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Estudio de la Interacción Bosón-Fermión en el Modelo de Bosones y Fermiones en Interacción

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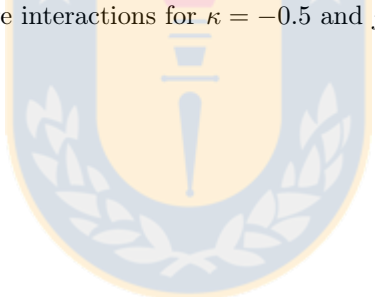
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Resumen

En este trabajo se presenta el desarrollo de la interacción bosón-fermión en el Modelo de Bosones y Fermiones en Interacción (IBFM-2) partiendo de la expresión general del operador de transferencia de un nucleón escrito como una expansión en términos de (hasta dos) operadores bosónicos y un operador fermiónico. Se encuentra el operador cuadrupolar fermiónico el cual se acopla al operador cuadrupolar bosónico para la obtención de la interacción cuadrupolo-cuadrupolo. También se encuentra que esta interacción contiene tres términos, uno directo considerando sólo la interacción del fermión con el core, y dos términos de intercambio que toman en cuenta la estructura de los bosones como pares de fermiones. Los coeficientes de esta interacción que aparecen a partir del operador de transferencia se obtienen por dos mapeos: OAI y GHP. En el caso de OAI, se encuentran los coeficientes de forma exacta. Se estudia esta interacción en el caso de una sola j para ambos mapeos, además de la contribución de los términos directo y de intercambio en el espectro. Finalmente se muestran distintos métodos para obtener las constantes de estructura de los operadores de creación de pares correlacionados S y D , de los cuales utilizamos uno de ellos, y calculamos la probabilidad de ocupación de protones en el ^{130}Te y ^{132}Xe .



Abstract

In this thesis the development of the boson-fermion interaction in the Interacting Boson-Fermion Model (IBFM-2) is shown. We start from the general expression of the one-nucleon transfer operator written as an expansion in terms of (up to two) boson operators and one fermion operator. We express the fermion quadrupole operator which is coupled to the boson quadrupole operator to obtain the quadrupole-quadrupole interaction. We find out that this interaction contains three interaction terms, one direct term which consider the fermion interaction with the core and two exchange interactions that take into account the structure of bosons as pair of fermions. The coefficients of this interactions come from the transfer operator, which are obtained using two mappings: OAI and GHP. In OAI all the coefficients are calculated exactly. The boson-fermion interaction is then studied in the single- j shell case using both mappings, moreover the contribution of the three interaction terms is studied. Finally we show different methods in order to obtain the structure constants of the correlated-pair creation operators S and D . We also use one method to obtain them and calculate the proton occupation probability in ^{130}Te and ^{132}Xe .



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Chapter 1

Introduction

The strong nuclear interaction is one of the four fundamental interactions in physics, along with the weak, electromagnetic and gravitational interactions. All of these interactions, except for the gravitational one, play an important role at the most profound level in the atomic nucleus, which is a highly quantal-mechanical, finite bound many-body system comprised of nucleons (protons and neutrons). The gravitational interaction is quite negligible in nuclei since the ratio of the gravitational and residual strong interactions is lesser than 10^{-35} . The strong interaction acts on color charges, so quarks and gluons which are inside a nucleon are affected by it mediated also by gluons according to quantum chromodynamics (QCD). Unfortunately, theoretically and numerically, it is quite intractable to manage for bigger systems than one or a few nucleons (light nuclei). Therefore, there remains no fundamental theory that can deal with the intricacies required, and for the forthcoming future, mere medium nuclei are beyond the scope of QCD. Considering this, a residual interaction called the nuclear force is used in different nuclear models. This force is experienced by nucleons only (without considering their inner structure) which combine nucleons together into an atomic nucleus. This residual interaction has two main features:

1. It's a short-range attractive interaction between two nucleons.
2. At very short distances it turns repulsive.

In these nuclear quantum systems the angular momentum is expected to be in general a good quantum number [1]. This is verified experimentally in various nuclei along the nuclear chart. The most successful and cited model that let us to study nuclear properties is the Nuclear Shell Model (SM) [2], whose foundations and interpretations are the most valuable. From it we assume that single particle levels in nuclei are organized in shells with various orbits. These shells are closed when protons or neutrons are equal to 2, 8, 20, 28, 50, 82, 126, . . . , the so-called magic numbers. Also, when we are in these circumstances, the nucleus has a spherical shape. In fact the nuclear interactions may be considered as a mean field caused by all nucleons, on the other hand when a nucleus has a number of nucleons far from a magic number, the nucleus gets deformed. This deformation breaks sphericity of the field and create correlations in nucleons producing configuration mixing in the orbital wave functions producing that orbital angular momentum is no-longer a good quantum number, although the total angular momentum of states is always a good quantum number. This also happens in nuclei with an odd number of protons and/or neutrons. The interactions with the single nucleon tends to deform the core from its symmetrical shape. Since the forces between the core and the single nucleons are attractive, it is easily seen that the core will undergo a deformation of the same type of the anisotropic distribution of the single nucleon. The quadrupolar moment is a good indicator of deformation. The more deformable the core, the larger the induced quadrupole moments.

The Bohr-Mottelson (geometric) model treats nuclei from a geometric point of view, where angular momentum algebra is no-longer used, but instead geometric coordinates. This model

also had an amazing success because led us to describe nuclei according to deformation parameters and also different algebra limits may be reached.

Both models have something in common even though they treat the nuclear interaction in different manners. They both consider short and long-range nuclear interactions. Since the nuclear interaction is attractive, the main participation at a low-lying energy regime would be that of joining two nucleons together. However, this interaction which is of a few fermis in extent, can also reach not so close nucleons. The mathematical treatment of it is developed by a multipole expansion of the interaction and presented in many texts [2,3]. Keeping the lower orders the monopole component keep sphericity, the dipole component do not preserve parity, which is a good quantum number in nuclear interactions. The next component is the quadrupole which causes deformations and represents the long-range attractive interactions. Therefore there are two types of main interactions that must be considered between nucleons, those are the pairing and the quadrupole-quadrupole interaction. Evidence for pairing interactions between nucleons is well-known and is long been established. The 0^+ ground state of all even-even nuclei is a good example, since this interaction pairs two protons or two neutrons to total angular momentum 0. Since it works in pairs of like particles, it is also responsible for the difference in mass between even and odd nuclei, as can be seen in the Bethe-Weizsäcker mass formula. The pairing interaction between like nucleons in nuclei is one aspect of a greater nuclear concept know as *collectivity* [4]. A rigorous treatment may describe pairing interaction between like nucleons as being the result of an attractive short-range potential. This interaction immediately is strong in near closed-shell nuclei, i.e. spherical nuclei, which are characterized by an energy vibrational spectrum. In other words, this interaction is the main ingredient of spherical nuclei. On the other hand, in the treatment of neutron-proton interaction the main ingredient is the quadrupole-quadrupole (attractive and effective) force, as can be seen in detail in [5]. This is also considered for empirical observations, where the quadrupole degree of freedom is the main ingredient of the collective model of Bohr.

All the features mentioned are caught in algebraic models, too. The most successful is certainly the Interacting Boson Model (BM) that was developed in 1974 by F. Iachello and A. Arima [6] aimed to reproduce low-lying energy levels and to be able to predict physical properties of nuclei with an even number of protons and neutrons. This is an algebraic model which may be considered as the result of a huge truncation of the basis space of the Nuclear Shell Model (SM) although it also may obtain the same classical limits as the collective model [7], and has originally a phenomenological character, that is, the model describes nuclei by fitting different parameters in order to reproduce observables. Since the number of parameters may increase for obtaining different observables, it becomes cumbersome to manage. A great advance in this model was developed later when studying the microscopic aspect of it in order to reduce the number of parameters. The parameters that appear may be obtained in several ways. For instance the quadrupole-quadrupole interaction terms may be obtained directly by a fitting using nuclear field theory, by equating their values with other models in order to obtain similar surface energy potentials [8], or by quasi-particle formalism [9]. The latter might be the most suitable due to its simplicity. It was first developed by Scholten [10] at the lower order using a particular mapping. There have been several mappings used in IBM like the Otsuka, Arima and Iachello (OAI) [11], Generalized Hostein-Primakoff (GHP) [12], Generalized Dyson and Schwinger [13] which are suitable for different situations. The important role of mappings is limited not only to the expressions for the coefficients, but also that by construction they respect Pauli exclusion principle. In this microscopic framework one considers pairs of nucleons (fermions) as bosons. This particular consideration has recently a strong theoretical support [14] allowing to study medium and heavy nuclei, which are very hard to study with the SM because the configuration space gets too bigger (matrix dimensions of order $\sim 10^{17}$) to be treated even with today's technology which puts to IBM as a fundamental model in nuclear physics.

For odd-mass system IBM is extended to the Interacting Boson-Fermion Model (IBFM) [15] by coupling the unpaired fermion degrees of freedom. In this model the fundamental interaction is that between bosons and the fermion which certainly is a quadrupole interaction. While quadrupole-quadrupole interaction is fundamental between protons and neutrons in nu-

clear models, it is also used between identical nucleons [16–18]. In this case it is known as the quadrupole pairing force, by which some models such as the IBM simplify the nuclear interaction by only two terms, the pure pairing and the quadrupole ones for the simplest cases. For this reason, in this work the quadrupole-quadrupole interaction term in the Hamiltonian of the Interacting Boson-Fermion Model-2 (IBFM-2) is studied after being constructed from the one-nucleon transfer operator in a general case, i.e. without considering at first hand a particular mapping.

In chapter 2 we present briefly the theory and features of the IBM and IBFM. In chapter 3 we develop the boson-fermion interaction using the quasi-particle formalism in a general form and also we use two mappings in order to obtain expressions for the coefficients that appear. Those mappings are discussed and the exact expressions for different needed quantities that show up in that formalisms are calculated. In chapter 4 we show the results of the study of the boson-fermion interaction in the single j -shell case for the OAI and GHP mappings. In chapter 5 we show different forms to obtain constants that appear naturally in the theory when we use the OAI mapping. Finally in chapter 6 we conclude our main ideas.



Chapter 2

Theoretical Aspects

2.1 Interacting Boson and Boson-Fermion Models

In the Interacting Boson Model the properties of low-lying collective states are calculated in terms of a system of interacting s and d bosons. This model was introduced originally without a physical interpretation for the bosons. But soon a connection with the SM was found through the microscopic theory brought by the new ideas to light different extensions of the IBM (IBM-1, ..., IBM-4). The two most important extensions for this thesis, IBM-1 and IBM-2 will be treated in detail. The rest of them IBM-3 and IBM-4 are not considered since they deal with lighter even-mass systems where protons and neutrons occupy the same valence shell in which case isospin becomes important.

2.1.1 IBM-1

In the low-lying energy regime the nucleons of an even-even nucleus stay together because of the pairing interaction which is responsible for the alignment in reverse-time of a pair of like-particles coupled to total angular momentum $J = 0$, where the distance between them is the minimum [14], even for protons. The residual nuclear interaction which has a longer extent than pairing's, is the responsible of a more energetic coupling ($J = 2$). Therefore the main interaction between pairs of nucleons are monopole and quadrupole, thus in the IBM-1 one considers only two bosons, the s and d bosons of angular momenta $L = 0$ and 2 , respectively. Here the bosons are interpreted as collective pairs of like-nucleons without distinguishing pair of protons or pair of neutrons. These pairs are only those that remain in the valence orbit, that is, the orbit outside a closed shell according to the SM. Since a pair of nucleons is considered a boson, the number of bosons is equal to half of the number of nucleons outside the nearest closed shell, because of all these reasons the boson number is finite and a conserved quantity.

The bosons are treated in second-quantization form [19], therefore we'll have creation and annihilation boson operators which transform as spherical tensor operators [1] (see Appendix A):

$$s^\dagger, \tilde{s}, d_\mu^\dagger, \tilde{d}_\mu, \quad \mu = \{\pm 2, \pm 1, 0\}, \quad (2.1.1)$$

where $s^\dagger(d_\mu^\dagger)$ is the $s(d)$ boson creation operator with angular momentum $L = 0(2)$, and $\tilde{s} = s(\tilde{d}_\mu = (-1)^{2+\mu}d_{-\mu})$ are the corresponding annihilation operators with μ the projection on the z -axis in the angular momentum space and $d = (d^\dagger)^\dagger$.

A compact notation for these operators is $b_i^\dagger = \{s^\dagger, d_{-2}^\dagger, \dots, d_2^\dagger\}$ where the angular momentum and the projection is considered in i . Due to the symmetric character of the wave functions,

the boson operators satisfy (see Appendix B):

$$[b_i^\dagger, b_{i'}^\dagger] = 0, \quad (2.1.2a)$$

$$[b_i, b_{i'}] = 0, \quad (2.1.2b)$$

$$[b_i, b_{i'}^\dagger] = \delta_{ii'}. \quad (2.1.2c)$$

With these operators we can already construct a Hamiltonian in a general form as an expansion in boson terms. When considering a Hamiltonian up to two body interactions and as a scalar operator under rotations we can write it as interactions among bosons coupled to certain angular momentum as

$$H = E_0 + \sum_l \varepsilon_l (b_l^\dagger \cdot \tilde{b}_l) + \sum_{LL'V''V'''} \frac{1}{2} u_{ll'v''v'''}^{(L)} \left[[b_l^\dagger \times b_{l'}^\dagger]^{(L)} \times [\tilde{b}_{v''} \times \tilde{b}_{v'''}]^{(L)} \right]_0^{(0)}. \quad (2.1.3)$$

The 16 parameters $u_{ll'v''v'''}^{(L)}$ reduce to 7 by using the fact that the Hamiltonian is Hermitian. When writing explicitly in terms of s and d bosons, Eq. (2.1.3) reads

$$\begin{aligned} H = & E_0 + \varepsilon_s (s^\dagger \cdot \tilde{s}) + \varepsilon_d (d^\dagger \cdot \tilde{d}) + \sum_{L=0,2,4} \frac{1}{2} \hat{L} c_L \left[[d^\dagger \times d^\dagger]^{(L)} \times [\tilde{d} \times \tilde{d}]^{(L)} \right]_0^{(0)} \\ & + \frac{1}{\sqrt{2}} v_2 \left[[d^\dagger \times d^\dagger]^{(2)} \times [\tilde{d} \times \tilde{s}]^{(2)} + [d^\dagger \times s^\dagger]^{(2)} \times [\tilde{d} \times \tilde{d}]^{(2)} \right]_0^{(0)} \\ & + \frac{1}{2} v_0 \left[[d^\dagger \times d^\dagger]^{(0)} \times [\tilde{s} \times \tilde{s}]^{(0)} + [s^\dagger \times s^\dagger]^{(0)} \times [\tilde{d} \times \tilde{d}]^{(0)} \right]_0^{(0)} \\ & + u_2 \left[[d^\dagger \times s^\dagger]^{(2)} \times [\tilde{d} \times \tilde{s}]^{(2)} \right]_0^{(0)} + \frac{1}{2} u_0 \left[[s^\dagger \times s^\dagger]^{(0)} \times [\tilde{s} \times \tilde{s}]^{(0)} \right]_0^{(0)}, \end{aligned} \quad (2.1.4)$$

where $\hat{L} = \sqrt{2L+1}$, E_0 is the lowest or the core energy, ε_s and ε_d are the binding energy of the s and d bosons, respectively, and also the two one-body terms. The seven two-body terms are given singly by three c_L parameters that specify the strength of the d boson interactions, u_0 that specifies the strength of the s boson interactions, and the strength of both boson interactions is given by u_2, v_0 and v_2 . These parameters are not independent since the boson number conservation is a constraint which entail to it. The different Hamiltonians that can be read in the literature are not shown in this work, but can be studied mainly in [6].

A fundamental feature of IBM-1 is that we can make use of group theory in this model since we are working with boson algebra, so we can obtain analytically the eigenvalues of the Hamiltonian (2.1.4) in certain particular cases, known as dynamical symmetries. This can be seen from Eq. (2.1.1), where using the operators b_i^\dagger of (2.1.2c) we can consider the set of bilinear products of the boson creation and annihilations operators:

$$\mathfrak{g} : G_{\alpha\beta} = b_\alpha^\dagger b_\beta. \quad (2.1.5)$$

These operators satisfy the commutation relations

$$[G_{\alpha\beta}, G_{\alpha'\beta'}] = G_{\alpha\beta'} \delta_{\beta\alpha'} - G_{\alpha'\beta} \delta_{\beta'\alpha}. \quad (2.1.6)$$

These operators also satisfy the Jacobi identity, therefore they form a Lie Algebra.

The 36 operators of the algebra \mathfrak{g} satisfy the commutation relations of the unitary algebra $\mathfrak{u}(6)$, and they are the generators of $U(6)$ group. For applications in nuclear physics it is more appropriate to use a coupled form,

$$G_k^{(K)}(l, l') = [b_l^\dagger \times \tilde{b}_{l'}]_k^{(K)}. \quad (2.1.7)$$

In this case the commutation relations of the operators (2.1.7) are

$$\begin{aligned}
& [G_k^{(K)}(l, l'), G_{k'}^{(K')}(l'', l''')] \\
&= \sum_{k'', K''} \sqrt{(2K+1)(2K'+1)} \langle KkK'k' | K''k'' \rangle (-1)^{K-K'} \\
& \times \left[(-1)^{K+K'+K''} \left\{ \begin{matrix} K & K' & K'' \\ l''' & l & l' \end{matrix} \right\} \delta_{l'l''} G_{k''}^{(K'')}(l, l''') - \left\{ \begin{matrix} K & K' & K'' \\ l'' & l' & l \end{matrix} \right\} \delta_{l'''} G_{k''}^{(K'')}(l'', l') \right],
\end{aligned} \tag{2.1.8}$$

where the bracket and the $\{\}$ terms are the Clebsch-Gordan coefficients and the $6j$ -symbols, respectively. This algebra has various sub-algebras. In nuclear physics the algebra $\mathfrak{so}(3)$ must be included since we want states characterized by a good value of angular momentum. There are three limit cases where the sub-algebras may get reduced to $\mathfrak{so}(3)$ ¹, those are the three chains denoted by the groups

$$\begin{aligned}
& U(5) \supset SO(5) \supset SO(3), \quad (I) \\
& \nearrow \\
U(6) & \rightarrow SU(3) \supset SO(3), \quad (II) \\
& \searrow \\
& SO(6) \supset SO(5) \supset SO(3), \quad (III).
\end{aligned} \tag{2.1.9}$$

With these chains we obtain the so-called dynamic symmetries. That is, instead of diagonalize numerically the Hamiltonian (2.1.4), we can find a solution of the eigenvalue problem in closed form under special circumstances (dynamic symmetries). They arise when H can be written in terms only of Casimir operators of each sub-algebra in a certain chain. Casimir operators $f(G)$ of a group $G \supset G'$ commute with all the generators of G and G' . A trivial Casimir operator is the total boson number operator

$$G_0^{(0)}(s, s) + \sqrt{5}G_0^{(0)}(d, d) = \hat{n}_s + \hat{n}_d = \hat{N}. \tag{2.1.10}$$

This operator commutes with all 36 operators of $U(6)$,

$$[\hat{N}, G_k^{(K)}(l, l')] = 0, \quad \text{for any } k, K, l, l', \tag{2.1.11}$$

and it is thus a Casimir operator of $U(6)$.

Since Casimir operator are diagonal in the respective chain the eigenvalue problem is solved as

$$E = \sum_{G \in \text{chain}} E_G \langle f(G) \rangle, \tag{2.1.12}$$

where $\langle f(G) \rangle$ is the eigenvalue of the Casimir operator. The presence of dynamical symmetries in the model is useful to construct basis states where more general situations can be solved and also for consistency checks in computer codes which numerically solve the more general Hamiltonians. More interestingly is the fact that there are regions in the Segrè nuclear chart where the three dynamical symmetries are present, which is shown in fig. 2.1.

2.1.2 IBM-2

This extension of IBM-1 distinguishes between protons and neutrons. Therefore, the microscopic basis of the model consider the s and d bosons as a correlated pair of like nucleons with positive

¹We want to mention that the last reduction in each chain is $SO(3) \supset SO(2)$, but it is omitted because the levels are degenerated in energy at the level of $SO(3)$.

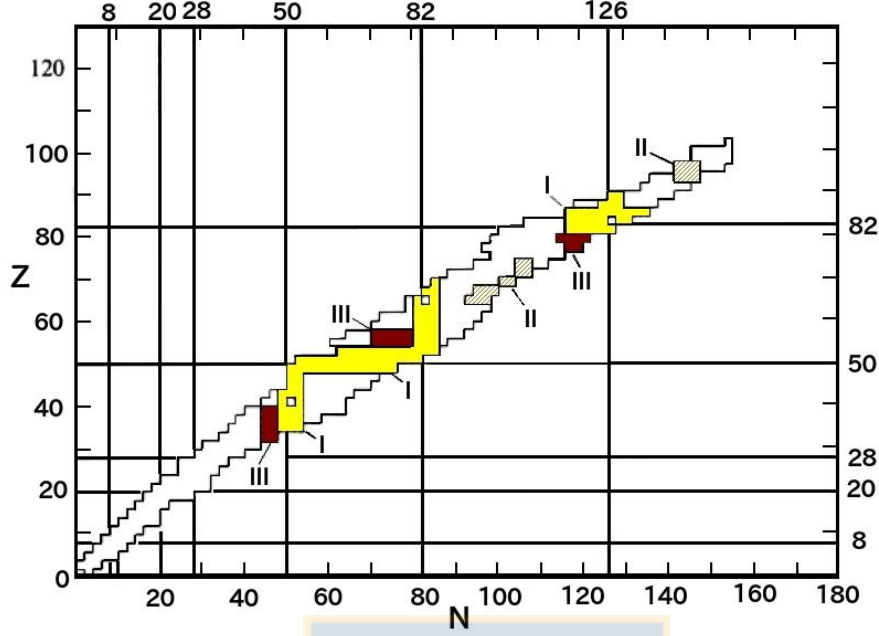


Figure 2.1: **Dynamic symmetries in the nuclear chart:** Regions in the nuclear chart where nuclei with dynamic symmetries are found. Taken from *The interacting boson model*, F. Iachello & A. Arima.

parity, i.e., there are bosons for protons (π) and for neutrons (ν) with angular momentum states $L = 0$ and $L = 2$. These operators satisfy the Bose commutation relations,

$$[b_{\rho,i}^\dagger, b_{\rho',i'}^\dagger] = 0, \quad (2.1.13a)$$

$$[b_{\rho,i}, b_{\rho',i'}] = 0, \quad (2.1.13b)$$

$$[b_{\rho,i}, b_{\rho',i'}^\dagger] = \delta_{\rho\rho'} \delta_{ii'}. \quad (2.1.13c)$$

The only difference with (2.1.2c) is the introduction of $\delta_{\rho\rho'}$ between creation and annihilation boson operators, where ρ stands for π or ν . It is interesting to see that there are not bosons formed by a proton and a neutron. This is because of the microscopy of the model: since in medium and heavy nuclei the numbers of neutrons and protons are very different, valence neutrons and protons occupy different major shells, what prevents the formation of correlated proton-neutron pairs.

A general form of the Hamiltonian in this model is given as

$$H_{\text{IBM-2}} = H_\pi + H_\nu + H_{\pi\nu}, \quad (2.1.14)$$

where H_ρ is that given in the IBM-1 for $\rho = \pi$ and ν , and $H_{\pi\nu} = H_{\nu\pi}$ is the interaction between proton and neutron bosons. This Hamiltonian preserves separately the number of proton and neutron bosons. Therefore the total number of bosons is again a conserved quantity. The term

$H_{\pi\nu}$, which is scalar under rotations and conserves total boson number, reads

$$\begin{aligned}
H_{\pi\nu} = & \sum_{L=0,1,2,3,4} w_L \left[\left[d_{\pi}^{\dagger} \times \tilde{d}_{\pi} \right]^{(L)} \times \left[d_{\nu}^{\dagger} \times \tilde{d}_{\nu} \right]^{(L)} \right]_0^{(0)} \\
& + w_5 \left[\left[s_{\pi}^{\dagger} \times \tilde{s}_{\pi} \right]^{(0)} \times \left[s_{\nu}^{\dagger} \times \tilde{s}_{\nu} \right]^{(0)} \right]_0^{(0)} + w_6 \left[\left[s_{\pi}^{\dagger} \times \tilde{s}_{\pi} \right]^{(0)} \times \left[d_{\nu}^{\dagger} \times \tilde{d}_{\nu} \right]^{(0)} \right]_0^{(0)} \\
& + w_7 \left[\left[d_{\pi}^{\dagger} \times \tilde{d}_{\pi} \right]^{(0)} \times \left[s_{\nu}^{\dagger} \times \tilde{s}_{\nu} \right]^{(0)} \right]_0^{(0)} \\
& + w_8 \left[\left[d_{\pi}^{\dagger} \times \tilde{s}_{\pi} + s_{\pi}^{\dagger} \times \tilde{d}_{\pi} \right]^{(2)} \times \left[d_{\nu}^{\dagger} \times \tilde{s}_{\nu} + s_{\nu}^{\dagger} \times \tilde{d}_{\nu} \right]^{(2)} \right]_0^{(0)} \\
& + w_9 \left[\left[d_{\pi}^{\dagger} \times \tilde{s}_{\pi} - s_{\pi}^{\dagger} \times \tilde{d}_{\pi} \right]^{(2)} \times \left[d_{\nu}^{\dagger} \times \tilde{s}_{\nu} - s_{\nu}^{\dagger} \times \tilde{d}_{\nu} \right]^{(2)} \right]_0^{(0)} \\
& + w_{10} \left[\left[d_{\pi}^{\dagger} \times \tilde{d}_{\pi} \right]^{(2)} \times \left[d_{\nu}^{\dagger} \times \tilde{s}_{\nu} + s_{\nu}^{\dagger} \times \tilde{d}_{\nu} \right]^{(2)} \right]_0^{(0)} \\
& + w_{11} \left[\left[d_{\pi}^{\dagger} \times \tilde{s}_{\pi} + s_{\pi}^{\dagger} \times \tilde{d}_{\pi} \right]^{(2)} \times \left[d_{\nu}^{\dagger} \times \tilde{d}_{\nu} \right]^{(2)} \right]_0^{(0)}. \tag{2.1.15}
\end{aligned}$$

There are ten parameters in H_{π} , ten more in H_{ν} and twelve terms in $H_{\pi\nu}$, given a total of 32 parameters to fit. The number of parameters is very big in order to be treated in a manageable and phenomenological way. However, using the fact that the residual nucleon-nucleon interactions in the SM is dominated by a pairing term between identical nucleons, which is given just by the d boson number operator $\varepsilon_d \hat{n}_d$, in addition to the quadrupole-quadrupole interaction between non-identical nucleons where the boson quadrupole operator is given by

$$\hat{Q}_{\rho} = \left[d_{\rho}^{\dagger} \times \tilde{s}_{\rho} + s_{\rho}^{\dagger} \times \tilde{d}_{\rho} \right]^{(2)} + \chi_{\rho} \left[d_{\rho}^{\dagger} \times \tilde{d}_{\rho} \right]^{(2)}, \quad \rho = \pi, \nu, \tag{2.1.16}$$

we can write the Hamiltonian $H_{\text{IBM-2}}$ through physical operators as

$$H_{\text{IBM-2}} = E_0 + \varepsilon_{d_{\pi}} \hat{n}_{\pi} + \varepsilon_{d_{\nu}} \hat{n}_{\nu} + \kappa \left(\hat{Q}_{\pi} \cdot \hat{Q}_{\nu} \right) + \lambda \hat{M}_{\pi\nu}. \tag{2.1.17}$$

This Hamiltonian was suggested by Talmi [20] and is known as Talmi's Hamiltonian. The first term E_0 is the lowest energy or the core energy, $\varepsilon_{d_{\pi}} \hat{n}_{\pi} + \varepsilon_{d_{\nu}} \hat{n}_{\nu}$ is the pairing interaction between identical particles, meanwhile $\kappa \left(\hat{Q}_{\pi} \cdot \hat{Q}_{\nu} \right)$ is the aforementioned quadrupole-quadrupole interaction between non-identical particles, where κ is the strength and is negative since it is an attractive interaction. The last terms is called the Majorana operator and is given by

$$\begin{aligned}
\hat{M}_{\pi\nu} = & \left(\left[s_{\nu}^{\dagger} \times d_{\pi}^{\dagger} - s_{\pi}^{\dagger} \times d_{\nu}^{\dagger} \right]^{(2)} \cdot \left[\tilde{s}_{\nu} \times \tilde{d}_{\pi} - \tilde{s}_{\pi} \times \tilde{d}_{\nu} \right]^{(2)} \right) \\
& - 2 \sum_{k=1,3} \xi_k \left(\left[d_{\nu}^{\dagger} \times d_{\pi}^{\dagger} \right]^{(k)} \cdot \left[\tilde{d}_{\nu} \times \tilde{d}_{\pi} \right]^{(k)} \right). \tag{2.1.18}
\end{aligned}$$

This term consider the asymmetry energy, which favours states in which the protons and neutrons move in phase. This term is responsible for the energetic splitting between the low-lying symmetric states and the antisymmetric neutron-proton boson states [21].

There are alternative forms of the Hamiltonian more general than the one shown above, for which computational programs were developed in order to numerically diagonalize it as NPBOS [22], or even more simple Hamiltonians which consider only the pairing and the quadrupole interactions [17, 23] in order to study in a simple way the energy spectrum among spherical, transitional and deformed nuclei.

2.1.3 IBFM

The IBM has great success because of its simplicity and the good results that brought with only the fitting of a few parameters. Also, since the matrices are not as big as in the case of the SM, the speed of the calculations is much shorter, but still accurate.

In order to study and reproduce experimental data of odd-mass nuclei (odd number of protons or neutrons), the most simple way to do this is by incorporating the degrees of freedom of the unpaired nucleon (fermion) to the IBM, through the coupling of this fermion with the even-even nucleus characterized by the Hamiltonian of IBM-2 with a particular fit of the parameters. This model is called The Interacting Boson-Fermion Model (IBFM) [15]. As in the case of IBM, there are several versions of this model which differ in their treatment of the proton and neutron degrees of freedom. The first version called IBFM-1 does not distinguish between protons and neutrons. It is similar to IBM-1 because no distinction between the two kinds of nucleons is made. Therefore the features of IBFM-1 will not be discussed in this work. We will only consider IBFM-2. It is worthy to mention that in analogy to IBM-3 and IBM-4 there are IBFM-3 and IBFM-4 which couple the degrees of freedom of one fermion to the even-mass systems described by the aforementioned models.

Several properties of IBM in their versions are preserved in IBFM by construction. First, the distinction in protons and neutrons is seen in the creation and annihilation boson operators introduced in the previous subsection, but in addition, there are now fermion creation and annihilation operators. These operators are also called the creation or annihilation operators of an "ideal" fermion since these operators satisfy Fermi anticommutation relations,

$$\{a_{\rho,i}^\dagger, a_{\rho',i'}^\dagger\} = 0, \quad (2.1.19a)$$

$$\{a_{\rho,i}, a_{\rho',i'}\} = 0, \quad (2.1.19b)$$

$$\{a_{\rho,i}, a_{\rho',i'}^\dagger\} = \delta_{\rho\rho'} \delta_{ii'}. \quad (2.1.19c)$$

In principle this model could be considered for odd-odd nuclei, thus we consider ρ and ρ' . However, this is not what we are going to use in this work since we will only consider an odd-even nuclear system.

Also boson and fermion operators are assumed to commute

$$[b_{\rho,i}, a_{\rho',i'}] = 0, \quad (2.1.20a)$$

$$[b_{\rho,i}, a_{\rho',i'}^\dagger] = 0, \quad (2.1.20b)$$

$$[b_{\rho,i}^\dagger, a_{\rho',i'}] = 0, \quad (2.1.20c)$$

$$[b_{\rho,i}^\dagger, a_{\rho',i'}^\dagger] = 0. \quad (2.1.20d)$$

The form of the Hamiltonian, is akin to that of IBM-2 in form, which reads

$$H_{\text{IBFM}} = H_B + H_F + V_{BF}, \quad (2.1.21)$$

with

$$H_B = H_{\pi B} + H_{\nu B} + H_{\pi\nu B}, \quad (2.1.22)$$

$$H_F = \sum_j \epsilon_{j\rho} \hat{n}_{j\rho} = \sum_{j\rho, m\rho} \epsilon_{j\rho} a_{\rho, j\rho, m\rho}^\dagger a_{\rho, j\rho, m\rho} \quad (2.1.23)$$

where H_B can be Talmi's Hamiltonian (2.1.17) plus interactions between identical bosons like the terms with c_L in Eq. (2.1.4).

The part related to the fermion is described in terms of an effective nucleon-nucleon interaction. As we said before, in most of the calculations only one proton or neutron is unpaired. In this case H_F is used as presented, where ϵ_j are the single-particle energies.

The most important part of the Hamiltonian for odd-even nuclei is the boson-fermion interaction V_{BF} , which contains in principle the interaction between the bosons and the fermion

in a similar form to $H_{\pi\nu}$ of Eq. (2.1.15) in IBM-2. However, considering again the microscopy theory in this model, this interaction has three important terms used until today. They are the monopole, the quadrupole and the exchange interaction which may account for most of the observed properties. These term are given by

$$V_{BF} = V_{BF}^{\text{MON}} + V_{BF}^{\text{QUAD}} + V_{BF}^{\text{EXC}}, \quad (2.1.24)$$

where

$$V_{BF}^{\text{MON}} = \sum_{j_k} A_{j_k} \hat{d}_\rho \hat{n}_{j_k, \rho'}, \quad (2.1.25a)$$

$$V_{BF}^{\text{QUAD}} = \sum_{j_k, j_{k'}} \Gamma_{j_k j_{k'}} \left(\hat{Q}_\rho \cdot \left[a_{j_k, \rho'}^\dagger \times \tilde{a}_{j_{k'}, \rho'} \right]^{(2)} \right), \quad (2.1.25b)$$

$$V_{BF}^{\text{EXC}} = \sum_{j_k, j_{k'}, j_{k''}} \Lambda_{j_k j_{k'}}^{j_{k''}} \left(\left\{ \left[\left[a_{\rho'}^\dagger \times \tilde{a}_{j_{k'}, \rho'} \right]^{(j_k)} \times \left[\tilde{s}_{\rho'} \times a_{j_{k'}, \rho'}^\dagger \right]^{(j_{k'})} \right]^{(2)} \right\} \cdot \left[s_\rho^\dagger \times \tilde{d}_\rho \right]^{(2)} \right), \quad (2.1.25c)$$

where $\rho \neq \rho'$ in this interactions and $::$ represent normal order. The $\Gamma_{j_k j_{k'}}$ and $\Lambda_{j_k j_{k'}}^{j_{k''}}$ coefficients were obtained by applying a microscopic derivation [9, 10] and hitherto are given by

$$\Gamma_{j_k j_{k'}} = -\sqrt{5}(u_{j_k} u_{j_{k'}} - v_{j_k} v_{j_{k'}}) Q_{j_k j_{k'}} \Gamma, \quad (2.1.26)$$

$$\Lambda_{j_k j_{k'}}^{j_{k''}} = -\beta_{j_k j_{k'}} \beta_{j_{k''} j_k} \sqrt{\frac{10}{N_{\rho'}(2j_k + 1)}} \Lambda \quad (2.1.27)$$

here Γ and Λ are parameters to fit and $u_{j_k}^2$ and $v_{j_k}^2$ are the vacancy and occupancy of the fermion in the j_k orbit and they satisfy

$$u_{j_k}^2 + v_{j_k}^2 = 1, \quad \forall k. \quad (2.1.28)$$

In addition $\beta_{j_k j_{k'}}$ is given by

$$\beta_{j_k j_{k'}} = -\sqrt{5}(u_{j_k} v_{j_{k'}} + v_{j_k} u_{j_{k'}}) Q_{j_k j_{k'}}, \quad (2.1.29)$$

where $Q_{j_k j_{k'}} = -\frac{1}{\sqrt{5}} \langle \frac{1}{2} l_k j_k || r^2 Y_2(\hat{r}) || \frac{1}{2} l_{k'} j_{k'} \rangle$ are the single particle reduced matrix elements of the quadrupole operator, and $Y_2(\hat{r})$ is the spherical harmonic of degree 2. Along with the radial contribution these reduced matrix elements may be written explicitly [5] as

$$\langle \frac{1}{2} l_k j_k || Y_2(\hat{r}) || \frac{1}{2} l_{k'} j_{k'} \rangle = (-1)^{j_k - \frac{1}{2}} \sqrt{\frac{5(2j_k + 1)(2j_{k'} + 1)}{4\pi}} \begin{pmatrix} j_k & 2 & j_{k'} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \frac{1}{2} [1 + (-1)^{l_k + l_{k'}}], \quad (2.1.30)$$

where the term in the big parentheses is a $3j$ -symbol, the j 's are the values of the total angular momenta, the l 's are the values of the orbital angular momenta. The states $|\frac{1}{2} l j\rangle$ have good angular momentum j and are obtained from the coupling between spin $\frac{1}{2}$ and orbital angular momentum l of the particle, which specifies an orbit in the SM.

An important property that (2.1.30) satisfies is that

$$Q_{j_k j_{k'}} = (-1)^{j_k - j_{k'}} Q_{j_{k'} j_k}. \quad (2.1.31)$$

Since the parity of an orbit in the SM is given by $\pi = (-1)^l$, it is clear that the factor $[1 + (-1)^{l_k + l_{k'}}]$ assures that parity is conserved by the interaction.

Number Operator Approximation

The $\Gamma_{j_k j_{k'}}$ and $\Lambda_{j_k j_{k'}}^{j_{k''}}$ coefficients shown above are obtained using the so-called Number Operator Approximation (NOA) [9]. It was introduced by Otsuka [24] as a way of using the $\mathfrak{su}(2)$ Lie

Algebra that doesn't appear when we treat correlated pair creation operators in non-degenerate orbits. When we treat a degenerate system, it is equivalent to treat a system of a single j -shell. In this case the creation operator of $J = 0$ angular momentum pairs S_j^\dagger , along with S_j^- and $S_{0,j}$, defined as

$$S_j^\dagger = \sqrt{\Omega_j} A_{jj}^{\dagger(00)}, \quad (2.1.32)$$

$$S_j^- = (S_j^\dagger)^\dagger, \quad (2.1.33)$$

$$S_j^0 = \frac{1}{2} (\hat{n}_j - \Omega_j), \quad (2.1.34)$$

$$\hat{n}_j = \sum_m C_{jm}^\dagger C_{jm}, \quad (2.1.35)$$

where \hat{n}_j is the number operator in the j -orbit, $\Omega_j = j + 1/2$ is half of the orbit and the A operator is the nucleon pair creation operator of total angular momentum J and projection μ on its z -axis, defined as

$$A_{jj'}^{\dagger(J\mu)} = \frac{1}{\sqrt{1 + \delta_{jj'}}} [C_j^\dagger \times C_{j'}^\dagger]_\mu^{(J)}, \quad (2.1.36)$$

and C_j^\dagger is the SM single-nucleon creation operator. The operators shown above satisfy the so-called quasi-spin algebra

$$[S_j^\dagger, S_{j'}^-] = 2S_j^0 \delta_{jj'}, \quad (2.1.37)$$

$$[S_{j'}^0, S_j^\dagger] = S_j^\dagger \delta_{jj'}, \quad (2.1.38)$$

$$[S_{j'}^0, S_j^-] = -S_j^- \delta_{jj'} \quad (2.1.39)$$

This is the same algebra of ladder operators in the quantum mechanical treatment of angular momentum. If we extend this to nondegenerate orbits, we must consider all the valence orbits in a shell, but not on the same footing, since for every orbit the nucleons may have different single-particles energies, and also different occupation probabilities, thus real constants α 's are introduced to account for what was aforementioned for each orbit such that a correlated pair of angular momentum 0 can be created using the operator

$$S^\dagger = \sum_j \alpha_j S_j^\dagger, \quad (2.1.40)$$

with its corresponding annihilation operator of pairs $S^- = (S^\dagger)^\dagger$.

In this case the algebra becomes

$$\begin{aligned} [S^\dagger, S^-] &= \sum_{j,j'} \alpha_j \alpha_{j'} [S_j^\dagger, S_{j'}^-] \\ &= \sum_{j,j'} \alpha_j \alpha_{j'} \delta_{jj'} 2S_j^0 \\ &= \sum_j \alpha_j^2 (\hat{n}_j - \Omega_j) \\ &= \hat{n}_\alpha - \Omega_e, \end{aligned} \quad (2.1.41)$$

where \hat{n}_α is defined as

$$\hat{n}_\alpha := \sum_j \alpha_j^2 \hat{n}_j, \quad (2.1.42)$$

and $\Omega_e = \sum_j \alpha_j^2 \Omega_j$ is the effective degeneration of the shell. The \hat{n}_α operator does not longer coincide with the number operator and the operators do not close under the quasispin algebra. The NOA consists in making

$$\hat{n} = \sum_j \hat{n}_j \approx \sum_j \alpha_j^2 \hat{n}_j = \hat{n}_\alpha. \quad (2.1.43)$$

Therefore the α_j coefficients are normalized such that (2.1.43) is satisfied. This is an approximation which has terrible consequences. For example it implies that the occupancies of nucleons in the orbits are linear in n (which is not). Even worse it predicts that the occupation probability in an orbit may be bigger than 1 [25].



Chapter 3

The Boson-Fermion Interaction in the IBFM-2

As it has been pointed out in the foregoing chapters 1 and 2, the boson-fermion interaction is the main goal in this thesis. The dominant interaction in the coupling of the unpaired particle and the boson degrees of freedom is the quadrupole interaction between protons and neutrons. So the first step in order to construct this interaction is to obtain a suitable quadrupole operator image in the IBFM. For this purpose, the quasi-particle (or pseudo-particle) method [9] will be used. In this method we express the transfer operator, which is the IBFM image for the shell model single-nucleon creation operator, through an expansion in tensor products of boson and fermion operators truncated to second order in boson operators. The second step is to couple the transfer operator with the annihilation operator to total angular momentum 2. Finally, the scalar product between this operator and the quadrupole operator of the IBM-2 given in Eq. (2.1.16) is used to write down the quadrupole interaction.

3.1 The one-nucleon transfer operator

The single-nucleon creation operator C_j^\dagger in the Shell Model is used in order to construct correlated pairs creation operators S^\dagger and D^\dagger to total angular momentum 0 and 2, respectively. Since this operator is used to write interactions in second quantization formalism, we may write the quadrupole operator in the SM scheme as

$$Q_{\text{SM}}^{(2)} = \sum_{j_1, j_2} Q_{j_1 j_2} \left[C_{j_1}^\dagger \times \tilde{C}_{j_2} \right]^{(2)}, \quad (3.1.1)$$

where $Q_{j_1 j_2}$ was defined in Eq. (2.1.30). In order to obtain the IBFM image for the Shell Model quadrupole operator we obtain the IBFM image for the SM single-nucleon creation operator of a ρ particle, i.e. the one-nucleon transfer operator for one kind of nucleon (proton or neutron). This operator is the one-nucleon creation operator in the i shell specified by the standard single-particle level quantum numbers n_i , l_i , $1/2$, j_i and m_i . We will replace them by just one label for simplicity and denote this operator by $c_{j_i m_i}^\dagger$. This pseudo-particle operator in the IBFM is defined as the equivalent of the single-nucleon operator C_j^\dagger in the SM space. The annihilation operator with good tensor character is given by

$$\tilde{c}_{j_i m_i} = (-1)^{j_i - m_i} c_{j_i - m_i}, \quad (3.1.2)$$

where $c_{j_i m_i} = (c_{j_i m_i}^\dagger)^\dagger$. For convenience, we will omit the sub-index i , and denote the operator just as c_{jm}^\dagger and we'll write down the transfer operator as an expansion in boson operators s^\dagger , \tilde{s} , d^\dagger and \tilde{d} only, and an "ideal" fermion creation operator [17] denoted as a^\dagger , which commute with

all boson operators and fulfils the usual fermion commutation relations given in Appendix B. Because of that, in Generalized Seniority (GS) scheme it's considered as the seniority raising operator [9] as it creates states with good GS quantum number in 1. The transfer operator in this expansion becomes

$$\begin{aligned}
c_{jm}^\dagger &= A_j a_{jm}^\dagger + B_j [s^\dagger \times \tilde{a}_j]_m^{(j)} + \sum_{j'} C_{jj'} [d^\dagger \times \tilde{a}_{j'}]_m^{(j)} \\
&+ \sum_{j'} D_{jj'} \left[[s^\dagger \times \tilde{d}]^{(2)} \times a_{j'}^\dagger \right]_m^{(j)} + \sum_{j',L} E_{jj'L} \left[[d^\dagger \times \tilde{d}]^{(L)} \times a_{j'}^\dagger \right]_m^{(j)} \\
&+ \sum_{j'} F_{jj'} \left[[d^\dagger \times \tilde{s}]^{(2)} \times a_{j'}^\dagger \right]_m^{(j)} + G_j \left[[s^\dagger \times \tilde{s}]^{(0)} \times a_j^\dagger \right]_m^{(j)} + \dots \quad (3.1.3)
\end{aligned}$$

where we cut off the expression to only two boson operators, and the j 's runs over all the valence orbits in the shell of interest. Due to the great number of terms in this operator, working with it in this form may become very awkward, so it is more pleasant if we reduce it to only three terms,

$$c_{jm}^\dagger = \mathfrak{A}_j a_{jm}^\dagger + \sum_{l,j'} \mathfrak{B}_{jj'}^l [\gamma_l^\dagger \times \tilde{a}_{j'}]_m^{(j)} + \sum_{l_1, l_2, L, j'} \mathfrak{C}_{jj'}^{l_1 l_2 L} \left[[\gamma_{l_1}^\dagger \times \tilde{\gamma}_{l_2}]^{(L)} \times a_{j'}^\dagger \right]_m^{(j)}, \quad (3.1.4)$$

where l 's are equal to 0 or 2^1 , and L is an integer number which may runs from 0 to 4, depending on the values of l 's. $\gamma_{l_i}^\dagger$ ($\tilde{\gamma}_{l_i}$) is a creation(annihilation) boson operator of angular momentum l_i , and the coefficients in (3.1.4) are defined as follows

$$\mathfrak{A}_j = A_j \quad (3.1.5a)$$

$$\mathfrak{B}_{jj'}^l = \begin{cases} B_j \delta_{jj'} & , l = 0 \\ C_{jj'} & , l = 2 \end{cases} \quad (3.1.5b)$$

$$\mathfrak{C}_{jj'}^{l_1 l_2 L} = \begin{cases} D_{jj'} \delta_{L2} & , l_1 = 0, l_2 = 2 \\ F_{jj'} \delta_{L2} & , l_1 = 2, l_2 = 0 \\ E_{jj'L} & , l_1 = l_2 = 2 \\ G_j & , l_1 = l_2 = 0 \end{cases} \quad (3.1.5c)$$

The Kronecker delta $\delta_{jj'}$ in $\mathfrak{B}_{jj'}^0$ is written for convenience despite it appears naturally in the angular momentum coupling, something which also happens in $\mathfrak{C}_{jj'}^{022}$ and $\mathfrak{C}_{jj'}^{202}$.

Having the one-nucleon transfer operator at hand, accordingly to B.2.1 and B.2.2 the corresponding annihilation operator is given by

$$\tilde{c}_{jm} = \mathfrak{A}_j \tilde{a}_{jm} - \sum_{l,j'} \mathfrak{B}_{jj'}^l [\tilde{\gamma}_l \times a_{j'}^\dagger]_m^{(j)} + \sum_{l_1, l_2, L, j'} \mathfrak{C}_{jj'}^{l_1 l_2 L} (-1)^{l_1 + l_2 + L} \left[[\gamma_{l_2}^\dagger \times \tilde{\gamma}_{l_1}]^{(L)} \times \tilde{a}_{j'} \right]_m^{(j)}. \quad (3.1.6)$$

It must be pointed out that all integer numbers are written as l or L with different subscript and superscripts to differentiate the numbers in the summations. Also, even if l_1 and l_2 are 0 or 2, we write explicitly the phase $(-1)^{l_1 + l_2}$ since a generalization with odd angular momentum boson operators may be straightforward and easier in the above expression.

3.2 The quadrupole operator

Now that we have defined the one-nucleon transfer operator in (3.1.4) in the IBFM, we construct the fermion quadrupole operator image of (3.1.1) by replacing the creation operators by (3.1.4).

¹Note that, in this work we only use s and d bosons, but a generalization to add a g boson with angular momentum 4 or higher is straightforward considering the form of (3.1.4).

This yields to

$$q_M^{(2)} = \sum_{j_1, j_2} Q_{j_1 j_2} \left[c_{j_1}^\dagger \times \tilde{c}_{j_2} \right]_M^{(2)} \quad (3.2.1)$$

Using the linearity of tensor product, we replace the transfer operator of (3.1.3) and the corresponding annihilation operator (3.1.6) into $q^{(2)}$ obtaining:

$$\left[c_{j_1}^\dagger \times \tilde{c}_{j_2} \right]^{(2)} = \mathfrak{A}_{j_1} \mathfrak{A}_{j_2} \left[a_{j_1}^\dagger \times \tilde{a}_{j_2} \right]^{(2)} \quad (3.2.2a)$$

$$- \sum_{l, j_2'} \mathfrak{A}_{j_1} \mathfrak{B}_{j_2 j_2'}^l \left[a_{j_1}^\dagger \times [\tilde{\gamma}_l \times a_{j_2'}^\dagger]^{(j_2)} \right]^{(2)} \quad (3.2.2b)$$

$$+ \sum_{l_2, l_2', L_2, j_2'} \mathfrak{A}_{j_1} \mathfrak{C}_{j_2 j_2'}^{l_2 l_2' L_2} (-1)^{l_2 + l_2' + L_2} \left[a_{j_1}^\dagger \times \left[[\gamma_{l_2}^\dagger \times \tilde{\gamma}_{l_2}]^{(L_2)} \times \tilde{a}_{j_2'} \right]^{(j_2)} \right]^{(2)} \quad (3.2.2c)$$

$$+ \sum_{l, j_1'} \mathfrak{B}_{j_1 j_1'}^l \mathfrak{A}_{j_2} \left[[\gamma_l^\dagger \times \tilde{a}_{j_1'}]^{(j_1)} \times \tilde{a}_{j_2} \right]^{(2)} \quad (3.2.2d)$$

$$- \sum_{l_1, j_1', l_2, j_2'} \mathfrak{B}_{j_1 j_1'}^{l_1} \mathfrak{B}_{j_2 j_2'}^{l_2} \left[[\gamma_{l_1}^\dagger \times \tilde{a}_{j_1'}]^{(j_1)} \times [\tilde{\gamma}_{l_2} \times a_{j_2'}^\dagger]^{(j_2)} \right]^{(2)} \quad (3.2.2e)$$

$$+ \sum_{\substack{l_1, j_1', l_2, \\ l_2', L_2, j_2'}} \mathfrak{B}_{j_1 j_1'}^{l_1} \mathfrak{C}_{j_2 j_2'}^{l_2 l_2' L_2} (-1)^{l_2 + l_2' + L_2} \left[[\gamma_{l_1}^\dagger \times \tilde{a}_{j_1'}]^{(j_1)} \times \left[[\gamma_{l_2'}^\dagger \times \tilde{\gamma}_{l_2}]^{(L_2)} \times \tilde{a}_{j_2'} \right]^{(j_2)} \right]^{(2)} \quad (3.2.2f)$$

$$+ \sum_{l_1, l_1', L_1, j_1'} \mathfrak{C}_{j_1 j_1'}^{l_1 l_1' L_1} \mathfrak{A}_{j_2} \left[\left[[\gamma_{l_1}^\dagger \times \tilde{\gamma}_{l_1'}]^{(L_1)} \times a_{j_1'}^\dagger \right]^{(j_1)} \times \tilde{a}_{j_2} \right]^{(2)} \quad (3.2.2g)$$

$$- \sum_{l_1, l_1', L_1, j_1', l_2, j_2'} \mathfrak{C}_{j_1 j_1'}^{l_1 l_1' L_1} \mathfrak{B}_{j_2 j_2'}^{l_2} \left[\left[[\gamma_{l_1}^\dagger \times \tilde{\gamma}_{l_1'}]^{(L_1)} \times a_{j_1'}^\dagger \right]^{(j_1)} \times [\tilde{\gamma}_{l_2} \times a_{j_2'}^\dagger]^{(j_2)} \right]^{(2)} \quad (3.2.2h)$$

$$+ \sum_{\substack{l_1, l_1', L_1, \\ j_1', j_2', \\ l_2, l_2', L_2}} \mathfrak{C}_{j_1 j_1'}^{l_1 l_1' L_1} \mathfrak{C}_{j_2 j_2'}^{l_2 l_2' L_2} (-1)^{l_2 + l_2' + L_2} \times \left[\left[[\gamma_{l_1}^\dagger \times \tilde{\gamma}_{l_1'}]^{(L_1)} \times a_{j_1'}^\dagger \right]^{(j_1)} \times \left[[\gamma_{l_2'}^\dagger \times \tilde{\gamma}_{l_2}]^{(L_2)} \times \tilde{a}_{j_2'} \right]^{(j_2)} \right]^{(2)}. \quad (3.2.2i)$$

At this stage we must consider the features of the IBFM space. In the above expression we have a total of 9 terms, however, the four terms (3.2.2b), (3.2.2d), (3.2.2f) and (3.2.2h) contain two $a_{j_i}^\dagger$ or two \tilde{a}_{j_i} operators. Those terms create or annihilate two particles. Since in the very beginning of the IBFM it is imposed that we only have a single unpaired particle, which is a quasi-particle whose degrees of freedom are coupled to the core of the system, those four terms are discarded, because they are beyond the model space in which the IBFM is built. Also, those terms change a quasi-fermion pair into a boson or a boson into a quasi-fermion pair, and may be referred as mixing terms [12]. Therefore, we keep all the rest of the terms of the former expression which effectively contribute to the operator, remaining only five terms in total.

Before writing the quadrupole operator in a correct and tractable way, we note two comments:

1. Let us consider the operator in the term (3.2.2e). Since we want to write all ideal fermion creation operator at the left side, and its annihilation operator at the right, that is, write the quadrupole operator in normal order with respect to the ideal fermion operator, we

rewrite this term by a simple angular momentum recoupling to obtain

$$\begin{aligned}
\left[[\gamma_{l_1}^\dagger \times \tilde{a}_{j_1'}]^{(j_1)} \times [\tilde{\gamma}_{l_2} \times a_{j_2'}^\dagger]^{(j_2)} \right]^{(J)} &= -(-1)^{j_1+j_2-J} \left[[\tilde{\gamma}_{l_2} \times a_{j_2'}^\dagger]^{(j_2)} \times [\gamma_{l_1}^\dagger \times \tilde{a}_{j_1'}]^{(j_1)} \right]^{(J)} \\
&+ \delta_{l_1 l_2} (-1)^{j_1-j_1'+l_1} \hat{j}_1 \hat{j}_2 \begin{Bmatrix} j_2' & j_1' & J \\ j_1 & j_2 & l_1 \end{Bmatrix} \left[a_{j_2'}^\dagger \times \tilde{a}_{j_1'} \right]^{(J)} \\
&+ \delta_{j_1' j_2'} (-1)^{j_1'+j_2+J-l_1} \hat{j}_1 \hat{j}_1' \hat{j}_2 \begin{Bmatrix} l_1 & j_1 & j_1' \\ j_2 & l_2 & J \end{Bmatrix} \left[\gamma_{l_1}^\dagger \times \tilde{\gamma}_{l_2} \right]^{(J)}.
\end{aligned} \tag{3.2.3}$$

The last contribution of (3.2.2e) written in this form is a pure boson part, which is already considered in the boson quadrupole operator. Therefore it's discarded. The other two terms contribute in a direct way and by changing bosons by fermions, which belongs to an exchange interaction.

2. Let us consider the term in (3.2.2i). This term has already four boson operators, which will produce three-body interactions in the boson-fermion interaction. Since we stay in the lowest order in number of boson operators, this term is also discarded.

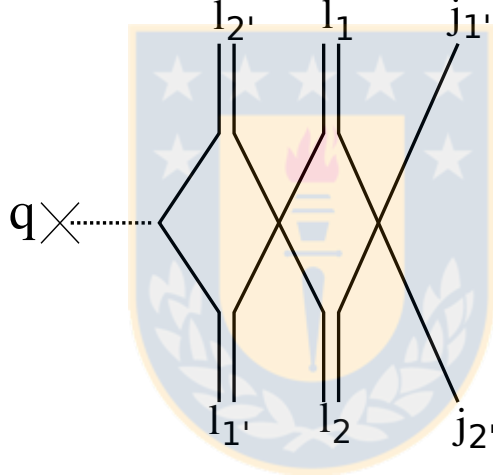


Figure 3.1: **Diagrammatic representation of the boson two-body term in the fermion quadrupole operator**, i.e., Eq. (3.2.2i).

With all these remarks, we are in position to write

$$\left[c_{j_1}^\dagger \times \tilde{c}_{j_2} \right]^{(2)} = \sum_{j_1', j_2'} \left(\mathfrak{A}_{j_1} \mathfrak{A}_{j_2} \delta_{j_1 j_1'} \delta_{j_2 j_2'} - \sum_l \mathfrak{B}_{j_1 j_2'}^l \mathfrak{B}_{j_2 j_1'}^l (-1)^{j_1-j_2'+l} \hat{j}_1 \hat{j}_2 \begin{Bmatrix} j_1' & j_2' & 2 \\ j_1 & j_2 & l \end{Bmatrix} \right) \left[a_{j_1'}^\dagger \times \tilde{a}_{j_2'} \right]^{(2)} \tag{3.2.4a}$$

$$+ \sum_{l_2, l_2', L_2, j_2'} \mathfrak{A}_{j_1} \mathfrak{C}_{j_2 j_2'}^{l_2 l_2' L_2} (-1)^{l_2+l_2'+L_2} \left[a_{j_1}^\dagger \times \left[[\gamma_{l_2}^\dagger \times \tilde{\gamma}_{l_2}]^{(L_2)} \times \tilde{a}_{j_2'} \right]^{(j_2)} \right]^{(2)} \tag{3.2.4b}$$

$$+ (-1)^{j_1+j_2} \sum_{j_1', j_2', l_1, l_2} \mathfrak{B}_{j_1 j_1'}^{l_1} \mathfrak{B}_{j_2 j_2'}^{l_2} \left[[\tilde{\gamma}_{l_2} \times a_{j_2'}^\dagger]^{(j_2)} \times [\gamma_{l_1}^\dagger \times \tilde{a}_{j_1'}]^{(j_1)} \right]^{(2)} \tag{3.2.4c}$$

$$+ \sum_{l_1, l_1', L_1, j_1'} \mathfrak{C}_{j_1 j_1'}^{l_1 l_1' L_1} \mathfrak{A}_{j_2} \left[\left[[\gamma_{l_1}^\dagger \times \tilde{\gamma}_{l_1'}]^{(L_1)} \times a_{j_1'}^\dagger \right]^{(j_1)} \times \tilde{a}_{j_2} \right]^{(2)}. \tag{3.2.4d}$$

Those terms are the only contributions to the operator since we have already eliminated the pure boson term and the two-quasiparticle mixing terms. The term in (3.2.4a) is a pure fermion term, which is called the quadrupole or direct term, while all the remaining ones, (3.2.4b)-(3.2.4d) are all exchange terms. Those give rise to the exchange force, which accounts for the fact that the bosons themselves are built up of fermions which can occupy some of the same single-particle orbits of the unpaired fermion. All the terms, separately conserve both the number of bosons and the number of fermions.

The boson-fermion image of the quadrupole operator is obtained when one substitute equations (3.2.4a)-(3.2.4d) into (3.2.1). This yields straightforwardly to a sum of four terms:

$$q^{(2)} = q_D + q_A^a + q_A^b + q_B \quad (3.2.5)$$

where q_D is the direct term, and the rest are the exchange terms. They are listed below:

Direct quadrupole

$$q_D = \sum_{j_1, j_2} \tilde{\Gamma}_{j_1 j_2} [a_{j_1}^\dagger \times \tilde{a}_{j_2}]^{(2)}, \quad (3.2.6a)$$

Exchange 1

$$q_A^a = \sum_{j' s, l' s, L} \Lambda_{j_1 j_2 j_2'}^{l' L} (-1)^{l+l'+L} \left[a_{j_1}^\dagger \times \left[[\gamma_{l'}^\dagger \times \tilde{\gamma}_l]^{(L)} \times \tilde{a}_{j_2'} \right]^{(j_2)} \right]^{(2)}, \quad (3.2.6b)$$

Exchange 2

$$q_A^b = \sum_{j' s, l' s, L} (-1)^{j_2 - j_1} \Lambda_{j_2 j_1 j_1'}^{l' L} \left[\left[[\gamma_l^\dagger \times \tilde{\gamma}_{l'}]^{(L)} \times a_{j_1'}^\dagger \right]^{(j_1)} \times \tilde{a}_{j_2} \right]^{(2)}, \quad (3.2.6c)$$

Exchange 3

$$q_B = \sum_{j' s, L' s} \Delta_{j_1 j_2 j_1' j_2'}^{l' L} \left[[\tilde{\gamma}_{l'} \times a_{j_2'}^\dagger]^{(j_2)} \times [\gamma_l^\dagger \times \tilde{a}_{j_1'}]^{(j_1)} \right]^{(2)}. \quad (3.2.6d)$$

where the coefficients of each term are given by

$$\tilde{\Gamma}_{j_1 j_2} = Q_{j_1 j_2} \mathfrak{A}_{j_1} \mathfrak{A}_{j_2} - \sum_{j_1', j_2', L} \left[(-1)^{j_1' - j_2 + L} Q_{j_1' j_2'} \mathfrak{B}_{j_1' j_2}^L \mathfrak{B}_{j_2' j_1}^L \hat{j}_1' \hat{j}_2' \left\{ \begin{matrix} j_1 & j_2 & 2 \\ j_1' & j_2' & L \end{matrix} \right\} \right], \quad (3.2.6e)$$

$$\Lambda_{j_1 j_2 j_2'}^{l' L} = Q_{j_1 j_2} \mathfrak{A}_{j_1} \mathfrak{C}_{j_2 j_2'}^{l' L}, \quad (3.2.6f)$$

$$\Delta_{j_1 j_2 j_1' j_2'}^{l' L} = (-1)^{j_1 + j_2} Q_{j_1 j_2} \mathfrak{B}_{j_1 j_1'}^l \mathfrak{B}_{j_2 j_2'}^{l'}, \quad (3.2.6g)$$

and $\hat{j} = \sqrt{2j+1}$. The terms (3.2.6a)-(3.2.6d) are illustrated diagrammatically in fig. 3.2. The operator $q^{(2)}$ in (3.2.5) is the most general form of the fermion quadrupole operator in the IBFM-2 of one type of nucleon (ρ) up to two boson operators.

3.2.1 Direct and Exchange interactions

Until now we have constructed the quadrupole operator in the IBFM using the quasi-particle method by means of the one-nucleon transfer operator. Now we find the expression for the

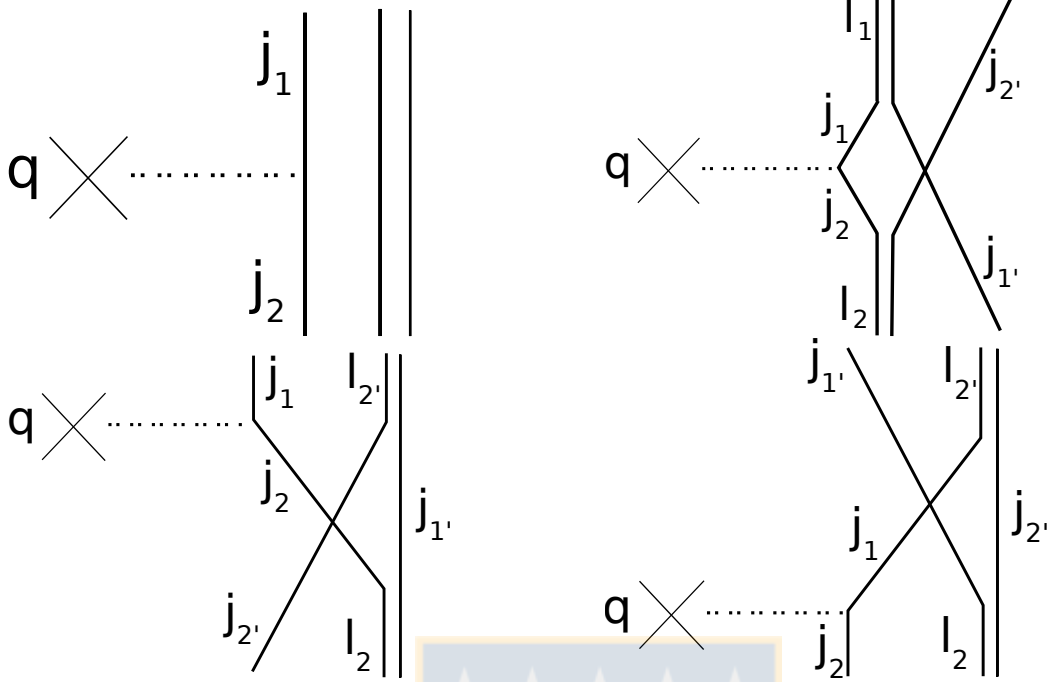


Figure 3.2: **Diagrammatic representation of the different terms of the quadrupole operator** (Eqs. (3.2.6a)-(3.2.6d)).

quadrupole-quadrupole interaction between unlike nucleons (ρ' and ρ) which is given just by the scalar product of both boson and fermion quadrupole operators

$$V_{\text{BF}} = \kappa \left(Q_{\rho'}^{(2)} \cdot q_{\rho}^{(2)} \right), \quad (3.2.7)$$

where $\kappa < 0$ is the boson-fermion interaction strength (since this interaction is attractive the sign of κ is negative) and $Q_{\rho'}^{(2)}$ is the boson quadrupole operator associated to ρ' given by

$$Q_{\rho'}^{(2)} = \left[d_{\rho'}^{\dagger} \times \tilde{s}_{\rho'} + s_{\rho'}^{\dagger} \times \tilde{d}_{\rho'} \right]^{(2)} + \chi_{\rho'} \left[d_{\rho'}^{\dagger} \times \tilde{d}_{\rho'} \right]^{(2)} = \sum_{\substack{l, l' \\ l=l' \neq 0}} q_{ll'}^{\rho'} \mathcal{B}_{ll', \rho'}^{(2)} \quad (3.2.8)$$

where $q_{20}^{\rho'} = q_{02}^{\rho'} = 1$, $q_{22}^{\rho'} = \chi_{\rho'}$, and

$$\mathcal{B}_{ll', M}^{(L)} = [\gamma_l^{\dagger} \times \tilde{\gamma}_{l'}]_M^{(L)}. \quad (3.2.9)$$

We replace $q_{\rho}^{(2)}$ of (3.2.5) and $Q_{\rho'}^{(2)}$ of (3.2.8) in (3.2.7), obtaining four terms, the first one is obtained considering q_{D} in (3.2.6a), whose expression is straightforwardly obtained,

$$\left(Q_{\rho'}^{(2)} \cdot q_{\text{D}} \right) = \sum_{j_1, j_2} \tilde{\Gamma}_{j_1 j_2} \left(Q_{\rho'}^{(2)} \cdot [a_{j_1}^{\dagger} \times \tilde{a}_{j_2}]_{\rho}^{(2)} \right), \quad (3.2.10)$$

while the three exchange terms in the scalar product with the boson quadrupole operator will be modified in order to treat the expression in computational programs for realistic calculations. This modification consists on letting boson(fermion) terms at the left(right) in the expressions, recoupling them using all properties in Appendix A. Hence, we will consider only the boson and

fermion operators in (3.2.6b)-(3.2.6d), which can be written through $\mathcal{B}_{l'l'}^{(L)}$ as

$$\left(\mathcal{B}_{l'l',\rho'}^{(2)} \cdot \left[a_{j_1}^\dagger \times \left[\mathcal{B}_{l_2' l_2}^{(L)} \times \tilde{a}_{j_2'} \right]_{\rho}^{(j_2)} \right]_{\rho}^{(2)} \right) = \sum_{L'} (-1)^{j_1+j_2+L} \hat{j}_2 \sqrt{5} \left\{ \begin{matrix} j_1 & 2 & j_2 \\ L & j_2' & L' \end{matrix} \right\} \left(\left[\mathcal{B}_{l'l',\rho'}^{(2)} \times \mathcal{B}_{l_2' l_2, \rho}^{(L)} \right]^{(L')} \cdot \left[a_{j_1}^\dagger \times \tilde{a}_{j_2'} \right]_{\rho}^{(L')} \right), \quad (3.2.11)$$

$$\left(\mathcal{B}_{l'l',\rho'}^{(2)} \cdot \left[[\tilde{\gamma}_{L_2} \times a_{j_2'}^\dagger]^{(j_2)} \times [\gamma_{L_1}^\dagger \times \tilde{a}_{j_1'}]^{(j_1)} \right]_{\rho}^{(2)} \right) = \delta_{L_1 L_2} \hat{j}_1 \hat{j}_2 (-1)^{j_1'+j_2+L_1} \left\{ \begin{matrix} j_1 & j_2 & 2 \\ j_2' & j_1' & L_1 \end{matrix} \right\} \left(\mathcal{B}_{l'l',\rho'}^{(2)} \cdot \left[a_{j_2'}^\dagger \times \tilde{a}_{j_1'} \right]_{\rho}^{(2)} \right) + \sum_{L, L'} (-1)^{L_1+L_2-L-L'} \hat{j}_1 \hat{j}_2 \hat{L} \sqrt{5} \left\{ \begin{matrix} L_2 & L_1 & L \\ j_2' & j_1' & L' \end{matrix} \right\} \left(\left[\mathcal{B}_{l'l',\rho'}^{(2)} \times \mathcal{B}_{L_1 L_2, \rho}^{(L)} \right]^{(L')} \cdot \left[a_{j_2'}^\dagger \times \tilde{a}_{j_1'} \right]_{\rho}^{(L')} \right), \quad (3.2.12)$$

$$\left(\mathcal{B}_{l'l',\rho'}^{(2)} \cdot \left[\left[\mathcal{B}_{l_1' l_1'}^{(L)} \times a_{j_1}^\dagger \right]^{(j_1)} \times \tilde{a}_{j_2} \right]_{\rho}^{(2)} \right) = \sum_{L'} (-1)^{j_1'+j_2+L-L'} \hat{j}_1 \sqrt{5} \left\{ \begin{matrix} j_1 & 2 & j_2 \\ L' & j_1' & L \end{matrix} \right\} \left(\left[\mathcal{B}_{l'l',\rho'}^{(2)} \times \mathcal{B}_{l_1' l_1', \rho}^{(L)} \right]^{(L')} \cdot \left[a_{j_1}^\dagger \times \tilde{a}_{j_2} \right]_{\rho}^{(L')} \right). \quad (3.2.13)$$

The exchange interaction terms are obtained after summing over the valence orbit and multiplying with the corresponding coefficients. There is an interesting relation, however, between (3.2.11) and (3.2.13). To see this relation, let us consider the following relations of Hermitian conjugation of operators acting on the space of the same ρ nucleon :

$$\left(\mathcal{B}_{l'l', M}^{(L)} \right)^\dagger = (-1)^{l'-l+M} \mathcal{B}_{l'l', -M}^{(L)}, \quad (3.2.14)$$

$$\left(\left[a_j^\dagger \times \tilde{a}_{j'} \right]_M^{(J)} \right)^\dagger = (-1)^{j'-j-M} \left[a_{j'}^\dagger \times \tilde{a}_j \right]_{-M}^{(J)}, \quad (3.2.15)$$

$$\left(\left[\mathcal{B}_{l'l}^{(L)} \times a_{j'}^\dagger \right]_M^{(j)} \right)^\dagger = -(-1)^{l-l'+L-(j-M)} \left[\mathcal{B}_{l'l}^{(L)} \times \tilde{a}_{j'} \right]_{-M}^{(j)}, \quad (3.2.16)$$

with these relations it is easy to prove that

$$\left(\left[\left[\mathcal{B}_{l'l}^{(L)} \times a_{j_1}^\dagger \right]^{(j_1)} \times \tilde{a}_{j_2} \right]_M^{(2)} \right)^\dagger = (-1)^{j_2-j_1+l'-l+L-M} \left[a_{j_2}^\dagger \times \left[\mathcal{B}_{l'l}^{(L)} \times \tilde{a}_{j_1} \right]^{(j_1)} \right]_{-M}^{(2)}. \quad (3.2.17)$$

We use this result in (3.2.6c) to obtain the following property,

$$\begin{aligned} (q_{A, M}^b)^\dagger &= \sum_{j' s, l' s, L} (-1)^{j_2-j_1} \Lambda_{j_2 j_1 j_1'}^{l' l L} (-1)^{j_2-j_1+l'-l+L-M} \left[a_{j_2}^\dagger \times \left[\mathcal{B}_{l'l}^{(L)} \times \tilde{a}_{j_1} \right]^{(j_1)} \right]_{-M}^{(2)} \\ &= (-1)^{-M} \sum_{j' s, l' s, L} \Lambda_{j_1 j_2 j_2'}^{l' l L} (-1)^{l+l'+L} \left[a_{j_1}^\dagger \times \left[\mathcal{B}_{l'l}^{(L)} \times \tilde{a}_{j_2} \right]^{(j_2)} \right]_{-M}^{(2)} \\ &= (-1)^{-M} q_{A, -M}^a, \end{aligned} \quad (3.2.18)$$

that is, there is a relation between q_A^a and q_A^b , they can be considered as the Hermitian conjugated of each other. Moreover, we want to construct a hermitian interaction in other to be used in the

Hamiltonian to obtain real eigenvalues. Hence, when considering the scalar product in (3.2.7) using (A.1.8), we can see that

$$\begin{aligned}
(Q_{\rho'}^{(2)} \cdot q_A^b)^\dagger &= \sum_m (-1)^m (Q_{\rho',m}^{(2)} q_{A,-m}^b)^\dagger \\
&= \sum_m (-1)^m (-1)^m q_{A,m}^a (-1)^m Q_{\rho',-m}^{(2)} \\
&= \sum_m (-1)^m Q_{\rho',-m}^{(2)} q_{A,m}^a \\
&= (Q_{\rho'}^{(2)} \cdot q_A^a).
\end{aligned} \tag{3.2.19}$$

The last expression assures that $(Q_{\rho'}^{(2)} \cdot (q_A^a + q_A^b))$ is a Hermitian operator. Therefore we define

$$V_{BF}^A = (Q_{\rho'}^{(2)} \cdot (q_A^a + q_A^b)) \tag{3.2.20}$$

$$\begin{aligned}
&= \sum_{j's, l's, L's} \Lambda_{j_1 j_2 j_2'}^{l'l'L} (-1)^{l+l'+j_1+j_2} \hat{j}_2 \sqrt{5} \left\{ \begin{matrix} j_1 & 2 & j_2 \\ L & j_2' & L' \end{matrix} \right\} \left([Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)}]^{(L')} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2'}]_{\rho}^{(L')} \right) \\
&+ \sum_{j's, l's, L's} \Lambda_{j_2 j_1 j_1'}^{l'l'L} (-1)^{j_1+j_1+L-L'} \hat{j}_1 \sqrt{5} \left\{ \begin{matrix} j_1 & 2 & j_2 \\ L' & j_1' & L \end{matrix} \right\} \left([Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)}]^{(L')} \cdot [a_{j_1'}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(L')} \right)
\end{aligned} \tag{3.2.21}$$

$$\begin{aligned}
&= \sum_{\substack{j_1, j_2, j_1' \\ l, l', L, L'}} \Lambda_{j_1 j_1' j_2}^{l'l'L} \sqrt{5} \hat{j}' \left\{ \begin{matrix} j_1 & 2 & j_1' \\ L & j_2 & L' \end{matrix} \right\} (-1)^{j_1+j'+l+l'} \left\{ \left([Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)}]^{(L')} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(L')} \right) \right. \\
&\quad \left. - (-1)^{j_1+j_2+L-L'-l-l'} \left([Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)}]^{(L')} \cdot [a_{j_2}^\dagger \times \tilde{a}_{j_1}]_{\rho}^{(L')} \right) \right\}
\end{aligned} \tag{3.2.22}$$

$$= \sum_{j's, l's, L's} \xi_{j_1 j_2}^{l'l'LL'} \left([Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)}]^{(L')} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(L')} \right), \tag{3.2.23}$$

where

$$\xi_{j_1 j_2}^{l'l'LL'} = \sum_{j'} (-1)^{j_1+j'} \sqrt{5} \hat{j}' \left[(-1)^{l+l'} \Lambda_{j_1 j_1' j_2}^{l'l'L} \left\{ \begin{matrix} j_1 & 2 & j_1' \\ L & j_2 & L' \end{matrix} \right\} + (-1)^{L'-L} \Lambda_{j_2 j_2' j_1}^{l'l'L} \left\{ \begin{matrix} j_2 & 2 & j_1' \\ L & j_1 & L' \end{matrix} \right\} \right]. \tag{3.2.24}$$

These different forms for V_{BF}^A show first that this operator is Hermitian and only one particular coefficient is needed to compute this term, which is clear in the expression with ξ , whose form is elegant in fact, but is not very useful since it requires more computational time to calculate the coefficients than to calculate the matrix elements, sum the coefficients and then calculate its Hermitian conjugated. Because of the properties mentioned above, we will refer to this term as the Exchange Interaction (A).

On the other hand, when summing the corresponding coefficients on (3.2.12) we obtain two additional terms

$$\begin{aligned}
(Q_{\rho'}^{(2)} \cdot q_B) &= \sum_{j's, L} \Delta_{j_1 j_2 j_1' j_2'}^{LL} \hat{j}_1 \hat{j}_2 (-1)^{j_1'+j_2+L} \left\{ \begin{matrix} j_1 & j_2 & 2 \\ j_2' & j_1' & L \end{matrix} \right\} (Q_{\rho'}^{(2)} \cdot [a_{j_2'}^\dagger \times \tilde{a}_{j_1'}]_{\rho}^{(2)}) \\
&+ \sum_{j's, L's} \Delta_{j_1 j_2 j_1' j_2'}^{l'l'L} (-1)^{l+l'-L-L'} \hat{j}_1 \hat{j}_2 \hat{L} \sqrt{5} \left\{ \begin{matrix} l' & l & L \\ j_2' & j_1' & L' \\ j_2 & j_1 & 2 \end{matrix} \right\} \\
&\times \left([Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)}]^{(L')} \cdot [a_{j_2'}^\dagger \times \tilde{a}_{j_1'}]_{\rho}^{(L')} \right).
\end{aligned} \tag{3.2.25}$$

The first term on the rhs of the last expression is related with the direct interaction term. This is evident by changing indices in order to rewrite this term as

$$\sum_{j's,L} \Delta_{j'_1 j'_2 j_2 j_1}^{LL} \hat{j}_1 \hat{j}_2 (-1)^{j_2 + j'_2 + L} \begin{Bmatrix} j_1 & j_2 & 2 \\ j'_1 & j'_2 & L \end{Bmatrix} \left(Q_{\rho'}^{(2)} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(2)} \right), \quad (3.2.26)$$

and noticing that $\tilde{\Gamma}_{j_1 j_2}$ may also be rewritten in terms of Δ ,

$$\tilde{\Gamma}_{j_1 j_2} = Q_{j_1 j_2} \mathfrak{A}_{j_1} \mathfrak{A}_{j_2} - \sum_{j'_1, j'_2, L} \left[\Delta_{j'_1 j'_2 j_2 j_1}^{LL} (-1)^{j_2 + j'_2 + L} \hat{j}_1 \hat{j}_2 \begin{Bmatrix} j_1 & j_2 & 2 \\ j'_1 & j'_2 & L \end{Bmatrix} \right]. \quad (3.2.27)$$

We can see clearly now that the direct term has already considered the first term of $(Q_{\rho'}^{(2)} \cdot q_B)$ on it, but with opposite sign. That is because those terms come from the same operator in (3.2.3) which is recoupled twice, once for obtaining the unpaired particle creation(annihilation) operator at the left(right) of the expression, and the second time when we used the scalar product for obtaining the quadrupole-quadrupole interaction. Since both terms cancel each other, the outcome of this sum is

$$\sum_{j's,L} \Delta_{j'_1 j'_2 j_2 j_1}^{LL} \hat{j}_1 \hat{j}_2 (-1)^{j_2 + j'_2 + L} \begin{Bmatrix} j_1 & j_2 & 2 \\ j'_1 & j'_2 & L \end{Bmatrix} \left(Q_{\rho'}^{(2)} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(2)} \right) + \sum_{j_1, j_2} \tilde{\Gamma}_{j_1 j_2} \left(Q_{\rho'}^{(2)} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(2)} \right) \quad (3.2.28)$$

$$= \sum_{j_1, j_2} Q_{j_1 j_2} \mathfrak{A}_{j_1} \mathfrak{A}_{j_2} \left(Q_{\rho'}^{(2)} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(2)} \right). \quad (3.2.29)$$

We call

$$\Gamma_{j_1 j_2} = Q_{j_1 j_2} \mathfrak{A}_{j_1} \mathfrak{A}_{j_2}, \quad (3.2.30)$$

since $Q_{j_1 j_2}$ may change sign under permutation of indices, it follows that

$$\Gamma_{j_1 j_2} = (-1)^{j_1 - j_2} \Gamma_{j_2 j_1}, \quad (3.2.31)$$

with this property it is easy to notice that the last term in (3.2.29) is Hermitian. That is

$$\begin{aligned} \left(\sum_{j_1, j_2} \Gamma_{j_1 j_2} \left(Q_{\rho'}^{(2)} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(2)} \right) \right)^\dagger &= \sum_{j_1, j_2} \Gamma_{j_1 j_2} \sum_m (-1)^m ([a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho, -m}^{(2)})^\dagger (Q_{\rho', m}^{(2)})^\dagger \\ &= \sum_{j_1, j_2} \Gamma_{j_1 j_2} \sum_m (-1)^m (-1)^{j_2 - j_1 + m} [a_{j_2}^\dagger \times \tilde{a}_{j_1}]_{\rho, m}^{(2)} (-1)^{-m} Q_{\rho', -m}^{(2)} \\ &= \sum_{j_1, j_2} \Gamma_{j_1 j_2} (-1)^{j_2 - j_1} \sum_m (-1)^m Q_{\rho', -m}^{(2)} [a_{j_2}^\dagger \times \tilde{a}_{j_1}]_{\rho, m}^{(2)} \\ &= \sum_{j_1, j_2} \Gamma_{j_2 j_1} \sum_m (-1)^m Q_{\rho', -m}^{(2)} [a_{j_2}^\dagger \times \tilde{a}_{j_1}]_{\rho, m}^{(2)} \\ &= \sum_{j_1, j_2} \Gamma_{j_2 j_1} \left(Q_{\rho'}^{(2)} \cdot [a_{j_2}^\dagger \times \tilde{a}_{j_1}]_{\rho}^{(2)} \right). \end{aligned} \quad (3.2.32)$$

Since we have dummy indices, the demonstration is done. In order to avoid redundancies in the sum of the operators, and for preserving the Hermiticity of the interaction we call the above term as the Direct Interaction and we denote it by V_{BF}^D .

Finally we must consider the remaining term in (3.2.33) which we call V_{BF}^B and is defined as

$$V_{BF}^B = \sum_{j's, L's} \Delta_{j_1 j_2 j'_1 j'_2}^{l'l'} (-1)^{l+l'-L-L'} \hat{j}_1 \hat{j}_2 \hat{L} \sqrt{5} \begin{Bmatrix} l' & l & L \\ j'_2 & j'_1 & L' \\ j_2 & j_1 & 2 \end{Bmatrix} \left(\left[Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l', \rho}^{(L)} \right]^{(L')} \cdot [a_{j_2}^\dagger \times \tilde{a}_{j_1}]_{\rho}^{(L')} \right), \quad (3.2.33)$$

which is also Hermitian. This can be seen by using the following two properties

$$\left(\left[Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)} \right]^{(L')} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(L')} \right)^\dagger = (-1)^{j_2-j_1+l'+l'+L-L'} \left(\left[Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)} \right]^{(L')} \cdot [a_{j_2}^\dagger \times \tilde{a}_{j_1}]_{\rho}^{(L')} \right), \quad (3.2.34)$$

$$\Delta_{j_1 j_2 j_1' j_2'}^{l'l'} = (-1)^{j_1-j_2} \Delta_{j_2 j_1 j_2' j_1'}^{l'l}, \quad (3.2.35)$$

along with the phase shift of the $9j$ -symbol, we have that

$$\begin{aligned} (V_{BF}^B)^\dagger &= \sum_{j's, L's} \Delta_{j_1 j_2 j_1' j_2'}^{l'l'} (-1)^{l+l'-L-L'} \hat{j}_1 \hat{j}_2 \hat{L} \sqrt{5} \begin{Bmatrix} l' & l & L \\ j_2' & j_1' & L' \\ j_2 & j_1 & 2 \end{Bmatrix} (-1)^{j_1'-j_2'+l'+l'+L-L'} \\ &\quad \left(\left[Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)} \right]^{(L')} \cdot [a_{j_2}^\dagger \times \tilde{a}_{j_1}]_{\rho}^{(L')} \right) \\ &= \sum_{j's, L's} (-1)^{j_1-j_2} \Delta_{j_2 j_1 j_2' j_1'}^{l'l'} (-1)^{j_1'-j_2'} \hat{j}_1 \hat{j}_2 \hat{L} \sqrt{5} \begin{Bmatrix} l & l' & L \\ j_1 & j_2 & L' \\ j_1 & j_2 & 2 \end{Bmatrix} (-1)^{j_1'+j_2'+j_1+j_2+l+l'+L+L'+2} \\ &\quad \left(\left[Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)} \right]^{(L')} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(L')} \right) \\ &= \sum_{j's, L's} \Delta_{j_2 j_1 j_2' j_1'}^{l'l'} (-1)^{l+l'+L+L'} \hat{j}_1 \hat{j}_2 \hat{L} \sqrt{5} \begin{Bmatrix} l & l' & L \\ j_1' & j_2' & L' \\ j_1 & j_2 & 2 \end{Bmatrix} \left(\left[Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)} \right]^{(L')} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(L')} \right) \\ &= V_{BF}^B \end{aligned} \quad (3.2.36)$$

Again, this term is Hermitian and is the last term that stem from (3.2.10), (3.2.11)-(3.2.13). Therefore we call this term the Exchange Interaction (B). With these results, we can write finally the quadrupole-quadrupole boson-fermion interaction (3.2.7) as

$$V_{BF} = \kappa (V_{BF}^D + V_{BF}^A + V_{BF}^B), \quad (3.2.37)$$

which is hermitian and described in function of a Direct and two Exchange Interactions. We write down below these interaction in order to summarize this section,

$$V_{BF}^D = \sum_{j_1, j_2} \Gamma_{j_1 j_2} \left(Q_{\rho'}^{(2)} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(2)} \right), \quad (3.2.38)$$

$$V_{BF}^A = \sum_{\substack{j_1, j_2, j' \\ l, l', L, L'}} \Lambda_{j_1 j_1' j_2}^{l'l' \lambda} \sqrt{5} \hat{j}' \begin{Bmatrix} j_1 & 2 & j' \\ L & j_2 & \lambda \end{Bmatrix} (-1)^{j_1+j'+l+l'} \left(\left[Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)} \right]^{(\lambda)} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(\lambda)} \right) + \text{H.c.}, \quad (3.2.39)$$

$$V_{BF}^B = \sum_{j's, L's} \Delta_{j_2 j_1' j_2 j_1}^{l'l'} (-1)^{l+l'-L-L'} \hat{j}_1' \hat{j}_2 \hat{L} \sqrt{5} \begin{Bmatrix} l' & l & L \\ j_1 & j_2 & \lambda \\ j_1' & j_2' & 2 \end{Bmatrix} \left(\left[Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)} \right]^{(\lambda)} \cdot [a_{j_1}^\dagger \times \tilde{a}_{j_2}]_{\rho}^{(\lambda)} \right). \quad (3.2.40)$$

We would like to mention that the coefficients attached to each interaction are different to those given by Scholten [10], since he only considers the transfer operator at the lower order with generalized seniority $\tilde{\nu} \leq 2$ and explicitly the s and d bosons in the interaction. Thus there are several terms in the boson-fermion interaction that do not appear in his expressions and are included in our treatment. In addition he uses only the OAI procedure 3.3 to obtain the values of the coefficients of the transfer operator, which we have rewritten in a simple and elegant form using different methods. For instance, in our expressions (3.2.39) and (3.2.40), different terms of the form $[s^\dagger \times \tilde{s}]$ appears, which may be rewritten according to the boson number operator relation,

$$(s^\dagger \cdot \tilde{s} + d^\dagger \cdot \tilde{d}) = \hat{N}. \quad (3.2.41)$$

Then, many terms could be considered as related to the Direct Interaction, directly proportional to N . Since we do not use this expression the different coefficients Γ , Λ and Δ are not equal to Scholten's and also may not be related so easily. In this sense we have developed a generalization of the quadrupole-quadrupole boson-fermion interaction considering up to two-body interaction terms and without using any particular mapping to obtain the coefficients that appear on it. This is particularly useful because there are different available mappings with different features to obtain these coefficients [13]. In the following section we will consider two mappings to obtain these coefficients in order to see and compare the different characteristics that inherit both mappings. They are the OAI and the GHP.

3.3 OAI and GHP Mappings of the transfer operator

In this work we have considered two different alternative schemes in order to obtain an expression for each coefficient of the one-nucleon transfer operator (3.1.4), they are the OAI [11], and the GHP [12,26] mappings. In this section we will discuss briefly both mappings and we will express the \mathfrak{A} , \mathfrak{B} and \mathfrak{C} coefficients in these mappings.

3.3.1 OAI mapping

The OAI method is based on the generalized Seniority scheme in the SM. In this scheme the SM space is truncated to the SD pairs space, i.e. to states constructed from the collective nucleon pair creation operators

$$S^\dagger = \sum_j \alpha_j \sqrt{\Omega_j} A_{jj}^{\dagger(00)}, \quad D_\mu^\dagger = \sum_{\substack{j,j' \\ j \leq j'}} \beta_{jj'} A_{jj'}^{\dagger(2\mu)}, \quad (3.3.1)$$

where $\Omega_j = j+1/2$ is half the occupancy of the orbit, and α 's and β 's are the structure constants of the S and D pairs, with 0 and 2 total angular momentum, respectively. These coefficients are straightforwardly related to the occupation probabilities in valence orbits.

The states of the SD space are tagged by the generalized seniority quantum number $\tilde{\nu}$, which by definition, counts the number of particles not in correlated S pairs.

Here, $A_{jj'}^{\dagger(2\mu)}$ is the pair creation operator defined in Eq. (2.1.36). We do a mapping between fermion states in SD space

$$|S^{N_s} D^{N_d} \omega J\rangle_F, \quad (3.3.2)$$

such that $N_s + N_d = N$ is the number of pairs, and boson states (Marumori mapping [13])

$$|s^{N_s} d^{N_d} \omega J\rangle_B, \quad (3.3.3)$$

where ω accounts for all the quantum numbers necessary to identify uniquely the states. The strategy is to equate the matrix elements of any fermion operator O between fermion states to matrix elements of the boson image of the fermion operator O^B between the corresponding boson states:

$$\langle S^{N_s} D^{N_d} \omega J | O | S^{N_s} D^{N_d} \omega J \rangle_F = \langle s^{N_s} d^{N_d} \omega J | O^B | s^{N_s} d^{N_d} \omega J \rangle_B. \quad (3.3.4)$$

The orthonormalized boson states of lower GS, that is, states with $\tilde{\nu} \leq 2$ are shown below,

$$|s^N\rangle = \frac{1}{\sqrt{N!}} (s^\dagger)^N |0\rangle_B, \quad (3.3.5a)$$

$$|s^N, jm\rangle = \frac{1}{\sqrt{N!}} (s^\dagger)^N a_{jm}^\dagger |0\rangle_B, \quad (3.3.5b)$$

$$|s^{N-1} d, 2\mu\rangle = \frac{1}{\sqrt{(N-1)!}} (s^\dagger)^{N-1} d_\mu^\dagger |0\rangle_B, \quad (3.3.5c)$$

where $|0\rangle_B$ is the state of a nucleus in closed shell for a type of nucleon. Since we will consider states with higher $\tilde{\nu}$, we will first start to construct the fermion states in order to set a one-to-one correspondence between the states in fermion and boson spaces. The states considered in this thesis are only those with low generalized seniority and those in the SD space. States with higher $\tilde{\nu}$ are used to obtain coefficients of higher order. In the OAI philosophy, only the states with lowest fermion $\tilde{\nu}$ are used in Eq. (3.3.4). Higher order states are more cumbersome to manage because they are not orthogonal.

Basis state in the IBFM

The set of Shell Model normalized states with $\tilde{\nu} \leq 2$ are constructed as:

$$|S^N\rangle := |2N, \tilde{\nu} = 0, 00\rangle = \eta_{2N,0,0}^{-1} (S^\dagger)^N |0\rangle, \quad (3.3.6)$$

$$|S^N, jm\rangle := |2N, \tilde{\nu} = 1, jm\rangle = \eta_{2N,1,j}^{-1} C_{jm}^\dagger (S^\dagger)^N |0\rangle, \quad (3.3.7)$$

$$|S^{N-1}D, 2\mu\rangle := |2N, \tilde{\nu} = 2, 2\mu\rangle = \eta_{2N,2,2}^{-1} D_\mu^\dagger (S^\dagger)^{N-1} |0\rangle. \quad (3.3.8)$$

It's evident that the sum of the exponents on the S and D operators is equal to the number of N pairs of one kind of particles, and the $\eta_{2N,\tilde{\nu},J}$ are the normalization constants for states with N pairs of particles of one kind (protons or neutrons), generalized seniority $\tilde{\nu}$ and total angular momentum J . It is worth stressing that in the literature it is often found the position of the single-nucleon creation operator and the S or D operators exchanged in the definition of the states (3.3.6)-(3.3.8). This is not a problem because it is ready to see that all these operators commute, i.e.:

$$[(S^\dagger)^N, D_\mu^\dagger] = 0, \quad (3.3.9)$$

$$[(S^\dagger)^N, C_{jm}^\dagger] = 0, \quad (3.3.10)$$

$$[D_\mu^\dagger, C_{jm}^\dagger] = 0, \quad (3.3.11)$$

for any integer numbers N , and for all j, m and μ quantum numbers.

Since we are not using any approximation to find a fancy form of these normalization constants, such as occurs in NOA, these constants are treated in an exact way. Also they were given without approximation for the first time by Pittel, Duval and Barret [27]. Their leading idea is the use of the multinomial theorem to expand $(S^\dagger)^N$ in terms of single-nucleon creation operators. They found the norm for the $\tilde{\nu} = 0$ states,

$$\eta_{2N,0,0}^2 = (N!)^2 \sum_{\substack{m_1, m_2, m_3, \dots, m_k \\ (\sum_j m_j = N)}} \prod_{i=1}^k \alpha_i^{2m_i} \binom{\Omega_i}{m_i}, \quad (3.3.12)$$

where the valence orbits i are enumerated from 1 to k and the m_i 's are nonnegative integers that represent the components of a composition² of N pairs of nucleons in k orbits. However, beyond equation (3.3.12) PDB's procedure becomes unwieldy and incorrect. The work of Lipas and others [29] have shown the correct expression of them, which we have used along this work.

²A composition [28] of a number N into k parts, is the separation of N into k elements such that the sum of all the elements is N . For example the composition of $N = 3$ in $k=2$ is

3	0
2	1
1	2
0	3

The number of these separations is $\binom{n+k-1}{n}$.

For more details see [30, 31].

$$\eta_{2N,2,2}^2 = \sum_{\substack{j,j' \\ j \leq j'}} \beta_{jj'}^2 \eta_{2N,2,2}(jj'), \quad (3.3.13)$$

$$\eta_{2N,2,2}(jj')^2 = \Delta(jj'2) \sum_{p=0}^{N-1} \left[\frac{(N-1)!}{p!} \right]^2 (-1)^{N-1-p} \eta_{2p,0,0}^2 \sum_{q=0}^{N-1-q} \alpha_j^{2N-2(1+p+q)} \alpha_{j'}^{2q}, \quad (3.3.14)$$

$$\eta_{2N,1,j}^2 = (-1)^N \alpha_j^{2N} (N!)^2 + \sum_{m=1}^N \eta_{2m,0,0}^2 (-1)^{N-m} \left[\frac{N!}{m!} \right]^2 \alpha_j^{2(N-m)}, \quad (3.3.15)$$

where

$$\Delta(jj'2) = \begin{cases} 1 & , \text{ if } |j - j'| \leq 2 \leq j + j' \\ 0 & , \text{ in other case} \end{cases} \quad (3.3.16)$$

The problem arises when dealing with states with generalized seniority $\tilde{\nu} > 2$, like those that are necessary to obtain $E_{jj'L}$.

The state

$$\left[C_j^\dagger \times |2N, \tilde{\nu} = 2, 2\rangle \right]_M^{(J)} = \eta_{2N,2,2}^{-1} \left[C_j^\dagger \times D^\dagger \right]_M^{(J)} (S^\dagger)^{N-1} |0\rangle$$

is not purely a $\tilde{\nu} = 3$ state, since it contains components of $\tilde{\nu} = 3$ and also $\tilde{\nu} = 1$. The normalized states are obtained by taking away the component associated with $\tilde{\nu} = 1$. Thus, the pure $\tilde{\nu} = 3$ state is

$$|2N + 1, \tilde{\nu} = 3, jJM\rangle = \eta_{2N,3,jJ}^{-1} \left[\left[C_j^\dagger \times |2N, \tilde{\nu} = 2, 2\rangle \right]_M^{(J)} - \chi_j^J |2N + 1, \tilde{\nu} = 1, JM\rangle \right], \quad (3.3.17)$$

where $\eta_{2N,3,jJ}$ is the normalization constant of this state, and the value of χ_j^J is obtained by requiring that

$$\langle 2N + 1, \tilde{\nu} = 1, JM | 2N + 1, \tilde{\nu} = 3, jJM \rangle = 0, \quad (3.3.18)$$

which yields to

$$\chi_j^J = \frac{1}{\eta_{2N,1,J} \eta_{2N,2,2}} \langle 0 | \left\{ (S^\dagger)^N C_{JM}^\dagger \right\}^\dagger \left[C_j^\dagger \times D^\dagger \right]_M^{(J)} (S^\dagger)^{N-1} |0\rangle. \quad (3.3.19)$$

On the other hand, the states with $\tilde{\nu} = 3$ are not orthogonal. Actually, it is ready to see that for two states with $\tilde{\nu} = 3$, but different angular momenta j and j' , the overlap of these two states is

$$\begin{aligned} \omega_{j'j}^J \mathfrak{P}_{Njj'J} &:= \mathfrak{P}_{Njj'J} \langle 2N + 1, \tilde{\nu} = 3, j'JM | 2N + 1, \tilde{\nu} = 3, jJM \rangle \\ &= \eta_{2N,2,2}^{-2} \langle 0 | \left\{ \left[C_{j'}^\dagger \times D^\dagger \right]_M^{(J)} (S^\dagger)^{N-1} \right\}^\dagger \left[C_j^\dagger \times D^\dagger \right]_M^{(J)} (S^\dagger)^{N-1} |0\rangle \\ &\quad - \chi_{j'}^J \eta_{2N,2,2}^{-1} \langle 0 | \left\{ \left[C_{j'}^\dagger \times D^\dagger \right]_M^{(J)} (S^\dagger)^{N-1} \right\}^\dagger |2N + 1, \tilde{\nu} = 1, JM\rangle \\ &\quad - \chi_j^J \eta_{2N,2,2}^{-1} \langle 2N + 1, \tilde{\nu} = 1, JM | \left[C_j^\dagger \times D^\dagger \right]_M^{(J)} (S^\dagger)^{N-1} |0\rangle \\ &\quad + \chi_{j'}^J \chi_j^J \langle 2N + 1, \tilde{\nu} = 1, JM | 2N + 1, \tilde{\nu} = 1, JM \rangle \\ &= \left(\frac{1}{\eta_{2N,2,2}^2} \langle 0 | \left\{ \left[C_{j'}^\dagger \times D^\dagger \right]_M^{(J)} (S^\dagger)^{N-1} \right\}^\dagger \left[C_j^\dagger \times D^\dagger \right]_M^{(J)} (S^\dagger)^{N-1} |0\rangle + \chi_{j'}^J \chi_j^J \right), \end{aligned} \quad (3.3.20)$$

where $\mathfrak{P}_{Njj'J} = \eta_{2N,3,j'J} \eta_{2N,3,jJ}$. We write the tensor products of the matrix element of (3.3.20) as (A.1.2) in Appendix A, and then through the commutation relation of fermion operators we

get,

$$\begin{aligned} \langle 0 | \left\{ \left[C_{j'}^\dagger \times D^\dagger \right]_M^{(J)} (S^\dagger)^{N-1} \right\}^\dagger \left[C_j^\dagger \times D^\dagger \right]_M^{(J)} (S^\dagger)^{N-1} | 0 \rangle &= \sum_{\substack{m_1, m_2, \\ m'_1, m'_2}} \langle j' m_1 2 m_2 | JM \rangle \langle j m'_1 2 m'_2 | JM \rangle \\ &\times \left\{ \delta_{jj'} \delta_{m_1 m'_1} \langle 0 | \left(D_{m_2}^\dagger (S^\dagger)^{N-1} \right)^\dagger D_{m'_2}^\dagger (S^\dagger)^{N-1} | 0 \rangle - \langle 0 | \left(D_{m_2}^\dagger (S^\dagger)^{N-1} \right)^\dagger C_{j m'_1}^\dagger C_{j' m_1} D_{m'_2}^\dagger (S^\dagger)^{N-1} | 0 \rangle \right\}. \end{aligned} \quad (3.3.21)$$

The first matrix element on the rhs of (3.3.21) is just $\delta_{m_2 m'_2} \eta_{2N,2,2}^2$ according to (3.3.8). With the product of the δ 's, the sum of the Clebsch-Gordan coefficients is 1, thus

$$\sum_{\substack{m_1, m_2, \\ m'_1, m'_2}} \langle j' m_1 2 m_2 | JM \rangle \langle j m'_1 2 m'_2 | JM \rangle \delta_{jj'} \delta_{m_1 m'_1} \langle 0 | \left(D_{m_2}^\dagger (S^\dagger)^{N-1} \right)^\dagger D_{m'_2}^\dagger (S^\dagger)^{N-1} | 0 \rangle = \delta_{jj'} \eta_{2N,2,2}^2. \quad (3.3.22)$$

In order to obtain the remaining matrix element of (3.3.21), we will consider the following matrix element,

$$\langle S^{N-1} D_M | \left[C_j^\dagger \times \tilde{C}_{j'} \right]_k^{(K)} | S^{N-1} D_{M'} \rangle = \eta_{2N,2,2}^{-2} \langle 0 | \left(S^{N-1} D_M^\dagger \right)^\dagger \left[C_j^\dagger \times \tilde{C}_{j'} \right]_k^{(K)} (S^\dagger)^{N-1} D_{M'}^\dagger | 0 \rangle. \quad (3.3.23)$$

Again, by writing the tensor product through the Clebsch-Gordan coefficients, and using the Wigner-Eckart theorem, we obtain

$$\begin{aligned} \sum_{m, m'} \langle j m j' m' | K k \rangle (-1)^{j' - m'} \langle 0 | \left(S^{N-1} D_M^\dagger \right)^\dagger C_{j m}^\dagger C_{j' - m'} (S^\dagger)^{N-1} D_{M'}^\dagger | 0 \rangle \\ = (-1)^{2-M} \eta_{2N,2,2}^2 \begin{pmatrix} 2 & K & 2 \\ -M & k & M' \end{pmatrix} \langle S^{N-1} D | \left[C_j^\dagger \times \tilde{C}_{j'} \right]^{(K)} || S^{N-1} D \rangle. \end{aligned} \quad (3.3.24)$$

Now, we multiply the former expression by $\langle j m_\alpha j' m_\beta | K k \rangle$ and we sum over $|j - j'| \leq K \leq j + j'$ and $-K \leq k \leq K$. Since in the lhs of (3.3.24) the only dependence on k and K is in the Clebsch-Gordan coefficients, by its orthogonality relation we get

$$\sum_{K, k} \langle j m j' m' | K k \rangle \langle j m_\alpha j' m_\beta | K k \rangle = \delta_{m m_\alpha} \delta_{m' m_\beta}.$$

Thus, after summing on m 's, we write the lhs of (3.3.24) as

$$(-1)^{j' - m_\beta} \langle 0 | \left(S^{N-1} D_M^\dagger \right)^\dagger C_{j m_\alpha}^\dagger C_{j' - m_\beta} (S^\dagger)^{N-1} D_{M'}^\dagger | 0 \rangle, \quad (3.3.25)$$

the result of the (3.3.24) after the previous operation becomes

$$\langle 0 | \left(S^{N-1} D_M^\dagger \right)^\dagger C_{j m_\alpha}^\dagger C_{j' m_\beta} (S^\dagger)^{N-1} D_{M'}^\dagger | 0 \rangle = \eta_{2N,2,2}^2 \mathfrak{M}_{m_\alpha m_\beta M M'}^{jj'}, \quad (3.3.26)$$

where

$$\begin{aligned} \mathfrak{M}_{m_\alpha m_\beta M M'}^{jj'} := & (-1)^{j' - M + m_\beta} \sum_{K, k} \langle j m_\alpha j' - m_\beta | K k \rangle \begin{pmatrix} 2 & K & 2 \\ -M & k & M' \end{pmatrix} \\ & \times \langle S^{N-1} D | \left[C_j^\dagger \times \tilde{C}_{j'} \right]^{(K)} || S^{N-1} D \rangle. \end{aligned} \quad (3.3.27)$$

Returning to our calculation, we recall $\{m_\alpha, m_\beta, M, M'\}$ from the former expression to $\{m'_1, m_1, m_2, m'_2\}$, and we use the property (A.3.6), as follows. We rewrite the Clebsch-Gordan coefficient in $\mathfrak{M}_{m'_1 m_1 m_2 m'_2}^{jj'}$ as a 3j-symbol. Summing on k , this yields to

$$\begin{aligned} \sum_k \langle jm'_1 j' - m_1 | K k \rangle \begin{pmatrix} 2 & K & 2 \\ -m_2 & k & m'_2 \end{pmatrix} &= \sum_k \hat{K} (-1)^{j'-j-k+K} \begin{pmatrix} j & j' & K \\ m'_1 & -m_1 & -k \end{pmatrix} \begin{pmatrix} 2 & 2 & K \\ -m_2 & m'_2 & k \end{pmatrix} \\ &= \hat{K} (-1)^{j+j+K+m'_1-m_1} \sum_{j, \mathbf{m}} (-1)^{K+j-m_2+m'_1} \hat{j}^2 \\ &\quad \times \begin{Bmatrix} 2 & 2 & K \\ j & j' & j \end{Bmatrix} \begin{pmatrix} j & 2 & j \\ m'_1 & m'_2 & \mathbf{m} \end{pmatrix} \begin{pmatrix} 2 & j' & j \\ m_2 & m_1 & \mathbf{m} \end{pmatrix}. \end{aligned} \quad (3.3.28)$$

Since we want to find an explicit expression for $\sum_{m'_s} \langle j' m_1 2 m_2 | JM \rangle \langle j m'_1 2 m'_2 | JM \rangle \mathfrak{M}_{m'_1 m_1 m_2 m'_2}^{jj'}$ to replace it on (3.3.21), we rewrite the 3j-symbols of (3.3.28) as Clebsch-Gordan coefficients. Then, because of their orthogonality relation we obtain,

$$\begin{aligned} \sum_{\substack{m'_1, m'_2 \\ m_1, m_2}} \langle j' m_1 2 m_2 | JM \rangle \langle j m'_1 2 m'_2 | JM \rangle \mathfrak{M}_{m'_1 m_1 m_2 m'_2}^{jj'} &= - \sum_{K=|j-j'|}^{j+j'} (-1)^{j'+J} \hat{K} \begin{Bmatrix} 2 & 2 & K \\ j & j' & J \end{Bmatrix} \\ &\quad \times \langle S^{N-1} D || [C_j^\dagger \times \tilde{C}_{j'}]^{(K)} || S^{N-1} D \rangle. \end{aligned} \quad (3.3.29)$$

So, after all the calculation, we obtain finally the desired matrix element, which is

$$\begin{aligned} \langle 0 | \left\{ [C_{j'}^\dagger \times D^\dagger]_M^{(J)} (S^\dagger)^{N-1} \right\}^\dagger [C_j^\dagger \times D^\dagger]_M^{(J)} (S^\dagger)^{N-1} | 0 \rangle &= \\ \eta_{2N,2,2}^2 \left\{ \delta_{jj'} + \sum_K (-1)^{J+j'} \hat{K} \begin{Bmatrix} 2 & 2 & K \\ j & j' & J \end{Bmatrix} \langle S^{N-1} D || [C_j^\dagger \times \tilde{C}_{j'}]^{(K)} || S^{N-1} D \rangle \right\}, \end{aligned} \quad (3.3.30)$$

where the sum on K runs over all the possible couplings of j and j' . The reduced matrix element is given in [29] and has the following recursive form

$$\begin{aligned} \langle S^{N-1} D || [C_j^\dagger \times \tilde{C}_{j'}]^{(K)} || S^{N-1} D \rangle &\equiv \langle 2N22 || [C_j^\dagger \times \tilde{C}_{j'}]^{(K)} || 2N22 \rangle \\ &= -\eta_{2N,2,2}^{-2} \left\{ 5\hat{K} \sum_i \beta_{ij} \beta_{j'i} \sqrt{(1+\delta_{ij})(1+\delta_{ij'})} \right. \\ &\quad \times [\alpha_j \alpha_{j'} (N-1)^2 \eta_{2(N-1),2,2}^2(ij) - (-1)^K \eta_{2N,2,2}^2(ij)] \\ &\quad \left. \begin{Bmatrix} j & K & j' \\ 2 & i & 2 \end{Bmatrix} + \eta_{2(N-1),2,2}^2 (N-1)^2 \alpha_{j'}^2 \right. \\ &\quad \left. \times \left[\sqrt{5} \hat{j} \delta_{jj'} \delta_{K0} + \langle 2(N-1)22 || [C_j^\dagger \times \tilde{C}_{j'}]^{(K)} || 2(N-1)22 \rangle \right] \right\}. \end{aligned} \quad (3.3.31)$$

This expression is valid for $N \geq 2$ pairs, since for only one pair ($N = 1$) of identical particles, the very last term vanishes, thereby we have a readily applicable recursion formula.

On the other hand, the expression for χ_j^J is found considering the corresponding matrix element which we call $\tilde{\chi}_j^J$ just for simplicity, and that is given explicitly as

$$\tilde{\chi}_j^J = \sum_{m, \mu} \langle jm2\mu | JM \rangle \langle 0 | S^N C_{JM} C_j^\dagger m D_\mu^\dagger (S^\dagger)^{N-1} | 0 \rangle. \quad (3.3.32)$$

Now, by the commutation relation of fermion operators, we can write the matrix element of the rhs of (3.3.32) as,

$$\begin{aligned} \langle 0|S^N C_{JM} C_{jm}^\dagger D_\mu^\dagger (S^\dagger)^{N-1}|0\rangle &= \delta_{jJ} \delta_{mM} \langle 0|S^N D_\mu^\dagger (S^\dagger)^{N-1}|0\rangle \\ &\quad - \langle 0|S^N C_{jm}^\dagger C_{JM} D_\mu^\dagger (S^\dagger)^{N-1}|0\rangle, \end{aligned} \quad (3.3.33)$$

the first term vanishes, because it is proportional to $\langle 2N, 0, 0|2N, 2, 2\mu\rangle$ which is null. Now we use the relation (C.0.3) of the Appendix C, by which we can rewrite the last matrix element as

$$\begin{aligned} \langle 0|S^N C_{jm}^\dagger C_{JM} D_\mu^\dagger (S^\dagger)^{N-1}|0\rangle &= \langle 0|\left\{N\alpha_j S^{N-1} \tilde{C}_{jm} + C_{jm}^\dagger S^{N-1}\right\} C_{JM} D_\mu^\dagger (S^\dagger)^{N-1}|0\rangle \\ &= N\alpha_j (-1)^{j-m} \langle 0|S^{N-1} C_{j-m} C_{JM} D_\mu^\dagger (S^\dagger)^{N-1}|0\rangle. \end{aligned} \quad (3.3.34)$$

To find the expression of the last term we use the following formula [29] for convenience :

$$\begin{aligned} &[\delta_{jj_1} \delta_{j'j'_1} - (-1)^{j_1+j'_1-K} \delta_{jj'_1} \delta_{j'j_1}] \delta_{KK'} \delta_{kk'} \eta_{2N,2,K}^2(jj') \\ &\equiv \langle 0|\left\{(S^\dagger)^{N-1} \left[C_j^\dagger \times C_{j'}^\dagger\right]_k^{(K)}\right\}^\dagger (S^\dagger)^{N-1} \left[C_{j_1}^\dagger \times C_{j'_1}^\dagger\right]_{k'}^{(K')}|0\rangle. \end{aligned} \quad (3.3.35)$$

Making $K = 2$, $k = \mu$ and multiplying both sides by $\langle j\tilde{m}j'\tilde{m}'|K'k'\rangle$, and summing over K' and k' we get in the rhs of (3.3.35),

$$\begin{aligned} &\sum_{K',k',m,m'} \langle j\tilde{m}j'\tilde{m}'|K'k'\rangle \langle jmj'm'|K'k'\rangle \langle 0|\left\{(S^\dagger)^{N-1} \left[C_{j_1}^\dagger \times C_{j_2}^\dagger\right]_\mu^{(2)}\right\}^\dagger (S^\dagger)^{N-1} C_{jm}^\dagger C_{j'm'}^\dagger|0\rangle \\ &= \sum_{m,m'} \delta_{m\tilde{m}} \delta_{m'\tilde{m}'} \langle 0|\left\{(S^\dagger)^{N-1} \left[C_{j_1}^\dagger \times C_{j_2}^\dagger\right]_\mu^{(2)}\right\}^\dagger (S^\dagger)^{N-1} C_{jm}^\dagger C_{j'm'}^\dagger|0\rangle \\ &= \langle 0|\left\{(S^\dagger)^{N-1} \left[C_{j_1}^\dagger \times C_{j_2}^\dagger\right]_\mu^{(2)}\right\}^\dagger (S^\dagger)^{N-1} C_{j\tilde{m}}^\dagger C_{j'\tilde{m}'}^\dagger|0\rangle, \end{aligned} \quad (3.3.36)$$

whilst on the lhs of (3.3.35) we get

$$\begin{aligned} &\sum_{K',k'} \langle j\tilde{m}j'\tilde{m}'|K'k'\rangle [\delta_{j_1j} \delta_{j_2j'} - (-1)^{j+j'} \delta_{j_1j'} \delta_{j_2j}] \delta_{2K'} \delta_{\mu k'} \eta_{2N,2,2}^2(j_1j_2) \\ &= \langle j\tilde{m}j'\tilde{m}'|2k\rangle \eta_{2N,2,2}^2(j_1j_2) [\delta_{j_1j} \delta_{j_2j'} - (-1)^{j+j'} \delta_{j_1j'} \delta_{j_2j}]. \end{aligned} \quad (3.3.37)$$

Since (3.3.36) is equal to (3.3.37), we multiply both sides for $\frac{\beta_{j_1j_2}}{\sqrt{1+\delta_{j_1j_2}}}$ obtaining for one side

$$\begin{aligned} &\frac{\beta_{j_1j_2}}{\sqrt{1+\delta_{j_1j_2}}} \eta_{2N,2,2}^2(j_1j_2) [\delta_{j_1j} \delta_{j_2j'} - (-1)^{j+j'} \delta_{j_1j'} \delta_{j_2j}] \\ &= \eta_{2N,2,2}^2(jj') \frac{\beta_{jj'}}{\sqrt{1+\delta_{jj'}}} \delta_{j_1j} \delta_{j_2j'} - (-1)^{j+j'} \eta_{2N,2,2}^2(j'j) \frac{\beta_{jj'} (-1)^{j-j'}}{\sqrt{1+\delta_{j'j}}} \delta_{j_1j'} \delta_{j_2j} \\ &= \eta_{2N,2,2}^2(jj') \frac{\beta_{jj'}}{\sqrt{1+\delta_{jj'}}} [\delta_{j_1j} \delta_{j_2j'} + \delta_{j_1j'} \delta_{j_2j}]. \end{aligned} \quad (3.3.38)$$

In the last expression we used the symmetry property $\eta_{2N,2,2}^2(jj') = \eta_{2N,2,2}^2(j'j)$ and that $\beta_{j'j} = (-1)^{j-j'} \beta_{jj'}$. Now summing over $j_1 \leq j_2$, it follows that one of the δ 's will be always equal to 1, while the other one will become necessarily into a $\delta_{jj'}$. Thus,

$$\begin{aligned} &\sum_{\substack{j_1, j_2 \\ j_1 \leq j_2}} \frac{\beta_{jj'}}{\sqrt{1+\delta_{jj'}}} \eta_{2N,2,2}^2(jj') [\delta_{j_1j} \delta_{j_2j'} + \delta_{j_1j'} \delta_{j_2j}] = \eta_{2N,2,2}^2(jj') \frac{\beta_{jj'}}{\sqrt{1+\delta_{jj'}}} [1 + \delta_{jj'}] \\ &= \eta_{2N,2,2}^2(jj') \beta_{jj'} \sqrt{1+\delta_{jj'}}. \end{aligned} \quad (3.3.39)$$

Thereby we obtain finally

$$\langle 0 | \{ (S^\dagger)^{N-1} D_\mu^\dagger \}^\dagger (S^\dagger)^{N-1} C_{j\tilde{m}}^\dagger C_{j'\tilde{m}'}^\dagger | 0 \rangle = \langle j\tilde{m}j'\tilde{m}' | 2\mu \rangle \eta_{2N,2,2}^2(jj') \beta_{jj'} \sqrt{1 + \delta_{jj'}}. \quad (3.3.40)$$

Thus, taking the Hermitian conjugate of the operators in the overlap, and changing the notation we obtain,

$$\langle 0 | S^{N-1} C_{j-m} C_{JM} D_\mu^\dagger (S^\dagger)^{N-1} | 0 \rangle = - \langle j - mJM | 2\mu \rangle \eta_{2N,2,2}^2(jJ) \beta_{jJ} \sqrt{1 + \delta_{jJ}}. \quad (3.3.41)$$

Replacing (3.3.41) into (3.3.34) and (3.3.33) we obtain that

$$\tilde{\chi}_j^J = \sum_{m,\mu} \langle jm2\mu | JM \rangle \langle j - mJM | 2\mu \rangle N \alpha_j (-1)^{j-m} \eta_{2N,2,2}^2(jJ) \beta_{jJ} \sqrt{1 + \delta_{jJ}}. \quad (3.3.42)$$

Since

$$(-1)^{j-m} \langle jm2\mu | JM \rangle \langle j - mJM | 2\mu \rangle = (-1)^{j+J} \sqrt{\frac{5}{2J+1}} \langle jm2\mu | JM \rangle^2, \quad (3.3.43)$$

it yields to

$$\tilde{\chi}_j^J = (-1)^{j+J} N \alpha_j \eta_{2N,2,2}^2(jJ) \beta_{jJ} \sqrt{\frac{5(1 + \delta_{jJ})}{2J+1}}. \quad (3.3.44)$$

Now χ_j^J is given just for the relation

$$\chi_j^J = \frac{1}{\eta_{2N,1,J} \eta_{2N,2,2}} \tilde{\chi}_j^J. \quad (3.3.45)$$

We still need the normalization constant of $|2N+1, \tilde{\nu}=3, jJ\rangle$. In order to obtain it, we impose that $\omega_{jj}^J = 1$, which leads precisely to normalized $\tilde{\nu}=3$ states. This condition yields to

$$\eta_{2N,3,jJ}^2 = 1 + (\chi_j^J)^2 + \sum_K (-1)^{J+j} \hat{K} \begin{Bmatrix} j & j & K \\ 2 & 2 & J \end{Bmatrix} \langle S^{N-1} D | [c_j^\dagger \times \tilde{c}_j]^{(K)} | | S^{N-1} D \rangle. \quad (3.3.46)$$

Thus, χ_j^J is defined, and we have completely defined $\omega_{j'j}^J$.

As we mentioned before, the states with $\tilde{\nu}=3$ and total angular momentum J are not orthogonal since $\omega_{j'j}^J$ has off diagonal terms different to zero. When we face this situation, it is necessary to carry out a orthogonalization process of each one of these groups of states characterized by a certain angular momentum J . This process can be done at least in two different ways. One of them is the outright known Gram-Schmidt method and the other one is a democratic mapping proposed by Skouras *et al.* [32]. For both cases, a very brief discussion about them will be given in the next sections.

Gram-Schmidt method

Let $|2N+1, \tilde{\nu}=3, \alpha_i JM\rangle_\perp$ ($i = 1, 2, 3, \dots, n$, where n is the number of valence orbitals) be the states of generalized seniority number $\tilde{\nu}=3$ and total angular momentum J orthogonal to each other, that can be constructed from suitable lineal combinations between the states in (3.3.17):

$$|2N+1, \tilde{\nu}=3, \tilde{\alpha}_1 JM\rangle_\perp = P_{11}^J |2N+1, \tilde{\nu}=3, j_1 JM\rangle, \quad (3.3.47)$$

$$|2N+1, \tilde{\nu}=3, \tilde{\alpha}_2 JM\rangle_\perp = P_{21}^J |2N+1, \tilde{\nu}=3, j_1 JM\rangle + P_{22}^J |2N+1, \tilde{\nu}=3, j_2 JM\rangle, \quad (3.3.48)$$

\vdots

$$|2N+1, \tilde{\nu}=3, \tilde{\alpha}_n JM\rangle_\perp = \sum_{k=1}^n P_{nk}^J |2N+1, \tilde{\nu}=3, j_k JM\rangle. \quad (3.3.49)$$

In this way, the P^J matrix is lower triangular by construction and is obtained through the overlap of the non-orthogonal states with the new ones as it is known.

As the components P_{ij}^J are obtained, we have as result the set of orthogonal states with $\tilde{\nu} = 3$ and total angular momentum J . Thus the respective correspondence with the states in the IBFM space may be done as:

$$|2N + 1, \tilde{\nu} = 3, \alpha_i JM\rangle_{\perp} \longrightarrow |2N + 1, \tilde{\nu} = 3, j_i JM\rangle_{\text{IBFM}}. \quad (3.3.50)$$

As it can be seen, the biggest issue when treating this method is that the correspondence is not unique since the choice is arbitrary at the beginning of the procedure. Even though the choice of the very first vector and the following ones may be random, we have $n!$ different possibilities for the number of correspondences. Consequently, in principle the E terms in (3.1.3) would be different for each correspondence. There is, however, a way to obtain this term unambiguously, which will be discussed next.

Democratic Correspondence Method

The method of democratic correspondence was introduced by L.D. Skouras *et al.* [32] as an alternative mapping to the OAI's. This method is based on the properties of the overlap matrix ω^J , whose expression was derived in (3.3.20), in the basis of non-orthogonal Shell Model states onto which the boson states are mapped. In contrast to OAI mapping, which assume a hierarchy of states according to the number of correlated S pairs, this method treats all mapped Shell Model states on an equal footing, i.e., it is democratic in that sense. In this way, we obtain orthogonal states with $\tilde{\nu} = 3$.

Let C^J be the $n \times n$ orthogonal matrix which contains the normalized eigenvectors (in columns) of ω^J , where n is the dimension of the space with $\tilde{\nu} = 3$ and total angular momentum J . Let λ^J be the diagonal matrix with the eigenvalues of ω^J . Thus, it satisfies

$$\omega^J C^J = C^J \lambda^J. \quad (3.3.51)$$

The orthonormal states are constructed as

$$|2N + 1, \tilde{\nu} = 3, \tilde{\alpha}_k JM\rangle_{\perp} = \frac{1}{\sqrt{\lambda_{kk}^J}} \sum_i^n C_{ik}^J |2N + 1, \tilde{\nu} = 3, j_i JM\rangle. \quad (3.3.52)$$

We can already do the correspondence of these states to the IBFM states in the following way:

$$|2N + 1, \tilde{\nu} = 3, \tilde{\alpha}_k JM\rangle_{\perp} \longrightarrow \sum_i^n C_{ik}^J |2N + 1, \tilde{\nu} = 3, j_i JM\rangle_{\text{IBFM}}, \quad k = 1, \dots, n. \quad (3.3.53)$$

where these states create a new set of orthonormal states as C^J is a orthogonal matrix when treating with normalized eigenvectors of ω^J .

At this point, some considerations must be pointed out.

1. When creating the orthogonal set of states in this method, the ambiguity attached in the Gram-Schmidt method disappears since the result of the diagonalization of ω^J is totally independent of the chosen state order before constructing it, thus, the correspondence between the states is unique.
2. As it can be seen from (3.3.53), we cannot work with states of a certain value of the angular momentum of single particle. This also occurred in Gram-Schmidt method where the correspondence is one to one, with boson states of definite angular momentum of single particle. However, the one to one correspondence with the boson state in this method is lost because of the diagonalization of ω^J .

For a better understanding of this orthonormalization method see [30, 32].

Transfer Operator Coefficients in OAI

In order to obtain the value of the coefficients in the general expression (until second order in boson operators) in the transfer operator, which has the form given by equation (3.1.3), we use Eq. (3.3.4) which equals matrix elements of the SM single-nucleon creation operator C_{jm}^\dagger to matrix elements of its image in the IBFM c_{jm}^\dagger . First, we will find those coefficients that can be obtained by lower GS ($\tilde{\nu} \leq 2$) states (3.3.6)-(3.3.8) along with the corresponding boson-unpaired-particle states (3.3.5a)-(3.3.5c), and secondly with states of $\tilde{\nu} > 2$. Since we do not care about the m projection of each state, we use the corresponding reduced matrix elements. For the lower seniority states, the coefficients are given by

$$A_j + NG_j = \frac{\langle S^N, j || C_j^\dagger || S^N \rangle}{{}_B \langle s^N, j || a_j^\dagger || s^N \rangle_B}, \quad (3.3.54)$$

$$B_j = \frac{\langle S^N || C_j^\dagger || S^{N-1}, j \rangle}{{}_B \langle s^N || s^\dagger \tilde{a}_j || s^{N-1}, j \rangle_B}, \quad (3.3.55)$$

$$C_{jj'} = \frac{\langle S^{N-1} D, 2 || C_j^\dagger || S^{N-1}, j' \rangle}{{}_B \langle s^{N-1} d, 2 || [d^\dagger \times \tilde{a}_{j'}]^{(j)} || s^{N-1}, j' \rangle_B}, \quad (3.3.56)$$

$$D_{jj'} = \frac{\langle S^N, j' || C_j^\dagger || S^{N-1} D, 2 \rangle}{{}_B \langle s^N, j' || [s^\dagger [\tilde{d} \times a_{j'}]^{(j)}] || s^{N-1} d, 2 \rangle_B}. \quad (3.3.57)$$

$$(3.3.58)$$

In our first expression we obtain a relation between A_j and G_j while the desirable outcomes is the expression for each coefficient individually, which is clearly not obtained here. An alternative method to obtain G_j would be using the relation

$$\left(s^\dagger \cdot \tilde{s} + d^\dagger \cdot \tilde{d} \right)_\rho = N_\rho, \quad (3.3.59)$$

where N_ρ is the number of ρ bosons, which is fixed for all the states in the model. With this tentative method we redefine the operators whose coefficients are attached to them, that is, we exchange two s bosons into two d bosons coupled to zero. However this method is useless since we can redefine the coefficients in (3.1.3) in the following way

$$A_j \rightarrow A'_j = A_j + G_j N, \quad (3.3.60)$$

$$E_{jj'L} \rightarrow E'_{jj'L} = E_{jj'L} - G_j \sqrt{5} \delta_{L0}. \quad (3.3.61)$$

Through this redefinition, we are setting $\mathfrak{C}_{jj'}^{000} = 0$. Clearly, this method gives only the values for our redefined coefficients, which depend on G_j . Thus, in OAI we cannot obtain an explicit expression for G_j , but, as it's seen, it is unimportant and also could be set to zero. Application of this is shown in section 4.2.1.

On the other hand, $F_{jj'}$ and $E_{jj'L}$ can only be found by matrix elements where states of $\tilde{\nu} > 2$ are involved. In this mapping

$$F_{jj'} = \frac{\langle S^{N-1} D; j' || C_j^\dagger || S^N \rangle}{\langle s^{N-1} d; j' || [[d^\dagger \tilde{s}]^2 a_{j'}]^{(j)} || s^N \rangle_B}, \quad (3.3.62)$$

through the bracket between a state of $\tilde{\nu} = 0$ and another state of $\tilde{\nu} = 3$. Since C_j^\dagger connects states with $\Delta\tilde{\nu} = 1$, this matrix element is null, that is, in OAI

$$F_{jj'} = 0, \quad \forall j \text{ of the valence orbits.} \quad (3.3.63)$$

Finally, the remaining term $E_{jLj'}$ is obtained by analogy with the orthonormalized states with the democratic correspondence method. The way for obtaining this coefficient is extensively discussed in [30], however it's obtained within the NOA frame. We did the same procedure

without NOA and obtained the coefficient exactly. Also, the coefficients A_j - $D_{jj'}$ are also obtained without NOA and with the same states in [25], therefore we proceed to show the results already for each coefficient,

$$A'_j = \frac{\eta_{2N,1,j}}{\eta_{2N,0,0}}, \quad (3.3.64a)$$

$$B_j = \sqrt{N}\alpha_j \frac{\eta_{2(N-1),1,j}}{\eta_{2N,0,0}}, \quad (3.3.64b)$$

$$C_{jj'} = \frac{\sqrt{5}}{\hat{j}} \frac{\eta_{2N,2,2}^2(jj')}{\eta_{2N,2,2}\eta_{2(N-1),1,j'}} \beta_{j'j} \sqrt{1 + \delta_{jj'}}, \quad (3.3.64c)$$

$$D_{jj'} = -\frac{\sqrt{5}}{\hat{j}} \frac{\eta_{2N,2,2}^2(jj')}{\eta_{2N,2,2}\eta_{2N,1,j'}} \sqrt{N}\alpha_j \beta_{j'j} \sqrt{1 + \delta_{jj'}}, \quad (3.3.64d)$$

$$F_{jj'} = 0, \quad (3.3.64e)$$

$$E'_{jj'L} = (-1)^{L+j+j'} \sum_J (2J+1) \frac{\hat{L}}{\hat{j}} \left\{ \begin{matrix} 2 & 2 & L \\ j & j' & J \end{matrix} \right\} A'_j [\delta_{jj'} - \{C^J \sqrt{\lambda^J} (C^J)^{-1}\}_{jj'}]. \quad (3.3.64f)$$

The matrices C^J and λ^J are the same of Eq. (3.3.51). Also it is interesting to see that (3.3.64a)-(3.3.64d) are related by

$$\alpha_{j'} A'_j D_{jj'} + \alpha_j B_{j'} C_{jj'} = 0. \quad (3.3.65)$$

The values of α and β structure constants are needed because they appear in other calculations, like probability occupations, which is treated in chapter 5, and β -decay [33] among others.

3.3.2 GHP mapping

The GHP scheme (sometimes also called Beliaev-Zelevinsky expansion) is an operators mapping where the operator commutation relation is preserved, i.e. the operators algebra for all fermion operators is conserved. Strictly speaking, a systematic perturbation expansion is done on a small parameter, and the structure is determined by requiring that any commutation rule must be fulfilled in each order of the expansion. The lowest order are the most important because of its physical interpretation, while the higher orders give rise to anharmonicities [34]. In this mapping the usual properties on operators are fulfilled,

$$(c_j^\dagger c_i^\dagger)^\dagger_{\text{GHP}} = (c_i c_j)_{\text{GHP}} \quad , \quad (c_j^\dagger)^\dagger_{\text{GHP}} = (c_j)_{\text{GHP}}. \quad (3.3.66)$$

In order to preserve the annihilation-creation fermion operators algebra, the transfer operator has the form

$$(c_j^\dagger)_{\text{GHP}} = \sum_i a_i^\dagger \left(\sqrt{\mathbb{I} - (B^\dagger B)^T} \right)_{ij} + \sum_i B_{ji}^\dagger a_i, \quad (3.3.67)$$

where B is the matrix of operators B_{ij} , the boson annihilation operators, which takes the place of the fermion pair $C_i C_j$. T operation represents the transpose of the matrix. The square-root operator in (3.3.67) is the hallmark of GHP expansions. This square root must be expanded in a Taylor series in order to be used. When the corresponding boson operators are written as collective boson operators, the coefficients of (3.1.3) are obtained and are given in [12]:

$$\mathfrak{A}_j = u_j \left(1 + \frac{v_j^2}{2u_j^2} \right), \quad (3.3.68a)$$

$$\mathfrak{B}_{jj'}^L = \mathbf{X}_{j'j}^L \frac{\hat{L}}{\hat{j}}, \quad (3.3.68b)$$

$$\mathfrak{C}_{jj'}^{l'l} = -\frac{1}{2u_j} \sum_{j''} (-1)^{j'+j''} \mathbf{X}_{jj''}^l \mathbf{X}_{j'j''}^{l'} \frac{\hat{l}' \hat{L}}{\hat{j}} \left\{ \begin{matrix} j & j' & L \\ l' & l & j'' \end{matrix} \right\}, \quad (3.3.68c)$$

where $X_{jj'}^l$ are structure coefficients of the collective boson operators used in GHP [12], and $v_j^2(u_j^2)$ is the occupation(vacancy) probability of the single-particle state for an orbit j ($v_j^2 + u_j^2 = 1$).

This mapping is suitable for deformed nuclei [35,36], where GS is not a good quantum number and OAI becomes worse since GS breaks down in deformed nuclei [17]. Also we would like to point out that relations between both mapping may be obtained [37].



Chapter 4

Single j -shell case

In this chapter we want to study the quadrupole-quadrupole operator of Eq. (3.2.37) in the case of a single j -shell, that is the case when all valence orbits considered are degenerate, i.e., they have the same energy, and the nucleons have the same probability occupation in any of them. This case is important for two reasons, first, the expressions of chapter 3 can be analytically treated since they get reduced in a simple and tractable form. And secondly, the aforementioned expressions in the single j -shell are known. Therefore we can compare both results from our general case to the specific case.

Since the coefficients of the boson-fermion interaction depend on the chosen scheme, we consider the application of the OAI and GHP mappings to study the interaction restricted to a single j -shell in the coefficients listed in (3.3.64a)-(3.3.64f), and (3.3.68a)-(3.3.68c), respectively. After expressing them in the single j -shell case, we'll compare, in both mappings, the behaviour of the Γ , Λ and Δ coefficients of the direct and exchange interactions, respectively. Finally, we'll set a toy Hamiltonian in order to see the behaviour, features and the effects of each one in the energy spectrum, along with the spectrum *per se*.

It is known that the single j -shell model has some deficiencies [23] since there is only one pair for each angular momentum, and also the value of j is usually taken to be large in order to represent a large shell, therefore there is a possibility that some uncanny effects appear because of the great value of j which do not appear in realistic cases.

4.1 Boson-Fermion Interaction

In the single j -shell case, we cannot sum over any j since all of them are equivalent, therefore our transfer operator reduces to

$$\begin{aligned} c_{jm}^\dagger &= A_j a_{jm}^\dagger + B_j [s^\dagger \times \tilde{a}_j]_m^{(j)} + C_{jj} [d^\dagger \times \tilde{a}_j]_m^{(j)} \\ &+ D_{jj} \left[[s^\dagger \times \tilde{d}]^{(2)} \times a_j^\dagger \right]_m^{(j)} + \sum_L E_{jjL} \left[[d^\dagger \times \tilde{d}]^{(L)} \times a_j^\dagger \right]_m^{(j)} \\ &+ F_{jj} \left[[d^\dagger \times \tilde{s}]^{(2)} \times a_j^\dagger \right]_m^{(j)} + G_j \left[[s^\dagger \times \tilde{s}]^{(0)} \times a_j^\dagger \right]_m^{(j)} + \dots \end{aligned}$$

With this single-nucleon transfer operator, our quadrupole-quadrupole boson-fermion interaction becomes

$$V_{BF}^D = \Gamma_{jj} \left(Q_{\rho'}^{(2)} \cdot [a_j^\dagger \times \tilde{a}_j]_{\rho}^{(2)} \right), \quad (4.1.1a)$$

$$V_{BF}^A = - \sum_{l,l',L,L'} \Lambda_{jjj}^{l'l'L} \sqrt{5} \hat{j} \left\{ \begin{matrix} j & 2 & j \\ L & j & L' \end{matrix} \right\} (-1)^{l+l'} \left([Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)}]^{(L')} \cdot [a_j^\dagger \times \tilde{a}_j]_{\rho}^{(L')} \right) + \text{H.c.}, \quad (4.1.1b)$$

$$V_{BF}^B = \sum_{l',s,L's} \Delta_{jjj}^{l'l'} (-1)^{l+l'-L-L'} (2j+1) \hat{L} \sqrt{5} \left\{ \begin{matrix} l' & l & L \\ j & j & L' \\ j & j & 2 \end{matrix} \right\} \left([Q_{\rho'}^{(2)} \times \mathcal{B}_{l'l,\rho}^{(L)}]^{(L')} \cdot [a_j^\dagger \times \tilde{a}_j]_{\rho}^{(L')} \right). \quad (4.1.1c)$$

where the expressions for the coefficients in the single j -shell may be taken from (3.2.6f),(3.2.6g) and (3.2.30). Also, for studying the trends of these coefficients, we will consider the following expressions,

$$\Gamma_{jj} = Q_{jj} A_j^2, \quad (4.1.2a)$$

$$\Lambda_{jjj} := \sum_{l',l,L} \Lambda_{jjj}^{l'l'L} = Q_{jj} A_j \left[D_{jj} + F_{jj} + G_j + \sum_L E_{jjL} \right], \quad (4.1.2b)$$

$$\Delta_{jjj} := \sum_{l'} \Delta_{jjj}^{l'l'} = -Q_{jj} [B_j + C_{jj}]^2. \quad (4.1.2c)$$

4.2 Coefficients in OAI in a Single j -shell

All the coefficients of the transfer operator (3.1.3) (except the E coefficient) in OAI are defined according to the normalization constants of the lower generalized seniority states. In this section we'll obtain the value of this normalization constants in the single j -case.

4.2.1 Transfer operator coefficients

We take from the definition (3.3.12) the normalization constant of the state with $\tilde{\nu} = 0$,

$$\eta_{2N,0,0}^2 = [N!]^2 \sum_{\substack{m_1, m_2, \dots, m_k \\ \sum_{i=1}^k m_i = N}} \prod_{i=1}^k \alpha_i^{2m_i} \binom{\Omega_i}{m_i},$$

we must stress that in a single j -shell all valence orbits are degenerate and all nucleons have the same occupation probability in any orbits, therefore $\alpha_j = \beta_{jj} = 1$, so the sum and the product vanish. This yields to

$$\eta_{2N,0,0}^2 = N!^2 \binom{\Omega_j}{N},$$

on the other hand, the $\tilde{\nu} = 1$ state normalization constant reads

$$\begin{aligned} \eta_{2N,1,j}^2 &= (-1)^N \alpha_j^{2N} (N!)^2 + \sum_{m=1}^N \eta_{2m,0,0}^2 (-1)^{N-m} \left[\frac{N!}{m!} \right]^2 \alpha_j^{2(N-m)} \\ &= (-1)^N (N!)^2 + \sum_{m=1}^N (m!)^2 \binom{\Omega_j}{m} (-1)^{N-m} \left(\frac{N!}{m!} \right)^2 \\ &= (-1)^N (N!)^2 \sum_{m=0}^N (-1)^m \binom{\Omega_j}{m}. \end{aligned} \quad (4.2.1)$$

It's easy to prove that for any $0 \leq m \leq n$

$$\sum_{m=0}^K \binom{n}{m} (-1)^m = (-1)^K \binom{n-1}{K}, \quad \text{for } 0 \leq K \leq n-1, \quad (4.2.2)$$

then, it's straightforward to obtain

$$\eta_{2N,1,j}^2 = (N!)^2 \binom{\Omega_j - 1}{N}. \quad (4.2.3)$$

The last normalization constant that we need for obtaining the coefficients is only $\eta_{2N,2,2}(jj)$. Taking out from its definition we got

$$\begin{aligned} \eta_{2N,2,2}^2(jj) &= \sum_{p=0}^{N-1} \left[\frac{(N-1)!}{p!} \right]^2 (-1)^{N-1-p} \eta_{2p,0,0}^2 \sum_{q=0}^{N-1-p} \alpha_j^{n-2-2p-2q} \alpha_j^{2q} \\ &= \sum_{p=0}^{N-1} [(N-1)!]^2 (-1)^{N-1-p} \binom{\Omega_j}{p} (N-p) \\ &= [(N-1)!]^2 \left[N \sum_{p=0}^{N-1} (-1)^{N-1-p} \binom{\Omega_j}{p} - \sum_{p=0}^{N-1} (-1)^{N-1-p} p \binom{\Omega_j}{p} \right] \end{aligned}$$

by using again the relation (4.2.2) and the property of the binomial coefficients that

$$p \binom{\Omega_j}{p} = \Omega_j \binom{\Omega_j - 1}{p-1},$$

then, we obtain

$$\begin{aligned} \eta_{2N,2,2}^2(jj) &= [(N-1)!]^2 \left[N \binom{\Omega_j - 1}{N-1} - \Omega_j \binom{\Omega_j - 2}{N-2} \right] \\ &= [(N-1)!]^2 \frac{\Omega_j - N}{\Omega_j - 1} \binom{\Omega_j - 1}{N-1} \\ &= (N-1)! \frac{(\Omega_j - 2)!}{(\Omega_j - N - 1)!}. \end{aligned}$$

Also, since there is no sum over any j , we obtain

$$\eta_{2N,2,2}^2 = \eta_{2N,2,2}^2(jj) \quad (4.2.4)$$

In summary we have finally,

$$\eta_{2N,0,0}^2 = N!^2 \binom{\Omega_j}{N}, \quad (4.2.5a)$$

$$\eta_{2N,1,j}^2 = N!^2 \binom{\Omega_j - 1}{N}, \quad (4.2.5b)$$

$$\eta_{2N,2,2}^2 = \eta_{2N,2,2}^2(jj) = (N-1)! \frac{(\Omega_j - 2)!}{(\Omega_j - N - 1)!}, \quad (4.2.5c)$$

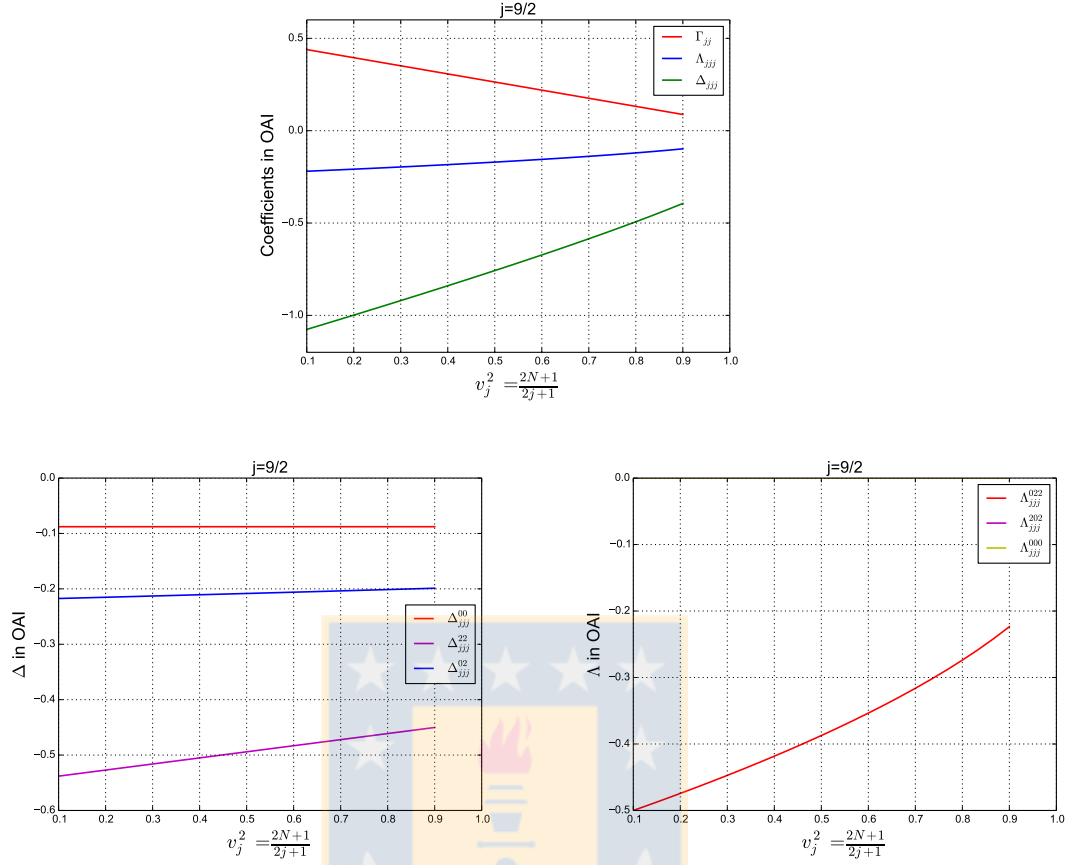


Figure 4.1: Values of Γ_{jj} , Λ_{jjj} and Δ_{jjj} . Also, in the above graphics at the right, $\Lambda_{jjj}^{000} = \Lambda_{jjj}^{202} = \Lambda_{jjj}^{22L} = 0$.

one readily notice that E'_{jjL} is exactly zero. Therefore, in this case the transfer operator reduces to

$$\begin{aligned}
c_{jm}^\dagger = & \sqrt{\frac{\Omega_j - N}{\Omega_j}} a_{jm}^\dagger + \frac{1}{\sqrt{\Omega_j}} [s^\dagger \times \tilde{a}_j]_m^{(j)} + \frac{1}{j} \sqrt{10} \sqrt{\frac{\Omega_j - N}{\Omega_j - 1}} [d^\dagger \times \tilde{a}_j]_m^{(j)} \\
& - \frac{1}{j} \sqrt{\frac{10}{\Omega_j - 1}} [s^\dagger \times \tilde{d}]^{(2)} \times a_j^\dagger \Big|_m^{(j)}. \tag{4.2.11}
\end{aligned}$$

We can see the trends of the coefficients defined in (4.1.2a), (4.1.2b) and (4.1.2c), $\Lambda_{jjj}^{l'l}$ and of $\Delta_{jjj}^{l'l}$ for the case of $j = 9/2$ in fig. 4.1, where we plot the coefficients in relation to the odd nucleon occupation probability v_j^2 which is defined as

$$v_j^2 = \frac{\langle n \rangle}{2j+1} = \frac{2N+1}{2j+1}, \tag{4.2.12}$$

where $\langle n \rangle$ is the expectation value of the number operator. Since we're dealing with an odd nucleon, $\langle n \rangle = 2N + 1$, where N is the number of bosons whose minimum value is zero, and maximum value is $j - 1/2$. For convenience we will consider the occupation probability as our independent variable, which takes the values

$$1/(2j+1) \leq v_j^2 \leq 1 - 1/(2j+1).$$

4.3 Coefficients in GHP

In the case of GHP mapping, there is no null coefficient as can be seen from (3.3.68a)-(3.3.68c). Moreover, the only reduction of them is $\mathbf{X}_{jj}^L = \sqrt{2}$ [12]. Therefore the transfer operator coefficients take the form

$$\mathfrak{A}_j = u_j \left(1 + \frac{v_j^2}{2u_j^2} \right), \quad (4.3.1a)$$

$$\mathfrak{B}_{jj}^L = \sqrt{2} \frac{\hat{L}}{\hat{j}}, \quad (4.3.1b)$$

$$\mathfrak{C}_{jj}^{l'l} = \frac{1}{u_j} \frac{\hat{l}\hat{L}}{\hat{j}} \begin{Bmatrix} j & j & L \\ l' & l & j \end{Bmatrix}. \quad (4.3.1c)$$

We see that $\mathfrak{B}_{jj}^0 = \Omega_{jj}^{-1/2}$ in OAI and GHP. So we expect that V_{BF}^B does not vary much between GHP and OAI. The plots of (4.1.2a)-(4.1.2c), $\Lambda_{jj}^{l'l}$ and of $\Delta_{jjj}^{l'l}$ in GHP are shown in fig. 4.2.

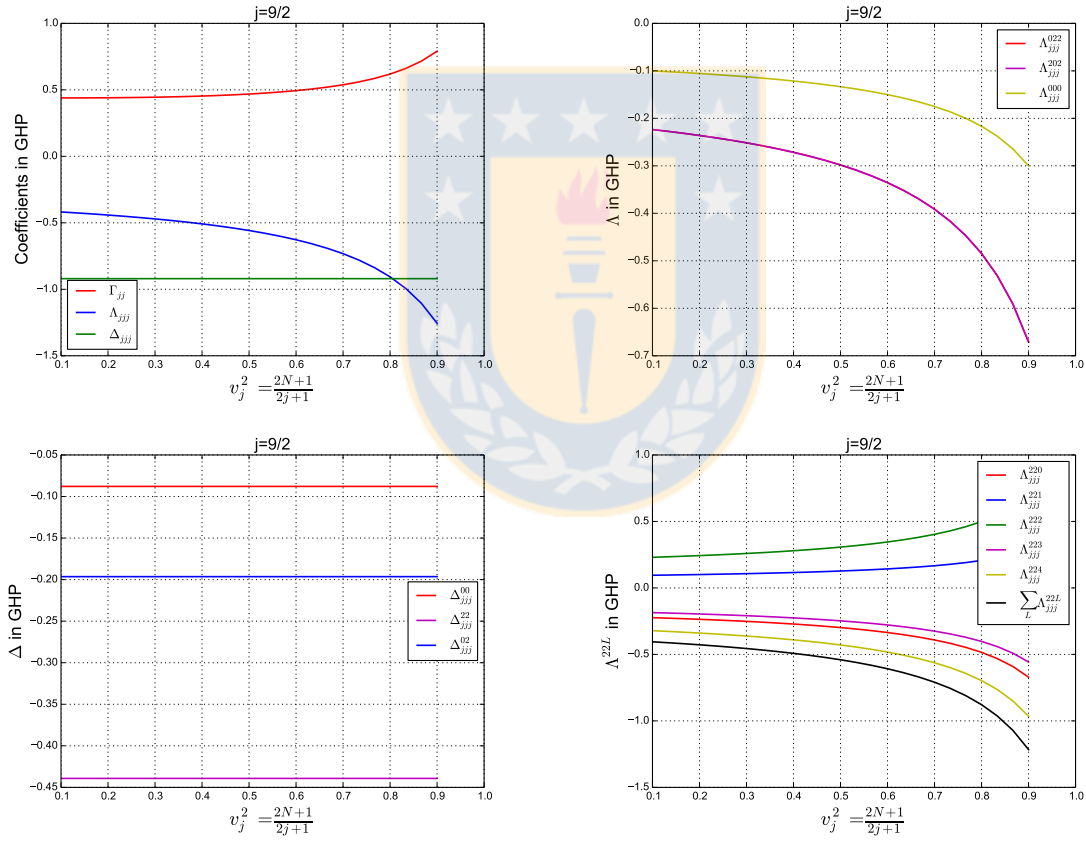


Figure 4.2: Same that fig. 4.1 including Λ_{jjj}^{22L} which are not null (contrary to OAI scheme).

4.4 Study of the Boson-Fermion Interaction in OAI and GHP

In order to study the derived boson-fermion interaction V_{BF} and separately the direct and the exchange terms in (4.1.1a)-(4.1.1c), we will consider a simple Hamiltonian where the energy

of the d bosons is the same for protons and neutrons. This term in the Hamiltonian plays the role of a monopole interaction, and the quadrupole-quadrupole interaction acts between the odd nucleon, which we chose as a neutron arbitrarily, and the alike bosons, protons in our case.

$$H = \epsilon \sum_{p=\pi,\nu} \hat{n}_{d_p} + \kappa \left(Q_\pi^{(2)} \cdot q_\nu^{(2)} \right). \quad (4.4.1)$$

We have set $\epsilon = 0.7$ MeV, $\chi_\pi = -\sqrt{7}/2$ and $N_\pi = 5$ bosons in the core. We have done calculations for $\kappa = -0.2$ and -0.5 MeV only. In the case $\kappa = 0$ there is no boson-fermion interaction since it would be the extreme case of a harmonic oscillation case which considers a spherical nucleus. For lower to higher values of κ we treat a nucleus from spherical to transitional and finally a deformed nucleus. The Hilbert space of the states is given by

$$H_{BF} = H_\pi^B \otimes H_\nu^B \otimes H_\nu^F, \quad (4.4.2)$$

where H_ρ^K is the Hilbert space for bosons of type ρ ($K = B$)/fermions ($K = F$). We have considered for this study up to eight basis vectors shown below¹,

$$|\psi_1; jm\rangle = |s_\pi^{N_\pi} s_\nu^{N_\nu} j_\nu; jm\rangle, \quad (4.4.3)$$

$$|\psi_2 j; JM\rangle = \left[| [s_\pi^{N_\pi} \times (ds^{N_\nu-1})_\nu]^{(2)} \rangle \times |j_\nu\rangle \right]_M^{(J)}, \quad (4.4.4)$$

$$|\psi_3 j; JM\rangle = \left[| [(ds^{N_\pi-1})_\pi \times s_\nu^{N_\nu}]^{(2)} \rangle \times |j_\nu\rangle \right]_M^{(J)}, \quad (4.4.5)$$

$$|\psi_4(L)j; JM\rangle = \left[| [(ds^{N_\pi-1})_\pi \times (ds^{N_\nu-1})_\nu]^{(L)} \rangle \times |j_\nu\rangle \right]_M^{(J)} \quad (4.4.6)$$

As in $|\psi_4(L)j; JM\rangle$ we have coupled d bosons of π and ν , the M-scheme assures us that the possible angular momentum coupling values are $0 \leq L \leq 4$.

The diagonalization of the hamiltonian (4.4.1) through the states of generalized seniority $\tilde{\nu} \leq 5$ up to two d bosons listed above for OAI and GHP mappings of the coefficients brings us the spectrum of it, which is obtained for $j = \frac{5}{2}, \frac{9}{2}$ and $\frac{13}{2}$ angular momenta and for $\kappa = -0.2$ and -0.5 MeV. The excitation energies for different final angular momentum J are shown in figure 4.3 in terms of the neutron occupation probability v_j^2 defined in (4.2.12). Also, we plot the result of calculations considering each contribution to the boson-fermion interaction separately (we set two of the three interaction terms to zero in (4.4.1)) in figures 4.4 to 4.9. In each figure the spectrum is given for final states $J = \pm 1$ and ± 2 only since the states $J = \pm 3$ and ± 4 are higher in energy at least by $\Delta E = \epsilon$ with respect to the former states. In all these six cases the lowest energy state is $J = j$, i.e., the ground state. It is worthy to mention that the so-called $J = j - 1$ anomaly [39], that is, when the first excited state has $J = j - 1$ is very well seen. However, the limit situation when the ground state is $J = j - 1$ is not reproduced.

In figure 4.3 we can see a general trend in which the energy differences among levels decrease as the value of j increases and, at the same time, their distances from the ground state increase, i.e., the levels become more excited. When we compare the results for both values of the strength κ , we observe that the separation between states is larger when $|\kappa|$ increases, since the weak coupling scheme becomes less important. In the OAI case with $\kappa = -0.2$ MeV, all the states are very close in energy having an energy gap $\Delta E \approx 0.1$ MeV. However, when $\kappa = -0.5$ MeV, $\Delta E \approx 0.4$ MeV for $j = 5/2$ and decreases to $\Delta E \approx 0.2$ MeV for $j = 13/2$. In the GHP case the energy difference ΔE is always bigger than OAI. There is also an increase in the excitation energy of the levels when the value of j increases and specially when the occupation probability in the orbit approaches its maximum value $\max\{v_j^2\}$. This effect is much pronounced in GHP than in OAI case and can be understood naively because of the predominance of collective degrees of freedom as the nucleus becomes more deformed as a consequence of the increasing

¹The addition of the last wave functions would be enough to describe the interactions in the Hamiltonian, since the matrix elements obtained with them reach the order 10^{-3} in comparison to those matrix elements obtained by the first three wave functions.

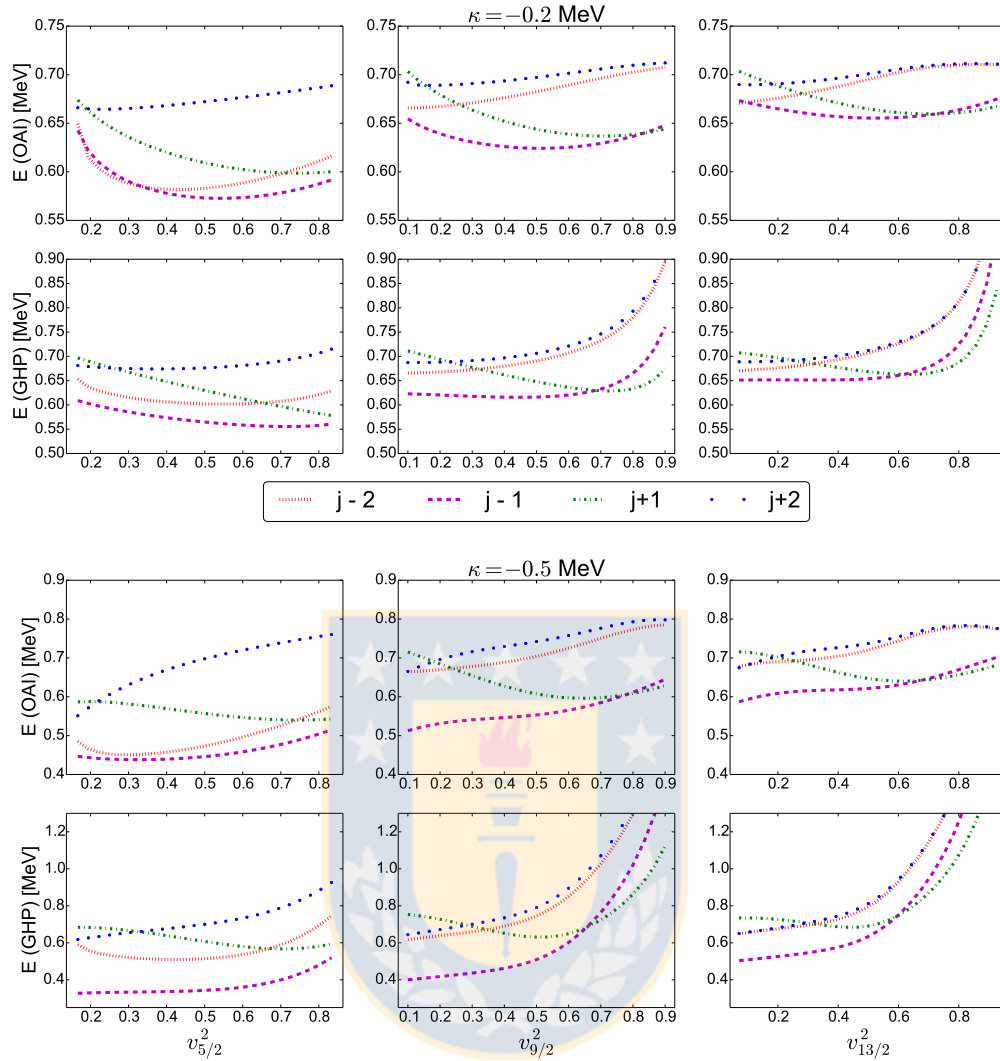


Figure 4.3: **Boson-fermion interaction energy spectrum in single j case:** Plot for $j = \frac{5}{2}, \frac{9}{2}$ and $\frac{13}{2}$ with $\kappa = -0.2$ MeV on the top and $\kappa = -0.5$ MeV on the bottom. The different colors stand for $J = j - 2, j - 1, j + 1$ and $j + 2$. The ground state is fixed to zero, and is not shown to visualize better the energies.

boson-fermion interaction strength and the number of particles that interact. The $J = j \pm 2$ levels are closer in energy and in the particular case of GHP they are almost degenerate when j increases. This effect is present also in the OAI case, but at a lesser extent. Finally, there are some level crossings which take place at smaller values of the occupation probability as the value of j increases between the levels with $J = j - 2$ and $J = j + 1$. For larger values of the occupation probability the crossings occur between $J = j - 1$ and $J = j + 1$ levels.

It is quite interesting to notice that for a low strength and below $\max\{v_j^2\}/2$, GHP and OAI show almost the same behaviour in the energy levels, which indicates that for vibrational nuclei both mapping work on the same footing. These results do not apply when the orbit is very filled or when $|\kappa|$ is not quite small, that is, when the nuclei are very deformed.

In figures 4.4 to 4.9 the contributions of the different terms in V_{BF} separately are shown for the six cases mentioned above. In general terms, the behaviour is rather different between OAI

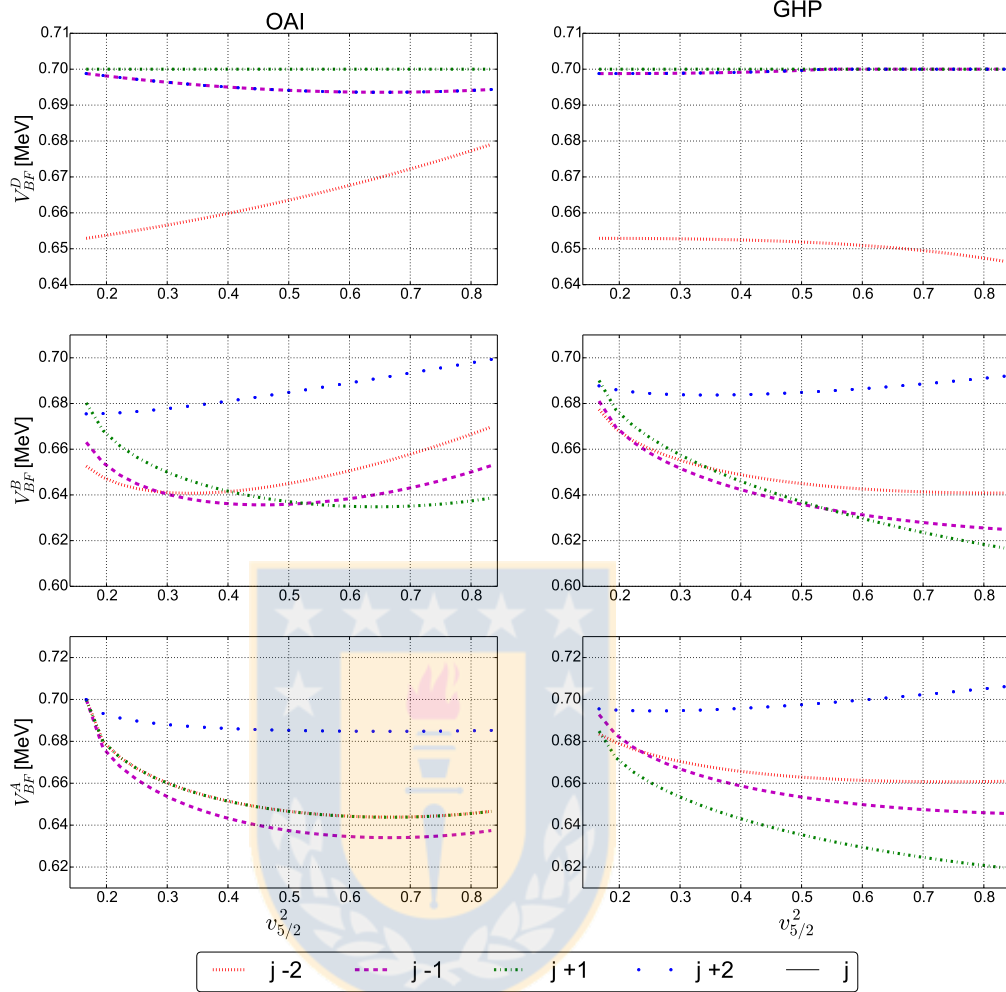


Figure 4.4: Direct and exchange interactions for $\kappa = -0.2$ and $j = 5/2$ in OAI and GHP.

and GHP, except for V_{BF}^B , where both approaches provide very similar results, since $\Delta_{jjjj(\text{OAI})}^{u'} \approx \Delta_{jjjj(\text{GHP})}^{u'}$. When j is high, an appreciable gap between $J = \pm 2$ levels (which are the highest in energy) with $J = \pm 1$ appears, which get reduced when $|\kappa|$ increases. Also, in all cases for OAI and GHP there is a level crossing between $J = j - 1$ and $J = j + 1$. In the case of the direct term, V_{BF}^D , a degeneration among some levels with different J happens for both OAI and GHP. The level with $J = j - 2$ is the only one which escapes from the degeneration above mentioned, but the values of its energy differ between OAI and GHP. Also the $J = j + 2$ level loses the degeneration condition in OAI and GHP. We can see that the exchange term V_{BF}^A is responsible for the observed rise or the level energies when the occupancy increases in the GHP case. The origin of this behaviour is in the presence of factors of the type $\sim u_j^{-1}$ in the coefficients of the GHP expansion in the one-nucleon transfer operator. In the OAI case the dependence is different and the energy spectra are bounded. This term remains essentially constant for any J .

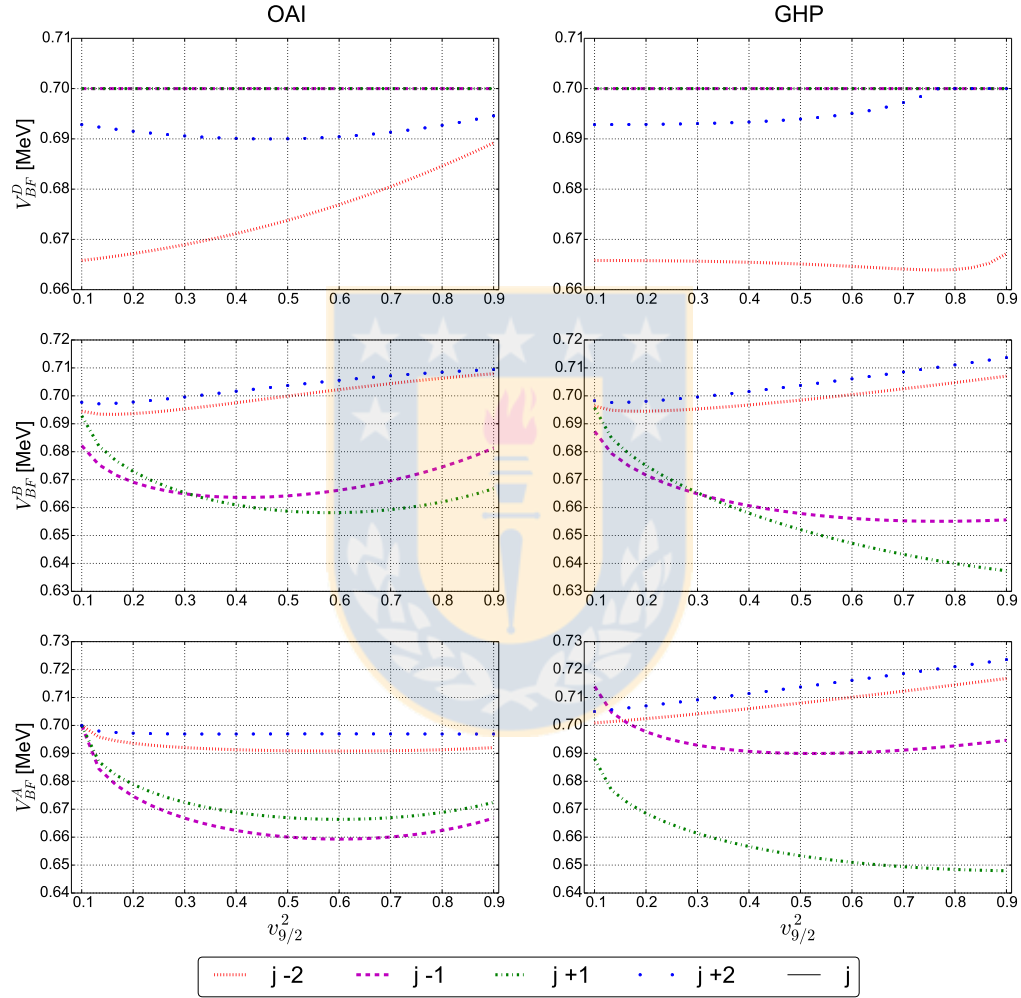


Figure 4.5: Direct and exchange interactions for $\kappa = -0.2$ and $j = 9/2$ in OAI and GHP.

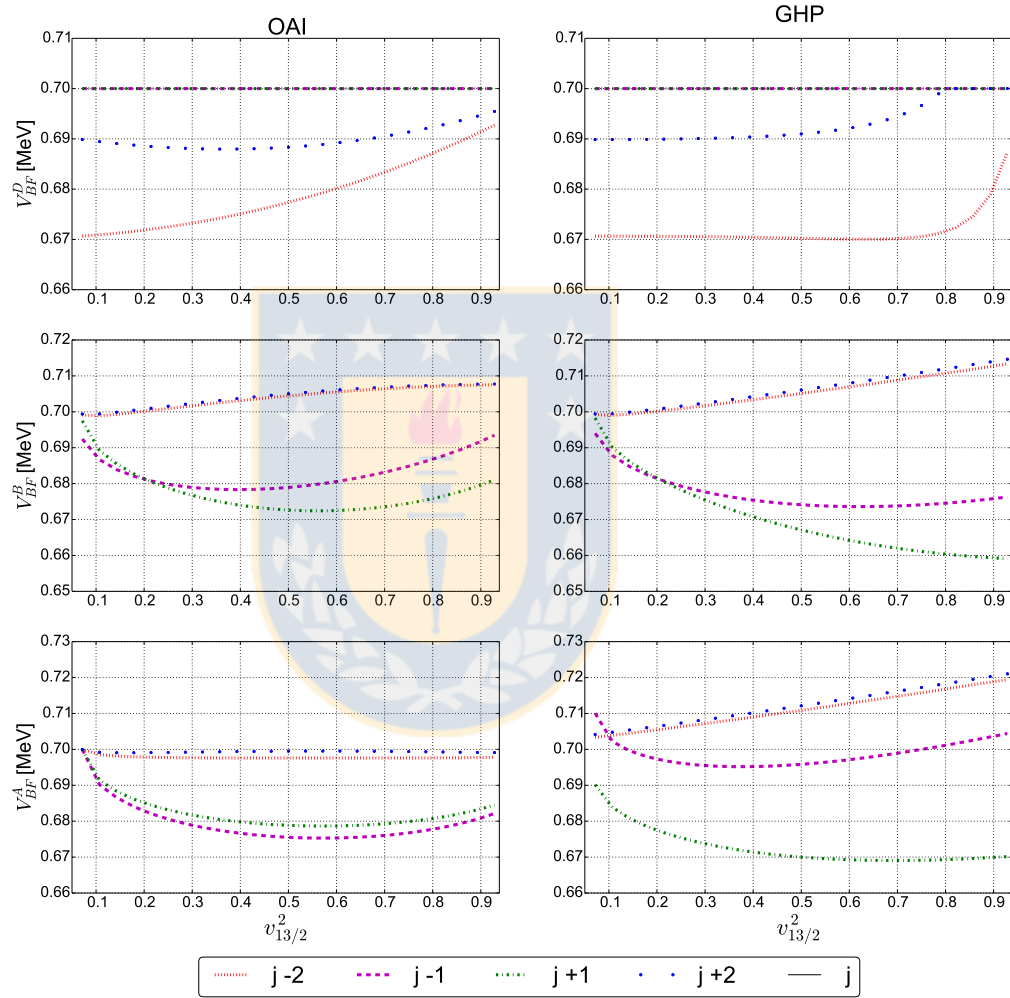


Figure 4.6: Direct and exchange interactions for $\kappa = -0.2$ and $j = 13/2$ in OAI and GHP.

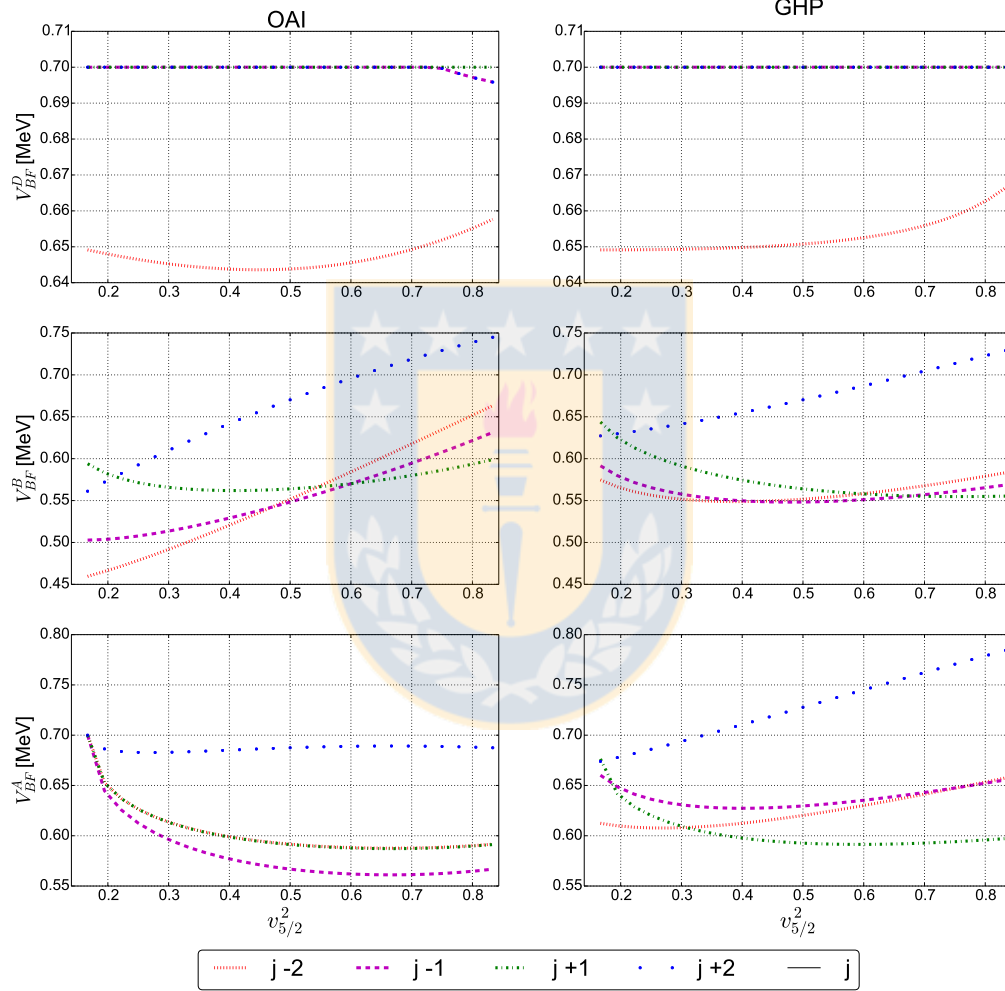


Figure 4.7: Direct and exchange interactions for $\kappa = -0.5$ and $j = 5/2$ in OAI and GHP.

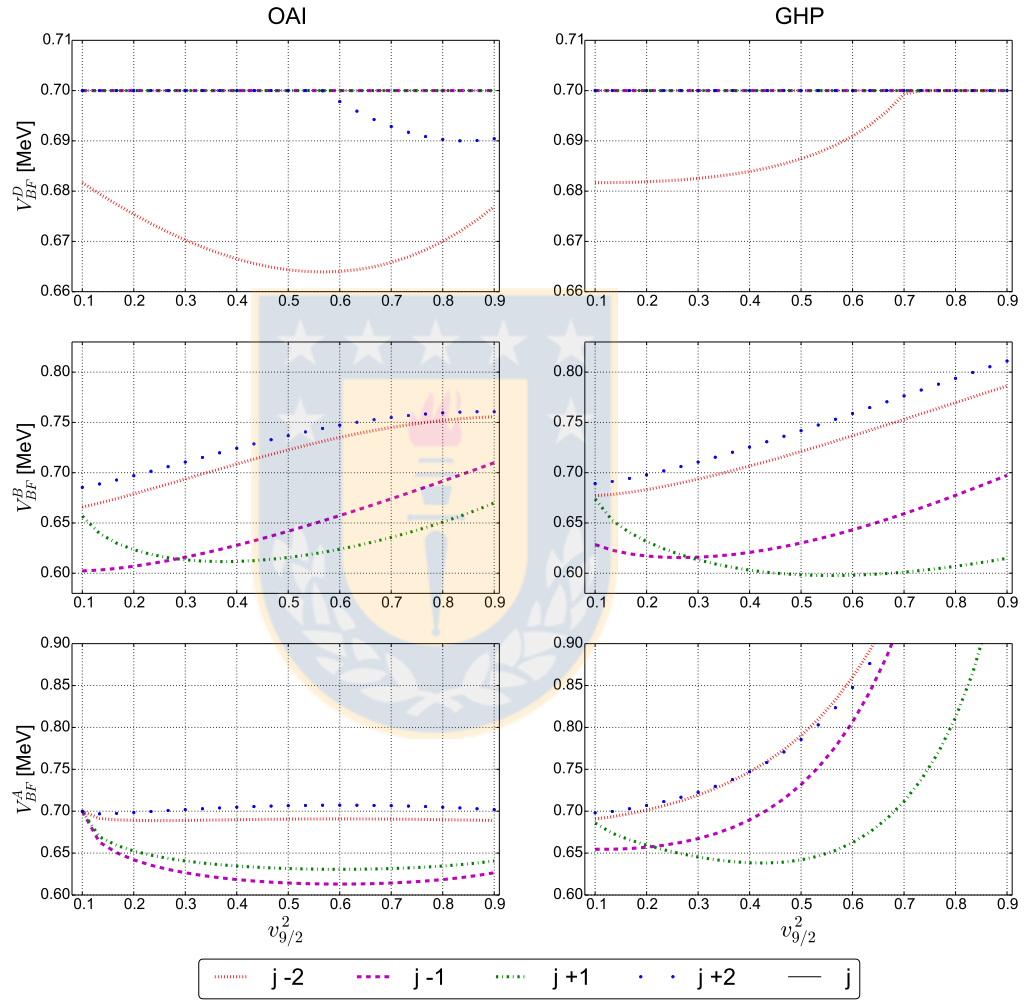


Figure 4.8: Direct and exchange interactions for $\kappa = -0.5$ and $j = 9/2$ in OAI and GHP.

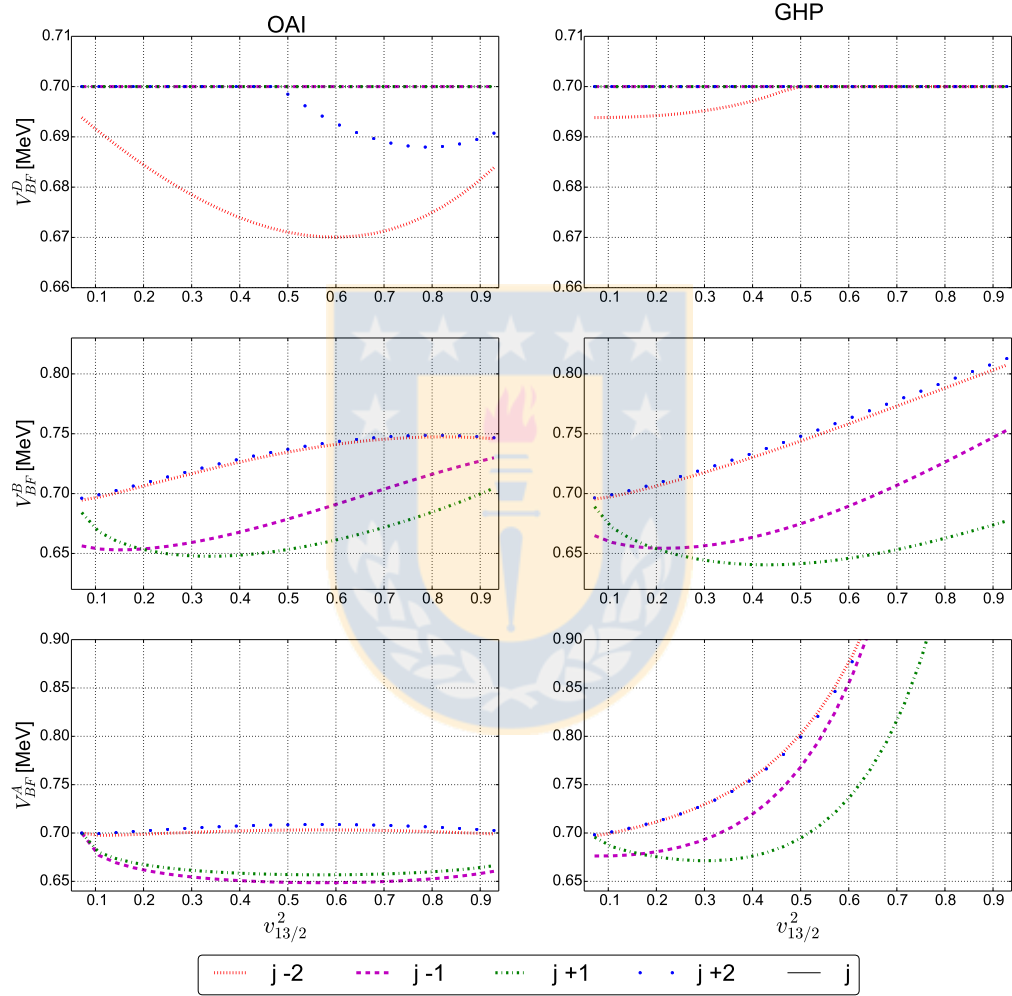


Figure 4.9: Direct and exchange interactions for $\kappa = -0.5$ and $j = 13/2$ in OAI and GHP.

Chapter 5

Structure of S and D nucleon pairs

As it was shown in chapter 3, the structure constants, i.e., α 's and β 's are needed in the definition of the correlated pair-creation operators S^\dagger and D^\dagger to construct the S and D pairs. Their knowledge is crucial to calculate some quantities of interest within the IBM, and also the boson-fermion interaction depends on them directly in the IBFM. There are several procedures to obtain their values, and depend mainly on the nucleus under study. One procedure for getting the α 's and β 's is to diagonalize a Hamiltonian in the space of two identical particles (holes) and to retain the lowest 0^+ and 2^+ eigenvectors [27]. Then we find the structure coefficient by imposing that the $2_1^+-0_1^+$ energy is equal to the experimental one in the corresponding nucleus with two valence particles (holes). Such a procedure neglects possible renormalization (polarization) effects that can arise in many-particle systems due to interactions between the pairs [40]. A proper treatment of such a renormalization effects requires a more complex variational principle. This kind of treatment was studied and applied in realistic cases by Yoshinaga *et al.* [17, 23, 41], where the α 's are found by requiring that

$$\delta \langle S^N | H | S^N \rangle = 0, \quad (5.0.1)$$

and the β 's are determined by requiring

$$\delta \langle S^{N-1} D | H | S^{N-1} D \rangle = 0, \quad (5.0.2)$$

where H may be the original SM Hamiltonian, or a schematic Hamiltonian up to two-body interactions (including one-body term). In the many j -shells case, this procedure is impossible to be solved by hand. Therefore it is resolved using numerical methods.

Another attractive idea is the one put forth by Klein and Vallieres [42] in the 80's, whereby the correlated pairs are chosen so as to minimize the trace of the Hamiltonian in the SD collective subspace, i.e., they impose that

$$\delta(\text{Tr}H) = 0 \quad (5.0.3)$$

where H is the Hamiltonian over the subspace considered. The reason is that they claim that the S and D pairs of the SM subspace have an average energy which is lower than that of the remaining states of the given nucleus. As the average energy of a set of states is proportional to the trace of the Hamiltonian over the subspace considered, Eq. (5.0.3) holds. The advantage of this method is that the invariance of the trace under a change of basis allows us to implement (5.0.3) without having to know the exact eigenstates. In addition, this trace variational principle is directly related with the time-invariance principle of the Schrödinger Equation as a basis for a theory of collective motion according to Klein [43].

The implementation of (5.0.3) can be carried out first, defining the normalized average

$$F(\alpha_{j_1}, \alpha_{j_2}, \dots, \beta_{j_1 j_1}, \beta_{j_1 j_2}, \dots, N) = N^{-1} \text{Tr} H_B = N^{-1} \text{Tr} H \quad (5.0.4)$$

where F depends on the α 's, the β 's, the number of bosons N , in addition to the parameters of the original Hamiltonian. Here H_B is the boson Hamiltonian. F is derived with respect to α and β , and then the solution to the non-linear equation is solved.

Another alternative procedure is to choose these pairs so as to minimize the energies of only the lowest-lying states.

In this chapter we follow the first method to obtain the structure coefficients, and use them to calculate occupation and vacancy probabilities in nuclei. In this procedure the Hamiltonian $H = H_D + V$, consists of H_D , which is a diagonal Hamiltonian containing the single-particle energies of the nucleons, and a residual interaction V between the two particles. We analyse the case when this interaction is a Surface Delta Interaction (SDI) [44]. First, we give a brief description of it. Then, we show how the probabilities of occupation(vacancies) can be calculated in the IBM, and two realistic examples are worked out. Finally we compare the experimental data with our outcomes.

5.1 Getting structure coefficients

As it was mentioned at the beginning of this chapter, there are different ways to get the structure coefficients. We will diagonalize a Hamiltonian with the single-particle energies plus the SDI in a two-particle space to obtain them. We will now show some basic notions about the SDI and how we obtain the α 's and β 's for constructing the S^\dagger and D^\dagger correlated pair-creation operators with the lowest eigenstates.

5.1.1 Use of surface delta interaction

The SDI is simple to handle and describes many nuclear properties reasonably well. However some assumptions about the effective interaction are made when it is used. These assumptions are explained in detail in [45] and here we just quote them:

1. The interaction takes place at the nuclear surface only.
2. The two-body interaction is a delta force.
3. The probability of finding a particle at the nuclear surface is approximately independent of the shell-model orbit in which the particle moves.

Even when these assumptions are not held, the justification to use this interaction is based on its success to reproduce experimental data. Taking into account the previous assumptions we can write the following expression for the SDI between particles(hole) 1 and 2

$$V^{\text{SDI}}(1, 2) = -4\pi g_T \delta(\vec{r}(1) - \vec{r}(2)) \delta(r(1) - R_0), \quad (5.1.1)$$

where R_0 is the nuclear radius, g_T is the SDI-strength which depend on the isospin quantum number T , and the factor 4π is introduced rather arbitrarily in order to avoid a similar factor $(4\pi)^{-1}$ in the final expression of a two-body matrix element, as will be shown below.

It can be shown (it is a tedious and unnecessary calculation to be proved in this work) that the two-body matrix elements are given in [45] and read

$$\begin{aligned} \langle j_a j_b | V^{\text{SDI}}(1, 2) | j_c j_d \rangle_{JT} = & (-1)^{n_a + n_b + n_c + n_d} \frac{G_T}{2(2J+1)} \sqrt{\frac{(2j_a+1)(2j_b+1)(2j_c+1)(2j_d+1)}{(1+\delta_{a,b})(1+\delta_{c,d})}} \\ & \times \left\{ (-1)^{j_b + j_d + l_b + l_d} \langle j_b - \frac{1}{2} j_a \frac{1}{2} | J0 \rangle \langle j_d - \frac{1}{2} j_c \frac{1}{2} | J0 \rangle [1 - (-1)^{l_a + l_b + J+T}] \right. \\ & \left. - \langle j_b \frac{1}{2} j_a \frac{1}{2} | J1 \rangle \langle j_d \frac{1}{2} j_c \frac{1}{2} | J1 \rangle [1 + (-1)^T] \right\}. \quad (5.1.2) \end{aligned}$$

For brevity of notation j_k on the states stands for the complete set of single-particle quantum number n_k, l_k and of course j_k *per se*. Also the matrix element does not depend on the orientation

in coordinate space and isospin. Therefore, the projection quantum numbers M and T_z are suppressed.

Here G_T , possessing the dimension of an energy, includes the product of the strength g_T and the value of the radial integrals, such that

$$G_T = g_T R_{n_a l_a}^4(R_0) R_0^2,$$

and is generally quoted as the strength of the SDI. The common procedure to obtain G_T is by finding the value such that it enables us to reproduce the $2_1^+ - 0_1^+$ splitting in the various two-valence-particles and two-valence-hole systems. From the above, the S^\dagger and D^\dagger correlated pair-creation operators are obtained by assuming a SDI between identical nucleons with a strength parameter G_T and an isospin value $T = 1$.

For a type of ρ -particle(hole), the lowest 0^+ and 2^+ eigenvectors that emerge from the diagonalization of $H_{\rho\rho}^F$, can be expressed as

$$|0_1^+\rangle = \sum_j \frac{a_j}{\sqrt{2}} [C_j^\dagger \times C_j^\dagger]^{(0)} |\tilde{0}\rangle, \quad (5.1.3)$$

$$|2_1^+\rangle = \sum_{\substack{j,j' \\ j \leq j'}} \frac{b_{jj'}}{(1 + \delta_{jj'})^{1/2}} [C_j^\dagger \times C_{j'}^\dagger]^{(2)} |\tilde{0}\rangle. \quad (5.1.4)$$

Here, $|\tilde{0}\rangle$ is the state of the core, that is, the state of a nucleus with closed shell. The α_j and $\beta_{jj'}$ coefficients of the S^\dagger and D^\dagger pair-creation operators are then given by

$$\alpha_j = \sqrt{\frac{\Omega}{\Omega_j}} a_j, \quad (5.1.5)$$

$$\beta_{jj'} = b_{jj'}, \quad (5.1.6)$$

where $\Omega = \sum_j \Omega_j$ and the relation $\beta_{jj'} = (-1)^{j-j'} \beta_{j'j}$ also is fulfilled for all j and j' . The normalization used for the α_j is that given in [46].

We obtain the single-particle energies from a semi-magical nucleus with an extra particle(hole). Then we fit G_T according to the $2_1^+ - 0_1^+$ splitting of the nucleus with two particles(holes) and we obtain the structure constants. Those constants are used in various nuclei where their particles(holes) are in the same shell because their variation is almost negligible [17].

5.2 Occupation probabilities

With the extraction of the structure coefficients of the S^\dagger and D^\dagger correlated pair-creation operators, we can obtain the occupation probabilities(vacancies) of particles(holes) of one kind of particle. First we will obtain the boson image of the number operator using the OAI method at the lowest order in boson operators. Then we use the afore discussed procedure with a SDI interaction to obtain the structure coefficients. We choose as example of this procedure ^{130}Te and ^{132}Xe since they have two and four protons in the 50-82 shell, and we can compare their proton occupancies with recent experimental results [47].

5.2.1 Boson image of number operator in a j -orbit

We need to find the boson image of the number operator

$$\hat{n}_{j,\rho} = \left(C_j^\dagger \cdot \tilde{C}_j \right)_\rho = -\hat{j} [C_j^\dagger \times \tilde{C}_j]_\rho^{(0)}, \quad (5.2.1)$$

in a j -orbit for a particular type of nucleon ρ . Let $\hat{n}_{j,\rho}^B$ be its boson image, then

$$\hat{n}_{\rho,j} \xrightarrow{\text{Bosonization}} \hat{n}_{\rho,j}^B = A_{j,\rho} (s^\dagger \cdot s)_\rho + B_{j,\rho} (d^\dagger \cdot \tilde{d})_\rho \quad (5.2.2)$$

$$= A_{j,\rho} \hat{n}_{s,\rho} + B_{j,\rho} \hat{n}_{d,\rho}, \quad (5.2.3)$$

the constants $A_{j,\rho}$ and $B_{j,\rho}$ are specified for each orbit j and they depend only on the shell of the system and the number of pairs of particles (holes) N of one kind of particle (ρ). These constants are directly related with the occupation probability of the particles in the system of $2N$ particles. Their values are calculated below.

Values of $A_{j,\rho}$ and $B_{j,\rho}$

Let's consider a system composed of N pairs of identical particles, then

$$\langle s^N | \hat{n}_{j,\rho}^B | s^N \rangle_B = \langle s^N | (A_{j,\rho}(s^\dagger \cdot s)_\rho + B_{j,\rho}(d^\dagger \cdot \tilde{d})_\rho) | s^N \rangle_B \quad (5.2.4)$$

$$\begin{aligned} &= A_{j,\rho} \langle s^N | (s^\dagger \cdot s)_\rho | s^N \rangle_B + B_{j,\rho} \langle s^N | (d^\dagger \cdot \tilde{d})_\rho | s^N \rangle_B \\ &= A_{j,\rho} N_\rho \end{aligned} \quad (5.2.5)$$

$$\stackrel{(\cdot \text{OAI})}{=} \langle S^N | \hat{n}_{j,\rho} | S^N \rangle. \quad (5.2.6)$$

Hereafter the ρ index is omitted, but we must stress that all the operators work only on a type of particle. Making equal equations (5.2.6) and (5.2.5) we obtain

$$\begin{aligned} A_j N &= \frac{1}{\eta_{2N,00}^2} \langle \tilde{0} | (S^-)^N \hat{n}_j (S^+)^N | \tilde{0} \rangle \\ &= \frac{-\sqrt{2j+1}}{\eta_{2N,00}^2} \langle \tilde{0} | (S^-)^N [C_j^\dagger \times \tilde{C}_j]^{(0)} (S^+)^N | \tilde{0} \rangle \\ &= \frac{-\sqrt{2j+1}}{\eta_{2N,00}^2} \eta_{2N,00}^2 \langle 2N00 || [C_j^\dagger \times \tilde{C}_j]^{(0)} || 2N00 \rangle. \end{aligned} \quad (5.2.7)$$

Thus,

$$A_j = -\frac{\sqrt{2j+1}}{N} \langle 2N00 || [C_j^\dagger \times \tilde{C}_j]^{(0)} || 2N00 \rangle. \quad (5.2.8)$$

From Frank and van Isacker [38] the reduced matrix element has the following explicit form

$$\langle 2N00 || [C_j^\dagger \times \tilde{C}_j]^{(0)} || 2N00 \rangle = \eta_{2N,00}^{-2} \sqrt{2j+1} \sum_{p=1}^N \left[\frac{(N)!}{(N-p)!} \right]^2 (-1)^p \alpha_j^{2p} \eta_{2(N-p),00}^2. \quad (5.2.9)$$

On the other hand, in order to obtain B_j we consider the following relation,

$$\langle ds^{N-1} | \hat{n}_j^B | ds^{N-1} \rangle_B = A_j (N-1) + B_j \quad (5.2.10)$$

$$= -\frac{\sqrt{2j+1}}{\sqrt{5}} \langle S^{N-1} D || [C_j^\dagger \times \tilde{C}_j]^{(0)} || S^{N-1} D \rangle, \quad (5.2.11)$$

the last expression is rewritten in shorter form as $\langle 2N22 || [C_j^\dagger \times \tilde{C}_j]^{(0)} || 2N22 \rangle$, and a general expression of that term such as $\langle S^{N-1} D || [C_j^\dagger \times \tilde{C}_j]^{(K)} || S^{N-1} D \rangle$ is given explicitly in equation (3.3.31). Thus, from (5.2.10) we find that

$$B_j = -(N-1)A_j - \frac{\sqrt{2j+1}}{\sqrt{5}} \langle 2N22 || [C_j^\dagger \times \tilde{C}_j]^{(0)} || 2N22 \rangle. \quad (5.2.12)$$

Conditions on $A_{j,\rho}$ and $B_{j,\rho}$

Let us consider the total boson number operator \hat{N}_B of a particle ρ defined as

$$\hat{N} = (s^\dagger \cdot s) + (d^\dagger \cdot \tilde{d}) \quad (5.2.13)$$

$$= \hat{n}_s + \hat{n}_d. \quad (5.2.14)$$

Its expectation value through the wave function of the system $|\psi\rangle_B$ will give the total number of bosons of type ρ ,

$$\begin{aligned} N_\rho &= \langle \psi | \hat{N}_\rho | \psi \rangle_B \\ &= \langle \psi | (s^\dagger \cdot s)_\rho | \psi \rangle_B + \langle \psi | (d^\dagger \cdot \tilde{d})_\rho | \psi \rangle_B, \end{aligned} \quad (5.2.15)$$

from this equation we see immediately that $\langle \psi | (s^\dagger \cdot s)_\rho | \psi \rangle_B = N_\rho - \langle \psi | (d^\dagger \cdot \tilde{d})_\rho | \psi \rangle_B$. Now, we find the expectation value of the boson image of the number operator with the same wave function (omitting the index ρ),

$$\begin{aligned} \langle \psi | \hat{n}_j | \psi \rangle_B &= A_j \langle \psi | (s^\dagger \cdot s) | \psi \rangle_B + B_j \langle \psi | (d^\dagger \cdot \tilde{d}) | \psi \rangle_B \\ &= A_j \left(N - \langle \psi | (d^\dagger \cdot \tilde{d}) | \psi \rangle_B \right) + B_j \langle \psi | (d^\dagger \cdot \tilde{d}) | \psi \rangle_B \\ &= A_j N + (B_j - A_j) \langle \psi | (d^\dagger \cdot \tilde{d}) | \psi \rangle_B. \end{aligned} \quad (5.2.16)$$

When summing over all j 's in (5.2.16), we find that

$$2N = N \left(\sum_j A_j \right) + \sum_j (B_j - A_j) \langle \psi | (d^\dagger \cdot \tilde{d}) | \psi \rangle_B, \quad (5.2.17)$$

where we use

$$\sum_j \langle \psi | \hat{n}_j | \psi \rangle_B = 2N, \quad (5.2.18)$$

since the boson image of the number operator must fulfil the same relation that the number operator in the SM. Since the lhs of the Eq. (5.2.17) is independent of the matrix element, the following identity fulfils,

$$\sum_j A_j = \sum_j B_j = 2 \quad (5.2.19)$$

$$(5.2.20)$$

5.2.2 Application to ^{130}Te and ^{132}Xe

Now we calculate the values of α 's and β 's using the values of the single-particle energies for protons in the 50-82 shell obtained from ^{133}Sb . They are given in table 5.1¹. Since ^{130}Te and ^{132}Xe have two and four protons after closed shell, they are systems one and two proton bosons, respectively. Also their matrix elements for protons ${}_B \langle \psi | (d^\dagger \cdot \tilde{d})_\rho | \psi \rangle_B$ is =0.0580707 and 0.18235134, respectively [49].

Orbital	Particle energies for π
3s _{1/2}	2.9900
2d _{3/2}	2.4396
2d _{5/2}	0.9623
1g _{7/2}	0.0
1h _{11/2}	2.7915

Table 5.1: Proton Single-Particle Energies in MeV for $A \sim 130$.

The values for α 's and β 's are shown in Table 5.2. They were obtained with a SDI strength $G_1 = 0.222$, which yield to a $2_1^+ - 0_1^+$ splitting value of 1.280133 MeV against the experimental value of 1.27911(10) MeV for ^{134}Te which has only two protons as valence nucleons.

¹The Energy of 3s_{1/2} is estimated as can be seen in [48]. Thus, this energy is not experimentally obtained as the rest of the energies given in the table.

$\alpha_{1/2}$	-0.384121
$\alpha_{3/2}$	-0.448745
$\alpha_{5/2}$	-0.818225
$\alpha_{7/2}$	-1.764673
$\alpha_{11/2}$	0.405164
$\beta_{1/2\ 3/2}$	0.0575473
$\beta_{1/2\ 5/2}$	0.0944939
$\beta_{3/2\ 3/2}$	0.0449477
$\beta_{3/2\ 5/2}$	-0.0578527
$\beta_{3/2\ 7/2}$	0.1900099
$\beta_{5/2\ 5/2}$	0.1341775
$\beta_{5/2\ 7/2}$	0.1328504
$\beta_{7/2\ 7/2}$	0.9512438
$\beta_{11/2\ 11/2}$	-0.0759644

Table 5.2: α_j and $\beta_{jj'}$ coefficients in 50-82 shell.

Now we can compare the occupancies for protons with experimental data [47, 50, 51]. The former two references are relatively old and lack information in $1h_{11/2}$ orbit for ^{130}Te and have no information about ^{132}Xe . Meanwhile the later is recent and provide more reliable information for both nuclei.

In table 5.3 we show the obtained $A_{j,\pi}$ and $B_{j,\pi}$ and occupation probabilities for ^{130}Te with its experimental data. For ^{132}Xe , we show the same results in table 5.4.

We can see from the tables that in ^{130}Te is method brings results that agree very well with experimental data. This was expected because this nucleus has precisely two valence protons only. In ^{132}Xe the results show certain agreement but they are not as good as in ^{130}Te . This could be because ^{132}Xe has 4 proton particles and 4 neutron holes in valence shell. That implies that the calculated structure constants with SDI would not fit the occupation probabilities. However, the experimental-theoretical occupancies are quite similar, which indicates as a first approach that this procedure is very useful and reliable in spite of its simplicity.

Before finishing this chapter, it's worthy to mention a possible method in order to improve the numerical results in IBFM using the structure constants. The IBFM programs like ODDPAR [52] hitherto use the α 's obtained from BCS calculations for the core nucleus (even-even nucleus), and then the β 's using Eq. (2.1.29). This method is practical in fact, but has not a direct relation with the core nucleus. We propose to obtain these constants for the core nucleus by requiring that the occupation probabilities of protons or neutrons are well reproduced. This assures us that one-nucleon transfer operator coefficients are related to the core, and the unpaired particle degrees of freedom are well coupled to the core.

Orbital	$A_{j,\pi}$	$B_{j,\pi}$	Exp. Occ.	Norm. Occ.	Calculated Occ.
$3s_{1/2}$	0.0184	0.0122	0.011(10)	0.01	0.0181
$2d_{3/2}$	0.0503	0.0468	***		0.0501
$2d_{5/2}$	0.2511	0.0659	0.32(3)	0.34	0.2403
$1g_{7/2}$	1.5570	1.8635	1.32(10)	1.40	1.5748
$1h_{11/2}$	0.1231	0.0115	0.24(3)	0.25	0.1166
Σ	1.9999	1.9999	1.89(11)	2	1.9999

Table 5.3: **Values of $A_{j,\pi}$, $B_{j,\pi}$ and probability occupation of protons in ^{130}Te .** The *** mark indicates that the experimental occupation probability in the value of $2d_{3/2}$ and $2d_{5/2}$ is summed. Norm. Occ. represents normalized experimental occupation utilizing the Macfarlane-French sum rules.



Orbital	$A_{j,\pi}$	$B_{j,\pi}$	Exp. Occ.	Norm. Occ.	Calculated Occ.
$3s_{1/2}$	0.0217	0.0255	0.13(2)	0.13	0.0441
$2d_{3/2}$	0.0590	0.0800	***		0.1219
$2d_{5/2}$	0.2856	0.1935	0.94(5)	0.92	0.5545
$1g_{7/2}$	1.4890	1.6339	2.60(10)	2.55	3.0043
$1h_{11/2}$	0.1447	0.0671	0.41(4)	0.40	0.2752
Σ	2.0000	2.0000	4.07(12)	4	4.0000

Table 5.4: Same as table 5.3, but for ^{132}Xe .

Chapter 6

Final discussion

In this thesis the boson-fermion interaction used in IBFM was studied deeply along with the quasi-particle formalism used originally by Scholten in order to construct a fermion quadrupole operator and microscopically deduce its coefficients using two methods: OAI and GHP. Also, different mapping procedures along with their features were studied. For this purpose a fundamental study in angular momentum algebra techniques and spherical tensor operators was carried out.

In the development of this thesis a general expression for the boson-fermion interaction was obtained from the one-nucleon transfer operator using the quasi-particle formalism. For this, the first task was to obtain the correct expression for a general one-nucleon transfer operator in the IBFM up to two boson operators and one ideal fermion operator. Then we constructed the fermion quadrupole operator to finally couple to the boson quadrupole operator. We studied the effects of the boson-fermion interaction using two mappings, OAI and GHP in the single j -shell case. The results are very interesting. When the strength of the interaction is not quite big both mappings bring similar results on a particular orbit that is half or less filled. That is comprehensible since OAI is well characterized by generalized seniority scheme where the number of particles not coupled to zero is not big, meanwhile GHP is more suitable for deformed nuclei because of the collectivity induced. Also, in this thesis we wrote all the expressions of the coefficients in OAI in exact form, that is, without using the Number Operator Approximation. In order to do that, we had to introduce the structure constants of the S and D correlated-pair creation operators. Different ways to obtain them were discussed. We used a SDI interaction as residual interaction in order to obtain these constants, which were used in real calculations in order to obtain the proton occupation probability in ^{130}Te and ^{132}Xe . The theoretical results coincide very well with experimental data.

Thanks to the work of this thesis the student was able to improve his computational skills in Fortran to obtain the occupation probabilities in ^{130}Te , ^{132}Xe and also in Python to diagonalize the Hamiltonian shown in chapter 4. Since all the results are generalizations of previous works, we expect to improve the numerical results in realistic calculations of odd A nuclei.

Appendix A

Mathematical relations

A.1 Tensor Product

Let $T_{m_1}^{(j_1)}$ and $T_{m_2}^{(j_2)}$ be two irreducible tensors¹ of degree j_1 and j_2 , respectively. In general, the typical product of irreducible tensors is not an irreducible tensor, but one way to obtain an irreducible tensor of degree j_3 is through the bilinear combination

$$T_{m_3}^{(j_3)} = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle T_{m_1}^{(j_1)} T_{m_2}^{(j_2)}, \quad (\text{A.1.2})$$

where $\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle$ are the Clebsch-Gordan coefficients. $T_{m_3}^{(j_3)}$ is called *tensor product* of degree j_3 out of $T_{m_1}^{(j_1)}$ and $T_{m_2}^{(j_2)}$ and it will be denoted as

$$T_{m_3}^{(j_3)} = \left[T^{(j_1)} \times T^{(j_2)} \right]_{m_3}^{(j_3)}. \quad (\text{A.1.3})$$

It is common to overlook the components of these tensors, so it is often to used the following notation

$$T^{(j_3)} = \left[T^{(j_1)} \times T^{(j_2)} \right]^{(j_3)}. \quad (\text{A.1.4})$$

In the same way, because of the orthogonality relation of the Clebsch-Gordan coefficients, it is also possible to retrieve the typical product of tensors given by

$$T_{m_1}^{(j_1)} T_{m_2}^{(j_2)} = \sum_{j_3=|j_1-j_2|}^{j_1+j_2} \sum_{m_3=-j_3}^{j_3} \langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle T_{m_3}^{(j_3)}. \quad (\text{A.1.5})$$

On the other hand, we can define the *scalar product* between two irreducible tensors $T^{(k)}$ and $U^{(k)}$ of the same degree as

$$(T^{(k)} \cdot U^{(k)}) = (-1)^k \sqrt{2k+1} [T^{(k)} \times U^{(k)}]_0^{(0)} \quad (\text{A.1.6})$$

$$= (-1)^k \sum_{m_1, m_2} \sqrt{2k+1} \langle k m_1 k m_2 | 00 \rangle T_{m_1}^{(k)} U_{m_2}^{(k)} \quad (\text{A.1.7})$$

$$= \sum_m (-1)^m T_m^{(k)} U_{-m}^{(k)}. \quad (\text{A.1.8})$$

¹An irreducible tensor of degree k is a set of $2k+1$ components $T_m^{(k)}$ with $-k \leq m \leq k$, which transforms under rotations according to

$$T_{m'}^{(k)} = \sum_m T_m^{(k)} D_{mm'}^{(k)}(R), \quad (\text{A.1.1})$$

where $D_{mm'}^{(k)}(R)$ are the Wigner matrices which depend on the rotation R .

When k is an semi-integer number the phase $(-1)^k$ is replaced by -1 in the above equation. In the particular case where $T^k/U^{(k)} = C_k^\dagger/\tilde{C}_k$ is a/an creation/annihilation operator, then

$$(C_k^\dagger \cdot \tilde{C}_k) = \sum_m C_{km}^\dagger C_{km}. \quad (\text{A.1.9})$$

A.2 Clebsch-Gordan coefficients and $3j$ -symbol

A.2.1 Properties

Let $\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle$ be the Clebsch-Gordan coefficients². They fulfil the next symmetry properties,

$$\langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{j_1+j_2-j_3} \langle j_1 - m_1 j_2 - m_2 | j_3 - m_3 \rangle \quad (\text{A.2.1})$$

$$= (-1)^{j_1+j_2-j_3} \langle j_2 m_2 j_1 m_1 | j_3 m_3 \rangle \quad (\text{A.2.2})$$

$$= (-1)^{j_1-m_1} \sqrt{\frac{2j_3+1}{2j_2+1}} \langle j_1 m_1 j_3 - m_3 | j_2 - m_2 \rangle \quad (\text{A.2.3})$$

$$= (-1)^{j_2+m_2} \sqrt{\frac{2j_3+1}{2j_1+1}} \langle j_3 - m_3 j_2 m_2 | j_1 - m_1 \rangle. \quad (\text{A.2.4})$$

A convenient way to derive these relations is by converting the Clebsch-Gordan coefficients to $3j$ -symbols. The symmetry properties of $3j$ -symbols are much simpler. Care is needed when simplifying phase factors, because the quantum numbers can be integer or half integer, e.g., $(-1)^{2j}$ is equal to 1 for integer j and equal to -1 for half-integer j . The following relations, however, are valid in either case:

$$(-1)^{4j} = (-1)^{2(j-m)} = 1,$$

and for j_1, j_2 , and j_3 appearing in the same Clebsch-Gordan coefficient:

$$(-1)^{2(j_1+j_2+j_3)} = (-1)^{2(m_1+m_2+m_3)} = 1.$$

In addition two important relations must be written. Those are

$$\langle j_1 m_1 j_2 m_2 | 00 \rangle = \delta_{j_1, j_2} \delta_{m_1, -m_2} \frac{(-1)^{j_1-m_1}}{\sqrt{2j_2+1}}, \quad (\text{A.2.5})$$

$$\sum_m (-1)^{j-m} \langle j m j - m | J 0 \rangle = \sqrt{2j+1} \delta_{J,0}. \quad (\text{A.2.6})$$

An easy way to obtain desirable properties and outcomes in this thesis is through the use of the $3j$ -symbols, which are defined as

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\hat{j}_3} \langle j_1 m_1 j_2 m_2 | j_3 - m_3 \rangle. \quad (\text{A.2.7})$$

Also, for any permutation of two columns yields to a phase,

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix}. \quad (\text{A.2.8})$$

Also, when the change of sign of all the m 's is performed, the same phase appears,

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \quad (\text{A.2.9})$$

²It is also common to find this coefficients as $\langle j_1 m_1 j_2 m_2 | j_1 j_2 j_3 m_3 \rangle$.

A.3 Properties of Wigner's $6j$ -symbol

A.3.1 Symmetry relations

The Wigner's $6j$ -symbol is invariant under the permutation of any two columns:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_5 & j_4 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_3 & j_2 \\ j_4 & j_6 & j_5 \end{Bmatrix} = \begin{Bmatrix} j_3 & j_2 & j_1 \\ j_6 & j_5 & j_4 \end{Bmatrix}. \quad (\text{A.3.1})$$

The $6j$ -symbol is also invariant if upper and lower arguments are interchanged in any two columns:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_4 & j_5 & j_3 \\ j_1 & j_2 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_5 & j_6 \\ j_4 & j_2 & j_3 \end{Bmatrix} = \begin{Bmatrix} j_4 & j_2 & j_6 \\ j_1 & j_5 & j_3 \end{Bmatrix}. \quad (\text{A.3.2})$$

A.3.2 Special case

When one of the j in the $6j$ -symbol is zero, it reduces to

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ 0 & l_3 & l_2 \end{Bmatrix} = \frac{(-1)^{j_1+j_2+j_3}}{\sqrt{(2j_2+1)(2j_3+1)}} \delta_{j_2,l_2} \delta_{j_3,l_3} \Delta(j_1 j_2 j_3), \quad (\text{A.3.3})$$

where

$$\Delta(j_1 j_2 j_3) = \begin{cases} 1 & \text{when } |j_1 - j_2| \leq j_3 \leq j_1 + j_2 \\ 0 & \text{in other case.} \end{cases} \quad (\text{A.3.4})$$

A.3.3 Relation between the $3j$ and $6j$ -symbols

We know from Talmi [5, 53] the following useful relations

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} &= \sum_{m_4, m_5, m_6} (-1)^{j_4+j_5+j_6+m_4+m_5+m_6} \\ &\times \begin{pmatrix} j_1 & j_5 & j_6 \\ m_1 & m_5 & -m_6 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_6 \\ -m_4 & m_2 & m_6 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ m_4 & -m_5 & m_3 \end{pmatrix}, \end{aligned} \quad (\text{A.3.5})$$

$$\begin{aligned} \sum_{m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ m_4 & m_5 & -m_3 \end{pmatrix} &= \sum_{j_6, m_6} (-1)^{j_3+j_6+m_1+m_4} (2j_6+1) \\ &\times \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \begin{pmatrix} j_4 & j_2 & j_6 \\ m_4 & m_2 & m_6 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ m_1 & m_5 & -m_6 \end{pmatrix}. \end{aligned} \quad (\text{A.3.6})$$

A.4 Properties of Wigner's $9j$ -symbols

In physics, Wigner's $9j$ -symbols were introduced by Eugene Paul Wigner in 1937. They appear in the re-coupling of four angular momentum vector,

$$\psi[j_1 j_4(j_7) j_2 j_5(j_8) | j_9 m_9] = \sum_{j_3, j_6} \sqrt{(2j_3+1)(2j_6+1)(2j_7+1)(2j_8+1)} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix} \quad (\text{A.4.1})$$

$$\times \psi[j_1 j_2(j_3) j_4 j_5(j_6) | j_9 m_9]. \quad (\text{A.4.2})$$

A.4.1 Symmetry relations

A $9j$ -symbol is invariant under reflection in either diagonal:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_4 & j_7 \\ j_2 & j_5 & j_8 \\ j_3 & j_6 & j_9 \end{Bmatrix} = \begin{Bmatrix} j_9 & j_6 & j_3 \\ j_8 & j_5 & j_2 \\ j_7 & j_4 & j_1 \end{Bmatrix}. \quad (\text{A.4.3})$$

The permutation of any two rows or any two columns yields a phase factor $(-1)^\Sigma$, where

$$\Sigma = \sum_{i=1}^9 j_i$$

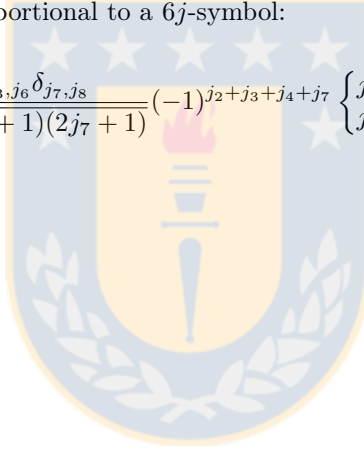
For example:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix} = (-1)^\Sigma \begin{Bmatrix} j_4 & j_5 & j_6 \\ j_1 & j_2 & j_3 \\ j_7 & j_8 & j_9 \end{Bmatrix} = (-1)^\Sigma \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_5 & j_4 & j_6 \\ j_8 & j_7 & j_9 \end{Bmatrix} = \begin{Bmatrix} j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \\ j_1 & j_2 & j_3 \end{Bmatrix}. \quad (\text{A.4.4})$$

A.4.2 Special case

When $j_9 = 0$ the $9j$ -symbol is proportional to a $6j$ -symbol:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & 0 \end{Bmatrix} = \frac{\delta_{j_3, j_6} \delta_{j_7, j_8}}{\sqrt{(2j_3 + 1)(2j_7 + 1)}} (-1)^{j_2 + j_3 + j_4 + j_7} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_5 & j_4 & j_7 \end{Bmatrix}. \quad (\text{A.4.5})$$



Appendix B

Relations between boson and fermion operators

B.1 Commutation relations

It is known that for fermion creation/annihilation operators the commutation relations are given by

$$\{f_{j,m}, f_{j',m'}^\dagger\} = \delta_{j,j'}\delta_{m,m'}, \quad (\text{B.1.1})$$

$$\{f_{j,m}^\dagger, f_{j',m'}^\dagger\} = \{f_{j,m}, f_{j',m'}\} = 0. \quad (\text{B.1.2})$$

where $\{\}$ is the anti-commutation relation¹, j is a positive semi-integer number and $-l \leq m \leq l$. Since the annihilation operator with good tensor character is defined as $\tilde{f}_{jm} = (-1)^{j-m} f_{j,-m}$, we will work with the following commutation relation

$$\begin{aligned} \{\tilde{f}_{jm}, f_{j'm'}^\dagger\} &= \{(-1)^{j-m} f_{j,-m}, f_{j'm'}^\dagger\} \\ &= (-1)^{j-m} \{f_{j,-m}, f_{j'm'}^\dagger\} \\ &= (-1)^{j-m} \delta_{jj'} \delta_{m'-m}. \end{aligned} \quad (\text{B.1.4})$$

For the boson operator we can proceed in the same way. By starting with the boson operators commutation relations

$$[b_{l,m}, b_{l',m'}^\dagger] = \delta_{l,l'} \delta_{m,m'}, \quad (\text{B.1.5})$$

$$[b_{l,m}^\dagger, b_{l',m'}^\dagger] = [b_{l,m}, b_{l',m'}] = 0, \quad (\text{B.1.6})$$

where l, l' are an integer number and m, m' satisfy the triangular condition. Also, it is easy to see that

$$[\tilde{b}_{l,m}, b_{l',m'}^\dagger] = (-1)^{l+m} \delta_{l,l'} \delta_{m'-m}, \quad (\text{B.1.7})$$

since

$$\tilde{b}_{l,m} = (-1)^{l+m} b_{l,-m}.$$

¹For any two operators A and B we have by definition:

$$\{A, B\} = AB + BA. \quad (\text{B.1.3})$$

B.2 Coupled operators

In general we want to write all operators in normal order, that is, a creation operator on the left and an annihilation operator on its right, thus we need to construct an operator with a certain angular momentum J out of boson and/or fermion operators. So now, we will analyse the different forms of those operators.

B.2.1 Boson-fermion operator

Let A_{LjJM}^\dagger be a creation operator of the form

$$A_{LjJM}^\dagger = [\gamma_L^\dagger \times \tilde{a}_j]_M^{(J)}, \quad (\text{B.2.1})$$

where γ_L^\dagger is the boson creation operator with integer angular momentum L . By definition its associated annihilation operator with good tensor character is constructed as $\tilde{A}_{JM} = (-1)^{(J-M)} A_{J-M}$. So, using the definition above we see that

$$\begin{aligned} A_{LjJM} &= \left(A_{LjJM}^\dagger \right)^\dagger = \left(\sum_{m_L, m_j} \langle L m_L j m_j | J M \rangle \gamma_{L, m_L}^\dagger (-1)^{j-m_j} a_{j, -m_j} \right)^\dagger \\ &= \sum_{m_L, m_j} \langle L m_L j m_j | J M \rangle (-1)^{j-m_j} a_{j, -m_j}^\dagger \gamma_{L, m_L}. \end{aligned}$$

We make the following change of variables $m_L = -M_L$ and $m_j = -M_j$ and using the fact the Clebsch-Gordan coefficient is zero unless $m_L + m_j = M$, then $-m_j = M_j = -M_L - M$, we obtain

$$\begin{aligned} A_{LjJM} &= \sum_{M_L, M_j} \langle L - M_L j - M_j | J M \rangle (-1)^{j-M_L-M} a_{j, M_j}^\dagger \gamma_{L, -M_L} \\ &= \sum_{M_L, M_j} \langle L M_L j M_j | J - M \rangle (-1)^{L+j-J} (-1)^{j-M_L-M} a_{j, M_j}^\dagger \gamma_{L, -M_L} \\ &= (-1)^{2j-J-M} \sum_{M_L, M_j} \langle L M_L j M_j | J - M \rangle (-1)^{L-M_L} \gamma_{L, -M_L} a_{j, M_j}^\dagger \\ &= (-1)^{2j-(J+M)} [\tilde{\gamma}_L \times a_j^\dagger]_{-M}^{(J)}. \end{aligned} \quad (\text{B.2.2})$$

So, it is straightforward to see that

$$(-1)^{J-M} A_{LjJ-M} = (-1)^{2j} [\tilde{\gamma}_L \times a_j^\dagger]_M^{(J)},$$

since j is a semi-integer number, then $(-1)^{2j} = -1$. Thus

$$\tilde{A}_{LjJM} \equiv -[\tilde{\gamma}_L \times a_j^\dagger]_M^{(J)}. \quad (\text{B.2.3})$$

B.2.2 Boson-boson-fermion operator

In this case we have an operator of the form

$$B_{l_1 l_2 j J M}^\dagger = \left[[\gamma_{l_1}^\dagger \times \tilde{\gamma}_{l_2}]^{(L)} \times a_j^\dagger \right]_M^{(J)}. \quad (\text{B.2.4})$$

Also, we can consider

$$\mathcal{B}_{l_1 l_2, M}^{(L)} := [\gamma_{l_1}^\dagger \times \tilde{\gamma}_{l_2}]_M^{(L)} \quad (\text{B.2.5})$$

because, it is easy to prove that

$$\left(\mathcal{B}_{l_1 l_2, M}^{(L)} \right)^\dagger = (-1)^{l_2-l_1+M} \mathcal{B}_{l_2 l_1, -M}^{(L)}. \quad (\text{B.2.6})$$

So we can write $B_{l_1 l_2 j J M}^\dagger$ as

$$B_{l_1 l_2 j J M}^\dagger = \left[\mathcal{B}_{l_1 l_2}^{(L)} \times a_j^\dagger \right]_M^{(J)}, \quad (\text{B.2.7})$$

then

$$\begin{aligned} B_{l_1 l_2 j J - M} &= (B_{l_1 l_2 j J - M}^\dagger)^\dagger = \sum_{m_L m_j} \langle L m_L j m_j | J - M \rangle a_{j, m_j} \left(\mathcal{B}_{l_1 l_2, m_L}^{(L)} \right)^\dagger \\ &= \sum_{m_L m_j} \langle L m_L j m_j | J - M \rangle (-1)^{l_2 - l_1 + m_L} [\gamma_{l_2}^\dagger \times \tilde{\gamma}_{l_1}]_{-m_L}^{(L)} a_{j, m_j}. \end{aligned}$$

Changing the sum indices m_L by $-M_L$, and m_j by $-M_j$, then we got

$$\begin{aligned} B_{l_1 l_2 j J - M} &= \sum_{M_L M_j} \langle L - M_L j - M_j | J - M \rangle (-1)^{l_2 - l_1 - M + M_j} [\gamma_{l_2}^\dagger \times \tilde{\gamma}_{l_1}]_{M_L}^{(L)} a_{j, -M_j} \\ &= \sum_{M_L M_j} \langle L M_L j M_j | J M \rangle (-1)^{L + j - J} (-1)^{l_2 - l_1 - M + M_j + (M_j - M_j)} [\gamma_{l_2}^\dagger \times \tilde{\gamma}_{l_1}]_{M_L}^{(L)} a_{j, -M_j} \\ &= -(-1)^{L - J + l_2 - l_1 - M} \sum_{M_L M_j} \langle L M_L j M_j | J M \rangle \mathcal{B}_{l_2 l_1, m_L}^{(L)} (-1)^{j - M_j} a_{j, -M_j} \\ &= -(-1)^{l_2 - l_1 + L - (J + M)} \left[\mathcal{B}_{l_2 l_1}^{(L)} \times \tilde{a}_j \right]_M^{(J)}. \end{aligned} \quad (\text{B.2.8})$$

The minus sign in the last expression is because $(-1)^{2M_j} = -1$. Finally, it is straightforward to obtain

$$\tilde{B}_{l_1 l_2 j J M} = (-1)^{J - M} B_{l_1 l_2 j J - M} = (-1)^{l_1 + l_2 + L} \left[\mathcal{B}_{l_2 l_1}^{(L)} \times \tilde{a}_j \right]_M^{(J)}. \quad (\text{B.2.9})$$

At the end, the minus sign cancels with the $(-1)^{J - M}$.

Appendix C

Commutation relations

$$\left\{ \left[C_j^\dagger \times \tilde{C}_{j'} \right]_k^{(K)} \right\}^\dagger = (-1)^{j-j'-k} \left[C_{j'}^\dagger \times \tilde{C}_j \right]_{-k}^{(K)}, \quad (\text{C.0.1})$$

$$\left[(S^\dagger)^N, \tilde{C}_{jm} \right] = N \alpha_j C_{jm}^\dagger (S^\dagger)^{N-1}, \quad (\text{C.0.2})$$

$$\left[(S^-)^N, C_{jm}^\dagger \right] = N \alpha_j (S^-)^{N-1} \tilde{C}_{jm}, \quad (\text{C.0.3})$$

$$\left[(S^\dagger)^N, \left[C_j^\dagger \times \tilde{C}_{j'} \right]_k^{(K)} \right] = N \alpha_{j'} \left[C_j^\dagger \times C_{j'}^\dagger \right]_k^{(K)} (S^\dagger)^{N-1}, \quad (\text{C.0.4})$$

$$\left[(S^\dagger)^N, \hat{n}_\alpha \right] = N M^\dagger (S^\dagger)^{N-1}, \text{ where } M^\dagger = \sum_j \alpha_j^3 (C_j^\dagger \cdot C_j^\dagger), \quad (\text{C.0.5})$$

$$\left[S^-, S^\dagger \right] = \Omega_e - \hat{n}_\alpha, \quad (\text{C.0.6})$$

$$\left[S^-, (S^\dagger)^N \right] = N (S^\dagger)^{N-1} [\Omega_e - \hat{n}_\alpha] + \frac{N(N-1)}{2} (S^\dagger)^{N-2} M^\dagger, \text{ for } N \geq 2. \quad (\text{C.0.7})$$

Bibliography

- [1] Amos de Shalit and Igal Talmi: *Nuclear Shell Theory*. Academic Press, dover edition, 1963.
- [2] Maria Goeppert Mayer and J. Hans D. Jensen: *Elementary theory of nuclear shell structure*. John Wiley & Sons, first edition, 1956.
- [3] J.P. Elliot: *The interacting boson model of nuclear structure*. Rep. Prog. Phys., **48**: 171–221, 1985.
- [4] Sean Mcallister: *The Neutrinoless Double-Beta Decay Candidates ^{128}Te and ^{130}Te : Single-Particle and Ground-State Pairing Characteristics*. PhD thesis, University of Manchester, 2010.
- [5] Igal Talmi: *Simple Models of Complex Nuclei: Shell Model and Interacting Boson Model*. Harwood Academic, contemporary concepts in physics edition, 1993.
- [6] F. Iachello and A. Arima: *The interacting boson model*. Cambridge University, Cambridge, 1987.
- [7] P. Van Isacker and Jin Quan Chen: *Classical limit of the interacting boson Hamiltonian*. Phys. Rev. C, **24**: 684, 1991.
- [8] Kosuke Nomura: *Interacting Boson Model from Energy Density Functionals*. PhD thesis, University of Tokyo, 2012.
- [9] Olaf Scholten: *Single particle degrees of freedom in the interacting boson model*. Progress in Particle and Nuclear Physics., **14**: 189–229, 1985.
- [10] Olaf Scholten: *The Interacting Boson Approximation Model and Applications*. PhD thesis, University of Groningen, 1980.
- [11] A. Arima T. Otsuka and F. Iachello: *Nuclear shell model and interacting bosons*. Nucl. Phys. A, **309**: 1–33, 1978.
- [12] J. Dukelsky C.E. Alonso, J.M. Arias and S. Pittel: *Generalized Holstein-Primakoff images of fermion operators*. Nucl. Phys. A, **539**: 391–402, 1992.
- [13] E.R. Marshalek: *How microscopic boson models work*. Nucl. Phys. A, **347**: 253–271, 1980.
- [14] M.A. Bentley D.D. Warner and P. Van Isacker: *The role of isospin symmetry in collective nuclear structure*. Nature Phys, **2**: 311, 2006.
- [15] F. Iachello and P. Van Isacker: *The interacting boson-fermion model*. Cambridge University, Cambridge, 1991.
- [16] Olaf Scholten: *The effective quadrupole force between like IBA-bosons*. Phys. Lett. B., **119**: 5, 1981.
- [17] A. Arima N. Yoshinaga and Y.D. Devi: *Microscopic derivation of interacting boson-fermion model Hamiltonian and its application to singly magic nuclei*. Phys. Lett. C., **62**, 2000.

- [18] P.V. Isacker A. Arima T. Otsuka, N. Yoshinaga and O. Scholten: *Shell-model evidence for the exchange term in the interacting-boson-fermion Hamiltonian*. Phys. Lett. C., **35**: 328, 1987.
- [19] D.D. Warner T. Otsuka K. Heyde Richard F. Casten, Pertti O. Lipas and Jerry P. Draayer: *Algebraic Approaches to Nuclear Structure*. Harwood Academic, 1993.
- [20] F. Iachello O. Otsuka, A. Arima and I. Talmi: *Shell model description of interacting bosons*. Phys. Lett. B., **76**: 139, 1978.
- [21] W. Steffen A.E.L. Dieperink N. Lo Iudice F. Palumbo O. Scholten D. Bohle, A. Richter: *New magnetic dipole excitation mode studied in the heavy deformed nucleus ^{156}Gd by inelastic electron scattering*. Phys. Lett. B., **137**: 27, 1984.
- [22] Takaharu Otsuka and Nobuaki Yoshida: *User manual of the program NPBOS*, 1985.
- [23] A. Arima N. Yoshinaga, T. Mizusaki and Y.D. Devi: *Microscopic Foundations of the Interacting Boson Model from the Shell-Model Point of View*. Prog. Theor. Phys. Suppl., **125**: 65, 1996.
- [24] Takaharu Otsuka. PhD thesis, University of Tokyo, 1979.
- [25] C.E. Alonso J. Barea and J.M. Arias: *The one nucleon transfer operator in the microscopic IBM without NOA*. Phys. Lett. B., **737**: 205–209, 2014.
- [26] S.T. Belyaev and V.G. Zelevinsky: *Anharmonic effects of quadrupole oscillations of spherical nuclei*. Nucl. Phys., **39**: 582, 1962.
- [27] P.D. Duval S. Pittel and B. R. Barret: *The microscopic Interacting Boson Model for non-degenerate orbits*. Annals of Physics, **144**: 168–169, 1982.
- [28] A. Nijenhuis and H.S. Wilf: *Combinatorial Algorithms for computers and calculations*. Academic Press, INC. (London) LTD., second edition edition, 1978.
- [29] P.O. Lipas and M. Koskinen: *Microscopic IBA calculations of (e, e') M1 form factors*. Nuclear Phys. A, **509**: 509–540, 1990.
- [30] C. E. Alonso J. Barea and J. M. Arias: *Boson-conserving one-nucleon transfer operator in the interacting boson model*. Phys. Rev. C, **65**: 034328, 2002.
- [31] A. Frank and P. Van Isacker: *Commutator algebra for the microscopic interacting boson model with nondegenerate orbits*. Phys. Rev. C, **26**: 1661, 1982.
- [32] M.A.Nagarajan L.D.Skouras, P.Van Isacker: *A democratic mapping between the shell model and the interacting boson model*. Nucl.Phys. A, **516**: 255, 1990.
- [33] C. E. Alonso E. Mardones, J. Barea and J. M. Arias: *β -decay rates of $^{121-131}\text{Cs}$ in the microscopic interacting boson-fermion model*. Phys. Rev. C, **95**: 034332, 2016.
- [34] E.R. Marshalek: *A new boson expansion for odd-particle systems*. Phys. Lett. B, **44**: 5–8, 1973.
- [35] H.M. Sofia J. Dukelsky, S. Pittel and C. Lima: *Boson mapping of one-body operators in deformed nuclei*. Nucl. Phys. A, **456**: 75–88, 1986.
- [36] T. Cohen: *Self-consistency and boson mappings for deformed systems*. Phys. Lett. B, **158**: 1–6, 1985.
- [37] E.R. marshalek: *On the relation between Beliaev-Zelevinsky and Marumori boson expansions*. Nucl. Phys. A, **161**: 401, 1971.

- [38] A. Frank and P. Van Isacker: *Commutator algebra for the microscopic interacting boson model with nondegenerate orbits*. Phys. Lett. C, **26**: 1661, 1982.
- [39] L.S. Kisslinger: *A note on coupling schemes in odd-mass nuclei*. Nucl. Phys., **78**: 341–352, 1966.
- [40] S. Pittel P. Federman and: *Unified shell-model description of nuclear deformation*. Phys. Rev. C, **20**: 820, 1979.
- [41] Niels R. Walet: *Microscopic foundation and geometric interpretation of the interacting-boson model*. PhD thesis, University of Utrecht, 1987.
- [42] A. Klein and M. Vallieres: *New method for studying the microscopic foundations of the interacting boson model*. Nucl. Phys. B, **98**: 5–10, 1981.
- [43] A. Klein and K. Tanabe: *Variational principles for collective motion: Relation between invariance principle of the Schrödinger equation and the trace variational principle*. Phys. Lett. B, **135**: 255–257, 1984.
- [44] A. Faessler and A. Plastino: *On the foundation of the surface delta-interaction*. Il Nuovo Cimento B, **47**: 297–300, 1967.
- [45] P.J. Broussard and P.W.M. Glaudemans: *Shell-model applications in nuclear spectroscopy*. North-Holland, 1977.
- [46] P.D. Duval and B. R. Barretl: *Shell-model determination of the interacting boson model parameters for two nondegenerate j shells*. Phys. Rev. C, **24**: 1272–1282, 1981.
- [47] J. P. Entwisle *et al.*: *Change of nuclear configurations in the neutrinoless double- β decay of $^{130}\text{Te} \rightarrow ^{130}\text{Xe}$ and $^{136}\text{Xe} \rightarrow ^{136}\text{Ba}$* . Phys. Rev. C, **93**: 064312, 2016.
- [48] W. J. Baldridge: *Shell-model studies for the ^{132}Sn region. I. Few proton cases*. Phys. Rev. C, **18**: 530, 1978.
- [49] J. Barea: *Private communication*.
- [50] J.B. Ball R.L. Auble and C.B. Fulmer: *The $(d, ^3\text{He})$ reaction on ^{126}Te , ^{128}Te and ^{130}Te nuclei*. Nucl. Phys. A, **116**: 14–26, 1968.
- [51] M. Caballero M. Conjeaud, S. Harar and N. Cindro: *The (t, α) reaction on even tellurium nuclei*. Nucl. Phys. A, **215**: 383–396, 1973.
- [52] Roelof Bijker: *code oddpar*. Unpublished, 1984.
- [53] Jouni Suhonen: *From Nucleons to Nucleus: Concepts of Microscopic Nuclear Theory*. Springer, 2007.

