L_2*-sub-Markovian semigroup.*

Proof. By Theorem 2.3.5 we know that the operator $\left(A_{\tilde{\lambda}}, H^{\psi,2}(\mathbb{R}^d)\right)$ generates a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L_2(\mathbb{R}^d)$. Hence, its resolvent satisfies

$$(\tau - A_{\tilde{\lambda}})^{-1}f = \int_0^\infty e^{-\tau s}T_s f ds$$

for all $f \in L_2(\mathbb{R}^d)$ and $\tau > 0$ ([29, Lemma 4.1.18]). In a similar way, by Theorem 2.3.6,

$$(\tau - \bar{B})^{-1}g = \int_0^\infty e^{-\tau s}S_s g ds$$

for all $g \in C_\infty(\mathbb{R}^d)$ and $\tau > 0$.

On the other hand, as it was described above in the proof of Theorem 2.3.6,

$$(\tau - B) : H^{\psi,[\frac{d}{\beta}]+4}(\mathbb{R}^d) \to H^{\psi,[\frac{d}{\beta}]+2}(\mathbb{R}^d)$$

is bijective for any $\tau > 0$. Therefore, by construction it follows that

$$(\tau - A_{\tilde{\lambda}})^{-1}\left(H^{\psi,[\frac{d}{\beta}]+2}(\mathbb{R}^d)\right) = H^{\psi,[\frac{d}{\beta}]+4}(\mathbb{R}^d) \subset H^{\psi,[\frac{d}{\beta}]+2}(\mathbb{R}^d).$$

In particular, this means that $(\tau - A_{\tilde{\lambda}})^{-1}$ leaves the set $V := H^{\psi,[\frac{d}{\beta}]+2}(\mathbb{R}^d)$ invariant for all $\tau > 0$. Since $V \subset H^{\psi,2}(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d)$, we can conclude that it is a core operator for $\left(A_{\tilde{\lambda}}, H^{\psi,2}(\mathbb{R}^d)\right)$ in the sense of [30, Lemma 3.3.9].

Again, by construction we also have that

$$(\tau - \bar{B})^{-1}f = (\tau - A_{\tilde{\lambda}})^{-1}f, \quad f \in V,$$

and therefore

$$T_t f = S_t f \quad a.e.$$

for all $f \in V$. Using this equality and exploiting the fact that $(S_t)_{t\geq 0}$ is a Feller semigroup, we find that $A_{\tilde{\lambda}}$ satisfies the Dirichlet condition

$$\int_{\mathbb{R}^d} (A_{\tilde{\lambda}} v)(x)\left((v-1)^+\right)(x)dx \leq 0, \quad v \in H^{\psi,2}(\mathbb{R}^d),$$

taking $p = 2$ in [29, Definition 4.6.7]. Indeed, for $f \in V$ we note that

$$\begin{aligned}\int_{\mathbb{R}^d} (T_t f)(x)\left((f-1)^+\right)(x)dx &= \int_{\mathbb{R}^d} (S_t f)(x)\left((f-1)^+\right)(x)dx \\ &= \int_{\mathbb{R}^d} \left(S_t((f-1)^+ + \min(1,f))\right)(x)\left((f-1)^+\right)(x)dx \\ &= \int_{\mathbb{R}^d} \left(S_t((f-1)^+)\right)(x)\left((f-1)^+\right)(x)dx \\ &\quad + \int_{\mathbb{R}^d} \left(S_t(\min(1,f))\right)(x)\left((f-1)^+\right)(x)dx.\end{aligned}$$

Since S_t is positivity preserving, [29, Lemma 4.6.24 A.] yields

$$\begin{aligned}\int_{\mathbb{R}^d} \left(S_t(\min(1,f))\right)(x)\left((f-1)^+\right)(x)dx &\leq \int_{\mathbb{R}^d} (S_t(1))(x)\left((f-1)^+\right)(x)dx \\ &\leq \int_{\mathbb{R}^d} \left((f-1)^+\right)(x)dx.\end{aligned}$$

The result follows from similar arguments as in the proofs of [29, Lemma 4.6.6 and Theorem 4.6.20] and [29, Lemma 4.6.17]. □

2.4 Strong solutions

According to the results of the previous section, let $(T_t)_{t\geq 0}$ the $L_2(\mathbb{R}^d)$-sub-Markovian semigroup generated by $\left(A_{\tilde{\lambda}}, H^{\psi,2}(\mathbb{R}^d)\right)$ under the assumptions $(\mathcal{C}_1)$-$(\mathcal{C}_3)$.

We define the non-linear operator

$$\mathcal{M}(v)(t) := E_\alpha(A_{\tilde{\lambda}}t^\alpha)u_0 + \alpha\int_0^t (t-s)^{\alpha-1}E'_\alpha(A_{\tilde{\lambda}}(t-s)^\alpha)\left(\lambda|v|^{\gamma-1}v + \tilde{\lambda}v\right)(s)ds \tag{2.12}$$

on the Banach space

$$Y_T := C([0,T]; L_2(\mathbb{R}^d)\cap L_\infty(\mathbb{R}^d))$$

with the usual norm, that is,

$$\|v\|_{Y_T} = \sup_{t\in[0,T]}\left(\|v(t)\|_2 + \|v(t)\|_\infty\right).$$

Due to Pollard-Zolotarev's formula (see, e.g. [40, Formulas (8.5) and (8.6)]) and the fact that $0<\alpha<1$, the extended definition of the Mittag-Leffler functions to operators like $A_{\tilde{\lambda}}$ yields

$$E_\alpha(A_{\tilde{\lambda}}t^\alpha) = \frac{1}{\alpha}\int_0^\infty T_{t^\alpha r}r^{-1-\frac{1}{\alpha}}G_\alpha(1, r^{-\frac{1}{\alpha}})dr$$

and

$$E'_\alpha(A_{\tilde{\lambda}}(t-s)^\alpha) = \frac{1}{\alpha}\int_0^\infty T_{(t-s)^\alpha r}r^{-\frac{1}{\alpha}}G_\alpha(1, r^{-\frac{1}{\alpha}})dr.$$

Besides,

$$\int_0^\infty r^{-1-\frac{1}{\alpha}}G_\alpha(1, r^{-\frac{1}{\alpha}})dr = \alpha$$

and

$$\int_0^\infty r^{-\frac{1}{\alpha}}G_\alpha(1, r^{-\frac{1}{\alpha}})dr = \frac{1}{\Gamma(\alpha)},$$

due to the Mellin transform of G_α and the Zolotarev's formula (see [40, Proposition 8.1.1]). Therefore, we can show the following result of fixed point.

Theorem 2.4.1. *Let $\alpha\in(0,1)$ and $\beta\in(0,2)$. Assume the hypothesis of Theorem 2.3.7 holds. Let $\lambda\in\mathbb{R}$ and $\gamma>1$. If $u_0\in L_2(\mathbb{R}^d)\cap L_\infty(\mathbb{R}^d)$, then for some $0<T^*<T$ the operator $\mathcal{M}$ defined by (2.12) has a unique fixed point in Y_{T^*}.*

Proof. Whenever $(T_t)_{t\geq 0}$ is a sub-Markovian semigroup on $L_p(\mathbb{R}^d)$, $1\leq p<\infty$, it is well known that

$$\|T_t(v)\|_p \leq \|v\|_p,$$

for all $v\in L_p(\mathbb{R}^d)$ and that

$$|T_t(v)| \leq |v|$$

and

$$\|T_t(v)\|_\infty \leq \|v\|_\infty$$

hold for all $v \in L_p(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$. Taking $p = 2$, we can conclude that the operator $\mathcal{M}$ is well defined. Indeed,

$$\begin{aligned}\|\mathcal{M}(v)(t)\|_2 &\leq \|u_0\|_2 + \int_0^t (t-s)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \left(|\lambda| \|v(s)\|_\infty^{\gamma-1} \|v(s)\|_2 + \tilde{\lambda} \|v(s)\|_2 \right) ds \\ &\leq \|u_0\|_2 + \frac{1}{\alpha\Gamma(\alpha)} \|v\|_{Y_T} t^\alpha \left(|\lambda| \|v\|_{Y_T}^{\gamma-1} + \tilde{\lambda} \right) < \infty\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{M}(v)(t)\|_\infty &\leq \|u_0\|_\infty + \int_0^t (t-s)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \left(|\lambda| \|v(s)\|_\infty^{\gamma} + \tilde{\lambda} \|v(s)\|_\infty \right) ds \\ &\leq \|u_0\|_\infty + \frac{1}{\alpha\Gamma(\alpha)} \|v\|_{Y_T} t^\alpha \left(|\lambda| \|v\|_{Y_T}^{\gamma-1} + \tilde{\lambda} \right) < \infty.\end{aligned}$$

We note that $\mathcal{M}(v)(0) = u_0$ and that the continuity of $t \mapsto E_\alpha(A_{\tilde{\lambda}} t^\alpha) u_0$ in $[0, T]$ follows from the strong continuity of $(T_t)_{t\geq 0}$ and the continuity of $(\cdot)^\alpha$, $0 < \alpha < 1$. Besides,

$$\|E_\alpha(A_{\tilde{\lambda}}(\cdot)^\alpha) u_0\|_{Y_T} \leq \|u_0\|_2 + \|u_0\|_\infty = \|u_0\|_{L_2(\mathbb{R}^d)\cap L_\infty(\mathbb{R}^d)}. \tag{2.13}$$

The continuity of

$$t \mapsto \int_0^t (t-s)^{\alpha-1} E'_\alpha(A_{\tilde{\lambda}}(t-s)^\alpha) \left(\lambda |v|^{\gamma-1} v + \tilde{\lambda} v \right)(s) ds$$

at $t = 0$ is straightforward. For the continuity in $(0, T]$, let $0 < t_0 < t \leq T$ without loss of generality. We see that

$$\begin{aligned}&\int_0^t s^{\alpha-1} E'_\alpha(A_{\tilde{\lambda}} s^\alpha) \left(\lambda |v|^{\gamma-1} v + \tilde{\lambda} v \right)(t-s) ds \\ &- \int_0^{t_0} s^{\alpha-1} E'_\alpha(A_{\tilde{\lambda}} s^\alpha) \left(\lambda |v|^{\gamma-1} v + \tilde{\lambda} v \right)(t_0-s) ds \\ &= \int_0^{t_0} s^{\alpha-1} E'_\alpha(A_{\tilde{\lambda}} s^\alpha) \left(\lambda |v|^{\gamma-1} v(t-s) - \lambda |v|^{\gamma-1} v(t_0-s) + \tilde{\lambda} v(t-s) - \tilde{\lambda} v(t_0-s) \right) ds \\ &\quad + \int_{t_0}^t s^{\alpha-1} E'_\alpha(A_{\tilde{\lambda}} s^\alpha) \left(\lambda |v|^{\gamma-1} v + \tilde{\lambda} v \right)(t-s) ds \\ &= \lambda \int_0^{t_0} s^{\alpha-1} E'_\alpha(A_{\tilde{\lambda}} s^\alpha) \left(|v|^{\gamma-1} v(t-s) - |v|^{\gamma-1} v(t_0-s) \right) ds \\ &\quad + \tilde{\lambda} \int_0^{t_0} s^{\alpha-1} E'_\alpha(A_{\tilde{\lambda}} s^\alpha) \left(v(t-s) - v(t_0-s) \right) ds \\ &\quad + \int_{t_0}^t s^{\alpha-1} E'_\alpha(A_{\tilde{\lambda}} s^\alpha) \left(\lambda |v|^{\gamma-1} v + \tilde{\lambda} v \right)(t-s) ds\end{aligned}$$

and using the property

$$|\, |a|^c a - |b|^c b \,| \lesssim |\, a - b \,| (|a|^c + |b|^c) \lesssim |\, a - b \,| (\, |a| + |b| \,)^c, \quad a, b \in \mathbb{R}, c > 0, \tag{2.14}$$

$$\begin{aligned}
&\left|\int_0^t s^{\alpha-1}E'_\alpha(A_{\tilde\lambda}s^\alpha)\left(\lambda|v|^{\gamma-1}v+\tilde\lambda v\right)(t-s)ds\right.\\
&\left.-\int_0^{t_0} s^{\alpha-1}E'_\alpha(A_{\tilde\lambda}s^\alpha)\left(\lambda|v|^{\gamma-1}v+\tilde\lambda v\right)(t_0-s)ds\right|\\
&\lesssim|\lambda|\int_0^{t_0} s^{\alpha-1}\left||v|^{\gamma-1}v(t-s)-|v|^{\gamma-1}v(t_0-s)\right|ds\\
&+\tilde\lambda\int_0^{t_0} s^{\alpha-1}\left|v(t-s)-v(t_0-s)\right|ds+\int_{t_0}^t s^{\alpha-1}\left(|\lambda||v|^{\gamma-1}|v|+\tilde\lambda|v|\right)(t-s)ds\\
&\lesssim|\lambda|\int_0^{t_0} s^{\alpha-1}\left|v(t-s)-v(t_0-s)\right|\left(|v|^{\gamma-1}(t-s)+|v|^{\gamma-1}(t_0-s)\right)ds\\
&+\tilde\lambda\int_0^{t_0} s^{\alpha-1}\left|v(t-s)-v(t_0-s)\right|ds+\int_{t_0}^t s^{\alpha-1}\left(|\lambda||v|^{\gamma-1}|v|+\tilde\lambda|v|\right)(t-s)ds.
\end{aligned}$$

This shows that

$$\begin{aligned}
&\left\|\int_0^t s^{\alpha-1}E'_\alpha(A_{\tilde\lambda}s^\alpha)\left(\lambda|v|^{\gamma-1}v+\tilde\lambda v\right)(t-s)ds\right.\\
&\left.-\int_0^{t_0} s^{\alpha-1}E'_\alpha(A_{\tilde\lambda}s^\alpha)\left(\lambda|v|^{\gamma-1}v+\tilde\lambda v\right)(t_0-s)ds\right\|_\infty\\
&\lesssim|\lambda|\int_0^{t_0} s^{\alpha-1}\left\|v(t-s)-v(t_0-s)\right\|_\infty\left(\|v\|_{Y_T}^{\gamma-1}+\|v\|_{Y_T}^{\gamma-1}\right)ds\\
&+\tilde\lambda\int_0^{t_0} s^{\alpha-1}\left\|v(t-s)-v(t_0-s)\right\|_\infty ds\\
&+\left(|\lambda|\|v\|_{Y_T}^\gamma+\tilde\lambda\|v\|_{Y_T}\right)\int_{t_0}^t s^{\alpha-1}ds\to 0
\end{aligned}$$

whenever $t\to t_0$.

The continuity with respect to the norm $\|\cdot\|_2$ is proved in a similar way.

For all $v,w\in Y_T$ we also find that

$$\begin{aligned}
&\|\mathcal{M}(v)(t)-\mathcal{M}(w)(t)\|_\infty\\
&\lesssim|\lambda|\int_0^t(t-s)^{\alpha-1}\|v(s)-w(s)\|_\infty\left(\|v\|_{Y_T}^{\gamma-1}+\|w\|_{Y_T}^{\gamma-1}\right)ds\\
&\quad+\tilde\lambda\int_0^t(t-s)^{\alpha-1}\|v(s)-w(s)\|_\infty ds\\
&\lesssim\|v-w\|_{Y_T}T^\alpha\left[|\lambda|\left(\|v\|_{Y_T}^{\gamma-1}+\|w\|_{Y_T}^{\gamma-1}\right)+\tilde\lambda\right]
\end{aligned}$$

and the estimate

$$\|\mathcal{M}(v)(t)-\mathcal{M}(w)(t)\|_\infty\lesssim T^\alpha\max(|\lambda|,\tilde\lambda)\|v-w\|_{Y_T}\left(\|v\|_{Y_T}^{\gamma-1}+\|w\|_{Y_T}^{\gamma-1}+1\right)$$

holds. Similarly, we obtain

$$\|\mathcal{M}(v)(t)-\mathcal{M}(w)(t)\|_2\lesssim T^\alpha\max(|\lambda|,\tilde\lambda)\|v-w\|_{Y_T}\left(\|v\|_{Y_T}^{\gamma-1}+\|w\|_{Y_T}^{\gamma-1}+1\right)$$

which yields

$$\|\mathcal{M}(v)-\mathcal{M}(w)\|_{Y_T} \leq CT^{\alpha}\max(|\lambda|,\tilde{\lambda})\,\|v-w\|_{Y_T}\left(\|v\|_{Y_T}^{\gamma-1}+\|w\|_{Y_T}^{\gamma-1}+1\right). \quad (2.15)$$

Let $T^* \in (0,T)$ and $R = 2\|u_0\|_{L_2(\mathbb{R}^d)\cap L_\infty(\mathbb{R}^d)}$. We define the closed ball on Y_{T^*} by

$$B_{Y_{T^*}} = \{v \in Y_{T^*} : \|v\|_{Y_{T^*}} \leq R\}.$$

Using estimates (2.13) and (2.15), with $w=0$, it follows that

$$\|\mathcal{M}(v)\|_{Y_{T^*}} \leq \|u_0\|_{L_2(\mathbb{R}^d)\cap L_\infty(\mathbb{R}^d)} + C(T^*)^{\alpha}\max(|\lambda|,\tilde{\lambda})\,\|v\|_{Y_{T^*}}\left(\|v\|_{Y_{T^*}}^{\gamma-1}+1\right).$$

For all $v \in B_{Y_{T^*}}$ we see that

$$\|\mathcal{M}(v)\|_{Y_{T^*}} \leq \frac{R}{2} + C(T^*)^{\alpha}\max(|\lambda|,\tilde{\lambda})R\left(R^{\gamma-1}+1\right)$$

and therefore

$$\|\mathcal{M}(v)\|_{Y_{T^*}} \leq R$$

for sufficiently small T^* such that

$$C(T^*)^{\alpha}\max(|\lambda|,\tilde{\lambda})R\left(R^{\gamma-1}+1\right) \leq \frac{R}{2}.$$

From (2.15) we can also derive that a sufficiently small T^* yields

$$C(T^*)^{\alpha}\max(|\lambda|,\tilde{\lambda})\left(2R^{\gamma-1}+1\right) < 1$$

for all $v,w \in B_{Y_{T^*}}$. This shows that $\mathcal{M}$ is a contraction as an operator $B_{Y_{T^*}} \to B_{Y_{T^*}}$ if T^* is chosen small enough. Now, the existence of the unique fixed point $\tilde{u} \in B_{Y_{T^*}}$ follows from the Banach contraction principle.

The uniqueness of $\tilde{u}$ in Y_{T^*} is proved as follows. We suppose that there exists another fixed point v of $\mathcal{M}$ in the Banach space Y_T. The previous analysis leads to

$$\|\mathcal{M}(v)(t)-\mathcal{M}(\tilde{u})(t)\|_\infty \lesssim \left(|\lambda|\left(\|v\|_{Y_T}^{\gamma-1}+\|\tilde{u}\|_{Y_T}^{\gamma-1}\right)+\tilde{\lambda}\right)\int_0^t (t-s)^{\alpha-1}\|v(s)-\tilde{u}(s)\|_\infty\,ds$$

which is equivalent to

$$\|v(t)-\tilde{u}(t)\|_\infty \lesssim \left(|\lambda|\left(\|v\|_{Y_T}^{\gamma-1}+\|\tilde{u}\|_{Y_T}^{\gamma-1}\right)+\tilde{\lambda}\right)\int_0^t (t-s)^{\alpha-1}\|v(s)-\tilde{u}(s)\|_\infty\,ds$$

by definition of fixed point. The Gronwall's inequality given in Lemma 1.2.1, with $\vartheta = 0$, shows that $v(t) = \tilde{u}(t)$ for all $t \in [0,T^*]$. □

Next, our aim is to prove that $\tilde{u}$ satisfies (2.1) in $[0,T^*]$. For this purpose, we already know that the operator $(A_{\tilde{\lambda}}, H^{\psi,2}(\mathbb{R}^d))$ has the following properties: it is L_2-dissipative, it is closed, its domain is a dense subspace of $L_2(\mathbb{R}^d)$ and it generates an L_2-sub-Markovian semigroup (Theorems 2.3.3, 2.3.4 and 2.3.7). Moreover, $(0,\infty) \subset \rho(A_{\tilde{\lambda}})$ ([29, Lemma 4.1.18]), hence $(-\infty,0) \subset \rho(-A_{\tilde{\lambda}})$.

Let $v \in H^{\psi,2}(\mathbb{R}^d)$ and denote $X = L_2(\mathbb{R}^d)$. The dissipativity of $A_{\tilde{\lambda}}$ and the fact that any $\tau > 0$ belongs to $\rho(A_{\tilde{\lambda}})$, yield

$$\begin{aligned}\|\tau(\tau - A_{\tilde{\lambda}})^{-1}(\tau - A_{\tilde{\lambda}})v\|_X &= \tau\|v\|_X \\ &\leq \|(\tau - A_{\tilde{\lambda}})v\|_X\end{aligned}$$

which implies that

$$\|\tau(\tau - A_{\tilde{\lambda}})^{-1}\|_{\mathbf{B}(X)} \leq 1$$

for all $\tau > 0$.

We also recall that in the proof of Theorem 2.3.4 we found that

$$-A_{\tilde{\lambda}} : H^{\psi,2}(\mathbb{R}^d) \to X$$

is bijective. That is, $-A_{\tilde{\lambda}}$ is a *sectorial* operator in the sense of [57, Section 8.1].

In addition, the operator $-A_{\tilde{\lambda}}$ belongs to the *class* $\mathcal{BIP}(X)$ and $\theta_{-A_{\tilde{\lambda}}} = \frac{\pi}{2}$ ([57, Definition 8.1 and Section 8.7 c)(ii)]). Hence, it satisfies [57, Theorem 8.7 part (i)] with $\omega_{-A_{\tilde{\lambda}}} = 0$.

On the other hand, the Laplace-transform of $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$, is $\hat{g_\alpha}(s) = s^{-\alpha}$, $\mathrm{Re}(s) > 0$ ([57, Example 2.1]). This yields

$$\lim_{s\to\infty} |\hat{g_\alpha}(s)| < \infty.$$

The kernel g_α is also 1-regular ([57, Definition 3.4 and Proposition 3.3]) and θ_a-sectorial with $\theta_a = \alpha\frac{\pi}{2}$ ([57, Definition 3.2]). Therefore, $\theta_a + \theta_{-A} < \pi$ because $0 < \alpha < 1$. This shows that g_α satisfies [57, Theorem 8.7 parts (ii), (iv) and (v)], with $\omega_a = 0$.

We recall that $X_{A_{\tilde{\lambda}}}$ denotes the space $H^{\psi,2}(\mathbb{R}^d)$ equipped with the graph norm $\|\cdot\|_{A_{\tilde{\lambda}}} = \|\cdot\|_X + \|A_{\tilde{\lambda}}(\cdot)\|_X$. In fact, norms $\|\cdot\|_{H^{\psi,2}}$ and $\|\cdot\|_{A_{\tilde{\lambda}}}$ are equivalent. Let $u_0 \in X_{A_{\tilde{\lambda}}}$.

Taken together, since X belongs to the *class* $\mathcal{HT}$ (see [57, definition in page 216, a characterization in page 217 and page 234]), above arguments allow to consider the Volterra equations

$$u(t) - \int_0^t g_\alpha(t-s)A_{\tilde{\lambda}}u(s)ds = (g_\alpha * g)(t), \quad t \in [0, T^*],$$

and

$$u(t) - \int_0^t g_\alpha(t-s)A_{\tilde{\lambda}}u(s)ds = u_0, \quad t \in [0, T^*],$$

where $g(t)(x) := \lambda|\tilde{u}|^{\gamma-1}\tilde{u}(t,x) + \tilde{\lambda}\tilde{u}(t,x)$.

Since $\tilde{u} \in C([0,T^*]; X \cap L_\infty(\mathbb{R}^d))$, it follows that $g \in L_2([0,T^*];X)$ and that $g_\alpha * g \in L_2([0,T^*];X)$. Consequently, [57, Theorem 8.7 parts (a) and (b)] imply that these equations have a strong solution u_1 and u_2, respectively. Therefore,

$$u(t) - \int_0^t g_\alpha(t-s)A_{\tilde{\lambda}}u(s)ds = u_0 + (g_\alpha * g)(t), \quad t \in [0, T^*], \tag{2.16}$$

has a unique a.e. strong solution $u := u_1 + u_2$ belonging to $L_2([0,T^*]; X_{A_{\tilde{\lambda}}})$.

However, the equation (2.16) is equivalent to the Cauchy problem

$$\begin{aligned} \partial_t^\alpha (u-u_0)(t) - A_{\tilde{\lambda}} u(t) &= g(t), \quad 0 < t \le T^*, \\ u(t)|_{t=0} &= u_0, \end{aligned} \tag{2.17}$$

where we have put $u(t) = u(t,\cdot)$ for simplicity.

Besides, the equation in (2.17) can be written in the form

$$u(t) = u_0 + J^\alpha \left(A_{\tilde{\lambda}} u(t) + g(t) \right), \quad 0 < t \le T^*,$$

and $A_{\tilde{\lambda}}$ has Yosida approximation ([29, Theorem 4.1.29]) given by

$$A_\tau := \tau A_{\tilde{\lambda}} (\tau - A_{\tilde{\lambda}})^{-1} = \tau A_{\tilde{\lambda}} R_\tau, \quad \tau > 0,$$

with the following properties.

(P_1) : A_τ is bounded on $L_2(\mathbb{R}^d)$ and the semigroup $(e^{tA_\tau})_{t\ge 0}$ is a strongly continuous contraction semigroup, for all $\tau > 0$.

(P_2) : $\lim_{\tau\to\infty} \|A_\tau f - A_{\tilde{\lambda}} f\|_2 = 0$ for all $f \in D(A_{\tilde{\lambda}})$.

(P_3) : $T_t f := e^{tA_{\tilde{\lambda}}} f := \lim_{\tau\to\infty} e^{tA_\tau} f$ for all $f \in L_2(\mathbb{R}^d), t \ge 0$.

Consequently,

$$\begin{aligned} u(t) &= u_0 + J^\alpha A_\tau u(t) + J^\alpha (A_{\tilde{\lambda}} - A_\tau) u(t) + J^\alpha g(t) \\ &= u_0 + A_\tau J^\alpha u(t) + J^\alpha (A_{\tilde{\lambda}} - A_\tau) u(t) + J^\alpha g(t). \end{aligned}$$

By replacing u in the second term of the r.h.s. of the last equality and using the semigroup property of the fractional integral J^α, we find that

$$\begin{aligned} A_\tau J^\alpha u(t) &= A_\tau J^\alpha u_0(t) + A_\tau J^{2\alpha} A_\tau u(t) + A_\tau J^{2\alpha} (A_{\tilde{\lambda}} - A_\tau) u(t) + A_\tau J^{2\alpha} g(t) \\ &= A_\tau J^\alpha u_0(t) + A_\tau^2 J^{2\alpha} u(t) + A_\tau J^{2\alpha} (A_{\tilde{\lambda}} - A_\tau) u(t) + A_\tau J^{2\alpha} g(t). \end{aligned}$$

Repeating this procedure recursively k-times, we obtain

$$u(t) = \sum_{m=0}^{k} (A_\tau J^\alpha)^m \left(u_0 + J^\alpha g \right)(t) + \sum_{m=0}^{k} (A_\tau J^\alpha)^m J^\alpha (A_{\tilde{\lambda}} - A_\tau) u(t) + (A_\tau J^\alpha)^{k+1} u(t),$$

where the last term has the form

$$\begin{aligned} (A_\tau J^\alpha)^{k+1} u(t) &= A_\tau^{k+1} J^{\alpha k + \alpha} u(t) \\ &= A_\tau^{k+1} \int_0^t \frac{(t-s)^{\alpha k + \alpha - 1}}{\Gamma(\alpha k + \alpha)} u(s) ds. \end{aligned}$$

The fact that $u \in L_2([0,T^*]; X_{A_{\tilde{\lambda}}})$ and Hölder's inequality imply that

$$\begin{aligned} \|(A_\tau J^\alpha)^{k+1} u(t)\|_2 &\le \|A_\tau\|_{\mathbf{B}(X)}^{k+1} \int_0^t \frac{(t-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k + \alpha)} \|u(s)\|_2 ds \\ &\le \|A_\tau\|_{\mathbf{B}(X)}^{k+1} \int_0^t \frac{(t-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k + \alpha)} \|u(s)\|_{A_{\tilde{\lambda}}} ds \\ &\le \|A_\tau\|_{\mathbf{B}(X)}^{k+1} \left(\int_0^t \frac{(t-s)^{2(\alpha k+\alpha-1)}}{\Gamma^2(\alpha k + \alpha)} ds \right)^{\frac{1}{2}} \left(\int_0^{T^*} \|u(s)\|_{A_{\tilde{\lambda}}}^2 ds \right)^{\frac{1}{2}} \\ &= \|A_\tau\|_{\mathbf{B}(X)}^{k+1} \|u\|_{L_2([0,T^*];X_{A_{\tilde{\lambda}}})} \frac{t^{2\alpha k + 2\alpha - 1}}{\Gamma(\alpha k + \alpha)(2\alpha k + 2\alpha - 1)} \end{aligned}$$

for some k large enough. Therefore,

$$\lim_{k\to\infty}\|(A_\tau J^\alpha)^{k+1}u(t)\|_2 = 0$$

and

$$u(t) = \sum_{m=0}^{\infty}(A_\tau J^\alpha)^m\left(u_0 + J^\alpha g\right)(t) + \sum_{m=0}^{\infty}(A_\tau J^\alpha)^m J^\alpha (A_{\tilde\lambda} - A_\tau)u(t)$$

is well defined for all $0 < t \leq T^*$.

Definitions of J^α and Mittag-Leffler functions lead to

$$\begin{aligned} u(t) &= E_\alpha(t^\alpha A_\tau)u_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A_\tau)g(s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A_\tau)(A_{\tilde\lambda} - A_\tau)u(s)ds \\ &= E_\alpha(t^\alpha A_\tau)u_0 + \alpha\int_0^t (t-s)^{\alpha-1}E'_\alpha((t-s)^\alpha A_\tau)g(s)ds \\ &\quad + \alpha\int_0^t (t-s)^{\alpha-1}E'_\alpha((t-s)^\alpha A_\tau)(A_{\tilde\lambda} - A_\tau)u(s)ds. \end{aligned}$$

Again, by [40, Formulas (8.5) and (8.6)] one gets the representation

$$\begin{aligned} u(t) =& \frac{1}{\alpha}\int_0^\infty e^{t^\alpha A_\tau r}\ u_0 r^{-1-\frac{1}{\alpha}}G_\alpha(1, r^{-\frac{1}{\alpha}})dr \\ &+ \int_0^t (t-s)^{\alpha-1}\int_0^\infty e^{(t-s)^\alpha A_\tau r}\ g(s)r^{-\frac{1}{\alpha}}G_\alpha(1, r^{-\frac{1}{\alpha}})drds \\ &+ \int_0^t (t-s)^{\alpha-1}\int_0^\infty e^{(t-s)^\alpha A_\tau r}(A_{\tilde\lambda} - A_\tau)u(s)r^{-\frac{1}{\alpha}}G_\alpha(1, r^{-\frac{1}{\alpha}})drds. \end{aligned}$$

On the other hand, from [52, Section 1.3 Lemma 3.2] it follows that

$$\|A_\tau u(t)\|_2 \leq \|A_{\tilde\lambda}u(t)\|_2, \quad \tau > 0, \quad 0 \leq t \leq T^*,$$

and property (P_1) implies that the last integral is dominated (in $\|\cdot\|_2$) by

$$\int_0^t (t-s)^{\alpha-1}\|u(s)\|_{A_{\tilde\lambda}}ds.$$

Here, we use that $u \in L_2([0, T^*]; X_{A_{\tilde\lambda}})$ and Hölder's inequality as above, together with properties (P_2)-(P_3) and the dominated convergence. Therefore,

$$\begin{aligned} u(t) =& \frac{1}{\alpha}\int_0^\infty T_{t^\alpha r}\ u_0 r^{-1-\frac{1}{\alpha}}G_\alpha(1, r^{-\frac{1}{\alpha}})dr \\ &+ \int_0^t (t-s)^{\alpha-1}\int_0^\infty T_{(t-s)^\alpha r}\ g(s)r^{-\frac{1}{\alpha}}G_\alpha(1, r^{-\frac{1}{\alpha}})drds \end{aligned}$$

whenever $\tau \to \infty$ and $\frac{1}{2} < \alpha < 1$.

This and the definition of g, show that the strong solution u has the form of the r.h.s. of the operator (2.12) and thus $u = \tilde{u}$.

Now, for the case $0 < \alpha \leq \frac{1}{2}$ we may repeat previous arguments, in particular [57, Theorem 8.7] using the space $L_{\bar{p}}([0, T^*]; X_{A_{\tilde{\lambda}}})$ with some $\frac{1}{\alpha} < \bar{p} < \infty$. In this situation, we note that $\bar{p} > 2$ and again $u = \tilde{u}$.

Finally, we note that the problem (2.17) is equivalent to the original (2.1).

We are now ready to state the main result of this section.

Theorem 2.4.2. *Let $\alpha \in (0,1)$, $\beta \in (0,2)$, $\lambda \in \mathbb{R}$ and $\gamma > 1$. Assume that $(\mathcal{C}_1)$-$(\mathcal{C}_3)$ are satisfied. Suppose $\frac{1}{\alpha} < \bar{p} < \infty$. If $u_0 \in H^{\psi,2}(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$, then there exists $T > 0$ such that the Cauchy problem (2.1) has a solution $u \in C([0,T]; L_2(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d))$. Moreover, u is strong in the sense that $u \in L_2([0,T]; X_{A_{\tilde{\lambda}}})$ whenever $\frac{1}{2} < \alpha < 1$ and $u \in L_{\bar{p}}([0,T]; X_{A_{\tilde{\lambda}}})$ whenever $0 < \alpha \leq \frac{1}{2}$.*

Chapter 3

The semi-linear problem with constant coefficients

This chapter deals with the solvability of the semi-linear Cauchy problem

$$\begin{aligned} \partial_t^\alpha (u-u_0)(t,x) + \Psi_\beta(-i\nabla)u(t,x) &= \lambda |u(t,x)|^{\gamma-1}u(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ u(t,x)|_{t=0} &= u_0(x), \quad x\in\mathbb{R}^d, \end{aligned} \tag{3.1}$$

where the symbol of the operator $\Psi_\beta(-i\nabla)$ is independent of x, that is,

$$\psi(\xi) = \|\xi\|^\beta \omega_\mu\left(\frac{\xi}{\|\xi\|}\right), \quad \xi\in\mathbb{R}^d,$$

and

$$\omega_\mu(\theta) := \int_{S^{d-1}} |\theta\cdot\eta|^\beta \mu(d\eta), \quad \theta\in S^{d-1}. \tag{3.2}$$

Again, $\gamma > 1$ and $\lambda \in \mathbb{R}$ are parameters of the non-linear term. As it was set in Chapter 2 for the dependent case of x, $\mu(d\eta)$ is a centrally symmetric finite (non-negative) Borel measure defined on S^{d-1}, the so-called spectral measure, and $\mu(d\eta) = \varrho(\eta)d\eta$ with the density ϱ. Some restrictions on the function ϱ may be required for the lower bound and behaviour of the fundamental solutions; see e.g., [38, Section 5.2]. More precisely, our basic hypothesis throughout this chapter is the following:

($\mathcal{H}_1$) The spectral measure μ has a strictly positive density, such that the function ω_μ is strictly positive and $(d+1+[\beta])$-times continuously differentiable on S^{d-1}.

We denote by ($\mathcal{H}_2$) to refer to ($\mathcal{H}_1$) whenever we need to assume that ω_μ is $(d+2+[\beta])$-times continuously differentiable on S^{d-1}. The considerations just made above have been taken from [40, Proposition 4.5.1] and [40, Theorem 4.5.1], for $d=1$ and $d>1$ respectively. We want to point out that the condition of strict positivity on ω_μ in $\mathcal{H}_1$, guarantees that the support of the measure μ on S^{d-1} is not contained in any hyperplane of $\mathbb{R}^d$ ([40, Section 4.5]).

Evolutionary problems like (3.1) can be considered as a generalization of the classical *rigid ignition model*. The behaviour of the combustion processes, involving non-linear source terms, has become a challenging field for mathematical analysis in the last decades (see, e.g. [10, Chapter 3],[23]). In the case $\beta = 2$ and $\omega_\mu \equiv 1$ we see that the operator, namely $\Psi_2(-i\nabla)$, becomes the negative Laplacian $(-\Delta)$ with symbol $\psi(\xi) = \|\xi\|^2$. The corresponding fundamental solution with $\alpha \in (0,1)$ has been studied (see, e.g. [17, Chapter 5]) and bounds can be found in [18]. These bounds are used in [33] in order to study the fundamental solution Z for the subdiffusion problem

$$\partial_t^\alpha (u - u_0) - \Delta u = 0, \quad t > 0,\ x \in \mathbb{R}^d,$$
$$u|_{t=0} = u_0, \quad x \in \mathbb{R}^d.$$

In this context, an important concept associated with α is the *mean squared displacement* (*MSD*) or the *centred second moment*, which describes how fast is the dispersion of the particles in a random process. In [33, Lemma 2.1], the authors proved that the MSD governed by the preceding equation specifically turns out to be $\frac{2d}{\Gamma(1+\alpha)} t^\alpha$, $t > 0$, $0 < \alpha < 1$. In the literature one traditionally finds that anomalous diffusion refers to this power-law. See, e.g. [64], [45], [43], [3] and references therein. However, in our case, the Cauchy problem (3.1) does not possess a finite MSD. This can be directly checked by using the definition of MSD ([33, expression (6)]) and similar arguments as in the proof of [33, Lemma 2.1] or Theorem 3.1.1 below. Models with infinite MSD involving equation (3.1) could be an open problem for future research, because we know from the existing literature that in some cases of *Lévy flights* there is also a divergent MSD and its physical meaning is still not very clear (see, e.g. [49] and [65]).

3.1 Fundamental solutions

Whenever we study evolution equations it is natural to ask about the fundamental solutions associated with the evolutionary problem. For equations like (3.1) we know that if an operator A generates a strongly continuous semigroup T_t in a Banach space, under suitable conditions, the solution of the Cauchy problem

$$\partial_t^\alpha (u - u_0)(t,x) = Au(t,x) + g(t,x), \quad t > 0,\ x \in \mathbb{R}^d,$$
$$u(t,x)|_{t=0} = u_0(x), \quad x \in \mathbb{R}^d,$$

has the representation

$$u(t) = E_\alpha(At^\alpha)u_0 + \alpha \int_0^t (t-s)^{\alpha-1} E'_\alpha(A(t-s)^\alpha) g(s)\,ds$$

in terms of the Mittagf-Leffler functions E_α and E'_α, respectively for $\alpha \in (0,1)$ (see e.g.,[40, Theorem 8.2.1]). This representation is also called the *mild solution* of the evolution equation. Considering $A = -\Psi_\beta(-i\nabla)$, the term corresponding to the operator $E_\alpha(-\Psi_\beta(-i\nabla)t^\alpha)$ has been studied extensively in [32, Section 2], where the authors found that one of the fundamental solutions is

$$Z(t,x) := \frac{1}{\alpha} \int_0^\infty G(t^\alpha s, x) s^{-1-\frac{1}{\alpha}} G_\alpha(1, s^{-\frac{1}{\alpha}})\,ds. \tag{3.3}$$

Here, G stands for the Green function that solves the equation

$$\partial_t v(t,x) + \Psi_\beta(-i\nabla)v(t,x) = 0, \quad t>0,\ x\in\mathbb{R}^d,$$

with the initial condition

$$G(t,x)|_{t=0} = \delta_0(x), \quad x\in\mathbb{R}^d,$$

δ_0 being the Dirac delta distribution. In this case, $G_\alpha(\cdot,\cdot)$ is the Green function that solves the problem

$$\partial_t v(t,s) + \frac{d^\alpha}{ds^\alpha}v(t,s) = 0, \quad t>0,\ s\in\mathbb{R},\ G_\alpha(0,s) = \delta(s),$$

and

$$\frac{d^\alpha}{ds^\alpha}f(s) := \frac{1}{\Gamma(-\alpha)}\int_0^\infty \frac{f(s-\tau)-f(s)}{\tau^{1+\alpha}}d\tau,$$

see [40, Formulas (1.111) and (2.74)]. Besides, in [32, Theorem 2] the authors established that the fundamental solution Z admits the following bounds. In what follows, we employ the notation $\Omega = \|x\|^\beta t^{-\alpha}$ for $x\in\mathbb{R}^d$ and $t>0$.

Proposition 3.1.1. *Let $\alpha\in(0,1)$ and $\beta\in(0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Then there exists a positive constant C such that for $(t,x)\in(0,\infty)\times\mathbb{R}^d$ the following two-sided estimates for Z hold. For $\Omega\le 1$,*

$$Z(t,x) \asymp Ct^{-\frac{\alpha d}{\beta}} \qquad \text{if} \quad d<\beta, \tag{3.4}$$

$$Z(t,x) \asymp Ct^{-\alpha}(|\log(\Omega)|+1) \qquad \text{if} \quad d=\beta, \tag{3.5}$$

$$Z(t,x) \asymp Ct^{-\frac{\alpha d}{\beta}}\Omega^{1-\frac{d}{\beta}} \qquad \text{if} \quad d>\beta. \tag{3.6}$$

For $\Omega\ge 1$,

$$Z(t,x) \asymp Ct^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}. \tag{3.7}$$

In the same way we have derived a second fundamental solution Y, as follows. From [40, Formula (8.8)] we know that

$$E'_\alpha(At^\alpha) = \frac{t^{1-\alpha}}{\alpha}\int_0^\infty e^{As}G_\alpha(s,t)ds.$$

By [40, Formula (2.77)] we also have

$$G_\alpha(s,t) = s^{-\frac{1}{\alpha}}G_\alpha\left(1, s^{-\frac{1}{\alpha}}t\right),$$

which produces

$$E'_\alpha(At^\alpha) = \frac{t^{1-\alpha}}{\alpha}\int_0^\infty e^{As}s^{-\frac{1}{\alpha}}G_\alpha(1,s^{-\frac{1}{\alpha}}t)ds.$$

Together with the transition density G, we see that

$$\begin{aligned}
&\alpha \int_0^t (t-s)^{\alpha-1} E'_\alpha(A(t-s)^\alpha) g(s) ds \\
&= \int_0^t \int_0^\infty e^{A\tau} g(s) \tau^{-\frac{1}{\alpha}} G_\alpha(1, \tau^{-\frac{1}{\alpha}}(t-s)) d\tau ds \\
&= \int_0^t \int_0^\infty T_\tau g(s) \tau^{-\frac{1}{\alpha}} G_\alpha(1, \tau^{-\frac{1}{\alpha}}(t-s)) d\tau ds \\
&= \int_0^t \int_0^\infty \left[\int_{\mathbb{R}^d} G(\tau, \cdot - z) g(s)(z) dz \right] \tau^{-\frac{1}{\alpha}} G_\alpha(1, \tau^{-\frac{1}{\alpha}}(t-s)) d\tau ds \\
&= \int_0^t \int_{\mathbb{R}^d} \left[\int_0^\infty G(\tau, \cdot - z) \tau^{-\frac{1}{\alpha}} G_\alpha(1, \tau^{-\frac{1}{\alpha}}(t-s)) d\tau \right] g(s)(z) dz ds,
\end{aligned}$$

i.e., we have the convolution of $g(s)$ with a function given in the square brackets. This allows us to define

$$Y(t-s, x) := \int_0^\infty G(\tau, x) \tau^{-\frac{1}{\alpha}} G_\alpha(1, \tau^{-\frac{1}{\alpha}}(t-s)) d\tau$$

and changing the integration variable to $\tau = (t-s)^\alpha r$ we obtain that

$$Y(t, x) := \int_0^\infty t^{\alpha-1} G(t^\alpha s, x) s^{-\frac{1}{\alpha}} G_\alpha(1, s^{-\frac{1}{\alpha}}) ds. \tag{3.8}$$

Therefore, we find the following bounds for the fundamental solution Y.

Proposition 3.1.2. *Under the same assumptions as Proposition 3.1.1, the following two-sided estimates for Y hold. For $\Omega \leq 1$,*

$$Y(t, x) \asymp C t^{-\frac{\alpha d}{\beta} + \alpha - 1} \quad \text{if} \quad d < 2\beta, \tag{3.9}$$
$$Y(t, x) \asymp C t^{-\alpha-1}(|\log(\Omega)| + 1) \quad \text{if} \quad d = 2\beta, \tag{3.10}$$
$$Y(t, x) \asymp C t^{-\frac{\alpha d}{\beta} + \alpha - 1} \Omega^{2-\frac{d}{\beta}} \quad \text{if} \quad d > 2\beta. \tag{3.11}$$

For $\Omega \geq 1$,

$$Y(t, x) \asymp C t^{-\frac{\alpha d}{\beta} + \alpha - 1} \Omega^{-1-\frac{d}{\beta}}. \tag{3.12}$$

Proof. The assertions follow from straightforward computations made in the proof of the estimates for Z, in [32, Theorem 2]. There, the authors used the fact that the asymptotic behaviour of G_α is the same as for the density w_α given in [32, Proposition 1] (with the skewness of the distribution that equals to 0) by

$$w_\alpha(\tau) \sim C \begin{cases} \tau^{-1-\alpha} & \text{as} \quad \tau \to \infty, \\ f_\alpha(\tau) := \tau^{-\frac{2-\alpha}{2(1-\alpha)}} e^{-c_\alpha \tau^{-\frac{\alpha}{1-\alpha}}} & \text{as} \quad \tau \to 0, \end{cases}$$

where $c_\alpha = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}$. See e.g., [40, Proposition 2.4.1] and [72, Theorem 2.5.2] for more details.

Keeping this in mind, we see that a difference between the functions Z and Y, given by (3.3) and (3.8) respectively, is the factor s^{-1} inside the improper Riemann integral of Z. Thus, we only need to check the corresponding two-sided estimates for

$$\int_0^\infty G(t^\alpha s, x) s^{-\frac{1}{\alpha}} G_\alpha(1, s^{-\frac{1}{\alpha}}) ds,$$

which can be written equivalently as

$$\int_0^\infty G(t^\alpha s, x) s^{-1-\frac{1}{\alpha}} G_\alpha(1, s^{-\frac{1}{\alpha}}) s\, ds \asymp I_1 + I_2.$$

Here, similar to the integrals that are used in [32, expression (33)],

$$I_1 := \int_0^1 \min\left(t^{-\frac{\alpha d}{\beta}} \Omega^{-1-\frac{d}{\beta}} s, t^{-\frac{\alpha d}{\beta}} s^{-\frac{d}{\beta}}\right) s\, ds$$

and

$$I_2 := \int_1^\infty \min\left(t^{-\frac{\alpha d}{\beta}} \Omega^{-1-\frac{d}{\beta}} s, t^{-\frac{\alpha d}{\beta}} s^{-\frac{d}{\beta}}\right) s^{-1-\frac{1}{\alpha}} f_\alpha(s^{-\frac{1}{\alpha}}) s\, ds,$$

where

$$\min\left(t^{-\frac{\alpha d}{\beta}} \Omega^{-1-\frac{d}{\beta}} s, t^{-\frac{\alpha d}{\beta}} s^{-\frac{d}{\beta}}\right) = \begin{cases} t^{-\frac{\alpha d}{\beta}} \Omega^{-1-\frac{d}{\beta}} s, & \text{for } s < \Omega, \\ t^{-\frac{\alpha d}{\beta}} s^{-\frac{d}{\beta}}, & \text{for } s \geq \Omega, \end{cases}$$

as in [32, expression (32)]. Next, we need to analyse the two-sided estimates for I_j, $j = 1, 2$. The case $\Omega \leq 1$ yields

$$\begin{aligned} I_1 &= t^{-\frac{\alpha d}{\beta}} \Omega^{-1-\frac{d}{\beta}} \int_0^\Omega s^2 ds + t^{-\frac{\alpha d}{\beta}} \int_\Omega^1 s^{-\frac{d}{\beta}} s\, ds \\ &= \frac{1}{3} t^{-\frac{\alpha d}{\beta}} \Omega^{2-\frac{d}{\beta}} + t^{-\frac{\alpha d}{\beta}} \int_\Omega^1 s^{1-\frac{d}{\beta}} ds. \end{aligned}$$

The last integral requires the sub-cases $d < 2\beta$, $d = 2\beta$ and $d > 2\beta$:

$$t^{-\frac{\alpha d}{\beta}} \int_\Omega^1 s^{1-\frac{d}{\beta}} ds = \begin{cases} t^{-\frac{\alpha d}{\beta}} \frac{1}{2-\frac{d}{\beta}} \left(1 - \Omega^{2-\frac{d}{\beta}}\right), & \text{for } d < 2\beta, \\ t^{-2\alpha} |log(\Omega)|, & \text{for } d = 2\beta, \\ t^{-\frac{\alpha d}{\beta}} \frac{1}{\frac{d}{\beta}-2} \left(\Omega^{2-\frac{d}{\beta}} - 1\right), & \text{for } d > 2\beta. \end{cases}$$

For I_2, since $\Omega \leq 1$, we have that

$$\begin{aligned} I_2 &= t^{-\frac{\alpha d}{\beta}} \int_1^\infty s^{-\frac{d}{\beta}} s^{-1-\frac{1}{\alpha}} f_\alpha(s^{-\frac{1}{\alpha}}) s\, ds \\ &= t^{-\frac{\alpha d}{\beta}} \int_1^\infty s^{-\frac{d}{\beta}-\frac{1}{\alpha}} s^{\frac{2-\alpha}{2\alpha(1-\alpha)}} e^{-(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} s^{\frac{1}{1-\alpha}}} ds \\ &= t^{-\frac{\alpha d}{\beta}} \int_1^\infty s^{-\frac{d}{\beta}+\frac{1}{2(1-\alpha)}} e^{-(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} s^{\frac{1}{1-\alpha}}} ds \\ &= Ct^{-\frac{\alpha d}{\beta}}. \end{aligned}$$

We point out that the improper integral is convergent due to the Laplace method for integrals (see e.g., [32, (A1)]). Therefore, if $d < 2\beta$ we find that

$$Ct^{-\frac{\alpha d}{\beta}} = I_2 \le I_1 + I_2 = \frac{1}{3}t^{-\frac{\alpha d}{\beta}}\Omega^{2-\frac{d}{\beta}} + t^{-\frac{\alpha d}{\beta}}\frac{1}{2-\frac{d}{\beta}}\left(1-\Omega^{2-\frac{d}{\beta}}\right) + Ct^{-\frac{\alpha d}{\beta}} \lesssim t^{-\frac{\alpha d}{\beta}},$$

if $d = 2\beta$ we obtain

$$I_1 + I_2 = \frac{1}{3}t^{-2\alpha} + t^{-2\alpha}|log(\Omega)| + Ct^{-2\alpha}$$

and $d > 2\beta$ implies that

$$\frac{1}{3}t^{-\frac{\alpha d}{\beta}}\Omega^{2-\frac{d}{\beta}} \le I_1 \le I_1 + I_2 = \frac{1}{3}t^{-\frac{\alpha d}{\beta}}\Omega^{2-\frac{d}{\beta}} + t^{-\frac{\alpha d}{\beta}}\frac{1}{\frac{d}{\beta}-2}\left(\Omega^{2-\frac{d}{\beta}}-1\right) + Ct^{-\frac{\alpha d}{\beta}} \lesssim t^{-\frac{\alpha d}{\beta}}\Omega^{2-\frac{d}{\beta}}.$$

Since the additional factor $t^{\alpha-1}$ is a constant for the integral of Y, the estimates (3.9)-(3.11) hold. Now, for the case $\Omega \ge 1$ we have that

$$\begin{aligned} I_1 &= t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}\int_0^1 s^2 ds \\ &= \frac{1}{3}t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}} \end{aligned}$$

and

$$I_2 = t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}\int_1^\Omega s^{1-\frac{1}{\alpha}}f_\alpha(s^{-\frac{1}{\alpha}})ds + t^{-\frac{\alpha d}{\beta}}\int_\Omega^\infty s^{-\frac{d}{\beta}-\frac{1}{\alpha}}f_\alpha(s^{-\frac{1}{\alpha}})ds.$$

We see that

$$\begin{aligned} I_2 &\le t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}\int_1^\Omega s^{1-\frac{1}{\alpha}}f_\alpha(s^{-\frac{1}{\alpha}})ds + t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}\int_\Omega^\infty s^{1-\frac{1}{\alpha}}f_\alpha(s^{-\frac{1}{\alpha}})ds \\ &= t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}\int_1^\infty s^{1-\frac{1}{\alpha}}f_\alpha(s^{-\frac{1}{\alpha}})ds \\ &= C_1 t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} I_2 &\ge t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}\int_1^\Omega s^{-\frac{d}{\beta}-\frac{1}{\alpha}}f_\alpha(s^{-\frac{1}{\alpha}})ds + t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}\int_\Omega^\infty s^{-\frac{d}{\beta}-\frac{1}{\alpha}}f_\alpha(s^{-\frac{1}{\alpha}})ds \\ &= t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}\int_1^\infty s^{-\frac{d}{\beta}-\frac{1}{\alpha}}f_\alpha(s^{-\frac{1}{\alpha}})ds \\ &= C_2 t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}. \end{aligned}$$

These bounds show that $I_1 + I_2 \asymp t^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}}$. The factor $t^{\alpha-1}$ completes the proof of the estimate (3.12). □

Remark 3.1.1. *We note a singularity at the origin with respect to the spatial variable for Z, whenever $d \geq \beta$, and for Y whenever $d \geq 2\beta$. It is well known that this type of singularities occurs in the equations of fractional evolution in time, even if $\beta = 2$ and $\omega_\mu \equiv 1$.*

As we will see later, the two-sided estimates of fundamental solutions (Z, Y) play a key role for our main results in the present chapter. Now, we continue by showing some properties of the fundamental solutions Z and Y.

Lemma 3.1.1. *Under the same assumptions as Proposition 3.1.1, there exists a positive constant C for all $t_1, t_2 > 0$ and $x \in \mathbb{R}^d$, such that there exists $t_c > 0$, between t_1 and t_2, and the following estimates for Z hold with $\Omega_c = \|x\|^\beta t_c^{-\alpha}$. For $\Omega_c \leq 1$,*

$$|Z(t_1, x) - Z(t_2, x)| \leq C|t_1 - t_2| \begin{cases} t_c^{-\frac{\alpha d}{\beta}-1} & \text{if} \quad d < \beta, \\ t_c^{-\alpha-1}(|\log(\Omega_c)| + 1) & \text{if} \quad d = \beta, \\ t_c^{-\frac{\alpha d}{\beta}-1}\Omega_c^{1-\frac{d}{\beta}} & \text{if} \quad d > \beta, \end{cases}$$

and for $\Omega_c \geq 1$,

$$|Z(t_1, x) - Z(t_2, x)| \leq C|t_1 - t_2| t_c^{-\frac{\alpha d}{\beta}-1}\Omega_c^{-1-\frac{d}{\beta}}.$$

Proof. From (3.3) it follows that

$$Z(t_1, x) - Z(t_2, x) = \frac{1}{\alpha}\int_0^\infty [G(t_1^\alpha s, x) - G(t_2^\alpha s, x)]\, s^{-1-\frac{1}{\alpha}} G_\alpha(1, s^{-\frac{1}{\alpha}})ds.$$

It is known ([40, Theorem 4.5.1]) that G is differentiable with respect to $t > 0$ and satisfies

$$|G(t, x)| \leq C \min\left(t^{-\frac{d}{\beta}}, \frac{t}{\|x\|^{d+\beta}}\right),$$
$$\left|t\frac{\partial G}{\partial t}(t, x)\right| \leq C \min\left(t^{-\frac{d}{\beta}}, \frac{t}{\|x\|^{d+\beta}}\right).$$

In these estimates, C depends on β, d and the bounds for ω_μ. Using this and the mean-value theorem, we have that for some t_c between t_1 and t_2,

$$\begin{aligned}
&|Z(t_1, x) - Z(t_2, x)| \\
&\leq \frac{1}{\alpha}\int_0^\infty \left|\frac{\partial G(t^\alpha s, x)}{\partial t}\right|_{t=t_c} |t_1 - t_2| s^{-1-\frac{1}{\alpha}} G_\alpha(1, s^{-\frac{1}{\alpha}})ds \\
&\leq \frac{|t_1 - t_2|}{t_c}\int_0^\infty |t_c^\alpha s G'(t_c^\alpha s, x)|\, s^{-1-\frac{1}{\alpha}} G_\alpha(1, s^{-\frac{1}{\alpha}})ds \\
&\leq C\frac{|t_1 - t_2|}{t_c}\int_0^\infty \min\left((t_c^\alpha s)^{-\frac{d}{\beta}}, \frac{t_c^\alpha s}{\|x\|^{d+\beta}}\right) s^{-1-\frac{1}{\alpha}} G_\alpha(1, s^{-\frac{1}{\alpha}})ds.
\end{aligned}$$

Therefore, we proceed in the same way from [32, Theorem 2], i.e.,

$$\begin{aligned}&\int_0^\infty \min\left((t_c^\alpha s)^{-\frac{d}{\beta}}, \frac{t_c^\alpha s}{\|x\|^{d+\beta}}\right) s^{-1-\frac{1}{\alpha}} G_\alpha(1, s^{-\frac{1}{\alpha}}) ds \\ &\asymp \int_0^1 \min\left((t_c^\alpha s)^{-\frac{d}{\beta}}, \frac{t_c^\alpha s}{\|x\|^{d+\beta}}\right) s^{-1-\frac{1}{\alpha}} w_\alpha(s^{-\frac{1}{\alpha}}) ds \\ &\quad + \int_1^\infty \min\left((t_c^\alpha s)^{-\frac{d}{\beta}}, \frac{t_c^\alpha s}{\|x\|^{d+\beta}}\right) s^{-1-\frac{1}{\alpha}} w_\alpha(s^{-\frac{1}{\alpha}}) ds.\end{aligned}$$

With the asymptotic behaviour of w_α, the first integral reduces to

$$C \int_0^1 \min\left((t_c^\alpha s)^{-\frac{d}{\beta}}, \frac{t_c^\alpha s}{\|x\|^{d+\beta}}\right) ds$$

and we note that the improper integral $\int_0^1 (t_c^\alpha s)^{-\frac{d}{\beta}} ds$ appears whenever $\Omega_c < s < 1$. Here, we need to check the cases $d = \beta$, $d < \beta$ and $d > \beta$, respectively. Thus, we get the desired bounds. ☐

Lemma 3.1.2. *Under the same assumptions as Proposition 3.1.1, there exists a positive constant C for all $t_1, t_2 > 0$ and $x \in \mathbb{R}^d$, such that there exists $t_c > 0$, between t_1 and t_2, and the following estimates for Y hold with $\Omega_c = \|x\|^\beta t_c^{-\alpha}$. For $\Omega_c \le 1$,*

$$|Y(t_1, x) - Y(t_2, x)| \le C|t_1 - t_2| \begin{cases} t_c^{-\frac{\alpha d}{\beta}+\alpha-2} & \text{if} \quad d < 2\beta, \\ t_c^{-\alpha-2}(|\log(\Omega_c)| + 1) & \text{if} \quad d = 2\beta, \\ t_c^{-\frac{\alpha d}{\beta}+\alpha-2}\Omega_c^{2-\frac{d}{\beta}} & \text{if} \quad d > 2\beta, \end{cases}$$

and for $\Omega_c \ge 1$,

$$|Y(t_1, x) - Y(t_2, x)| \le C|t_1 - t_2| t_c^{-\frac{\alpha d}{\beta}+\alpha-2} \Omega_c^{-1-\frac{d}{\beta}}.$$

Proof. The assertions follow from straightforward computations made in the proof of the previous estimates for Z, but using (3.8). ☐

Lemma 3.1.3. *Let $\alpha \in (0, 1)$ and $\beta \in (0, 2)$. Assume the hypothesis $(\mathcal{H}_2)$ holds. Then there exists a positive constant C for all $t > 0$ and $x_1, x_2 \in \mathbb{R}^d$, such that there exists ζ in the open segment connecting x_1 and x_2, and the following estimates for Z hold with $\Omega_\zeta = \|\zeta\|^\beta t^{-\alpha}$. For $\Omega_\zeta \le 1$,*

$$|Z(t, x_1) - Z(t, x_2)| \le C\|x_1 - x_2\| t^{-\frac{\alpha(d+1)}{\beta}} \Omega_\zeta^{1-\frac{d+1}{\beta}}$$

and for $\Omega_\zeta \ge 1$,

$$|Z(t, x_1) - Z(t, x_2)| \le C\|x_1 - x_2\| t^{-\frac{\alpha(d+1)}{\beta}} \Omega_\zeta^{-1-\frac{d+1}{\beta}}.$$

Proof. From (3.3) it follows that

$$Z(t,x_1)-Z(t,x_2)=\frac{1}{\alpha}\int_0^\infty [G(t^\alpha s,x_1)-G(t^\alpha s,x_2)]\,s^{-1-\frac{1}{\alpha}}G_\alpha(1,s^{-\frac{1}{\alpha}})ds.$$

From ([40, Theorem 4.5.1]) we know that G is one time continuously differentiable in x and satisfies, for any $j=1,\cdots,d$,

$$\left|\frac{\partial G}{\partial x_j}(t,x)\right|\le C\min\left(t^{-\frac{d+1}{\beta}},\frac{t}{\|x\|^{d+\beta+1}}\right).$$

We recall that C depends on β, d and the bounds for ω_μ. Let $DG(t^\alpha s,x)$ the Jacobian of $G(t^\alpha s,\cdot)$ in the point x. By the mean-value inequality, we have that for some ζ in the open segment between x_1 and x_2,

$$\begin{aligned}
&|Z(t,x_1)-Z(t,x_2)|\\
&\le\frac{1}{\alpha}\int_0^\infty |DG(t^\alpha s,\zeta)(x_1-x_2)|s^{-1-\frac{1}{\alpha}}G_\alpha(1,s^{-\frac{1}{\alpha}})ds\\
&\le\frac{\|x_1-x_2\|}{\alpha}\int_0^\infty\sum_{j=1}^d\left|\frac{\partial G}{\partial x_j}(t^\alpha s,\zeta)\right|s^{-1-\frac{1}{\alpha}}G_\alpha(1,s^{-\frac{1}{\alpha}})ds\\
&\lesssim\|x_1-x_2\|\int_0^\infty\min\left((t^\alpha s)^{-\frac{d+1}{\beta}},\frac{t^\alpha s}{\|\zeta\|^{d+\beta+1}}\right)s^{-1-\frac{1}{\alpha}}G_\alpha(1,s^{-\frac{1}{\alpha}})ds.
\end{aligned}$$

Now, we proceed in the same way from [32, Theorem 2]. □

Lemma 3.1.4. *Under the same assumptions as Lemma 3.1.3, then there exists a positive constant C for all $t>0$ and $x_1,x_2\in\mathbb{R}^d$, such that there exists ζ in the open segment connecting x_1 and x_2, and the following estimates for Y hold with $\Omega_\zeta=\|\zeta\|^\beta t^{-\alpha}$. For $\Omega_\zeta\le 1$,*

$$|Y(t,x_1)-Y(t,x_2)|\le C\|x_1-x_2\|\begin{cases}t^{-\frac{\alpha(d+1)}{\beta}+\alpha-1} & \text{if}\quad d+1<2\beta,\\ t^{-\alpha-1}(|\log(\Omega_\zeta)|+1) & \text{if}\quad d+1=2\beta,\\ t^{-\frac{\alpha(d+1)}{\beta}+\alpha-1}\Omega_\zeta^{2-\frac{d+1}{\beta}} & \text{if}\quad d+1>2\beta,\end{cases}$$

and for $\Omega_\zeta\ge 1$,

$$|Y(t,x_1)-Y(t,x_2)|\le C\|x_1-x_2\|t^{-\frac{\alpha(d+1)}{\beta}+\alpha-1}\Omega_\zeta^{-1-\frac{d+1}{\beta}}.$$

Proof. This is similar to the proof of the previous lemma for Z. □

Remark 3.1.2. *It is worth mentioning that all these estimates have been thoroughly investigated using the Zolotarev-Pollard formula for Mittag-Leffler functions E_α, which is valid for the case $0<\alpha<1$ (see [32, Section 2] and [40, Proposition 8.1.1]). To our knowledge this type of representation has not been explored explicitly in the literature for the case $\alpha>1$, however, we refer the reader to [9] and [4] for the study of evolution equations with a Caputo fractional derivative of order $1<\alpha<2$.*

Next, we estimate the L_p-norm of Z. Let $p \geq 1$ and $t \in (0,\infty)$. We begin by splitting the integral on $\mathbb{R}^d$ according to the conditions for Ω given by (3.4)-(3.7). That is,

$$\int_{\mathbb{R}^d} Z^p(t,x)dx = \int_{\{\Omega\geq 1\}} Z^p(t,x)dx + \int_{\{\Omega\leq 1\}} Z^p(t,x)dx.$$

In the case of $\Omega \geq 1$, the integral on this set has two-sided estimates for all $d \geq 1$ and all $\beta \in (0,2)$, given by (3.7). Therefore

$$\int_{\{\Omega\geq 1\}} Z^p(t,x)dx \asymp \int_{\{\Omega\geq 1\}} t^{-\frac{\alpha dp}{\beta}} \Omega^{(-1-\frac{d}{\beta})p} dx.$$

Setting $r = \|x\|$, we obtain

$$\begin{aligned}\int_{\{\Omega\geq 1\}} Z^p(t,x)dx &\asymp \int_{t^{\frac{\alpha}{\beta}}}^{\infty} t^{-\frac{\alpha dp}{\beta}} (r^\beta t^{-\alpha})^{(-1-\frac{d}{\beta})p} r^{d-1} dr \\ &= \int_{t^{\frac{\alpha}{\beta}}}^{\infty} t^{\alpha p} r^{(-\beta-d)p+d-1} dr \\ &= \int_{t^{\frac{\alpha}{\beta}}}^{\infty} t^{\alpha p} (t^{\frac{\alpha}{\beta}} \underbrace{t^{-\frac{\alpha}{\beta}} r}_{s})^{(-\beta-d)p+d-1} dr \\ &= \int_{1}^{\infty} t^{-\frac{\alpha dp}{\beta}+\frac{\alpha d}{\beta}} s^{-\beta p-1-(p-1)d} ds \\ &= t^{-\frac{\alpha dp}{\beta}(1-\frac{1}{p})} \int_{1}^{\infty} s^{-\beta p-1-(p-1)d} ds.\end{aligned}$$

The last integral converges if and only if $\beta p + 1 + (p-1)d > 1$, which holds true for all $d \geq 1$, $\beta \in (0,2)$ and $1 \leq p < \infty$. Consequently, we obtain the estimate

$$\int_{\{\Omega\geq 1\}} Z^p(t,x)dx \asymp t^{-\frac{\alpha dp}{\beta}(1-\frac{1}{p})}. \tag{3.13}$$

Now, for $\Omega \leq 1$, we consider the following cases separately: $\beta \in (0,1)$, $\beta = 1$ and $\beta \in (1,2)$. For $\beta \in (0,1)$, we employ the bounds given by (3.6) and we set again $r = \|x\|$ and the substitution $r = t^{\frac{\alpha}{\beta}} s$, obtaining

$$\begin{aligned}\int_{\{\Omega\leq 1\}} Z^p(t,x)dx &\asymp \int_{\{\Omega\leq 1\}} t^{-\frac{\alpha dp}{\beta}} \Omega^{(1-\frac{d}{\beta})p} dx \\ &\asymp \int_0^{t^{\frac{\alpha}{\beta}}} t^{-\frac{\alpha dp}{\beta}} (r^\beta t^{-\alpha})^{(1-\frac{d}{\beta})p} r^{d-1} dr \\ &= \int_0^{t^{\frac{\alpha}{\beta}}} t^{-\alpha p} r^{(\beta-d)p+d-1} dr \\ &= \int_0^1 t^{-\frac{\alpha dp}{\beta}+\frac{\alpha d}{\beta}} s^{(\beta-d)p+d-1} ds \\ &= t^{-\frac{\alpha dp}{\beta}(1-\frac{1}{p})} \int_0^1 s^{(\beta-d)p+d-1} ds.\end{aligned}$$

The last integral converges if and only if $1-d+(d-\beta)p<1$, which is equivalent with $p<\dfrac{d}{d-\beta}$. This together with (3.13) gives

$$\|Z(t,\cdot)\|_p \asymp t^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)},\ 1\le p<\frac{d}{d-\beta},\ d\ge 1,\ \beta\in(0,1). \tag{3.14}$$

For $\beta=1$, we check out $d=\beta=1$ and $d>\beta$. We employ the bounds given by (3.5) and (3.6), respectively. In the case of $d=1$, we obtain

$$\begin{aligned}
\int_{\{\Omega\le 1\}} Z^p(t,x)dx &\asymp \int_{\{\Omega\le 1\}} t^{-\alpha p}\left(|\log\Omega|+1\right)^p dx\\
&\asymp \int_0^{t^\alpha} t^{-\alpha p}\left(|\log(rt^{-\alpha})|+1\right)^p dr\\
&= \int_0^1 t^{-\alpha p+\alpha}\left(|\log(s)|+1\right)^p ds\\
&= t^{-\alpha p\left(1-\frac{1}{p}\right)}\int_0^1\left(|\log(s)|+1\right)^p ds.
\end{aligned}$$

The last integral converges for all $1\le p<\infty$. Together with (3.13) yields

$$\|Z(t,\cdot)\|_p \asymp t^{-\alpha\left(1-\frac{1}{p}\right)},\ 1\le p<\infty,\ d=1,\ \beta=1. \tag{3.15}$$

Now, for $d\ge 2$ the corresponding estimate is similar to the case of $\beta\in(0,1)$. Thus,

$$\|Z(t,\cdot)\|_p \asymp t^{-\alpha d\left(1-\frac{1}{p}\right)},\ 1\le p<\frac{d}{d-1},\ d\ge 2,\ \beta=1. \tag{3.16}$$

Finally, for $\beta\in(1,2)$, we check out $d<\beta$ and $d>\beta$. We employ the bounds given by (3.4) and (3.6), respectively. For $d=1$,

$$\begin{aligned}
\int_{\{\Omega\le 1\}} Z^p(t,x)dx &\asymp \int_{\{\Omega\le 1\}} t^{-\frac{\alpha dp}{\beta}}dx\\
&\asymp \int_0^{t^{\frac{\alpha}{\beta}}} t^{-\frac{\alpha p}{\beta}}dr\\
&= t^{-\frac{\alpha p}{\beta}}t^{\frac{\alpha}{\beta}}\\
&= t^{-\frac{\alpha p}{\beta}\left(1-\frac{1}{p}\right)}.
\end{aligned}$$

Together with (3.13) and the fact that

$$\sup_{x\in\mathbb{R}} Z(t,\cdot)\asymp t^{-\frac{\alpha}{\beta}},$$

we conclude

$$\|Z(t,\cdot)\|_p \asymp t^{-\frac{\alpha}{\beta}\left(1-\frac{1}{p}\right)},\ 1\le p\le\infty,\ d=1,\ \beta\in(1,2). \tag{3.17}$$

In the case of $d\ge 2$, the corresponding estimate is similar to that of $\beta\in(0,1)$. Thus,

$$\|Z(t,\cdot)\|_p \asymp t^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)},\ 1\le p<\frac{d}{d-\beta},\ d\ge 2,\ \beta\in(1,2). \tag{3.18}$$

Gathering the two-sided estimates from (3.14) to (3.18), we have proved the following result.

Theorem 3.1.1. *Let $d \in \mathbb{N}$, $\alpha \in (0,1)$ and $\beta \in (0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. The kernel $Z(t,\cdot)$ belongs to $L_p(\mathbb{R}^d)$ for all $t > 0$ if, and only if, $1 \le p < \kappa_1$, where*

$$\kappa_1 = \kappa_1(d,\beta) := \begin{cases} \frac{d}{d-\beta} & \text{if } d > \beta, \\ \infty & \text{otherwise.} \end{cases}$$

Moreover, the two-sided estimate

$$\|Z(t,\cdot)\|_p \asymp t^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)}, \; t > 0, \tag{3.19}$$

holds for every $1 \le p < \kappa_1$. In the case of $d < \beta$, (3.19) remains true for $p = \infty$.

Remark 3.1.3. *As we can see, unlike the Gaussian fundamental solutions here we can not ensure the L_p-integrability of $Z(t,\cdot)$ for all $1 \le p \le \infty$, $t > 0$.*

We next examine the critical case $p = \frac{d}{d-\beta}$ for $d > \beta$, in the L_p-weak space with the quasi-norm $|\cdot|_{p,\infty}$ defined by

$$|f|_{p,\infty} := \sup_{\lambda > 0}\{\lambda \; d_f(\lambda)^{\frac{1}{p}}\},$$

where

$$d_f(\lambda) = |\{x \in \mathbb{R}^d : f(x) > \lambda\}|$$

stands for the distribution function of f.

Theorem 3.1.2. *Let $d \in \mathbb{N}$, $\alpha \in (0,1)$ and $\beta \in (0,2)$ such that $d > \beta$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Then $Z(t,\cdot) \in L_{\frac{d}{d-\beta},\infty}(\mathbb{R}^d)$ and satisfies*

$$|Z(t)|_{\frac{d}{d-\beta},\infty} \lesssim t^{-\alpha}, \; t > 0.$$

Proof. Let $t > 0$ and denote $Z(t) = Z(t,\cdot)$. Set $p = \dfrac{d}{d-\beta}$. By definition, $Z(t) \in L_{p,\infty}(\mathbb{R}^d)$ if the quasi-norm

$$|Z(t)|_{p,\infty} = \sup_{\lambda > 0}\{\lambda \; d_{Z(t)}(\lambda)^{\frac{1}{p}}\} < \infty.$$

As above we use the similarity variable $\Omega = \|x\|^\beta t^{-\alpha}$ and we split $Z(t)$ as $Z(t) = Z(t)\chi(t)_{\{\Omega \le 1\}} + Z(t)\chi(t)_{\{\Omega \ge 1\}}$. Then

$$|Z(t)|_{p,\infty} \le 2\left(|Z(t)\chi(t)_{\{\Omega\le 1\}}|_{p,\infty} + |Z(t)\chi(t)_{\{\Omega\ge 1\}}|_{p,\infty}\right).$$

By (3.13) and the L_p version Tchebyshev's inequality ([68, Formula (5.49)]), we obtain

$$|Z(t)\chi(t)_{\{\Omega\ge 1\}}|_{p,\infty} \le \|Z(t)\chi(t)_{\{\Omega\ge 1\}}\|_p \le Ct^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)} = Ct^{-\alpha}.$$

Now, employing (3.6) we have

$$\begin{aligned} d_{Z(t)\chi(t)_{\{\Omega\leq 1\}}}(\lambda) &= |\{x\in\mathbb{R}^d : Z(t,x)>\lambda \quad \text{y} \quad \Omega\leq 1\}| \\ &\leq |\{x\in\mathbb{R}^d : \lambda < Ct^{-\frac{\alpha d}{\beta}}\Omega^{1-\frac{d}{\beta}}\}| \\ &= |\{x\in\mathbb{R}^d : \lambda < Ct^{-\frac{\alpha d}{\beta}}(\|x\|^{\beta}t^{-\alpha})^{1-\frac{d}{\beta}}\}| \\ &= |\{x\in\mathbb{R}^d : \lambda < Ct^{-\alpha}\|x\|^{\beta-d}\}| \\ &\leq C_1\left(t^{-\alpha}\lambda^{-1}\right)^{\frac{d}{d-\beta}}. \end{aligned}$$

Thereby, we find that

$$\lambda\, d_{Z(t)\chi(t)_{\{\Omega\leq 1\}}}(\lambda)^{\frac{1}{p}} \leq C_1^{\frac{1}{p}} t^{-\alpha},$$

and thus

$$|Z(t)\chi(t)_{\{\Omega\leq 1\}}|_{p,\infty} \lesssim t^{-\alpha}.$$

The proof is complete. □

Analogously to the analysis done for the L_p-integrability of Z, we can obtain the corresponding results for Y using the bounds given by (3.9)-(3.12).

Theorem 3.1.3. *Let $d\in\mathbb{N}$, $\alpha\in(0,1)$ and $\beta\in(0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. The kernel $Y(t,\cdot)$ belongs to $L_p(\mathbb{R}^d)$ for all $t>0$ if, and only if, $1\leq p<\kappa_2$, where*

$$\kappa_2 = \kappa_2(d,\beta) := \begin{cases} \frac{d}{d-2\beta} & \text{if } d>2\beta, \\ \infty & \text{otherwise.} \end{cases}$$

Moreover, the two-sided estimate

$$\|Y(t,\cdot)\|_p \asymp t^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+(\alpha-1)},\ t>0, \tag{3.20}$$

holds for every $1\leq p<\kappa_2$. In the case of $d<2\beta$, (3.20) remains true for $p=\infty$.

Theorem 3.1.4. *Let $d\in\mathbb{N}$, $\alpha\in(0,1)$ and $\beta\in(0,2)$ such that $d>2\beta$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Then $Y(t,\cdot)\in L_{\frac{d}{d-2\beta},\infty}(\mathbb{R}^d)$ and satisfies*

$$|Y(t)|_{\frac{d}{d-2\beta},\infty} \lesssim t^{-\alpha-1},\ t>0.$$

Now, we can establish that $Z(t,\cdot)$ is a locally Lipschitz function in L_p and it is an approximation of the identity. To this end, we first show that Z and Y satisfy the following scaling property.

Lemma 3.1.5. *Let $t>0$, $x\in\mathbb{R}^d$. Then*

$$Z(t,x) = t^{-\frac{\alpha d}{\beta}} Z\left(1, t^{-\frac{\alpha}{\beta}}x\right)$$

and

$$Y(t,x) = t^{-\frac{\alpha d}{\beta}+\alpha-1} Y\left(1, t^{-\frac{\alpha}{\beta}}x\right).$$

Proof. It is well-known that the Fourier transform in the spatial variable of Z, denoted by $\hat{Z}(t,\xi)$, is

$$\hat{Z}(t,\xi) = E_{\alpha,1}(-t^{\alpha}\psi(\xi)), \quad t>0, \tag{3.21}$$

see, e.g. [34, Sub-section 8.4] and [44, Section 5]).

Using this and changing the integration variables, where $\bar{\xi} = \frac{\xi}{\|\xi\|}$ and ω_μ is given by (3.2), we get

$$\begin{aligned}
&Z(t,x)\\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}E_{\alpha,1}(-t^{\alpha}\psi(\xi))d\xi\\
&= \frac{1}{(2\pi)^d}\int_0^\infty\int_{S^{d-1}} e^{i\left(\frac{x}{t^{\frac{\alpha}{\beta}}}\cdot\bar{\xi}\right)t^{\frac{\alpha}{\beta}}\|\xi\|} E_{\alpha,1}(-(t^{\frac{\alpha}{\beta}}\|\xi\|)^{\beta}\omega_\mu(\bar{\xi}))\|\xi\|^{d-1}\theta(d\bar{\xi})d\|\xi\|\\
&= \frac{1}{(2\pi)^d}\int_0^\infty\int_{S^{d-1}} e^{i\left(\frac{x}{t^{\frac{\alpha}{\beta}}}\cdot\bar{\xi}\right)t^{\frac{\alpha}{\beta}}\|\xi\|} E_{\alpha,1}(-(t^{\frac{\alpha}{\beta}}\|\xi\|)^{\beta}\omega_\mu(\bar{\xi}))(t^{-\frac{\alpha}{\beta}}\underbrace{t^{\frac{\alpha}{\beta}}\|\xi\|}_{r})^{d-1}\theta(d\bar{\xi})d\|\xi\|\\
&= t^{-\frac{\alpha d}{\beta}}\frac{1}{(2\pi)^d}\int_0^\infty\int_{S^{d-1}} e^{i\left(\frac{x}{t^{\frac{\alpha}{\beta}}}\cdot\bar{\xi}\right)r} E_{\alpha,1}(-r^{\beta}\omega_\mu(\bar{\xi}))r^{d-1}\theta(d\bar{\xi})dr\\
&= t^{-\frac{\alpha d}{\beta}} Z\left(1, t^{-\frac{\alpha}{\beta}}x\right).
\end{aligned}$$

On the other hand, it is also known that the Fourier transform in the spatial variable of Y is given by

$$\hat{Y}(t,\xi) = t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}\psi(\xi)), \quad t>0. \tag{3.22}$$

Therefore, the previous argument can be applied to

$$Y(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}\psi(\xi))d\xi.$$

□

Theorem 3.1.5. *Let $d\in\mathbb{N}$, $\alpha\in(0,1)$, $\beta\in(0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Then*

(i) $Z\in C((0,\infty);L_p(\mathbb{R}^d))$ for $1\le p<\kappa_1$. Further, for each $\epsilon>0$ there exists a constant $C>0$ depending on $\epsilon, d, p, \alpha, \beta$ such that

$$\|Z(t,\cdot)-Z(s,\cdot)\|_p \le C|t-s|, \tag{3.23}$$

holds for all $t,s\ge\epsilon$. In the case of $d<\beta$, (3.23) remains true for $p=\infty$.

(ii) For any $v\in L_p(\mathbb{R}^d)$, with $1\le p<\infty$, we have

$$\lim_{t\to 0}\|Z(t,\cdot)\star v - v\|_p = 0.$$

Proof. Let $0 < \epsilon \leq s, t$. From Lemma 3.1.1 we know that there exists $\tau > 0$ between s and t, with $\Omega_c = \|x\|^{\beta}\tau^{-\alpha}$, such that

$$|Z(t,x) - Z(s,x)| \leq C|t-s| \begin{cases} \tau^{-\frac{\alpha d}{\beta}-1} & \text{if} \quad d < \beta, \\ \tau^{-\alpha-1}(|\log(\Omega_c)|+1) & \text{if} \quad d = \beta, \\ \tau^{-\frac{\alpha d}{\beta}-1}\Omega_c^{1-\frac{d}{\beta}} & \text{if} \quad d > \beta, \end{cases}$$

for $\Omega_c \leq 1$ and

$$|Z(t,x) - Z(s,x)| \leq C|t-s|\tau^{-\frac{\alpha d}{\beta}-1}\Omega_c^{-1-\frac{d}{\beta}}$$

for $\Omega_c \geq 1$. Similar arguments as in the proof of Theorem 3.1.1, with the only difference that there appears Ω instead of Ω_c in the spatial integral respect to x, show that

$$\|Z(t,\cdot) - Z(s,\cdot)\|_p \leq C_1|t-s|\tau^{-\frac{\alpha d}{\beta}(1-\frac{1}{p})-1}, \ t > 0,$$

if and only if $1 \leq p < \kappa_1$. Recall that the condition *if and only if* guarantees the existence of the improper Riemann integrals in the proof.

Indeed, let $\Omega_s = \|x\|^{\beta}s^{-\alpha}$ and $\Omega_t = \|x\|^{\beta}t^{-\alpha}$. Without loss of generality we suppose that $s < \tau < t$, which implies that $\Omega_t < \Omega_c < \Omega_s$. When $\Omega_c \geq 1$ we note that $\Omega_s \geq 1$ and if $\Omega_c \leq 1$ then $\Omega_t \leq 1$. Keeping this in mind we have that

$$\begin{aligned}
\int_{\{\Omega_c \geq 1\}} |Z(t,x) - Z(s,x)|^p dx &\leq C^p|t-s|^p \int_{\{\Omega_c \geq 1\}} \tau^{(-\frac{\alpha d}{\beta}-1)p}\Omega_c^{(-1-\frac{d}{\beta})p} dx \\
&\lesssim C^p|t-s|^p \int_{\{\Omega_s \geq 1\}} \tau^{(\alpha-1)p}\|x\|^{(-\beta-d)p} dx \\
&\lesssim C^p|t-s|^p s^{(\alpha-1)p} \int_{s^{\frac{\alpha}{\beta}}}^{+\infty} r^{(-\beta-d)p}r^{d-1} dr \\
&= C^p|t-s|^p s^{-\frac{\alpha dp}{\beta}(1-\frac{1}{p})-p}.
\end{aligned}$$

For $\Omega_c \leq 1$ we need to check the cases $d < \beta$, $d = \beta$ and $d > \beta$, respectively. If $d < \beta$ we see that

$$\begin{aligned}
\int_{\{\Omega_c \leq 1\}} |Z(t,x) - Z(s,x)|^p dx &\leq C^p|t-s|^p \int_{\{\Omega_c \leq 1\}} \tau^{(-\frac{\alpha d}{\beta}-1)p} dx \\
&\lesssim C^p|t-s|^p s^{(-\frac{\alpha d}{\beta}-1)p} \int_{\{\Omega_t \leq 1\}} dx \\
&= C^p|t-s|^p s^{(-\frac{\alpha d}{\beta}-1)p} t^{\frac{\alpha d}{\beta}} \\
&= C^p|t-s|^p s^{(-\frac{\alpha d}{\beta}-1)p} (s + (t-s))^{\frac{\alpha d}{\beta}} \\
&\lesssim C^p|t-s|^p s^{(-\frac{\alpha d}{\beta}-1)p} \left(s^{\frac{\alpha d}{\beta}} + (t-s)^{\frac{\alpha d}{\beta}}\right) \\
&= C^p|t-s|^p s^{-\frac{\alpha dp}{\beta}(1-\frac{1}{p})-p} \left(1 + (s^{-1}(t-s))^{\frac{\alpha d}{\beta}}\right).
\end{aligned}$$

If $d = \beta$ we obtain

$$\begin{aligned}\int_{\{\Omega_c \le 1\}} |Z(t,x) - Z(s,x)|^p dx &\le C^p |t-s|^p \int_{\{\Omega_c \le 1\}} \tau^{(-\alpha-1)p} \left(|log(\Omega_c)| + 1\right)^p dx \\ &\lesssim C^p |t-s|^p s^{(-\alpha-1)p} \int_{\{\Omega_t \le 1\}} \left(|log(\Omega_t)| + 1\right)^p dx \\ &= C^p |t-s|^p s^{(-\alpha-1)p} t^{\alpha} \\ &\lesssim C^p |t-s|^p s^{(-\alpha-1)p} \left(s^{\alpha} + (t-s)^{\alpha}\right) \\ &= C^p |t-s|^p s^{-\alpha p \left(1-\frac{1}{p}\right) - p} \left(1 + (s^{-1}(t-s))^{\alpha}\right)\end{aligned}$$

and $d > \beta$, with $p < \frac{d}{d-\beta}$, yields

$$\begin{aligned}\int_{\{\Omega_c \le 1\}} |Z(t,x) - Z(s,x)|^p dx &\le C^p |t-s|^p \int_{\{\Omega_c \le 1\}} \tau^{\left(-\frac{\alpha d}{\beta} - 1\right)p} \Omega_c^{\left(1-\frac{d}{\beta}\right)p} dx \\ &\lesssim C^p |t-s|^p \int_{\{\Omega_t \le 1\}} \tau^{(-\alpha-1)p} \|x\|^{(\beta-d)p} dx \\ &\lesssim C^p |t-s|^p s^{(-\alpha-1)p} \int_0^{t^{\frac{\alpha}{\beta}}} r^{(\beta-d)p} r^{d-1} dr \\ &= C^p |t-s|^p s^{(-\alpha-1)p} t^{\alpha p - \frac{\alpha d p}{\beta} + \frac{\alpha d}{\beta}} \\ &\lesssim C^p |t-s|^p s^{(-\alpha-1)p} \left(s^{\alpha p - \frac{\alpha d p}{\beta} + \frac{\alpha d}{\beta}} + (t-s)^{\alpha p - \frac{\alpha d p}{\beta} + \frac{\alpha d}{\beta}}\right) \\ &= C^p |t-s|^p s^{-\frac{\alpha d p}{\beta}\left(1-\frac{1}{p}\right) - p} \left(1 + (s^{-1}(t-s))^{\alpha p - \frac{\alpha d p}{\beta} + \frac{\alpha d}{\beta}}\right).\end{aligned}$$

This implies that

$$\|Z(t,\cdot) - Z(s,\cdot)\|_p \le C_1 |t-s| s^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right) - 1} \left(1 + (s^{-1}(t-s))^k\right),$$

with $k = \frac{\alpha d}{\beta p}$ whenever $d < \beta$, $k = \frac{\alpha}{p}$ whenever $d < \beta$ and $k = \alpha - \frac{\alpha d}{\beta}\left(1 - \frac{1}{p}\right)$ whenever $d > \beta$. Nevertheless, the factor $s^{-1}(t-s)$ does not affect the results of our work and therefore it will not be considered. Thus, it makes sense to assume that τ, between s and t, is independent of x.

In order to get (3.23), we use the fact that $\tau > \epsilon$. Thus,

$$\|Z(t,\cdot) - Z(s,\cdot)\|_p \le C_1 |t-s| \epsilon^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right) - 1}$$

for all $t, s \ge \epsilon$ and we take $C = C_1 \epsilon^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right) - 1}$. This proves (i).

To prove (ii), let $v \in L_p(\mathbb{R}^d)$ with $p \ge 1$. From (3.21) and since $\psi(0) = 0$, we have

$$\int_{\mathbb{R}^d} Z(t,x) dx = \hat{Z}(t,0) = 1, \quad t > 0. \tag{3.24}$$

Now, for $t > 0$ define $\phi_t(x) := t^{-\frac{\alpha d}{\beta}} Z\left(1, t^{-\frac{\alpha}{\beta}} x\right)$, $x \in \mathbb{R}^d$. By (3.24) it follows that $\phi_t \star v \in L_p(\mathbb{R}^d)$ for all $p \ge 1$. By applying the Minkowski's integral inequality we obtain

$$\begin{aligned}
\|\phi_t \star v - v\|_p &= \left(\int_{\mathbb{R}^d} |(\phi_t \star v)(x) - v(x)|^p dx \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} v(x-y)\phi_t(y)dy - \int_{\mathbb{R}^d} v(x)\phi_t(y)dy \right|^p dx \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (v(x-y) - v(x))\phi_t(y)dy \right|^p dx \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (v(x - t^{\frac{\alpha}{\beta}} y) - v(x)) Z(1,y)dy \right|^p dx \right)^{\frac{1}{p}} \\
&\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |v(x - t^{\frac{\alpha}{\beta}} y) - v(x)|^p Z^p(1,y)dx \right)^{\frac{1}{p}} dy \\
&= \int_{\mathbb{R}^d} Z(1,y) \|v(\cdot - t^{\frac{\alpha}{\beta}} y) - v(\cdot)\|_p \, dy.
\end{aligned}$$

Since $\|v(\cdot - t^{\frac{\alpha}{\beta}} y) - v(\cdot)\|_p \to 0$ as $t \to 0$ and $\|v(\cdot - t^{\frac{\alpha}{\beta}} y) - v(\cdot)\|_p \leq 2\|v\|_p$, we apply the dominated convergence theorem to the last integral concluding $Z(t,\cdot) \star v \to v$ in $L_p(\mathbb{R}^d)$ as $t \to 0$. □

In the same way, we can establish that $Y(t,\cdot)$ is locally Lipschitz function in L_p and that $\frac{1}{g_\alpha(t)} Y(t,\cdot)$ is an approximation of the identity in L_p as $t \to 0$.

Theorem 3.1.6. *Let $d \in \mathbb{N}$, $\alpha \in (0,1)$, $\beta \in (0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Then*

(i) $Y \in C((0,\infty); L_p(\mathbb{R}^d))$ for $1 \leq p < \kappa_2$. Further, for each $\epsilon > 0$ there exists a constant $C > 0$ depending on $\epsilon, d, p, \alpha, \beta$ such that

$$\|Y(t,\cdot) - Y(s,\cdot)\|_p \leq C|t-s|, \tag{3.25}$$

holds for all $t, s \geq \epsilon$. In the case of $d < 2\beta$, (3.25) remains true for $p = \infty$.

(ii) For any $v \in L_p(\mathbb{R}^d)$, with $1 \leq p < \infty$, we have that

$$\lim_{t\to 0} \left\| \frac{1}{g_\alpha(t)} Y(t,\cdot) \star v - v \right\|_p = 0.$$

Proof. The proof of (i) is similar to the one we used in Theorem 3.1.5 part (i). Such arguments and Lemma 3.1.2 imply that

$$\|Y(t,\cdot) - Y(s,\cdot)\|_p \leq C_1 |t-s| s^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-2} \left(1 + (s^{-1}(t-s))^k\right),$$

with $k = \frac{\alpha d}{\beta p}$ whenever $d < 2\beta$, $k = \frac{2\alpha}{p}$ whenever $d = 2\beta$ and $k = 2\alpha - \frac{\alpha d}{\beta}\left(1 - \frac{1}{p}\right)$ whenever $d > 2\beta$. Again, the factor $s^{-1}(t-s)$ does not affect the results of our work and therefore it will not be considered.

The key point to prove (ii) is Lemma 3.1.5 and (3.22), because

$$\frac{1}{g_\alpha(t)}\int_{\mathbb{R}^d} Y(t,x)dx = 1, \quad t>0.$$

□

We finish this section by proving the following relation between the fundamental solutions Z and Y.

Lemma 3.1.6. *Let fixed $x\in\mathbb{R}^d\backslash\{0\}$. Under the same assumptions as in Proposition 3.1.1, Z and Y satisfy*

$$Y(\cdot,x)=\frac{d}{dt}(g_\alpha * Z(\cdot,x)), \quad t>0.$$

Proof. From (3.22) and Fubini, we have that

$$\begin{aligned}(g_{1-\alpha} * Y(\cdot,x))(t) &= \int_0^t g_{1-\alpha}(t-s)Y(s,x)ds\\ &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\left[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}s^{\alpha-1}E_{\alpha,\alpha}(-\psi(\xi)s^\alpha)d\xi\right]ds\\ &= \frac{1}{\Gamma(1-\alpha)(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\int_0^t \frac{s^{\alpha-1}}{(t-s)^\alpha}E_{\alpha,\alpha}(-\psi(\xi)s^\alpha)ds\,d\xi.\end{aligned}$$

On the other hand, using the definition of $E_{\alpha,\alpha}$ given in Section 1.2, it follows that

$$\frac{s^{\alpha-1}}{(t-s)^\alpha}E_{\alpha,\alpha}(-\psi(\xi)s^\alpha)=\sum_{k=0}^{\infty}\frac{(-\psi(\xi))^k s^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}\frac{1}{(t-s)^\alpha}.$$

By integrating respect to s, we obtain

$$\int_0^t \frac{s^{\alpha-1}}{(t-s)^\alpha}E_{\alpha,\alpha}(-\psi(\xi)s^\alpha)ds=\sum_{k=0}^{\infty}\frac{(-\psi(\xi))^k}{\Gamma(k\alpha+\alpha)}\int_0^t\frac{s^{k\alpha+\alpha-1}}{(t-s)^\alpha}ds.$$

In the last integral, the substitution $s=t\tau$ yields

$$\int_0^t\frac{s^{k\alpha+\alpha-1}}{(t-s)^\alpha}ds=\int_0^1\frac{t^{k\alpha}\tau^{k\alpha+\alpha-1}}{(1-\tau)^\alpha}d\tau.$$

We note that this improper integral exists because $\alpha\in(0,1)$. By using Beta B and Euler Gamma Γ functions, we get

$$\int_0^1 \tau^{k\alpha+\alpha-1}(1-\tau)^{1-\alpha-1}d\tau = B(k\alpha+\alpha,1-\alpha)=\frac{\Gamma(k\alpha+\alpha)\Gamma(1-\alpha)}{\Gamma(k\alpha+1)}$$

and thus

$$\begin{aligned}\int_0^t \frac{s^{\alpha-1}}{(t-s)^\alpha}E_{\alpha,\alpha}(-\psi(\xi)s^\alpha)ds &= \sum_{k=0}^{\infty}\frac{(-\psi(\xi))^k}{\Gamma(k\alpha+\alpha)}t^{k\alpha}\frac{\Gamma(k\alpha+\alpha)\Gamma(1-\alpha)}{\Gamma(k\alpha+1)}\\ &=\Gamma(1-\alpha)E_{\alpha,1}(-\psi(\xi)t^\alpha).\end{aligned}$$

Using this in the first part of the proof, we conclude that

$$Z(t,x)=(g_{1-\alpha}*Y(\cdot,x))(t).$$

The convolution with g_α and the derivative w.r.t. the time complete the proof. □

3.2 Local well-posedness

Due to the properties of the pair $(g_\alpha, g_{1-\alpha})$, with $0 < \alpha < 1$ (see Section 1.4), and under suitable conditions, the problem (3.1) can be rewritten as the *semi-linear Volterra equation*

$$u + g_\alpha * \Psi_\beta(-i\nabla)u = u_0 + \lambda\, g_\alpha * |u|^{\gamma-1}u.$$

This fact is particularly exploited in Section 3.4. However, in this section we deal with the corresponding integral representation for *mild solutions in the sense of Volterra*, which leads to fixed points of the integral equation

$$u(t,x) = \int_{\mathbb{R}^d} Z(t,x-y)u_0(y)dy + \lambda \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)|u(s,y)|^{\gamma-1}u(s,y)dyds. \tag{3.26}$$

In this section, by a local solution of the Cauchy problem (3.1) we understand the solution u of the corresponding integral equation (3.26) (the so-called *mild solution*) belonging to the Banach space

$$E_T := C([0,T]; L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)) \cap C((0,T]; L_\infty(\mathbb{R}^d)),$$

with the norm

$$\|v\|_{E_T} := \sup_{t\in[0,T]} \Big(\|v(t,\cdot)\|_p + \|v(t,\cdot)\|_1 \Big) + \sup_{t\in(0,T]} t^{\frac{\alpha d}{\beta p}} \|v(t,\cdot)\|_\infty .$$

We define on E_T the operator $\mathcal{M}$ given by

$$\mathcal{M}(v)(t,x) := \int_{\mathbb{R}^d} Z(t,x-y)u_0(y)dy + \lambda \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)|v(s,y)|^{\gamma-1}v(s,y)dyds \tag{3.27}$$

where $u_0 \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ is a given data and $v \in E_T$. The space $L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ is equipped with the usual norm $\|\cdot\|_1 + \|\cdot\|_p$.

We also need to define the number $\kappa := \begin{cases} \frac{d}{\beta}, & d > \beta, \\ 1, & \textit{otherwise}. \end{cases}$

Theorem 3.2.1. *Let $\alpha \in (0,1)$ and $\beta \in (0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Let $\lambda \in \mathbb{R}$ and $\gamma > 1$. Suppose that $\max\Big(1, \kappa, \frac{d(\gamma-1)}{\beta}\Big) < p < \infty$. If $u_0 \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$, then for some $0 < T^* < T$ the operator $\mathcal{M}$ defined by (3.27) has a unique fixed point in E_{T^*}.*

Proof. Since $u_0 \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ and $Z(t,\cdot) \in L_1(\mathbb{R}^d)$, it follows that $Z(t,\cdot) \star u_0 \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ for each $t \in [0,T]$. Condition $\max(1,\kappa) < p < \infty$ and Young's inequality for convolutions imply that $Z(t,\cdot) \star u_0 \in L_\infty(\mathbb{R}^d)$ for each $t \in (0,T]$, since there exists a q such that $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < q < \kappa_1$, thus $Z(t,\cdot) \in L_q(\mathbb{R}^d)$. We recall that κ_1 was introduced in Theorem 3.1.1. Furthermore,

$$\begin{aligned} \|Z(t,\cdot) \star u_0\|_1 &\le \|Z(t,\cdot)\|_1 \|u_0\|_1 = \|u_0\|_1 , \\ \|Z(t,\cdot) \star u_0\|_p &\le \|Z(t,\cdot)\|_1 \|u_0\|_p = \|u_0\|_p , \\ \|Z(t,\cdot) \star u_0\|_\infty &\le \|Z(t,\cdot)\|_q \|u_0\|_p \lesssim t^{-\frac{\alpha d}{\beta p}} \|u_0\|_p . \end{aligned} \tag{3.28}$$

The continuity of $t \mapsto Z(t,\cdot) \star u_0$ in $[0,T]$ with respect to the norm topology on $L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ follows from Theorem 3.1.5. The continuity in $(0,T]$ with respect to the norm topology on $L_\infty(\mathbb{R}^d)$ follows from the same considerations mentioned above for p and from Theorem 3.1.5 part (i).

Let us consider $t \mapsto \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds$, where $0 < t \leq T$.

Previous arguments together with the Minkowski's integral inequality and the fact that $\||v(s,\cdot)|^{\gamma-1}v(s,\cdot)\|_1 \leq s^{-\frac{\alpha d(\gamma-1)}{\beta p}} \|v\|_{E_T}^{\gamma-1} \|v(s,\cdot)\|_1$ for $s > 0$, produce

$$\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_1 \leq \|v\|_{E_T}^{\gamma} \int_0^t \|Y(t-s,\cdot)\|_1 s^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds.$$

From Theorem 3.1.3 it follows that

$$\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_1 \lesssim \|v\|_{E_T}^{\gamma} \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds.$$

Condition $\frac{d(\gamma-1)}{\beta} < p < \infty$ yields

$$\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_1 \lesssim \|v\|_{E_T}^{\gamma} t^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}}$$

with

$$0 < \alpha - \frac{\alpha d(\gamma-1)}{\beta p} < 1.$$

Similarly,

$$\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_p \lesssim \|v\|_{E_T}^{\gamma} t^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}}.$$

Therefore,

$$\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_1 + \left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_p$$
$$\lesssim \|v\|_{E_T}^{\gamma} t^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}}.$$

This shows that

$$\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_{L_p(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)} \to 0$$

whenever $t \to 0$, and that

$$\int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$$

for $t \in [0,T]$.

Again, condition $\max(1,\kappa) < p < \infty$ and Young's convolution inequality imply that for any $t \in (0,T]$, $Y(t,\cdot) \in L_q(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < q < \kappa_2$ (see Theorem

3.1.3 and note that $\kappa_2 \geq \kappa_1$). Hence, $Y(t-s,\cdot)\star|v(s,\cdot)|^{\gamma-1}v(s,\cdot) \in L_\infty(\mathbb{R}^d)$, $0<s<t$.

Further, condition $\max(1,\kappa) < p < \infty$ also implies that $0 < \frac{d}{\beta p} < 1$. Thus,

$$\begin{aligned}
&\left\| \int_0^t Y(t-s,\cdot)\star|v(s,\cdot)|^{\gamma-1}v(s,\cdot)ds \right\|_\infty \\
&\leq \int_0^t \|Y(t-s,\cdot)\|_{\frac{p}{p-1}} \||v(s,\cdot)|^{\gamma-1}v(s,\cdot)\|_p \, ds \\
&\lesssim \|v\|_{E_T}^\gamma \int_0^t (t-s)^{-\frac{\alpha d}{\beta p}+\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds \\
&\lesssim \|v\|_{E_T}^\gamma \left[\int_0^{\frac{t}{2}} (t-s)^{-\frac{\alpha d}{\beta p}+\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds + \int_{\frac{t}{2}}^t (t-s)^{-\frac{\alpha d}{\beta p}+\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds \right] \\
&\lesssim \|v\|_{E_T}^\gamma \left[\left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta p}+\alpha-1} \int_0^{\frac{t}{2}} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds + \left(\frac{t}{2}\right)^{-\frac{\alpha d(\gamma-1)}{\beta p}} \int_{\frac{t}{2}}^t (t-s)^{-\frac{\alpha d}{\beta p}+\alpha-1} ds \right] \\
&\lesssim \|v\|_{E_T}^\gamma \left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta p}+\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} < \infty.
\end{aligned}$$

The continuity of $t \mapsto \int_0^t Y(t-s,\cdot)\star|v(s,\cdot)|^{\gamma-1}v(s,\cdot)ds$ in $(0,T]$ with respect to the norm topology on $L_p(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)$, it follows from conditions on p, continuity of v and the property (2.14),

$$\big|\,|a|^c a - |b|^c b\,\big| \lesssim |\,a-b\,|(|a|^c+|b|^c) \lesssim |\,a-b\,|(\,|a|+|b|\,)^c, \quad a,b\in\mathbb{R}, c>0.$$

Indeed, suppose $0 < t_0 < t$ without loss of generality. We have that

$$\begin{aligned}
&\left\| \int_0^t Y(t-s,\cdot)\star|v(s,\cdot)|^{\gamma-1}v(s,\cdot)ds - \int_0^{t_0} Y(t_0-s,\cdot)\star|v(s,\cdot)|^{\gamma-1}v(s,\cdot)ds \right\|_1 \\
&\leq \int_0^{t_0} \|Y(s,\cdot)\|_1 \||v(t-s,\cdot)|^{\gamma-1}v(t-s,\cdot) - |v(t_0-s,\cdot)|^{\gamma-1}v(t_0-s,\cdot)\|_1 ds \\
&\quad + \int_{t_0}^t \|Y(s,\cdot)\|_1 \||v(t-s,\cdot)|^{\gamma-1}v(t-s,\cdot)\|_1 ds
\end{aligned}$$

$$\begin{aligned}
&\lesssim \|v\|_{E_T}^{\gamma-1}\varepsilon \int_0^{t_0} s^{\alpha-1}\left((t-s)^{-\frac{\alpha d(\gamma-1)}{\beta p}} + (t_0-s)^{-\frac{\alpha d(\gamma-1)}{\beta p}}\right) ds \\
&\quad + \|v\|_{E_T}^\gamma \int_{t_0}^t s^{\alpha-1}(t-s)^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds \\
&\lesssim \|v\|_{E_T}^{\gamma-1}\varepsilon \int_0^{t_0} s^{\alpha-1}(t_0-s)^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds + \|v\|_{E_T}^\gamma \int_{t_0}^t s^{\alpha-1}(t-s)^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds \\
&\lesssim \|v\|_{E_T}^{\gamma-1}\varepsilon t_0^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} + \|v\|_{E_T}^\gamma (t-t_0)^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}}.
\end{aligned}$$

Similarly, we obtain that

$$\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds - \int_0^{t_0} Y(t_0-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_p \to 0$$

whenever $t \to t_0$.

Now, for the continuity of $t \mapsto \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds$ in $(0,T]$ with respect to the norm topology on $L_\infty(\mathbb{R}^d)$, we find that

$$\begin{aligned}
&\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds - \int_0^{t_0} Y(t_0-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_\infty \\
&\leq \int_0^{t_0} \|Y(s,\cdot)\|_q \| |v(t-s,\cdot)|^{\gamma-1} v(t-s,\cdot) - |v(t_0-s,\cdot)|^{\gamma-1} v(t_0-s,\cdot) \|_p \, ds \\
&\quad + \int_{t_0}^t \|Y(s,\cdot)\|_q \| |v(t-s,\cdot)|^{\gamma-1} v(t-s,\cdot) \|_p \, ds \\
&\lesssim \|v\|_{E_T}^{\gamma-1} \varepsilon \int_0^{t_0} s^{-\frac{\alpha d}{\beta p}+\alpha-1} \left((t-s)^{-\frac{\alpha d(\gamma-1)}{\beta p}} + (t_0-s)^{-\frac{\alpha d(\gamma-1)}{\beta p}} \right) ds \\
&\quad + \|v\|_{E_T}^{\gamma} \int_{t_0}^t s^{-\frac{\alpha d}{\beta p}+\alpha-1} (t-s)^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds \\
&\lesssim \|v\|_{E_T}^{\gamma-1} \varepsilon \int_0^{t_0} s^{-\frac{\alpha d}{\beta p}+\alpha-1} (t_0-s)^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds + \|v\|_{E_T}^{\gamma} \int_{t_0}^t s^{-\frac{\alpha d}{\beta p}+\alpha-1} (t-s)^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds \\
&\lesssim \|v\|_{E_T}^{\gamma-1} \varepsilon t_0^{-\frac{\alpha d}{\beta p}+\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} + \|v\|_{E_T}^{\gamma} t_0^{-\frac{\alpha d}{\beta p}} (t-t_0)^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}}.
\end{aligned}$$

Up to this point, we have proved that the operator $\mathcal{M}$ given by (3.27) is well defined. Now, let $v, w \in E_T$. Previous arguments show that

$$\begin{aligned}
&\|\mathcal{M}(v)(t,\cdot) - \mathcal{M}(w)(t,\cdot)\|_1 \\
&\leq |\lambda| \int_0^t \|Y(t-s,\cdot)\|_1 \left\| |v(s,\cdot)|^{\gamma-1} v(s,\cdot) - |w(s,\cdot)|^{\gamma-1} w(s,\cdot) \right\|_1 ds \\
&\lesssim (\|v\|_{E_T} + \|w\|_{E_T})^{\gamma-1} \|v-w\|_{E_T} \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} ds \\
&\lesssim (\|v\|_{E_T} + \|w\|_{E_T})^{\gamma-1} \|v-w\|_{E_T} t^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} \\
&\lesssim (\|v\|_{E_T} + \|w\|_{E_T})^{\gamma-1} \|v-w\|_{E_T} T^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}}.
\end{aligned}$$

In the same way we estimate

$$\|\mathcal{M}(v)(t,\cdot) - \mathcal{M}(w)(t,\cdot)\|_p \lesssim (\|v\|_{E_T} + \|w\|_{E_T})^{\gamma-1} \|v-w\|_{E_T} T^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}}.$$

Similarly,

$$\|\mathcal{M}(v)(t,\cdot) - \mathcal{M}(w)(t,\cdot)\|_\infty \lesssim (\|v\|_{E_T} + \|w\|_{E_T})^{\gamma-1} \|v-w\|_{E_T} t^{\alpha-\frac{\alpha d}{\beta p}-\frac{\alpha d(\gamma-1)}{\beta p}}$$

and we have

$$\begin{aligned}
t^{\frac{\alpha d}{\beta p}} \|\mathcal{M}(v)(t,\cdot) - \mathcal{M}(w)(t,\cdot)\|_\infty &\lesssim (\|v\|_{E_T} + \|w\|_{E_T})^{\gamma-1} \|v-w\|_{E_T} t^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} \\
&\lesssim (\|v\|_{E_T} + \|w\|_{E_T})^{\gamma-1} \|v-w\|_{E_T} T^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}}.
\end{aligned}$$

This shows that

$$\|\mathcal{M}(v)-\mathcal{M}(w)\|_{E_T} \le C_2 T^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} \|v-w\|_{E_T} \left(\|v\|_{E_T}+\|w\|_{E_T}\right)^{\gamma-1}. \tag{3.29}$$

Besides, we derive from (3.28) that

$$\|Z \star u_0\|_{E_T} \le C_1(\|u_0\|_1 + \|u_0\|_p). \tag{3.30}$$

Let $T^* \in (0,T)$ and $R = 2C_1\|u_0\|_{L_p(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)}$. We consider the closed ball

$$B_{T^*,R} := \{w \in E_{T^*} : \|w\|_{E_{T^*}} \le R\}.$$

Our aim now is to obtain a suitable T^* such that $\mathcal{M}$ is a contraction as an operator $B_{T^*,R} \to B_{T^*,R}$, thereby existence and uniqueness of a fixed point of this operator follow from the Banach fixed-point theorem.

First, we find a condition on T^* such that $\mathcal{M}(B_{T^*,R}) \subset B_{T^*,R}$. Let $w \in B_{T^*,R}$. Using (3.29) with $v \equiv 0$ and (3.30), we obtain that

$$\begin{aligned}
\|\mathcal{M}(w)\|_{E_{T^*}} &\le C_1\|u_0\|_{L_p(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)} + C_2 (T^*)^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} \|w\|_{E_{T^*}} \left(\|w\|_{E_{T^*}}\right)^{\gamma-1}\\
&= \frac{R}{2} + C_2 (T^*)^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} \|w\|^{\gamma}_{E_{T^*}}\\
&\le \frac{R}{2} + C_2 (T^*)^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} R^{\gamma}.
\end{aligned}$$

Therefore, we need to set the condition

$$C_2 (T^*)^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} R^{\gamma} \le \frac{R}{2}. \tag{3.31}$$

On the other hand, let $v, w \in B_{T^*,R}$. Using (3.29) we get

$$\begin{aligned}
\|\mathcal{M}(v)-\mathcal{M}(w)\|_{E_{T^*}} &\le C_2 (T^*)^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} \|v-w\|_{E_{T^*}} \left(\|v\|_{E_{T^*}}+\|w\|_{E_{T^*}}\right)^{\gamma-1}\\
&\le C_2 (T^*)^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} \|v-w\|_{E_{T^*}} (2R)^{\gamma-1}.
\end{aligned}$$

Consequently, we also need

$$C_2 (T^*)^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} (2R)^{\gamma-1} < 1. \tag{3.32}$$

Thus, for sufficiently small T^*, the requirements (3.31) and (3.32) are satisfied. For the uniqueness, we want to conclude that the integral equation (3.26),

$$u(t,x) = \int_{\mathbb{R}^d} Z(t,x-y)u_0(y)dy + \lambda \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)|u(s,y)|^{\gamma-1}u(s,y)dyds,$$

can only have at most one solution (fixed point) in the Banach space

$$E_T := C([0,T]; L_p(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)) \cap C((0,T]; L_\infty(\mathbb{R}^d)),$$

with the norm

$$\|v\|_{E_T} := \sup_{t\in[0,T]} \Big(\|v(t,\cdot)\|_p + \|v(t,\cdot)\|_1\Big) + \sup_{t\in(0,T]} t^{\frac{\alpha d}{\beta p}} \|v(t,\cdot)\|_\infty,$$

whenever $\frac{d(\gamma-1)}{\beta p} < 1$.

Indeed, we suppose that there are two solutions, u_1 and u_2, of (3.26). Using the property given by (2.14), it follows that

$$\|u_1(t) - u_2(t)\|_1 \leq |\lambda| \int_0^t \|Y(t-s,\cdot)\|_1 \||u_1(s,\cdot)|^{\gamma-1}u_1(s,\cdot) - |u_2(s,\cdot)|^{\gamma-1}u_2(s,\cdot)\|_1 ds$$
$$\lesssim |\lambda| \left(\|u_1\|_{E_T} + \|u_2\|_{E_T}\right)^{\gamma-1} \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} \|u_1(s) - u_2(s)\|_1 ds.$$

By applying Lemma 1.2.1, with $g = 0$ and $\vartheta = \frac{\alpha d(\gamma-1)}{\beta p}$, we obtain the desired result. $\square$

3.3 Global solution with small initial condition

In this section, by a global solution of the Cauchy problem (3.1) we understand the solution u of the integral equation (3.26) belonging to the Banach space

$$E := C([0,\infty); L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)) \cap C((0,\infty); L_\infty(\mathbb{R}^d)),$$

with the norm

$$\|v\|_E := \sup_{t\geq 0} \left(\langle t \rangle^{\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)} \|v(t,\cdot)\|_p + \|v(t,\cdot)\|_1 \right) + \sup_{t>0} \{t\}^{\frac{\alpha d}{\beta p}} \langle t \rangle^{\frac{\alpha d}{\beta p'}} \|v(t,\cdot)\|_\infty ,$$

where $1 \leq p' < p$, $\langle t \rangle := \sqrt{1+t^2}$ and $\{t\} := \dfrac{t}{\sqrt{1+t^2}}$.

As in the previous section, we define on E the operator

$$\mathcal{M}(v)(t,x) := \int_{\mathbb{R}^d} Z(t,x-y)u_0(y)dy + \lambda \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)|v(s,y)|^{\gamma-1}v(s,y)dyds,$$

$u_0 \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ is a given data and $v \in E$.

Let $\lambda \in \mathbb{R}$ and $\gamma > 1$. Let $1 = p' < \frac{d}{\beta}(\gamma-1)$ whenever $d < \beta$, or $\frac{d}{\beta} < p' < \frac{d}{\beta}(\gamma-1)$ whenever $d \geq \beta$. Suppose that $\max\left(1, \kappa, \frac{d(\gamma-1)}{\beta}\right) < p < \infty$. These conditions on p and p' guarantee the existence of $1 \leq q < \kappa_1$ such that

$$\frac{1}{l} + \frac{1}{q} = 1 + \frac{1}{r},$$

considering $r \in \{1, p, \infty\}$, where l, q and r are related to Young's convolution inequality, that is,

$$\|Z(t,\cdot) \star u_0\|_r \leq \|Z(t,\cdot)\|_q \|u_0\|_l, \quad t \geq 0.$$

We note that for $r = 1$, this is possible only if $l = q = 1$. Similar situation we have for Y. For $r = 1$ we can not get a factor of time estimating $\|Z \star u_0\|_1$ because $\|Z(t,\cdot)\|_1 = 1$ por all $t \geq 0$.

These conditions, together with Young's convolution inequality and a result of interpolation, yield the following bounds.

$\|Z(t,\cdot)\star u_0\|_1 \le \|Z(t,\cdot)\|_1 \|u_0\|_1 = \|u_0\|_1$ for all $t \ge 0$.

$\|Z(t,\cdot)\star u_0\|_p \le \|Z(t,\cdot)\|_1 \|u_0\|_p = \|u_0\|_p$ for all $0 \le t \le 1$.

$\|Z(t,\cdot)\star u_0\|_p \le \|Z(t,\cdot)\|_q \|u_0\|_{p'} \lesssim t^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)} \max(\|u_0\|_1, \|u_0\|_p)$ for all $t > 1$.

$\|Z(t,\cdot)\star u_0\|_\infty \le \|Z(t,\cdot)\|_q \|u_0\|_p \lesssim t^{-\frac{\alpha d}{\beta p}} \|u_0\|_p$ for all $0 < t \le 1$.

$\|Z(t,\cdot)\star u_0\|_\infty \le \|Z(t,\cdot)\|_q \|u_0\|_{p'} \lesssim t^{-\frac{\alpha d}{\beta p'}} \|u_0\|_{p'} \lesssim t^{-\frac{\alpha d}{\beta p'}} \max(\|u_0\|_1, \|u_0\|_p)$ for $t > 1$.

Therefore,

$$\sup_{t\ge 0}\left(\langle t\rangle^{\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)} \|Z(t,\cdot)\star u_0\|_p + \|Z(t,\cdot)\star u_0\|_1\right) + \sup_{t>0}\{t\}^{\frac{\alpha d}{\beta p}} \langle t\rangle^{\frac{\alpha d}{\beta p'}} \|Z(t,\cdot)\star u_0\|_\infty$$
$$\lesssim (\|u_0\|_1 + \|u_0\|_p).$$

For $0 \le t \le 1$ we have that

$$\begin{aligned}
&\left\|\int_0^t Y(t-s,\cdot)\star |v(s,\cdot)|^{\gamma-1}v(s,\cdot)ds\right\|_1\\
&\le \int_0^t \|Y(t-s,\cdot)\|_1 \||v(s,\cdot)|^{\gamma-1}v(s,\cdot)\|_1 ds\\
&\lesssim \|v\|_E^\gamma \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds\\
&\lesssim \|v\|_E^\gamma \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} ds\\
&\lesssim \|v\|_E^\gamma t^{\alpha-\frac{\alpha d}{\beta p}(\gamma-1)} \lesssim \|v\|_E^\gamma.
\end{aligned}$$

For $t > 1$, we use the fact that $\alpha - \frac{\alpha d(\gamma-1)}{\beta p} > 0$ and that $\alpha - \frac{\alpha d(\gamma-1)}{\beta p'} < 0$. Choosing

$0 < c < \dfrac{t}{2}$ we get that

$$
\begin{aligned}
&\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_1 \\
&\leq \int_0^t \|Y(t-s,\cdot)\|_1 \||v(s,\cdot)|^{\gamma-1} v(s,\cdot)\|_1 ds \\
&\lesssim \|v\|_E^\gamma \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s \rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&\lesssim \|v\|_E^\gamma \left[\int_0^{\frac{t}{2}} (t-s)^{\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s \rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \right. \\
&\qquad \left. + \int_{\frac{t}{2}}^t (t-s)^{\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s \rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \right] \\
&\lesssim \|v\|_E^\gamma \left[\int_0^c s^{\alpha-1-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s \rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \right. \\
&\qquad + \int_c^{\frac{t}{2}} s^{\alpha-1-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s \rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&\qquad \left. + \int_{\frac{t}{2}}^t (t-s)^{\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s \rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \right] \\
&\lesssim \|v\|_E^\gamma \left[\int_0^c s^{\alpha-1-\frac{\alpha d}{\beta p}(\gamma-1)} ds + \int_c^{\frac{t}{2}} s^{\alpha-1-\frac{\alpha d}{\beta p'}(\gamma-1)} ds \right. \\
&\qquad \left. + \left\langle \frac{t}{2} \right\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} \int_{\frac{t}{2}}^t (t-s)^{\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} ds \right] \\
&\lesssim \|v\|_E^\gamma \left[\frac{c^{\alpha-\frac{\alpha d}{\beta p}(\gamma-1)}}{\alpha - \frac{\alpha d(\gamma-1)}{\beta p}} - \frac{c^{\alpha-\frac{\alpha d}{\beta p'}(\gamma-1)}}{\alpha - \frac{\alpha d(\gamma-1)}{\beta p'}} + \left\langle \frac{t}{2} \right\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} \left\langle \frac{t}{2} \right\rangle^{\alpha-\frac{\alpha d}{\beta p}(\gamma-1)} \right] \\
&\lesssim \|v\|_E^\gamma \left[C + \left\langle \frac{t}{2} \right\rangle^{\alpha-\frac{\alpha d}{\beta p'}(\gamma-1)} \right] \lesssim \|v\|_E^\gamma .
\end{aligned}
$$

For $0 \leq t \leq 1$ we get

$$
\begin{aligned}
&\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_p \\
&\leq \int_0^t \|Y(t-s,\cdot)\|_1 \||v(s,\cdot)|^{\gamma-1} v(s,\cdot)\|_p ds \\
&\lesssim \|v\|_E^\gamma \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s \rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} \langle s \rangle^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&\lesssim \|v\|_E^\gamma \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} ds \lesssim \|v\|_E^\gamma .
\end{aligned}
$$

For $t > 1$, the fact that $\|v(s,\cdot)\|_{p'} \leq \max(\|v(s,\cdot)\|_1, \|v(s,\cdot)\|_p)$, $s > 0$, together with the condition $\frac{d}{\beta p'} < 1$, yields

$$\begin{aligned}
&\left\|\int_0^t Y(t-s,\cdot)\star|v(s,\cdot)|^{\gamma-1}v(s,\cdot)ds\right\|_p\\
&\leq \int_0^t \|Y(t-s,\cdot)\|_q \||v(s,\cdot)|^{\gamma-1}v(s,\cdot)\|_{p'}ds\\
&\lesssim \|v\|_E^{\gamma-1}\int_0^t (t-s)^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha-1}s^{-\frac{\alpha d}{\beta p}(\gamma-1)}\langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)}\|v(s,\cdot)\|_{p'}ds\\
&\lesssim \|v\|_E^{\gamma}\int_0^t (t-s)^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha-1}s^{-\frac{\alpha d}{\beta p}(\gamma-1)}\langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)}ds\\
&\lesssim \|v\|_E^{\gamma}t^{\frac{\alpha d}{\beta p}}\int_0^t (t-s)^{-\frac{\alpha d}{\beta p'}+\alpha-1}s^{-\frac{\alpha d}{\beta p}(\gamma-1)}\langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)}ds\\
&\lesssim \|v\|_E^{\gamma}t^{\frac{\alpha d}{\beta p}}\left[\int_0^{\frac{t}{2}} (t-s)^{-\frac{\alpha d}{\beta p'}+\alpha-1}s^{-\frac{\alpha d}{\beta p}(\gamma-1)}\langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)}ds\right.\\
&\left.+\int_{\frac{t}{2}}^t (t-s)^{-\frac{\alpha d}{\beta p'}+\alpha-1}s^{-\frac{\alpha d}{\beta p}(\gamma-1)}\langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)}ds\right]\\
&\leq \|v\|_E^{\gamma}t^{\frac{\alpha d}{\beta p}}\left[\left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta p'}}\int_0^{\frac{t}{2}} (t-s)^{\alpha-1}s^{-\frac{\alpha d}{\beta p}(\gamma-1)}\langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)}ds\right.\\
&\left.+\left\langle\frac{t}{2}\right\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)}\left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta p}(\gamma-1)}\int_{\frac{t}{2}}^t (t-s)^{-\frac{\alpha d}{\beta p'}+\alpha-1}ds\right]\\
&\lesssim \|v\|_E^{\gamma}t^{\frac{\alpha d}{\beta p}}\left[C\left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta p'}}+\left\langle\frac{t}{2}\right\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)}\left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta p}(\gamma-1)-\frac{\alpha d}{\beta p'}+\alpha}\right]\\
&\lesssim \|v\|_E^{\gamma}t^{\frac{\alpha d}{\beta p}-\frac{\alpha d}{\beta p'}}\left[C+\left\langle\frac{t}{2}\right\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)}\left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta p}(\gamma-1)+\alpha}\right]\\
&\lesssim \|v\|_E^{\gamma}t^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)}.
\end{aligned}$$

Now, for all $0 < t \leq 1$, as in the proof of Theorem 3.2.1, we obtain that

$$\begin{aligned}
&\left\|\int_0^t Y(t-s,\cdot)\star|v(s,\cdot)|^{\gamma-1}v(s,\cdot)ds\right\|_\infty\\
&\leq \int_0^t \|Y(t-s,\cdot)\|_{\frac{p}{p-1}}\||v(s,\cdot)|^{\gamma-1}v(s,\cdot)\|_p ds\\
&\lesssim \|v\|_E^{\gamma}\int_0^t (t-s)^{-\frac{\alpha d}{\beta p}+\alpha-1}s^{-\frac{\alpha d}{\beta p}(\gamma-1)}\langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)}ds\\
&\lesssim \|v\|_E^{\gamma}\int_0^t (t-s)^{-\frac{\alpha d}{\beta p}+\alpha-1}s^{-\frac{\alpha d}{\beta p}(\gamma-1)}ds\\
&\lesssim \|v\|_E^{\gamma}t^{-\frac{\alpha d}{\beta p}}.
\end{aligned}$$

For all $t > 1$ we find

$$\begin{aligned}
&\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_\infty \\
&\leq \int_0^t \|Y(t-s,\cdot)\|_{\frac{p'}{p'-1}} \||v(s,\cdot)|^{\gamma-1} v(s,\cdot)\|_{p'} ds \\
&\lesssim \|v\|_E^\gamma \int_0^t (t-s)^{-\frac{\alpha d}{\beta p'}+\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s \rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&\lesssim \|v\|_E^\gamma t^{-\frac{\alpha d}{\beta p'}}.
\end{aligned}$$

Using these bounds and from a straightforward inspection of the proof of Theorem 3.2.1, it follows that the operator $\mathcal{M}$ is well defined on E. From the beginning of this section, we also get the estimate

$$\|Z \star u_0\|_E \leq C_1(\|u_0\|_1 + \|u_0\|_p).$$

In the same way, as in the proof of Theorem 3.2.1, the following estimate

$$\|\mathcal{M}(v) - \mathcal{M}(w)\|_E \leq C_2 \|v - w\|_E \left(\|v\|_E + \|w\|_E\right)^{\gamma-1}$$

holds, considering as before the cases $0 \leq t \leq 1$ and $t > 1$, respectively.

Finally, if $u_0 \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ is sufficiently small, then the operator $\mathcal{M}$ also satisfies the proof of Theorem 3.2.1 with the corresponding closed ball on E, that is, the conditions (3.31) and (3.32) are satisfied without the restriction on the time.

As the uniqueness of the local solution was shown, similar situation happens for a global solution of (3.26) in the Banach space

$$E := C([0,\infty); L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)) \cap C((0,\infty); L_\infty(\mathbb{R}^d)),$$

with the norm

$$\|v\|_E := \sup_{t \geq 0} \left(\langle t \rangle^{\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)} \|v(t,\cdot)\|_p + \|v(t,\cdot)\|_1 \right) + \sup_{t>0} \{t\}^{\frac{\alpha d}{\beta p}} \langle t \rangle^{\frac{\alpha d}{\beta p'}} \|v(t,\cdot)\|_\infty .$$

Recall that $1 \leq p' < p$, $\langle t \rangle := \sqrt{1+t^2}$ and $\{t\} := \dfrac{t}{\sqrt{1+t^2}}$. Again, we use that $\frac{d(\gamma-1)}{\beta p} < 1$ for applying Lemma 1.2.1.

Consequently, we have proved the following result.

Theorem 3.3.1. *Let $\alpha \in (0,1)$ and $\beta \in (0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Let $\lambda \in \mathbb{R}$ and $\gamma > 1$. Let $1 = p' < \frac{d}{\beta}(\gamma-1)$ whenever $d < \beta$, or $\frac{d}{\beta} < p' < \frac{d}{\beta}(\gamma-1)$ whenever $d \geq \beta$. Suppose that $\max\left(1, \kappa, \frac{d(\gamma-1)}{\beta}\right) < p < \infty$. If $u_0 \in L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ is sufficiently small, then the operator $\mathcal{M}$ has a unique fixed point u in E and the optimal time decay estimate*

$$\|u(t)\|_1 + t^{\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)} \|u(t)\|_p + t^{\frac{\alpha d}{\beta p'}} \|u(t)\|_\infty \lesssim (\|u_0\|_1 + \|u_0\|_p)$$

is true for all $t \geq 1$.

3.4 Global solution with non-negative initial condition

In this section we consider the operator $(-\Psi_\beta(-i\nabla), C_0^\infty(\mathbb{R}^d))$ on the Hilbert space $X = (L_2(\mathbb{R}^d), \|\cdot\|_2)$. First, we have that $(-\Psi_\beta(-i\nabla), C_0^\infty(\mathbb{R}^d))$ satisfies the positive maximum principle since its symbol ψ (independent of x) is a continuous and negative definite function on $\mathbb{R}^d$ (Corollary 2.1.1, [29, Theorem 4.5.6]). It is also symmetric because ψ is real. Consequently, $(-\Psi_\beta(-i\nabla), C_0^\infty(\mathbb{R}^d))$ is closable ([63, Theorem 3.6]).

Defining the domain of $-\Psi_\beta(-i\nabla)$ as $D(-\Psi_\beta(-i\nabla)) = \overline{C_0^\infty(\mathbb{R}^d)}^{\|\cdot\|_{\Psi_\beta,L_2}} =: H_2^\beta(\mathbb{R}^d)$, where the closure is respect to the graph norm $\|\cdot\|^2_{\Psi_\beta,L_2} = \|\cdot\|_2^2 + \|\Psi_\beta(-i\nabla)(\cdot)\|_2^2$ ([29, Theorems 2.7.14 and 3.10.3]) and $H_2^\beta(\mathbb{R}^d)$ is an *anisotropic function space* ([29, Section 3.10 and Example 4.1.16]), we get that $(-\Psi_\beta(-i\nabla), H_2^\beta(\mathbb{R}^d)) =: A$ generates a symmetric sub-Markovian semigroup on $L_2(\mathbb{R}^d)$, ([29, Examples 4.1.13 and 4.3.9]).

As we know (see Chapter 1), a sub-Markovian semigroup is a strongly continuous contraction semigroup. Therefore, A is closed and $H_2^\beta(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d)$ ([29, Corollary 4.1.15]). Moreover, A is a self-adjoint operator ([29, Section 4.7]).

Since A is closed, linear, densely defined and self-adjoint on the Hilbert space X, it follows that $-A$ is a normal operator ([59, Definition 13.29]) and $\sigma(-A) \subset \mathbb{R}$. Moreover, $\sigma(-A) \subset [0,\infty)$ because A satisfies [13, Theorem 8.3.2 (i)]. From Parseval's theorem we also have that $-A$ is strictly positive, hence it is 1-1 and satisfies [59, Theorem 13.11 b)]. Therefore, $-A$ is sectorial ([57, Section 8.1]). Besides, X belongs to the class $\mathcal{HT}$ (see [57, definition in page 216, a characterization in page 217 and page 234]).

This shows that the operator $-A$ belongs to $\mathcal{BIP}(X)$ and $\theta_{-A} = 0$ ([57, Definition 8.1 and Section 8.7 c)(i)]), furthermore, it satisfies [57, Theorem 8.7 (i)] with $\omega_{-A} = 0$.

On the other hand, the Laplace-transform of $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$, is $\hat{g_\alpha}(s) = s^{-\alpha}$, $\mathrm{Re}(s) > 0$ ([57, Example 2.1]). This yields

$$\lim_{s\to\infty} |\hat{g_\alpha}(s)| < \infty.$$

The kernel g_α is also 1-regular ([57, Definition 3.4 and Proposition 3.3]) and θ_a-sectorial with $\theta_a = \alpha\frac{\pi}{2}$ ([57, Definition 3.2]). Therefore, $\theta_a + \theta_{-A} < \pi$.

This shows that g_α satisfies [57, Theorem 8.7 parts (ii), (iv) and (v)], with $\omega_a = 0$.

Now, we consider the Volterra equation

$$u(t) = f(t) + \int_0^t g_\alpha(t-s)Au(s)ds, \quad t \in [0,T]. \tag{3.33}$$

The family of bounded linear operators $\{S(t)\}_{t\geq 0}$ on $L_2(\mathbb{R}^d)$, given by

$$S(t)v := Z(t,\cdot) \star v, \tag{3.34}$$

is a **resolvent** for (3.33). That is, S satisfies the following conditions ([57, Definition

1.3]).

$$(S_1): S(0) = I \text{ and } S(t) \text{ is strongly continuous on } [0, \infty),$$
$$(S_2): S(t)v \in D(A) \text{ and } AS(t)v = S(t)Av, \text{ for all } v \in D(A) \text{ and } t \geq 0,$$
$$(S_3): S(t)v = v + \int_0^t g_\alpha(t-s)AS(s)vds, \text{ for all } v \in D(A) \text{ and } t \geq 0.$$

Indeed, it is easy to see that $S(t)$ is a bounded linear operator and (S_1) is satisfied from Theorem 3.1.5. For (S_2), let $t \geq 0$ and $v \in H_2^\beta(\mathbb{R}^d)$. We have that

$$\|S(t)v\|_2 \leq \|v\|_2 < \infty$$

and

$$\begin{aligned}
\int_{\mathbb{R}^d} (1+\|\xi\|^2)^\beta |\widehat{S(t)v}(\xi)|^2 d\xi &= \int_{\mathbb{R}^d} (1+\|\xi\|^2)^\beta E_{\alpha,1}^2(-t^\alpha \psi(\xi))|\widehat{v}(\xi)|^2 d\xi \\
&\leq \int_{\mathbb{R}^d} (1+\|\xi\|^2)^\beta |\widehat{v}(\xi)|^2 d\xi < \infty.
\end{aligned}$$

Besides, denoting by $\langle \cdot, \cdot \rangle$ the usual Euclidean inner product on $\mathbb{R}^d$, we obtain

$$\begin{aligned}
AS(t)v &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot, \xi \rangle} (-\psi(\xi)) \widehat{S(t)v}(\xi) d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot, \xi \rangle} (-\psi(\xi)) E_{\alpha,1}(-t^\alpha \psi(\xi)) \widehat{v}(\xi) d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot, \xi \rangle} E_{\alpha,1}(-t^\alpha \psi(\xi)) (-\psi(\xi)) \widehat{v}(\xi) d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot, \xi \rangle} E_{\alpha,1}(-t^\alpha \psi(\xi)) \widehat{Av}(\xi) d\xi \\
&= S(t)Av.
\end{aligned}$$

The last condition (S_3) also holds because

$$\begin{aligned}
&\int_0^t g_\alpha(t-s)AS(s)vds \\
&= \int_0^t g_\alpha(t-s) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot, \xi \rangle} (-\psi(\xi)) E_{\alpha,1}(-s^\alpha \psi(\xi)) \widehat{v}(\xi) d\xi ds \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot, \xi \rangle} (-\psi(\xi)) \widehat{v}(\xi) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} E_{\alpha,1}(-s^\alpha \psi(\xi)) ds d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot, \xi \rangle} (-\psi(\xi)) \widehat{v}(\xi) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{(-\psi(\xi))^k s^{\alpha k}}{\Gamma(\alpha k+1)} ds d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot, \xi \rangle} (-\psi(\xi)) \widehat{v}(\xi) \sum_{k=0}^{\infty} (-\psi(\xi))^k \int_0^t \frac{(t-s)^{\alpha-1} s^{\alpha k}}{\Gamma(\alpha)\Gamma(\alpha k+1)} ds d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot,\xi\rangle}(-\psi(\xi))\widehat{v}(\xi) \sum_{k=0}^{\infty} (-\psi(\xi))^k \frac{t^{\alpha+\alpha k}}{\Gamma(\alpha k+1+\alpha)} d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot,\xi\rangle}\widehat{v}(\xi) \sum_{k=0}^{\infty} \frac{(-\psi(\xi))^{k+1}(t^\alpha)^{k+1}}{\Gamma(\alpha(k+1)+1)} d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot,\xi\rangle}\widehat{v}(\xi)\left[E_{\alpha,1}(-t^\alpha\psi(\xi)) - 1\right] d\xi \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot,\xi\rangle} E_{\alpha,1}(-t^\alpha\psi(\xi))\widehat{v}(\xi) d\xi - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle \cdot,\xi\rangle}\widehat{v}(\xi) d\xi \\
&= S(t)v - v.
\end{aligned}
$$

Since a strong solution u of (3.33) is also a mild solution, from [57, Proposition 1.2 (i)] it follows that u satisfies

$$
u(t) = \frac{d}{dt} \int_0^t S(s) f(t-s) ds, \quad t \in [0,T]. \tag{3.35}
$$

The equation (3.35) is called the **variation of parameters formula** for the Volterra equation (3.33).

Now, let $\widetilde{u}$ the unique local mild solution of (3.1) given by (3.26) on $[0,T]$, under the assumptions of Theorem 3.2.1. Define

$$
g(t)(x) = g(t,x) := \lambda |\widetilde{u}(t,x)|^{\gamma-1}\widetilde{u}(t,x), \quad 0 \le t \le T, \quad x \in \mathbb{R}^d. \tag{3.36}
$$

We claim that $g_\alpha * g(t) \in L_2(\mathbb{R}^d)$ for $0 \le t \le T$, whenever $\frac{d\gamma}{\beta p} < 1$. Indeed,

$$
\begin{aligned}
\|g_\alpha * g(t)\|_{L_2(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} |(g_\alpha * g(t))(x)|^2 \, dx \right)^{\frac{1}{2}} \\
&\le \left(\int_{\mathbb{R}^d} \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |g(s,x)| ds \right)^2 dx \right)^{\frac{1}{2}} \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_{\mathbb{R}^d} |g(s,x)|^2 dx \right)^{\frac{1}{2}} ds \\
&\lesssim \|\widetilde{u}\|_{E_T}^{\gamma-1} \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} \left(\int_{\mathbb{R}^d} |\widetilde{u}(s,x)|^2 dx \right)^{\frac{1}{2}} ds \\
&\lesssim \|\widetilde{u}\|_{E_T}^{\gamma-1} \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} \|\widetilde{u}(s,\cdot)\|_2 ds \\
&\lesssim \|\widetilde{u}\|_{E_T}^{\gamma-1} \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} \max(\|\widetilde{u}(s,\cdot)\|_1, \|\widetilde{u}(s,\cdot)\|_\infty) ds \\
&\lesssim \|\widetilde{u}\|_{E_T}^{\gamma-1} \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} (\|\widetilde{u}(s,\cdot)\|_1 + \|\widetilde{u}(s,\cdot)\|_\infty) ds \\
&\lesssim \|\widetilde{u}\|_{E_T}^{\gamma} \int_0^t (t-s)^{\alpha-1} s^{-\frac{\alpha d(\gamma-1)}{\beta p}} \left(1 + s^{-\frac{\alpha d}{\beta p}}\right) ds \\
&\lesssim \|\widetilde{u}\|_{E_T}^{\gamma} \left(t^{\alpha-\frac{\alpha d(\gamma-1)}{\beta p}} + t^{\alpha-\frac{\alpha d\gamma}{\beta p}} \right) < \infty.
\end{aligned}
$$

From here, we get

$$g_\alpha * g(0) \equiv 0.$$

In the same way it can be shown that $g_\alpha * g \in L_2([0,T]; L_2(\mathbb{R}^d))$. We additionally have that $g \in L_2([0,T]; L_2(\mathbb{R}^d))$ whenever $\frac{2\alpha d\gamma}{\beta p} < 1$.

Let $u_0 \in D(A)$. Consider the equations of Volterra

$$u(t) = g_\alpha * g(t) + \int_0^t g_\alpha(t-s)Au(s)ds \tag{3.37}$$

and

$$u(t) = u_0 + \int_0^t g_\alpha(t-s)Au(s)ds, \tag{3.38}$$

for $t \in [0,T]$.

Conclusions (a) and (b) of [57, Theorem 8.7], using $B \equiv 0$ and the Banach space $X_A = (H_2^\beta(\mathbb{R}^d), \|\cdot\|_{\Psi_\beta, L_2})$, imply that (3.37) and (3.38) have a unique a.e. strong solution u_1 and u_2, respectively.

Therefore, $u := u_1 + u_2 \in L_2([0,T]; X_A)$ is the unique a.e. strong solution of the Volterra equation

$$u(t) = u_0 + g_\alpha * g(t) + \int_0^t g_\alpha(t-s)Au(s)ds \tag{3.39}$$

and satisfies (3.35) with $f(t) := u_0 + g_\alpha * g(t)$, $t \in [0,T]$.

Next, we need to prove the following result using the operators $S(t)$ given by (3.34).

Lemma 3.4.1. *Let $\alpha \in (0,1)$ and $\beta \in (0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Let $\lambda \in \mathbb{R}$ and $\gamma > 1$. Assume that u is a fixed point of the operator given by (3.27) on E_T with $u_0 \in L_2(\mathbb{R}^d)$. If $\frac{d\gamma}{\beta p} < 1$, then u satisfies*

$$u(t) = \frac{d}{dt}\int_0^t S(s)f(t-s)ds,$$

where $f(t) := u_0 + g_\alpha * g(t)$ and g like (3.36), $t \in [0,T]$.

Proof. Let $x \in \mathbb{R}^d$. We define

$$\begin{aligned}
F(t,s) &:= \int_{\mathbb{R}^d} g_\alpha * Z(\cdot, x-y)(t-s)g(s,y)dy \\
&= \int_{\mathbb{R}^d} g(s,y)\left[\int_0^{t-s} \frac{(t-s-\tau)^{\alpha-1}}{\Gamma(\alpha)} Z(\tau, x-y)d\tau\right]dy \\
&= \int_0^{t-s} \frac{(t-s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \int_{\mathbb{R}^d} Z(\tau, x-y)g(s,y)dyd\tau \\
&\lesssim \|u\|_{E_T}^\gamma \int_0^{t-s} \frac{(t-s-\tau)^{\alpha-1}}{\Gamma(\alpha)} s^{-\frac{\alpha d\gamma}{\beta p}} \underbrace{\int_{\mathbb{R}^d} Z(\tau, x-y)dyd\tau}_{1} \\
&\lesssim \|u\|_{E_T}^\gamma s^{-\frac{\alpha d\gamma}{\beta p}}(t-s)^\alpha.
\end{aligned}$$

That is,

$$g_\alpha * Z(\cdot, x - \cdot\cdot)(t-s)g(s, \cdot\cdot) \in L_1(\mathbb{R}^d), \quad 0 < s \le t, \quad t > 0. \tag{3.40}$$

Without loss of generality, let $0 < t_0 < t$. The same arguments as in the proof of Theorem 3.1.5, part (i), but using the bounds of Y given in Lemma 3.1.2, yield

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} Y(t-s, x-y) g(s,y) dy - \int_{\mathbb{R}^d} Y(t_0 - s, x-y) g(s,y) dy \right| \\
&\le \int_{\mathbb{R}^d} |Y(t-s, x-y) - Y(t_0 - s, x-y)|\, |g(s,y)| dy \\
&\lesssim \|u\|_{E_T}^{\gamma} s^{-\frac{\alpha d \gamma}{\beta p}} \int_{\mathbb{R}^d} |Y(t-s, x-y) - Y(t_0-s, x-y)|\, dy \\
&\lesssim \|u\|_{E_T}^{\gamma} s^{-\frac{\alpha d \gamma}{\beta p}} \|Y(t-s, x-\cdot) - Y(t_0 - s, x - \cdot)\|_1 \\
&\lesssim \|u\|_{E_T}^{\gamma} s^{-\frac{\alpha d \gamma}{\beta p}} |t - t_0| (t_0 - s)^{\alpha - 2} \to 0
\end{aligned}$$

whenever $t \to t_0$.

This, together with (3.40) and Lemma 3.1.6, proves that

$$\frac{\partial F}{\partial t}(t,s) = \int_{\mathbb{R}^d} Y(t-s, x-y) g(s,y) dy, \tag{3.41}$$

which is continuous w.r.t. $t \in (0, T]$, for $0 < s < t$. Therefore,

$$\int_0^t \int_{\mathbb{R}^d} Y(t-s, x-y) g(s,y) dy ds = \int_0^t \frac{\partial F}{\partial t}(t,s) ds.$$

From the previous work to (3.40) we also obtain

$$\left| \int_{t_0}^t F(t,s) ds \right| \lesssim \int_{t_0}^t s^{-\frac{\alpha d \gamma}{\beta p}} (t-s)^{\alpha} ds \lesssim t_0^{-\frac{\alpha d \gamma}{\beta p}} (t - t_0)^{\alpha + 1}$$

and thus

$$\lim_{t \to t_0} \frac{1}{t - t_0} \int_{t_0}^t F(t,s) ds = 0.$$

The mean-value theorem and the dominated convergence theorem yield

$$\begin{aligned}
\frac{d}{dt} \int_0^t F(t,s) ds \bigg|_{t = t_0} &= \lim_{t \to t_0} \frac{1}{t - t_0} \left(\int_0^t F(t,s) ds - \int_0^{t_0} F(t_0, s) ds \right) \\
&= \lim_{t \to t_0} \frac{1}{t - t_0} \int_0^{t_0} [F(t,s) - F(t_0, s)]\, ds + \lim_{t \to t_0} \frac{1}{t - t_0} \int_{t_0}^t F(t,s) ds \\
&= \lim_{t \to t_0} \int_0^{t_0} \frac{\partial F}{\partial t}(t_c, s) ds \\
&= \int_0^{t_0} \lim_{t \to t_0} \frac{\partial F}{\partial t}(t_c, s) ds \\
&= \int_0^{t_0} \frac{\partial F}{\partial t}(t_0, s) ds < \infty,
\end{aligned}$$

because $t_0 < t_c < t$ and the equality (3.41) implies

$$\begin{aligned}
\left|\frac{\partial F}{\partial t}(t_c, s)\right| &\lesssim \int_{\mathbb{R}^d} Y(t_c - s, x - y)|g(s, y)|dy \\
&\lesssim \|u\|_{E_T}^{\gamma} s^{-\frac{\alpha d\gamma}{\beta p}} \int_{\mathbb{R}^d} Y(t_c - s, x - y)dy \\
&\lesssim \|u\|_{E_T}^{\gamma} s^{-\frac{\alpha d\gamma}{\beta p}} \frac{(t_c - s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\lesssim \|u\|_{E_T}^{\gamma} s^{-\frac{\alpha d\gamma}{\beta p}} (t_0 - s)^{\alpha-1}.
\end{aligned}$$

Consequently,

$$\frac{d}{dt}\int_0^t F(t, s)ds = \int_0^t \frac{\partial F}{\partial t}(t, s)ds.$$

In view of (3.41), the non-linear part of (3.27) can be written as

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y)g(s, y)dyds &= \int_0^t \frac{\partial}{\partial t} F(t, s)ds \\
&= \frac{d}{dt}\int_0^t F(t, s)ds \\
&= \frac{d}{dt}\int_0^t \int_{\mathbb{R}^d} g_\alpha * Z(\cdot, x - y)(t - s)g(s, y)dyds \\
&= \frac{d}{dt}\int_0^t \int_{\mathbb{R}^d} g_\alpha * Z(\cdot, x - y)(s)g(t - s, y)dyds.
\end{aligned}$$

Using Fubini and convolution respect to the time, the integral over $\mathbb{R}^d$ is

$$\int_0^s \int_{\mathbb{R}^d} \frac{(s - \tau)^{\alpha-1}}{\Gamma(\alpha)} Z(\tau, x - y)g(t - s, y)dyd\tau$$

and hence

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} g_\alpha * Z(\cdot, x - y)(s)g(t - s, y)dyds \\
&= \int_0^t \int_0^s \int_{\mathbb{R}^d} \frac{(s - \tau)^{\alpha-1}}{\Gamma(\alpha)} Z(\tau, x - y)g(t - s, y)dyd\tau ds \\
&= \int_0^t \int_\tau^t \int_{\mathbb{R}^d} \frac{(s - \tau)^{\alpha-1}}{\Gamma(\alpha)} Z(\tau, x - y)g(t - s, y)dydsd\tau \\
&= \int_0^t \int_{\mathbb{R}^d} Z(\tau, x - y) \int_\tau^t \frac{(s - \tau)^{\alpha-1}}{\Gamma(\alpha)} g(t - s, y)dsdyd\tau \\
&= \int_0^t \int_{\mathbb{R}^d} Z(\tau, x - y) \int_0^{t-\tau} \frac{(t - \tau - s)^{\alpha-1}}{\Gamma(\alpha)} g(s, y)dsdyd\tau \\
&= \int_0^t \int_{\mathbb{R}^d} Z(\tau, x - y)g_\alpha * g(\cdot, y)(t - \tau)dyd\tau \\
&= \int_0^t Z(\tau, \cdot\cdot) \star (g_\alpha * g(\cdot, \cdot\cdot)(t - \tau))(x)d\tau \\
&= \int_0^t S(\tau)(g_\alpha * g(\cdot, \cdot\cdot)(t - \tau))(x)d\tau.
\end{aligned}$$

Therefore, the operator (3.27) evaluated in u has the form

$$u(t) = \frac{d}{dt}\int_0^t S(s)\,(u_0 + g_\alpha * g(\cdot,\cdot\cdot)(t-s))\,ds, \quad t \in [0,T].$$

□

By Lemma 3.4.1 and uniqueness, we obtain $\widetilde{u} = u$. Besides, (3.39) is equivalent to

$$\begin{aligned} \partial_t^\alpha (u-u_0)(t,x) + \Psi_\beta(-i\nabla)u(t,x) = \lambda |u(t,x)|^{\gamma-1}u(t,x), \quad t \in (0,T],\ x \in \mathbb{R}^d,\\ u(t,x)|_{t=0} = u_0(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{3.42}$$

Indeed,

$$\begin{aligned} &u(t) - \int_0^t g_\alpha(t-s)Au(s)ds = u_0 + g_\alpha * g(t)\\ &\Leftrightarrow g_{1-\alpha} * u(t) + g_{1-\alpha} * g_\alpha * \Psi_\beta(-i\nabla)u(t) = g_{1-\alpha} * (u_0 + g_\alpha * g)(t)\\ &\Leftrightarrow g_{1-\alpha} * (u-u_0)(t) + 1 * \Psi_\beta(-i\nabla)u(t) = 1 * g(t)\\ &\Leftrightarrow \frac{d}{dt} g_{1-\alpha} * (u-u_0)(t) + \Psi_\beta(-i\nabla)u(t) = g(t). \end{aligned}$$

Now, we define $u^+(t,x) = \max(u(t,x),0)$ and $u^-(t,x) = \max(-u(t,x),0)$.

From [70, Section 2], it is known that if $v \in L_2([0,T];\mathbb{R})$, $g_{1-\alpha} * v \in W_2^1([0,T];\mathbb{R})$ and $(g_{1-\alpha} * v)(0) = 0$, then the operator $\partial_t^\alpha v := \frac{d}{dt}(g_{1-\alpha} * v)$ has a Yosida approximation $\frac{d}{dt}(g_{1-\alpha,n} * v)$ in $L_2([0,T];\mathbb{R})$ as $n \to \infty$, with nonnegative and nonincreasing $g_{1-\alpha,n} \in W_1^1([0,T];\mathbb{R})$ for all $n \in \mathbb{N}$. From this work, one can also derive

$$v^- \frac{d}{dt}(g_{1-\alpha,n} * v)(t) \le -\frac{1}{2}\frac{d}{dt}(g_{1-\alpha,n} * (v^-)^2)(t) \quad a.e.\ t \in (0,T), \quad n \in \mathbb{N}. \tag{3.43}$$

Next, we use these results to prove that u is a local positive solution a.e. of (3.1), whenever $u_0 \ge 0$, but non zero, and $\lambda < 0$. By contradiction, suppose that $u < 0$ somewhere on $(0,T] \times \mathbb{R}^d$. Let $x \in \mathbb{R}^d$. In order to apply (3.43) to $u(\cdot,x)$, we need the condition $\frac{2\alpha d}{\beta p} < 1$ to get $u(\cdot,x) \in L_2([0,T];\mathbb{R})$. For the requirement $g_{1-\alpha} * u(\cdot,x) \in L_2([0,T];\mathbb{R})$, we have that

$$\begin{aligned} \|g_{1-\alpha} * u(\cdot,x)\|_2^2 &= \int_0^T |g_{1-\alpha} * u(t,x)|^2 dt\\ &= \int_0^T \left| \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} u(s,x)ds \right|^2 dt\\ &\lesssim \|u\|_{E_T}^2 \int_0^T \left[\int_0^t (t-s)^{-\alpha} s^{-\frac{\alpha d}{\beta p}} ds \right]^2 dt \lesssim \int_0^T \left[t^{1-\alpha-\frac{\alpha d}{\beta p}} \right]^2 dt \end{aligned}$$

which is finite because $0 < \frac{2\alpha d}{\beta p} < 1$ and $0 < \alpha < 1$.

We also find conditions such that $\frac{d}{dt}(g_{1-\alpha} * u)(\cdot, x) \in L_2([0,T];\mathbb{R})$. For this purpose, we already know that $u \in L_2([0,T]; X_A)$ is a strong solution of (3.39) and satisfies the equation (3.42). Conclusions (a) and (b) of [57, Theorem 8.7]) yield

$$\int_0^T |\Psi_\beta(-i\nabla)u(t,x)|^2 dt < \infty \quad a.e., \quad x \in \mathbb{R}^d.$$

If $\frac{2\alpha d\gamma}{\beta p} < 1$, we also get that $|u|^{\gamma-1}u(\cdot,x) \in L_2([0,T];\mathbb{R})$ and $\frac{d}{dt}(g_{1-\alpha} * u_0)(\cdot,x) \in L_2([0,T];\mathbb{R})$ whenever $0 < \alpha < \frac{1}{2}$. Finally, it can be readily checked that $(g_{1-\alpha} * u)(0,x) = 0$ whenever $\alpha + \frac{\alpha d}{\beta p} < 1$.

This work allows one to employ the Yosida approximation of $g_{1-\alpha}$ in $L_2([0,T];\mathbb{R})$ with u. Using (3.43) we obtain

$$u^- \frac{d}{dt}(g_{1-\alpha,n} * u)(t,x) \leq -\frac{1}{2}\frac{d}{dt}(g_{1-\alpha,n} * (u^-)^2)(t,x) \quad a.e.\ t \in (0,T).$$

Besides, $(g_{1-\alpha,n} * (u^-)^2)(0,x) = 0$ for all $n \in \mathbb{N}$ (see e.g., [70, Formula 8] and [33, Formula 10]). Consequently,

$$\begin{aligned}
&\int_0^T u^- \frac{d}{dt}(g_{1-\alpha,n} * u)(t,x)dt \\
&\leq -\frac{1}{2}\int_0^T \frac{d}{dt}(g_{1-\alpha,n} * (u^-)^2)(t,x)dt \\
&= -\frac{1}{2}(g_{1-\alpha,n} * (u^-)^2)(T,x) + \frac{1}{2}(g_{1-\alpha,n} * (u^-)^2)(0,x) \\
&= -\frac{1}{2}(g_{1-\alpha,n} * (u^-)^2)(T,x) \\
&\leq 0.
\end{aligned}$$

Thus,

$$\int_0^T u^- \frac{d}{dt}(g_{1-\alpha} * u)(t,x)dt \leq \int_0^T u^- \frac{d}{dt}((g_{1-\alpha} - g_{1-\alpha,n}) * u)(t,x)dt$$

and applying Hölder we conclude that

$$\begin{aligned}
&\int_0^T u^- \frac{d}{dt}(g_{1-\alpha} * u)(t,x)dt \\
&\leq \left(\int_0^T (u^-)^2(t,x)dt\right)^{\frac{1}{2}} \left(\int_0^T \left|\frac{d}{dt}((g_{1-\alpha} - g_{1-\alpha,n}) * u)(t,x)\right|^2 dt\right)^{\frac{1}{2}} \\
&< \|u^-(\cdot,x)\|_2 \varepsilon.
\end{aligned}$$

This shows that

$$\int_0^T u^- \frac{d}{dt}(g_{1-\alpha} * (u - u_0))(t,x)dt \leq 0 \tag{3.44}$$

a.e., $x \in \mathbb{R}^d$.

On the other hand, if $v, w \in E_T$ we have that $v(t,\cdot), w(t,\cdot) \in L_2(\mathbb{R}^d)$ whenever $t > 0$. Besides, $-\Psi_\beta(-i\nabla)$ satisfies [63, Theorem 3.6] and we can write

$$\langle \Psi_\beta(-i\nabla)v(t,\cdot), w(t,\cdot)\rangle_2 = \frac{1}{2}\int_{\mathbb{R}^{2d}} (v(t,y)-v(t,x))(w(t,y)-w(t,x))\nu(dy)dx \quad (3.45)$$

with a positive kernel ν on $\mathbb{R}^d$ such that $\displaystyle\int_{\mathbb{R}^d\setminus\{x\}} \frac{\|y-x\|^2}{1+\|y-x\|^2}\nu(dy) < \infty$.

Using (3.42), we obtain

$$\begin{aligned} u^-(t,x)\partial_t^\alpha(u-u_0)(t,x) &= -\, u^-(t,x)\Psi_\beta(-i\nabla)u(t,x) + \lambda u^-(t,x)|u(t,x)|^{\gamma-1}u(t,x) \\ &= -\, u^-(t,x)\Psi_\beta(-i\nabla)u^+(t,x) + u^-(t,x)\Psi_\beta(-i\nabla)u^-(t,x) \\ &\quad -\lambda |u(t,x)|^{\gamma-1}(u^-)^2(t,x). \end{aligned}$$

Integrating over $[0,T]\times\mathbb{R}^d$ and using Fubini in a convenient form, we get

$$\begin{aligned} \int_{\mathbb{R}^d}\int_0^T u^-(t,x)\partial_t^\alpha(u-u_0)(t,x)dtdx &= -\int_0^T\int_{\mathbb{R}^d} u^-(t,x)\Psi_\beta(-i\nabla)u^+(t,x)dxdt \\ &\quad + \int_0^T\int_{\mathbb{R}^d} u^-(t,x)\Psi_\beta(-i\nabla)u^-(t,x)dxdt \\ &\quad - \lambda\int_0^T\int_{\mathbb{R}^d} |u(t,x)|^{\gamma-1}(u^-)^2(t,x)dxdt. \end{aligned}$$

From (3.44) and (3.45) it follows that the left-hand side is non-positive and the right-hand side is strictly positive, respectively. This contradiction shows that $u \geq 0$ a.e. For the case $\frac{1}{2} \leq \alpha < 1$ we set the parameter $\bar{p}$, such that $1 < \bar{p} < \frac{1}{\alpha}$. Due to this choice, we have that $\bar{p} < \frac{\bar{p}}{\bar{p}-1}$ and that $2 < \frac{\bar{p}}{\bar{p}-1}$. Thereby, we can apply similar arguments, i.e., Yosida approximation of $g_{1-\alpha}$ in $L_{\bar{p}}([0,T];\mathbb{R})$ and [57, Theorem 8.7] in $L_{\bar{p}}([0,T];L_2(\mathbb{R}^d))$, for obtaining again that $u \geq 0$ a.e. Whenever $0 < \alpha < \frac{1}{2}$, we fix $\bar{p} = 2$. We are now in a position to show the following theorem.

Theorem 3.4.1. *Let $\alpha \in (0,1)$ and $\beta \in (0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Let $\lambda < 0$, $\gamma > 1$. Suppose that $\max\left(1, \kappa, \frac{d\gamma}{\beta}\right) < p < \infty$, $\frac{\bar{p}\alpha d}{(\bar{p}-1)\beta p} < 1$, $\frac{\bar{p}\alpha d\gamma}{\beta p} < 1$ and $\alpha + \frac{\alpha d}{\beta p} < 1$. If $u_0 \in L_p(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)\cap H_2^\beta(\mathbb{R}^d)$ is non-negative a.e., then there exists a unique non-negative global solution $u \in C([0,\infty);L_p(\mathbb{R}^d)\cap L_1(\mathbb{R}^d))\cap C((0,\infty);L_\infty(\mathbb{R}^d))$ to the Cauchy problem (3.1). Moreover, estimate*

$$\|u(t)\|_1 + t^{\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)}\|u(t)\|_p + t^{\frac{\alpha d}{\beta p'}}\|u(t)\|_\infty \lesssim (\|u_0\|_1 + \|u_0\|_p)$$

is true for all $t \geq 1$, with p' as in Theorem 3.3.1.

Proof. As in Section 3.2, one finds that there exists a unique local solution $u \in E_{T^*}$ for some $T^* > 0$. This solution is non-negative a.e., as discussed above, and satisfies

$$u(t,x) = \int_{\mathbb{R}^d} Z(t,x-y)u_0(y)dy + \lambda\int_0^t\int_{\mathbb{R}^d} Y(t-s,x-y)|u(s,y)|^{\gamma-1}u(s,y)dyds.$$

The positivity of Z and Y yields

$$u(t,x) \leq \int_{\mathbb{R}^d} Z(t,x-y)u_0(y)dy.$$

Therefore,

$\|u(t,\cdot)\|_1 \leq \|Z(t,\cdot)\star u_0\|_1 \leq \|u_0\|_1$ for $t\in[0,T^*]$,
$\|u(t,\cdot)\|_p \leq \|Z(t,\cdot)\star u_0\|_p \leq \|u_0\|_p$ for $t\in[0,T^*]$,
$\|u(t,\cdot)\|_\infty \leq \|Z(t,\cdot)\star u_0\|_\infty \lesssim t^{-\frac{\alpha d}{\beta p}}\|u_0\|_p$ for $t\in(0,T^*]$
and thus we obtain

$$\|u\|_{E_{T^*}} \leq C\|u_0\|_{L_p(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)}. \tag{3.46}$$

We note that the constant C is independent of T^*, that is, we can apply Theorem 3.2.1 on $[T^*,T_1]$, with $0<T^*<T_1$, and the extended solution $u\in E_{T_1}$ also satisfies (3.46).

In particular, $\|u(T^*)\|_{L_p(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)} \leq C\|u_0\|_{L_p(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)}$. This estimate allows to prolong the local solution for all times $t>0$ (see [24, Theorem 1.20]). Since $u(T^*)$ is non-negative and it is the new initial condition on $[T^*,T_1]$, it follows that $u\in E_{T_1}$ is also non-negative. Consequently, the global solution is non-negative.

On the other hand, as in proof of Theorem 3.3.1, we also get for all $t\geq 1$,

$$\|u(t,\cdot)\|_p \leq \|Z(t,\cdot)\star u_0\|_p \lesssim t^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)}\max(\|u_0\|_1,\|u_0\|_p),$$
$$\|u(t,\cdot)\|_\infty \leq \|Z(t,\cdot)\star u_0\|_\infty \lesssim t^{-\frac{\alpha d}{\beta p'}}\max(\|u_0\|_1,\|u_0\|_p).$$

Thus, we obtain the desired estimate. $\square$

3.5 Asymptotic behaviour for global solutions

In this section we study the L_p-decay of a global solution, which was obtained in Section 3.3 and in Section 3.4, respectively. We recall that in the first case (Theorem 3.3.1), a small initial data is required. In the other case (Theorem 3.4.1), $\lambda<0$ and initial data non-negative are required, which yield a global solution non-negative.

Lemma 3.5.1. *Let $\alpha\in(0,1)$ and $\beta\in(1,2)$. Assume the hypothesis $(\mathcal{H}_2)$ holds. Then there exists a positive constant C for all $t>0$ and $y\in\mathbb{R}^d$, such that the estimate*

$$\|Z(t,\cdot-y)-Z(t,\cdot)\|_q \leq C\|y\|\|\nabla Z(t,\cdot)\|_q \lesssim \|y\|t^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{q}\right)-\frac{\alpha}{\beta}}$$

is true for $1\leq q<\frac{d}{d+1-\beta}$.

Proof. By application of bounds given in Lemma 3.1.3, we have

$$|Z(t,x-y)-Z(t,x)| \leq C\|y\|\begin{cases} t^{-\alpha}\|x-\varepsilon y\|^{\beta-(d+1)} & \text{if} \quad \|x-\varepsilon y\|\leq t^{\frac{\alpha}{\beta}} \\ t^{\alpha}\|x-\varepsilon y\|^{-\beta-(d+1)} & \text{if} \quad \|x-\varepsilon y\|\geq t^{\frac{\alpha}{\beta}}\end{cases}$$
$$=: C\|y\|D(t,x-\varepsilon y),$$

where $\varepsilon \in (0,1)$. Using an argument as in Hardy's inequality proof, we get

$$\begin{aligned}
\|Z(t,\cdot - y) - Z(t,\cdot)\|_q &= \left(\int_{\mathbb{R}^d} |Z(t,x-y) - Z(t,x)|^q dx\right)^{\frac{1}{q}} \\
&\lesssim \|y\| \left(\int_{\mathbb{R}^d} D^q(t,x-\varepsilon y)dx\right)^{\frac{1}{q}} \\
&\lesssim \|y\| \left(\int_{\mathbb{R}^d} \left(\int_0^1 D(t,x-sy)ds\right)^q dx\right)^{\frac{1}{q}} \\
&\lesssim \|y\| \int_0^1 \left(\int_{\mathbb{R}^d} D^q(t,x-sy)dx\right)^{\frac{1}{q}} ds \\
&\lesssim \|y\| \|D(t,\cdot)\|_q.
\end{aligned}$$

We note that $\|D(t,\cdot)\|_q$ can be estimated in the same way as $Z(t,\cdot)$ in Theorem 3.1.1 and that $\|\nabla Z(t,\cdot)\|_q \lesssim \|D(t,\cdot)\|_q$. The mean-value inequality completes the proof. □

On the other hand, from [16] it is known the following decomposition lemma.

Lemma 3.5.2. *Assume that* $1 \leq r < \frac{d}{d-1}$, $f \in L_1(\mathbb{R}^d)$ *and* $\|\cdot\| f \in L_r(\mathbb{R}^d)$, *then there exists a vectorial function* $\boldsymbol{F} \in L_r(\mathbb{R}^d;\mathbb{R}^d)$ *such that*

$$f = \left(\int_{\mathbb{R}^d} f(y)dy\right)\delta_0 + div\boldsymbol{F}.$$

in the distributional sense and

$$\|\boldsymbol{F}\|_r \leq C(q,d)\|\|\cdot\| f\|_r.$$

Using this with $f = u_0$, we find that

$$\begin{aligned}
Z(t,\cdot) \star u_0(x) &= \left(\int_{\mathbb{R}^d} u_0(y)dy\right) Z(t,\cdot) \star \delta_0(x) + Z(t,\cdot) \star div\boldsymbol{F}(x) \\
&= \left(\int_{\mathbb{R}^d} u_0(y)dy\right) Z(t,x) + \nabla Z(t,\cdot) \star \boldsymbol{F}(x).
\end{aligned}$$

Let $A = \displaystyle\int_{\mathbb{R}^d} u_0(y)dy$. Young's inequality for convolutions yields

$$\begin{aligned}
\|Z(t,\cdot) \star u_0 - AZ(t,\cdot)\|_p &\lesssim \|\nabla Z(t,\cdot)\|_q \|\boldsymbol{F}\|_r \\
&\lesssim \|\nabla Z(t,\cdot)\|_q \|\|\cdot\| u_0\|_r
\end{aligned}$$

and we get

$$\|Z(t,\cdot) \star u_0 - AZ(t,\cdot)\|_p \lesssim t^{-\frac{\alpha d}{\beta}(1-\frac{1}{q})-\frac{\alpha}{\beta}} \|\|\cdot\| u_0\|_r. \tag{3.47}$$

Now, the idea is to find conditions such that

$$\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}, \quad 1 \leq q < \frac{d}{d+1-\beta}, \quad 1 \leq r < \frac{d}{d-1}. \tag{3.48}$$

First, we need $\beta \in (1,2)$ since Lemma 3.5.1 requires this for choices of q. Besides, in the case $d=1$ is enough $p<\infty$ but in the case $d \geq 2$ we need $p < \frac{d}{d-\beta}$. In this way it is possible to get the estimate (3.47) and the required upper bound on p allows one to achieve both the L_p-norm and the L_q-norm for kernel Y. Therefore, the estimate

$$\left\| \lambda \int_0^t Y(t-s,\cdot\cdot) \star |u(s,\cdot\cdot)|^{\gamma-1}u(s,\cdot\cdot)(\cdot)ds - BY(t,\cdot) \right\|_p \lesssim t^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha-1} \tag{3.49}$$

is true for $t \geq 1$, with $B = \lambda \displaystyle\int_0^\infty \int_{\mathbb{R}^d} |u(s,y)|^{\gamma-1}u(s,y)dyds$, whenever $\frac{\alpha d}{\beta p'}(\gamma-1) > 1$. Here, p' is the same as in Theorems 3.3.1 and 3.4.1.

Indeed, we have that

$$\begin{aligned}
&\int_0^t Y(t-s,\cdot\cdot) \star |u(s,\cdot\cdot)|^{\gamma-1}u(s,\cdot\cdot)(x)ds - Y(t,x)\int_0^\infty \int_{\mathbb{R}^d} |u(s,y)|^{\gamma-1}u(s,y)dyds \\
=&\int_0^{\frac{t}{2}} Y(t-s,\cdot\cdot) \star |u(s,\cdot\cdot)|^{\gamma-1}u(s,\cdot\cdot)(x)ds + \int_{\frac{t}{2}}^t Y(t-s,\cdot\cdot) \star |u(s,\cdot\cdot)|^{\gamma-1}u(s,\cdot\cdot)(x)ds \\
&- \int_0^{\frac{t}{2}} (Y(t,x) - Y(t-s,x) + Y(t-s,x)) \int_{\mathbb{R}^d} |u(s,y)|^{\gamma-1}u(s,y)dyds \\
&- \int_{\frac{t}{2}}^\infty Y(t,x) \int_{\mathbb{R}^d} |u(s,y)|^{\gamma-1}u(s,y)dyds \\
=&\int_0^{\frac{t}{2}} Y(t-s,\cdot\cdot) \star |u(s,\cdot\cdot)|^{\gamma-1}u(s,\cdot\cdot)(x)ds + \int_{\frac{t}{2}}^t Y(t-s,\cdot\cdot) \star |u(s,\cdot\cdot)|^{\gamma-1}u(s,\cdot\cdot)(x)ds \\
&- \int_0^{\frac{t}{2}} Y(t-s,x) \int_{\mathbb{R}^d} |u(s,y)|^{\gamma-1}u(s,y)dyds - \int_{\frac{t}{2}}^\infty Y(t,x) \int_{\mathbb{R}^d} |u(s,y)|^{\gamma-1}u(s,y)dyds \\
&- \int_0^{\frac{t}{2}} (Y(t,x) - Y(t-s,x)) \int_{\mathbb{R}^d} |u(s,y)|^{\gamma-1}u(s,y)dyds \\
=:&J_1(t,x) + J_2(t,x) - J_3(t,x) - J_4(t,x) - J_5(t,x).
\end{aligned}$$

Now, we estimate the L_p-norm for J_k, $k=1,\cdots,5$, recalling that $\frac{d}{\beta p}(\gamma-1) < 1$, $\frac{d}{\beta p'} < 1$ and $1 \leq p' < p$.

$$\begin{aligned}
\|J_1(t,\cdot)\|_p &\leq \int_0^{\frac{t}{2}} \|Y(t-s,\cdot\cdot) \star |u(s,\cdot\cdot)|^{\gamma-1}u(s,\cdot\cdot)\|_p ds \\
&\leq \int_0^{\frac{t}{2}} \|Y(t-s,\cdot\cdot)\|_p \||u(s,\cdot\cdot)|^{\gamma-1}u(s,\cdot\cdot)\|_1 ds \\
&\lesssim \|u\|_E^\gamma \int_0^{\frac{t}{2}} (t-s)^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s \rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&\lesssim \|u\|_E^\gamma \left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-1} \int_0^{\frac{t}{2}} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s \rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds.
\end{aligned}$$

Choosing $0 < c < \frac{t}{2}$ we get

$$\begin{aligned}
&\int_0^{\frac{t}{2}} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&= \int_0^{c} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds + \int_c^{\frac{t}{2}} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&\lesssim \int_0^{c} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} ds + \int_c^{\frac{t}{2}} s^{-\frac{\alpha d}{\beta p'}(\gamma-1)} ds \\
&\lesssim \frac{1}{1-\frac{\alpha d}{\beta p}(\gamma-1)} c^{1-\frac{\alpha d}{\beta p}(\gamma-1)} + \frac{1}{1-\frac{\alpha d}{\beta p'}(\gamma-1)} \left(\left(\frac{t}{2}\right)^{1-\frac{\alpha d}{\beta p'}(\gamma-1)} - c^{1-\frac{\alpha d}{\beta p'}(\gamma-1)} \right).
\end{aligned}$$

We note that $1 - \frac{\alpha d}{\beta p}(\gamma - 1) > 0$ and $1 - \frac{\alpha d}{\beta p'}(\gamma - 1) < 0$. Thus,

$$\|J_1(t,\cdot)\|_p \lesssim \|u\|_E^\gamma \left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-1} \lesssim t^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-1}.$$

However, if $t \geq 1$ we have that $t^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)} \leq t^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)}$ and we obtain

$$\|J_1(t,\cdot)\|_p \lesssim t^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha-1}. \tag{3.50}$$

We continue with J_2 and we estimate

$$\begin{aligned}
\|J_2(t,\cdot)\|_p &\leq \int_{\frac{t}{2}}^{t} \|Y(t-s,\cdot\cdot) \star |u(s,\cdot\cdot)|^{\gamma-1} u(s,\cdot\cdot)\|_p ds \\
&\leq \int_{\frac{t}{2}}^{t} \|Y(t-s,\cdot\cdot)\|_q \||u(s,\cdot\cdot)|^{\gamma-1} u(s,\cdot\cdot)\|_{p'} ds \\
&\lesssim \|u\|_E^\gamma \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha-1} s^{-\frac{\alpha d}{\beta p'}(\gamma-1)} ds \\
&\lesssim \|u\|_E^\gamma \left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta p'}(\gamma-1)} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha-1} ds \\
&\lesssim \|u\|_E^\gamma \left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta p'}(\gamma-1)} (t-s)^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha} \Bigg|_{\frac{t}{2}}^{t} \\
&\lesssim \|u\|_E^\gamma \left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta p'}(\gamma-1)} \left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha}.
\end{aligned}$$

Therefore,

$$\|J_2(t,\cdot)\|_p \lesssim t^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha-\frac{\alpha d}{\beta p'}(\gamma-1)}. \tag{3.51}$$

For J_3 we have that

$$\begin{aligned}
\|J_3(t,\cdot)\|_p &\le \int_0^{\frac{t}{2}} \|Y(t-s,\cdot)\|_p \int_{\mathbb{R}^d} |u(s,y)|^{\gamma-1}|u(s,y)|dy\, ds \\
&\lesssim \|u\|_E^\gamma \int_0^{\frac{t}{2}} (t-s)^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&\lesssim \|u\|_E^\gamma \left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-1} \int_0^{\frac{t}{2}} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds
\end{aligned}$$

and proceeding in the same way as in the estimate of J_1, we obtain

$$\|J_3(t,\cdot)\|_p \lesssim t^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha-1}. \tag{3.52}$$

For J_4 we find

$$\begin{aligned}
\|J_4(t,\cdot)\|_p &\le \|Y(t,\cdot)\|_p \int_{\frac{t}{2}}^{\infty} \int_{\mathbb{R}^d} |u(s,y)|^{\gamma-1}|u(s,y)|dyds \\
&\lesssim \|u\|_E^\gamma t^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-1} \int_{\frac{t}{2}}^{\infty} s^{-\frac{\alpha d}{\beta p'}(\gamma-1)} ds \\
&\lesssim \|u\|_E^\gamma t^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-1} \left(\frac{t}{2}\right)^{1-\frac{\alpha d}{\beta p'}(\gamma-1)}
\end{aligned}$$

and we get

$$\|J_4(t,\cdot)\|_p \lesssim t^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha-\frac{\alpha d}{\beta p'}(\gamma-1)}. \tag{3.53}$$

In order to estimate J_5, we follow the same arguments as in the proof of Theorem 3.1.5, first part, but using the bounds of Y given in Lemma 3.1.2, that is,

$$\begin{aligned}
\|J_5(t,\cdot)\|_p &\le \int_0^{\frac{t}{2}} \|Y(t,\cdot)-Y(t-s,\cdot)\|_p \int_{\mathbb{R}^d} |u(s,y)|^{\gamma-1}|u(s,y)|dyds \\
&\lesssim \|u\|_E^\gamma \int_0^{\frac{t}{2}} s(t-s)^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-2} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&\lesssim \|u\|_E^\gamma \left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-1} \int_0^{\frac{t}{2}} s(t-s)^{-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&\lesssim \|u\|_E^\gamma \left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-1} \int_0^{\frac{t}{2}} s\, s^{-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&\lesssim \|u\|_E^\gamma \left(\frac{t}{2}\right)^{-\frac{\alpha d}{\beta}\left(1-\frac{1}{p}\right)+\alpha-1} \int_0^{\frac{t}{2}} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds.
\end{aligned}$$

Now, we proceed in the same way as in the estimate of J_1 and thus

$$\|J_5(t,\cdot)\|_p \lesssim t^{-\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)+\alpha-1}. \tag{3.54}$$

Gathering estimates from (3.50) to (3.54), we have proved (3.49). This, together with (3.47), show the following result.

Theorem 3.5.1. *Let $\alpha \in (0,1)$ and $\beta \in (1,2)$. Assume the hypothesis $(\mathcal{H}_2)$ holds. Let $\lambda \in \mathbb{R}$ and $\gamma > 1$. Suppose that the initial data $u_0 \in L_1(\mathbb{R}^d)$, p and p' satisfy conditions as in Theorem 3.3.1. Let $p < \kappa_1$ as in Theorem 3.1.1 and $\frac{\alpha d}{\beta p'}(\gamma - 1) > 1$. Assume in addition that $\|\cdot\| u_0 \in L_r(\mathbb{R}^d)$, with some r and q as in (3.48). If $u \in E$ is a global solution to the Cauchy problem (3.1), then u has the asymptotic behavior*

$$\|u(t,\cdot) - AZ(t,\cdot) - BY(t,\cdot)\|_p \to 0$$

as $t \to \infty$, with the constants

$$A = \int_{\mathbb{R}^d} u_0(y)dy$$

and

$$B = \lambda \int_0^\infty \int_{\mathbb{R}^d} |u(s,y)|^{\gamma-1} u(s,y)dyds.$$

Chapter 4

A blow-up case

In this chapter we study the behaviour of solutions for the Cauchy problem

$$\begin{aligned} \partial_t^\alpha(u-u_0)(t,x)+\Psi_\beta(-i\nabla)u(t,x)=|u(t,x)|^{\gamma-1}u(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ u(t,x)|_{t=0}=u_0(x)\geq 0, \quad x\in\mathbb{R}^d, \end{aligned} \tag{4.1}$$

under the same definitions given in Chapter 3. In this case, we recall that the symbol of the operator $\Psi_\beta(-i\nabla)$ is independent of x and it has the form

$$\psi(\xi)=\|\xi\|^\beta\omega_\mu\left(\frac{\xi}{\|\xi\|}\right), \quad \xi\in\mathbb{R}^d,$$

with

$$\omega_\mu(\theta):=\int_{S^{d-1}}|\theta\cdot\eta|^\beta\mu(d\eta), \quad \theta\in S^{d-1}.$$

We also recall that the Borel measure $\mu(d\eta)$ is centrally symmetric, finite (non-negative), defined on S^{d-1}. The basic hypothesis throughout the chapter is the same as in Chapter 3:

($\mathcal{H}_1$) The spectral measure μ has a strictly positive density, such that the function ω_μ is strictly positive and $(d+1+[\beta])$-times continuously differentiable on S^{d-1}.

Again, we denote by ($\mathcal{H}_2$) to refer to ($\mathcal{H}_1$) whenever we need to assume that ω_μ is $(d+2+[\beta])$-times continuously differentiable on S^{d-1}.

Our aim here is to obtain a Fujita type blow-up result, together with Fujita's critical exponent in terms of the parameters of the stable non-Gaussian process. Besides, we want to show a result for global solutions. For this purpose, we introduce the following definitions.

Definition 4.0.1. *Let $\alpha\in(0,1)$, $\beta\in(0,2)$ and $\gamma>1$. Assume the hypothesis ($\mathcal{H}_1$) holds. Suppose that $1<p<\infty$ and that $u_0\in L_p(\mathbb{R}^d)$ is a non-negative function. A function u is called a* ***local solution*** *of (4.1), if there exists $T>0$ such that*

(i) $u\in C([0,T];L_p(\mathbb{R}^d))\cap L_\infty((0,T)\times\mathbb{R}^d)$,

(ii) u *satisfies (4.1) in* $[0,T]$.

A function u is called a ***global solution*** *of (4.1) if (i)-(ii) are satisfied for any $T > 0$. We say that u is a* ***mild solution*** *of (4.1) if $u \in C([0,T]; L_p(\mathbb{R}^d)) \cap L_\infty((0,T)\times\mathbb{R}^d)$ and it satisfies the integral equation*

$$u(t,x) = \int_{\mathbb{R}^d} Z(t,x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)|u(s,y)|^{\gamma-1}u(s,y)dyds \quad (4.2)$$

for all $x \in \mathbb{R}^d$ and $0 \le t < T$.

Definition 4.0.2. *Let $T > 0$. We say that a function $u : [0,T) \times \mathbb{R}^d \to \mathbb{R}$ blows-up at the finite time T if*

$$\lim_{t\to T^-} \|u(t)\|_\infty = +\infty,$$

Therefore, our main result is stated as follows.

Theorem 4.0.1. *Let $\alpha \in (0,1)$ and $\beta \in (0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Suppose that $\alpha = \frac{\beta}{2}$, that $1 < p < \infty$ and that $u_0 \in L_p(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ is a non-negative function. If $1 < \gamma < 1 + \frac{\beta}{d}$, then all non-trivial non-negative solutions of (4.1) that admit the representation (4.2) can only be local. If $\gamma = 1 + \frac{\beta}{d}$, then the non-trivial non-negative solutions can only be local whenever the initial condition is sufficiently large. Moreover, if additionally $u_0 \in L_\infty(\mathbb{R}^d)$, then any positive mild solution of (4.1) blows-up in finite time.*

Since the literature on blow-up theorems of Fujita type is quite extensive, we do not attempt to review it in this chapter. Nevertheless, let us emphasize that the relation $\alpha = \frac{\beta}{2}$ pays a crucial role in the proof of Theorem 4.0.1, which makes a similarity with what Fujita (1928-) found in 1966 for the case $\alpha = 1$ working in the Gaussian framework when $\beta = 2$ and $\omega_\mu \equiv 1$.

Before proving Theorem 4.0.1, we give some precise results for solutions of (4.1) in the sense of Definition 4.0.1.

4.1 Representation of solution in its integral form

In this section we analyse the conditions under which a local solution u of (4.1), in the sense of Definition 4.0.1, can be represented by (4.2).

Although the subordination principle employed here follows directly from [8, Chapter 3], for instance, the point we want to emphasize is the relation

$$Y(\cdot,x) = \frac{d}{dt}(g_\alpha * Z(\cdot,x)), \quad t > 0,$$

in the context of non-Gaussian process, which was proved in Lemma 3.1.6.

First, we recall from Corollary 2.1.1 that the symbol $\psi(\xi)$ is a continuous and negative definite function. Thereby, from [29, Example 4.6.29] we know that the operator $(-\Psi_\beta(-i\nabla), C_0^\infty(\mathbb{R}^d))$ satisfies, for any $1 < p < \infty$, the Dirichlet condition

$$\int_{\mathbb{R}^d} (-\Psi_\beta(-i\nabla)f)(x) \left((f-1)^+\right)^{p-1}(x)dx \le 0, \quad f \in C_0^\infty(\mathbb{R}^d),$$

and consequently it is $L_p(\mathbb{R}^d)$-dissipative ([29, Propositions 4.6.11 and 4.6.12]). In fact, the density of $C_0^\infty(\mathbb{R}^d)$ in $L_p(\mathbb{R}^d)$ implies that $(-\Psi_\beta(-i\nabla), C_0^\infty(\mathbb{R}^d))$ is closable and its closure $(A, D(A))$ generates a sub-Markovian semigroup $\{T_t\}_{t\geq 0}$ on $L_p(\mathbb{R}^d)$ which is a strongly continuous contraction semigroup ([29, Lemma 4.1.36, Theorems 4.1.33 and 4.6.17, Definition 4.1.6]). Besides, A is densely defined on $L_p(\mathbb{R}^d)$ ([29, Corollary 4.1.15]). On the other hand, it is well known that g_α is a completely positive function and belongs to $L_{1,loc}(\mathbb{R}^+)$ (see Section 1.4).

We denote $u(t) = u(t,\cdot)$ and $|u|^{\gamma-1}(t) = |u(t,\cdot)|^{\gamma-1}$. Since u and u_0 satisfy Definition 4.0.1, if $u_0 \in L_\infty(\mathbb{R}^d)$ we observe that $g_\alpha * |u|^{\gamma-1}u(t) \in L_p(\mathbb{R}^d)$ for $0 \leq t < T$.

Equation (4.1) can be written as the Volterra equation

$$u(t) = u_0 + g_\alpha * |u|^{\gamma-1}u(t) + g_\alpha * Au(t), \quad 0 < t < T, \tag{4.3}$$

which admits a resolvent $\{S(t)\}_{t\geq 0}$ in $L_p(\mathbb{R}^d)$ ([57, Theorems 4.1 and 4.2]). From [57, Corollary 4.5] we have that

$$S(t) = -\int_0^\infty T_\tau \, w(t; d\tau), \quad t > 0,$$

where w is the propagation function associated with g_α. In order to describe this resolvent we use the representation

$$T_t f(\cdot) = \int_{\mathbb{R}^d} G(t, \cdot - y) f(y) dy, \quad f \in D(A),$$

the function G being the fundamental solution of the homogeneous problem

$$\begin{aligned} \partial_t \, G(t,x) + \Psi_\beta(-i\nabla) G(t,x) = 0, \quad t > 0, \; x \in \mathbb{R}^d, \\ G(t,x)|_{t=0} = \delta_0(x), \quad x \in \mathbb{R}^d. \end{aligned}$$

See, e.g. [52, Section 1.2 Theorem 2.4 (c) and Section 4.1 Theorem 1.3]. For $v \in D(A)$ we see that

$$\begin{aligned} S(t)v &= -\int_0^\infty T_\tau v \, w(t; d\tau) \\ &= -\int_0^\infty G(t,\cdot) \star v \, w(t; d\tau) \end{aligned}$$

and using the Fourier transform we obtain

$$\begin{aligned} \mathcal{F}(S(t)v) &= -\int_0^\infty e^{-\tau\psi(\xi)} \widehat{v} \, w(t; d\tau) \\ &= s(t, \psi(\xi)) \widehat{v} \\ &= \widehat{Z}(t,\xi) \widehat{v} \end{aligned}$$

with the relaxation function s that comes via scalar Volterra equations (see Section 1.4, [57, Proposition 4.9], [33, Section 1]). This implies that

$$S(t)v = Z(t,\cdot) \star v$$

and the boundedness of $S(t)$ leads to an extension to all of $L_p(\mathbb{R}^d)$.

Let $0 < t < T$. If $u(s) \in D(A)$, $0 \leq s \leq t$, identity (4.3) and [57, Proposition 1.1, Definition 1.3] yield

$$\begin{aligned}
1 * u(t) &= \int_0^t u(s)ds \\
&= \int_0^t \left(S(t-s)u(s) - A(g_\alpha * S)(t-s)u(s)\right) ds \\
&= \int_0^t S(t-s)u(s)ds - \int_0^t (g_\alpha * S)(t-s)Au(s)ds \\
&= \int_0^t S(s)u(t-s)ds - \int_0^t S(s)(g_\alpha * Au)(t-s)ds \\
&= \int_0^t S(s)\left(u(t-s) - (g_\alpha * Au)(t-s)\right) ds \\
&= \int_0^t S(s)\left(u_0 + g_\alpha * |u|^{\gamma-1}u(t-s)\right) ds
\end{aligned}$$

and thus we get the *variation of parameters formula* for (4.3) given by

$$u(t) = \frac{d}{dt}\int_0^t S(s)\left(u_0 + g_\alpha * |u|^{\gamma-1}u\right)(t-s)ds.$$

We note that

$$\frac{d}{dt}\int_0^t S(s)u_0 ds = S(t)u_0 = Z(t,\cdot) \star u_0.$$

By proceeding as in the proof of Lemma 3.4.1 together with Lemma 3.1.6, but working with the $L_p(\mathbb{R}^d)$ space, the fact that $\sup\limits_{0\leq t<T} \||u|^{\gamma-1}u(t)\|_\infty < \infty$ leads to

$$\frac{d}{dt}\int_0^t S(s)\left(g_\alpha * |u|^{\gamma-1}u\right)(t-s)ds = \int_0^t Y(t-s,\cdot) \star |u|^{\gamma-1}u(s,\cdot)ds.$$

Theorem 4.1.1. *Let $\alpha \in (0,1)$ and $\beta \in (0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Let $\gamma > 1$ and suppose that $1 < p < \infty$. Let $u_0 \in D(A) \cap L_\infty(\mathbb{R}^d)$ be a non-negative function. If u is a local solution in the sense of Definition 4.0.1 for some $T > 0$ and $u(t) \in D(A)$ for all $0 \leq t < T$, then u admits the representation (4.2).*

4.2 Continuity and non-negativeness of solution in $[0,T) \times \mathbb{R}^d$

Let u be a local solution of (4.1). In this section we show that u is a continuous and non-negative function on $[0,T) \times \mathbb{R}^d$, for some $T > 0$. For this purpose, the representation (4.2) obtained in the previous section is particularly important. Besides, we need the following technical results.

Lemma 4.2.1. *Let $d\in\mathbb{N}$, $\alpha\in(0,1)$ and $\beta\in(0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. If f is a continuous and bounded function on $\mathbb{R}^d$, then $Z(t,\cdot)\star f\to f$ uniformly on compact sets whenever $t\to 0$.*

Proof. From Lemma 3.1.5 and Formula (3.24) we know that $g(x):=Z(1,x)$, $x\in\mathbb{R}^d$, satisfies all assumptions of [19, Theorem 1.6] with $\epsilon=t^{\frac{\alpha}{\beta}}$. □

Lemma 4.2.2. *Under the same assumptions as Lemma 3.1.3, then there exists a positive constant C for all $t>0$ and $x_1,x_2\in\mathbb{R}^d$, such that the estimate*

$$\|Y(t,\cdot-x_1)-Y(t,\cdot-x_2)\|_q\le C\|x_1-x_2\|\|\nabla Y(t,\cdot)\|_q\lesssim\|x_1-x_2\|t^{-\frac{\alpha d}{\beta}(1-\frac{1}{q})-\frac{\alpha}{\beta}+\alpha-1}\tag{4.4}$$

is true for $1\le q<\kappa'$, where $\kappa':=\begin{cases}\frac{d}{d+1-2\beta}, & d+1>2\beta \text{ and } \beta>\frac{1}{2},\\ \infty, & d+1\le 2\beta.\end{cases}$

In the case of $d+1<2\beta$, (4.4) remains true for $q=\infty$.

Proof. It follows from the same arguments as in Lemma 3.5.1 but using the bounds given in Lemma 3.1.4. □

In what follows we use the parameter $\kappa:=\begin{cases}\frac{d}{\beta}, & d>\beta,\\ 1, & \textit{otherwise}\end{cases}$ which sets a condition on p for the existence of some $q\ge 1$ such that

$$\frac{1}{p}+\frac{1}{q}=1$$

and the L_q-norm for $Y(t,\cdot)$, $t>0$, is reached. Indeed, by choosing $\kappa<p<\infty$ we obtain that $1<q<\infty$ whenever $\kappa=1$ and $1<q<\frac{d}{d-\beta}$ whenever $\kappa=\frac{d}{\beta}$. This implies that $q<\kappa_2$, with κ_2 as in Theorem 3.1.3.

Theorem 4.2.1. *Let $\alpha\in(0,1)$ and $\beta\in(1,2)$. Assume the hypothesis $(\mathcal{H}_2)$ holds. Let $\gamma>1$ and suppose that $\max(1,\kappa)<p<\infty$. Let $u_0\in D(A)\cap L_\infty(\mathbb{R}^d)\cap C(\mathbb{R}^d)$ be a non-negative function. If u is a local solution in the sense of Definition 4.0.1 for some $T>0$ and $u(t)\in D(A)$ for all $0\le t<T$, then $u\in C([0,T)\times\mathbb{R}^d)$.*

Proof. From Theorem 4.1.1 it follows that the local solution u has the form

$$u(t,x)=\int_{\mathbb{R}^d}Z(t,x-y)u_0(y)dy+\int_0^t\int_{\mathbb{R}^d}Y(t-s,x-y)|u|^{\gamma-1}u(s,y)dyds,$$

$x\in\mathbb{R}^d$, $0\le t<T$. We define

$$u_1(t,x):=\int_{\mathbb{R}^d}Z(t,x-y)u_0(y)dy$$

and

$$u_2(t,x):=\int_0^t\int_{\mathbb{R}^d}Y(t-s,x-y)|u|^{\gamma-1}u(s,y)dyds.$$

We shall show that for all $\epsilon>0$, there exists $\delta>0$ such that

$$|u_j(t,x)-u_j(t_0,x_0)|<\epsilon,\forall(t,x)\in B((t_0,x_0),\delta)\subset[0,T)\times\mathbb{R}^d,$$

for $j\in\{1,2\}$.

Let $x_0\in\mathbb{R}^d$ and $0<t_0<T$. We suppose $t_0<t<T$ without loss of generality. For u_1 we see that

$$\begin{aligned}
|u_1(t,x)-u_1(t_0,x_0)| &\le \int_{\mathbb{R}^d}|Z(t,x-y)-Z(t_0,x_0-y)|u_0(y)dy\\
&\le \int_{\mathbb{R}^d}|Z(t,x-y)-Z(t_0,x-y)|u_0(y)dy\\
&\quad+\int_{\mathbb{R}^d}|Z(t_0,x-y)-Z(t_0,x_0-y)|u_0(y)dy\\
&\lesssim \|u_0\|_\infty\int_{\mathbb{R}^d}|Z(t,x-y)-Z(t_0,x-y)|dy\\
&\quad+\|u_0\|_\infty\int_{\mathbb{R}^d}|Z(t_0,x-y)-Z(t_0,x_0-y)|dy\\
&\lesssim \|u_0\|_\infty|t-t_0|t_0^{-1}+\|u_0\|_\infty\|x-x_0\|t_0^{-\frac{\alpha}{\beta}},
\end{aligned}$$

where the last estimates follow from Theorem 3.1.5 and Lemma 3.5.1, respectively. Thus,

$$|u_1(t,x)-u_1(t_0,x_0)|\lesssim|t-t_0|t_0^{-1}+\|x-x_0\|t_0^{-\frac{\alpha}{\beta}}$$

and we can take a ball in $\mathbb{R}^d$ of radius $C^{-1}\epsilon t_0^{\frac{\alpha}{\beta}}$ centered at x_0, and an interval in $[0,T)$ of radius $C^{-1}\epsilon t_0$ centered at t_0, where C is the constant of the estimate.

For the continuity of u_1 in $(0,x_0)$ we have that

$$\begin{aligned}
|u_1(t,x)-u_1(0,x_0)| &= |u_1(t,x)-u_0(x_0)|\\
&= |u_1(t,x)-u_0(x)+u_0(x)-u_0(x_0)|\\
&\le |u_1(t,x)-u_0(x)|+|u_0(x)-u_0(x_0)|\\
&= \left|\int_{\mathbb{R}^d}Z(t,x-y)u_0(y)dy-u_0(x)\right|+|u_0(x)-u_0(x_0)|.
\end{aligned}$$

We note that, by Lemma 4.2.1, the continuity and boundedness of u_0 imply the uniform limit on compact subsets of $\mathbb{R}^d$ for the first term as $t\to0$. By choosing a sufficiently small δ we get the desired result.

Next, we analyse the continuity of u_2. We see that

$$\begin{aligned}
|u_2(t,x)| &\le \int_0^t\int_{\mathbb{R}^d}Y(t-s,x-y)|u(s,y)|^\gamma dyds\\
&\le \sup_{0\le s\le t}\|u(s)\|_\infty^\gamma\int_0^t\int_{\mathbb{R}^d}Y(t-s,x-y)dyds\\
&\le \sup_{0\le s\le t}\|u(s)\|_\infty^\gamma\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&\le \sup_{0\le s\le t}\|u(s)\|_\infty^\gamma\frac{t^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

This proves that

$$\lim_{t\to0}u_2(t,x)=0$$

uniformly on $\mathbb{R}^d$.

Now, let $x_0 \in \mathbb{R}^d$ and $0 < t_0 < T$. Again, we suppose $t_0 < t < T$ without loss of generality. We find that

$$\begin{aligned}
&|u_2(t,x) - u_2(t_0,x_0)| \\
&\leq |u_2(t,x) - u_2(t_0,x)| + |u_2(t_0,x) - u_2(t_0,x_0)| \\
&\leq \int_0^{t_0} \int_{\mathbb{R}^d} Y(s,x-y) \left| |u|^{\gamma-1} u(t-s,y) - |u|^{\gamma-1} u(t_0-s,y) \right| dyds \\
&\quad + \int_{t_0}^{t} \int_{\mathbb{R}^d} Y(s,x-y) |u(t-s,y)|^{\gamma} dyds \\
&\quad + \int_0^{t_0} \int_{\mathbb{R}^d} |Y(t_0-s,x-y) - Y(t_0-s,x_0-y)| |u(s,y)|^{\gamma} dyds \\
&\lesssim \gamma \sup_{0\leq s\leq t} \|u(s)\|_{\infty}^{\gamma-1} \int_0^{t_0} \int_{\mathbb{R}^d} Y(s,x-y) |u(t-s,y) - u(t_0-s,y)| dyds \\
&\quad + \sup_{0\leq s\leq t} \|u(s)\|_{\infty}^{\gamma} \int_{t_0}^{t} \int_{\mathbb{R}^d} Y(s,x-y) dyds \\
&\quad + \sup_{0\leq s\leq t} \|u(s)\|_{\infty}^{\gamma} \int_0^{t_0} \int_{\mathbb{R}^d} |Y(t_0-s,x-y) - Y(t_0-s,x_0-y)| dyds \\
&\lesssim \gamma \sup_{0\leq s\leq t} \|u(s)\|_{\infty}^{\gamma-1} \int_0^{t_0} \|Y(s,\cdot) \star |u(t-s) - u(t_0-s)|\|_{\infty} \, ds \\
&\quad + \sup_{0\leq s\leq t} \|u(s)\|_{\infty}^{\gamma} \int_{t_0}^{t} s^{\alpha-1} ds \\
&\quad + \sup_{0\leq s\leq t} \|u(s)\|_{\infty}^{\gamma} \int_0^{t_0} \|x - x_0\| (t_0-s)^{-\frac{\alpha}{\beta}+\alpha-1} ds,
\end{aligned}$$

where the last integral is estimated by Lemma 4.2.2. For estimating the first term, we use the continuity of u with respect to the norm topology on $L_p(\mathbb{R}^d)$, Young's convolution inequality and Theorem 3.1.3, i.e.,

$$\begin{aligned}
\int_0^{t_0} \|Y(s,\cdot) \star |u(t-s) - u(t_0-s)|\|_{\infty} \, ds &\lesssim \int_0^{t_0} \|Y(s,\cdot)\|_q \|u(t-s) - u(t_0-s)\|_p \, ds \\
&\lesssim \epsilon \int_0^{t_0} s^{-\frac{\alpha d}{\beta p}+\alpha-1} ds.
\end{aligned}$$

Thus,

$$|u_2(t,x) - u_2(t_0,x_0)| \lesssim \epsilon t_0^{\alpha - \frac{\alpha d}{\beta p}} + (t^{\alpha} - t_0^{\alpha}) + \|x - x_0\| t_0^{\alpha - \frac{\alpha}{\beta}}.$$

□

The second result of this section is the following.

Theorem 4.2.2. *Let $\alpha \in (0,1)$ and $\beta \in (0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Let $\gamma > 1$ and suppose that $1 < p < \infty$. Let $u_0 \in D(A) \cap L_{\infty}(\mathbb{R}^d)$ be a non-negative function. If u is a local solution in the sense of Definition 4.0.1 for some $T > 0$ and $u(t) \in D(A)$ for all $0 \leq t < T$, then there exists $0 < T^* \leq T$ such that u is non-negative in $[0, T^*) \times \mathbb{R}^d$.*

Proof. We define the operator

$$\mathcal{M}v(t,x) := \int_{\mathbb{R}^d} Z(t,x-y)v_0(y)dy + \int_0^t\int_{\mathbb{R}^d} Y(t-s,x-y)g(v(s,y))dyds$$

on the Banach space $L_\infty((0,T)\times\mathbb{R}^d)$, where g is a non-decreasing Lipschitz function with $g(0)=0$ and $v_0\in L_\infty(\mathbb{R}^d)$. As in the proof of [1, Lemma 1.3], we derive that the operator $\mathcal{M}$ has a unique fixed point v. Furthermore, $v\geq w$ whenever $v_0\geq w_0$, where w is the fixed point associated with $w_0\in L_\infty(\mathbb{R}^d)$. Our aim now is to apply this result to a sequence of functions g_n, such that for each $n\in\mathbb{N}$ they have the same properties as g but with the additional constraint that their structure approximates the non-linear term $(\cdot)^\gamma$ on $[0,\infty)$. In accordance with our particular situation with $\gamma>1$, we need a sequence that allows us to control the derivative of the function $(\cdot)^\gamma$. For that purpose, we define

$$g_n(r) := \begin{cases} 0 & \text{if} \quad r<0,\\ r^\gamma & \text{if} \quad 0\leq r\leq n,\\ a_n - b_n e^{-r} & \text{if} \quad r>n,\end{cases}$$

where a_n, b_n are positive constants that guarantee the existence of $g_n'\geq 0$ on $\mathbb{R}$ a.e. By construction we have that for all $n\in\mathbb{N}$ the constant Lipschitz of g_n is $\gamma n^{\gamma-1}$, $g_n(0)=0$ and $g_n(r)=r^\gamma$ for $0\leq r\leq n$. Therefore, there exists a unique function $u_n\in L_\infty((0,T)\times\mathbb{R}^d)$ such that $0\leq u_n$ and

$$u_n(t,x) = \int_{\mathbb{R}^d} Z(t,x-y)\left(u_0+\frac{1}{n}\right)(y)dy + \int_0^t\int_{\mathbb{R}^d} Y(t-s,x-y)g_n(u_n(s,y))dyds,$$

for $x\in\mathbb{R}^d$ and $0<t<T$. Since $\frac{1}{n}\geq\frac{1}{n+1}$, we have that $u_{n+1}\leq u_n$. Thus, for almost every $(t,x)\in(0,T)\times\mathbb{R}^d$, the sequence of real numbers $(u_n(t,x))_{n\in\mathbb{N}}$ is decreasing and bounded from below by zero. Consequently, we can define the function

$$\widetilde{u}(t,x) = \lim_{n\to\infty} u_n(t,x)$$

a.e. in $(0,T)\times\mathbb{R}^d$. On the other hand, we have that

$$\|u_n(t)\|_\infty \leq \left\|u_0+\frac{1}{n}\right\|_\infty + \frac{\gamma n^{\gamma-1}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u_n(s)\|_\infty ds$$

and Gronwall's inequality (see [69, Corollary 2]) yields

$$\begin{aligned}\|u_n(t)\|_\infty &\leq \left\|u_0+\frac{1}{n}\right\|_\infty E_{\alpha,1}\left(\gamma n^{\gamma-1}t^\alpha\right)\\ &\leq \left\|u_0+\frac{1}{n}\right\|_\infty E_{\alpha,1}\left(\gamma n^{\gamma-1}T^\alpha\right), \quad 0<t<T.\end{aligned}$$

Now, for small enough $0<T^*\leq T$ we can find $N\in\mathbb{N}$ such that

$$\left\|u_0+\frac{1}{N}\right\|_\infty E_{\alpha,1}\left(\gamma N^{\gamma-1}(T^*)^\alpha\right)\leq N.$$

Therefore, for all $n \geq N$ it follows that $u_n(t,x) \leq N$, for $x \in \mathbb{R}^d$ and $0 < t < T^*$. This shows that

$$u_n(t,x) = \int_{\mathbb{R}^d} Z(t,x-y)\left(u_0 + \frac{1}{n}\right)(y)dy + \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)u_n(s,y)^\gamma dyds, \quad n \geq N.$$

We note that the non-linear integral term is dominated by N^γ and the dominated convergence theorem implies that

$$\widetilde{u}(t,x) = \int_{\mathbb{R}^d} Z(t,x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)\widetilde{u}(s,y)^\gamma dyds.$$

Next, we show that $u = \widetilde{u}$ a.e. in $(0,T^*)$. Indeed,

$$\begin{aligned}
&|u(t,x) - \widetilde{u}(t,x)| \\
&\leq \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)\left||u|^{\gamma-1}u(s,y) - \widetilde{u}(s,y)^\gamma\right| dyds \\
&= \int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)\left||u|^{\gamma-1}u(s,y) - |\widetilde{u}|^{\gamma-1}\widetilde{u}(s,y)\right| dyds \\
&\lesssim \sup_{0\leq s<T^*}\left(\|u(s)\|_\infty^{\gamma-1} + \|\widetilde{u}(s)\|_\infty^{\gamma-1}\right)\int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)|u(s,y) - \widetilde{u}(s,y)|dyds \\
&\leq C(T^*)\int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)\|u(s) - \widetilde{u}(s)\|_\infty dyds \\
&\leq C(T^*)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|u(s) - \widetilde{u}(s)\|_\infty ds
\end{aligned}$$

and thus

$$\|u(t) - \widetilde{u}(t)\|_\infty \leq \frac{C(T^*)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s) - \widetilde{u}(s)\|_\infty ds.$$

By Gronwall's inequality we conclude the desired result. □

4.3 Existence of blow-up

In this section we prove Theorem 4.0.1. We start by obtaining some estimates. Let $t > 0$. Using the bounds given in Proposition 3.1.1, it is clear that

$$Z(t,x-y) \geq Ct^{-\frac{\alpha d}{\beta}}e^{-\frac{\|x-y\|^2}{4t}}, \quad \Omega \leq 1.$$

If $\Omega \geq 1$, we have that

$$\begin{aligned}
Z(t,x-y) &\geq Ct^{-\frac{\alpha d}{\beta}}\Omega^{-1-\frac{d}{\beta}} \\
&= Ct^{-\frac{\alpha d}{\beta}}t^{\alpha+\frac{\alpha d}{\beta}}\|x-y\|^{-\beta-d} \\
&= Ct^{-\frac{\alpha d}{\beta}}t^{\alpha+\frac{\alpha d}{\beta}}(2\sqrt{t})^{-\beta-d}\left(\frac{\|x-y\|}{2\sqrt{t}}\right)^{-\beta-d} \\
&\geq Ct^{-\frac{\alpha d}{\beta}}t^{\alpha+\frac{\alpha d}{\beta}}(2\sqrt{t})^{-\beta-d}e^{-\frac{\|x-y\|^2}{4t}},
\end{aligned}$$

whenever $d \leq 3$. For larger dimensions, it is always possible to find a suitable constant $K > 1$, depending on β and d, such that $\left(\frac{\|x-y\|}{2\sqrt{t}}\right)^{-\beta-d} \geq e^{-K\frac{\|x-y\|^2}{4t}}$. From the hypothesis $\alpha = \frac{\beta}{2}$, it follows that

$$Z(t, x-y) \geq C2^{-\beta-d}t^{-\frac{\alpha d}{\beta}}e^{-\frac{\|x-y\|^2}{4t}}, \quad \Omega \geq 1,$$

which means that

$$Z(t, x-y) \geq C_1 t^{-\frac{\alpha d}{\beta}}e^{-\frac{\|x-y\|^2}{4t}}, \tag{4.5}$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$, with $C_1 = \dfrac{C}{2^{\beta+d}}$.

We may assume without loss of generality that the constant C of the Proposition 3.1.2 is the same as that of the Proposition 3.1.1. In this way, we have also derived

$$Y(t-s, x-y) \geq C_1(t-s)^{-\frac{\alpha d}{\beta}+\alpha-1}e^{-\frac{\|x-y\|^2}{4(t-s)}}, \tag{4.6}$$

for all $0 \leq s < t$ and $x, y \in \mathbb{R}^d$.

Now, we proceed by contradiction. We suppose that there exists a global non-trivial non-negative solution u of (4.1), according to Definition 4.0.1. In this case, $u_0(y_0) > 0$ for some $y_0 \in \mathbb{R}^d$. The continuity of u_0 implies that

$$u_0(y) > C_0, \quad \forall y \in B(y_0, \delta),$$

with some $\delta > 0$ and $C_0 = \dfrac{u_0(y_0)}{2}$.

The representation (4.2) for u is

$$u(t,x) = \int_{\mathbb{R}^d} Z(t, x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} Y(t-s, x-y)u(s,y)^\gamma dyds$$

for all $x \in \mathbb{R}^d$ and $0 < t < T$. We note that, given the assumption made, T can be arbitrarily large. As in Section 4.2, we define

$$u_1(t,x) := \int_{\mathbb{R}^d} Z(t, x-y)u_0(y)dy$$

and

$$u_2(t,x) := \int_0^t \int_{\mathbb{R}^d} Y(t-s, x-y)u(s,y)^\gamma dyds.$$

Using (4.5), it follows that

$$\begin{aligned} u_1(t,x) &\geq C_1 t^{-\frac{\alpha d}{\beta}} \int_{\mathbb{R}^d} e^{-\frac{\|x-y\|^2}{4t}} u_0(y)dy \\ &\geq C_1 C_0 t^{-\frac{\alpha d}{\beta}} \int_{B(y_0,\delta)} e^{-\frac{\|x-y\|^2}{4t}} dy \\ &\geq C_1 C_0 t^{-\frac{\alpha d}{\beta}} e^{-\frac{\|x-y_0\|^2}{2t}} \int_{B(y_0,\delta)} e^{-\frac{\|y-y_0\|^2}{2t}} dy \end{aligned}$$

and we obtain

$$u_1(t,x) \geq C_2 t^{-\frac{\alpha d}{\beta}} e^{-\frac{\|x\|^2}{t}}, \quad t>1, \quad x \in \mathbb{R}^d. \tag{4.7}$$

Let H be the heat kernel

$$H(t,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{\|x\|^2}{4t}}, \quad t>0, \quad x \in \mathbb{R}^d.$$

Using the fact that

$$\int_{\mathbb{R}^d} H(t,x)dx = 1,$$

we define the function

$$F(t) = \int_{\mathbb{R}^d} H(t,x)u(t,x)dx, \quad t>0, \tag{4.8}$$

and splitting the integral into two parts we see that

$$F(t) = \int_{\mathbb{R}^d} H(t,x)u_1(t,x)dx + \int_{\mathbb{R}^d} H(t,x)u_2(t,x)dx.$$

In the first integral we use the estimate (4.7), for obtaining

$$F(t) \geq C_3 t^{-\frac{\alpha d}{\beta}} + \int_{\mathbb{R}^d} H(t,x)u_2(t,x)dx$$

whenever $t>1$.

In the second integral, we use the fact that (see Theorem 3.1.6)

$$\frac{1}{g_\alpha(t)} \int_{\mathbb{R}^d} Y(t,x)dx = 1, \quad t>0.$$

Jensen's inequality and Fubini's theorem yield

$$\begin{aligned}
&\int_{\mathbb{R}^d} H(t,x)u_2(t,x)dx \\
&= \int_{\mathbb{R}^d} H(t,x)\left[\int_0^t \int_{\mathbb{R}^d} Y(t-s,x-y)u(s,y)^\gamma dyds\right]dx \\
&= \int_0^t g_\alpha(t-s)\int_{\mathbb{R}^d} H(t,x)\left[\int_{\mathbb{R}^d} \frac{1}{g_\alpha(t-s)} Y(t-s,x-y)u(s,y)^\gamma dy\right]dxds \\
&\geq \int_0^t g_\alpha(t-s)\int_{\mathbb{R}^d} H(t,x)\left[\int_{\mathbb{R}^d} \frac{1}{g_\alpha(t-s)} Y(t-s,x-y)u(s,y)dy\right]^\gamma dxds \\
&= \int_0^t (g_\alpha(t-s))^{1-\gamma}\int_{\mathbb{R}^d} H(t,x)\left[\int_{\mathbb{R}^d} Y(t-s,x-y)u(s,y)dy\right]^\gamma dxds \\
&\geq \int_0^t (g_\alpha(t-s))^{1-\gamma}\left[\int_{\mathbb{R}^d} H(t,x)\int_{\mathbb{R}^d} Y(t-s,x-y)u(s,y)dy\,dx\right]^\gamma ds \\
&\geq \int_0^t (g_\alpha(t-s))^{1-\gamma}\left\{\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} H(t,x)Y(t-s,x-y)dx\right]u(s,y)dy\right\}^\gamma ds.
\end{aligned}$$

The expression in the square brackets can be estimated with (4.6), i.e.,

$$\begin{aligned}&\int_{\mathbb{R}^d} H(t,x)Y(t-s,x-y)dx\\ &\geq C_1(t-s)^{-\frac{\alpha d}{\beta}+\alpha-1}\int_{\mathbb{R}^d} H(t,x)e^{-\frac{\|x-y\|^2}{4(t-s)}}dx\\ &= C_1(4\pi s)^{-\frac{d}{2}}e^{-\frac{\|y\|^2}{4s}}\left(\frac{s}{t}\right)^{\frac{d}{2}}(t-s)^{-\frac{\alpha d}{\beta}+\alpha-1}\int_{\mathbb{R}^d} e^{\frac{\|y\|^2}{4s}-\frac{\|x\|^2}{4t}-\frac{\|x-y\|^2}{4(t-s)}}dx.\end{aligned}$$

Proceeding in the same way as in [27, page 42], with $\alpha = \frac{\beta}{2}$, we get

$$\int_{\mathbb{R}^d} H(t,x)Y(t-s,x-y)dx \geq C_4(4\pi s)^{-\frac{d}{2}}e^{-\frac{\|y\|^2}{4s}}\left(\frac{s}{t}\right)^{\frac{d}{2}}(t-s)^{\alpha-1}$$

and thus

$$\begin{aligned}&\left\{\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} H(t,x)Y(t-s,x-y)dx\right]u(s,y)dy\right\}^{\gamma}\\ &\geq C_4^{\gamma}(t-s)^{(\alpha-1)\gamma}\left(\frac{s}{t}\right)^{\frac{d}{2}\gamma}\left\{\int_{\mathbb{R}^d}(4\pi s)^{-\frac{d}{2}}e^{-\frac{\|y\|^2}{4s}}u(s,y)dy\right\}^{\gamma}\\ &= C_4^{\gamma}(t-s)^{(\alpha-1)\gamma}\left(\frac{s}{t}\right)^{\frac{d}{2}\gamma}F^{\gamma}(s)\end{aligned}$$

for $0 < s < t$. It follows that

$$\int_{\mathbb{R}^d} H(t,x)u_2(t,x)dx \geq C_4^{\gamma}\int_0^t (g_\alpha(t-s))^{1-\gamma}(t-s)^{(\alpha-1)\gamma}\left(\frac{s}{t}\right)^{\frac{d}{2}\gamma}F^{\gamma}(s)ds$$

and hence

$$F(t) \geq \frac{C_3}{t^{\frac{d}{2}}} + C_5\frac{t^{\alpha-1}}{t^{\frac{d}{2}\gamma}}\int_0^t s^{\frac{d}{2}\gamma}F^{\gamma}(s)ds$$

for all $t > 1$. Consequently,

$$t^{\frac{d}{2}\gamma}t^{1-\alpha}F(t) \geq C_3t^{\frac{d}{2}(\gamma-1)}t^{1-\alpha} + C_5\int_0^t s^{\frac{d}{2}\gamma}F^{\gamma}(s)ds. \tag{4.9}$$

Defining the r.h.s. of this expression as $f(t)$, $t > 1$, we have that

$$f(t) \geq C_3t^{\frac{d}{2}(\gamma-1)}t^{1-\alpha} \tag{4.10}$$

and that

$$f'(t) \geq C_5t^{\frac{d}{2}\gamma}F^{\gamma}(t). \tag{4.11}$$

From (4.9) it follows that

$$\begin{aligned}f'(t) &\geq C_5t^{\frac{d}{2}\gamma}\left(\frac{f(t)}{t^{\frac{d}{2}\gamma+1-\alpha}}\right)^{\gamma}\\ &= C_5t^{\frac{d}{2}\gamma(1-\gamma)-(1-\alpha)\gamma}f^{\gamma}(t).\end{aligned}$$

Therefore,
$$f'(t)f^{-\gamma}(t) \geq C_5 t^{\frac{d}{2}\gamma(1-\gamma)-(1-\alpha)\gamma}$$
and
$$\int_t^T f'(s)f^{-\gamma}(s)ds \geq C_5 \int_t^T s^{\frac{d}{2}\gamma(1-\gamma)-(1-\alpha)\gamma} ds$$
with $T > t$. From here, we get that
$$\frac{f^{1-\gamma}(t)}{\gamma-1} \geq C_5 \int_t^T s^{\frac{d}{2}\gamma(1-\gamma)-(1-\alpha)\gamma} ds$$
and using (4.10) we also obtain the estimate
$$\frac{f^{1-\gamma}(t)}{\gamma-1} \leq \frac{C_3^{1-\gamma}}{\gamma-1} t^{-\frac{d}{2}(1-\gamma)^2-(1-\alpha)(\gamma-1)}.$$
This implies that
$$\frac{C_3^{1-\gamma}}{\gamma-1} t^{-\frac{d}{2}(1-\gamma)^2-(1-\alpha)(\gamma-1)} \geq C_5 \int_t^T s^{-\frac{d}{2}\gamma(\gamma-1)-(1-\alpha)\gamma} ds. \tag{4.12}$$
Next we analyse the r.h.s. of (4.12), according to the following cases with $a := d - 2(1-\alpha)$.

For the case $1 < \gamma \leq \frac{a}{d} + \frac{2}{d\gamma}$, we have
$$\begin{aligned} \gamma \leq \frac{a}{d} + \frac{2}{d\gamma} &\Rightarrow d\gamma^2 \leq a\gamma + 2 \\ &\Leftrightarrow d\gamma^2 + 2(1-\alpha)\gamma - d\gamma - 2 \leq 0 \\ &\Leftrightarrow -\frac{d\gamma}{2}(\gamma-1) - (1-\alpha)\gamma + 1 \geq 0, \end{aligned}$$
which yields a contradiction for large enough T.

For the case $\frac{a}{d} + \frac{2}{d\gamma} < \gamma < \frac{a}{d} + \frac{2}{d}$, we write the expression (4.12) as
$$\frac{C_3^{1-\gamma}}{\gamma-1} t^{-\frac{d}{2}(1-\gamma)^2-(1-\alpha)(\gamma-1)} \geq C_5 \frac{t^{-\frac{d}{2}\gamma(\gamma-1)-(1-\alpha)\gamma+1} - T^{-\frac{d}{2}\gamma(\gamma-1)-(1-\alpha)\gamma+1}}{\frac{d}{2}\gamma(\gamma-1) + (1-\alpha)\gamma - 1}.$$
Besides
$$\begin{aligned} \gamma < \frac{a}{d} + \frac{2}{d} &\Rightarrow d\gamma < d - 2(1-\alpha) + 2 \\ &\Leftrightarrow -1 < -\frac{d}{2}(\gamma-1) - (1-\alpha) \\ &\Leftrightarrow \frac{d\gamma}{2}(\gamma-1) + (1-\alpha)\gamma - 1 < \frac{d}{2}(\gamma-1)^2 + (1-\alpha)(\gamma-1), \end{aligned}$$
which is a contradiction for large enough t and $T \to \infty$.

For the critical case $\gamma = 1 + \frac{\beta}{d}$, we use the facts that
$$u(t,x)^\gamma \geq u_1(t,x)^\gamma$$

and

$$u(t,x) \geq u_2(t,x),$$

together with the estimates (4.6) and (4.7). Therefore, for $t > 2$, we get

$$\begin{aligned}
u(t,x) &\geq \int_1^{\frac{t}{2}} \int_{\mathbb{R}^d} Y(t-s, x-y) u(s,y)^\gamma dy ds \\
&\geq C_1 C_2^\gamma \int_1^{\frac{t}{2}} (t-s)^{-\frac{\alpha d}{\beta}+\alpha-1} \int_{\mathbb{R}^d} e^{-\frac{\|x-y\|^2}{4(t-s)}} s^{-\frac{\alpha d\gamma}{\beta}} e^{-\frac{\gamma\|y\|^2}{s}} dy ds \\
&= \frac{C_1 C_2^\gamma}{t^{\frac{d}{2}}} e^{-\frac{\|x\|^2}{t}} \int_1^{\frac{t}{2}} \frac{(t-s)^{\alpha-1} t^{\frac{d}{2}}}{(t-s)^{\frac{d}{2}} s^{\frac{d+\beta}{2}}} \int_{\mathbb{R}^d} e^{\frac{\|x\|^2}{t} - \frac{\|x-y\|^2}{4(t-s)} - \frac{\gamma\|y\|^2}{s}} dy ds \\
&\geq \frac{C_1 C_2^\gamma}{t^{\frac{d}{2}}} e^{-\frac{\|x\|^2}{t}} \int_1^{\frac{t}{2}} \frac{(t-s)^{\alpha-1} t^{\frac{d}{2}}}{(t-s)^{\frac{d}{2}} s^{\frac{d}{2}+\alpha}} \int_{\mathbb{R}^d} e^{\frac{\|x\|^2}{t} - \frac{\|x\|^2}{t} - \frac{\|y\|^2}{2(t-s)} - \frac{\gamma\|y\|^2}{s}} dy ds \\
&\geq \frac{C_6}{t^{\frac{d}{2}}} e^{-\frac{\|x\|^2}{t}} \int_1^{\frac{t}{2}} \frac{(t-s)^{\alpha-1}}{s^\alpha} ds \\
&\geq \frac{C_6}{t^{\frac{d}{2}}} e^{-\frac{\|x\|^2}{t}} \int_1^{\frac{t}{2}} \frac{1}{t-s} ds
\end{aligned}$$

and hence

$$u(t,x) \geq \frac{C_6}{t^{\frac{d}{2}}} e^{-\frac{\|x\|^2}{t}} \ln\left(2 - \frac{2}{t}\right).$$

Using this and (4.8), we obtain that

$$F(t) \geq \frac{C_7}{t^{\frac{d}{2}}} \ln\left(2 - \frac{2}{t}\right). \tag{4.13}$$

Now,

$$\begin{aligned}
t^{\frac{d}{2}\gamma} t^{1-\alpha} F(t) &= \frac{1}{2} t^{\frac{d}{2}\gamma} t^{1-\alpha} F(t) + \frac{1}{2} t^{\frac{d}{2}\gamma} t^{1-\alpha} F(t) \\
&\geq \frac{C_7}{2} \frac{t^{\frac{d}{2}\gamma} t^{1-\alpha}}{t^{\frac{d}{2}}} \ln\left(2 - \frac{2}{t}\right) + \frac{C_5}{2} \int_0^t s^{\frac{d}{2}\gamma} F^\gamma(s) ds,
\end{aligned}$$

where (4.13) yields the bound for the first term and the second term comes from (4.9). The critical value of γ yields

$$t^{\frac{d}{2}\gamma} t^{1-\alpha} F(t) \geq \frac{C_7}{2} t \ln\left(2 - \frac{2}{t}\right) + \frac{C_5}{2} \int_0^t s^{\frac{d}{2}\gamma} F^\gamma(s) ds.$$

Defining the r.h.s. of this expression as the new $f(t)$, $t > 1$, we proceed as before but using

$$f(t) \geq C_8 t \ln\left(2 - \frac{2}{t}\right)$$

and

$$f'(t) \geq C_9 t^{\frac{d}{2}\gamma} F^\gamma(t)$$

instead of (4.10) and (4.11), respectively, with $C_8 = \frac{C_7}{2}$ and $C_9 = \frac{C_5}{2}$. The resulting expression, instead of (4.12), is

$$\frac{C_8^{1-\gamma}}{\gamma-1} t^{1-\gamma} \ln^{1-\gamma}\left(2-\frac{2}{t}\right) \geq C_9 \int_t^T s^{-\frac{d}{2}\gamma(\gamma-1)-(1-\alpha)\gamma} ds$$

or, in this case,

$$\frac{C_8^{1-\gamma}}{\gamma-1} t^{1-\gamma} \ln^{1-\gamma}\left(2-\frac{2}{t}\right) \geq C_9 \int_t^T s^{-\gamma} ds.$$

This implies, as $T \to \infty$, that

$$C_8^{1-\gamma} \ln^{1-\gamma}\left(2-\frac{2}{t}\right) \geq C_9,$$

which is a contradiction whenever the initial condition is sufficiently large at the point y_0.

So far we note that in this proof we do not require that u satisfies (4.1). Hence, any positive mild solution u can only be local under the assumptions of Theorem 4.0.1. In this context, let

$$\widetilde{T} = \sup\left\{T > 0 : u \in C([0,T]; L_p(\mathbb{R}^d)) \cap L_\infty((0,T)\times\mathbb{R}^d) \right.$$
$$\left. \text{is a positive mild solution of (4.1)}\right\}.$$

Previous work implies that $\widetilde{T} < +\infty$. Suppose that $\lim_{t\to\widetilde{T}^-} \|u(t)\|_\infty < +\infty$. Since $u_0 \in L_\infty(\mathbb{R}^d)$, it follows that there exists $M > 0$ such that $\|u(t)\|_\infty \leq M$ for all $t \in [0, \widetilde{T})$. We choose a sequence $t_n \to \widetilde{T}$ as $n \to \infty$, with $t_n < \widetilde{T}$ for all $n \in \mathbb{N}$. We suppose $\frac{1}{2}\widetilde{T} < t_m < t_n$ without loss of generality, with $n, m \geq N$ for some $N \in \mathbb{N}$. As in the proof of Theorem 3.2.1, we find that

$$\begin{aligned}\|u(t_n) - u(t_m)\|_p \lesssim &(t_n - t_m) t_m^{-1} \|u_0\|_p \\ &+ M^{\gamma-1} \int_0^{t_m} \|Y(t_n - s) - Y(t_m - s)\|_1 \|u(s)\|_p ds \\ &+ M^{\gamma-1} \int_{t_m}^{t_n} \|Y(t_n - s)\|_1 \|u(s)\|_p ds.\end{aligned}$$

On the other hand, for any $t \in [0, \widetilde{T})$ we see that

$$\|u(t)\|_p \leq \|u_0\|_p + \frac{M^{\gamma-1}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u(s)\|_p ds$$

and Gronwall's inequality ([69, Corollary 2]) yields

$$\|u(t)\|_p \leq \|u_0\|_p E_{\alpha,1}(M^{\gamma-1} t^\alpha), \quad 0 \leq t < \widetilde{T}.$$

This shows that $\|u(t)\|_p \leq \|u_0\|_p E_{\alpha,1}(M^{\gamma-1}\widetilde{T}^\alpha) =: K$ for all $t \in [0, \widetilde{T})$. Thus,

$$\begin{aligned}\|u(t_n) - u(t_m)\|_p \lesssim &(t_n - t_m) t_m^{-1} \|u_0\|_p \\ &+ M^{\gamma-1} K \int_0^{t_m} \|Y(t_n - s) - Y(t_m - s)\|_1 ds \\ &+ M^{\gamma-1} K \int_{t_m}^{t_n} \|Y(t_n - s)\|_1 ds.\end{aligned}$$

The integral over $[0, t_m]$ can be estimated, using Theorems 3.1.3 and 3.1.6, as follows:

$$\int_0^{t_m} \|Y(t_n - s) - Y(t_m - s)\|_1 ds = \int_0^{t_m} \|Y(t_n - t_m + s) - Y(s)\|_1 ds$$
$$\leq \int_0^{\infty} \|Y(t_n - t_m + s) - Y(s)\|_1 ds$$
$$= \int_0^{t_n - t_m} \|Y(t_n - t_m + s) - Y(s)\|_1 ds + \int_{t_n - t_m}^{\infty} \|Y(t_n - t_m + s) - Y(s)\|_1 ds$$
$$\leq \int_0^{t_n - t_m} \|Y(t_n - t_m + s)\|_1 ds + \int_0^{t_n - t_m} \|Y(s)\|_1 ds + \int_{t_n - t_m}^{\infty} \|Y(t_n - t_m + s) - Y(s)\|_1 ds$$
$$\lesssim \int_0^{t_n - t_m} (t_n - t_m + s)^{\alpha - 1} ds + \int_0^{t_n - t_m} s^{\alpha - 1} ds + \int_{t_n - t_m}^{\infty} (t_n - t_m) s^{\alpha - 2} ds$$
$$\lesssim (t_n - t_m)^{\alpha}.$$

Consequently,

$$\|u(t_n) - u(t_m)\|_p \lesssim (t_n - t_m)\widetilde{T}^{-1}\|u_0\|_p + M^{\gamma - 1} K (t_n - t_m)^{\alpha}$$

and thus $(u(t_n))_{n\in\mathbb{N}}$ represents a Cauchy sequence in $L_p(\mathbb{R}^d)$. We define $u(\widetilde{T}) := \lim_{t\to\widetilde{T}^-} u(t)$. From [60, Theorem 3.12] it follows that $\|u(\widetilde{T})\|_\infty \leq M$ and that $u(\widetilde{T}) \geq 0$. Next, as in the proof of [71, Theorem 3.2], we define the operator

$$\mathcal{M}v(t) := Z(t) \star u_0 + \int_0^{\widetilde{T}} Y(t-s) \star u^{\gamma}(s) ds + \int_{\widetilde{T}}^{t} Y(t-s) \star |v(s)|^{\gamma-1} v(s) ds$$

on the Banach space

$$E_\tau = C([\widetilde{T}, \widetilde{T} + \tau]; L_p(\mathbb{R}^d)) \cap L_\infty([\widetilde{T}, \widetilde{T} + \tau) \times \mathbb{R}^d),$$

with some $\tau > 0$ and the norm

$$\|v\|_{E_\tau} = \sup_{t\in[\widetilde{T}, \widetilde{T}+\tau]} \|v(t)\|_p + \sup_{(t,x)\in[\widetilde{T}, \widetilde{T}+\tau)\times\mathbb{R}^d} |v(t,x)|.$$

It is straightforward to see that $\mathcal{M} : E_\tau \to E_\tau$ is well defined and that $\mathcal{M}v(\widetilde{T}) = u(\widetilde{T})$. Besides, for $v, w \in E_\tau$ we have that

$$\begin{aligned}
|\mathcal{M}v(t,x) - \mathcal{M}w(t,x)| &\leq \|\mathcal{M}v(t) - \mathcal{M}w(t)\|_\infty \\
&\leq \int_{\widetilde{T}}^{t} \|Y(t-s)\|_1 \||v(s)|^{\gamma-1}v(s) - |w(s)|^{\gamma-1}w(s)\|_\infty ds \\
&\lesssim (\|v\|_{E_\tau} + \|w\|_{E_\tau})^{\gamma-1} \int_{\widetilde{T}}^{t} (t-s)^{\alpha-1} \|v(s) - w(s)\|_\infty ds \\
&\lesssim (\|v\|_{E_\tau} + \|w\|_{E_\tau})^{\gamma-1} \|v - w\|_{E_\tau} (t - \widetilde{T})^{\alpha},
\end{aligned}$$

and hence

$$\|\mathcal{M}v(t) - \mathcal{M}w(t)\|_\infty \lesssim (\|v\|_{E_\tau} + \|w\|_{E_\tau})^{\gamma-1} \|v - w\|_{E_\tau} \tau^{\alpha}, \quad t \in [\widetilde{T}, \widetilde{T} + \tau).$$

Similarly,

$$\|\mathcal{M}v(t) - \mathcal{M}w(t)\|_p \lesssim (\|v\|_{E_\tau} + \|w\|_{E_\tau})^{\gamma-1}\|v - w\|_{E_\tau}\tau^\alpha, \quad t \in [\widetilde{T}, \widetilde{T} + \tau].$$

Therefore, there exists $C_{10} > 0$ such that

$$\|\mathcal{M}v - \mathcal{M}w\|_{E_\tau} \leq C_{10}\tau^\alpha(\|v\|_{E_\tau} + \|w\|_{E_\tau})^{\gamma-1}\|v - w\|_{E_\tau}, \quad v, w \in E_\tau. \tag{4.14}$$

We also find that

$$\begin{aligned}\left\|Z(t) \star u_0 + \int_0^{\widetilde{T}} Y(t-s) \star u^\gamma(s)ds\right\|_\infty &\leq \|Z(t) \star u_0\|_\infty + \int_0^{\widetilde{T}} \|Y(t-s)\|_1\|u^\gamma(s)\|_\infty ds \\ &\lesssim \|u_0\|_\infty + M^\gamma \int_0^{\widetilde{T}} (t-s)^{\alpha-1}ds \\ &\lesssim \|u_0\|_\infty + M^\gamma(t^\alpha - (t - \widetilde{T})^\alpha) \\ &\lesssim \|u_0\|_\infty + M^\gamma \widetilde{T}^\alpha\end{aligned}$$

and that

$$\left\|Z(t) \star u_0 + \int_0^{\widetilde{T}} Y(t-s) \star u^\gamma(s)ds\right\|_p \lesssim \|u_0\|_p + M^{\gamma-1}K\widetilde{T}^\alpha,$$

that is, there exists $C_{11} > 0$ such that

$$\left\|Z(t) \star u_0 + \int_0^{\widetilde{T}} Y(t-s) \star u^\gamma(s)ds\right\|_{E_\tau} \leq C_{11}\left(\|u_0\|_\infty + \|u_0\|_p + M^{\gamma-1}(M+K)\widetilde{T}^\alpha\right). \tag{4.15}$$

Let $R = 2C_{11}\left(\|u_0\|_\infty + \|u_0\|_p + M^{\gamma-1}(M+K)\widetilde{T}^\alpha\right)$. If we consider the closed ball

$$B_{E_\tau} := \{w \in E_\tau : \|w\|_{E_\tau} \leq R\},$$

then estimates (4.14), with $v = 0$, and (4.15) show that $\mathcal{M} : B_{E_\tau} \to B_{E_\tau}$ is a contraction whenever τ is small enough (see Theorem 3.2.1), thus showing that $\mathcal{M}$ has a unique fixed point $w' \in B_{E_\tau}$. Moreover, since $u \geq 0$ we obtain that $w' \geq 0$ in $[\widetilde{T}, \widetilde{T} + \tau) \times \mathbb{R}^d$ following the same arguments as in the proof of Theorem 4.2.2, but one must now use the fact that

$$v_n(t) = Z(t)\star\left(u_0 + \frac{1}{n}\right) + \int_0^{\widetilde{T}} Y(t-s)\star\left(u + \frac{1}{n}\right)^\gamma(s)ds + \int_{\widetilde{T}}^t Y(t-s)\star g_n(v_n(s))ds,$$

for all $n \in \mathbb{N}$ and $v_n \in L_\infty(\widetilde{T}, \widetilde{T} + \tau) \times \mathbb{R}^d$. However, this leads to a contradiction with the definition of $\widetilde{T}$, and therefore $\lim_{t\to\widetilde{T}^-}\|u(t)\|_\infty = +\infty$.

□

The final result of this section deals with the case $\gamma > 1 + \frac{\beta}{d}$. For this purpose, as in Section 4.2, we set

$$\kappa = \begin{cases} \frac{d}{\beta}, & d > \beta, \\ 1, & otherwise \end{cases}.$$

We also define $H_2^\beta(\mathbb{R}^d) := \overline{C_0^\infty(\mathbb{R}^d)}^{\|\cdot\|_{\Psi_\beta,L_2}}$, with the closure being respect to the graph norm $\|\cdot\|^2_{\Psi_\beta,L_2} = \|\cdot\|_2^2 + \|\Psi_\beta(-i\nabla)(\cdot)\|_2^2$.

Theorem 4.3.1. *Let $\alpha \in (0,1)$ and $\beta \in (0,2)$. Assume the hypothesis $(\mathcal{H}_1)$ holds. Suppose that $\gamma > 1 + \frac{\beta}{d}$, that $\max\left(1, \kappa, \frac{d(\gamma-1)}{\beta}\right) < p < \infty$ and that $1 = p' < \frac{d}{\beta}(\gamma - 1)$ whenever $d < \beta$, or $\frac{d}{\beta} < p' < \frac{d}{\beta}(\gamma-1)$ whenever $d \geq \beta$. If $u_0 \in L_1(\mathbb{R}^d) \cap H_2^\beta(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$ is sufficiently small and non-negative, then there exists a global solution u to (4.1) in the sense of Definition 4.0.1 and the optimal time decay estimate*

$$\|u(t)\|_1 + t^{\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)}\|u(t)\|_p + t^{\frac{\alpha d}{\beta p'}}\|u(t)\|_\infty \lesssim (\|u_0\|_1 + \|u_0\|_p + \|u_0\|_\infty)$$

is true for all $t \geq 1$.

Remark 4.3.1. *Whenever $d \leq \beta$, the existence of parameter p' follows from the fact that $\gamma > 1 + \frac{\beta}{d}$. However, in the case $d > \beta$ one can not generally guarantee the existence of p'.*

Proof. We consider the Banach space

$$E := C([0,\infty); L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)) \cap L_\infty((0,\infty); L_\infty(\mathbb{R}^d)),$$

with the norm

$$\|v\|_E := \sup_{t\geq 0}\left(\langle t\rangle^{\frac{\alpha d}{\beta}\left(\frac{1}{p'}-\frac{1}{p}\right)}\|v(t,\cdot)\|_p + \|v(t,\cdot)\|_1\right) + \sup_{t>0}\{t\}^{\frac{\alpha d}{\beta p}}\langle t\rangle^{\frac{\alpha d}{\beta p'}}\|v(t,\cdot)\|_\infty,$$

where $\langle t\rangle := \sqrt{1+t^2}$ and $\{t\} := \dfrac{t}{\sqrt{1+t^2}}$. We define on E the operator

$$\mathcal{M}(v)(t,x) := \int_{\mathbb{R}^d} Z(t,x-y)u_0(y)dy + \int_0^t\int_{\mathbb{R}^d} Y(t-s,x-y)|v(s,y)|^{\gamma-1}v(s,y)dyds$$

and similar arguments as in Sections 3.2 and 3.3 show that

$$\mathcal{M}(v) \in C([0,\infty); L_p(\mathbb{R}^d) \cap L_1(\mathbb{R}^d))$$

and that

$$\|Z(t,\cdot) \star u_0\|_\infty \leq \|Z(t,\cdot)\|_1\|u_0\|_\infty = \|u_0\|_\infty, \quad t > 0.$$

For $0 < t \leq 1$ we have that

$$\begin{aligned}\left\|\int_0^t Y(t-s,\cdot)\star|v(s,\cdot)|^{\gamma-1}v(s,\cdot)ds\right\|_\infty &\leq \int_0^t \|Y(t-s,\cdot)\|_1\||v(s,\cdot)|^{\gamma-1}v(s,\cdot)\|_\infty ds \\ &\lesssim \sup_{(t,x)\in[0,1]\times\mathbb{R}^d}|v(t,x)|^\gamma\int_0^t(t-s)^{\alpha-1}ds \\ &\lesssim \sup_{(t,x)\in[0,1]\times\mathbb{R}^d}|v(t,x)|^\gamma\end{aligned}$$

and for $t > 1$ we obtain (see Section 3.3)

$$\begin{aligned}
&\left\| \int_0^t Y(t-s,\cdot) \star |v(s,\cdot)|^{\gamma-1} v(s,\cdot) ds \right\|_\infty \\
&\leq \int_0^t \|Y(t-s,\cdot)\|_{\frac{p'}{p'-1}} \| |v(s,\cdot)|^{\gamma-1} v(s,\cdot)\|_{p'} ds \\
&\lesssim \|v\|_E^\gamma \int_0^t (t-s)^{-\frac{\alpha d}{\beta p'}+\alpha-1} s^{-\frac{\alpha d}{\beta p}(\gamma-1)} \langle s\rangle^{-\frac{\alpha d}{\beta}(\gamma-1)\left(\frac{1}{p'}-\frac{1}{p}\right)} ds \\
&\lesssim \|v\|_E^\gamma t^{-\frac{\alpha d}{\beta p'}} \\
&\leq \|v\|_E^\gamma .
\end{aligned}$$

This proves that

$$\mathcal{M}(v) \in L_\infty((0,\infty); L_\infty(\mathbb{R}^d)).$$

Besides, as in Section 3.3 one finds that

$$\|Z \star u_0\|_E \leq C_1 \left(\|u_0\|_1 + \|u_0\|_p + \|u_0\|_\infty\right)$$

and that the operator $\mathcal{M}$ is a contraction in the closed ball $B_R = \{v \in E : \|v\|_E \leq R\}$ of radius $R = 2C_1 \left(\|u_0\|_1 + \|u_0\|_p + \|u_0\|_\infty\right)$. Consequently there exists a fixed point $\widetilde{u}$ which is unique in E because of Gronwall's inequality ([69, Corollary 2]).

Let $T > 0$. We define the Volterra equation

$$u(t) = u_0 + g_\alpha * |\widetilde{u}|^{\gamma-1}\widetilde{u}(t) + g_\alpha * Au(t), \quad 0 \leq t \leq T,$$

and by proceeding as in Section 3.4, since $u_0 \in H_2^\beta(\mathbb{R}^d)$, we find that there exists a unique strong solution $u \in L_2([0,T]; H_2^\beta(\mathbb{R}^d))$, and it satisfies the variation of parameters formula

$$u(t) = \frac{d}{dt}\int_0^t S(s)\left(u_0 + g_\alpha * |\widetilde{u}|^{\gamma-1}\widetilde{u}\right)(t-s)ds.$$

On the other hand, similar arguments as in Lemma 3.4.1 show that the fixed point $\widetilde{u}$ satisfies

$$\widetilde{u}(t) = \frac{d}{dt}\int_0^t S(s)\left(u_0 + g_\alpha * |\widetilde{u}|^{\gamma-1}\widetilde{u}\right)(t-s)ds$$

and therefore $\widetilde{u} = u$. This holds for any $T > 0$ which implies that u is global. □

Remark 4.3.2. *Since $u_0 \in L_\infty(\mathbb{R}^d)$, Theorem 4.2.2 guarantees the positivity of the global solution u on $[0,T)$ for some $T > 0$.*

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