UNIVERSIDAD DE CONCEPCIÓN
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE FÍSICA

Extension of the Poincaré group with half-integer spin generators: hypergravity, asymptotically flat structure and energy bounds

Thesis to obtain the degree of Doctor in Physics

by

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Concepción, Chile
October 2015

Para mis padres

Agradecimientos

Para empezar quiero agradecer a toda mi familia por siempre haberme brindado su amor e incondicional apoyo en mis ideas y proyectos. En particular le doy las gracias a mis primeros profesores; Marisol y Gustavo, quienes inculcaron en mí valores que no tienen precio, como la pasión por el trabajo y la responsabilidad con que uno debe afrontar nuevos desafíos.

Quiero agradecer a mis compañeros y amigos de la Universidad de Concepción (UdeC), con los cuales compartí gratos y entretenidos momentos, especialmente a Gabriel Araneda, Jorge Bernales, Miguel Campos, Nataly Cisternas y Cristián Jara.

Es un deber agradecer también a los profesores de la Universidad de Concepción. En particular al profesor Guillermo Rubilar por sus claras y bien preparadas clases. No puedo olvidar al profesor Patricio Salgado quien me entregó su desinteresada ayuda cuando decidí adentrarme en el fascinante mundo de la Física Teórica, siendo no solo un gran profesor si no también una excelente persona.

Debo destacar la excelente labor del profesor Jaime Araneda como el director del programa de Doctorado en Ciencias Físicas, quien siempre se mostró dispuesto a ayudarme en los distintos escollos burocráticos que tuve que sortear en este largo camino.

Agradezco también al Centro de Estudios Científicos (CECs) de Valdivia por darme la oportunidad de formar parte de este instituto de nivel mundial en investigación en Física Teórica. Especialmente a los profesores Ricardo Troncoso, Jorge Zanelli y Cristián Martínez quienes siempre han tenido sus puertas abiertas para cualquier consulta.

Es realmente un honor haber trabajado no solo con un excelente investigador y profesor sino también una gran persona, me refiero a mi maestro Ricardo Troncoso quien siempre tuvo la mejor disposición para guiarme durante mi investigación. Su rigurosidad e ingenio han sido y serán siempre una fuente de inspiración para la realización de trabajos del más alto estándar.

Deseo agradecer a los amigos y colegas que a lo largo de este viaje he conocido, con quienes he compartido interesantes conversaciones y han sido un pilar fundamental en mi desarrollo como investigador. En particular a Patrick Concha, Cristián Erices, Ernesto Frodden, Javier Matulich, Pablo Pais, Alfredo Pérez, Miguel Riquelme, Evelyn Rodríguez, Patricio Salgado, David Tempo y Adolfo Toloza. Un agradecimiento especial a Hernán A. González quien me brindó su amistad y casa en parte de mi pasantía en la Universidad Libre de Bruselas, Bélgica.

Mi estadía en la Universidad Libre de Bruselas no hubiera sido posible sin el apoyo financiero de la Comisión Nacional de Investigación Científica y Tecnológica CONICYT a través del beneficio para pasantías doctorales. En ella participé en la “Amsterdam-Brussels-Paris Doctoral School”, la cual me dio la oportunidad de aprender sobre los mas recientes avances que se han llevado a cabo en investigación en Física Teórica. En especial agradecer al profesor Marc Henneaux por la gentileza de invitarme a realizar esta enriquecedora estadía de investigación.

Por último, quiero mencionar a quien siempre me ha acompañado desde mi entrada a la universidad, a mi mejor amiga y compañera, Marcela Cárdenas. Has estado ahí en los buenos y malos momentos, siempre dispuesta a escucharme y a aconsejarme. Ha sido un placer trabajar y estar a tu lado. Te amo.

La dedicación exclusiva al programa de Doctorado ha sido posible gracias al financiamiento de la Escuela de Graduados de la UdeC y al proyecto de CONICYT “Southern Theoretical Physics Laboratory” ACT-91 durante el 2011 y a la beca de Doctorado Nacional de CONICYT en el periodo 2012-2015.

Abstract

This thesis presents a novel extension of the Poincaré group with half-integer spin generators and a reformulation of hypergravity. This theory describes fermionic higher spin fields minimally coupled to gravity in three spacetime dimensions. The subsequent studies of its asymptotically flat structure and energy bounds are also carried out.

We start our discussion in the case of three spacetime dimensions, where it is shown that the theory of hypergravity can be reformulated in order to incorporate this structure as its local gauge symmetry. Since the algebra admits a nontrivial Casimir operator, the theory can be described in terms of gauge fields associated to the extension of the Poincaré group with a Chern-Simons action. We also show that the Poincaré group can be extended with arbitrary half-integer spin generators for $d \geq 3$ dimensions.

The asymptotic structure of three-dimensional hypergravity is also analyzed. In the case of gravity minimally coupled to a spin-$5/2$ field, a consistent set of boundary conditions is proposed, being wide enough so as to accommodate a generic choice of chemical potentials associated to the global charges. The algebra of the canonical generators of the asymptotic symmetries is given by a hypersymmetric nonlinear extension of BMS_3. It is shown that the asymptotic symmetry algebra can be recovered from a subset of a suitable limit of the direct sum of the $W_{(2,4)}$ algebra with its hypersymmetric extension. The presence of hypersymmetry generators allows to construct bounds for the energy, which turn out to be nonlinear and saturate for spacetimes that admit globally-defined Killing vector-spinors. The null orbifold or Minkowski spacetime can then be seen as the corresponding ground state in the case of fermions that fulfill periodic or antiperiodic boundary conditions, respectively. The hypergravity theory is also explicitly extended so as to admit parity-odd terms in the action. It is then shown that the asymptotic symmetry algebra includes an additional central charge, being proportional to the coupling of the Lorentz-Chern-Simons form. The generalization of these results in the case of gravity minimally coupled to arbitrary half-integer spin fields is also carried out. The hypersymmetry bounds are found to be given by a suitable polynomial of degree $s + 1/2$ in the energy, where s is the spin of the fermionic generators.

This thesis captures the work and results that were presented in the following publications:

- *"Extension of the Poincaré group with half-integer spin generators: hypergravity and beyond"*
  Oscar Fuentealba, Javier Matulich and Ricardo Troncoso,
  JHEP 1509,003 (2015) [arXiv:1505.06173 [hep-th]]

- *"Asymptotically flat structure of hypergravity in three spacetime dimensions"*
  Oscar Fuentealba, Javier Matulich and Ricardo Troncoso,
  JHEP 1510, 009 (2015) [arXiv:1508.04663 [hep-th]]

Contents

Chapter 1

Introduction

Nowadays, we have the good fortune of witnessing the era in which the simplest minimal realistic versions of supersymmetric field theories are about to be either tested or falsified by the Large Hadron Collider (LHC). The underlying geometric structure of these kind of theories, as well as most of their widely studied extensions, relies on the super-Poincaré group. Its corresponding superalgebra has the following nonvanishing (anti)commutators

$$
\begin{aligned}
[J_{ab}, J_{cd}] &= J_{ad}\eta_{bc} - J_{bd}\eta_{ac} + J_{ca}\eta_{bd} - J_{cb}\eta_{ad} \,, \\
[J_{ab}, P_c] &= \eta_{ac}P_b - \eta_{bc}P_a \,, \\
[J_{ab}, Q_\alpha] &= \frac{1}{2}(\Gamma_{ab})^{\beta}{}_{\alpha} Q_\beta \,, \\
\{Q_\alpha, Q_\beta\} &= -\frac{1}{2}\left(C\Gamma^c\right)_{\alpha\beta} P_c \,.
\end{aligned}
\tag{1.1}
$$

A supersymmetric field theory exhibits a symmetry between bosonic and fermionic fields. In this kind of theories bosons and fermions occur always in pairs, and these supersymmetric partners can be accommodated within an irreducible representation of the super-Poincaré algebra (1.1) (see, e. g., [1, 2]). The case of the graviton is particularly interesting because there are two possible interacting theories with different sypersymmetric partners; one possibility is supergravity and has undergone spectacular developments [3, 4]. In this case the supersymmetric partner is a massless spin-3/2 field called gravitino (Rarita-Schwinger field). The consistency of this theory can be checked at least in two ways (see, e. g., [5]). The first one comes from the consistency of the Rarita-Schwinger field equation yielding that the background has to be curved, fulfilling the Einstein equations. The second one is the existence of an appropriate set of gauge transformations from which the invariance of the action can be verified. This is separately proved at the quadratic and quartic terms on fermions.

A second possibility for superpartner is considering a massless spin-5/2 field, a theory that has been called *hypergravity.* Hypergravity was explored in the early days of supergravity as a potentially interesting alternative, but it was soon realized that suffers from various inconsistencies. The heart of these obstructions relies on the fact that the action does not satisfy the appropriate higher-spin gauge invariance conditions, leading to field equations that are not even algebraically consistent [6, 7]. Specifically, the gauge variation of Einstein-Hilbert action is proportional to the Ricci-Tensor, but the variation of the minimally coupled higher spin action is proportional to the full Riemann tensor instead. Then the contribution of both actions cannot be generically cancelled out. However, in three spacetime dimension there is a well-known relation between the Riemann tensor and the Ricci tensor, which reads

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)R\,. \tag{1.2}$$

Thus in principle is possible to circumvent this difficulty, and hence to formulate a consistent interacting theory between gravity and the spin-5/2 field. Indeed, this theory exists and it was found by Aragone and Deser in 1984 [8]. The invariance of this theory was shown in the so-called "1.5 formalism" of standard supergravity where the equation associated to the spin connection, which fixes the value of the torsion, has to be fulfilled. Due to the complexities that come from the the theory in this formalism, the corresponding study of its asymptotic structure was hard to be carried out. In this point we would like to stress that a deeper understanding of the theory cannot be attained unless it is endowed with a consistent set of boundary conditions, which would allow to study the asymptotic structure of hypergravity. The algebra of the asymptotic symmetries allows the possibility of analyzing of establishing energy bounds. This is because of the presence of fermionic generators [9, 10]. For all these purposes it would be helpful to find the underlying geometrical structure of hypergravity which would serve for its formulation as a gauge theory, being this one of the main goals of the present work.

The plan of this thesis is the following:

Chapter 2 is devoted to a brief review of the canonical prescription of a three-dimensional Chern-Simons action and its relation with pure gravity in the presence of either negative or vanishing cosmological constant. We will summarize the formulation of the standard Brown-Henneaux boundary conditions for pure three-dimensional gravity with negative cosmological constant in terms of gauge fields. Its asymptotic symmetry algebra is also obtained, corresponding to an (infinite-dimensional) central extension of the conformal algebra in two dimensions, specifically, given by two copies of the Virasoro algebra endowed with a central charge

$c = 3\ell/2G$. Analogously, we show the standard boundary conditions for pure gravity with a vanishing cosmological constant in the language of gauge fields, obtaining that the asymptotic symmetry algebra of the canonical generators is the infinite-dimensional algebra with a central charge $c = 3/G$ known as BMS_3, being the algebra of the Bondi-Metzer-Sachs group in three dimensions. Finally, we recover the BMS_3 algebra from the two copies of the Virasoro algebra by performing a suitable Inönü-Wigner contraction.

In Chapter 3 a novel extension of the Poincaré group with half-integer fermionic generators in $d \geq 3$ dimensions, is introduced, which we dub it *the hyper-Poincaré group*. It is shown that the theory of Aragone and Deser of three-dimensional hypergravity can be reformulated so as to incorporate this structure as its local gauge symmetry. Since the algebra admits a nontrivial Casimir, the theory can be described in terms of gauge fields, where the dynamical fields go along the generators of the hyper-Poincaré group. The action is given by a three-dimensional Chern-Simons form. The results will be generalized to the case of General Relativity minimally coupled to arbitrary half-integer spin fields.

In Chapter 4 we summarize the recent results of the analysis of the asymptotic structure of hypergravity in three spacetime dimensions with negative cosmological constant, where consistency requires spin-4 fields [11]. This theory can be formulated through a Chern-Simons action for $OSp\,(1|4) \otimes OSp\,(1|4)$, which has one fermion per copy, i. e., $\mathcal{N} = (1,1)$. The asymptotic symmetry superalgebra of the canonical generators corresponds to two copies of the hypersymmetric extension $W_{(2,\frac{5}{2},4)}$ of $W_{(2,4)}$. The hypersymmetry bounds that come from the anticommutator of the hypersymmetry generators, turn out to be nonlinear. As explained in [11] they saturate for extremal higher spin black holes and $sp\,(4)$-solitonic solutions, where the former possess 1/4 of the hypersymmetries and the latter ones are maximally (hyper)symmetric.

In Chapter 5 we study the asymptotic structure of three-dimensional hypergravity with a vanishing cosmological constant. A consistent set of boundary conditions is proposed, being wide enough so as to accommodate a generic choice of chemical potentials associated to the global charges. In this case, the algebra of canonical generators of the asymptotic symmetries is given by a hypersymmetric nonlinear extension of BMS_3, which can also be recovered from a subset of a suitable Inönü-Wigner contraction of the direct sum of the $W_{(2,4)}$ algebra with its hypersymmetric extension $W_{(2,\frac{5}{2},4)}$. As explained above, the presence of hypersymmetry generators allows to construct bounds for the energy which, as in the case of negative cosmological constant, are nonlinear and saturate for spacetimes that admit globally-defined *Killing vector-spinors*, selecting the same spectrum as supergravity. The theory of hypergravity is extended so as to include parity-odd terms in the action. In this case the asymptotic symmetry algebra is also a nonlinear extension of BMS_3 but it acquires an additional central charge along the Virasoro subalgebra being proportional to the coupling constant of the Lorentz-Chern-Simons term. The generalization of these results in the case of gravity minimally coupled to *arbitrary* half-integer spin fields is also carried out.

Finally, Appendix A is devoted to our conventions and some useful identities. The fundamental (5×5) matrix representation of the generators of $OSp\,(1|4)$ is shown in Appendix B. In Appendix C the Lie-algebra-valued parameter and the transformation law of the fields for the case of the asymptotically AdS structure of hypergravity are explicitly written. In Appendix D, an alternative interesting form to obtain the explicit form of the Killing vector spinor is presented, while Appendix E includes the asymptotic hypersymmetry algebra in the case of fermionic fields of spin 3/2 (supergravity), as well as for fields of spin 7/2 and 9/2.

Chapter 2

Asymptotic structure of three-dimensional pure gravity

Three-dimensional gravity possesses an extraordinarily rich asymptotic structure, even more than its four-dimensional counterpart. In the former case, it was shown that the algebra of the canonical generators at space-like infinity of asymptotically anti-de Sitter (AdS) spacetimes turns out to be two copies of the infinte-dimensional Virasoro algebra endowed with a central charge [12]. On the other hand, in the latter case, corresponding to four-dimensional gravity the asymptotic symmetry algebra is only given by $so(3,2)$ [13]. The three-dimensional result has been very relevant and indeed it can be seen as the precursor of the so-called AdS/CFT correspondence (see e. g. [14]) and also has served to recover the Bekenstein-Hawking entropy for the BTZ black hole [15, 16] through a microscopical derivation [17].

It is well known that the topological roots of gravity in three spacetime dimensions allows to formulate it as a Chern-Simons theory for the (A)dS or Poincaré group in the case of (negative) positive or vanishing cosmological constant, respectively [18, 19]. This is reflected on the fact that all the solutions in vacuum have constant curvature which agrees with the lack of local degrees of freedom, meaning that the solutions are only sensitive to the topology of the spacetime manifold.

This chapter (based on the review article [20]) is devoted to reproduce the pure gravity asymptotic structure with negative and vanishing cosmological constant in terms of a Chern-Simons theory. Taking this into account, we first introduce the Hamiltonian formulation and global charges of a three-dimensional Chern-Simons theory through a very brief review. Then, we explicitly perform the formulation of three-dimensional gravity as a Chern-Simons theory, and finally we will recover the asymptotic symmetry algebra in the cases of asymptotically AdS and flat spacetimes.

2.1 Hamiltonian formulation and global charges

A three-dimensional Chern-Simons theory is a topological theory, i. e., it does not require the existence of a spacetime metric, being only sensitive to the topology of the manifold upon which is defined. Let us consider a manifold M, locally described by a set of coordinates x^μ. The topology of M is assumed to be $\Sigma \times R$, where Σ is a spacelike surface and R is a real timelike line. The Chern-Simons action reads

$$I_{CS} = \frac{k}{4\pi} \int\limits_M \left\langle AdA + \frac{2}{3} A^3 \right\rangle , \tag{2.1}$$

where k is a constant, $A = A_\mu^I T_I dx^\mu$ is the gauge field, and T_I stand for the generators of a Lie algebra $\mathfrak{g}$. This algebra is assumed to admit an invariant nondegenerate bilinear form $g_{IJ} = \langle T_I, T_J \rangle$. The field equations in this case imply that the connection becomes locally flat on-shell, namely, it has a vanishing curvature

$$F = dA + A^2 = 0 . \tag{2.2}$$

The theory is then devoid of local degrees of freedom. Note that the Chern-Simons action (2.1) is of first order and then is already in a Hamiltonian form, where

$$I_H = -\frac{k}{4\pi} \int\limits_{\Sigma \times R} dt d^2x \varepsilon^{ij} \left\langle A_i \dot{A}_j - A_t F_{ij} \right\rangle + B_H . \tag{2.3}$$

Here B_H is a boundary term that has to be included in order to ensure that the action attains an extremum for a suitable set of boundary conditions. It is straightforward to verify that A_i correspond to the dynamical fields, whose Poisson brackets are given by

$$\left\{ A_i^I (x) , A_j^J (x') \right\} = \frac{2\pi}{k} g^{IJ} \varepsilon_{ij} \delta (x - x') . \tag{2.4}$$

From the Hamiltonian form of the action (2.3), one can read that the Lagrange multiplier A_t is associated to the constraint

$$G = \frac{k}{4\pi} \varepsilon^{ij} F_{ij} . \tag{2.5}$$

An infinitesimal gauge transformation on the dynamical fields $\delta A_i = \partial_i \Lambda + [A_i, \Lambda]$ spanned by a Lie-algebra-valued parameter Λ is generated by the following smeared generator (see e. g. [21, 22, 23])

$$G(\Lambda) = \int\limits_\Sigma d^2x \langle \Lambda G \rangle . \tag{2.6}$$

Following the Regge-Teitelboim approach [24], in the case where the spacelike section Σ has a boundary ($\partial\Sigma \neq 0$), the generator of the gauge transformations has to be improved by a boundary term $Q(\Lambda)$, being such that its functional variation is well-defined everywhere , i. e.,

$$\tilde{G}(\Lambda) = G(\Lambda) + Q(\Lambda) \,, \tag{2.7}$$

where the variation of the conserved charge associated to the asymptotic gauge symmetry spanned by Λ is given by

$$\delta Q(\Lambda) = -\frac{k}{2\pi} \int\limits_{\Sigma} \langle \Lambda \delta A_\phi \rangle \, d\phi \,, \tag{2.8}$$

which is determined by the dynamical fields at a fixed time slice at the boundary $\partial\Sigma$. Note that if one requires the improved action to be invariant under gauge transformations, the infinitesimal gauge transformation of the Lagrange multiplier is recovered

$$\delta A_t = \partial_t \Lambda + [A_t, \Lambda] \,. \tag{2.9}$$

On the other hand, diffeomorphisms $\delta_\xi A_\mu = -\mathcal{L}_\xi A_\mu$, are equivalent to gauge transformations with parameter $\Lambda = -\xi^\mu A_\mu$, only on-shell, by virtue of the identity $\mathcal{L}_\xi A_\mu = \nabla_\mu (\xi^\nu A_\nu) + \xi^\nu F_{\nu\mu}$. Therefore, the variation of the asymptotic symmetry generator spanned by an asymptotic Killing vector reads

$$\delta Q(\xi) = \frac{k}{2\pi} \int\limits_{\partial\Sigma} \xi^\mu \langle A_\mu \delta A_\phi \rangle \, d\phi \,. \tag{2.10}$$

In order to integrate the variation of the canonical generators (2.8), a precise set of asymptotic conditions must be given. This will be performed in the case of pure gravity in Section 2.3. In the next section, we will explicitly show the relation between three-dimensional gravity and a Chern-Simons theory.

2.2 General Relativity in three-spacetime dimensions as a Chern-Simons theory

General Relativity in three spacetime dimensions shares important conceptual features with the four-dimensional theory, however in the former case some difficulties that arise in higher dimensions are avoided. In this sense, the absence of local degrees of freedom suggests that the theory can be formulated in terms of a topological theory, requiring less structure in order to formulate it. Indeed, this fact can be verified by a very simple counting argument; in d spacetime dimensions, the phase space of the theory is characterized by a spatial metric defined on a constant-time hypersurface of $d(d-1)/2$ components, but its conjugate momentum has another $d(d-1)/2$ components, then one has $d\,(d-1)$ degrees of freedom per spacetime point. Nonetheless, it is well-known that there are d first-class constraints, and then one can eliminate d degrees of freedom by coordinate choices. Therefore, as a result, one has $d\,(d-3)\,/2$ local degrees of freedom per spacetime point, being evident the absence of local degrees of freedom in three dimensions.

As mentioned above, General Relativity in vacuum can be formulated as a Chern-Simons theory [18, 19]. In the case of negative cosmological constant, the dreibein, and dualized spin connection $\left(\omega^a = \frac{1}{2}\varepsilon^{abc}\omega_{bc}\right)$ correspond to the components of the gauge connection given by

$$A = \frac{e^a}{\ell}P_a + \omega^a J_a \,, \tag{2.11}$$

that takes values in the $so\,(2,2)$ algebra, being spanned by the Lorentz generators J_a and the generators of the AdS boosts P_a. Here ℓ corresponds to the AdS radius. The commutation rules for $so(2,2)$ read

$$[J_a, J_b] = \varepsilon_{abc}J^c \quad , \quad [J_a, P_b] = \varepsilon_{abc}P^c \quad , \quad [P_a, P_b] = \varepsilon_{abc}J^c \,. \tag{2.12}$$

Since this algebra admits an invariant bilinear form, whose only nonvanishing components are given by $\langle J_a P_b \rangle = \eta_{ab}$, the Chern-Simons action (2.1) reduces, up to a boundary term, to

$$I\,[e,\omega] = \frac{k}{4\pi}\int 2R^a e_a + \frac{1}{3\ell^2}\varepsilon_{abc}e^a e^b e^c \,, \tag{2.13}$$

where $R^a = d\omega^a + \frac{1}{2}\varepsilon^{abc}\omega_b\omega_c$ is the dual of the curvature two-form and the level is fixed in terms of the AdS radius and the Newton constant as $k = \ell/4G$. Note that this is the precise form of the action of three-dimensional General Relativity with negative cosmological constant $-1/\ell^2$. The field equations $F = 0$ imply that

$$R^a = -\frac{1}{2\ell^2}\varepsilon^{abc}e_b e_c \quad , \quad T^a = De^a = 0 \,, \tag{2.14}$$

which means that the spacetime curvature is constant and it has a vanishing torsion. Here D denotes the Lorentz covariant derivative. By construction, the action changes by a boundary term under the following infinitesimal local gauge transformations spanned by the parameter $\lambda = \frac{\lambda^a}{\ell}P_a + \sigma^a J_a$,

$$\delta e^a = D\lambda^a - \varepsilon^{abc}\sigma_b e_c \quad , \quad \delta\omega^a = D\sigma^a - \frac{1}{\ell^2}\varepsilon^{abc}\lambda_b e_c \,. \tag{2.15}$$

It is worth noting that the AdS algebra in three dimensions $so\,(2,2)$ is isomorphic to two copies of the special linear algebra in two dimensions, $sl\,(2,R)$. Then $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, where $\mathfrak{g}_\pm$ stands for two copies of $sl\,(2,R)$. The $sl(2,R)$ algebra reads

$$[L_i, L_j] = (i-j)\, L_{i+j}\,, \tag{2.16}$$

where the generators L_i, with $i = -1, 0, 1$, are assumed to be same for both copies and are chosen to be

$$L_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad L_0 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad L_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\,. \tag{2.17}$$

Splitting the connection in two independent $sl\,(2,R)$-valued gauge fields, according to $A = A^+ + A^-$, the Chern-Simons action (2.1) reduces to

$$I_{CS} = I_{CS}\left[A^+\right] - I_{CS}\left[A^-\right]\,, \tag{2.18}$$

so that the invariant bilinear form corresponds to the trace in the representation (2.17), i. e., $\langle ... \rangle = tr\,(...)$. The relation between the Chern-Simons connections written in terms of the spin connection ω and the dreibein e is now given by

$$A^\pm = \omega \pm \frac{e}{\ell}\,. \tag{2.19}$$

The metric is recovered from $g_{\mu\nu} = 2tr\,(e_\mu e_\nu)$, which is manifestly invariant under the local Lorentz transformations which is the diagonal subgroup of $SL(2,R) \otimes SL(2,R)$.

On the other hand, General Relativity without cosmological constant is a Chern-Simons theory for the Poincaré group. In this case the gauge connection simply reads

$$A = e^a P_a + \omega^a J_a\,, \tag{2.20}$$

where the nonvanishing commutators of the Poincaré algebra reads

$$[J_a, J_b] = \varepsilon_{abc} J^c \quad , \quad [J_a, P_b] = \varepsilon_{abc} P^c\,. \tag{2.21}$$

By using the same invariant bilinear form that in the case of negative cosmological constant, it is straightforward to verify that the Chern-Simons action (2.1) now reduces, up to a boundary term, to

$$I\,[e,\omega] = \frac{k}{4\pi}\int 2R^a e_a\,, \tag{2.22}$$

where the level is now fixed only in terms of the Newton constant as $k = 1/4G$, and the Chern-Simons field equations imply now that

$$R^a = 0 \quad , \quad T^a = De^a = 0\,. \tag{2.23}$$

By construction, the action (2.22) is invariant under the following local gauge transformation generated by $\lambda = \lambda^a P_a + \sigma^a J_a$,

$$\delta e^a = D\lambda^a - \varepsilon^{abc}\sigma_b e_c \quad , \quad \delta\omega^a = D\sigma^a \, . \tag{2.24}$$

Note that, as expected, the results coincide with those after performing the flat space limit $\ell \to \infty$ for which the $so(2,2)$ algebra leads to the Poincaré algebra after an Inönü-Wigner contraction.

2.3 Asymptotic structure of General Relativity in three spacetime dimensions

In General Relativity and in other gauge theories formulated on spacetimes with boundaries, asymptotic symmetries play a fundamental role. Actually, they are the physical symmetries of the theory being necessary for a suitable definition of conserved charges [12, 24]. The asymptotic symmetries, in the context of General Relativity, are those gauge transformations that leave the spacetime configurations asymptotically invariant. The procedure for obtaining them is the following; one starts with a group of global symmetries which is related with a background solution, and one must find appropriate boundary conditions at infinity that should contain solutions of physical interest. These conditions must be unchanged under the action of the group of the global symmetries. Then one tries to find the most general set of transformations that leave invariant the asymptotic conditions. One would expect to recover the global symmetry group but in some cases the situation is rather different, obtaining a group much bigger than the group of isometries of the background configuration. These symmetry transformations must have well-defined canonical generators [24], and these should obey an isomorphic central extension of the Lie algebra of the infinitesimal symmetries with a possible central charge that cannot be eliminated by adding terms to the canonical generators. There are two important examples that reflects the effects aforementioned, the first was observed in the 60's in the case of four-dimensional asymptotically flat spacetimes, where the vacuum configuration is the Minkowski spacetime and its corresponding isometry is given by the Poincaré group. In this case the algebra of the canonical generators of the asymptotic symmetries turns out to be the infinite-dimensional Bondi-Metzer-Sachs (BMS_4) group [25, 26]. The second one was the aforementioned case of three-dimensional asymptotically AdS spacetimes addressed by Brown and Henneaux in 1986 [12], where the Poisson brackets of canonical generators of the asymptotic symmetry correspond to two copies of the Virasoro algebra with a central term. The last case will be analyzed in detail but from the Chern-Simons point of view, by writting the analog of the Brown-Henneaux boundary conditions in the language of gauge fields, which leads to the expected results. Analogously, by following the same procedure we will review how to find the asymptotic structure associated to three-dimensional asymptotically flat spacetimes, where we recover the infinite-dimensional BMS_3 algebra endowed with a central charge found in [27, 28].

2.3.1 Negative cosmological constant: Brown-Henneaux boundary conditions

The Brown-Henneaux boundary conditions of three-dimensional gravity with negative cosmological constant written in terms of gauge fields [29] read

$$A_\phi^\pm = L_{\pm1} - \frac{2\pi}{k}\mathcal{L}^\pm L_{\mp1} + \mathcal{O}\left(\frac{1}{r}\right) \quad , \quad A_r^\pm = \mathcal{O}\left(\frac{1}{r}\right) , \tag{2.25}$$

where $\mathcal{L}^\pm$ is a function of time t and the angular coordinate ϕ. It is worth to note that the radial coordinate can be entirely captured by a gauge transformation on the connection

$$A^\pm = g_\pm^{-1} a^\pm g_\pm + g_\pm^{-1} dg_\pm , \tag{2.26}$$

where the group element is given by $g_\pm = e^{\pm r L_0}$. The dynamical fields in this case are the leading terms of the asymptotic behaviour (2.25), i. e.,

$$a_\phi^\pm = L_{\pm1} - \frac{2\pi}{k}\mathcal{L}^\pm L_{\mp1} \quad , \quad a_r^\pm = 0 . \tag{2.27}$$

Then, the relevant terms of the dynamical field go only along the angular components of the gauge connection a. The asymptotic form of the dynamical fields must be preserved under gauge transformations of the form

$$\delta a_\phi^\pm = \partial_\phi \lambda^\pm + \left[a_\phi^\pm, \lambda^\pm\right] . \tag{2.28}$$

Thus, the Lie-algebra valued parameters have to be given by

$$\lambda^\pm\left[\epsilon_\pm\right] = \epsilon_\pm L_{\pm1} \mp \epsilon_\pm{}' L_0 + \frac{1}{2}\left(\epsilon_\pm{}'' - \frac{4\pi}{k}\epsilon_\pm \mathcal{L}^\pm\right) L_{\mp1} , \tag{2.29}$$

provided the functions $\mathcal{L}^\pm$ transform as

$$\delta\mathcal{L}_\pm = \epsilon_\pm \mathcal{L}^{\pm\prime} + 2\epsilon_\pm{}'\mathcal{L}^\pm - \frac{k}{4\pi}\epsilon_\pm{}''' , \tag{2.30}$$

where $\epsilon_\pm = \epsilon_\pm(t, \phi)$ are arbitrary functions and primes denote derivatives with respect to ϕ.

Considering that at spatial infinity, the time evolution of the dynamical fields of the connection $a_\phi^\pm$ is a gauge transformation with gauge parameters equal to the Lagrange multipliers $a_t^\pm$, the most general Lagrange multipliers that preserve the asymptotic conditions (2.27) are of the form [30]

$$a_t^\pm = \lambda^\pm \left[\xi_\pm\right] \, . \tag{2.31}$$

Here $\xi_\pm$ are also arbitrary functions of t and ϕ, and they are assumed to be fixed at the boundary. Consistency of the Lagrange multipliers under gauge transformations demands that the fields $\mathcal{L}^\pm$ satisfy the following field equations at the asymptotic region

$$\dot{\mathcal{L}}_\pm = \xi_\pm \mathcal{L}^{\pm\prime} + 2\xi_\pm{}' \mathcal{L}^\pm - \frac{k}{4\pi}\xi_\pm{}''' \, , \tag{2.32}$$

while the parameters of the asymptotic symmetries fulfill

$$\dot{\epsilon}_\pm = \epsilon_\pm{}' \xi_\pm - \epsilon_\pm \xi_\pm{}' \, . \tag{2.33}$$

These conditions are necessary in order to ensure the conservation of the global charges.

Replacing the gauge parameters (2.29) and the asymptotic conditions (2.27) in the variation of the canonical generators (2.8) we get that

$$\delta Q\left[\lambda\right] = \delta Q\left[\lambda^+\right] - \delta Q\left[\lambda^-\right] \, , \tag{2.34}$$

which can be readily integrated as

$$Q\left[\lambda^\pm\right] = -\int \epsilon_\pm \mathcal{L}_\pm d\phi \, . \tag{2.35}$$

Since the Poisson brackets fulfill

$$\left\{Q\left[\lambda_1^\pm\right], Q\left[\lambda_2^\pm\right]\right\} = \delta_{\lambda_2^\pm} Q\left[\lambda_1^\pm\right] \, , \tag{2.36}$$

the algebra of the canonical generators can be directly obtained from the transformation of the fields (2.30). Expanding in Fourier modes according to $X = \frac{1}{2\pi}\sum_n X_n e^{in\phi}$, the Poisson brackets reduce to two copies of the Virasoro algebra with the same central charge $c = 6k = 3\ell/2G$. The algebras explicitly read

$$i\left\{\mathcal{L}_m^\pm, \mathcal{L}_n^\pm\right\} = (m-n)\,\mathcal{L}_{m+n}^\pm + \frac{k}{2} m^3 \delta_{m+n,0} \, , \tag{2.37}$$

which coincide with the asymptotic symmetry algebra found in the metric formulation [12].

2.3.2 Vanishing cosmological constant

This subsection is devoted to the asymptotic structure in the case of three-dimensional asymptotically flat spacetimes. The asymptotic conditions in terms of gauge fields were first proposed in the context of flat higher spin gravity in [32, 33], and in the context of flat supergravity in [34]. For this purpose we are going to relabel the Poincaré generators according to

$$\begin{aligned} \hat{J}_{-1} = -2J_0 \quad &, \quad \hat{J}_1 = J_1 \quad , \quad \hat{J}_0 = J_2 \, , \\ \hat{P}_{-1} = -2P_0 \quad &, \quad \hat{P}_1 = P_1 \quad , \quad \hat{P}_0 = P_2 \, , \end{aligned} \tag{2.38}$$

so as the nonvanishing commutators of the Poincaré algebra (2.21) now read

$$\left[\hat{J}_m, \hat{J}_n\right] = (m-n)\,\hat{J}_{m+n} \quad , \quad \left[\hat{J}_m, \hat{P}_n\right] = (m-n)\,\hat{P}_{m+n} \, . \tag{2.39}$$

Considering this, the asymptotic behaviour in this case is given by

$$a_\phi = \hat{J}_1 - \frac{\pi}{k}\left(\mathcal{J}\hat{P}_{-1} + \mathcal{P}\hat{P}_{-1}\right) \, , \tag{2.40}$$

where the radial coordinate can be switched on by a gauge transformation with group element $g = e^{\frac{r}{2}\hat{P}_{-1}}$. Here $\mathcal{J}$ and $\mathcal{P}$ depend on the null coordinate u and the angular coordinate ϕ. The asymptotic form of the dynamical fields, in this case, is preserved under the action of gauge transformations spanned by the following parameter

$$\begin{aligned} \lambda\left[T, Y\right] \;&=\; T\hat{P}_1 + Y\hat{J}_1 + -T'\hat{P}_0 - Y'\hat{J}_0 \\ &\quad -\frac{1}{2}\left(\frac{2\pi}{k}Y\mathcal{P} - Y''\right)\hat{J}_{-1} - \frac{\pi}{k}\left(T\mathcal{P} + Y\mathcal{J} - \frac{k}{2\pi}T''\right)\hat{P}_{-1} \, , \end{aligned} \tag{2.41}$$

provided the transformation law of the fields is given by

$$\begin{aligned} \delta\mathcal{P} \;&=\; 2\mathcal{P}Y' + \mathcal{P}'Y - \frac{k}{2\pi}Y''' \, , \\ \delta\mathcal{J} \;&=\; 2\mathcal{J}Y' + \mathcal{J}'Y + 2\mathcal{P}T' + \mathcal{P}'T - \frac{k}{2\pi}T''' \, . \end{aligned} \tag{2.42}$$

where the parameters $T\left(u,\phi\right)$ and $Y\left(u,\phi\right)$ are arbitrary functions. The Lagrange multiplier reads [30]

$$a_u = \lambda\left[\mu_{\mathcal{P}}, \mu_{\mathcal{J}}\right] \, . \tag{2.43}$$

where $\mu_{\mathcal{P}}, \mu_{\mathcal{J}}$ also stand for arbitrary functions of u, ϕ and they are assumed to be fixed at the boundary. Consistency of preserving the asymptotic form of the

Lagrange multiplier now leads to the following field equations

$$\begin{aligned}\dot{\mathcal{P}} &= 2\mathcal{P}\mu_{\mathcal{J}}{}' + \mathcal{P}'\mu_{\mathcal{J}} - \frac{k}{2\pi}\mu_{\mathcal{J}}{}''' \,, \\ \dot{\mathcal{J}} &= 2\mathcal{J}\mu_{\mathcal{J}}{}' + \mathcal{J}'\mu_{\mathcal{J}} + 2\mathcal{P}\mu_{\mathcal{P}}{}' + \mathcal{P}'\mu_{\mathcal{P}} - \frac{k}{2\pi}\mu_{\mathcal{P}}{}''' \,, \end{aligned} \tag{2.44}$$

which have to be fulfilled in the asymptotic region. In turn, the parameters of the transformation satisfy the following conditions

$$\begin{aligned}\dot{Y} &= \mu_{\mathcal{J}}Y' - \mu_{\mathcal{J}}{}'Y \,, \\ \dot{T} &= \mu_{\mathcal{J}}T' - \mu_{\mathcal{J}}{}'T + \mu_{\mathcal{P}}Y' - \mu_{\mathcal{P}}{}'Y \,. \end{aligned} \tag{2.45}$$

The canonical generator in this case is also easily integrated, which reads

$$Q\left[T,Y\right] = -\int \left(T\mathcal{P} + Y\mathcal{P}\right) d\phi \,. \tag{2.46}$$

Expanding in Fourier modes, the nonvanishing components of the Poisson brackets are given by

$$\begin{aligned} i\left\{\mathcal{J}_m, \mathcal{J}_n\right\} &= (m-n)\,\mathcal{J}_{m+n} \,, \\ i\left\{\mathcal{J}_m, \mathcal{P}_n\right\} &= (m-n)\,\mathcal{P}_{m+n} + km^3\delta_{m+n,0} \,, \end{aligned} \tag{2.47}$$

which coincides with the infinite-dimensional BMS_3 algebra with a central charge $c = 3/G$ found in [27, 28].

It is worth pointing out that by making the following change of basis on the generators of the Virasoro algebras (2.37)

$$\mathcal{P}_n = \frac{1}{\ell}\left(\mathcal{L}_n^+ + \mathcal{L}_{-n}^-\right) \quad , \quad \mathcal{J}_n = \mathcal{L}_n^+ - \mathcal{L}_{-n}^- \,, \tag{2.48}$$

and rescaling the AdS level according to $k \to k\ell$, in the large AdS radius limit $\ell \to \infty$, the BMS_3 algebra (2.47) is recovered.

Chapter 3

Extension of the Poincaré group with half-integer spin generators and hypergravity

According to the Haag-Łopuszański-Sohnius theorem [35], the super-Poincaré group is a consistent extension of the Poincaré group that includes fermionic generators of spin 1/2. Indeed, in flat spacetimes of dimension greater than three, the addition of fermionic generators of spin $s \geq 3/2$ would imply that the irreducible representations necessarily contained higher spin fields, which are known to suffer from inconsistencies (see, e.g., [36, 6, 7, 37, 38, 39, 40],). However, in three spacetime dimensions, higher spin fields do not possess local propagating degrees of freedom, and as a consequence, it is possible to describe them consistently [41, 42, 43, 44, 45, 30, 46] even on locally flat spacetimes [32, 33, 47, 48]. Hence, in the latter context, since no-go theorems about massless higher spin fields can be circumvented, it is natural to look for an extension of the Poincaré group with fermionic half-integer spin generators. Results along these lines have already been explored in [49]. In what follows, we begin with the construction of the searched for extension of the Poincaré group in the case of spin 3/2 generators, that for short, hereafter we dub it the hyper-Poincaré group. It is shown that the algebra admits a nontrivial Casimir operator and, as an application, we explain how the hypergravity theory of Aragone and Deser [8] can be formulated so as to incorporate the hyper-Poincaré group as its local gauge symmetry. Concretely, we show how hypergravity can be described in terms of hyper-Poincaré-valued gauge fields with a Chern-Simons action. The results are then extended to the case of fermionic generators of spin $n + \frac{1}{2}$, as well as to the minimal coupling of General Relativity with gauge fields of spin $n + \frac{3}{2}$, so that the super-Poincaré group and supergravity are recovered for $n = 0$. The hyper-Poincaré algebra is also shown to admit an infinite-dimensional nonlinear extension that contains the BMS$_3$ algebra as it is shown in detail in Chapter 5, which in the case of spin-3/2 generators, reduces to a subset of a suitable contraction the direct sum of the $W_{(2,4)}$ algebra with its hypersymmetric extension $W_{(2,\frac{5}{2},4)}$. We conclude explaining how the hyper-Poincaré group is extended to the case of $d \geq 3$ dimensions.

3.1 Fermionic spin-3/2 generators

In three spacetime dimensions, the nonvanishing commutators of the Poincaré algebra can be written as

$$[J_a, J_b] = \varepsilon_{abc} J^c \quad , \quad [J_a, P_b] = \varepsilon_{abc} P^c \, . \tag{3.1}$$

The additional fermionic generators are assumed to transform in an irreducible spin-3/2 representation of the Lorentz group, so that they are described by "Γ-traceless" vector-spinors that fulfill

$$Q^a \Gamma_a = 0 \, , \tag{3.2}$$

where Γ_a stand for the Dirac matrices. Their corresponding commutation rules with the Lorentz generators are then given by

$$[J_a, Q_{\alpha b}] = \frac{1}{2} \left(\Gamma_a\right)^{\beta}{}_{\alpha} Q_{\beta b} + \varepsilon_{abc} Q^c_{\alpha} \, . \tag{3.3}$$

Therefore, requiring consistency of the closure as well as the Jacobi identity, implies that the only remaining nonvanishing (anti)commutators of the algebra read

$$\left\{Q^a_\alpha, Q^b_\beta\right\} = -\frac{2}{3} \left(C\Gamma^c\right)_{\alpha\beta} P_c \eta^{ab} + \frac{5}{6} \varepsilon^{abc} C_{\alpha\beta} P_c + \frac{1}{6} (C\Gamma^{(a})_{\alpha\beta} P^{b)} \, , \tag{3.4}$$

where C is the charge conjugation matrix (see Apenndix A). It is then simple to verify that apart from $I_1 = P^a P_a$, the algebra admits another Casimir operator given by

$$I_2 = 2 J^a P_a + Q^a_\alpha C^{\alpha\beta} Q_{\beta a} \, , \tag{3.5}$$

which implies the existence of an invariant (anti)symmetric bilinear form, whose only nonvanishing components are of the form

$$\langle J_a, P_b \rangle = \eta_{ab} \quad , \quad \left\langle Q^a_\alpha, Q^b_\beta \right\rangle = \frac{2}{3} C_{\alpha\beta} \eta^{ab} - \frac{1}{3} \varepsilon^{abc} (C\Gamma_c)_{\alpha\beta} \, . \tag{3.6}$$

It is worth highlighting that the inclusion of the higher spin generators Q^a_α does not jeopardize the causal structure, since there is no need to enlarge the Lorentz group.

3.1.1 Hypergravity

In order to describe a massless spin-$\frac{5}{2}$ field minimally coupled to General Relativity, let us consider a connection 1-form that takes values in the hyper-Poincaré algebra described above, which reads

$$A = e^a P_a + \omega^a J_a + \psi_a^\alpha Q_\alpha^a \,, \tag{3.7}$$

where e^a, ω^a and ψ_a^α stand for the dreibein, the dualized spin connection

$$\omega^a = \frac{1}{2}\varepsilon^{abc}\omega_{bc} \,, \tag{3.8}$$

and the Γ-traceless spin-5/2 field ($\Gamma^a\psi_a = 0$), respectively. The components of the field strength $F = dA + A^2$ are then given by

$$F = R^a J_a + \tilde{T}^a P_a + D\psi_a^\alpha Q_\alpha^a \,, \tag{3.9}$$

where $R^a = d\omega^a + \frac{1}{2}\varepsilon^{abc}\omega_b\omega_c$ is the dualized curvature 2-form and the covariant derivative of the spin-5/2 field reads

$$D\psi^a = d\psi^a + \frac{1}{2}\omega^b\Gamma_b\psi^a + \varepsilon^{abc}\omega_b\psi_c \,, \tag{3.10}$$

which by virtue of the Fierz expansion of the product of three Γ-matrices (see Appendix A) can be written as

$$D\psi^a = d\psi^a + \frac{3}{2}\omega^b\Gamma_b\psi^a - \omega_b\Gamma^a\psi^b \,. \tag{3.11}$$

Note that in the eq. (3.11) the covariant derivative of the vector-spinor field ψ^a is manifestly Γ-traceless. The hypercovariant torsion 2-form then reads

$$\tilde{T}^a := T^a - \frac{3}{4}i\bar{\psi}_b\Gamma^a\psi^b \,, \tag{3.12}$$

with $T^a = de^a + \varepsilon^{abc}\omega_b e_c$, and $\bar{\psi}_{a\alpha} = \psi_a^\beta C_{\beta\alpha}$ is the Majorana conjugate.

Note that under an infinitesimal gauge transformation $\delta A = d\lambda + [A,\lambda]$, spanned by a hyper-Poincaré-valued zero-form given by

$$\lambda = \lambda^a P_a + \sigma^a J_a + \epsilon_a^\alpha Q_\alpha^a \,, \tag{3.13}$$

the components of the gauge field transform according to

$$\begin{aligned} \delta e^a &= D\lambda^a - \varepsilon^{abc}\sigma_b e_c + \frac{3}{2}i\bar{\epsilon}_b\Gamma^a\psi^b \,, \\ \delta\omega^a &= D\sigma^a \,, \\ \delta\psi^a &= -\frac{3}{2}\sigma^b\Gamma_b\psi^a + \sigma_b\Gamma^a\psi^b + D\epsilon^a \,. \end{aligned} \tag{3.14}$$

The invariant bilinear form (3.6) then allows to construct a Chern-Simons action for the gauge field (3.7), given by

$$I = \frac{k}{4\pi}\int \left\langle AdA + \frac{2}{3}A^3 \right\rangle , \tag{3.15}$$

which up to a boundary term, reduces to

$$I = \frac{k}{4\pi}\int 2R^a e_a + i\bar{\psi}_a D\psi^a \, . \tag{3.16}$$

It is worth pointing out that, despite the action (3.16) is formally the same as the one considered by Aragone and Deser in [8], it does possess a different local structure. Indeed, note that under local hypersymmetry transformations spanned by $\lambda = \epsilon_a^\alpha Q_\alpha^a$, the nonvanishing transformation rule for the spin connection considered in [8], agrees with ours only on-shell. Actually, by construction, as in the case of supergravity [50], here the algebra of the local gauge symmetries (3.14) closes off-shell according to the hyper-Poincaré group, without the need of auxiliary fields.

In the case of negative cosmological constant, it can be seen that hypergravity requires the presence of additional spin-4 fields [51, 52, 11]. This will be addressed in detail in Chapter 4.

3.2 Fermionic generators of arbitrary half-integer spin

In this case, the fermionic generators correspond to tensor-spinors $Q_\alpha^{a_1 \ldots a_n}$, transforming in an irreducible representation of the Lorentz group, so that they are completely symmetric in the vector indices, as well as Γ-traceless, i. e.,

$$Q^{a_1 \ldots a_n}\Gamma_{a_1} = 0 \, . \tag{3.17}$$

These conditions imply that their anticommutation rules acquire a somehow cumbersome expression, and it is then more convenient to write the hyper-Poincaré algebra in the Maurer-Cartan formalism. The Maurer-Cartan 1-form is given by

$$\Omega = \rho^a P_a + \tau^a J_a + \chi_{a_1 \ldots a_n}^\alpha Q_\alpha^{a_1 \ldots a_n} , \tag{3.18}$$

where $\chi_{a_1\dots a_n}$ is Γ-traceless and completely symmetric in the vector indices, which can be seen as a flat connection that fulfills

$$\begin{aligned} d\tau^a &= -\frac{1}{2}\epsilon^{abc}\tau_b\tau_c\,, \\ d\rho^a &= -\epsilon^{abc}\tau_b\rho_c + \frac{1}{2}\left(n+\frac{1}{2}\right)i\bar{\chi}_{a_1\dots a_n}\Gamma^a\chi^{a_1\dots a_n}\,, \\ d\chi^{a_1\dots a_n} &= -\left(n+\frac{1}{2}\right)\tau^b\Gamma_b\chi^{a_1\dots a_n} + \tau_b\Gamma^{(a_1}\chi^{a_2\dots a_n)b}\,. \end{aligned} \tag{3.19}$$

Note that the Jacobi identity now translates into the consistency of the nilpotence of the exterior derivative ($d^2 = 0$), which for the algebra (3.19) is clearly satisfied.

The nontrivial Casimir operator now reads

$$I_2 = 2J^aP_a + Q_{\alpha a_1\dots a_n}C^{\alpha\beta}Q_\beta^{a_1\dots a_n}. \tag{3.20}$$

It is also worth pointing out that the super-Poincaré algebra corresponds to the case of $n = 0$, while the hyper-Poincaré algebra described in Section 3.1 is recovered for $n = 1$.

3.2.1 Hypergravity in the generic case

The minimal coupling of General Relativity with a massless fermionic field of spin $s = n + \frac{3}{2}$, described by a completely symmetric Γ-traceless 1-form $\psi_{a_1\dots a_n}$, can then be formulated in terms of a gauge field for the hyper-Poincaré algebra, which now reads

$$A = e^aP_a + \omega^aJ_a + \psi^\alpha_{a_1\dots a_n}Q_\alpha^{a_1\dots a_n}\,. \tag{3.21}$$

The components of the curvature 2-form are then given by

$$F = R^aJ_a + \tilde{T}^aP_a + D\psi^\alpha_{a_1\dots a_n}Q_\alpha^{a_1\dots a_n}, \tag{3.22}$$

where the covariant derivative of the spin-$\left(n+\frac{3}{2}\right)$ field can be written as

$$D\psi^{a_1\dots a_n} = d\psi^{a_1\dots a_n} + \left(n+\frac{1}{2}\right)\omega^b\Gamma_b\psi^{a_1\dots a_n} - \omega_b\Gamma^{(a_1}\psi^{a_2\dots a_n)b}\,, \tag{3.23}$$

and

$$\tilde{T}^a = T^a - \frac{1}{2}\left(n+\frac{1}{2}\right)i\bar{\psi}_{a_1\dots a_n}\Gamma^a\psi^{a_1\dots a_n}\,. \tag{3.24}$$

The transformation rules of the fields under local hypersymmetry can then be obtained from a gauge transformation of the connection (3.21) with a fermionic parameter given by $\lambda = \epsilon^{\alpha}_{a_1 \dots a_n} Q^{a_1 \dots a_n}_{\alpha}$, so that they read

$$\begin{aligned} \delta e^a &= \left(n + \frac{1}{2}\right) i\bar{\epsilon}_{a_1 \dots a_n} \Gamma^a \psi^{a_1 \dots a_n} , \\ \delta \omega^a &= 0 , \\ \delta \psi^{a_1 \dots a_n} &= D\epsilon^{a_1 \dots a_n} . \end{aligned} \tag{3.25}$$

The Casimir operator (3.20) then implies the existence of an (anti)symmetric tensor of rank 2, which once contracted with the wedge product of two curvatures, gives

$$\begin{aligned} \langle F^2 \rangle &= 2R^a \tilde{T}_a + iD\bar{\psi}_{a_1 \dots a_n} D\psi^{a_1 \dots a_n} \\ &= d\left(2R^a e_a + i\bar{\psi}_{a_1 \dots a_n} D\psi^{a_1 \dots a_n}\right) , \end{aligned} \tag{3.26}$$

being an exact form that is manifestly invariant under the hypersymmetry transformations (3.25). Therefore, as in the case of (super)gravity [18, 19], the action can also be written as a Chern-Simons one (3.15), which up to a boundary term reduces to

$$I = \frac{k}{4\pi} \int 2R^a e_a + i\bar{\psi}_{a_1 \dots a_n} D\psi^{a_1 \dots a_n} , \tag{3.27}$$

so that the field equations now read $F = 0$, with F given by (3.22).

Note that the standard supergravity action in [53, 54, 55] is recovered for $n = 0$; and as it occurs in the spin-5/2 case, the generic theory agrees with the one of Aragone and Deser only on-shell.

3.3 Remarks

We would like to stress that a deeper understanding of the theory cannot be attained unless it is endowed with a consistent set of boundary conditions. In this sense, one of the advantages of formulating hypergravity as a Chern-Simons theory is that the analysis of its asymptotic structure can be directly performed in a canonical form, as in the case of negative cosmological constant [11]. Indeed, in analogy with the case of three-dimensional flat supergravity [34], in Chapter 5 it is shown that the mode expansion of the asymptotic symmetry algebra of hypergravity with a spin-5/2 fermionic field is defined through the following Poisson brackets

$$\begin{aligned}
i\{\mathcal{J}_m,\mathcal{J}_n\} &= (m-n)\,\mathcal{J}_{m+n}\,,\\
i\{\mathcal{J}_m,\mathcal{P}_n\} &= (m-n)\,\mathcal{P}_{m+n}+km\left(m^2-1\right)\delta_{m+n,0}\,,\\
i\{\mathcal{P}_m,\mathcal{P}_n\} &= 0 \quad , \quad i\{\mathcal{P}_m,\psi_n\}=0\,,\\
i\{\mathcal{J}_m,\psi_n\} &= \left(\frac{3m}{2}-n\right)\psi_{m+n}\,,\\
i\{\psi_m,\psi_n\} &= \frac{1}{4}\left(6m^2-8mn+6n^2-9\right)\mathcal{P}_{m+n}+\frac{9}{4k}\sum_q\mathcal{P}_{m+n-q}\mathcal{P}_q\\
&\quad +k\left(m^2-\frac{9}{4}\right)\left(m^2-\frac{1}{4}\right)\delta_{m+n,0}\,,
\end{aligned} \tag{3.28}$$

which describe a nonlinear hypersymmetric extension of the BMS_3 algebra [27, 28, 56]. This algebra corresponds to a subset of a suitable contraction the direct sum of the $W_{(2,4)}$ algebra with its hypersymmetric extension $W_{(2,\frac{5}{2},4)}$ [57, 58, 11], as it is explicitly shown in Section 5.3.1.

When fermions fulfill antiperiodic boundary conditions, the modes of the fermionic global charges ψ_m are labelled by half-integers, so that the wedge algebra of (3.28) reduces to the one of the hyper-Poincaré group. In fact, dropping the nonlinear terms, and restricting the modes according to $|n|<\Delta$, where Δ stands for the conformal weight of the generators, the hyper-Poincaré algebra is manifestly recovered provided the modes in (3.28) are identified with the generators J_a, P_a, $Q_{\alpha a}$, according to

$$\begin{aligned}
&\mathcal{J}_{-1}=-\sqrt{2}J_0 \quad , \quad \mathcal{J}_1=\sqrt{2}J_1 \quad , \quad \mathcal{J}_0=J_2\,,\\
&\mathcal{P}_{-1}=-\sqrt{2}P_0 \quad , \quad \mathcal{P}_1=\sqrt{2}P_1 \quad , \quad \mathcal{P}_0=P_2\,,\\
&\psi_{-\frac{3}{2}}=2^{\frac{5}{4}}\sqrt{3}Q_{+0} \quad , \quad \psi_{-\frac{1}{2}}=2^{\frac{3}{4}}\sqrt{3}Q_{-0}\,,\\
&\psi_{\frac{1}{2}}=-2^{\frac{1}{4}}\sqrt{3}Q_{+1} \quad , \quad \psi_{\frac{3}{2}}=-2^{-\frac{1}{4}}\sqrt{3}Q_{-1}\,.
\end{aligned} \tag{3.29}$$

It is also worth noting that (3.28) can then be regarded as a hypersymmetric extension of the Galilean conformal algebra in two dimensions [59, 60], which is isomorphic to BMS_3 and turns out to be relevant in the context of non-relativistic holography.

Another advantage of formulating hypergravity in terms of a Chern-Simons action is that, as in case of supergravity [61, 34], the theory can be readily extended to include parity odd terms in the Lagrangian, as will be explicitly performed in Section 5.5. This is made by a simple modification of the invariant bilinear form, so that it acquires an additional component given by $\langle J_a, J_b\rangle=\mu\eta_{ab}$, followed by a shift in the spin connection of the form $\omega^a \to \omega^a + \gamma e^a$, so that the constants μ, γ parametrize the new allowed couplings in the action. As a consecuence, when hypergravity is extended in this way, the hyper-BMS$_3$ algebra (3.28) acquires an additional nontrivial central extension along its Virasoro subgroup.

The hyper-Poincaré group admits a consistent generalization to the case of $d \geq 3$ spacetime dimensions. In the case of fermionic Γ-traceless spin-$\frac{3}{2}$ generators, the nonvanishing (anti)commutators of the algebra are given by

$$\begin{aligned}
[J_{ab}, J_{cd}] &= J_{ad}\eta_{bc} - J_{bd}\eta_{ac} + J_{ca}\eta_{bd} - J_{cb}\eta_{ad} \,, \\
[J_{ab}, P_c] &= \eta_{ac}P_b - \eta_{bc}P_a \,, \\
[J_{ab}, Q^\alpha_c] &= -\frac{1}{2}(\Gamma_{ab})^\alpha{}_\beta Q^\beta_c + \eta_{ac}Q^\alpha_b - \eta_{bc}Q^\alpha_a \,, \\
\left[J_{ab}, \bar{Q}_{\alpha c}\right] &= \frac{1}{2}(\Gamma_{ab})^\beta{}_\alpha \bar{Q}_{\beta c} + \eta_{ac}\bar{Q}_{\alpha b} - \eta_{bc}\bar{Q}_{\alpha a} \,, \\
\left\{Q^{\alpha a}, \bar{Q}^b_\beta\right\} &= \frac{3\,(d-2)}{d^2} i \left[(d+1)\,(\Gamma^c)^\alpha{}_\beta P_c \eta^{ab} - \frac{d+2}{d-2}(\Gamma^{abc})^\alpha{}_\beta P_c - (\Gamma^{(a|})^\alpha{}_\beta P^{|b)}\right] ,
\end{aligned} \tag{3.30}$$

where $\bar{Q}_a = Q^\dagger_a \Gamma^0$ stands for the Dirac conjugate.

In the generic case, the spin-$\left(n + \frac{1}{2}\right)$ generators correspond to completely symmetric Γ-traceless tensor-spinors that fulfill $\Gamma^{a_1} Q_{a_1 \ldots a_n} = 0$. In order to avoid the intricacies related to the latter condition, as well as with the suitable (anti)symmetrization of the (anti)commutation rules of the generators, it is better to express the algebra in terms of its Maurer-Cartan form. It is now given by

$$\Omega = \rho^a P_a + \frac{1}{2}\tau^{ab} J_{ab} + \bar{\chi}^{a_1\ldots a_n}_\alpha Q^\alpha_{a_1\ldots a_n} - \bar{Q}^{a_1\ldots a_n}_\alpha \chi^\alpha_{a_1\ldots a_n} \,, \tag{3.31}$$

where $\chi_{a_1 \ldots a_n}$ is Γ-traceless and completely symmetric in the vector indices, so that its components fulfill[1]

$$\begin{aligned}
d\tau^{ab} &= -\tau^a{}_c \tau^{cb} \,, \\
d\rho^a &= -\tau^a{}_b \rho^b + \frac{1}{2}\left(n + \frac{1}{2}\right) i\bar{\chi}_{a_1\ldots a_n}\Gamma^a \chi^{a_1\ldots a_n} \,, \\
d\chi^{a_1\ldots a_n} &= -\frac{1}{4}\tau^{ab}\Gamma_{ab}\chi^{a_1\ldots a_n} - \tau^{(a_1}{}_b \chi^{a_2\ldots a_n)b} \,, \\
d\bar{\chi}^{a_1\ldots a_n} &= -\frac{1}{4}\bar{\chi}^{a_1\ldots a_n}\tau^{ab}\Gamma_{ab} - \tau^{(a_1}{}_b \bar{\chi}^{a_2\ldots a_n)b} \,.
\end{aligned} \tag{3.32}$$

This algebra can be easily written in terms of Majorana spinors when they exist, and it reduces to super-Poincaré for $n = 0$.

[1]In the case of $d = 2$ spacetime dimensions the algebra is consistent. However, the subset spanned by translations and the fermionic generators is an abelian ideal.

Note that there was no need to enlarge the Lorentz group in order to accommodate the higher spin generators, so that the additional symmetries do not seem to interfere with the causal structure. Indeed, as in the case of supersymmetry, the quotient of the hyper-Poincaré group over the Lorentz subgroup now defines a hyperspace which is an extension of Minkowski spacetime with additional Γ-traceless tensor-spinor coordinates. However, as anticipated by Haag, Łopuszański and Sohnius, the irreducible representations, which could be obtained from suitable hyperfields, necessarily contain higher spin fields. Nevertheless, it would be worth to explore whether the hyper-Poincaré algebra may manifest itself through theories or models whose fundamental fields do not transform as linear multiplets, as it would be the case of nonlinear realizations, hyper-Poincaré-valued gauge fields, or extended objects.

Chapter 4

Asymptotically anti-de Sitter structure of hypergravity

The three-dimensional hypergravity theory in presence of negative cosmological constant needs by consistency additional spin-4 fields besides the spin-2 and spin-5/2 fields [11]. Then, this theory turns out to be within the framework of three-dimensional higher spin gravity [41, 42, 43] which recently has attracted a great deal of interest. The asymptotic symmetry algebra in these cases is given by two copies of a nonlinear W-algebra, where the detailed structure depends on the spin content of the model [44, 45]. In the particular case of hypergravity, it contains fermionic generators of (conformal) spin 5/2 and bosonic generators of (conformal) spin 4, which extends the two copies of the infinite-dimensional Virasoro algebra of pure gravity with negative cosmological constant [12], with the same central charge. Specifically, the theory is described by a Chern-Simons action for two copies of the $OSp\left(1|4\right)$ group. The asymptotic symmetry superalgebra is the direct sum of two copies of the hypersymmetric extension $W_{(2,\frac{5}{2},4)}$ of $W_{(2,4)}$. A remarkable feature of this theory is the presence of black holes endowed with spin-4 charges, where its thermodynamics can be studied by considering the Euclidean continuation of the theory, even though the causal structure of this kind of black holes is difficult to define because the metric is not invariant under higher spin gauge transformations. It is known that in four dimensions extreme black holes have interesting properties, which can often be related to the fact that they possess unbroken supersymmetries, saturating the Bogomol'nyi–Prasad–Sommerfield (BPS) bounds [62, 63, 64, 65, 66, 67, 68, 69, 70]. This was also shown in the three-dimensional case of the extreme BTZ black hole by Coussaert and Henneaux [71]. Taking this into account, in this chapter the recent results of [11] are revisited related to the study of the asymptotic AdS structure of hypergravity in three spacetime dimensions, which allowed the analysis of the hypersymmetry bounds on higher spin black holes. In this case, supersymmetry is naturally enlarged to the case of *hypersymmetry generators*, which are fermionic generators of half-integer spins greater than 1/2. Then, along the lines of supergravity, one can derive bounds for the conserved charges from the anticommutator of these hypersymmetry generators, which turn out to be nonlinear, and as explained in detail in [11], they are saturated by extreme higher spin black holes which have 1/4 of the hypersymmetries and also by maximally (hyper)symmetric $sp\,(4)$-solitonic solutions.

The theory of hypergravity in three spacetime dimensions with negative cosmological constant is described by a three-dimensional Chern-Simons theory for $OSp\,(1|4) \oplus OSp\,(1|4)$. The gauge connection is given by $A = A^+ + A^-$, where

$$A^{\pm} = \left(\omega^i \pm \frac{e^i}{\ell}\right) L_i + \left(W^m \pm \frac{E^m}{\ell}\right) U_m + \psi_{\pm}^p \mathcal{S}_p \,. \tag{4.1}$$

Here ω and e are the spin-2 dualized spin connection and dreibein, respectively. In turn, W is the spin-4 spin connection and E is the spin-4 dreiben, and ψ stands for the spin-5/2 field. These connections take values in the $osp\,(1|4)$ algebra, being spanned by L_i, with $i = 0, \pm 1$, which stand for the (conformal) spin-2 generators of the $sl\,(2, \mathbb{R})$ subalgebra, while U_m and $\mathcal{S}_p$, with $m = 0, \pm 1, \pm 2, \pm 3$ and $p = \pm\frac{1}{2}, \pm\frac{3}{2}$, correspond to the (conformal) spin-4 and fermionic (conformal) spin-5/2 generators, respectively. The (anti)commutation rules of $osp\,(1|4)$ read

$$\begin{aligned}
[L_i, L_j] &= (i - j)\, L_{i+j} \,, \\
[L_i, U_m] &= (3i - m)\, U_{i+m} \,, \\
[L_i, \mathcal{S}_p] &= \left(\frac{3i}{2} - p\right) \mathcal{S}_{i+p} \,, \\
[U_m, U_n] &= \frac{1}{2^2 3} (m - n) \left(\left(m^2 + n^2 - 4\right) \left(m^2 + n^2 - \frac{2}{3} mn - 9\right) - \frac{2}{3} (mn - 6)\, mn \right) L_{m+n} \\
&\quad + \frac{1}{6} (m - n) \left(m^2 - mn + n^2 - 7\right) U_{m+n} \,, \\
[U_m, \mathcal{S}_p] &= \frac{1}{2^3 3} \left(2m^3 - 8m^2 p + 20mp^2 + 82p - 23m - 40p^3\right) \mathcal{S}_{i+p} \,, \\
\{\mathcal{S}_p, \mathcal{S}_q\} &= U_{p+q} + \frac{1}{2^2 3} \left(6p^2 - 8pq + 6q^2 - 9\right) L_{p+q} \,.
\end{aligned} \tag{4.2}$$

The action is given by the difference of two Chern-Simons forms

$$I = I_{CS}\left[A^+\right] - I_{CS}\left[A^-\right] \,, \tag{4.3}$$

with

$$I_{CS}\,[A] = \frac{k_4}{4\pi} \int str \left[A dA + \frac{2}{3} A^3 \right] \,, \tag{4.4}$$

where $k_4 = \kappa/10$, and $str\,[...]$ stands for the supertrace of the fundamental matrix representation given in Appendix B. As shown in Chapter 2, it is possible to gauge away the radial dependence of the asymptotic form of the connections by virtue of a suitable gauge group element $g_{\pm}(r)$, so that

$$A^{\pm} = g_{\pm}^{-1} a^{\pm} g_{\pm} + g_{\pm}^{-1} d g_{\pm}\,, \tag{4.5}$$

with

$$a^{\pm} = a_t^{\pm} dt + a_{\phi}^{\pm} d\phi\,, \tag{4.6}$$

where the components of the gauged connection depend on the temporal coordinate t and the angular coordinate ϕ. The asymptotic behaviour of the dynamical fields go along the highest weight generators [44, 45, 72, 73], such that

$$a_{\phi}^{\pm} = L_{\pm 1} - \frac{2\pi}{k}\left(\mathcal{L}^{\pm} L_{\mp 1} - \frac{1}{10}\mathcal{U}^{\pm} U_{\mp 3} + \Psi^{\pm} \mathcal{S}_{\mp 3}\right)\,. \tag{4.7}$$

Here $\mathcal{L}^{\pm} = \mathcal{L}^{\pm}(t,\phi), \mathcal{U}^{\pm} = \mathcal{U}^{\pm}(t,\phi)$, and $\psi^{\pm} = \psi^{\pm}(t,\phi)$. The asymptotic conditions are preserved under gauge transformations $\delta a_{\phi}^{\pm} = d\lambda^{\pm} + \left[a_{\phi}^{\pm}, \lambda^{\pm}\right]$ spanned by the Lie-algebra-valued parameters $\lambda^{\pm} = \lambda^{\pm}[\epsilon_{\pm}, \chi_{\pm}, \vartheta_{\pm}]$. They depend on two bosonic functions $\epsilon_{\pm}(t,\phi)$, $\chi_{\pm}(t,\phi)$ and one Grassmann-valued function $\vartheta_{\pm}(t,\phi)$. Moreover, the fields have to transform in a precise way under the transformations generated by these parameters for having so. The Lie-algebra-valued parameters and the transformation law of the fields are explicitly shown in Appendix C.

As explained above, the asymptotic form of the Lagrange multipliers have to be of the form

$$a_t^{\pm} = \pm\lambda^{\pm}\left[\mu_{\mathcal{L}}^{\pm}, \mu_{\mathcal{U}}^{\pm}, \mu_{\Psi}^{\pm}\right]\,. \tag{4.8}$$

This is in order to preserve the asymptotic symmetries under time evolution. The arbitrary functions $\mu_{\mathcal{L}}^{\pm}(t,\phi)$, $\mu_{\mathcal{U}}^{\pm}(t,\phi)$, $\mu_{\psi}^{\pm}(t,\phi)$ are fixed at the boundary. Note that consistency of the asymptotic form of the Lagrangian multipliers under the asymptotic symmetries implies that the gauge parameters $\epsilon_{\pm}, \chi_{\pm}, \vartheta_{\pm}$ have to satisfy suitable "deformed chirality conditions", which are differential equations of first order in time (see e. g. [30, 46]).

By replacing the Lie-algebra-valued-parameters (C.1) and the asymptotic conditions for the dynamical fields (4.7) in the variation of the canonical generators (2.8), it is straightforward to integrate the value of the global charges, which reads

$$Q = Q^{+}[\epsilon_{+}, \chi_{+}, \vartheta_{+}] - Q^{-}[\epsilon_{-}, \chi_{-}, \vartheta_{-}]\,, \tag{4.9}$$

where

$$Q^{\pm}[\epsilon_{\pm}, \chi_{\pm}, \vartheta_{\pm}] = -\int\left(\epsilon_{\pm}\mathcal{L}^{\pm} + \chi_{\pm}\mathcal{U}^{\pm} - i\vartheta_{\pm}\Psi^{\pm}\right) d\phi\,. \tag{4.10}$$

By using that the Poisson brackets fulfill

$$\{Q[\lambda_1], Q[\lambda_2]\} = \delta_{\lambda_2} Q[\lambda_1] \ , \tag{4.11}$$

it is possible to compute the algebra of the canonical generators from the transformation law of the fields given in (C.3). The asymptotic symmetry algebra of one copy then reads

$$\begin{aligned}
i\{\mathcal{L}_m, \mathcal{L}_n\} &= (m-n)\,\mathcal{L}_{m+n} + \frac{\kappa}{2} m^3 \delta_{m+n,0} \ , \\
i\{\mathcal{L}_m, \mathcal{U}_n\} &= (3m-n)\,\mathcal{U}_{m+n} \ , \\
i\{\mathcal{L}_m, \Psi_n\} &= \left(\frac{3m}{2} - n\right)\Psi_{m+n} \ , \\
i\{\mathcal{U}_m, \mathcal{U}_n\} &= \frac{1}{2^2 3^2}(m-n)\left(3m^4 - 2m^3 n + 4m^2 n^2 - 2mn^3 + 3n^4\right)\mathcal{L}_{m+n} \\
&\quad + \frac{1}{6}(m-n)\left(m^2 - mn + n^2\right)\mathcal{U}_{m+n} - \frac{2^3 3\pi}{\kappa}(m-n)\,\Lambda^{(6)}_{m+n} \\
&\quad - \frac{7^2\pi}{3^2\kappa}(m-n)\left(m^2 + 4mn + n^2\right)\Lambda^{(4)}_{m+n} + \frac{\kappa}{2^3 3^2} m^7 \delta_{m+n,0} \ , \\
i\{\mathcal{U}_m, \Psi_n\} &= \frac{1}{2^2 3}\left(m^3 - 4m^2 n + 10mn^2 - 20n^3\right)\Psi_{m+n} - \frac{23\pi}{3\kappa} i\Lambda^{(11/2)}_{m+n} \\
&\quad + \frac{\pi}{3\kappa}(23m - 82n)\,\Lambda^{(9/2)}_{m+n} \ , \\
i\{\Psi_m, \Psi_n\} &= \mathcal{U}_{m+n} + \frac{1}{2}\left(m^2 - \frac{4}{3}mn + n^2\right)\mathcal{L}_{m+n} + \frac{3\pi}{\kappa}\Lambda^{(4)}_{m+n} + \frac{\kappa}{6} m^4 \delta_{m+n,0} \ ,
\end{aligned} \tag{4.12}$$

where the coefficients $\Lambda^{(l)}_m$ correspond to the expansion of the terms in Appendix C, while the fermionic modes are labeled by integers or half-integers in the case of periodic or antiperiodic boundary conditions, respectively. Note that only in the case of antiperiodic boundary conditions, the wedge algebra of (4.12) reduces to one copy of $osp\,(1|4)$ (4.2). This is reached by dropping the nonlinear terms of (4.12), restricting the modes according to $|n| < s$ (where s in the conformal spin of the generators) and provided the zero modes of the $sl\,(2,\mathbb{R})$ generators are shifted as

$$L_0 \to L_0 + \frac{\kappa}{4\pi} \ . \tag{4.13}$$

Note that the asymptotic symmetry algebra (4.12) is the hypersymmetric extension of the $W_{(2,4)}$ known as $W_{(2,\frac{5}{2},4)}$. This algebra corresponds to a classical limit of WB_2 [57, 58].

Hypersymmetry should imply bounds for the bosonic conserved charges, then once the asymptotic symmetry algebra is obtained one could wonder which are the consequences of having hypersymmetry bounds and whether they are similar to the supersymmetry ones.

We will consider bosonic configurations carrying global charges with only zero modes for each copy,

$$\mathcal{L}_0 = 2\pi\mathcal{L} \quad , \quad \mathcal{U}_0 = 2\pi\mathcal{U}\,. \tag{4.14}$$

As the in the case of supergravity, the bounds can be derived from unitarity by following the semi-classical considerations [9, 74, 75, 10, 76, 77]. To derive these bounds, one starts from the anticommutators of the hypersymmetry generators in (4.12) with $m = -n = p \geq 0$, which are reduced to

$$\frac{1}{2\pi}\left(\hat{\psi}_p\hat{\psi}_{-p} + \hat{\psi}_{-p}\hat{\psi}_p\right) = \hat{\mathcal{U}} + \frac{5}{3}p^2\hat{\mathcal{L}} + \frac{3\pi}{\kappa}\hat{\mathcal{L}}^2 + \frac{\kappa}{12\pi}p^4\,. \tag{4.15}$$

Since the hermitian operator at the left hand side of eq. (4.15) is positive definite, in the classical limit, the global charges fulfill the following manifestly nonlinear bounds

$$\mathcal{U} + \frac{5}{3}p^2\mathcal{L} + \frac{3\pi}{\kappa}\mathcal{L}^2 + \frac{\kappa}{12\pi}p^4 \geq 0\,. \tag{4.16}$$

This expression can be factorized as

$$\left(p^2 + \lambda_{[+]}^2\right)\left(p^2 + \lambda_{[-]}^2\right) \geq 0\,, \tag{4.17}$$

where $\pm\lambda_{[\pm]}$ coincide with the eigenvalues of the dynamical field a_ϕ (by using the $sp\,(4,\mathbb{R})$ matrix representation that can be read from Appendix B), given by

$$\lambda_{[\pm]}^2 = \frac{10\pi}{\kappa}\left(\mathcal{L} \pm \frac{4}{5}\sqrt{\mathcal{L}^2 - \frac{3\kappa}{16\pi}\mathcal{U}}\right)\,, \tag{4.18}$$

which completely characterize the holonomies of a_ϕ along a circle,

$$\mathcal{H}_\phi = \mathcal{P}\exp\left[\int a_\phi d\phi\right]\,. \tag{4.19}$$

Then, the bounds on the conserved charges (4.16) can be seen as bounds on the holonomies. The reality condition of both eigenvalues implies that $\mathcal{L} \geq 0$ and moreover it leads to the following bounds for the charges

$$-\mathcal{L}^2 \leq \frac{\kappa}{3\pi}\mathcal{U} \leq \frac{2^4}{3^2}\mathcal{L}^2\,, \tag{4.20}$$

These bounds have remarkable consequences; first of all, as shown in [11], from the thermodynamical analysis (by solving the trivial holonomy condition along a thermal cycle and evaluating the entropy) one concludes that the bounds in (4.20) are saturated for different classes of extreme black holes. Only the class with spin-4 charge negative, i. e. $-\mathcal{L}^2 = \frac{\kappa}{3\pi}\mathcal{U}$, is hypersymmetric and has 1/4 of the hypersymmetries (per copy), and besides saturates the unitarity bounds. On the other hand, the other class of extreme black holes $\left(\frac{\kappa}{3\pi}\mathcal{U} = \frac{2^4}{3^2}\mathcal{L}^2\right)$ do satisfy the bound (4.16), but it does not saturate it. Hence, all the black holes that saturate the unitarity bounds are extremal, but not all extreme black holes are hypersymmetric. Second, in the Lorentzian sector of the theory, where contractible cycles go along of the angular coordinate, one finds solitonic-like solutions which are maximally (hyper)symmetric. These are analogs of the conical defects and surpluses of references [78, 79, 80, 81, 82]. In this case only one class of solitonic-like solutions is compatible with the unitarity bounds.

Chapter 5

Asymptotically flat structure of hypergravity and energy bounds

It has been shown that the inconsistencies arising in the minimal coupling of a massless spin-5/2 field to General Relativity [36, 6, 7, 38] can be successfully surmounted in three-dimensional spacetimes [8] with the theory of hypergravity of Aragone and Deser. As explained in Chapter 3, hypergravity can be reformulated as a Chern-Simons theory of an extension of the Poincaré group with fermionic generators of spin 3/2. As shown in Chapter 4, in the case of negative cosmological constant, additional spin-4 fields are required by consistency, and the anticommutator of the generators of the asymptotic hypersymmetries, associated to fermionic spin-3/2 parameters, leads to interesting nonlinear bounds for the bosonic global charges of spin 2 and 4 [11]. The bounds saturate provided the bosonic configurations admit globally-defined *Killing vector-spinors*. One of the main purposes of this chapter is to show how these results extend to the case of asymptotically flat spacetimes in hypergravity, also in the case of arbitrary half-integer spin fields. In the next section we briefly summarize the results of Chapter 3, showing the formulation of hypergravity as a Chern-Simons theory for the hyper-Poincaré group in the simplest case of fermionic spin-5/2 fields, while section 5.2 is devoted to explore the global hypersymmetry properties of cosmological spacetimes and solutions with conical defects. In the case of fermions that fulfill periodic boundary conditions, it is shown that the null orbifold possesses a single constant Killing vector-spinor. Analogously, for antiperiodic boundary conditions, Minkowski spacetime is singled out as the maximally (hyper)symmetric configuration, and the explicit expression of the globally-defined Killing vector-spinors is found. The asymptotically flat structure of hypergravity in three spacetime dimensions is analyzed in section 5.3, where a precise set of boundary conditions that includes "chemical potentials" associated to the global charges is proposed. The algebra of the canonical generators of the asymptotic symmetries is found to be given by a suitable hypersymmetric nonlinear extension of the BMS_3 algebra. It is also shown that this algebra corresponds to a subset of a suitable Inönü-Wigner contraction of the direct sum of the $W_{(2,4)}$ algebra with its hypersymmetric extension $W_{(2,\frac{5}{2},4)}$. The hypersymmetry bounds that arise from the anticommutator of the fermionic generators are found to be nonlinear, and are shown to saturate for spacetimes that admit unbroken hypersymmetries, like the ones aforementioned. This is explicitly carried out in section 5.4. In section 5.5, the previous analysis is performed in the case of an extension of the hypergravity theory that includes additional parity-odd terms in the action. It is found that the asymptotic symmetry algebra admits an additional central charge along the Virasoro subgroup. The results are then extended to the case of General Relativity minimally coupled to half-integer spin fields in section 5.6, including the asymptotically flat structure, and the explicit expression of the Killing tensor-spinors. The hypersymmetry bounds are shown to be described by a polynomial of degree $s+1/2$ in the energy, where s is the spin of the fermionic generators. We conclude in section 6 with some remarks, including the extension of these results to the case of hypergravity with additional parity-odd terms and fermions of arbitrary half-integer spin.

5.1 General Relativity minimally coupled to a spin-5/2 field

It was shown in Chapter 3 that the hypergravity theory of Aragone and Deser [8] can be reformulated as a gauge theory of a suitable extension of the Poincaré group with fermionic spin-3/2 generators. The action is described by a Chern-Simons form, so that the dreibein, the (dualized) spin connection, and the spin-5/2 field correspond to the components of a gauge field given by

$$A = e^a P_a + \omega^a J_a + \psi_a^\alpha Q_\alpha^a \,, \tag{5.1}$$

that takes values in the hyper-Poincaré algebra, being spanned by the set $\{P_a, J_a, Q_\alpha^a\}$. The fermionic fields and generators are assumed to be Γ-traceless, i. e., $\Gamma^a \psi_a = 0$, and $Q^a \Gamma_a = 0$, so that the nonvanishing (anti)commutation rules read

$$\begin{gathered}
[J_a, J_b] = \varepsilon_{abc} J^c \quad , \quad [J_a, P_b] = \varepsilon_{abc} P^c \,, \\
[J_a, Q_{\alpha b}] = \frac{1}{2} \left(\Gamma_a\right)^\beta{}_\alpha Q_{\beta b} + \varepsilon_{abc} Q_\alpha^c \,, \\
\{Q_\alpha^a, Q_\beta^b\} = -\frac{2}{3} \left(C\Gamma^c\right)_{\alpha\beta} P_c \eta^{ab} + \frac{5}{6} \varepsilon^{abc} C_{\alpha\beta} P_c + \frac{1}{6} (C\Gamma^{(a})_{\alpha\beta} P^{b)} \,,
\end{gathered} \tag{5.2}$$

where C stands for the charge conjugation matrix. The Majorana conjugate then reads $\bar{\psi}_{\alpha a} = \psi_a^\beta C_{\beta\alpha}$. Since the algebra admits an invariant bilinear form, whose only nonvanishing components are given by

$$\langle J_a, P_b \rangle = \eta_{ab} \quad , \quad \left\langle Q^a_\alpha, Q^b_\beta \right\rangle = \frac{2}{3} C_{\alpha\beta} \eta^{ab} - \frac{1}{3} \varepsilon^{abc} (C\Gamma_c)_{\alpha\beta} \,, \tag{5.3}$$

the action can be written as

$$I\,[A] = \frac{k}{4\pi} \int \left\langle AdA + \frac{2}{3} A^3 \right\rangle \,, \tag{5.4}$$

which up to a surface term reduces to

$$I = \frac{k}{4\pi} \int 2R^a e_a + i\bar{\psi}_a D\psi^a . \tag{5.5}$$

Here $R^a = d\omega^a + \frac{1}{2}\varepsilon^{abc}\omega_b\omega_c$ is the dual of the curvature two-form, and since the fermionic field is Γ-traceless, its Lorentz covariant derivative fulfills

$$D\psi^a \quad = \quad d\psi^a + \frac{3}{2}\omega^b \Gamma_b \psi^a - \omega_b \Gamma^a \psi^b \,.$$

The field equations are then given by $F = dA + A^2 = 0$, whose components read

$$R^a = 0 \quad , \quad T^a = \frac{3}{4} i \bar{\psi}_b \Gamma^a \psi^b \quad , \quad D\psi^a = 0 \,, \tag{5.6}$$

where $T^a = De^a$ corresponds to the torsion two-form.

Therefore, by construction, the action changes by a boundary term under local hypersymmetry transformations spanned by $\delta A = dA + [A, \lambda]$, with $\lambda = \epsilon^\alpha_a Q^a_\alpha$, so that the transformation law of the fields reduces to

$$\delta e^a = \frac{3}{2} i \bar{\epsilon}_b \Gamma^a \psi^b \quad , \quad \delta\omega^a = 0 \quad , \quad \delta\psi^a = D\epsilon^a \,. \tag{5.7}$$

Note that the transformation rules of the fields in [8] agree with the ones in (5.7), on-shell.

5.2 Unbroken hypersymmetries: Killing vector-spinors

It is interesting to explore the set of bosonic solutions that possess unbroken global hypersymmetries. According to the transformation rules of the fields in (5.7), this class of configurations has to fulfill the following Killing vector-spinor equation:

$$\delta\psi^a = d\epsilon^a + \frac{1}{2}\omega^b \Gamma_b \epsilon^a + \varepsilon^{abc}\omega_b \epsilon_c = 0 \,, \tag{5.8}$$

where the spin-3/2 parameter ϵ^a is Γ-traceless.

As it follows from the field equations (5.6), the spin connection is locally flat, and it can then be written as

$$\omega = \omega^a J_a = g^{-1} dg \,, \tag{5.9}$$

with $g = e^{\lambda^a J_a}$. Therefore, the general solution of the Killing vector-spinor equation (5.8) is given by

$$\epsilon_a^\alpha = \left(g_S^{-1}\right)_\beta^\alpha (g_V)_a^b \, \eta_b^\beta , \tag{5.10}$$

where η_b^β is a Γ-traceless constant vector-spinor. Here, g_S and g_V stand for the same group element g, but expressed in the spinor and the vector (adjoint) representations, respectively. Since the generators of the Lorentz group in the spinor and vector representations are given by

$$(J_a)_\beta^\alpha = \frac{1}{2} (\Gamma_a)_\beta^\alpha \quad , \quad (J_a)_{bc} = -\varepsilon_{abc} \,, \tag{5.11}$$

the group elements explicitly read

$$(g_S)_\beta^\alpha = \exp\left[\frac{1}{2}\lambda^a (\Gamma_a)_\beta^\alpha\right] \quad , \quad (g_V)_{bc} = \exp\left[-\lambda^a \varepsilon_{abc}\right] . \tag{5.12}$$

The proof of this fact is indeed straightforward. By introducing (5.10) in (5.8), we have that the first term is given by

$$d\epsilon^a = d\left(g_S^{-1}\right)_\beta^\alpha \left(g_V^{-1}\right)_b^a \eta^{b\beta} + \left(g_S^{-1}\right)_\beta^\alpha d\left(g_V^{-1}\right)_b^a \eta^{b\beta} \,, \tag{5.13}$$

the second term of (5.8) leads to the following result

$$\begin{aligned} \frac{1}{2}\omega^b \Gamma_b \epsilon^a &= \left(g_S^{-1}\right)_\beta^\alpha d\left(g_S\right)_\gamma^\beta \left(g_S^{-1}\right)_\delta^\gamma \left(g_V^{-1}\right)_b^a \eta^{b\delta} \\ &= -d\left(g_S^{-1}\right)_\delta^\alpha \left(g_V^{-1}\right)_b^a \eta^{b\delta} \,, \end{aligned} \tag{5.14}$$

while the third term of (5.8) gives that

$$\begin{aligned} \varepsilon^{abc}\omega_b \epsilon_c &= \left(g_V^{-1}\right)_b^a d\left(g_V\right)_c^b \left(g_S^{-1}\right)_\beta^\alpha \left(g_V^{-1}\right)_d^c \eta^{d\beta} \\ &= -\left(g_S^{-1}\right)_\beta^\alpha d\left(g_V^{-1}\right)_d^a \eta^{d\beta} \,. \end{aligned} \tag{5.15}$$

Here it is clear that (5.14) and (5.15) exactly cancel out the first and second terms of the right hand side of (5.13), respectively.

Hence, bosonic configurations that admit unbroken hypersymmetries possess Killing vector-spinors of the form (5.10) provided they are globally well-defined, either for periodic or antiperiodic boundary conditions.

5.2.1 Cosmological spacetimes and solutions with conical defects

Let us focus on an interesting class of circularly symmetric solutions that describe cosmological spacetimes as well as configurations with conical defects. The latter class was introduced in [83, 84], while the former one was explored in [85, 86, 87]. The thermodynamic properties of cosmological spacetimes have been analyzed in [88, 89, 90, 91]. As explained in [30, 46, 11], it is useful to express the solution for a fixed range of the coordinates, so that the Hawking temperature and the chemical potential for the angular momentum manifestly appear in the metric. Hereafter we follow the conventions of [91], and for latter purposes, it is convenient to write the line element in outgoing null coordinates, which reads

$$ds^2 = -\frac{4\pi}{k}\left(\frac{\pi\mathcal{J}^2}{kr^2}-\mathcal{P}\right)\mu_{\mathcal{P}}^2du^2 - 2\mu_{\mathcal{P}}dudr + r^2\left[d\phi + \left(\mu_{\mathcal{J}} + \frac{2\pi\mu_{\mathcal{P}}\mathcal{J}}{kr^2}\right)du\right]^2 . \tag{5.16}$$

Here $\mathcal{P}$ determines the mass, whose associated "chemical potential" relates to the inverse Hawking temperature according to $\mu_{\mathcal{P}} = -\beta^{-1}$. Analogously, $\mu_{\mathcal{J}}$ stands for the chemical potential associated to the angular momentum $\mathcal{J}$. We also assume a non-diagonal form for the Minkowski metric in a local frame, so that its nonvanishing components are given by $\eta_{01} = \eta_{10} = \eta_{22} = 1$. The dreibein can then be chosen as

$$e^0 = -dr + \frac{2\pi\mu_{\mathcal{P}}\mathcal{P}}{k}du + \frac{2\pi\mathcal{J}}{k}\left(d\phi + \mu_{\mathcal{J}}du\right) \quad , \quad e^1 = \mu_{\mathcal{P}}du \quad , \quad e^2 = r\left(d\phi + \mu_{\mathcal{J}}du\right) , \tag{5.17}$$

and hence, the components of the dualized spin connection are given by

$$\omega^0 = \frac{2\pi\mathcal{P}}{k}\left(d\phi + \mu_{\mathcal{J}}du\right) \quad , \quad \omega^1 = d\phi + \mu_{\mathcal{J}}du \quad , \quad \omega^2 = 0 . \tag{5.18}$$

As explained at the beginning of section 5.2, since the curvature two-form vanishes, the spin connection (5.18) is locally flat, and it can then be generically written as $\omega = g^{-1}dg$, with

$$g = \exp\left[\left(J_1 + \frac{2\pi\mathcal{P}}{k}J_0\right)\hat{\phi}\right] , \tag{5.19}$$

and $\hat{\phi} = \phi + \mu_{\mathcal{J}}u$.

Note that in the case of $\mathcal{P} \neq 0$, for the spinor and vector representations, the group element g in (5.19) exponentiates as

$$g_S = \cosh\left[\sqrt{\frac{\pi\mathcal{P}}{k}}\hat{\phi}\right]\mathbb{I}_{2\times2} + \sqrt{\frac{k}{\pi\mathcal{P}}}\sinh\left[\sqrt{\frac{\pi\mathcal{P}}{k}}\hat{\phi}\right]\left(J_1 + \frac{2\pi\mathcal{P}}{k}J_0\right) , \tag{5.20}$$

$$g_V = \mathbb{I}_{3\times3}+\frac{1}{2}\sqrt{\frac{k}{\pi\mathcal{P}}}\sinh\left[2\sqrt{\frac{\pi\mathcal{P}}{k}}\hat{\phi}\right]\left(J_1+\frac{2\pi\mathcal{P}}{k}J_0\right)+\frac{k}{2\pi\mathcal{P}}\sinh\left[\sqrt{\frac{\pi\mathcal{P}}{k}}\hat{\phi}\right]^2\left(J_1+\frac{2\pi\mathcal{P}}{k}J_0\right)^2\,, \tag{5.21}$$

respectively, while for $\mathcal{P}=0$, it reduces to

$$g_S = \mathbb{I}_{2\times2}+\hat{\phi}J_1 \quad , \quad g_V = \mathbb{I}_{3\times3}+\hat{\phi}J_1+\frac{1}{2}\hat{\phi}^2J_1^2\,. \tag{5.22}$$

One then concludes that cosmological spacetimes, for which $\mathcal{P}>0$, necessarily break all the hypersymmetries. Indeed, this class of solutions cannot admit globally-defined Killing vector-spinors because, according to (5.20) and (5.21), the (anti)periodic boundary conditions for the vector-spinor ϵ_a in (5.10) fail to be fulfilled.

In the case of configurations with $\mathcal{P}=0$, equations (5.10) and (5.22) imply that the Killing vector-spinor is constant and satisfies:

$$\frac{3}{2}\Gamma_1\epsilon_a-\Gamma_a\epsilon_1=0\,, \tag{5.23}$$

so that it fulfills periodic boundary conditions, and possesses a single nonvanishing component given by $\epsilon_0^-=\eta_0^-$.

For the remaining case,

$$\mathcal{P}:=-\frac{kj^2}{\pi}<0\,, \tag{5.24}$$

describing solutions with conical defects, the group element in both representations reduces to

$$g_S=\cos\left[j\hat{\phi}\right]\mathbb{I}_{2\times2}+\frac{1}{j}\sin\left[j\hat{\phi}\right]\left(J_1-2j^2J_0\right)\,, \tag{5.25}$$

$$g_V=\mathbb{I}_{3\times3}+\frac{1}{2j}\sin\left[2j\hat{\phi}\right]\left(J_1-2j^2J_0\right)+\frac{1}{2j^2}\sin\left[j\hat{\phi}\right]^2\left(J_1-2j^2J_0\right)^2\,. \tag{5.26}$$

Therefore, this class of configurations possesses four independent Killing vector-spinors that fulfill (anti)periodic boundary conditions provided j is a (half-)integer. The explicit form of the Killing vector-spinors is then obtained from (5.10), where g_S and g_V are given by eqs. (5.25) and (5.26). Note that this is the maximum number of hypersymmetries. Indeed, for these configurations the holonomy of the spin connection becomes trivial, which in the spinor representation means that

$$g_S^{-1}(\hat{\phi})g_S(\hat{\phi}+2\pi)=-\mathbb{I}_{2\times2}\,, \tag{5.27}$$

while in the vector representation the condition reads

$$g_V^{-1}(\hat{\phi})g_V(\hat{\phi}+2\pi)=\mathbb{I}_{3\times3}\,. \tag{5.28}$$

It is worth pointing out that if j were different from a (half-)integer, the configurations would not solve the field equations in vacuum. This is because they would possess a conical singularity at the origin, and hence they should necessarily be supported by an external source.

As it occurs in the case of supersymmetry, it is natural to expect that the bosonic global charges fulfill suitable bounds that turn out to be saturated for configurations that possess unbroken hypersymmetries. Indeed, as shown in [34], the bounds that correspond to three-dimensional supergravity with asymptotically flat boundary conditions certainly do so. Actually, the bounds also exclude conical surplus solutions, in particular those whose angular coordinate ranges from zero to $4\pi j$, with $j > 1/2$, despite they are maximally supersymmetric. When a negative cosmological constant is considered, this is also the case not only for supergravity [71], but also for hypergravity [11], where in the latter case the bounds turn out to be nonlinear. Thus, one of the main purposes of the following sections is showing how these results can be extended to the case of hypergravity endowed with a suitable set of asymptotically flat boundary conditions, as well as how to recover them in the vanishing cosmological constant limit.

5.3 Asymptotically flat behaviour and the hyper-BMS$_3$ algebra

Let us introduce a suitable set of asymptotic conditions that allows to describe the dynamics of asymptotically flat spacetimes in hypergravity. The set must be relaxed enough so as to accommodate the solutions of interest that have been described in section 5.2.1, and simultaneously, restricted in an appropriate way in order to ensure finiteness of the canonical generators associated to the asymptotic symmetries. As shown above, in the case of pure General Relativity, a consistent set of boundary conditions exists, whose asymptotic symmetry algebra corresponds to BMS$_3$ with a nontrivial central extension [27, 28, 56]. These results have been extended to the case of supergravity [34], as well as for General Relativity coupled to higher spin fields [32, 33, 90, 91]. In order to carry out this task in hypergravity, we take advantage of the Chern-Simons formulation of the theory, depicted in section 5.1. Since the hypersymmetry generators are Γ-traceless, it is useful to get rid of

$$Q_2 = Q_1\Gamma_0 - Q_0\Gamma_1 \,, \tag{5.29}$$

so that once the remaining generators are relabeled according to

$$\begin{aligned}
\hat{J}_{-1} &= -2J_0 \quad , \quad \hat{J}_1 = J_1 \quad , \quad \hat{J}_0 = J_2 \, , \\
\hat{P}_{-1} &= -2P_0 \quad , \quad \hat{P}_1 = P_1 \quad , \quad \hat{P}_0 = P_2 \, , \\
\hat{\mathcal{Q}}_{-\frac{3}{2}} &= 2^{\frac{5}{4}}\sqrt{3}Q_{+0} \quad , \quad \hat{\mathcal{Q}}_{-\frac{1}{2}} = 2^{\frac{3}{4}}\sqrt{3}Q_{-0} \, , \\
\hat{\mathcal{Q}}_{\frac{1}{2}} &= -2^{\frac{1}{4}}\sqrt{3}Q_{+1} \quad , \quad \hat{\mathcal{Q}}_{\frac{3}{2}} = -2^{-\frac{1}{4}}\sqrt{3}Q_{-1} \, ,
\end{aligned} \tag{5.30}$$

the hyper-Poincaré algebra (5.2) reads

$$\begin{aligned}
\left[\hat{J}_m, \hat{J}_n\right] &= (m-n)\,\hat{J}_{m+n} \, , \\
\left[\hat{J}_m, \hat{P}_n\right] &= (m-n)\,\hat{P}_{m+n} \, , \\
\left[\hat{J}_m, \hat{\mathcal{Q}}_p\right] &= \left(\frac{3m}{2} - p\right)\hat{\mathcal{Q}}_{m+p} \, , \\
\left\{\hat{\mathcal{Q}}_p, \hat{\mathcal{Q}}_q\right\} &= \frac{1}{4}\left(6p^2 - 8pq + 6q^2 - 9\right)\hat{P}_{p+q} \, ,
\end{aligned} \tag{5.31}$$

with $m, n = \pm 1, 0$, and $p, q = \pm\frac{1}{2}, \pm\frac{3}{2}$.

Thus, following the lines of [29], and as explained in [33, 34], the radial dependence of the asymptotic form of the gauge field can be gauged away by a suitable group element of the form $h = e^{\frac{r}{2}\hat{P}_{-1}}$, so that

$$A = h^{-1}ah + h^{-1}dh \, , \tag{5.32}$$

and hence, the remaining analysis can be entirely performed in terms of the connection $a = a_u du + a_\phi d\phi$, that depends only on time and the angular coordinate. As explained in [30, 46], one starts prescribing the asymptotic form of the dynamical gauge field at a fixed time slice with $u = u_0$, so that the asymptotic fall-off of a_ϕ is assumed to be such that the deviations with respect to the reference background go along the highest weight generators of (5.31). Choosing the reference background to be given by the null orbifold [92], that corresponds to the configuration in (5.16) with $\mathcal{J} = P = 0$, the asymptotic form of the dynamical field reads

$$a_\phi = \hat{J}_1 - \frac{\pi}{k}\left(\mathcal{J}\hat{P}_{-1} + \mathcal{P}\hat{J}_{-1} - \frac{\psi}{3}\hat{\mathcal{Q}}_{-\frac{3}{2}}\right) \, , \tag{5.33}$$

where $\mathcal{J}$, $\mathcal{P}$ and ψ stand for arbitrary functions of u, ϕ. The asymptotic symmetries then correspond to gauge transformations $\delta a = d\lambda + [a, \lambda]$ that preserve the form of (5.33). Therefore, the hyper-Poincaré-valued parameter λ is found to depend on three arbitrary functions of u and ϕ, so that

$$\lambda = T\hat{P}_1 + Y\hat{J}_1 + \mathcal{E}\hat{\mathcal{Q}}_{\frac{3}{2}} + \eta_{\left(\frac{3}{2}\right)}\left[T, Y, \mathcal{E}\right] \, , \tag{5.34}$$

where $\mathcal{E}$ is Grassmann-valued, and

$$
\begin{aligned}
\eta_{\left(\frac{3}{2}\right)}\left[T,Y,\mathcal{E}\right] \;=\;& -T'\hat{P}_0 - Y'\hat{J}_0 - \mathcal{E}'\hat{\mathcal{Q}}_{\frac{1}{2}} - \frac{1}{2}\left(\frac{2\pi}{k}Y\mathcal{P} - Y''\right)\hat{J}_{-1} \\
& -\frac{\pi}{k}\left(T\mathcal{P} + Y\mathcal{J} - \frac{3}{2}i\psi\mathcal{E} - \frac{k}{2\pi}T''\right)\hat{P}_{-1} - \frac{1}{2}\left(\frac{3\pi}{k}\mathcal{E}\mathcal{P} - \mathcal{E}''\right)\hat{\mathcal{Q}}_{-\frac{1}{2}} \\
& -\frac{\pi}{3k}\left(Y\psi - \frac{7}{2}\mathcal{E}'\mathcal{P} - \frac{3}{2}\mathcal{E}\mathcal{P}' + \frac{k}{2\pi}\mathcal{E}'''\right)\hat{\mathcal{Q}}_{-\frac{3}{2}}\,;
\end{aligned}
\tag{5.35}
$$

while the transformation law of the fields reads

$$
\begin{aligned}
\delta\mathcal{P} &= 2\mathcal{P}Y' + \mathcal{P}'Y - \frac{k}{2\pi}Y'''\,, \\
\delta\mathcal{J} &= 2\mathcal{J}Y' + \mathcal{J}'Y + 2\mathcal{P}T' + \mathcal{P}'T - \frac{k}{2\pi}T''' + \frac{5}{2}i\psi\mathcal{E}' + \frac{3}{2}i\psi'\mathcal{E}\,, \\
\delta\psi &= \frac{5}{2}\psi Y' + \psi'Y - \frac{9\pi}{2k}\mathcal{P}^2\mathcal{E} + \frac{3}{2}\mathcal{P}''\mathcal{E} + 5\mathcal{P}'\mathcal{E}' + 5\mathcal{P}\mathcal{E}'' - \frac{k}{2\pi}\mathcal{E}''''\,.
\end{aligned}
\tag{5.36}
$$

Hereafter, prime stands for ∂_ϕ. Since the time evolution of a_ϕ corresponds to a gauge transformation parametrized by the Lagrange multiplier a_u, its asymptotic form will be maintained along different time slices provided a_u is of the allowed form, i. e.,

$$
a_u = \lambda\left[\mu_{\mathcal{P}}, \mu_{\mathcal{J}}, \mu_\psi\right]\,,
\tag{5.37}
$$

where the chemical potentials $\mu_{\mathcal{P}}, \mu_{\mathcal{J}}, \mu_\psi$ stand for arbitrary functions of u, ϕ, that are assumed to be fixed at the boundary. Consistency then demands that the field equations, which now reduce to

$$
\begin{aligned}
\dot{\mathcal{P}} &= 2\mathcal{P}{\mu_{\mathcal{J}}}' + \mathcal{P}'\mu_{\mathcal{J}} - \frac{k}{2\pi}{\mu_{\mathcal{J}}}'''\,, \\
\dot{\mathcal{J}} &= 2\mathcal{J}{\mu_{\mathcal{J}}}' + \mathcal{J}'\mu_{\mathcal{J}} + 2\mathcal{P}{\mu_{\mathcal{P}}}' + \mathcal{P}'\mu_{\mathcal{P}} - \frac{k}{2\pi}{\mu_{\mathcal{P}}}''' + \frac{5}{2}i\psi{\mu_\psi}' + \frac{3}{2}i\psi'\mu_\psi\,, \\
\dot{\psi} &= \frac{5}{2}\psi{\mu_{\mathcal{J}}}' + \psi'\mu_{\mathcal{J}} - \frac{9\pi}{2k}\mathcal{P}^2\mu_\psi + \frac{3}{2}\mathcal{P}''\mu_\psi + 5\mathcal{P}'{\mu_\psi}' + 5\mathcal{P}{\mu_\psi}'' - \frac{k}{2\pi}{\mu_\psi}''''\,,
\end{aligned}
\tag{5.38}
$$

have to hold in the asymptotic region, while the parameters of the asymptotic symmetries fulfill the following conditions

$$
\begin{aligned}
\dot{Y} &= \mu_{\mathcal{J}}Y' - {\mu_{\mathcal{J}}}'Y\,, \\
\dot{T} &= \mu_{\mathcal{J}}T' - {\mu_{\mathcal{J}}}'T + \mu_{\mathcal{P}}Y' - {\mu_{\mathcal{P}}}'Y + \frac{9\pi}{k}i\mu_\psi\mathcal{E}\mathcal{P} - \frac{3}{2}i{\mu_\psi}''\mathcal{E} + 2i{\mu_\psi}'\mathcal{E}' - \frac{3}{2}i\mu_\psi\mathcal{E}''\,, \\
\dot{\mathcal{E}} &= \frac{3}{2}\mu_\psi Y' - {\mu_\psi}'Y - \frac{3}{2}{\mu_{\mathcal{J}}}'\mathcal{E} + \mu_{\mathcal{J}}\mathcal{E}'\,,
\end{aligned}
\tag{5.39}
$$

which are needed in order to ensure that the global charges are conserved.[1]

Following the Regge-Teitelboim approach [24], the variation of the canonical generators is found to be generically given by

$$\delta Q\left[\lambda\right] = -\frac{k}{2\pi}\int\left\langle \lambda\delta a_{\phi}\right\rangle d\phi\,, \tag{5.40}$$

which by virtue of (5.33) and (5.34), up to an arbitrary constant without variation, integrate as

$$Q\left[T,Y,\mathcal{E}\right] = -\int\left(T\mathcal{P}+Y\mathcal{J}-i\mathcal{E}\psi\right)d\phi\,. \tag{5.41}$$

It is worth highlighting that the global charges are manifestly independent of the radial coordinate r. Therefore, the boundary can be located at an arbitrary fixed value $r=r_0$, and it corresponds to a timelike surface with the topology of a cylinder.

Since the Poisson brackets fulfill $\left\{Q\left[\lambda_1\right],Q\left[\lambda_2\right]\right\} = \delta_{\lambda_2}Q\left[\lambda_1\right]$, the algebra of the canonical generators can be directly obtained from the transformation law of the fields in (5.36). Expanding in Fourier modes, the nonvanishing Poisson brackets read

$$\begin{aligned}
i\left\{\mathcal{J}_m,\mathcal{J}_n\right\} &= \left(m-n\right)\mathcal{J}_{m+n}\,,\\
i\left\{\mathcal{J}_m,\mathcal{P}_n\right\} &= \left(m-n\right)\mathcal{P}_{m+n}+km^3\delta_{m+n,0}\,,\\
i\left\{\mathcal{J}_m,\psi_n\right\} &= \left(\frac{3m}{2}-n\right)\psi_{m+n}\,,\\
i\left\{\psi_m,\psi_n\right\} &= \frac{1}{2}\left(3m^2-4mn+3n^2\right)\mathcal{P}_{m+n}+\frac{9}{4k}\sum_q\mathcal{P}_{m+n-q}\mathcal{P}_q+km^4\delta_{m+n,0}\,,
\end{aligned} \tag{5.42}$$

where the modes of the generators ψ_m are labeled by (half-)integers when the fermions fulfill (anti)periodic boundary conditions.

It is then clear that, with respect to $\mathcal{J}_m$, the conformal weight of the generators $\mathcal{P}_m$ and ψ_n, is given by 2 and 5/2, respectively. Note that the subset spanned by $\mathcal{J}_m$ and $\mathcal{P}_m$ corresponds to the BMS_3 algebra of General Relativity with the same central extension, and hence (5.42) stands for its hypersymmetric extension that is manifestly nonlinear.

It is useful to perform the following shift in the generators:

$$\mathcal{P}_n\rightarrow\mathcal{P}_n-\frac{k}{2}\delta_{n,0}\,, \tag{5.43}$$

[1] Since global symmetries are necessarily contained within the asymptotic ones, these results provide an interesting alternative path to find the explicit expression of the Killing vector-spinors. See appendix D.

so that the algebra now reads

$$
\begin{aligned}
i\left\{\mathcal{J}_m,\mathcal{J}_n\right\} &= (m-n)\,\mathcal{J}_{m+n}\,,\\
i\left\{\mathcal{J}_m,\mathcal{P}_n\right\} &= (m-n)\,\mathcal{P}_{m+n}+km\left(m^2-1\right)\delta_{m+n,0}\,,\\
i\left\{\mathcal{J}_m,\psi_n\right\} &= \left(\frac{3m}{2}-n\right)\psi_{m+n}\,,\\
i\left\{\psi_m,\psi_n\right\} &= \frac{1}{4}\left(6m^2-8mn+6n^2-9\right)\mathcal{P}_{m+n}+\frac{9}{4k}\sum_q\mathcal{P}_{m+n-q}\mathcal{P}_q\\
&\quad +k\left(m^2-\frac{9}{4}\right)\left(m^2-\frac{1}{4}\right)\delta_{m+n,0}\,,
\end{aligned}
\tag{5.44}
$$

in agreement with the result that was anticipated in Chapter 3. Indeed, dropping the nonlinear terms in (5.44), when the fermions fulfill antiperiodic boundary conditions, the wedge algebra, which is spanned by the subset of $\{\mathcal{J}_m,\mathcal{P}_m,\psi_n\}$ with $m=\pm1,0$ and $n=\pm3/2,\pm1/2$, reduces to the hyper-Poincaré algebra in eq. (5.31).

It can also be seen that the hyper-BMS$_3$ algebra (5.42) turns out to be a subset of a precise Inönü-Wigner contraction of the direct sum of the $W_{(2,4)}$ algebra with its hypersymmetric extension $W_{\left(2,\frac{5}{2},4\right)}$. This is the main subject of the next subsection.

5.3.1 Flat limit of the asymptotic symmetry algebra from the case of negative cosmological constant

As explained in Chapter 4, it has been recently shown that the asymptotic symmetries of three-dimensional hypergravity with negative cosmological constant are spanned by two copies of the classical limit of the WB_2 algebra [11]. This algebra is also known as $W_{(2,\frac{5}{2},4)}$ and corresponds to the hypersymmetric extension of $W_{(2,4)}$ [57, 58]. The hypergravity theory that was discussed in [11] possesses the minimum number of hypersymmetries in each sector, so that the gauge group is given by $OSp\,(1|4)\otimes OSp\,(1|4)$. In analogy with the case of three-dimensional supergravity [18], one may say that the theory aforementioned corresponds to the $\mathcal{N}=(1,1)$ AdS$_3$ hypergravity. In this sense, there are two inequivalent minimal locally hypersymmetric extensions of General Relativity with negative cosmological constant, which correspond to the $(1,0)$ and the $(0,1)$ theories. It is then simple to verify that both minimal theories possess the same vanishing cosmological constant limit, and hence in order to proceed with the analysis we will consider the $(0,1)$ one, whose gauge group is given by $Sp\,(4)\otimes OSp\,(1|4)$. According to [11], the asymptotic symmetry algebra of the minimal hypergravity theory with negative cosmological constant then corresponds to $W_{(2,4)}\oplus W_{(2,\frac{5}{2},4)}$.

The classical limit of the $W_{(2,\frac{5}{2},4)}$ algebra reads

$$
\begin{aligned}
i\{\mathcal{L}_m,\mathcal{L}_n\} &= (m-n)\,\mathcal{L}_{m+n}+\frac{\kappa}{2}m^3\delta_{m+n,0}\,,\\
i\{\mathcal{L}_m,\mathcal{U}_n\} &= (3m-n)\,\mathcal{U}_{m+n}\,,\\
i\{\mathcal{L}_m,\Psi_n\} &= \left(\frac{3m}{2}-n\right)\Psi_{m+n}\,,\\
i\{\mathcal{U}_m,\mathcal{U}_n\} &= \frac{1}{2^2 3^2}(m-n)\left(3m^4-2m^3n+4m^2n^2-2mn^3+3n^4\right)\mathcal{L}_{m+n}\\
&\quad+\frac{1}{6}(m-n)\left(m^2-mn+n^2\right)\mathcal{U}_{m+n}-\frac{2^3 3\pi}{\kappa}(m-n)\,\Lambda^{(6)}_{m+n}\\
&\quad-\frac{7^2\pi}{3^2\kappa}(m-n)\left(m^2+4mn+n^2\right)\Lambda^{(4)}_{m+n}+\frac{\kappa}{2^3 3^2}m^7\delta_{m+n,0}\,,\\
i\{\mathcal{U}_m,\Psi_n\} &= \frac{1}{2^2 3}\left(m^3-4m^2n+10mn^2-20n^3\right)\Psi_{m+n}-\frac{23\pi}{3\kappa}i\Lambda^{(11/2)}_{m+n}\\
&\quad+\frac{\pi}{3\kappa}(23m-82n)\,\Lambda^{(9/2)}_{m+n}\,,\\
i\{\Psi_m,\Psi_n\} &= \mathcal{U}_{m+n}+\frac{1}{2}\left(m^2-\frac{4}{3}mn+n^2\right)\mathcal{L}_{m+n}+\frac{3\pi}{\kappa}\Lambda^{(4)}_{m+n}+\frac{\kappa}{6}m^4\delta_{m+n,0}\,,
\end{aligned}
\tag{5.45}
$$

where the fermionic modes are labeled by (half-)integers in the case of (anti)periodic boundary conditions, and $\Lambda^{(l)}_m=\int\Lambda^{(l)}e^{-im\phi}d\phi$ stand for the mode expansion of the nonlinear terms, given by

$$\Lambda^{(4)} = \mathcal{L}^2\,, \tag{5.46}$$

$$\Lambda^{(9/2)} = \mathcal{L}\Psi\,, \tag{5.47}$$

$$\Lambda^{(11/2)} = \frac{27}{23}\mathcal{L}'\Psi\,, \tag{5.48}$$

$$\Lambda^{(6)} = -\frac{7}{18}\mathcal{U}\mathcal{L}-\frac{8\pi}{3\kappa}\mathcal{L}^3+\frac{295}{432}(\mathcal{L}')^2+\frac{22}{27}\mathcal{L}''\mathcal{L}+\frac{25}{12}i\Psi\Psi'\,. \tag{5.49}$$

The bosonic generators $\mathcal{L}_m$ and $\mathcal{U}_m$ span the $W_{(2,4)}$ subalgebra.

In order to take the vanishing cosmological constant limit of the asymptotic symmetry algebra of the minimal theory, given by $W_{(2,4)}\oplus W_{(2,\frac{5}{2},4)}$, it is useful to perform the following change of basis:

$$\mathcal{P}_n=\frac{1}{\ell}\left(\mathcal{L}^+_n+\mathcal{L}^-_{-n}\right)\quad,\quad \mathcal{J}_n=\mathcal{L}^+_n-\mathcal{L}^-_{-n}\,,$$

$$\mathcal{W}_n=\frac{1}{\sqrt{\ell}}\left(\mathcal{U}^+_n+\mathcal{U}^-_{-n}\right)\quad,\quad \mathcal{V}_n=\frac{1}{\sqrt{\ell}}\left(\mathcal{U}^+_n-\mathcal{U}^-_{-n}\right)\quad,\quad \psi_n=\sqrt{\frac{6}{\ell}}\Psi^+_n\,, \tag{5.50}$$

where $\mathcal{L}_n^-, \mathcal{U}_n^-$ stand for the generators of the (left) $W_{(2,4)}$ algebra, and $\mathcal{L}_n^+, \mathcal{U}_n^+, \psi_n^+$ span the (right) $W_{(2,\frac{5}{2},4)}$ algebra. Therefore, rescaling the level according to $\kappa = k\ell$, in the large AdS radius limit, $\ell \to \infty$, one obtains that the nonvanishing brackets of the contracted algebra read

$$\begin{aligned}
i\{\mathcal{J}_m, \mathcal{J}_n\} &= (m-n)\,\mathcal{J}_{m+n}\,, \\
i\{\mathcal{J}_m, \mathcal{P}_n\} &= (m-n)\,\mathcal{P}_{m+n} + km^3\delta_{m+n,0}\,, \\
i\{\mathcal{J}_m, \mathcal{W}_n\} &= (3m-n)\,\mathcal{W}_{m+n}\,, \\
i\{\mathcal{J}_m, \mathcal{V}_n\} &= (3m-n)\,\mathcal{V}_{m+n}\,, \\
i\{\mathcal{V}_m, \mathcal{W}_n\} &= \frac{1}{2^2 3^2}(m-n)\left(3m^4 - 2m^3 n + 4m^2n^2 - 2mn^3 + 3n^4\right)\mathcal{P}_{m+n} \\
&\quad -\frac{2^3\pi}{k}(m-n)\,\tilde{\Lambda}^{(6)}_{m+n} - \frac{7^2}{3^2 4k}(m-n)\left(m^2+4mn+n^2\right)\sum_q \mathcal{P}_{m+n-q}\mathcal{P}_q \\
&\quad +\frac{k}{2^2 3^2} m^7 \delta_{m+n,0}\,, \\
i\{\mathcal{J}_m, \psi_n\} &= \left(\frac{3m}{2} - n\right)\psi_{m+n}\,, \\
i\{\psi_m, \psi_n\} &= \frac{1}{2}\left(3m^2 - 4mn + 3n^2\right)\mathcal{P}_{m+n} + \frac{9}{4k}\sum_q \mathcal{P}_{m+n-q}\mathcal{P}_q + km^4\delta_{m+n,0}\,,
\end{aligned} \tag{5.51}$$

with

$$\tilde{\Lambda}^{(6)} = -\frac{7}{12}\mathcal{W}\mathcal{P} - \frac{2\pi}{k}\mathcal{P}^3 + \frac{295}{288}\left(\mathcal{P}'\right)^2 + \frac{11}{9}\mathcal{P}\mathcal{P}''\,. \tag{5.52}$$

It is then apparent that one can consistently get rid of the (conformal) spin-4 generators $\mathcal{V}_m$, $\mathcal{W}_n$, since the Inönü-Wigner contraction of $W_{(2,4)} \oplus W_{(2,\frac{5}{2},4)}$ in eq. (5.51) possesses a subset spanned by $\{\mathcal{P}_m, \mathcal{J}_m, \psi_m\}$, which precisely corresponds to the hyper-BMS$_3$ algebra in (5.42). Note that this is just a reflection of the fact that in the vanishing cosmological constant limit, the hypergravity theory can be consistently formulated without the need of spin-4 fields.

5.4 Hypersymmetry bounds

In the case of hypergravity with negative cosmological constant, it has been recently shown that the anticommutator of the generators of the asymptotic hypersymmetries implies the existence of interesting nonlinear bounds for the bosonic charges, that saturate for configurations that admit unbroken hypersymmetries [11]. In this section, following these lines, we explicitly show that this is also the case for hypergravity with asymptotically flat boundary conditions. In order to perform this task, it is useful to assume that the bosonic global charges are just determined by the zero modes. Indeed, as explained in [46], a generic bosonic configuration can be brought to the "rest frame" through the action of suitable elements of the asymptotic symmetry algebra. The searched for bounds can then be found along the same semi-classical reasoning as in the case of supergravity [9, 74, 75, 10, 76, 77, 71]. Hence, the fermionic bracket in (5.42) becomes an anticommutator, which in the rest frame, and for $m = -n = p$, reads

$$\frac{1}{2\pi}\left(\hat{\psi}_p\hat{\psi}_{-p} + \hat{\psi}_{-p}\hat{\psi}_p\right) = 5p^2\hat{\mathcal{P}} + \frac{9\pi}{2k}\hat{\mathcal{P}}^2 + \frac{k}{2\pi}p^4 \geq 0\,, \tag{5.53}$$

with $\hat{\mathcal{P}}_0 = 2\pi\hat{\mathcal{P}}$. Thus, since the left-hand side of (5.53) is a positive-definite hermitian operator, in the classical limit, and for any value of the (half-)integer p, the energy has to fulfill the following bounds:

$$\left(p^2 + \frac{9\pi}{k}\mathcal{P}\right)\left(p^2 + \frac{\pi}{k}\mathcal{P}\right) \geq 0\,, \tag{5.54}$$

which are manifestly nonlinear.

Note that for any configuration with $\mathcal{P} > 0$, the bounds in (5.54) are automatically fulfilled, but never saturate. Indeed, this is the case of the cosmological spacetimes in (5.16), which goes by hand with the fact that they do not admit globally-defined Killing vector-spinors, and hence, break all the hypersymmetries.

These bounds are also clearly fulfilled in the case of $\mathcal{P} = 0$, and for fermions with periodic boundary conditions, the one for $p = 0$ is saturated. This relates to the fact that this class of configurations, that includes the null orbifold, possesses a single unbroken hypersymmetry spanned by a constant Killing vector-spinor.

In the case of $\mathcal{P} < 0$, the class of smooth configurations are the ones for which the holonomy of the connection around an angular cycle is trivial. This means that they are maximally (hyper)symmetric, and then possess four Killing vector-spinors. As explained in section 5.2.1, their energy is given by $\mathcal{P} = -kj^2/\pi$, and the bounds in (5.54) then reduce to

$$\left(p^2 - 9j^2\right)\left(p^2 - j^2\right) \geq 0\,. \tag{5.55}$$

Remarkably, the bounds are only fulfilled in the case of $j^2 = 1/4$, so that four of them saturate, corresponding to $p = \pm 1/2$, and $p = \pm 3/2$. This is the case of Minkowski spacetime ($\mathcal{P} = -k/4\pi$), with Killing vector-spinors that fulfill antiperiodic boundary conditions. Hence, in spite of being maximally hypersymmetric, smooth solutions whose energy is lower than the one of Minkowski spacetime are excluded by the hypersymmetry bounds.

It is worth highlighting that one arrives to similar conclusions in the case of asymptotically flat spacetimes in supergravity [34]. In fact, despite the analysis is fairly different, the supersymmetry bounds precisely select the same spectrum, including the corresponding ground states who saturate the bounds for spinors that fulfill different periodicity conditions.

5.5 Hypergravity reloaded

Let us look for a different theory of three-dimensional (hyper)gravity that is still compatible with the asymptotically flat boundary conditions described above, but now allowing the presence of spacetime torsion even in vacuum. For simplicity, we consider modifications such that the field equations are still of first order for the dreibein and the spin connection. One interesting possibility is to include additional terms, so that the action is given by

$$I_\gamma = \frac{k}{4\pi}\int 2R^a e_a + \gamma^2 \epsilon_{abc} e^a e^b e^c + 2\gamma T^a e_a \,. \tag{5.56}$$

Remarkably, despite the second (volume) term in the action looks like a cosmological constant $\Lambda = -3\gamma^2$, the field equations actually imply that the Riemann curvature vanishes. Indeed, the presence of the last (parity-odd) term in the action has the effect of making the volume term to act as the source for a fully antisymmetric torsion in vacuum, being proportional to the volume element, given by

$$T^a = -\gamma \varepsilon^{abc} e_b e_c \,, \tag{5.57}$$

so that the remaining field equations fix the curvature two-form according to

$$R^a = \frac{1}{2}\gamma^2 \varepsilon^{abc} e_b e_c \,. \tag{5.58}$$

Therefore, eq. (5.57) implies that the spin connection splits as

$$\omega^a = \bar{\omega}^a + \kappa^a \,, \tag{5.59}$$

where $\bar{\omega}^a$ is the (torsionless) Levi-Civita connection, and the contorsion reads $\kappa^a = -\gamma e^a$. The curvature two-form is then given by

$$R^a = \bar{R}^a + \frac{1}{2}\gamma^2 \varepsilon^{abc} e_b e_c \,, \tag{5.60}$$

and hence, equation (5.58) implies the vanishing of the Riemann tensor, i. e.,

$$\bar{R}_a = \frac{1}{4}\varepsilon_{a\rho\tau} R^{\rho\tau}{}_{\mu\nu} dx^\mu dx^\nu = 0 \,. \tag{5.61}$$

The most general theory that possesses the features described above is obtained by considering the addition of the Lorentz-Chern-Simons form,

$$L(\omega) = \omega^a d\omega_a + \frac{1}{3}\varepsilon_{abc}\omega^a\omega^b\omega^c \,, \tag{5.62}$$

with an independent coupling μ, provided the remaining couplings in (5.56) are suitable shifted. The action is then given by

$$I_{\mu,\gamma} = \frac{k}{4\pi}\int 2\left(1+\mu\gamma\right)R^a e_a + \gamma^2\left(1+\mu\frac{\gamma}{3}\right)\epsilon_{abc}e^a e^b e^c + \mu L(\omega) + \gamma\left(2+\mu\gamma\right)T^a e_a \ . \tag{5.63}$$

Noteworthy, despite the fact that the Lorentz-Chern-Simons form is not a boundary term, the shifts in the other couplings are such that the field equations in vacuum just become reshuffled, coinciding with the previous ones for $\mu = 0$, given by (5.57) and (5.58). Actually, one should highlight that both actions, (5.56) and (5.63), differ off-shell, which reflects through the fact that the canonical generators do not have the same form. Consequently, as in the case of supergravity [34], the asymptotic symmetry algebra of the latter acquires an additional central extension with respect to the former (see below).

The locally hypersymmetric extension of the theory described by (5.63) is given by the following action

$$I_{\mu,\gamma,\psi} = I_{\mu,\gamma} + \frac{k}{4\pi}\int i\bar{\psi}_a\left(D + \frac{3}{2}\gamma e^b\Gamma_b\right)\psi^a \,, \tag{5.64}$$

which is invariant under the following local hypersymmetry transformations:

$$\delta e^a = \frac{3}{2}i\bar{\epsilon}_b\Gamma^a\psi^b \quad , \quad \delta\omega^a = -\frac{3}{2}i\gamma\bar{\epsilon}_b\Gamma^a\psi^b \quad , \quad \delta\psi^a = D\epsilon^a + \frac{3}{2}\gamma e^b\Gamma_b\epsilon^a - \gamma e^b\Gamma^a\epsilon_b \, . \tag{5.65}$$

The field equations now read

$$R^a = \frac{1}{2}\gamma^2\varepsilon^{abc}e_b e_c - \frac{3}{4}i\gamma\bar{\psi}_b\Gamma^a\psi^b \quad , \quad T^a = -\gamma\varepsilon^{abc}e_b e_c + \frac{3}{4}i\bar{\psi}_b\Gamma^a\psi^b \,,$$

$$D\psi^a = -\frac{3}{2}\gamma e^b\Gamma_b\psi^a + \gamma e^b\Gamma^a\psi_b \, . \tag{5.66}$$

Note that in the case of $\mu = \gamma = 0$, the action (5.64), the transformations rules (5.65), and the field equations (5.66), reduce to the ones of the locally hypersymmetric extension of General Relativity, given by eqs. (5.5), (5.7), and (5.6), respectively.

As outlined in Section 3.2.1, in analogy with the case of supergravity [61], [34], the action (5.63) can be formulated as a Chern-Simons one for the hyper-Poincaré group (5.2) by virtue of a simple modification of the invariant bilinear form, and a suitable shift of the spin connection. Indeed, the invariant bilinear form in (5.3) can be consistently modified to admit an additional nonvanishing component given by

$$\langle J_a, J_b \rangle = \mu \eta_{ab} \,, \tag{5.67}$$

so that the Chern-Simons action (5.4) now depends on a different hyper-Poincaré-algebra-valued gauge field, defined as

$$A = e^a P_a + \hat{\omega}^a J_a + \psi_a^\alpha Q_\alpha^a \ , \tag{5.68}$$

with $\hat{\omega}^a := \omega^a + \gamma e^a$. Therefore, in terms of the covariant derivative with respect to $\hat{\omega}^a$ and its corresponding curvature, given by $\hat{D}$ and $\hat{R}^a$, respectively, up to a surface term, the Chern-Simons action reduces to

$$I_{\mu,\gamma,\psi} = \frac{k}{4\pi} \int 2\hat{R}^a e_a + \mu L(\hat{\omega}) + i\bar{\psi}_a \hat{D}\psi^a \ , \tag{5.69}$$

which precisely agrees with (5.64). Note that the field equations (5.66) correspond to the vanishing of the components of the curvature associated to (5.68), so that they can be compactly written as $F = dA + A^2 = 0$, being manifestly covariant under the full hyper-Poincaré group.

One of the advantages of having formulated the extension of hypergravity with parity-odd terms as a Chern-Simons theory, is that its asymptotically flat structure can be directly obtained along the lines of the results in section 5.3.

The asymptotically flat boundary conditions for the connection (5.68) are then proposed to be precisely as in eqs. (5.32), (5.33), and (5.37), so that the asymptotic fall-off of the spin connection ω^a becomes modified. Therefore, the asymptotic symmetries remain the same as in section 5.3, being spanned by the hyper-Poincaré algebra valued parameter $\lambda = \lambda\left[T, Y, \mathcal{E}\right]$ given by (5.34). The global charges are then found to acquire a correction due to the additional component of the invariant bilinear form in (5.67), so that they now read

$$Q\left[T, Y, \mathcal{E}\right] = -\int \left(T\mathcal{P} + Y\tilde{\mathcal{J}} - i\mathcal{E}\psi\right) d\phi \,, \tag{5.70}$$

with $\tilde{\mathcal{J}}{=}\mathcal{J}{+}\mu\mathcal{P}$, and do not depend on the parameter γ. Note that the shift in the canonical generator associated to Y implies that in the extended theory, even static configurations, as it is the case of Minkowski spacetime, may carry angular momentum.

It is then simple to verify that, once the canonical generators are expanded in modes, their nonvanishing Poisson brackets are given by

$$\begin{aligned}
i\left\{\tilde{\mathcal{J}}_m,\tilde{\mathcal{J}}_n\right\} &= (m-n)\,\tilde{\mathcal{J}}_{m+n}+\mu k m^3\delta_{m+n,0}\,,\\
i\left\{\tilde{\mathcal{J}}_m,\mathcal{P}_n\right\} &= (m-n)\,\mathcal{P}_{m+n}+k m^3\delta_{m+n,0}\,,\\
i\left\{\tilde{\mathcal{J}}_m,\psi_n\right\} &= \left(\frac{3m}{2}-n\right)\psi_{m+n}\,,\\
i\left\{\psi_m,\psi_n\right\} &= \frac{1}{2}\left(3m^2-4mn+3n^2\right)\mathcal{P}_{m+n}+\frac{9}{4k}\sum_q \mathcal{P}_{m+n-q}\mathcal{P}_q+k m^4\delta_{m+n,0}\,,
\end{aligned} \tag{5.71}$$

which corresponds to a hypersymmetric extension of the BMS_3 algebra, with an additional central extension along its Virasoro subalgebra.

5.6 General Relativity minimally coupled to half-integer spin fields

In the generic case of fermionic fields of spin $n+\frac{3}{2}$, and in the absence of cosmological constant, the hypergravity action reads [8],

$$I=\frac{k}{4\pi}\int 2R^a e_a+i\bar{\psi}_{a_1\ldots a_n}D\psi^{a_1\ldots a_n}\,, \tag{5.72}$$

where $\psi_{a_1\ldots a_n}$ describes a Grassmann-valued 1-form that is $\Gamma-$traceless, i. e., $\Gamma^{a_1}\psi_{a_1\ldots a_n}=0$, and completely symmetric in its vector indices. Its covariant derivative can be conveniently written as

$$D\psi^{a_1\ldots a_n}=d\psi^{a_1\ldots a_n}+\left(n+\frac{1}{2}\right)\omega^b\Gamma_b\psi^{a_1\ldots a_n}-\omega_b\Gamma^{(a_1}\psi^{a_2\ldots a_n)b}\,. \tag{5.73}$$

The standard supergravity action in [93, 94, 95], is then recovered for $n=0$, while the theory discussed in section 5.1 corresponds to $n=1$.

The generic theory can also be formulated in terms of a Chern-Simons action for a gauge field that takes values in the hyper-Poincaré algebra, given by

$$A=e^aP_a+\omega^aJ_a+\psi^\alpha_{a_1\ldots a_n}Q^{a_1\ldots a_n}_\alpha\,. \tag{5.74}$$

Here $Q^{a_1\ldots a_n}_\alpha$ correspond to $\Gamma-$traceless fermionic generators of spin $n+\frac{1}{2}$. The explicit expression of the generic hyper-Poincaré algebra can be compactly written in terms of its Maurer-Cartan form (see Section 3.2). The field equations then read $F = dA + A^2 = 0$, with

$$F = R^a J_a + \tilde{T}^a P_a + D\psi^\alpha_{a_1\ldots a_n} Q_\alpha^{a_1\ldots a_n} \,, \tag{5.75}$$

where the hypercovariant torsion is now given by

$$\tilde{T}^a = T^a - \frac{1}{2}\left(n + \frac{1}{2}\right) i\bar{\psi}_{a_1\ldots a_n}\Gamma^a\psi^{a_1\ldots a_n} \,. \tag{5.76}$$

Thus, by construction, the action is invariant under gauge transformations generated by $\lambda = \epsilon^\alpha_{a_1\ldots a_n} Q_\alpha^{a_1\ldots a_n}$, so that

$$\begin{aligned} \delta e^a &= \left(n + \frac{1}{2}\right) i\bar{\epsilon}_{a_1\ldots a_n}\Gamma^a\psi^{a_1\ldots a_n} \,, \\ \delta\omega^a &= 0 \,, \\ \delta\psi^{a_1\ldots a_n} &= D\epsilon^{a_1\ldots a_n} \,. \end{aligned} \tag{5.77}$$

5.6.1 Killing tensor-spinors

According to (5.77), a purely bosonic configuration is invariant under local hypersymmetry transformations provided the following "Killing tensor-spinor equation" is fulfilled:

$$d\epsilon_{a_1\ldots a_n} + \left(n + \frac{1}{2}\right)\omega^b\Gamma_b\epsilon_{a_1\ldots a_n} - \omega^b\Gamma_{(a_1}\epsilon_{a_2\ldots a_n)b} = 0 \,. \tag{5.78}$$

Since the field equations imply the vanishing of the curvature two-form R^a, the general solution of (5.78) is now given by

$$\epsilon^\alpha_{a_1\cdots a_n} = \left(g_S^{-1}\right)^\alpha_\beta (g_V)^{b_1}_{a_1} \cdots (g_V)^{b_n}_{a_n} \, \eta^\beta_{b_1\cdots b_n} \,, \tag{5.79}$$

where g_S and g_V are defined in eq. (5.12). As explained in section 5.2, both stand for the same group element g that determines the spin connection, $\omega = g^{-1}dg$, but expressed in the spinor and the vector representations, respectively. In the generic case, $\eta^\beta_{b_1\cdots b_n}$ is a constant $\Gamma-$traceless tensor-spinor. Unbroken hypersymmetries then correspond to Killing tensor-spinors of the form (5.79), that are globally well-defined.

The hypersymmetry properties of the class of solutions discussed in section 5.2.1, describing cosmological spacetimes and configurations with conical defects, then go as follows. For any configuration with $\mathcal{P} \neq 0$, g_S and g_V are given by (5.20) and (5.21), respectively; while in the case of $\mathcal{P} = 0$, they read as in eq. (5.22). Therefore, in the case of $\mathcal{P} > 0$ the solutions cannot possess globally-defined Killing tensor-spinors, because $\epsilon^{\alpha}_{a_1\cdots a_n}$ in (5.79) do not fulfill neither periodic nor antiperiodic boundary conditions. This means that hypersymmetries are necessarily broken for cosmological spacetimes.

By virtue of (5.79) and (5.22), configurations with $\mathcal{P}=0$ only admit constant Killing tensor-spinors that fulfill the following condition:

$$\left(n+\frac{1}{2}\right)\Gamma_1\epsilon_{a_1\ldots a_n}-\Gamma_{(a_1}\epsilon_{a_2\ldots a_n)1}=0\,, \tag{5.80}$$

which implies that they have a single nonvanishing component given by $\epsilon^{-}_{00\cdots 0}=\eta^{-}_{00\cdots 0}$. Therefore, this class of spacetimes possesses just one unbroken hypersymmetry, which relates to the fact that there is only one hypersymmetry bound that saturates for fermions with periodic boundary conditions (see below).

As explained in section 5.2.1, smooth solutions with conical defects are maximally hypersymmetric and their energy is determined by $\mathcal{P}=-kj^2/\pi<0$, where j is a (half-)integer. For this class of configurations, the explicit form of the Killing tensor-spinors is then given by (5.79), with g_S and g_V being described by eqs. (5.25) and (5.26), respectively. It will also be shown below that conical surpluses are excluded by the hypersymmetry bounds, which are fulfilled only for $j^2=1/4$, which corresponds to the case of Minkowski spacetime.

5.6.2 Asymptotically flat structure and hypersymmetry bounds

In order to describe the asymptotically flat behaviour of hypergravity in the generic case, it is convenient to make use of the $\Gamma-$traceless condition of the fields and the generators, which amounts to reduce the number of independent components. The hyper-Poincaré algebra in (3.19) can then be alternatively written as

$$\begin{aligned}
\left[\hat{J}_m,\hat{J}_n\right] &= (m-n)\,\hat{J}_{m+n}\\
\left[\hat{J}_m,\hat{P}_n\right] &= (m-n)\,\hat{P}_{m+n}\,,\\
\left[\hat{J}_m,\hat{\mathcal{Q}}_p\right] &= (sm-p)\,\hat{\mathcal{Q}}_{m+p}\,,\\
\left\{\hat{\mathcal{Q}}_p,\hat{\mathcal{Q}}_q\right\} &= f^{(s)}_{p,q}\hat{P}_{p+q}\,,
\end{aligned} \tag{5.81}$$

with $m,n=0,\pm1$, and $p,q=\pm\frac{1}{2},\pm\frac{3}{2}\ldots,\pm s$, where s stands for the spin of the fermionic generators $\hat{\mathcal{Q}}_p$. The structure constants fulfill $f^{(s)}_{p,q}=f^{(s)}_{q,p}=f^{(s)}_{-p,-q}$, and the nonvanishing ones are given by

$$f^{(s)}_{p,-p}=-\frac{2p}{s+p+1}f^{(s)}_{p,-p-1}=(-1)^{p+\frac{1}{2}}\,2p\prod_{k=\frac{1}{2}}^{|p|}\frac{(2s+2k)}{(2s-2\,(k-1))}\,, \tag{5.82}$$

provided $|p+q| \leq 1$. Here the fermionic generators have been normalized according to $f^{(s)}_{\frac{1}{2},-\frac{1}{2}} = -1$.

It is amusing to verify that the Jacobi identity now translates into the fact that the structure constants $f^{(s)}_{p,q}$ solve the following recursion relation:

$$(m-(p+q))\, f^{(s)}_{q,p} - (sm-p)\, f^{(s)}_{q,m+p} - (sm-q)\, f^{(s)}_{p,m+q} = 0\,. \tag{5.83}$$

For later purposes it is useful to note that

$$f^{(s)}_{s,-s} = (-1)^{s+\frac{1}{2}}\, \frac{2s}{2s+1} \frac{(4s)!!}{(2s-1)!!^2}\,. \tag{5.84}$$

The structure constants can also be conveniently written as

$$f^{(s)}_{m,n} = \sum_{l=0}^{s-\frac{1}{2}} h^{(l)}_{m,n}\,, \tag{5.85}$$

where $h^{(l)}_{m,n}$ stand for homogeneous polynomials of degree $2l$ in m, n, i. e., $h^{(l)}_{\lambda m,\lambda n} = \lambda^{2l} h^{(l)}_{m,n}$. Indeed, as it is shown below, the asymptotic symmetry algebra can be naturally expressed in terms of $h^{(l)}_{m,n}$, where m, n are extended to be arbitrary (half-)integers. Note that in the case of supergravity $f^{(1/2)}_{m,n} = -1$, while for fermionic generators of spin $s = 3/2$, the form of $f^{(3/2)}_{m,n}$ can be read from eq. (5.31). In the case of fermionic generators with $s = 5/2$, $7/2$ the explicit form of $f^{(5/2)}_{m,n}$ and $f^{(7/2)}_{m,n}$ is given in Appendices E.2 and E.3, respectively.

Following the lines of Section 5.3, the asymptotic form of the gauge field can be written as in eq. (5.32), so that at a fixed time slice, the dynamical field is proposed to be given by

$$a_\phi = \hat{J}_1 - \frac{\pi}{k}\left(\mathcal{J}\hat{P}_{-1} + \mathcal{P}\hat{J}_{-1} + \alpha_s \psi \hat{\mathcal{Q}}_{-s}\right)\,, \tag{5.86}$$

with

$$\alpha_s = \left(f^{(s)}_{-s,s+1}\right)^{-1} = -2s\left(f^{(s)}_{s,-s}\right)^{-1}\,, \tag{5.87}$$

and $f^{(s)}_{s,-s}$ can be read from eq. (5.84).

The asymptotic symmetries are then generically spanned by a hyper-Poincaré-valued parameter of the form

$$\lambda = T\hat{P}_1 + Y\hat{J}_1 + \mathcal{E}\hat{\mathcal{Q}}_s + \eta_{(s)}\left[T, Y, \mathcal{E}\right]\,, \tag{5.88}$$

where $\eta_{(s)}\left[T, Y, \mathcal{E}\right]$ goes along all but the lowest weight generators, provided the fields $\mathcal{J}$, $\mathcal{P}$, ψ transform in a suitable way.

The asymptotic form of the Lagrange multiplier can then be written in terms of the chemical potentials according to

$$a_u = \lambda \left[\mu_{\mathcal{P}}, \mu_{\mathcal{J}}, \mu_{\psi}\right] . \tag{5.89}$$

Its form is preserved under evolution in time as long as the field equations are fulfilled in the asymptotic region, and the parameters are subject to appropriate conditions, being described by first order equations in time.

In order to integrate the variation of the canonical generators in (5.40), one needs the relevant fermionic component of the invariant bilinear form, which is given by

$$\left\langle \hat{\mathcal{Q}}_s, \hat{\mathcal{Q}}_{-s} \right\rangle = 2\alpha_s^{-1} , \tag{5.90}$$

so that the global charges in the generic case acquire the same form as in eq. (5.41), i. e.,

$$Q\left[T, Y, \mathcal{E}\right] = -\int \left(T\mathcal{P} + Y\mathcal{J} - i\mathcal{E}\psi\right) d\phi \,. \tag{5.91}$$

Once expanded in Fourier modes, the nonvanishing Poisson brackets of the canonical generators are given by

$$\begin{aligned}
i\left\{\mathcal{J}_m, \mathcal{J}_n\right\} &= (m-n)\,\mathcal{J}_{m+n} \,, \\
i\left\{\mathcal{J}_m, \mathcal{P}_n\right\} &= (m-n)\,\mathcal{P}_{m+n} + km^3\delta_{m+n,0} \,, \\
i\left\{\mathcal{J}_m, \psi_n\right\} &= (sm-n)\,\psi_{m+n} \,, \\
i\left\{\psi_m, \psi_n\right\} &= \sum_{q=0}^{s-1/2} \frac{(-1)^{2s-q}}{s-q+\frac{1}{2}} \left(\frac{2}{k}\right)^{s-q-\frac{1}{2}} h_{m,n}^{(q)} \mathcal{P}_{m+n}^{s-q-\frac{1}{2}} + (-1)^{s-\frac{1}{2}} \frac{2km^{2s+1}}{\alpha_s\,(2s)!}\delta_{m+n,0} + \Xi_{m+n}^{(s)} \,.
\end{aligned} \tag{5.92}$$

The conformal weight of the fermionic generators ψ_n with respect to $\mathcal{J}_m$ is given by $\Delta = s+1$. Here $h_{m,n}^{(q)}$ stand for the homogeneous polynomials defined through eq. (5.85), extended to the case of (half-)integers, and

$$\mathcal{P}_{m+n}^{r} := \sum_{i_1,\cdots i_r} \mathcal{P}_{m+n-i_1\cdots-i_r}\mathcal{P}_{i_1}\cdots\mathcal{P}_{i_r} \,. \tag{5.93}$$

Here $\Xi_{m+n}^{(s)}$ stands for the mode expansion of nonlinear terms that contains derivatives of $\mathcal{P}$, and becomes nontrivial provided $s > 3/2$. Indeed, according to eqs. (E.9) and (5.42), in the case of supergravity ($s = 1/2$), and for $s = 3/2$, one finds that $\Xi_{m+n}^{(1/2)} = \Xi_{m+n}^{(3/2)} = 0$; while for $s = 5/2$ it is proportional to the mode expansion of $\left(\mathcal{P}'\right)^2$ (see eq. (E.20)). The explicit form of $\Xi^{(7/2)}$ is given in eq. (E.30).

As in Section 5.4, the asymptotic symmetry algebra (5.92) also implies the existence of nonlinear bounds for the energy. Indeed, making the same assumptions, for $m=-n=p$, the (fermionic) anticommutator is manifestly positive-definite. Furthermore, since in the "rest frame" the bosonic global charges just correspond to $\mathcal{P}_0$, the nonlinear terms described by $\Xi^{(s)}_{m+n}$ in the fermionic anticommutator do not contribute. Therefore, in the generic case the bounds are given by

$$\prod_{i=0}^{n}\left(p^2+(2i+1)^2\,\frac{\pi\mathcal{P}}{k}\right)\geq 0\,, \tag{5.94}$$

where p is a (half-)integer for the case of fermionic fields of spin $s=n+3/2$ that fulfill (anti)periodic boundary conditions.

It is then clear that the bounds are fulfilled for configurations with $\mathcal{P}>0$, as it is the case of cosmological spacetimes. The fact that they never saturate agrees with the nonexistence of globally-defined Killing tensor-spinors. Note that in the case of $\mathcal{P}=0$ the bounds are also satisfied, while the one with $p=0$ saturates, which corresponds to the fact that configurations of this sort admit a single unbroken hypersymmetry, being generated by a constant Killing tensor-spinor.

For the class of maximally hypersymmetric smooth solutions with negative energy ($\mathcal{P}=-kj^2/\pi$) described in section 5.2.1, the bounds (5.94) read

$$\prod_{i=0}^{n}\left(p^2-(2i+1)^2\,j^2\right)\geq 0\,, \tag{5.95}$$

which implies that the only case that fulfills all of them, also saturate the ones for $p=\pm(2i+1)/2$, with $i=0,1,\ldots,n$, and corresponds to $j^2=1/4$. Thus, Minkowski spacetime becomes naturally selected at the ground state in the case of fermions that satisfy antiperiodic boundary conditions, possessing the maximum number of Killing tensor-spinors described by (5.79), with (5.25) and (5.26).

5.7 Remarks

In sum, in the case of fermions that fulfill periodic boundary conditions the energy spectrum is nonnegative ($\mathcal{P}\geq 0$), so that the allowed class of solutions is generically characterized by the cosmological spacetimes described in section 5.2.1. The ground state is then given by a configuration of vanishing energy that saturates only one of the bounds ($p=0$). This corresponds to the null orbifold which, as shown in Section 5.6.1, possesses a single Killing tensor-spinor. If the fermions satisfy antiperiodic boundary conditions, the spectrum becomes enlarged since the bounds now imply that $\mathcal{P}\geq -k/4\pi$. Nonetheless, since conical defects and surpluses generically do not fulfill the field equation in vacuum, they are discarded unless they are smooth. According to (5.94), in this case the ground state saturates as many bounds as the maximum number of Killing tensor-spinors, and it can be identified with Minkowski spacetime, so that the spectrum acquires a gap.

In the case of fermionic fields of spin $s = n + \frac{3}{2}$, the locally hypersymmetric extension of the action $I_{\mu,\gamma}$ in (5.63), that includes parity-odd terms, can also be formulated as a Chern-Simons theory for the hyper-Poincaré group in (3.19). In order to carry out this task, the invariant bilinear form has to be suitably modified, so that it acquires additional components being determined by eq. (5.67). The gauge field reads

$$A = e^a P_a + \hat{\omega}^a J_a + \psi^\alpha_{a_1 \dots a_n} Q^{a_1 \dots a_n}_\alpha \,, \tag{5.96}$$

where, as in section 5.5, $\hat{\omega}^a = \omega^a + \gamma e^a$. Therefore, up to a boundary term, the action of the extended hypergravity theory reduces to

$$I_{\mu,\gamma,\psi_n} = I_{\mu,\gamma} + \frac{k}{4\pi} \int i\bar{\psi}_{a_1 \dots a_n} \left[D + \left(n + \frac{1}{2} \right) \gamma e^b \Gamma_b \right] \psi^{a_1 \dots a_n} \,, \tag{5.97}$$

being by construction locally invariant under

$$\begin{aligned}
\delta e^a &= \left(n + \frac{1}{2} \right) i\bar{\epsilon}_{a_1 \dots a_n} \Gamma^a \psi^{a_1 \dots a_n} \,, \\
\delta \omega^a &= - \left(n + \frac{1}{2} \right) i\gamma\bar{\epsilon}_{a_1 \dots a_n} \Gamma^a \psi^{a_1 \dots a_n} \,, \\
\delta \psi^{a_1 \dots a_n} &= \left[D + \left(n + \frac{1}{2} \right) \gamma e^b \Gamma_b \right] \epsilon^{a_1 \dots a_n} - \gamma e_b \Gamma^{(a_1} \epsilon^{a_2 \dots a_n)b} \,.
\end{aligned} \tag{5.98}$$

Note that the extended hypergravity action (5.97), and its corresponding local hypersymmetry transformations (5.98), agree with the corresponding ones for the locally hypersymmetric extension of General Relativity, given by (5.72) and (5.77), respectively, in the case of $\mu = \gamma = 0$. Consequently, a suitable set of asymptotically flat boundary conditions for the extended theory is also proposed to be described by gauge fields of the form (5.32), (5.86), and (5.89). The canonical generators of the asymptotic symmetries then reduce to the ones in eq. (5.70), with $\tilde{\mathcal{J}}=\mathcal{J}+\mu\mathcal{P}$, so that their algebra is readily found to be described by (5.92), but with an additional central extension along the Virasoro subalgebra, precisely as in eq. (5.71).

Chapter 6

Conclusions and forthcoming results

In this thesis, a novel extension of the Poincaré group with half-integers spin generators was explored. Specifically, in the case of three spacetime dimensions, it was shown that the theory of Aragone and Deser of hypergravity can be reformulated in order to incorporate this new group as its local gauge symmetry. The nontrivial Casimir operator for this algebra allowed to express the theory in terms of hyper-Poincaré-valued gauge fields, where the action of the theory is given by a three-dimensional Chern-Simons form. It was also shown that it is possible to extend this algebra for $d \geq 3$ dimensions and for arbitrary half-integer spin generators through the Maurer-Cartan form of the algebra. A consistent set of boundary conditions was proposed, being wide enough in order to accommodate a generic choice of chemical potentials associated to the global charges. The asymptotic symmetry algebra of the canonical generators turned out to be a hypersymmetric nonlinear extension of BMS_3, which could be recovered from a subset of a suitable limit of $W_{(2,4)} \oplus W_{(2,\frac{5}{2},4)}$. The bounds for the energy are nonlinear and it was explicitly shown that saturate for spacetimes that admit globally-defined Killing vector-spinors, selecting the same spectrum as supergravity. The extended theory admitting parity-odd terms in the action and the consequences on its asymptotic structure were analyzed. Finally, the generalization for arbitrary half-integer spin was also carried out, determining the explicit form of the hyper-BMS_3 algebra and hypersymmetry bounds.

It is worth pointing out that prescribing the asymptotic behaviour of gauge fields to be described by deviations with respect to a reference background that go along the highest weight generators of the algebra, turns out to be a very successful strategy. Indeed, this is not only the case of General Relativity in three spacetime dimensions [29], but it is also so for its locally supersymmetric extension with or without cosmological constant [72, 34], or even when the theory is nonminimally coupled to higher spin fields [44, 45, 73, 32, 33, 30, 96, 46, 90, 91, 97].

One of the interesting features of dealing with hypersymmetry, is that nonlinear bounds for the energy have been shown to naturally emerge from the anticommutator of fermionic generators. In the case of vanishing cosmological constant, the hypersymmetry bounds for the theory in vacuum turn out to exclude solutions that describe conical defects and surpluses [83, 84], despite the latter are maximally (hyper)symmetric. In presence of higher spin fields, the analogue of this class of configurations has been discussed in [78, 98, 51, 79, 80, 81, 82, 11]. It is then worth highlighting that, according to the results that have been recently obtained for hypergravity with negative cosmological constant [11], one is naturally led to expect that only a suitable subset of asymptotically flat solitonic-like solutions might fulfill the hypersymmetry bounds, for which the higher spin charges become tuned in terms of the mass. Indeed, it is amusing to verify that the gauge group $Sp\left(4\right)\otimes OSp\left(1|4\right)$ admits an inequivalent Inönü-Wigner contraction as compared with the one described in section 5.3.1, so that the electric-like spin-4 charges cannot be consistently decoupled in this alternative flat limit of the $(0,1)$ theory. This contraction is defined through a different change of basis:[1]

$$
\begin{gathered}
\hat{P}_i=\frac{1}{\ell}\left(L_i^{+}+L_{-i}^{-}\right) \quad , \quad \hat{J}_i=L_i^{+}-L_{-i}^{-} , \\
\hat{W}_n=\frac{1}{\ell}\left(U_n^{+}+U_{-n}^{-}\right) \quad , \quad \hat{V}_n=U_n^{+}-U_{-n}^{-} \quad , \quad \hat{\mathcal{Q}}_p=\sqrt{\frac{6}{\ell}}\mathcal{S}_p^{+} , \qquad (6.1)
\end{gathered}
$$

where L_i^{-}, U_n^{-} stand for the generators of the (left) $sp\left(4\right)$ algebra, and L_i^{+}, U_n^{+}, S_p^{+} span the (right) $osp\left(1|4\right)$ algebra. Thus, in the limit of large AdS radius, $\ell\to\infty$, the nonvanishing components of the (anti)commutators of the new algebra read

$$
\begin{aligned}
\left[\hat{J}_i,\hat{J}_j\right] &= (i-j)\,\hat{J}_{i+j} \quad , \quad \left[\hat{J}_i,\hat{P}_j\right]=(i-j)\,\hat{P}_{i+j} , \\
\left[\hat{J}_i,\hat{W}_n\right] &= (3i-n)\,\hat{W}_{i+n} \quad , \quad \left[\hat{J}_i,\hat{V}_n\right]=(3i-n)\,\hat{V}_{i+n} , \\
\left[\hat{\mathcal{P}}_i,\hat{V}_n\right] &= (3i-n)\,\hat{W}_{i+n} \quad , \quad \left[\hat{J}_i,\hat{\mathcal{Q}}_p\right]=\left(\frac{3i}{2}-p\right)\hat{\mathcal{Q}}_{i+p} , \\
\left[\hat{V}_m,\hat{V}_n\right] &= \frac{1}{2^2 3}(m-n)\left(\left(m^2+n^2-4\right)\left(m^2+n^2-\frac{2}{3}mn-9\right)-\frac{2}{3}\left(mn-6\right)mn\right)\hat{J}_{m+n} \\
&\quad +\frac{1}{6}(m-n)\left(m^2-mn+n^2-7\right)\hat{V}_{m+n} , \qquad (6.2)
\end{aligned}
$$

[1] Note that $OSp\left(1|4\right)$, corresponding to the super-AdS$_4$ group, as well as the superconformal group in three spacetime dimensions, admits two interesting consistent "flat limits" ($\ell\to\infty$), which can be obtained rescaling the generators either as in eq. (5.50), or in (6.1), provided the left copy is switched off.

$$\left[\hat{V}_m, \hat{W}_n\right] = \frac{1}{2^2 3}(m-n)\left(\left(m^2+n^2-4\right)(m^2+n^2-\frac{2}{3}mn-9)-\frac{2}{3}(mn-6)\,mn\right)\hat{P}_{m+n}$$
$$+\frac{1}{6}(m-n)\left(m^2-mn+n^2-7\right)\hat{W}_{m+n}\,,$$
$$\left[\hat{V}_m, \hat{\mathcal{Q}}_p\right] = \frac{1}{2^3 3}\left(2m^3-8m^2p+20mp^2+82p-23m-40p^3\right)\hat{\mathcal{Q}}_{m+p}\,,$$
$$\left\{\hat{\mathcal{Q}}_p, \hat{\mathcal{Q}}_q\right\} = 3\hat{W}_{p+q}+\frac{1}{2^2}\left(6p^2-8pq+6q^2-9\right)\hat{P}_{p+q}\,,$$

which is to be regarded to span the gauge group of flat hypergravity coupled to spin-4 fields. Here, $i,j=0,\pm1$, $m,n=0,\pm1,\pm2,\pm3$, and $p,q=\pm\frac{1}{2},\pm\frac{3}{2}$. Preliminary results point out that the mode expansion of the asymptotically flat symmetry algebra of hypergravity with a fermionic spin-5/2 field, being coupled to spin-4 fields, is then expected to be such that the nonvanishing Poisson brackets are given by

$$\begin{aligned}
i\{\mathcal{J}_m,\mathcal{J}_n\} &= (m-n)\,\mathcal{J}_{m+n} \quad , \quad i\{\mathcal{J}_m,\mathcal{P}_n\}=(m-n)\,\mathcal{P}_{m+n}+km^3\delta_{m+n,0}\,,\\
i\{\mathcal{J}_m,\mathcal{W}_n\} &= (3m-n)\,\mathcal{W}_{m+n} \quad , \quad i\{\mathcal{J}_m,\mathcal{V}_n\}=(3m-n)\,\mathcal{V}_{m+n}\,,\\
i\{\mathcal{P}_m,\mathcal{V}_n\} &= (3m-n)\,\mathcal{W}_{m+n} \quad , \quad i\{\mathcal{J}_m,\psi_n\}=\left(\frac{3m}{2}-n\right)\psi_{m+n}\,,\\
i\{\mathcal{V}_m,\mathcal{V}_n\} &= \frac{1}{2^2 3^2}(m-n)\left(3m^4-2m^3n+4m^2n^2-2mn^3+3n^4\right)\mathcal{J}_{m+n}\\
&\quad+\frac{1}{6}(m-n)\left(m^2-mn+n^2\right)\mathcal{V}_{m+n}-\frac{2^3 3\pi}{k}(m-n)\,\Theta^{(6)}_{m+n}\\
&\quad-\frac{7^2\pi}{3^2 k}(m-n)\left(m^2+4mn+n^2\right)\Theta^{(4)}_{m+n}\\
i\{\mathcal{V}_m,\mathcal{W}_n\} &= \frac{1}{2^2 3^2}(m-n)\left(3m^4-2m^3n+4m^2n^2-2mn^3+3n^4\right)\mathcal{P}_{m+n}\\
&\quad+\frac{1}{6}(m-n)\left(m^2-mn+n^2\right)\mathcal{W}_{m+n}-\frac{2^3 3\pi}{k}(m-n)\,\Omega^{(6)}_{m+n}\\
&\quad-\frac{7^2\pi}{3^2 k}(m-n)\left(m^2+4mn+n^2\right)\Omega^{(4)}_{m+n}+\frac{k}{2^2 3^2}m^7\delta_{m+n,0}\,,\\
i\{\mathcal{V}_m,\psi_n\} &= \frac{1}{2^2 3}\left(m^3-4m^2n+10mn^2-20n^3\right)\psi_{m+n}-\frac{23\pi}{3k}i\Omega^{(11/2)}_{m+n}\\
&\quad+\frac{\pi}{3k}(23m-82n)\,\Omega^{(9/2)}_{m+n}\,,\\
i\{\psi_m,\psi_n\} &= 3\mathcal{W}_{m+n}+\frac{3}{2}\left(m^2-\frac{4}{3}mn+n^2\right)\mathcal{P}_{m+n}+\frac{9\pi}{k}\Omega^{(4)}_{m+n}+km^4\delta_{m+n,0}\,,
\end{aligned} \tag{6.3}$$

where $\Omega_m^{(l)}$, and $\Theta_m^{(l)}$ stand for the mode expansion of the following nonlinear terms:[2]

$$\begin{aligned}
\Omega^{(4)} &= \frac{1}{2}\mathcal{P}^2\,,\\
\Theta^{(4)} &= \mathcal{J}\mathcal{P}\,,\\
\Omega^{(9/2)} &= \frac{1}{2}\mathcal{P}\psi\,,\\
\Omega^{(11/2)} &= \frac{27}{46}\mathcal{P}'\psi\,,\\
\Omega^{(6)} &= -\frac{7}{36}\mathcal{W}\mathcal{P}-\frac{2\pi}{3k}\mathcal{P}^3+\frac{295}{864}\left(\mathcal{P}'\right)^2+\frac{11}{27}\mathcal{P}\mathcal{P}''\,,\\
\Theta^{(6)} &= -\frac{7}{36}\left(\mathcal{V}\mathcal{P}+\mathcal{W}\mathcal{J}\right)-\frac{2\pi}{k}\mathcal{P}^2\mathcal{J}+\frac{295}{432}\mathcal{J}'\mathcal{P}'+\frac{11}{27}\left(\mathcal{J}\mathcal{P}''+\mathcal{P}\mathcal{J}''\right)+\frac{25}{72}i\psi\psi'\,.
\end{aligned} \tag{6.4}$$

Indeed, this asymptotic symmetry algebra is recovered from a contraction that corresponds to a different flat limit of $W_{(2,4)}\oplus W_{(2,\frac{5}{2},4)}$, as compared with the one in 5.3.1. The flat limit is now defined according to

$$\mathcal{P}_n=\frac{1}{\ell}\left(\mathcal{L}_n^+ + \mathcal{L}_{-n}^-\right)\quad,\quad \mathcal{J}_n=\mathcal{L}_n^+ - \mathcal{L}_{-n}^-\,,$$

$$\mathcal{W}_n=\frac{1}{\ell}\left(\mathcal{U}_n^+ + \mathcal{U}_{-n}^-\right)\quad,\quad \mathcal{V}_n=\mathcal{U}_n^+ - \mathcal{U}_{-n}^-\quad,\quad \psi_n=\sqrt{\frac{6}{\ell}}\Psi_n^+\,, \tag{6.5}$$

where the level also rescales as $\kappa = k\ell$.

Moreover, once the modes $\mathcal{P}_m$ are shifted according to (5.43), it is simple to verify that the wedge algebra of (6.3) reduces to the algebra of the gauge group in (6.2).

From the anticommutator of the fermionic generators in (6.3), one then finds that the zero modes of the energy and the electric-like spin-4 charge, $2\pi\mathcal{P}=\mathcal{P}_0$, $2\pi\mathcal{W}=\mathcal{W}_0$, fulfill the following bounds

$$3\mathcal{W}+\frac{9\pi}{2k}\mathcal{P}^2+5p^2\mathcal{P}+\frac{k}{2\pi}p^4\geq 0\,, \tag{6.6}$$

which agree with the bounds in [11] in the flat limit. It would then be interesting to explore different classes of solutions endowed with electric-like spin-4 charge, including cosmological spacetimes and solitonic-like configurations that fulfill the

[2] The infinite-dimensional nonlinear algebras in eqs. (5.51) and (6.3), correspond to different hypersymmetric extensions of the BMS_3 algebra, being isomorphic to the Galilean conformal algebra in two dimensions, and then relevant in the context of non-relativistic holography [60, 59, 89, 99, 100].

bounds (6.6), as well as the hypersymmetric ones that should saturate them. Note that since the bounds (6.6) factorize as

$$\left(p^2+\lambda_{[+]}^2\right)\left(p^2+\lambda_{[-]}^2\right)\geq 0\,, \tag{6.7}$$

it is natural to expect that the eigenvalues of the holonomy of the dynamical gauge field a_ϕ along an angular cycle, for the class of solutions aforementioned, have to be given by

$$\lambda_{[\pm]}^2=\frac{5\pi}{k}\left(\mathcal{P}\pm\frac{4}{5}\sqrt{\mathcal{P}^2-\frac{3k}{8\pi}\mathcal{W}}\right)\,. \tag{6.8}$$

In the case of solitonic-like solutions, these eigenvalues should then correspond to a couple of purely imaginary integers, that become related for the class of configurations that fulfill the bounds (6.7), saturating just some of them.

As a final remark, since the hyper-Poincaré group actually exists for $d\geq 3$ spacetime dimensions, it would be interesting to explore whether similar results as the ones obtained here could extend to higher-dimensional spacetimes. In this sense, despite the no-go results in four dimensions [6, 7, 37, 39, 40], some interesting results have been recently found in the case of hypergravity at the noninteracting level [101]. Whether these results correspond to a suitable weak field limit of Vasiliev higher spin gravity [102, 103], or another theory that has yet to be found, remains as an open question.

Appendices

Appendix A

Conventions

The orientation is chosen to be such that the Levi-Civita symbol fulfills $\varepsilon_{012} = 1$, and the Minkowski metric is assumed to be non-diagonal, whose only nonvanishing components read $\eta_{01} = \eta_{10} = \eta_{22} = 1$. Round brackets correspond to symmetrization of the indices enclosed by them, so that

$$X^{(a|}Y^{b}Z^{|c)} = X^{a}Y^{b}Z^{c} + X^{c}Y^{b}Z^{a} \,. \tag{A.1}$$

Note that the three-dimensional Dirac matrices, that satisfy the Clifford algebra $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$, fulfill the following identity:

$$\Gamma^{a}\Gamma^{b}\Gamma^{c} = \varepsilon^{abc} + \eta^{ab}\Gamma^{c} + \eta^{bc}\Gamma^{a} - \eta^{ac}\Gamma^{b} \,. \tag{A.2}$$

Note also that the presence of the imaginary unit "i" in the product of real Grassmann variables is because we assume that $(\theta_1\theta_2)^* = -\theta_1\theta_2$.

The generators of the Lorentz group, in the spinorial and vector (adjoint) representations, are assumed to be given by $(J_a)^{\alpha}_{\beta} = \frac{1}{2}(\Gamma_a)^{\alpha}_{\beta}$, and $(J_a)^{b}_{c} = -\varepsilon_{a\ \ c}^{\ b}$, respectively. The three-dimensional Γ-matrices are chosen as

$$\Gamma_0 = \frac{1}{\sqrt{2}}(\sigma_1 + i\sigma_2) \quad , \quad \Gamma_1 = \frac{1}{\sqrt{2}}(\sigma_1 - i\sigma_2) \quad , \quad \Gamma_2 = \sigma_3 \,, \tag{A.3}$$

where σ_i stand for the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.4}$$

For a vector-spinor ψ^{α}_{a}, with $\alpha = +, -$, and $a = 0, 1, 2$, the Majorana conjugate is defined as $\bar{\psi}_{\alpha a} = \psi^{\beta}_{a} C_{\beta\alpha}$, where the charge conjugation matrix C, and its inverse are chosen as

$$C_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad C^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{A.5}$$

so that $C^T = -C$, and $(C\Gamma_a)^T = C\Gamma_a$.

Appendix B

Fundamental representation of the $OSp\,(1|4)$ generators

The fundamental matrix representation of the $OSp\,(1|4)$ generators is explicitly given by

$$L_{-1}=\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad L_{1}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$L_{0}=\frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$U_{-3}=\begin{pmatrix} 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad U_{-2}=\frac{5}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$U_{-1}=\frac{2}{\sqrt{3}}\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad U_{3}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$U_2 = \frac{5}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad U_1 = \frac{2}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$U_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$\mathcal{S}_{-\frac{3}{2}} = \frac{2}{\sqrt{3}} \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathcal{S}_{-\frac{1}{2}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \end{pmatrix} ,$$

$$\mathcal{S}_{\frac{1}{2}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \mathcal{S}_{\frac{3}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

Appendix C

Asymptotically AdS structure: Lie-algebra-valued parameters and transformation law of the fields

The Lie-algebra-valued parameters that maintain the asymptotic form of the $osp\,(1|4)$-valued gauge connections (4.7) are given by

$$\lambda^{\pm}\left[\epsilon_{\pm},\chi_{\pm},\vartheta_{\pm}\right]=\epsilon_{\pm}L_{\pm1}-\chi_{\pm}U_{\pm3}\mp\vartheta_{\pm}\mathcal{S}_{\pm3}+\eta^{\pm}\left[\epsilon_{\pm},\chi_{\pm},\vartheta_{\pm}\right]\,, \tag{C.1}$$

with

$$\begin{aligned}
\eta^{\pm}\left[\epsilon_{\pm},\chi_{\pm},\vartheta_{\pm}\right] \;=\;& -\frac{3\pi}{\kappa}\left(i\psi^{\pm}\vartheta_{\pm}+\frac{2}{3}\epsilon_{\pm}\mathcal{L}^{\pm}+2\chi_{\pm}\mathcal{U}^{\pm}-\frac{\kappa}{6\pi}\epsilon_{\pm}''\right)L_{\mp1}\mp\epsilon_{\pm}'L_{0}\\
&+\frac{6\pi}{\kappa}\left(\chi_{\pm}\mathcal{L}^{\pm}-\frac{\kappa}{12\pi}\chi_{\pm}'\right)U_{\pm1}\mp\frac{2\pi}{\kappa}\left(\chi_{\pm}\mathcal{L}^{\pm\prime}+\frac{8}{3}\chi_{\pm}'\mathcal{L}^{\pm}-\frac{\kappa}{12\pi}\chi_{\pm}'''\right)U_{0}\\
&-\frac{\pi}{2\kappa}\left[i\psi^{\pm}\vartheta_{\pm}+2\left(\mathcal{U}^{\pm}-\frac{1}{2}\mathcal{L}^{\pm\prime\prime}+\frac{12\pi}{\kappa}\left(\mathcal{L}^{\pm}\right)^{2}\right)\chi-\frac{11}{3}\chi_{\pm}'\mathcal{L}^{\pm\prime}\right.\\
&\left.-\frac{14}{3}\chi_{\pm}''\mathcal{L}^{\pm}+\frac{\kappa}{12\pi}\chi_{\pm}^{(4)}\right]U_{\mp1}\pm\chi_{\pm}'U_{\pm2}\pm\frac{\pi}{2\kappa}\left[i\psi^{\pm}\vartheta_{\pm}'+\frac{1}{5}i\psi^{\pm\prime}\vartheta_{\pm}\right.\\
&-\frac{5}{3}\chi_{\pm}''\mathcal{L}^{\pm\prime}-\frac{4}{3}\mathcal{L}^{\pm}\chi_{\pm}'''+\frac{2}{5}\left(\mathcal{U}^{\pm}-\frac{1}{2}\mathcal{L}^{\pm\prime\prime}+\frac{18\pi}{\kappa}\left(\mathcal{L}^{\pm}\right)^{2}\right)'\chi_{\pm}\\
&\left.+\frac{6}{5}\left(\mathcal{U}^{\pm}-\frac{7}{9}\mathcal{L}^{\pm\prime\prime}+\frac{44\pi}{3\kappa}\left(\mathcal{L}^{\pm}\right)^{2}\right)\chi_{\pm}'+\frac{\kappa}{60\pi}\chi_{\pm}^{(5)}\right]U_{\mp2} \qquad\qquad (\text{C.2})\\
&-\frac{\pi}{4\kappa}\left\{i\psi^{\pm}\vartheta_{\pm}''+\frac{1}{15}i\left(\psi^{\pm\prime\prime}-2^{4}\frac{5\pi}{\kappa}\mathcal{L}^{\pm}\psi^{\pm}\right)\vartheta_{\pm}+\frac{2}{5}i\psi^{\pm\prime}\vartheta_{\pm}'\right.
\end{aligned}$$

$$
\begin{aligned}
&-\chi'''_{\pm}\mathcal{L}^{\pm\prime}-\frac{4}{5}\epsilon_{\pm}\mathcal{U}^{\pm}+\frac{2}{3}\left(\mathcal{U}^{\pm}-\frac{13}{10}\mathcal{L}^{\pm\prime\prime}+\frac{272\pi}{15\kappa}\left(\mathcal{L}^{\pm}\right)^{2}\right)\chi''_{\pm}\\
&+\frac{8}{15}\left(\mathcal{U}^{\pm}-\frac{17}{24}\mathcal{L}^{\pm\prime\prime}+\frac{241\pi}{12\kappa}\left(\mathcal{L}^{\pm}\right)^{2}\right)'\chi'_{\pm}+\frac{40\pi}{3\kappa}\left[i\psi^{\pm}\psi^{\pm\prime}-\frac{11}{5^{2}}\mathcal{U}^{\pm}\mathcal{L}^{\pm}\right.\\
&-\frac{12\pi}{5\kappa}\left(\mathcal{L}^{\pm}\right)^{3}+\frac{\kappa}{10^{2}\pi}\left(\mathcal{U}^{\pm}-\frac{1}{2}\mathcal{L}^{\pm\prime\prime}\right)''+\frac{3^{2}}{5^{2}}\left(\mathcal{L}^{\pm\prime}\right)^{2}+\left.\frac{23}{50}\mathcal{L}^{\pm\prime\prime}\mathcal{L}^{\pm}\right]\chi_{\pm}\\
&-\frac{5}{9}\chi^{(4)}_{\pm}\mathcal{L}^{\pm}+\left.\frac{\kappa}{180\pi}\chi^{(6)}_{\pm}\right\}U_{\mp3}-\frac{2\pi}{\kappa}\left[\epsilon_{\pm}\psi^{\pm}+\frac{1}{2}\vartheta_{\pm}\mathcal{L}^{\pm\prime}+\frac{7}{6}\vartheta'_{\pm}\mathcal{L}^{\pm}\right.\\
&-\frac{5}{3}\left(\psi^{\pm\prime\prime}+\frac{52\pi}{5\kappa}\mathcal{L}^{\pm}\psi^{\pm}\right)\chi_{\pm}-\frac{25}{6}\chi'_{\pm}\psi^{\pm\prime}-\frac{17}{6}\chi''_{\pm}\psi^{\pm}-\left.\frac{\kappa}{12\pi}\vartheta'''_{\pm}\right]\mathcal{S}_{\mp\frac{3}{2}}\\
&\pm\frac{3\pi}{\kappa}\left(\vartheta_{\pm}\mathcal{L}^{\pm}-\frac{10}{3}\chi_{\pm}\psi^{\pm\prime}-5\chi'_{\pm}\psi^{\pm}-\frac{\kappa}{6\pi}\vartheta''_{\pm}\right)\mathcal{S}_{\mp\frac{1}{2}}\\
&+\frac{20\pi}{\kappa}\left(\chi_{\pm}\psi^{\pm}+\frac{\kappa}{20\pi}\vartheta'_{\pm}\right)\mathcal{S}_{\pm\frac{1}{2}}\,.
\end{aligned}
$$

The transformation law of the fields $\mathcal{L}^{\pm}(t,\phi)$, $\mathcal{U}^{\pm}(t,\phi)$, $\psi^{\pm}(t,\phi)$ read

$$
\begin{aligned}
\delta\mathcal{L}^{\pm} &= 2\epsilon'_{\pm}\mathcal{L}^{\pm}+\epsilon_{\pm}\mathcal{L}^{\pm\prime}-\frac{\kappa}{4\pi}\epsilon'''_{\pm}+3\mathcal{U}^{\pm\prime}\chi_{\pm}+4\mathcal{U}^{\pm}\chi'_{\pm}+\frac{5}{2}i\psi^{\pm}\vartheta'_{\pm}+\frac{3}{2}i\psi^{\pm\prime}\vartheta_{\pm}\,,\\
\delta\psi^{\pm} &= \frac{5}{2}\epsilon'_{\pm}\psi^{\pm}+\epsilon_{\pm}\psi^{\pm\prime}-\left(\mathcal{U}^{\pm}-\frac{1}{2}\mathcal{L}^{\pm\prime\prime}+\frac{3\pi}{\kappa}\Lambda^{(4)}_{\pm}\right)\vartheta_{\pm}+\frac{5}{3}\left(\mathcal{L}^{\pm}\vartheta'_{\pm}-\frac{\kappa}{20\pi}\vartheta'''_{\pm}\right)'\\
&\quad+\frac{82\pi}{3\kappa}\left(\Lambda^{(9/2)\prime}_{\pm}-\frac{23}{82}\Lambda^{(11/2)}_{\pm}-\frac{5\kappa}{82\pi}\psi^{\pm\prime\prime\prime}\right)\chi_{\pm}+\frac{35\pi}{\kappa}\left(\Lambda^{(9/2)}_{\pm}-\frac{\kappa}{6\pi}\psi^{\pm\prime\prime}\right)\chi'_{\pm}\\
&\quad-7\chi''_{\pm}\psi^{\pm\prime}-\frac{35}{12}\chi'''_{\pm}\psi^{\pm}\,, \qquad\qquad (C.3)\\
\delta\mathcal{U}^{\pm} &= 4\epsilon'_{\pm}\mathcal{U}^{\pm}+\epsilon_{\pm}\mathcal{U}^{\pm\prime}+\frac{23\pi}{3\kappa}i\left(\Lambda^{(11/2)}_{\pm}+\Lambda^{(9/2)\prime}_{\pm}-\frac{\kappa}{92\pi}\psi^{\pm\prime\prime\prime}\right)\vartheta_{\pm}-\frac{7}{4}i\psi^{\pm\prime}\vartheta''_{\pm}\\
&\quad-\frac{35}{12}i\psi^{\pm}\vartheta'''_{\pm}+\frac{35\pi}{\kappa}i\left(\Lambda^{(9/2)}_{\pm}-\frac{\kappa}{60\pi}\psi^{\pm\prime\prime}\right)\vartheta'_{\pm}-\frac{1}{6}\left[\left(\mathcal{U}^{\pm}-\frac{1}{2}\mathcal{L}^{\pm\prime\prime}\right)''\right.\\
&\quad+\left.\frac{144}{\kappa}\left(\Lambda^{(6)}_{\pm}-\frac{49}{216}\Lambda^{(4)\prime\prime}_{\pm}\right)\right]'\chi_{\pm}-\frac{5}{6}\left[\left(\mathcal{U}^{\pm}-\frac{2}{3}\mathcal{L}^{\pm\prime\prime}\right)''+\frac{288}{5\kappa}\Lambda^{(6)}_{\pm}\right]\chi'_{\pm}\\
&\quad+\frac{14}{9}\left(\mathcal{L}^{\pm\prime\prime}-\frac{27}{28}\mathcal{U}^{\pm}-\frac{21\pi}{\kappa}\Lambda^{(4)}_{\pm}\right)'\chi''_{\pm}+\frac{7}{3}\left(\mathcal{L}^{\pm\prime\prime}-\frac{3}{7}\mathcal{U}^{\pm}-\frac{28\pi}{3\kappa}\Lambda^{(4)}_{\pm}\right)\chi'''\\
&\quad+\frac{35}{18}\mathcal{L}^{\pm\prime}\chi^{(4)}_{\pm}+\frac{7}{9}\mathcal{L}^{\pm}\chi^{(5)}_{\pm}-\frac{\kappa}{144\pi}\chi^{(7)}_{\pm}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_{\pm}^{(4)} &= \left(\mathcal{L}^{\pm}\right)^2 \,, \\
\Lambda_{\pm}^{(9/2)} &= \mathcal{L}^{\pm}\psi^{\pm} \,, \\
\Lambda_{\pm}^{(11/2)} &= \frac{27}{23}\mathcal{L}^{\pm\prime}\psi^{\pm} \,, \\
\Lambda_{\pm}^{(6)} &= -\frac{7}{18}\mathcal{U}^{\pm}\mathcal{L}^{\pm} - \frac{8\pi}{3\kappa}\left(\mathcal{L}^{\pm}\right)^3 + \frac{295}{432}\left(\mathcal{L}^{\pm\prime}\right)^2 + \frac{22}{27}\mathcal{L}^{\pm\prime\prime}\mathcal{L}^{\pm} + \frac{25}{12}i\psi^{\pm}\psi^{\pm\prime} \,.
\end{aligned}
\tag{C.4}
$$

Appendix D

Killing vector-spinors from an alternative approach

Killing vector-spinors ϵ_a^α in (5.10) that are globally well defined can be recovered as a particular case of the asymptotic symmetries discussed in section 5.3. Indeed, they are of the form $\epsilon_a^\alpha Q_\alpha^a = \lambda\,[0,0,\mathcal{E}]$, with λ given by (5.34). Hence, for the class of bosonic configurations discussed in section 5.2.1, that only carries the zero modes of the global charges, and their corresponding chemical potentials are constant, the components of the Killing vector-spinors can be written as

$$\lambda\,[0,0,\mathcal{E}] = \mathcal{E}\hat{\mathcal{Q}}_{\frac{3}{2}} - \mathcal{E}'\hat{\mathcal{Q}}_{\frac{1}{2}} - \frac{1}{2}\left(\frac{3\pi}{k}\mathcal{E}\mathcal{P} - \mathcal{E}''\right)\hat{\mathcal{Q}}_{-\frac{1}{2}} - \frac{\pi}{3k}\left(-\frac{7}{2}\mathcal{E}'\mathcal{P} + \frac{k}{2\pi}\mathcal{E}'''\right)\hat{\mathcal{Q}}_{-\frac{3}{2}}\,. \tag{D.1}$$

The requirements of invariance under hypersymmetry can then be read from eqs. (5.36), (5.39), so that the Killing vector-spinor equation reduces to

$$\delta\psi = -\frac{9\pi}{2k}\mathcal{P}^2\mathcal{E} + 5\mathcal{P}\mathcal{E}'' - \frac{k}{2\pi}\mathcal{E}'''' = 0\,, \tag{D.2}$$

$$\dot{\mathcal{E}} = \mu_{\mathcal{J}}\mathcal{E}'\,, \tag{D.3}$$

which can be readily integrated. In fact, the solution of eq. (D.2) is generically given by

$$\mathcal{E} = \mathcal{A}_1 e^{\sqrt{\frac{\pi\mathcal{P}}{k}}\phi} + \mathcal{A}_2 e^{-\sqrt{\frac{\pi\mathcal{P}}{k}}\phi} + \mathcal{A}_3 e^{3\sqrt{\frac{\pi\mathcal{P}}{k}}\phi} + \mathcal{A}_4 e^{-3\sqrt{\frac{\pi\mathcal{P}}{k}}\phi}\,, \tag{D.4}$$

where $\mathcal{A}_I = \mathcal{A}_I\,(u)$ stand for four arbitrary functions.

In the case of $\mathcal{P} > 0$, $\mathcal{E}$ clearly cannot fulfill neither antiperiodic nor periodic boundary conditions, and therefore, cosmological spacetimes break all the hypersymmetries.

Note that if the energy vanishes ($\mathcal{P}=0$), eq. (D.2) integrates in a different way:

$$\mathcal{E}=\mathcal{A}_0+\mathcal{A}_1\phi+\mathcal{A}_2\phi^2+\mathcal{A}_3\phi^3\,. \tag{D.5}$$

Periodicity then implies that $\mathcal{E}=\mathcal{A}_0\left(u\right)$, while the remaining equation (D.3) fixes the arbitrary function to be a constant. Hence, vanishing energy configurations, as the null orbifold, admit a single constant Killing vector-spinor.

Finally, for $\mathcal{P}=-kn^2/\pi<0$, eq. (D.4) reads

$$\mathcal{E}=\mathcal{A}_1e^{in\phi}+\mathcal{A}_1^*e^{-in\phi}+\mathcal{A}_3e^{3in\phi}+\mathcal{A}_3^*e^{-3in\phi}\,, \tag{D.6}$$

so that n turns out to be a (half-)integer for fermions fulfilling (anti)periodic boundary conditions. The remaining equation (D.3) then fixes the form of the arbitrary functions $\mathcal{A}_I\left(u\right)$, and hence there are four independent Killing vector-spinors, determined by

$$\mathcal{E}=\mathcal{E}_1e^{in\left(\mu_{\mathcal{J}}u+\phi\right)}+\mathcal{E}_1^*e^{-in\left(\mu_{\mathcal{J}}u+\phi\right)}+\mathcal{E}_3e^{3in\left(\mu_{\mathcal{J}}u+\phi\right)}+\mathcal{E}_3^*e^{-3in\left(\mu_{\mathcal{J}}u+\phi\right)}\,. \tag{D.7}$$

It is worth pointing out that Minkowski spacetime, which corresponds to $n^2=1/4$, is the only one of this class that fulfills all the hypersymmetry bounds (5.55), saturating precisely four of them in the case of antiperiodic boundary conditions. Indeed, the remaining solutions of this sort, in spite of possessing the maximum number of hypersymmetries, become manifestly excluded by the bounds. In the case of periodic boundary conditions, the null orbifold also satisfies the bounds, but it saturates only one of them.

Appendix E

Asymptotic hypersymmetry algebra

E.1 Spin-$3/2$ fields (supergravity)

In our conventions, the super-Poincaré algebra with $\mathcal{N}=1$ reads

$$
\begin{aligned}
[J_a, J_b] &= \varepsilon_{abc} J^c\,, \\
[J_a, P_b] &= \varepsilon_{abc} P^c\,, && \text{(E.1)}\\
[J_a, Q_\alpha] &= \frac{1}{2}\left(\Gamma_a\right)^{\beta}{}_{\alpha}\, Q_{\beta b}\,, && \text{(E.2)}\\
\{Q_\alpha, Q_\beta\} &= -\frac{1}{2}\left(C\Gamma^c\right)_{\alpha\beta} P_c\,,
\end{aligned}
$$

so that changing the basis according to

$$
\begin{gathered}
\hat{J}_{-1} = -2J_0 \quad , \quad \hat{J}_1 = J_1 \quad , \quad \hat{J}_0 = J_2\,, \\
\hat{P}_{-1} = -2P_0 \quad , \quad \hat{P}_1 = P_1 \quad , \quad \hat{P}_0 = P_2\,, \\
\hat{\mathcal{Q}}_{-\frac{1}{2}} = 2^{\frac{3}{4}} Q_+ \quad , \quad \hat{\mathcal{Q}}_{\frac{1}{2}} = 2^{\frac{1}{4}} Q_-\,,
\end{gathered}
\tag{E.3}
$$

it acquires the following form

$$
\begin{aligned}
\left[\hat{J}_m, \hat{J}_n\right] &= (m-n)\,\hat{J}_{m+n}\,, \\
\left[\hat{J}_m, \hat{P}_n\right] &= (m-n)\,\hat{P}_{m+n}\,, \\
\left[\hat{J}_m, \hat{\mathcal{Q}}_p\right] &= \left(\frac{m}{2}-p\right)\hat{\mathcal{Q}}_{m+p}\,, && \text{(E.4)}\\
\left\{\hat{\mathcal{Q}}_p, \hat{\mathcal{Q}}_q\right\} &= -\hat{P}_{p+q}\,.
\end{aligned}
$$

with $m,n=0,\pm1$, and $p,q=\pm\frac{1}{2}$.

The asymptotic form of the dynamical gauge field then reads

$$a_\phi = \hat{J}_1 - \frac{\pi}{k}\left(\mathcal{P}\hat{J}_{-1} + \mathcal{J}\hat{P}_{-1} + \psi\hat{\mathcal{Q}}_{-\frac{1}{2}}\right) , \tag{E.5}$$

which is mapped into itself under gauge transformations generated by

$$\lambda = T\hat{P}_1 + Y\hat{J}_1 + \mathcal{E}\hat{\mathcal{Q}}_{\frac{1}{2}} + \eta_{\left(\frac{1}{2}\right)}\left[T,Y,\mathcal{E}\right] , \tag{E.6}$$

with

$$\begin{aligned}\eta_{\left(\frac{1}{2}\right)}\left[T,Y,\mathcal{E}\right] &= -T'\hat{P}_0 - Y'\hat{J}_0 - \left(\mathcal{E}' + \frac{\pi Y\psi}{k}\right)\hat{\mathcal{Q}}_{-\frac{1}{2}} + \frac{1}{2}\left(Y'' - \frac{2\pi Y\mathcal{P}}{k}\right)\hat{J}_{-1} \\ &\quad + \frac{1}{2}\left(T'' - \frac{2\pi T\mathcal{P}}{k} - \frac{2\pi Y\mathcal{J}}{k} + \frac{i\pi\mathcal{E}\psi}{k}\right)\hat{P}_{-1} ,\end{aligned} \tag{E.7}$$

provided the fields transform according to:

$$\begin{aligned}\delta\mathcal{P} &= 2\mathcal{P}Y' + \mathcal{P}'Y - \frac{k}{2\pi}Y''' , \\ \delta\mathcal{J} &= 2\mathcal{J}Y' + \mathcal{J}'Y + 2\mathcal{P}T' + \mathcal{P}'T - \frac{k}{2\pi}T''' + \frac{3}{2}i\psi\mathcal{E}' + \frac{1}{2}i\psi'\mathcal{E} , \\ \delta\psi &= \frac{3}{2}\psi Y' + \psi'Y - \mathcal{P}\mathcal{E} + \frac{k}{\pi}\mathcal{E}'' .\end{aligned} \tag{E.8}$$

Once expanded in modes, the asymptotic symmetry algebra is found to be given by

$$\begin{aligned}i\left\{\mathcal{J}_m,\mathcal{J}_n\right\} &= (m-n)\,\mathcal{J}_{m+n} , \\ i\left\{\mathcal{J}_m,\mathcal{P}_n\right\} &= (m-n)\,\mathcal{P}_{m+n} + km^3\delta_{m+n,0} , \\ i\left\{\mathcal{J}_m,\psi_n\right\} &= \left(\frac{m}{2}-n\right)\psi_{m+n} , \\ i\left\{\psi_m,\psi_n\right\} &= \mathcal{P}_{m+n} + 2km^2\delta_{m+n,0} ,\end{aligned} \tag{E.9}$$

and hence in the case of fermions that fulfill periodic boundary conditions, the energy $\mathcal{P}=\frac{\mathcal{P}_0}{2\pi}$ is bounded to be nonnegative,

$$\mathcal{P} \geq 0 , \tag{E.10}$$

while in the case of fermions subject to antiperiodic boundary conditions, the energy fulfills

$$\frac{1}{4} + \frac{\pi\mathcal{P}}{k} \geq 0 . \tag{E.11}$$

These results agree with the ones in [34].

E.2 Spin-$7/2$ fields

In the case of fermionic generators of (conformal) spin $\Delta = 7/2$, the hyper-Poincaré algebra can be written as

$$\begin{aligned}
\left[\hat{J}_m, \hat{J}_n\right] &= (m-n)\,\hat{J}_{m+n}\,, \\
\left[\hat{J}_m, \hat{P}_n\right] &= (m-n)\,\hat{P}_{m+n}\,, \qquad \text{(E.12)}\\
\left[\hat{J}_m, \hat{\mathcal{Q}}_p\right] &= \left(\frac{5m}{2} - p\right)\hat{\mathcal{Q}}_{m+p}\,, \qquad \text{(E.13)}\\
\left\{\hat{\mathcal{Q}}_p, \hat{\mathcal{Q}}_q\right\} &= f^{(5/2)}_{p,q}\,\hat{P}_{p+q}\,,
\end{aligned}$$

with

$$\begin{aligned}
f^{(5/2)}_{p,q} &= -\frac{1}{192}\left[80\left(p^4+q^4\right) - 128\left(p^3 q + p q^3\right)\right. \\
&\left. +144 p^2 q^2 - 620\left(p^2+q^2\right) + 832 pq + 675\right]\,, \qquad \text{(E.14)}
\end{aligned}$$

where $m, n = 0, \pm 1$, and $p, q = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$.

The asymptotic behaviour of the dynamical gauge field now reads

$$a_\phi = \hat{J}_1 - \frac{\pi}{k}\left(\mathcal{J}\hat{P}_{-1} + \mathcal{P}\hat{J}_{-1} + \frac{\psi}{10}\hat{\mathcal{Q}}_{-\frac{5}{2}}\right)\,, \qquad \text{(E.15)}$$

so that the asymptotic symmetries are now spanned by

$$\lambda = T\hat{P}_1 + Y\hat{J}_1 + \mathcal{E}\hat{\mathcal{Q}}_{\frac{1}{2}} + \eta_{\left(\frac{5}{2}\right)}\left[T, Y, \mathcal{E}\right]\,, \qquad \text{(E.16)}$$

with

$$\begin{aligned}
\eta_{\left(\frac{5}{2}\right)}\left[T, Y, \mathcal{E}\right] &= -T'\hat{P}_0 - Y'\hat{J}_0 - \mathcal{E}'\hat{\mathcal{Q}}_{\frac{3}{2}} + \frac{1}{2}\left(T'' - \frac{2\pi T\mathcal{P}}{k} - \frac{2\pi Y\mathcal{J}}{k} - \frac{5i\pi\mathcal{E}\psi}{k}\right)\hat{P}_{-1} \\
&+\frac{1}{2}\left(Y'' - \frac{2\pi Y\mathcal{P}}{k}\right)\hat{J}_{-1} + \frac{1}{2}\left(\mathcal{E}'' - \frac{5\pi\mathcal{E}\mathcal{P}}{k}\right)\hat{\mathcal{Q}}_{\frac{1}{2}} \\
&-\frac{1}{6}\left(\mathcal{E}''' - \frac{13\pi\mathcal{E}'\mathcal{P}}{k} - \frac{5\pi\mathcal{E}\mathcal{P}'}{k}\right)\hat{\mathcal{Q}}_{-\frac{1}{2}} \qquad \text{(E.17)}\\
&+\frac{1}{24}\left(\mathcal{E}^{(4)} + \frac{45\pi^2\mathcal{E}\mathcal{P}^2}{k^2} - \frac{22\pi\mathcal{E}''\mathcal{P}}{k} - \frac{18\pi\mathcal{E}'\mathcal{P}'}{k} - \frac{5\pi\mathcal{E}\mathcal{P}''}{k}\right)\hat{\mathcal{Q}}_{-\frac{3}{2}} \\
&-\frac{1}{120}\left(\mathcal{E}^{(5)} + \frac{149\pi^2\mathcal{E}'\mathcal{P}^2}{k^2} + \frac{12\pi Y\psi}{k} - \frac{30\pi\mathcal{E}'''\mathcal{P}}{k}\right. \\
&\left.+\frac{130\pi^2\mathcal{E}\mathcal{P}\mathcal{P}'}{k^2} - \frac{40\pi\mathcal{E}''\mathcal{P}'}{k} - \frac{23\pi\mathcal{E}'\mathcal{P}''}{k} - \frac{5\pi\mathcal{E}\mathcal{P}'''}{k}\right)\hat{\mathcal{Q}}_{-\frac{5}{2}}\,,
\end{aligned}$$

and the transformation law of the fields is given by

$$\begin{aligned}
\delta\mathcal{P} &= 2\mathcal{P}Y' + \mathcal{P}'Y - \frac{k}{2\pi}Y''' , \\
\delta\mathcal{J} &= 2\mathcal{J}Y' + \mathcal{J}'Y + 2\mathcal{P}T' + \mathcal{P}'T - \frac{k}{2\pi}T''' + \frac{7}{2}i\psi\mathcal{E}' + \frac{5}{2}i\psi'\mathcal{E} , \\
\delta\psi &= \frac{7}{2}\psi Y' + \psi' Y + \left(-\frac{75\pi^2\mathcal{P}^3}{4k^2} + \frac{65\pi\left(\mathcal{P}'\right)^2}{6k} + \frac{155\pi\mathcal{P}\mathcal{P}''}{12k} - \frac{5}{12}\mathcal{P}^{(4)}\right)\mathcal{E} \\
&\quad + \left(\frac{259\pi\mathcal{P}\mathcal{P}'}{6k} - \frac{7}{3}\mathcal{P}'''\right)\mathcal{E}' + \left(\frac{259\pi\mathcal{P}^2}{12k} - \frac{21}{4}\mathcal{P}''\right)\mathcal{E}'' - \frac{35}{6}\mathcal{E}'''\mathcal{P}' - \frac{35}{12}\mathcal{E}^{(4)}\mathcal{P} + \frac{k\mathcal{E}^{(6)}}{12\pi} .
\end{aligned} \tag{E.18}$$

The Poisson brackets of the asymptotic symmetry algebra in this case read

$$\begin{aligned}
i\left\{\mathcal{J}_m, \mathcal{J}_n\right\} &= (m-n)\,\mathcal{J}_{m+n} , \\
i\left\{\mathcal{J}_m, \mathcal{P}_n\right\} &= (m-n)\,\mathcal{P}_{m+n} + km^3\delta_{m+n,0} , \\
i\left\{\mathcal{J}_m, \psi_n\right\} &= \left(\frac{5m}{2} - n\right)\psi_{m+n} , \\
i\left\{\psi_m, \psi_n\right\} &= \frac{1}{12}\left(5m^4 - 8m^3n + 9m^2n^2 - 8mn^3 + 5n^4\right)\mathcal{P}_{m+n} \\
&\quad + \frac{1}{48k}\left(155m^2 - 208mn + 155n^2\right)\sum_q \mathcal{P}_{m+n-q}\mathcal{P}_q \\
&\quad + \frac{75}{16k^2}\sum_q \mathcal{P}_{m+n-q-r}\mathcal{P}_q\mathcal{P}_r + \frac{k}{6}m^6\delta_{m+n,0} + \Xi^{(5/2)}_{m+n} ,
\end{aligned} \tag{E.19}$$

where $\Xi^{(5/2)}_m = \int \Xi^{(5/2)} e^{-im\phi}d\phi$ stands for the mode expansion of

$$\Xi^{(5/2)} = \frac{25\pi\left(\mathcal{P}'\right)^2}{12k} . \tag{E.20}$$

The anticommutator of the fermionic charges then implies that the energy has to fulfill the following bounds

$$\left(p^2 + \frac{25\pi\mathcal{P}}{k}\right)\left(p^2 + \frac{9\pi\mathcal{P}}{k}\right)\left(p^2 + \frac{\pi\mathcal{P}}{k}\right) \geq 0 , \tag{E.21}$$

with p given by a (half-)integer for fermions that fulfill (anti)periodic boundary conditions.

E.3 Spin-9/2 fields

The hyper-Poincaré algebra with fermionic generators of (conformal) spin $\Delta = 9/2$ is described by

$$\begin{aligned}
\left[\hat{J}_m, \hat{J}_n\right] &= (m-n)\,\hat{J}_{m+n}\,, \\
\left[\hat{J}_m, \hat{P}_n\right] &= (m-n)\,\hat{P}_{m+n}\,, \qquad \text{(E.22)}\\
\left[\hat{J}_m, \hat{\mathcal{Q}}_p\right] &= \left(\frac{7m}{2}-p\right)\hat{\mathcal{Q}}_{m+p}\,, \qquad \text{(E.23)}\\
\left\{\hat{\mathcal{Q}}_p, \hat{\mathcal{Q}}_q\right\} &= f^{(7/2)}_{p,q}\,\hat{P}_{p+q}\,,
\end{aligned}$$

where

$$\begin{aligned}
f^{(7/2)}_{p,q} = \; & \frac{1}{2304}\left[112\left(p^6+q^6\right) - 192\left(p^5q+pq^5\right) + 240\left(p^4q^2+p^2q^4\right) - 256p^3q^3\right.\\
& -2240\left(p^4+q^4\right) + 3616\left(p^3q+pq^3\right) - 4080p^2q^2 \qquad \text{(E.24)}\\
& \left. +11578\left(p^2+q^2\right) - 15592pq - 11025\right]\,,
\end{aligned}$$

and with $m, n = 0, \pm1$, and $p, q = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{7}{2}$.

The asymptotic fall-off of the dynamical gauge field is now given by

$$a_\phi = \hat{J}_1 - \frac{\pi}{k}\left(\mathcal{J}\hat{P}_{-1} + \mathcal{P}\hat{J}_{-1} - \frac{\psi}{35}\hat{\mathcal{Q}}_{-\frac{7}{2}}\right)\,, \qquad \text{(E.25)}$$

and the asymptotic symmetries turn out to be parametrized according to

$$\lambda = T\hat{P}_1 + Y\hat{J}_1 + \mathcal{E}\hat{\mathcal{Q}}_{\frac{1}{2}} + \eta_{\left(\frac{7}{2}\right)}\left[T, Y, \mathcal{E}\right]\,, \qquad \text{(E.26)}$$

with

$$
\begin{aligned}
\eta_{\left(\frac{7}{2}\right)}[T,Y,\mathcal{E}] \quad = \quad & -T'\hat{P}_0 - Y'\hat{J}_0 - \mathcal{E}'\hat{\mathcal{Q}}_{\frac{5}{2}} + \frac{1}{2}\left(T'' - \frac{2\pi T\mathcal{P}}{k} - \frac{2\pi Y\mathcal{J}}{k} - \frac{7i\pi\mathcal{E}\psi}{k}\right)\hat{P}_{-1} \\
& + \frac{1}{2}\left(Y'' - \frac{2\pi Y\mathcal{P}}{k}\right)\hat{J}_{-1} + \frac{1}{2}\left(\mathcal{E}'' - \frac{7\pi\mathcal{E}\mathcal{P}}{k}\right)\hat{\mathcal{Q}}_{\frac{3}{2}} \\
& - \frac{1}{6}\left(\mathcal{E}''' - \frac{19\pi\mathcal{E}'\mathcal{P}}{k} - \frac{7\pi\mathcal{E}\mathcal{P}'}{k}\right)\hat{\mathcal{Q}}_{\frac{1}{2}} \\
& + \frac{1}{24}\left(\mathcal{E}^{(4)} + \frac{105\pi^2\mathcal{E}\mathcal{P}^2}{k^2} - \frac{34\pi\mathcal{E}''\mathcal{P}}{k} - \frac{26\pi\mathcal{E}'\mathcal{P}'}{k} - \frac{7\pi\mathcal{E}\mathcal{P}''}{k}\right)\hat{\mathcal{Q}}_{-\frac{1}{2}} \\
& - \frac{1}{120}\left(\mathcal{E}^{(5)} + \frac{409\pi^2\mathcal{E}'\mathcal{P}^2}{k^2} + \frac{322\pi^2\mathcal{E}\mathcal{P}\mathcal{P}'}{k^2} - \frac{50\pi\mathcal{E}'''\mathcal{P}}{k} - \frac{60\pi\mathcal{E}''\mathcal{P}'}{k}\right. \\
& \left. - \frac{33\pi\mathcal{E}'\mathcal{P}''}{k} - \frac{7\pi\mathcal{E}\mathcal{P}'''}{k}\right)\hat{\mathcal{Q}}_{-\frac{3}{2}} + \frac{1}{720}\left(\mathcal{E}^{(6)} - \frac{1575\pi^3\mathcal{E}\mathcal{P}^3}{k^3}\right. \qquad \text{(E.27)} \\
& + \frac{919\pi^2\mathcal{E}''\mathcal{P}^2}{k^2} + \frac{1530\pi^2\mathcal{E}'\mathcal{P}\mathcal{P}'}{k^2} + \frac{427\pi^2\mathcal{E}\mathcal{P}\mathcal{P}''}{k^2} + \frac{322\pi^2\mathcal{E}\left(\mathcal{P}'\right)^2}{k^2} \\
& \left. - \frac{65\pi\mathcal{E}^{(4)}\mathcal{P}}{k} - \frac{110\pi\mathcal{E}^{(3)}\mathcal{P}'}{k} - \frac{93\pi\mathcal{E}''\mathcal{P}''}{k} - \frac{40\pi\mathcal{E}'\mathcal{P}'''}{k} - \frac{7\pi\mathcal{E}\mathcal{P}^{(4)}}{k}\right)\hat{\mathcal{Q}}_{-\frac{5}{2}} \\
& - \frac{1}{5040}\left(\mathcal{E}^{(7)} - \frac{6483\pi^3\mathcal{E}''\mathcal{P}^3}{k^3} - \frac{8589\pi^3\mathcal{E}\mathcal{P}^2\mathcal{P}'}{k^3} + \frac{1519\pi^2\mathcal{E}'''\mathcal{P}^2}{k^2}\right. \\
& + \frac{4088\pi^2\mathcal{E}''\mathcal{P}\mathcal{P}'}{k^2} + \frac{2353\pi^2\mathcal{E}'\mathcal{P}\mathcal{P}''}{k^2} + \frac{1852\pi^2\mathcal{E}'\left(\mathcal{P}'\right)^2}{k^2} \\
& + \frac{511\pi^2\mathcal{E}\mathcal{P}\mathcal{P}'''}{k^2} + \frac{1071\pi^2\mathcal{E}\mathcal{P}'\mathcal{P}''}{k^2} - \frac{144\pi Y\psi}{k} - \frac{77\pi\mathcal{E}^{(5)}\mathcal{P}}{k} - \frac{175\pi\mathcal{E}^{(4)}\mathcal{P}'}{k} \\
& \left. - \frac{203\pi\mathcal{E}'''\mathcal{P}''}{k} - \frac{133\pi\mathcal{E}''\mathcal{P}'''}{k} - \frac{47\pi\mathcal{E}'\mathcal{P}^{(4)}}{k} - \frac{7\pi\mathcal{E}\mathcal{P}^{(5)}}{k}\right)\hat{\mathcal{Q}}_{-\frac{7}{2}}\,,
\end{aligned}
$$

so that the fields transform as

$$\begin{aligned}
\delta\mathcal{P} &= 2\mathcal{P}Y' + \mathcal{P}'Y - \frac{k}{2\pi}Y''' \,, \\
\delta\mathcal{J} &= 2\mathcal{J}Y' + \mathcal{J}'Y + 2\mathcal{P}T' + \mathcal{P}'T - \frac{k}{2\pi}T''' + \frac{9}{2}i\psi\mathcal{E}' + \frac{7}{2}i\psi'\mathcal{E} \,, \\
\delta\psi &= \frac{9}{2}\psi Y' + \psi'Y + \left(-\frac{1225\pi^3\mathcal{P}^4}{16k^3} + \frac{5789\pi^2\mathcal{P}''\mathcal{P}^2}{72k^2} + \frac{2429\pi^2\left(\mathcal{P}'\right)^2\mathcal{P}}{18k^2}\right. \\
&\quad \left. -\frac{35\pi\mathcal{P}^{(4)}\mathcal{P}}{9k} - \frac{119\pi\left(\mathcal{P}''\right)^2}{16k} - \frac{791\pi\mathcal{P}'\mathcal{P}'''}{72k} + \frac{7}{144}\mathcal{P}^{(6)}\right)\mathcal{E} + \\
&\quad + \left(\frac{3229\pi^2\mathcal{P}'\mathcal{P}^2}{12k^2} - \frac{131\pi\mathcal{P}'''\mathcal{P}}{6k} - \frac{99\pi\mathcal{P}'\mathcal{P}''}{2k} + \frac{3}{8}\mathcal{P}^{(5)}\right)\mathcal{E}' \\
&\quad + \left(\frac{3229\pi^2\mathcal{P}^3}{36k^2} - \frac{197\pi\mathcal{P}''\mathcal{P}}{4k} - \frac{165\pi\left(\mathcal{P}'\right)^2}{4k} + \frac{5}{4}\mathcal{P}^{(4)}\right)\mathcal{E}'' \\
&\quad + \left(\frac{7}{3}\mathcal{P}''' - \frac{329\pi\mathcal{P}\mathcal{P}'}{6k}\right)\mathcal{E}^{(3)} + \left(\frac{21}{8}\mathcal{P}'' - \frac{329\pi\mathcal{P}^2}{24k}\right)\mathcal{E}^{(4)} \\
&\quad + \frac{7}{4}\mathcal{E}^{(5)}\mathcal{P}' + \frac{7}{12}\mathcal{E}^{(6)}\mathcal{P} - \frac{k\mathcal{E}^{(8)}}{144\pi} \,.
\end{aligned} \tag{E.28}$$

The mode expansion of the asymptotic symmetry algebra is then given by

$$\begin{aligned}
i\left\{\mathcal{J}_m, \mathcal{J}_n\right\} &= (m-n)\,\mathcal{J}_{m+n} \,, \\
i\left\{\mathcal{J}_m, \mathcal{P}_n\right\} &= (m-n)\,\mathcal{P}_{m+n} + km^3\delta_{m+n,0} \,, \\
i\left\{\mathcal{J}_m, \psi_n\right\} &= \left(\frac{7m}{2} - n\right)\psi_{m+n} \,, \\
i\left\{\psi_m, \psi_n\right\} &= \frac{1}{144}\left(7m^6 - 12m^5n + 15m^4n^2 - 16m^3n^3 + 15m^2n^4 - 12mn^5 + 7n^6\right)\mathcal{P}_{m+n} \\
&\quad + \frac{1}{144k}\left(140m^4 - 226m^3n + 255m^2n^2 - 226mn^3 + 140n^4\right)\sum_q \mathcal{P}_{m+n-q}\mathcal{P}_q \\
&\quad + \frac{1}{864k^2}\left(5789m^2 - 7796mn + 5789n^2\right)\sum_{q,r}\mathcal{P}_{m+n-q-r}\mathcal{P}_q\mathcal{P}_r \\
&\quad + \frac{1225}{128k^3}\sum_{q,r,t}\mathcal{P}_{m+n-q-r}\mathcal{P}_q\mathcal{P}_r\mathcal{P}_t + \frac{k}{72}m^8\delta_{m+n,0} + \Xi^{(7/2)}_{m+n} \,,
\end{aligned} \tag{E.29}$$

where

$$\Xi^{(7/2)}_{m+n} = 7\Theta_{m+n} + \left(329m^2 - 494mn + 329n^2\right)\chi_{m+n} \,, \tag{E.30}$$

and Θ_m and χ_m correspond to the mode expansion of

$$\Theta = \frac{1596\pi^2 \mathcal{P} \left(\mathcal{P}'\right)^2}{432k^2} + \frac{661\pi \left(\mathcal{P}''\right)^2}{432k} \quad , \quad \chi = \frac{\pi \left(\mathcal{P}'\right)^2}{144k} \,, \tag{E.31}$$

respectively.

The energy is then found to fulfill the following bounds

$$\left(p^2 + \frac{49\pi\mathcal{P}}{k}\right)\left(p^2 + \frac{25\pi\mathcal{P}}{k}\right)\left(p^2 + \frac{9\pi\mathcal{P}}{k}\right)\left(p^2 + \frac{\pi\mathcal{P}}{k}\right) \geq 0 \,, \tag{E.32}$$

where according to the (anti)periodicity conditions for the fermions, p corresponds to a (half-)integer.

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