Una mejora a la tomografía cuántica autoguiada
An enhancement to self-guided quantum tomography

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RESUMEN

En esta tesis estudiamos una técnica tomográfica que busca minimizar la infidelidad entre el estado cuántico del sistema y una aproximación a dicho estado. Para ello, proponemos un método de optimización para funciones de variables complejas no-holomórficas, basandonos en un método similar para funciones de variables reales. Demostramos la validez de nuestra propuesta tanto teóricamente como comparandola con técnicas anteriores mediante simulaciones computacionales.

ABSTRACT

In this thesis we study a tomographic technique that seeks to minimize the infidelity between the quantum state of the system and an estimate of such state. For that, we propose an optimization method for non-holomorphic functions with complex variables, based on a similar method for functions with real variables. We prove the validity of our proposal both theoretically and by comparing it to previous techniques using computational simulations.

Chapter 1

Introduction

A physical system can be characterized by its state, a complete mathematical description of the system [1]. If the state is known, the result of any measurement on the system can be predicted. It is then of great interest to devise a technique for experimentally determining the state of an unknown system.

Classically, such a procedure can always—theorically, if not practically—be designed. However, in quantum mechanics this is not the case, as there are two important results that prevent it. First, because of Heisenberg uncertainty principle [2], performing a measurement to determine one property will make following measurents to give no information on other properties. Second, the no-cloning theorem [3] states it is impossible to perfectly copy the system without knowing its state.

On the other hand, it is possible to make an estimate of the state of a system if we prepare many systems in the same state, and we perform different measurements on each one. The problem of obtaining a good enough estimate is called quantum tomography, first proposed by Fano in 1957 [4], but it was not experimentally realized until 1993 [5–7].

Because of the importance of characterizing quantum systems, the development of better tomographic schemes is of great interest to many areas, like quantum computing [8, 9], quantum information [10], quantum metrology [11], quantum simulators [12], quantum error correction [13], boson sampling [14, 15], and characterizing optical signals [16], cavity fields [17] and trapped ions [18], among many other applications.

Most tomographic schemes collect a large quantity of experimental data and postprocess it to estimate the state of the system. However, in 2014, Ferrie [19] proposed a quantum tomography technique that aproaches the target state in real time, needing less computational resources.

In this work, we will improve the perfomance of Ferrie's method by adapting the stochatic approximation algorithm used in the original article to perform directly on functions of complex variables through Wirtinger calculus. As Ferrie's method has already been compared with standard tomographic schemes [20], it will only be necessary to compare our results with those of Ferrie.

In chapter 2, we will introduce the mathematical and physical tools and concepts used in this thesis.

In chapter 3, we will derive the adapted algorithm for functions of complex variables, proving it satisfies under some basic requirements to be well behaved.

In chapter 4, we will apply the previously obtained algortihm in the problem of quantum tomography and perform computational simulations to contrast our results with the original technique.

Chapter 2

Fundamental concepts

2.1 Introduction to Classical Statistics

Throughout this work we will make use of certain concepts of classical statistics that we will present in this section. Quantum mechanics itself is built around statistical elements, and conditional probabilities are vital for understanding the main results of this thesis.

2.1.1 Events, probabilities and conditional probabilities

The *sample space* Ω is the set of all possible outcomes ω for a given experiment such that performing the experiment once makes it so that one and only one outcome in the sample space occurs. The experiment may be any procedure: from the toss of a coin and the roll of a die to the measuring of a physical property of a system. Depending on the experiment, the sample space may be finite or infinite, and countable or uncountable.

We call subsets of the sample space *events* E_i. The n events in the set $\{E_i\}_{i=1}^n$ are called *mutually exclusive* if, for any two of them E_j and E_k distinct, $E_j \cap E_k = \varnothing$. If an outcome belonging to an event occurs, we say that event occurs.

The *probability* p of an event is a measure acting on a subset of the sample space Ω of likely it is for that event to occur. Probability must satisfy the following conditions [21]:

1. $p(E) \geq 0$ for all events E,
2. If $E = \Omega$, $p(E) = 1$,
3. If the events $\{E_i\}_{i=1}^n$ are mutually exclusive, $p\left(\bigcup_{i=1}^n p(E_i)\right) = \sum_{i=1}^n p(E_i)$.

If $p(E) = 0$, we say event E is *impossible*. It can be shown that probability satisfies other properties, most importantly the following two

$$p(E) = 1 - p(E^c), \tag{2.1}$$

$$p(A \cup B) = p(A) + p(B) - p(A \cap B), \tag{2.2}$$

where E^c is the *complement* of E, the set of all outcomes not in E. For simplicity, we will define $A,B = A \cap B$.

It may be the case that we are interested in the probability of a certain event A given that another event B had already occurred. In this situation, the sample space Ω is replaced by I, as only the outcomes in B can occur now. Given that $p(B) > 0$, we define the *conditional probability* of A given B as $p(A|B) = p(A,B)/p(B)$.

2.1.2 Probabilities on random variables

A *random variable* X is a function that, to each outcome ω in the sample space Ω, asigns it a real number $X(\omega) = x$. If the set of all possible values for X is countable we say X is *discrete*, otherwise we say is *continuous*.

We can define the probability of obtaining x as the probability of the event of all ω such that $X(\omega) = x$. If X is discrete, we can express this as

$$p(X = x) = p(\{\omega : X(\omega) = x\}). \tag{2.3}$$

However, if X is continuous, it may be the case that $\{\omega : X(\omega) = x\}$ has measure zero, thus $p(X = x) = 0$. We can, nonetheless, obtain the probability for the random variable obtaining a value within an interval. For the random variable X to be between the real values a and b, $a < b$, the probability is

$$p(a < X \leq b) = p(\{\omega : a < X(\omega) \leq b\}). \tag{2.4}$$

However, there is a better to express this probability. We define the *probability density* f of X as a real-valued function that satisfies

1. $f(x) \geq 0$, $\forall x \in \mathbb{R}$,

2. $\int_{-\infty}^{\infty} f(x)dx = 1,$

3. $p(a < X \leq b) = \int_{a}^{b} f(x)dx.$

The *expected value* of a random variable is the weighted mean of the values of the random variable, using the probabilities of those values as weights. The expected value is not necessarily a possible value for the random variable. For a discrete random variable X, the expected value is

$$\mathbb{E}(X) = \sum_{x \in I} xp(X = x). \tag{2.5}$$

where I is the set of all possible values for X. On the other hand, if X is continuous with density probability f, its expected value will be

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)dx. \tag{2.6}$$

We say that two random variables X and Y are *independent* if

$$p(X = x, Y = y) = p(X = x)p(Y = y). \tag{2.7}$$

From this, it can be proven that, if X and Y are independent random variables,

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y). \tag{2.8}$$

Indeed, let us consider the expectation value of a discrete random variable $Z = XY$

$$\mathbb{E}(Z) = \sum_z zp(Z=z) \tag{2.9}$$
$$= \sum_z \sum_{x,y:xy=z} xyp(X=x,Y=y) \tag{2.10}$$
$$= \sum_{x,y} xyp(X=x)p(Y=y) \tag{2.11}$$
$$= \sum_x xp(X=x) \sum_y yp(Y=y) \tag{2.12}$$
$$= \mathbb{E}(X)\mathbb{E}(Y). \tag{2.13}$$

We can condition the probability of a random variable upon a fixed value of another random variable. If both X and Y are discrete random variables, the *conditional probability* of X given $Y = y$, with y fixed and $p(Y = y) > 0$, is

$$p(X=x|Y=y) = \frac{p(X=x,Y=y)}{p(Y=y)}. \tag{2.14}$$

If, instead, X and Y are continuous random variables, we must consider the probability densities $f(x,y)$ of XY and $f_Y(y)$ of Y. The *conditional probability density* of X given $Y = y$, with y fixed and $f_Y(y) > 0$, is

$$f_X(x|y) = \frac{f(x,y)}{f_Y(y)}. \tag{2.15}$$

With conditional probabilities we can also condition the expected value of a random variable upon another. The *conditional expected value* of X given $Y = y$ is defined in the case of X and Y discrete as

$$\mathbb{E}(X|Y=y) = \sum_x xp(X=x|Y=y). \tag{2.16}$$

and in the case of X and Y continuous as

$$\mathbb{E}(X|Y=y) = \int_{-\infty}^{\infty} xf_X(x|y)dx. \tag{2.17}$$

2.2 Introduction to Quantum Mechanics

Quantum mechanics is a mathematical model useful for developing physical theories. It doesn't, by itself, provides us with the physical laws that rule a system, but instead creates a framework where such laws can be formulated [22, 23]. This framework is built on four postulates, which we will enunciate shortly.

2.2.1 Postulates of Quantum Mechanics

The first of these postulates makes reference to the manner used to describe a physical system.

Postulate 1. *A physical system is associated to a complex vector space with inner product (called a* Hilbert space*). A* state vector *is a ray in said Hilbert space that completely describes the system.*

Following Dirac's notation, we will represent a state vector by a *ket*, $|\psi\rangle$. The inner products between two vectors, $|\psi\rangle$ and $|\phi\rangle$, will be denoted as $\langle\phi|\psi\rangle$ and $\langle\psi|\phi\rangle$. Notice that, due to the Hilbert space being complex, $\langle\phi|\psi\rangle \neq \langle\psi|\phi\rangle$; actually it is verified that $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$. State vectors are also *normalized*, that is to say, $\langle\psi|\psi\rangle = 1$.

A transformation $\hat{U}$ is called *unitary* if it is invertible and its inverse is the self-adjoint of $\hat{U}$, that is, $\hat{U}\hat{U}^\dagger = \hat{I}$, where $\hat{I}$ is the identity transformation, that is, $\hat{I}|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle$ in the Hilbert space.

The second postulate describes the change of a quantum system with time.

Postulate 2. *A closed quantum system evolves according to a unitary transformation $\hat{U}$. For an initial time t_0, at any later time t the relationship between the initial and final states is given by:*

$$|\psi(t)\rangle = \hat{U}(t,t_0)|\psi(t_0)\rangle. \tag{2.18}$$

Postulate 2 describes the evolution of a closed quantum system, that is, a system that does not interact with any other system. However, to perform a measurement on the system we need to interact with it. The third postulate addresses this situation.

Postulate 3. *A quantum measurement consists of a set $\{\hat{M}_m\}$ of* measurement operators *that act on the Hilbert space of the quantum system. The index m labels the outcome of the measurement, and it ranges to include all posible results of said measurement. The probability of obtaining the result m when the system is in the state $|\psi\rangle$ inmediately before the measurement is*

$$p(m) = \langle\psi|\hat{M}_m^\dagger \hat{M}_m|\psi\rangle. \tag{2.19}$$

After the measurement, the new state of the system is

$$|\psi_m\rangle = \frac{\hat{M}_m|\psi\rangle}{\sqrt{\langle\psi|\hat{M}_m^\dagger \hat{M}_m|\psi\rangle}}. \tag{2.20}$$

The measurement operators must satisfy the completeness condition*,*

$$\sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{I}. \tag{2.21}$$

In some cases, we may be interested in the state of a system composed by two or more subsystems. The fourth and last postulate describes such a system in terms of the subsystems.

Postulate 4. *For a physical system composed of two or more distinct subsystems, its state space is the tensor product of the state spaces of the subsystems. If the subsystems are*

numbered from 1 to n, with subsystem number i prepared in the state $|\psi_i\rangle$*, then the state of the system is* $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

It is important to remark that not all states in the composite space are of this form; Postulate 4 tells us what the composite state is if we know the state for each subsystem, not the other way round. When the composite state can be written as in Postulate 4 for some choice of bases of the subsystem, we say that the system is prepared in a *separable* state. Otherwise, we say it is *entangled*.

2.2.2 The Density Matrix

Let us consider a system whose state vector is unknown to us, but we know in which state vectors the system may be, and with what probabilities. Let us call those states $|\psi_i\rangle$ and the respective probabilities p_i. The set of all this states with their corresponding probabilities, $\{p_i, |\psi_i\rangle\}$ is called an *ensemble of pure states*. We can then characterize the state of the system with a *density operator*, also called *density matrix*, defined by

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{2.22}$$

From Postulates 2 and 3, it is possible to show how evolution and measurement works with the density matrix representation of the system. Let us consider an ensemble in an initial time t_0, $\{p_i, |\psi_i(t_0)\rangle\}$. At a later time t, if the system was initially in the state $|\psi_i(t_0)\rangle$, it would now be in the state $|\psi_i(t)\rangle = \hat{U}(t,t_0)|\psi_i(t_0)\rangle$, so the new ensemble is $\{p_i, \hat{U}(t,t_0)|\psi_i(t_0)\rangle\}$. The new density matrix is

$$\rho(t) = \sum_i p_i \hat{U}(t,t_0)|\psi_i(t_0)\rangle\langle\psi_i(t_0)|\hat{U}^\dagger(t,t_0) = \hat{U}(t,t_0)\rho(t_0)\hat{U}^\dagger(t,t_0). \tag{2.23}$$

For a measurement $\{\hat{M}_m\}$ on the ensemble $\{p_i, |\psi_i\rangle\}$, we have that the probability of obtaining outcome m is

$$p(m) = \sum_i p(m|i)p_i \tag{2.24}$$

$$= \sum_i p_i \langle\psi_i|\hat{M}_m^\dagger \hat{M}_m|\psi_i\rangle \tag{2.25}$$

$$= \sum_i p_i \mathrm{tr}(\hat{M}_m^\dagger \hat{M}_m|\psi_i\rangle\langle\psi_i|) \tag{2.26}$$

$$= \mathrm{tr}(\hat{M}_m^\dagger \hat{M}_m \rho). \tag{2.27}$$

Also, if the system was in the state $|\psi_i\rangle$ before the measurement and outcome m was obtained, after the measurement the system is in the state

$$|\psi_i^m\rangle = \frac{\hat{M}_m|\psi_i\rangle}{\sqrt{\mathrm{tr}(\hat{M}_m^\dagger \hat{M}_m|\psi_i\rangle\langle\psi_i|)}}. \tag{2.28}$$

Therefore, the density matrix after the measurement is

$$\rho_m = \sum_i p(i|m)|\psi_i^m\rangle\langle\psi_i^m| = \sum_i p(i|m)\frac{\hat{M}_m|\psi_i\rangle\langle\psi_i|\hat{M}_m^\dagger}{\mathrm{tr}(\hat{M}_m^\dagger \hat{M}_m|\psi_i\rangle\langle\psi_i|)}. \tag{2.29}$$

Because $p(m,i) = p(m|i)p_i = p(i|m)p(m)$, we have

$$p(i|m) = \frac{p_i p(m|i)}{p(m)} = \frac{p_i \mathrm{tr}(\hat{M}_m^\dagger \hat{M}_m|\psi_i\rangle\langle\psi_i|)}{\mathrm{tr}(\hat{M}_m^\dagger \hat{M}_m \rho)}. \tag{2.30}$$

Then,

$$\rho_m = \sum_i p_i \frac{\hat{M}_m|\psi_i\rangle\langle\psi_i|\hat{M}_m^\dagger}{\mathrm{tr}(\hat{M}_m^\dagger \hat{M}_m \rho)} = \frac{\hat{M}_m \rho \hat{M}_m^\dagger}{\mathrm{tr}(\hat{M}_m^\dagger \hat{M}_m \rho)}. \tag{2.31}$$

We are interested in a way to characterize an operator as a density matrix. We will show that an operator ρ is a density matrix if and only if it satisfies the following conditions:

1. $\mathrm{tr}(\rho) = 1$,

2. $\rho > 0$ (ρ is a positive operator).

Let us consider ρ a density matrix arising from the ensemble $\{p_i, |\psi_i\rangle\}$, that is, $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Then $\text{tr}(\rho) = \sum_i p_i \text{tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1$. Also, for any $|\phi\rangle$, $\langle\phi|\rho|\phi\rangle = \sum_i p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle = \sum_i p_i |\langle\phi|\psi_i\rangle|^2 \geq 0$, so $\rho > 0$. To show that the equivalence goes both ways, let us consider an operator ρ such that $\text{tr}(\rho) = 1$ and $\rho > 0$. Then we have that $\rho = \sum_i a_i |\phi_i\rangle\langle\phi_i|$ with the $|\phi_i\rangle$ orthonormal and the a_i are real and non-negative, because $\rho > 0$. Also, $\text{tr}(\rho) = \sum_i a_i = 1$, so we can consider the a_i as probabilities and the operator ρ as a density matrix arising from the ensemble $\{a_i, |\phi_i\rangle\}$.

Our interest in the density matrix representation of the quantum state comes from the fact that there are states that can be represented by a density matrix but not by a state vector. It is easy to show that every vector is associated with a density matrix, in particular, to the projector of such vector.

For a state represented by a density matrix ρ, if there exists a vector $|\psi\rangle$ such that $\rho = |\psi\rangle\langle\psi|$, then we say that the system is in a *pure state*. Otherwise, we say it is in a *mixed state* and it can not be represented by a state vector.

A state ρ can be determined as pure or mixed by evaluating the trace of ρ^2. Let us express ρ as its spectral descomposition, $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with the $|\psi_i\rangle$ orthonormal, so $\rho^2 = \sum_i p_i^2 |\psi_i\rangle\langle\psi_i|$ and $\text{tr}(\rho^2) = \sum_i p_i^2 \leq 1$. Because $p_i^2 \leq p_i$, equality will be achieved if and only if one of the p_i equals 1 and the rest are 0. This corresponds to the case of a pure state, so a state ρ is pure if and only if $\text{tr}(\rho^2) = 1$.

2.2.3 Quantum Measurement Operators

In postulate 3 we introduced the measurement operators $\{\hat{M}_m\}$. We will present some of the most important classes of measurements.

The simplest type of measurement is the *projective measurement*. A projective measurement is associated to an hermitian operator called an *observable*. An observable $\hat{O}$ can be spectrally decomposed as $\hat{O} = \sum_m m\hat{\pi}_m$, with $\hat{\pi}_m$ projectors onto the eigenstates of $\hat{O}$, $|\phi_m\rangle$, and m their respectives eigenvalues. We call operator $\hat{O}$ an observable because the possible outcomes of performing the measurement associated to $\{\hat{\pi}_m\}$ are the eigenstates m.

Considering that projectors are hermitian and idempotent, the probability to obtain outcome m upon measuring the state vector $|\psi\rangle$ is, according to (2.19),

$$p(m) = \langle\psi|\hat{\pi}_m|\psi\rangle, \tag{2.32}$$

and the state after the measurement is, by (2.20)

$$|\psi_m\rangle = \frac{\hat{\pi}_m|\psi\rangle}{\sqrt{\langle\psi|\hat{\pi}_m|\psi\rangle}}. \tag{2.33}$$

For a density matrix ρ, we have, from (2.27) and (2.31), the probability of outcome m and the new state after measurement are

$$p(m) = \mathrm{tr}(\rho\hat{\pi}_m), \tag{2.34}$$

$$\rho_m = \frac{\hat{\pi}_m\rho\hat{\pi}_m}{\mathrm{tr}(\rho\hat{\pi}_m)}. \tag{2.35}$$

Another, more general, choice of measurement is the *Positive Operator-Valued Measure* (or *POVM*). A POVM is a set $\{\hat{E}_m\}$ such that the operators $\hat{E}_m$ are all positive and satisfy the relation $\sum_m \hat{E}_m = \hat{I}$. A POVM can indeed be used to describe a measurement; because the $\hat{E}_m$ are positive, we can consider $\hat{M}_m = \sqrt{\hat{E}_m}$ such that $\hat{M}_m^\dagger\hat{M}_m = \hat{E}_m$. Then, $\sum_m \hat{E}_m = \sum_m \hat{M}_m^\dagger\hat{M}_m = \hat{I}$, satisfying the completeness condition (2.21), thus, $\{M_m\}$ is a measurement.

It can be easily seen that the probability of obtaining outcome m when the system is in state $|\psi\rangle$ is

$$p(m) = \langle\psi|\hat{E}_m|\psi\rangle,, \tag{2.36}$$

and when it is in state ρ is

$$p(m) = \mathrm{tr}(\rho\hat{E}_m). \tag{2.37}$$

A particular case of POVM we are interested in is the *symetrically informatically complete POVM* (*SIC-POVM*). For a Hilbert space of dimension d, a SIC-POVM is a set of d^2 operators $\hat{E}_m = \hat{\pi}_m/d$, where the $\hat{\pi}_m$ are projectors that satisfy

$$\mathrm{tr}(\hat{\pi}_i\hat{\pi}_j) = \frac{1}{d+1} \quad \text{for all } i \neq j. \tag{2.38}$$

It is not known if SIC-POVMs exist in all dimensions, but it has been conjectured so [24–27].

2.2.4 Mutually Unbiased Bases

For a Hilbert space of dimension d, let us consider two or more orthonormal bases of state vectors $\{|\alpha_i\rangle\}$, where α labels each basis and i labels each element of a given basis. We call those bases a set of *mutually unbiased bases* (*MUBs*) if and only if they satisfy the condition [28]

$$|\langle\alpha_i|\beta_j\rangle|^2 = \frac{1}{d}(1-\delta_{i,j}) + \delta_{i,j}\delta_{\alpha,\beta}. \tag{2.39}$$

From this equation we have that, for $\alpha \neq \beta$, i.e., different bases, $|\langle\alpha_i|\beta_j\rangle|^2 = 1/d$

For the case where the dimension d is a prime power, the system can have a set of MUBs with up to $d+1$ bases [28–30]. For spaces with dimension not a prime power, it is not

known how many MUBs the system allows.

2.3 Tomography of Quantum States via Postprocessed State Estimation

These postulates provide us with the method for predicting the results of a measurement on a system —*if* we know the state of the system. As shown in postulate 3, measurements on the system give us a randomly chosen outcome, and each outcome transforms the state of the system after the measurement. It is not possible to directly obtain the state of a system experimentally.

What is, in fact, possible, is to obtain the probabilities for each outcome by performing a convenient set measurements on multiple systems, all prepared in the same way, and with this information attempt to reconstruct the density matrix of the system. This procedure is known as *quantum tomography* and it is the main focus of this thesis.

We will present some common tomography schemes that postprocess the gathered data to accurately estimate the state of the system.

2.3.1 Standard Quantum Tomography

For a d-dimensional system, we will call its state a *qudit*. The density matrix ρ_d of a qudit has d^2 coefficients, so we need a set of d^2 operators, however, because $\mathrm{tr}(\rho_d) = 1$, only $d^2 - 1$ measurements will be necessary to reconstruct the state. We start by defining the d-dimensional elementary matrices e^i_j

$$(e^i_j)_{\mu,\nu} = \delta_{i,\mu}\delta_{j,\nu}, \tag{2.40}$$

with $1 \leq \mu, \nu \leq d$, μ, ν labeling the position within the matrix. Clearly, e^i_j is the matrix with a single 1 in the ith row, jth column, and 0s in every other position. Also,

$\mathrm{tr}(e^i_j) = \sum_{\mu=1}^{d}(e^i_j)_{\mu,\mu} = \sum_{\mu=1}^{d}\delta_{i,\mu}\delta_{j,\mu} = \delta_{i,j}$. The product of elementary matrices is given by $(e^i_j e^k_l)_{\mu,\nu} = \sum_{\lambda=1}^{d}(e^i_j)_{\mu,\lambda}(e^k_l)_{\lambda,\nu} = \sum_{\lambda=1}^{d}\delta_{i,\mu}\delta_{j,\lambda}\delta_{k,\lambda}\delta_{l,\nu} = \delta_{i,\mu}\delta_{j,k}\delta_{l,\nu} = (e^i_l)_{\mu,\nu}\delta_{j,k}$, thus

$$e^i_j e^k_l = e^i_l \delta_{j,k}. \tag{2.41}$$

From here, we can express the generators of the *special unitary group of degree d* (or SU(d)), composed by all unitary $d \times d$ matrices of determinant 1, as [31]

$$\Theta^i_j = e^i_j + e^j_i, \tag{2.42}$$

$$\beta^i_j = -i(e^i_j - e^j_i), \tag{2.43}$$

$$\eta^k_k = \sqrt{\frac{2}{k(k+1)}}\sum_{j=1}^{k}(e^j_j - e^{k+1}_{k+1}), \tag{2.44}$$

with $1 \leq i < j \leq d$ and $1 \leq k \leq d-1$. There are $d(d-1)$ Θ^i_j matrices, as well as β^i_j, and $d-1$ η^k_k matrices. All of them are traceless. The η^k_k are diagonal matrices, while the Θ^i_j and the β^i_j have no diagonal elements.

We can label the generators of SU(d) using a single letter by

$$\hat{\sigma}_{(j-1)^2+2(i-1)} = \Theta^i_j, \tag{2.45}$$

$$\hat{\sigma}_{(j-1)^2+2i-1} = \beta^i_j, \tag{2.46}$$

$$\hat{\sigma}_{j^2-1} = \eta^{j-1}_{j-1}. \tag{2.47}$$

We also define $\hat{\sigma}_0 = \hat{I}_d$. We can then write the density matrix as

$$\rho_d = \frac{1}{d}\sum_{j=0}^{d^2-1} r_j \hat{\sigma}_j, \tag{2.48}$$

where $r_j \in \mathbb{R}$. To find r_0, we just need to trace (2.48), so $1 = \frac{1}{d}\sum_{j=0}^{d^2-1} r_j \mathrm{tr}(\hat{\sigma}_j) = \frac{1}{d} r_0 \mathrm{tr}(\hat{\sigma}_0) = r_0$. Obtaining the other r_j is trickier. From (2.48), we have

$$\mathrm{tr}(\rho_d \hat{\sigma}_k) = \frac{1}{d}\sum_{j=0}^{d^2-1} r_j \mathrm{tr}(\hat{\sigma}_j \hat{\sigma}_k). \tag{2.49}$$

To determine the value of $\mathrm{tr}(\hat{\sigma}_j\hat{\sigma}_k)$, we will have to study each case individually.

- Both $\hat{\sigma}_j$ and $\hat{\sigma}_k$ are Θ matrices:

$$\mathrm{tr}(\Theta^i_j\Theta^k_l) = \mathrm{tr}(e^i_l\delta_{j,k} + e^i_k\delta_{j,l} + e^j_k\delta_{i,l} + e^j_l\delta_{i,k}) \tag{2.50}$$

$$= 2\delta_{i,l}\delta_{j,k} + 2\delta_{i,k}\delta_{j,l}. \tag{2.51}$$

  However, because $i < j$ and $k < l$, if $j = k$ it is impossible that $i = l$, therefore,

$$\mathrm{tr}(\Theta^i_j\Theta^k_l) = 2\delta_{i,k}\delta_{j,l}. \tag{2.52}$$

- Both $\hat{\sigma}_j$ and $\hat{\sigma}_k$ are β matrices:

$$\mathrm{tr}(\beta^i_j\beta^k_l) = -\mathrm{tr}(e^i_l\delta_{j,k} - e^i_k\delta_{j,l} + e^j_k\delta_{i,l} - e^j_l\delta_{i,k}) \tag{2.53}$$

$$= -2\delta_{i,l}\delta_{j,k} + 2\delta_{i,k}\delta_{j,l}. \tag{2.54}$$

  Again, if $j = k$ then $i \neq l$, so

$$\mathrm{tr}(\beta^i_j\beta^k_l) = 2\delta_{i,k}\delta_{j,l}. \tag{2.55}$$

- Both $\hat{\sigma}_j$ and $\hat{\sigma}_k$ are η matrices (we assume $k \leq l$):

$$\mathrm{tr}(\eta^k_k\eta^l_l) = \frac{2}{\sqrt{k(k+1)l(l+1)}} \times \sum_{i=1}^{k}\sum_{j=1}^{l} \mathrm{tr}(e^i_j\delta_{i,j} - e^i_{l+1}\delta_{i,l+1} - e^{k+1}_j\delta_{k+1,j} + e^{k+1}_{l+1}\delta_{k+1,l+1}) \tag{2.56}$$

$$= \frac{2}{\sqrt{k(k+1)l(l+1)}}(k - k(1-\delta_{k,l}) + kl\delta_{k,l}) \tag{2.57}$$

$$= \frac{2(kl+k)\delta_{k,l}}{\sqrt{k(k+1)l(l+1)}} \tag{2.58}$$

$$= \frac{2k(k+1)\delta_{k,l}}{k(k+1)} \tag{2.59}$$

$$= 2\delta_{k,l}. \tag{2.60}$$

- $\hat{\sigma}_j$ is a Θ matrix and $\hat{\sigma}_k$ is a β matrix:

$$\mathrm{tr}(\Theta^i_j \beta^k_l) = -i \cdot \mathrm{tr}(e^i_l \delta_{j,k} - e^i_k \delta_{j,l} - e^j_k \delta_{i,l} + e^j_l \delta_{i,k}) \tag{2.61}$$

$$= 0. \tag{2.62}$$

- $\hat{\sigma}_j$ is a Θ matrix and $\hat{\sigma}_k$ is a η matrix:

$$\mathrm{tr}(\Theta^i_j \eta^k_k) = \sqrt{\frac{2}{k(k+1)}} \sum_{l=1}^{k} \mathrm{tr}(e^i_l \delta_{j,l} - e^i_{k+1} \delta_{j,k+1} - e^j_{k+1} \delta_{i,k+1} + e^j_l \delta_{i,l}) \tag{2.63}$$

$$= \sqrt{\frac{2}{k(k+1)}} \sum_{l=1}^{k} (\delta_{i,l}\delta_{j,l} - \delta_{i,k+1}\delta_{j,k+1} - \delta_{j,k+1}\delta_{i,k+1} + \delta_{j,l}\delta_{i,l}) \tag{2.64}$$

. Because $i \neq j$:

$$\mathrm{tr}(\Theta^i_j \eta^k_k) = 0. \tag{2.65}$$

- $\hat{\sigma}_j$ is a β matrix and $\hat{\sigma}_k$ is a η matrix:

$$\mathrm{tr}(\beta^i_j \eta^k_k) = -i\sqrt{\frac{2}{k(k+1)}} \sum_{l=1}^{k} \mathrm{tr}(e^i_l \delta_{j,l} - e^i_{k+1} \delta_{j,k+1} + e^j_{k+1} \delta_{i,k+1} - e^j_l \delta_{i,l}) \tag{2.66}$$

$$= -i\sqrt{\frac{2}{k(k+1)}} \sum_{l=1}^{k} (\delta_{i,l}\delta_{j,l} - \delta_{i,k+1}\delta_{j,k+1} + \delta_{j,k+1}\delta_{i,k+1} - \delta_{j,l}\delta_{i,l}) \tag{2.67}$$

$$= 0. \tag{2.68}$$

We can conclude then that $\mathrm{tr}(\hat{\sigma}_j \hat{\sigma}_k) \neq 0$ if and only if $\hat{\sigma}_j = \hat{\sigma}_k$ and $\mathrm{tr}(\hat{\sigma}^2_j) = 2$. Therefore, in (2.49), all terms in the sum with $j \neq k$ vanish and we get, for $1 \leq k \leq d^2 - 1$

$$\mathrm{tr}(\rho_d \hat{\sigma}_k) = \frac{1}{d} r_k \mathrm{tr}(\hat{\sigma}^2_k), \tag{2.69}$$

$$r_k = \frac{d}{2} \mathrm{tr}(\rho_d \hat{\sigma}_k). \tag{2.70}$$

We can obtain $\mathrm{tr}(\rho_d \hat{\sigma}_k)$ experimentally, so we can reconstruct the density matrix with this procedure using $d^2 - 1$ measurements.

Let us study the case where $d = 2$. The generators of SU(2) are

$$\hat{\sigma}_1 = \Theta_2^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.71}$$

$$\hat{\sigma}_2 = \beta_2^1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \tag{2.72}$$

$$\hat{\sigma}_3 = \eta_1^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.73}$$

We also have that $r_k = \mathrm{tr}(\rho_2 \hat{\sigma}_k)$ for $k = 1,2,3$ amd $r_0 = 1$. Therefore, we can express ρ_2 as

$$\rho_2 = \frac{1}{2}\begin{bmatrix} 1 + \mathrm{tr}(\rho_2\hat{\sigma}_3) & \mathrm{tr}(\rho_2\hat{\sigma}_1) - i\cdot\mathrm{tr}(\rho_2\hat{\sigma}_2) \\ \mathrm{tr}(\rho_2\hat{\sigma}_1) + i\cdot\mathrm{tr}(\rho_2\hat{\sigma}_2) & 1 - \mathrm{tr}(\rho_2\hat{\sigma}_3) \end{bmatrix}. \tag{2.74}$$

2.3.2 MUB Tomography

Let us consider a Hilbert space with dimension d a prime power, so the system allows for a full set of MUBs $\{\{|\alpha_i\rangle\}_{i=1}^{d}\}_{\alpha=1}^{d+1}$. We denote the projector onto the state $|\alpha_i\rangle$ as $\hat{\Pi}_{\alpha,i} = |\alpha_i\rangle\langle\alpha_i|$ and the probability of obtaining the outcome associated to $|\alpha_i\rangle$ is $p_{\alpha,i} = \mathrm{tr}(\rho\hat{\Pi}_{\alpha,i})$. Becuase the $|\alpha_i\rangle$ for a fixed α form a basis of the Hilbert space, it is verified that

$$\sum_{i=1}^{d} \hat{\Pi}_{\alpha,i} = \hat{I} \tag{2.75}$$

and

$$\sum_{i=1}^{d} p_{\alpha,i} = 1 \tag{2.76}$$

for every α. Because of the dependence between the projectors on the same basis, it is possible to write the density matrix as

$$\rho = \frac{1}{d}\hat{I} + \sum_{\beta=1}^{d+1}\sum_{j=1}^{d-1} c_{\beta,j}\left(\hat{\Pi}_{\beta,j} - \frac{1}{d}\hat{I}\right), \tag{2.77}$$

for some parameters $c_{\beta,j}$. Following the proof in [28], we will show how to perform tomography with MUBs.

Multiplying by $\hat{\Pi}_{\alpha,i}$ and applying the trace in (2.77), we have a new expression for the probabilities

$$p_{\alpha,i} = \frac{1}{d}\sum_{\beta=1}^{d+1} + \sum_{j=1}^{d-1} M_{\alpha i,\beta j} c_{\beta,j}, \tag{2.78}$$

with $M_{\alpha i,\beta j} = \mathrm{tr}(\hat{\Pi}_{\alpha,i}\hat{\Pi}_{\beta,j}) - 1/d = \delta_{\alpha,\beta}(\delta_{i,j} - 1/d)$. We can reduce the index αi as a new one, $k = (\alpha-1)(d-1)+i$, $k = 1,...,d^2-1$. Therefore, we can rewrite (2.78) as a system of equations using the matrix M like this

$$\begin{bmatrix} p_1 - 1/d \\ \vdots \\ p_{d^2-1} - 1/d \end{bmatrix} = M \begin{bmatrix} c_1 \\ \vdots \\ c_{d^2-1} \end{bmatrix}, \tag{2.79}$$

where M is, by definition of the individual $M_{\alpha i,\beta j}$, equal to a $(d^2-1)\times(d^2-1)$ matrix

$$M = \begin{bmatrix} \mathscr{M} & 0 & \cdots & 0 \\ 0 & \mathscr{M} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathscr{M} \end{bmatrix}, \tag{2.80}$$

with the $\mathscr{M}$ being $(d-1)\times(d-1)$ submatrices

$$\mathscr{M} = \frac{1}{d}\begin{bmatrix} d-1 & -1 & \cdots & -1 \\ -1 & d-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & d-1 \end{bmatrix}. \tag{2.81}$$

It can be easily checked than its inverse matrix is

$$\mathscr{M}^{-1} = \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix}. \tag{2.82}$$

From this matrix and (2.79), we obtain

$$c_{\alpha,i} = 2p_{\alpha,i} - \frac{2}{d} + \sum_{\beta\neq\alpha}^{d-1}\left(p_{\beta,j} - \frac{1}{d}\right) \tag{2.83}$$

$$= p_{\alpha,i} - \frac{1}{d} + \sum_{\beta=1}^{d-1}\left(p_{\beta,j} - \frac{1}{d}\right) \tag{2.84}$$

$$= p_{\alpha,i} - \frac{1}{d} + 1 - p_{\beta,d} - \frac{d-1}{d} \tag{2.85}$$

$$= p_{\alpha,i} - p_{\beta,d}. \tag{2.86}$$

Substituting in (2.77), we have

$$\rho = \frac{1}{d}\hat{I} + \sum_{\alpha=1}^{d+1}\sum_{i=1}^{d-1}(p_{\alpha,i} - p_{\alpha,d})\left(\hat{\Pi}_{\alpha,i} - \frac{1}{d}\hat{I}\right) \tag{2.87}$$

$$= \frac{1}{d(d+1)}\sum_{\alpha=1}^{d+1}\sum_{i=1}^{d} p_{\alpha,i}\hat{I} + \sum_{\alpha=1}^{d+1}\sum_{i=1}^{d-1}(p_{\alpha,i} - p_{\alpha,d})\left(\hat{\Pi}_{\alpha,i} - \frac{1}{d}\hat{I}\right) \tag{2.88}$$

$$= \frac{1}{d(d+1)}\sum_{\alpha=1}^{d+1} p_{\alpha,d}\hat{I} + \sum_{\alpha=1}^{d}\sum_{i=1}^{d-1} p_{\alpha,i}\left(\hat{\Pi}_{\alpha,i} - \frac{1}{d}\hat{I} + \frac{1}{d(d+1)}\hat{I}\right)$$
$$- \sum_{\alpha=1}^{d+1} p_{\alpha,d}\sum_{i=1}^{d-1}\left(\hat{\Pi}_{\alpha,i} - \frac{1}{d}\hat{I}\right) \tag{2.89}$$

$$= \sum_{\alpha=1}^{d+1}\sum_{i=1}^{d-1} p_{\alpha,i}\left(\hat{\Pi}_{\alpha,i} - \frac{1}{d+1}\hat{I}\right)$$
$$- \sum_{\alpha=1}^{d+1} p_{\alpha,d}\left(\hat{I} - \hat{\Pi}_{\alpha,d} - \frac{(d-1)}{d}\hat{I} - \frac{1}{d(d+1)\hat{I}}\right) \tag{2.90}$$

$$= \sum_{\alpha=1}^{d+1}\sum_{i=1}^{d-1} p_{\alpha,i}\left(\hat{\Pi}_{\alpha,i} - \frac{1}{d+1}\hat{I}\right) - \sum_{\alpha=1}^{d+1} p_{\alpha,d}\left(\frac{1}{d+1}\hat{I} - \hat{\Pi}_{\alpha,d}\right) \tag{2.91}$$

$$= \sum_{\alpha=1}^{d+1}\sum_{i=1}^{d} p_{\alpha,i}\left(\hat{\Pi}_{\alpha,i} - \frac{1}{d+1}\hat{I}\right). \tag{2.92}$$

By estimating $p_{\alpha,i}$ experimentally, it is then possible to reconstruct the density matrix ρ.

2.3.3 SIC-POVM Tomography

Let us consider a Hilbert space of dimension d and let us suppose that for d, there exists a SIC-POVM $\{\hat{E}_i\}_{i=1}^{d^2} = \{\hat{\pi}_i/d\}_{i=1}^{d^2}$. Because the SIC-POVM contains d^2 elements, it is

enough to reconstruct the density matrix ρ as

$$\rho = \sum_{i=1}^{d^2} c_i \hat{\pi}_i, \tag{2.93}$$

where the c_i are real coefficients. From the trace of both sides of (2.93) we have that $\sum_{i=1}^{d^2} c_i = 1$. If we apply π_j on both sides of (2.93) and then we trace, we obtain

$$d\mathrm{tr}(\rho \hat{E}_j) = \sum_{i \neq j}^{d^2} \frac{c_i}{d+1} + c_j = \frac{1-c_j}{d+1} + c_j, \tag{2.94}$$

$$c_j = (d+1)\mathrm{tr}(\rho \hat{E}_j) - \frac{1}{d}. \tag{2.95}$$

And so

$$\rho = \sum_{i=1}^{d^2} \left[(d+1)\mathrm{tr}(\rho \hat{E}_i) - \frac{1}{d} \right] \hat{\pi}_i. \tag{2.96}$$

From here, it is possible to reconstruct the density matrix in a laboratory by estimating $\mathrm{tr}(\rho \hat{E}_i)$.

2.3.4 Maximum Likelihood Estimation

We have seen that it is possible to reconstruct the density matrix using experimental data, either the expectation values or probabilities for each outcome. However, it may be the case that this result, the tomographic density matrix, is not actually a physical density matrix. By construction, it's trace equals one and it is hermitian, but it is not always a positive operator. It is necessary to process the tomographic density matrix to derive a valid physical density matrix. One technique useful for this task is the maximum likelihood estimation [32].

To ensure that the physical density matrix ρ_p is positive, we will write it as

$$\rho_p(t) = \frac{\hat{T}^\dagger(t)\hat{T}(t)}{\mathrm{tr}(\hat{T}^\dagger(t)\hat{T}(t))}, \tag{2.97}$$

where $t = \{t_i\}_{i=1}^{d^2}$ is a set of real parameters and

$$\hat{T}(t) = \begin{bmatrix} t_1 & 0 & \cdots & 0 \\ t_{n+1} + it_{n+2} & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ t_{d^2-1} + it_{d^2} & t_{d^2-3} + it_{d^2-2} & \cdots & t_d \end{bmatrix}. \tag{2.98}$$

The maximum likelihood estimation consists in the optimization problem of finding the t that minimizes the function

$$\mathscr{L}(t) = \sum_{j=1}^{d^2} \frac{(\mathbb{E}[n_j|\rho_p(t)] - n_j)^2}{2\mathbb{E}[n_j|\rho_p(t)]}, \tag{2.99}$$

where n_j is the number of times we obtain outcome j. As our initial guess, we will consider the density matrix in (2.97) as our tomographic density matrix, thus determining t; because the parameters obtained may be complex, we will only consider the real part of t.

2.4 Tomography of Quantum States via Optimization

It is possible to design a tomography scheme that has no need of postprocessing. Instead, we define a distance in the Hilbert space, and we seek to minimize the function defined as the distance to the state of the system. For this purpose, we will briefly introduce the *descent methods* for minimizing a function.

2.4.1 Descent Methods

We are interested in finding $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, with n natural, such that, for $f : \mathbb{R}^n \to \mathbb{R}$ continuously differentiable,

$$f(\tilde{\boldsymbol{x}}) = \min_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}). \tag{2.100}$$

This implies that $\tilde{\boldsymbol{x}}$ is a local minimum, so it must satisfy

$$\nabla f(\tilde{\boldsymbol{x}}) = 0. \tag{2.101}$$

Finding $\tilde{\boldsymbol{x}}$ is usually impossible using analytic methods, so we will use a iterative algorithm that provide us with a sequence $\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \ldots$ such that $f(\hat{\boldsymbol{x}}_{k+1}) < f(\hat{\boldsymbol{x}}_k)$. We make the algorithm stop when $\nabla f(\hat{\boldsymbol{x}}_k)$ is close enough to zero, that is, $\nabla f(\hat{\boldsymbol{x}}_k) < \boldsymbol{\varepsilon}$ for a given $\boldsymbol{\varepsilon} > 0$.

We will consider iterative algorithms of the form

$$\hat{\boldsymbol{x}}_{k+1} = \hat{\boldsymbol{x}}_k + t_k \boldsymbol{\Lambda}_k, \tag{2.102}$$

where $\boldsymbol{\Lambda}_k \in \mathbb{R}^n$ is the *step direction* and $t_k \in \mathbb{R}$ is the *step size*. A valid descent method defines step directions and sizes such that $f(\hat{\boldsymbol{x}}_{k+1}) < f(\hat{\boldsymbol{x}}_k)$ is verified.

The simplest choice of step direction is the *gradient descent method*, where we choose $\boldsymbol{\Lambda}_k = -\nabla f(\hat{\boldsymbol{x}}_k)$. Some simple choices for the step size are *constant step size* ($t_k = t$) and *diminishing step size* ($\lim_{k\to\infty} t_k = 0$ while $\sum_{k=1}^{\infty} t_k$ diverges).

2.4.2 Simultaneous Perturbation Stochastic Approximation

The *simultaneous perturbation stochastic approximation* (*SPSA*), presented in [33], is a descent method that uses a diminishing step size and an approximation of the gradient as its step direction. As stated previously, we are looking for $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$ such that

$$\boldsymbol{g}(\tilde{\boldsymbol{x}}) = \nabla f(\tilde{\boldsymbol{x}}) = 0. \tag{2.103}$$

The iterative algorithm is of the form

$$\hat{\boldsymbol{x}}_{k+1} = \hat{\boldsymbol{x}}_k + a_k \hat{\boldsymbol{g}}_k(\hat{\boldsymbol{x}}_k), \tag{2.104}$$

where a_k is the step size and $\hat{\boldsymbol{g}}_k \in \mathbb{R}^n$ is an estimate of $\boldsymbol{g}$ at the k-th iteration of the algorithm, with $\hat{g}_{k,i}(\hat{\boldsymbol{x}}_k)$, the i-th term of $\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{x}}_k)$, given by

$$\hat{g}_{k,i}(\hat{\boldsymbol{x}}_k) = \frac{f(\hat{\boldsymbol{x}}_k + c_k \boldsymbol{\Delta}_k) + \varepsilon_{k,+} - f(\hat{\boldsymbol{x}}_k - c_k \boldsymbol{\Delta}_k) - \varepsilon_{k,-}}{2 c_k \Delta_{k,i}}, \tag{2.105}$$

where c_k is a positive parameter, $\boldsymbol{\Delta}_k \in \mathbb{R}^n$ is a randomly generated vector whose components are $\Delta_{k,i}$, and $\varepsilon_{k,+}$ and $\varepsilon_{k,-}$ are noise terms that satisfy $\mathbb{E}(\varepsilon_{k,+} - \varepsilon_{k,-} | \boldsymbol{\Delta}_k, \mathscr{F}_k) = 0$, $\mathscr{F}_k$ defined as the set of all estimates of $\boldsymbol{x}$ until the k-th iteration, $\{\hat{\boldsymbol{x}}_1, ..., \hat{\boldsymbol{x}}_k\}$.

We can identify two additional terms in the algorithm that will be of further interest. The first is the *bias* in the estimator $\hat{\boldsymbol{g}}_k$, defined as

$$\boldsymbol{b}_k(\hat{\boldsymbol{x}}_k) = \mathbb{E}(\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{x}}_k) - \boldsymbol{g}(\hat{\boldsymbol{x}}_k) | \hat{\boldsymbol{x}}_k). \tag{2.106}$$

The other is the *error* term,

$$\boldsymbol{e}_k(\hat{\boldsymbol{x}}_k) = \hat{\boldsymbol{g}}_k(\hat{\boldsymbol{x}}_k) - \mathbb{E}(\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{x}}_k) | \hat{\boldsymbol{x}}_k). \tag{2.107}$$

It is then possible to rewrite algorithm (2.104) as

$$\hat{\boldsymbol{x}}_{k+1} = \hat{\boldsymbol{x}}_k - a_k[\boldsymbol{g}(\hat{\boldsymbol{x}}_k) + \hat{\boldsymbol{b}}_k(\hat{\boldsymbol{x}}_k) + \hat{\boldsymbol{e}}_k(\hat{\boldsymbol{x}}_k)]. \tag{2.108}$$

We will present two results in [33] relevant to this work: Lemma 1 and Proposition 1, respectively

Theorem 1 (Bias in the estimator of the gradient). *For α_0, α_1 and α_2 positive real constants and $\Omega = \{\omega\}$ the sample space that generates the sequence $\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, ...$, consider all $k \geq K$ for some $K < \infty$, and suppose that for each such k the $\Delta_{k,i}$ are independent and identically distributed, and symetrically distributed about 0 (that is, $\mathbb{E}(\Delta_{k,i}) = 0$) with $|\Delta_{k,i}| \leq \alpha_0$ almost surely[1] and $\mathbb{E}(|\Delta_{k,i}^{-1}|) \leq \alpha_1$. For almost all $\hat{\boldsymbol{x}}_k$ (at each $k \geq K$) suppose that for all $\boldsymbol{x}$ in an open neighborhood of $\hat{\boldsymbol{x}}_k$, that is not an function of k or ω, $f \in \mathscr{C}^3(\mathbb{R}^n)$ with $|\partial_{x_i}\partial_{x_j}\partial_{x_l} f(\boldsymbol{x})| \leq \alpha_2$. Then for almost all $\omega \in \Omega$*

$$\boldsymbol{b}_k(\hat{\boldsymbol{x}}_k) = \mathscr{O}(c_k^2). \tag{2.109}$$

[1] A property is satisfied *almost surely* (*a.s*) if it is valid in all but a subset of zero measure of the sample space

Theorem 2 (Convergence of the SPSA algorithm). *Let the conditions of theorem 1 and the following assumptions hold:*

A1: $a_k, c_k > 0\ \forall k$; $\lim_{k\to\infty}(|a_k| + |c_k|) = 0$; $\sum_{k=1}^{\infty} a_k = \infty$; $\sum_{k=1}^{\infty}(a_k/c_k)^2 < \infty$,

A2: *For some* $\beta_0, \beta_1, \beta_2 > 0$, $\forall k$, $\mathbb{E}(\varepsilon_{k,\pm}^2) \leq \alpha_0$, $\mathbb{E}([f(\hat{\boldsymbol{x}}_k \pm c_k \boldsymbol{\Delta}_k)]^2)\alpha_1$, *and* $\mathbb{E}(\Delta_{k,i}^{-2}) \leq \alpha_2$,

A3: $\|\hat{\boldsymbol{x}}_k\| < \infty$ *a.s.* $\forall k$,

A4: $\boldsymbol{x}(t) = \tilde{\boldsymbol{x}}$ *is an asymptotically stable solution of the differential equation* $\mathrm{d}\boldsymbol{x}(t)/\mathrm{d}t = -\boldsymbol{g}(\boldsymbol{x})$,

A5: *Consider the domain of attraction* $D(\tilde{\boldsymbol{x}}) = \{\boldsymbol{x}_0 : \lim_{t\to\infty} \boldsymbol{x}(t|\boldsymbol{x}_0) = \tilde{\boldsymbol{x}}\}$ *where* $\boldsymbol{x}(t|\boldsymbol{x}_0)$ *denotes the solutions to the differential equation of A4 based on initial conditions* $\boldsymbol{x}(0) = \boldsymbol{x}_0$. *There exists a compact* $S \subseteq D(\tilde{\boldsymbol{x}})$ *such that* $\hat{\boldsymbol{x}}_k \in S$ *infinitely often for almost all sample points.*

Then

$$\lim_{k\to\infty} \hat{\boldsymbol{x}}_k = \tilde{\boldsymbol{x}}. \tag{2.110}$$

2.4.3 Self-Guided Quantum Tomography

Originally proposed in [19], *self-guided quantum tomography* (*SGQT*) is a tomography technique quite different from those already mentioned, mainly because it has no need of postprocessing the data using maximum likelihood estimation or another similar method. Instead, SGQT uses SPSA to minimize the distance between the state of the system and an estimate of it.

We consider the infidelity function between two pure states, $m(\psi, \phi) = 1 - |\langle\psi|\phi\rangle|^2$. In principle, any distance measure function between states can be used, but the infidelity of two pure states is easier to determine experimentally. If we denote the state of the system

as $|\psi\rangle$, we can define $f(|\phi\rangle_k) = m(\psi, \phi_k)$ and use the algorithm (2.104) with $|\phi_k\rangle$ instead of $\hat{x}_k$ and using $f(|\phi\rangle_k \pm c_k \Delta_k)$ in (2.105). Here, Δ_k is a vector of the same dimension as the Hilbert space of the quantum system, with all its terms randomly chosen as $+1$ or -1. The parameters a_k and c_k are defined as

$$a_k = \frac{A}{(k+1+B)^s}, \tag{2.111}$$

$$c_k = \frac{C}{(k+1)^r}, \tag{2.112}$$

where A, B, C, s and r are chosen numerically and depend on the target function, considering they must satisfy assumption A1 of theorem 2. According to [19, 34], good choices for these values are $A = 3$, $B = 0$, $C = 0.1$, $s = 0.602$ and $r = 0.101$.

It has been shown [19, 20] that SGQT can provide better aproximations of the quantum state of the system than standard quantum tomography using the same number of resources. Therefore, SGQT not only ends with the estimator for the state, thus making no use of postprocessing; it is a very robust tomography scheme as well. However, it needs to use more measurement bases compared with standard methods, specially at lower dimensions.

2.5 Wirtinger calculus

We define the sets $\mathfrak{R}^n = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\}$, $\mathfrak{C}^n = \{(\boldsymbol{z}, \boldsymbol{z}^*) : \boldsymbol{z} \in \mathbb{C}^n\}$ and $\mathfrak{C}^{*n} = \{(\boldsymbol{z}^*, \boldsymbol{z}) : \boldsymbol{z} \in \mathbb{C}^n\}$. Every $\boldsymbol{z} \in \mathbb{C}^n$ can be represented with a single element in $\mathfrak{R}^n$, $\mathfrak{C}^n$ and $\mathfrak{C}^{*n}$. For $\mathfrak{C}^n$ and $\mathfrak{C}^{*n}$, this representation is trivial; for R^n, we just need to see that $\boldsymbol{\chi} = (\mathrm{Re}(\boldsymbol{z}), \mathrm{Im}(\boldsymbol{z})) \in \mathfrak{R}^n$. We can also find an isomorphism between $\mathfrak{R}^n$ and $\mathfrak{C}^n$. Indeed, consider the matrix

$$J = \begin{bmatrix} \mathbb{I} & i\mathbb{I} \\ \mathbb{I} & -i\mathbb{I} \end{bmatrix}. \tag{2.113}$$

It is clear that $\boldsymbol{\mu} = (\boldsymbol{z}, \boldsymbol{z}^*) = J\boldsymbol{\chi}$. In a similar manner, we can find an isomorphism between $\mathfrak{C}^n$ and $\mathfrak{C}^{*n}$ using the matrix

$$S = \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}, \tag{2.114}$$

such that $\boldsymbol{\mu} = S\boldsymbol{\mu}^*$

Using this notation it is possible to define *cogradient operator* as

$$\frac{\partial}{\partial \boldsymbol{z}} = \partial_{\boldsymbol{z}} = \frac{1}{2}\begin{bmatrix} \dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial y_1} \\ \vdots \\ \dfrac{\partial}{\partial x_n} - i\dfrac{\partial}{\partial y_n} \end{bmatrix}, \tag{2.115}$$

and the *conjugate cogradient operator*

$$\frac{\partial}{\partial \boldsymbol{z}^*} = \partial_{\boldsymbol{z}^*} = \frac{1}{2}\begin{bmatrix} \dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial y_1} \\ \vdots \\ \dfrac{\partial}{\partial x_n} + i\dfrac{\partial}{\partial y_n} \end{bmatrix}, \tag{2.116}$$

where $\boldsymbol{x} = \mathrm{Re}(\boldsymbol{z})$ and $\boldsymbol{y} = \mathrm{Im}(\boldsymbol{z})$. From here we define the *complex gradient operator* as

$$\frac{\partial}{\partial \boldsymbol{\mu}} = \partial_{\boldsymbol{\mu}} = (\partial_{\boldsymbol{z}}, \partial_{\boldsymbol{z}^*}). \tag{2.117}$$

Expressing the real gradient operator as $\partial_{\boldsymbol{\chi}} = (\partial_{\boldsymbol{x}}, \partial_{\boldsymbol{y}})$, we can see that the two gradients are related by

$$\partial_{\boldsymbol{\chi}} = J^T \partial_{\boldsymbol{\mu}}. \tag{2.118}$$

The framework presented above constitutes the fundamentals of *Wirtinger calculus* [35–37]. Wirtinger calculus arose as a mean to determine the gradient of a non-holomorphic function, in particular, of a real-valued function with complex variables. For a function

to be holomorphic it must satisfy the Cauchy-Riemann conditions: considering $f(z) = u(x,y) + iv(x,y)$, then f is holomorphic if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{2.119}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{2.120}$$

Clearly, if f is real, $v = 0$ and the Cauchy-Riemann conditions are only satisfied if u is a constant function.

For an holomorphic function $f(z) = u(x,y) + iv(x,y)$, its complex derivative is defined as

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{2.121}$$

According to Wirtinger calculus, its cogradient is

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{\mathrm{d}f}{\mathrm{d}z}, \tag{2.122}$$

while its conjugate cogradient is

$$\frac{\partial f}{\partial z^*} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = 0. \tag{2.123}$$

We can see that, for holomorphic functions, Wirtinger calculus is equivalent to complex calculus, and the Cauchy-Riemann conditions are equivalent to $\partial_{z^*} f = 0$.

Let us consider a non-holomorphic function, for example, $f(z) = |z|^2 = zz^* = x^2 + y^2$. Because f is a real function, it does not satisfy the Cauchy-Riemann conditions and it does not have a complex derivative. However, the cogradient and conjugate cogradient are, respectively, $\partial_z f = x - iy = z^*$ and $\partial_{z^*} f = x + iy = z$. Indeed, the cogradient can be regarded as partial derivative with respect to z with z^* behaving as if it were a variable independent of z; similarly for the conjugate cogradient.

Chapter 3

Complex SPSA

SGQT, as presented in [19], has two important problems. First, the method on which SGQT is based, SPSA, is designed for real-valued functions of real variables. SGQT, however, aims to minimize the infidelity between two quantum states. Because the quantum states are complex vectors, the infidelity is a real-valued function of complex variables, thus, SPSA is not directly applicable. As for the second problem, in [19] the perfomance of SGQT is evaluated through the median. However, theorems 1 and 2 for SPSA make no reference to the median whatsoever; they state that the expected value is well-behaved.

In this section, we will address the first of these problems. For that purpose, we will develop a new method, the *complex simultaneous perturbation stochastic approximation* (or ℂ*SPSA*). ℂSPSA is quite similar to SPSA, but it works for functions of complex variables. To show that ℂSPSA works just as well as SPSA, we will prove theorems similar to 1 and 2.

3.1 ℂSPSA algortihm

Let us consider $f : \mathrm{Dom}(f) \subseteq \mathfrak{C}^n \to \mathbb{R}$. We are looking for $\tilde{\boldsymbol{\mu}} = (\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}}^*) \in \mathfrak{C}^n$ such that

$$f(\tilde{\boldsymbol{\mu}}) = \min_{\boldsymbol{\mu} \in \mathrm{Dom}(f)} f(\boldsymbol{\mu}). \tag{3.1}$$

Therefore, $\tilde{\boldsymbol{\mu}}$ satisfies

$$\frac{\partial f}{\partial \boldsymbol{\mu}}(\tilde{\boldsymbol{\mu}}) = 0. \tag{3.2}$$

This is equivalent to say

$$\boldsymbol{g}(\tilde{\boldsymbol{\mu}}) = \frac{\partial f}{\partial \boldsymbol{z}}(\tilde{\boldsymbol{\mu}}) = 0. \tag{3.3}$$

The iterative of the algorithm is of the form

$$\hat{\boldsymbol{z}}_{k+1} = \hat{\boldsymbol{z}}_k - a_k \hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k), \tag{3.4}$$

where $\hat{\boldsymbol{\mu}}_k = (\hat{\boldsymbol{z}}_k, \hat{\boldsymbol{z}}_k^*)$ is an estimate of $\tilde{\boldsymbol{\mu}}$ at the k-th iteration, $\hat{\boldsymbol{g}}_k$ is an estimate of $\boldsymbol{g}$ at the k-th iteration and the a_k are real parameters. We will define the terms of the estimator $\hat{\boldsymbol{g}}_k$ as

$$\hat{g}_{k,i}(\hat{\boldsymbol{\mu}}_k) = \frac{f(\hat{\boldsymbol{\mu}}_k + c_k\boldsymbol{\Lambda}_k) + \varepsilon_{k,+} - f(\hat{\boldsymbol{\mu}}_k - c_k\boldsymbol{\Lambda}_k) - \varepsilon_{k,-}}{2c_k\Delta_{k,i}^*}, \tag{3.5}$$

with $i = 1,...,n$, c_k real parameter, $\boldsymbol{\Lambda}_k = (\boldsymbol{\Delta}_k, \boldsymbol{\Delta}_k^*) \in \mathfrak{C}^n$ a randomly generated vector, components of $\boldsymbol{\Delta}_k$ are $\Delta_{k,i}$, and $\varepsilon_{k,\pm}$ are noise terms.

We can also define the bias and the error term in $\mathbb{C}$SPSA as

$$\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k) = \mathbb{E}(\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k) - \boldsymbol{g}(\hat{\boldsymbol{\mu}}_k)|\hat{\boldsymbol{\mu}}_k), \tag{3.6}$$

$$\boldsymbol{e}_k(\hat{\boldsymbol{\mu}}_k) = \hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k) - \mathbb{E}(\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k)|\hat{\boldsymbol{\mu}}_k). \tag{3.7}$$

So, just like before with SPSA, we can rewrite (3.4) as

$$\hat{\boldsymbol{z}}_{k+1} = \hat{\boldsymbol{z}}_k - a_k[\boldsymbol{g}(\hat{\boldsymbol{\mu}}_k) + \hat{\boldsymbol{b}}_k(\hat{\boldsymbol{\mu}}_k) + \hat{\boldsymbol{e}}_k(\hat{\boldsymbol{\mu}}_k)]. \tag{3.8}$$

So far, $\mathbb{C}$SPSA is quite similar to SPSA. It is to be expected, then, that results similar to theorems 1 and 2 can be derived for $\mathbb{C}$SPSA. These results will be now be obtained.

3.2 Bias in the estimator of the gradient in $\mathbb{C}$SPSA

In SPSA, according to theorem 1, the bias in the estimator of the gradient is of the order of c_k. Later, as part of the hypothesis of theorem 2, we assumed that $\lim_{k\to\infty} c_k = 0$, so the bias also tends to vanish after a great number of iterations of the algorithm. We are interested in deducing a similar theorem for $\mathbb{C}$SPSA. For that purpose we will need a previous result that describes the series expansion of functions with multiple complex variables.

3.2.1 Series expansion of functions with multiple real variables

Let V and W be finite-dimensional vector spaces on $\mathbb{R}$, with $U \subseteq V$ an open subset. Let $f : U \to W$ such that $f \in \mathscr{C}^K$, K≥ 1. Let us choose $\boldsymbol{x}_0 \in U$ and $r > 0$ such that $B_r(\boldsymbol{x}_0) \subset U$, with $B_r(\boldsymbol{x}_0)$ the ball centered in $\boldsymbol{x}_0$ and radius r for a choice of norm in V. Let us choose $\boldsymbol{\kappa} \in V$ such that $\|\boldsymbol{\kappa}\| < r$. Then, according to Eq. (7) in [38], for positive integers $j < K$ we have that the differential operator $D^j_{\boldsymbol{\kappa}}$ over f is

$$D^j_{\boldsymbol{\kappa}} f(\boldsymbol{x}_0) = \sum_{i_1,\ldots,i_j} \kappa_{i_1} \cdot \ldots \cdot \kappa_{i_j} \frac{\partial^j f}{\partial x_{i_1} \ldots \partial x_{i_j}}(\boldsymbol{x}_0). \tag{3.9}$$

With will need the following result, presented as theorem 5.1 in [38]:

Theorem 3 (Taylor's series expansion for functions with multiple real variables). *With notation as above,*

$$f(\boldsymbol{x}_0 + \boldsymbol{\kappa}) = \sum_{j=0}^{p} \frac{1}{j!} (D^{(j)}_{\boldsymbol{\kappa}} f)(\boldsymbol{x}_0) + P_{p,\boldsymbol{\kappa}}(\boldsymbol{x}_0), \tag{3.10}$$

where

$$P_{p,\boldsymbol{\kappa}}(\boldsymbol{x_0}) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} [(D^{(p)}_{\boldsymbol{\kappa}} f)(\boldsymbol{x}_0 + t\boldsymbol{\kappa}) - (D^{(p)}_{\boldsymbol{\kappa}} f)(\boldsymbol{x}_0)]. \tag{3.11}$$

Let us consider an example. The series expansion up to the second power for function f is

$$f(\boldsymbol{x}_0 + \boldsymbol{\kappa}) = (D^{(0)}_{\boldsymbol{\kappa}} f)(\boldsymbol{x}_0) + (D^{(1)}_{\boldsymbol{\kappa}} f)(\boldsymbol{x}_0) + \frac{1}{2}(D^{(2)}_{\boldsymbol{\kappa}} f)(\boldsymbol{x}_0) + P_{2,\boldsymbol{\kappa}}(\boldsymbol{x}_0), \tag{3.12}$$

with

$$P_{2,\boldsymbol{\kappa}}(\boldsymbol{x}_0) = \int_0^1 (1-t)[(D^{(2)}_{\boldsymbol{\kappa}} f)(\boldsymbol{x}_0 + t\boldsymbol{\kappa}) - (D^{(2)}_{\boldsymbol{\kappa}} f)(\boldsymbol{x}_0)], \tag{3.13}$$

and the derivatives of f are

$$(D^{(0)}_{\boldsymbol{\kappa}} f)(\boldsymbol{x}_0) = f(\boldsymbol{x}_0), \tag{3.14}$$

$$(D_{\boldsymbol{\kappa}}^{(1)} f)(\boldsymbol{x}_0) = \sum_i \kappa_i \frac{\partial f}{\partial x_i}(\boldsymbol{x}_0), \tag{3.15}$$

$$(D_{\boldsymbol{\kappa}}^{(2)} f)(\boldsymbol{x}_0) = \sum_{i,j} \kappa_i \kappa_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}_0). \tag{3.16}$$

If $\boldsymbol{x} = (x, y)$, we have

$$(D_{\boldsymbol{\kappa}}^{(1)} f)(\boldsymbol{x}_0) = \kappa_x \frac{\partial f}{\partial x}(\boldsymbol{x}_0) + \kappa_y \frac{\partial f}{\partial y}(\boldsymbol{x}_0), \tag{3.17}$$

$$(D_{\boldsymbol{\kappa}}^{(2)} f)(\boldsymbol{x}_0) = \kappa_x^2 \frac{\partial^2 f}{\partial x^2}(\boldsymbol{x}_0) + \kappa_y^2 \frac{\partial^2 f}{\partial y^2}(\boldsymbol{x}_0) + 2\kappa_x \kappa_y \frac{\partial^2 f}{\partial x \partial y}(\boldsymbol{x}_0). \tag{3.18}$$

Instead, if $\boldsymbol{x} = (x, y, z)$, we have

$$(D_{\boldsymbol{\kappa}}^{(1)} f)(\boldsymbol{x}_0) = \kappa_x \frac{\partial f}{\partial x}(\boldsymbol{x}_0) + \kappa_y \frac{\partial f}{\partial y}(\boldsymbol{x}_0) + \kappa_z \frac{\partial f}{\partial z}(\boldsymbol{x}_0), \tag{3.19}$$

$$\begin{aligned}(D_{\boldsymbol{\kappa}}^{(2)} f)(\boldsymbol{x}_0) = {} & \kappa_x^2 \frac{\partial^2 f}{\partial x^2}(\boldsymbol{x}_0) + \kappa_y^2 \frac{\partial^2 f}{\partial y^2}(\boldsymbol{x}_0) + \kappa_z^2 \frac{\partial^2 f}{\partial z^2}(\boldsymbol{x}_0) \\ & + 2\kappa_x \kappa_y \frac{\partial^2 f}{\partial x \partial y}(\boldsymbol{x}_0) + 2\kappa_z \kappa_x \frac{\partial^2 f}{\partial z \partial x}(\boldsymbol{x}_0) + 2\kappa_y \kappa_z \frac{\partial^2 f}{\partial y \partial z}(\boldsymbol{x}_0).\end{aligned} \tag{3.20}$$

3.2.2 Series expansion of functions with multiple complex variables

Let W be a vector space on $\mathbb{R}$, and $U \subseteq \mathfrak{C}^n$ an open subset. Let $f : U \to W$ such that $f \in \mathscr{C}^K$, $K \geq 1$. Let us choose $\boldsymbol{\mu}_0 \in U$ and $r > 0$ such that $B_r(\boldsymbol{\mu}_0) \subset U$. Let us choose $\boldsymbol{h} \in \mathfrak{C}^n$ such that $\|h\| < r$. For positive integers $j < K$, we define the differential operator $\mathscr{D}_{\boldsymbol{h}}^{(j)}$ over f as

$$(\mathscr{D}_{\boldsymbol{h}}^{(j)} f)(\boldsymbol{\mu}_0) = \sum_{i_1,\ldots,i_j} h_1 \cdot \ldots \cdot h_j \frac{\partial^j f}{\partial \mu_{i_1} \ldots \partial \mu_{i_j}}(\boldsymbol{\mu}_0). \tag{3.21}$$

It is important to remark that $\boldsymbol{\mu} = (\boldsymbol{z}, \boldsymbol{z}^*)$. For example, if $n = 2$, $\boldsymbol{\mu} = (z_1, z_2, z_1^*, z_2^*)$, that is, $\mu_1 = z_1$, $\mu_2 = z_2$, $\mu_3 = z_1^*$ and $\mu_4 = z_2^*$.

Theorem 4. *With notation as above, for*

$$f(\boldsymbol{\mu}_0+\boldsymbol{h}) = \sum_{j=0}^{2} \frac{1}{j!}(\mathscr{D}_{\boldsymbol{h}}^{(j)} f)(\boldsymbol{\mu}_0) + R_{2,\boldsymbol{h}}(\boldsymbol{\mu}_0), \tag{3.22}$$

where

$$R_{2,\boldsymbol{h}}(\boldsymbol{\mu}_0) = \int_0^1 (1-t)[(\mathscr{D}_{\boldsymbol{h}}^{(2)} f)(\boldsymbol{\mu}_0+t\boldsymbol{h}) - (\mathscr{D}_{\boldsymbol{h}}^{(2)} f)(\boldsymbol{\mu}_0)]\mathrm{d}t, \tag{3.23}$$

Proof. Let us consider $\boldsymbol{\chi}_0 = (\mathrm{Re}(\boldsymbol{z}_0), \mathrm{Im}(\boldsymbol{z}_0))$, such that $\boldsymbol{\mu}_0 = J\boldsymbol{\chi}_0$. Becuase of the isomorphism between $\mathfrak{R}^n$ and $\mathfrak{C}^n$, we can define a function f' over a subset of $\mathfrak{R}^n$ such that $f'(\boldsymbol{\chi}) = f'(J^{-1}\boldsymbol{\mu}) = f(\boldsymbol{\mu})$. Let us choose $\boldsymbol{\kappa} = J^{-1}\boldsymbol{h}$. Then, by theorem 3, we have that

$$f'(\boldsymbol{\chi}_0+\boldsymbol{\kappa}) = \sum_{j=0}^{2} \frac{1}{j!}(D_{\boldsymbol{\kappa}}^{(j)} f')(\boldsymbol{\chi}_0) + P_{2,\boldsymbol{\kappa}}(\boldsymbol{\chi}_0) \tag{3.24}$$

$$= f'(\boldsymbol{\chi}_0) + \sum_i \kappa_i \frac{\partial f'}{\partial \chi_i}(\boldsymbol{\chi}_0) + \frac{1}{2}\sum_{i,j} \kappa_i \kappa_j \frac{\partial^2 f'}{\partial \chi_i \partial \chi_j}(\boldsymbol{\chi}_0) + P_{2,\boldsymbol{\kappa}}(\boldsymbol{\chi}_0) \tag{3.25}$$

$$= f'(\boldsymbol{\chi}_0) + \boldsymbol{\kappa}^T \frac{\partial f'}{\partial \boldsymbol{\chi}}(\boldsymbol{\chi}_0) + \frac{1}{2}\boldsymbol{\kappa}^T \frac{\partial^2 f'}{\partial \boldsymbol{\chi} \partial \boldsymbol{\chi}^T}(\boldsymbol{\chi}_0)\boldsymbol{\kappa} + P_{2,\boldsymbol{\kappa}}(\boldsymbol{\chi}_0) \tag{3.26}$$

$$= f(\boldsymbol{\mu}_0) + (\boldsymbol{h}^T (J^{-1})^T) J^T \frac{\partial f}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}_0)$$

$$+ \frac{1}{2}(\boldsymbol{h}^T (J^{-1})^T) J^T \frac{\partial^2 f}{\partial \boldsymbol{\mu} \partial \boldsymbol{\mu}^T}(\boldsymbol{\mu}_0) J(J^{-1}\boldsymbol{h}) + P_{2,\boldsymbol{\kappa}}(\boldsymbol{\chi}_0) \tag{3.27}$$

$$= f(\boldsymbol{\mu}_0) + \boldsymbol{h}^T \frac{\partial f}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}_0) + \frac{1}{2}\boldsymbol{h}^T \frac{\partial^2 f}{\partial \boldsymbol{\mu} \partial \boldsymbol{\mu}^T}(\boldsymbol{\mu}_0)\boldsymbol{h} + P_{2,\boldsymbol{\kappa}}(\boldsymbol{\chi}_0) \tag{3.28}$$

$$= f(\boldsymbol{\mu}_0) + \sum_i h_i \frac{\partial f}{\partial \mu_i}(\boldsymbol{\mu}_0) + \frac{1}{2}\sum_{i,j} h_i h_j \frac{\partial^2 f}{\partial \mu_i \partial \mu_j}(\boldsymbol{\mu}_0) + P_{2,\boldsymbol{\kappa}}(\boldsymbol{\chi}_0) \tag{3.29}$$

$$= \sum_{j=0}^{2} \frac{1}{j!}(\mathscr{D}_{\boldsymbol{h}}^{(j)} f)(\boldsymbol{\mu}_0) + P_{2,\boldsymbol{\kappa}}(\boldsymbol{\chi}_0). \tag{3.30}$$

Meanwhile, we also have that

$$P_{2,\kappa}(\chi_0) = \int_0^1 (1-t)((D_\kappa^{(2)} f')(\chi_0 + t\kappa) - (D_\kappa^{(2)} f')(\chi_0))\mathrm{d}t \tag{3.31}$$

$$= \int_0^1 (1-t) \sum_{i,j} \kappa_i \kappa_j \left[\frac{\partial^2 f'}{\partial \chi_i \partial \chi_j}(\chi_0 + t\kappa) - \frac{\partial^2 f'}{\partial \chi_i \partial \chi_j}(\chi_0) \right] \tag{3.32}$$

$$= \int_0^1 (1-t) \kappa^T \left[\frac{\partial^2 f'}{\partial \chi \partial \chi^T}(\chi_0 + t\kappa) - \frac{\partial^2 f'}{\partial \chi \partial \chi^T}(\chi_0) \right] \kappa \mathrm{d}t \tag{3.33}$$

$$= \int_0^1 (1-t)(h^T (J^{-1})^T) J^T$$

$$\cdot \left[\frac{\partial^2 f}{\partial \mu \partial \mu^T}(\mu_0 + th) - \frac{\partial^2 f}{\partial \mu \partial \mu^T}(\mu_0) \right] J(J^{-1}h)\mathrm{d}t \tag{3.34}$$

$$= \int_0^1 (1-t) h^T \left[\frac{\partial^2 f}{\partial \mu \partial \mu^T}(\mu_0 + th) - \frac{\partial^2 f}{\partial \mu \partial \mu^T}(\mu_0) \right] h \mathrm{d}t \tag{3.35}$$

$$= \int_0^1 (1-t) \sum_{i,j} h_i h_j \left[\frac{\partial^2 f}{\partial \mu_i \partial \mu_j}(\mu_0 + th) - \frac{\partial^2 f}{\partial \mu_i \partial \mu_j}(\mu_0) \right] \mathrm{d}t \tag{3.36}$$

$$= \int_0^1 (1-t)((\mathscr{D}_h^{(2)} f)(\mu_0 + th) - (\mathscr{D}_h^{(2)} f)(\mu_0))\mathrm{d}t \tag{3.37}$$

$$= R_{2,h}(\mu_0). \tag{3.38}$$

And so,

$$f'(\chi_0 + \kappa) = \sum_{j=0}^{2} \frac{1}{j!} (\mathscr{D}_h^{(j)} f)(\mu_0) + R_{2,h}(\mu_0). \tag{3.39}$$

Clearly, $f'(\chi_0 + \kappa) = f'(J^{-1}(\mu_0 + h)) = f(\mu_0 + h)$, therefore,

$$f(\mu_0 + h) = \sum_{j=0}^{2} \frac{1}{j!} (\mathscr{D}_h^{(j)} f)(\mu_0) + R_{2,h}(\mu_0). \tag{3.40}$$

□

3.2.3 Theorem for the bias in the estimator

Theorem 5. *For α_0, α_1 and α_2 positive real constants and $\Omega = \{\omega\}$ the sample space that generates the sequence $\hat{z}_1, \hat{z}_2, \ldots$, consider all $k \geq K$ for some $K < \infty$, and suppose that for each such k the $\Delta_{k,i}$ are independent and identically distributed, and symetrically distributed about 0 (that is, $\mathbb{E}(\Delta_{k,i}) = 0$) with $|\Delta_{k,i}| \leq \alpha_0$ a.s., $\mathbb{E}(|\Delta_{k,i}^{-1}|) \leq \alpha_1$ and $\mathbb{E}[e^{2i\phi_{k,i}}] = 0$,*

where $\phi_{k,i}$ is the phase of $\Delta_{k,i}$ in its polar decomposition. For almost all $\hat{\boldsymbol{\mu}}_k$ (at each $k \geq K$) suppose that for all $\boldsymbol{\mu}$ in an open neighborhood of $\hat{\boldsymbol{\mu}}_k$, that is not an function of k or ω, $f \in \mathscr{C}^2(\mathfrak{C}^n)$ with $|\partial_{\mu_i}\partial_{\mu_j} f(\boldsymbol{\mu})| \leq \alpha_2$. Then for almost all $\omega \in \Omega$

$$\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k) = \mathscr{O}(c_k). \tag{3.41}$$

Proof. By definition of bias, we have, for all $l \in \{1, ..., n\}$,

$$b_{k,l}(\hat{\boldsymbol{\mu}}_k) = \mathbb{E}\left(\hat{g}_{k,l}(\hat{\boldsymbol{\mu}}_k) - g_l(\hat{\boldsymbol{\mu}}_k) \,|\, \hat{\boldsymbol{\mu}}_k\right) \tag{3.42}$$

$$= \mathbb{E}\left(\left[\frac{f(\hat{\boldsymbol{\mu}}_k + c_k \boldsymbol{\Lambda}_k) - f(\hat{\boldsymbol{\mu}}_k - c_k \boldsymbol{\Lambda}_k)}{2 c_k \Delta^*_{k,l}} - \frac{\partial f}{\partial z^*_l}(\hat{\boldsymbol{\mu}}_k) \right] \middle| \, \hat{\boldsymbol{\mu}}_k \right). \tag{3.43}$$

Because $|\partial_{\mu_i}\partial_{\mu_j} f(\boldsymbol{\mu})|$ is bounded, $\mathbb{E}(|\partial_{\mu_i}\partial_{\mu_j} f(\boldsymbol{\mu})|)$ is well defined. From theorem 4 we can expand $f(\hat{\boldsymbol{\mu}}_k \pm c_k \boldsymbol{\Delta}_k)$, so

$$
\begin{aligned}
b_{k,l}(\hat{\boldsymbol{\mu}}_k) = \mathbb{E}\Bigg(\Bigg[\frac{1}{2c_k\Delta^*_{k,l}} \Bigg\{ & f(\hat{\boldsymbol{\mu}}_k) + \sum_{i=1}^{2n} c_k \Lambda_{k,i} \frac{\partial f}{\partial \mu_i}(\hat{\boldsymbol{\mu}}_k) \\
& + \frac{1}{2} \sum_{i,j=1}^{2n} c_k \Lambda_{k,i} c_k \Lambda_{k,j} \frac{\partial^2 f}{\partial \mu_i \partial \mu_j}(\hat{\boldsymbol{\mu}}_k) + R_{2,c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k) - f(\hat{\boldsymbol{\mu}}_k) \\
& + \sum_{i=1}^{2n} c_k \Lambda_{k,i} \frac{\partial f}{\partial \mu_i}(\hat{\boldsymbol{\mu}}_k) - \frac{1}{2} \sum_{i,j=1}^{2n} c_k \Lambda_{k,i} c_k \Lambda_{k,j} \frac{\partial^2 f}{\partial \mu_i \partial \mu_j}(\hat{\boldsymbol{\mu}}_k) \\
& - R_{2,-c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k) \Big\} - \frac{\partial f}{\partial z^*_l}(\hat{\boldsymbol{\mu}}_k) \Bigg] \Bigg| \hat{\boldsymbol{\mu}}_k \Bigg) \qquad (3.44) \\
= \mathbb{E}\Bigg(\Bigg[& \sum_{i=1}^{2n} \frac{\Lambda_{k,i}}{\Delta^*_{k,l}} \frac{\partial f}{\partial \mu_i}(\hat{\boldsymbol{\mu}}_k) - \frac{\partial f}{\partial z^*_l}(\hat{\boldsymbol{\mu}}_k) \\
& + \frac{R_{2,c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k) - R_{2,-c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k)}{2c_k\Delta^*_{k,l}} \Bigg] \Bigg| \hat{\boldsymbol{\mu}}_k \Bigg) \qquad (3.45) \\
= \mathbb{E}\Bigg(\Bigg[& \sum_{i=1}^{n} \frac{\Delta_{k,i}}{\Delta^*_{k,l}} \frac{\partial f}{\partial z_i}(\hat{\boldsymbol{\mu}}_k) + \sum_{i=1}^{n} \frac{\Delta^*_{k,i}}{\Delta^*_{k,l}} \frac{\partial f}{\partial z^*_i}(\hat{\boldsymbol{\mu}}_k) - \frac{\partial f}{\partial z^*_l}(\hat{\boldsymbol{\mu}}_k) \\
& + \frac{R_{2,c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k) - R_{2,-c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k)}{2c_k\Delta^*_{k,l}} - \frac{\partial f}{\partial z^*_l}(\hat{\boldsymbol{\mu}}_k) \Bigg] \Bigg| \hat{\boldsymbol{\mu}}_k \Bigg) \qquad (3.46) \\
= \mathbb{E}\Bigg(\Bigg[& \sum_{i=1}^{n} \frac{\Delta_{k,i}}{\Delta^*_{k,l}} \frac{\partial f}{\partial z_i}(\hat{\boldsymbol{\mu}}_k) + \sum_{i\neq l}^{n} \frac{\Delta^*_{k,i}}{\Delta^*_{k,l}} \frac{\partial f}{\partial z^*_i}(\hat{\boldsymbol{\mu}}_k) \\
& + \frac{R_{2,c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k) - R_{2,-c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k)}{2c_k\Delta^*_{k,l}} \Bigg] \Bigg| \hat{\boldsymbol{\mu}}_k \Bigg). \qquad (3.47)
\end{aligned}
$$

Becuase $\partial_{z_i} f(\hat{\boldsymbol{\mu}}_k)$ is completely determined by $\hat{\boldsymbol{\mu}}_k$,

$$
\mathbb{E}\left(\sum_{i\neq l}^{n} \frac{\Delta^*_{k,i}}{\Delta^*_{k,l}} \frac{\partial f}{\partial z^*_i}(\hat{\boldsymbol{\mu}}_k) \middle| \hat{\boldsymbol{\mu}}_k \right) = \sum_{i\neq l}^{n} \mathbb{E}\left(\frac{\Delta^*_{k,i}}{\Delta^*_{k,l}} \middle| \hat{\boldsymbol{\mu}}_k \right) \frac{\partial f}{\partial z^*_i}(\hat{\boldsymbol{\mu}}_k) \qquad (3.48)
$$

and

$$
\mathbb{E}\left(\sum_{i=1}^{n} \frac{\Delta_{k,i}}{\Delta^*_{k,l}} \frac{\partial f}{\partial z_i}(\hat{\boldsymbol{\mu}}_k) \middle| \hat{\boldsymbol{\mu}}_k \right) = \sum_{i=1}^{n} \mathbb{E}\left(\frac{\Delta_{k,i}}{\Delta^*_{k,l}} \middle| \hat{\boldsymbol{\mu}}_k \right) \frac{\partial f}{\partial z_i}(\hat{\boldsymbol{\mu}}_k). \qquad (3.49)
$$

Besides, the $\{\Delta_{k,i}\}$ are independently distributed, and are independent of $\hat{\boldsymbol{\mu}}_k$, so

$$\mathbb{E}\left(\frac{\Delta_{k,i}^*}{\Delta_{k,l}^*}\middle|\hat{\boldsymbol{\mu}}_k\right) = \mathbb{E}((\Delta_{k,l}^*)^{-1})\mathbb{E}(\Delta_{k,i}^*) \tag{3.50}$$

and

$$\mathbb{E}\left(\frac{\Delta_{k,i}}{\Delta_{k,l}^*}\middle|\hat{\boldsymbol{\mu}}_k\right) = \mathbb{E}\left(\frac{\Delta_{k,i}}{\Delta_{k,l}^*}\right). \tag{3.51}$$

If $i \neq l$ we also have that

$$\mathbb{E}\left(\frac{\Delta_{k,i}}{\Delta_{k,l}^*}\right) = \mathbb{E}((\Delta_{k,l}^*)^{-1})\mathbb{E}(\Delta_{k,i}). \tag{3.52}$$

Then

$$\begin{aligned}|b_{k,l}(\hat{\boldsymbol{\mu}}_k)| \leq & \left|\mathbb{E}\left(\frac{\Delta_{k,l}}{\Delta_{k,l}^*}\right)\frac{\partial f}{\partial \boldsymbol{z}_l}(\hat{\boldsymbol{\mu}}_k)\right| \\ & + \left|\mathbb{E}((\Delta_{k,l}^*)^{-1})\sum_{i\neq l}^{n}\left(\mathbb{E}(\Delta_{k,i})\frac{\partial f}{\partial z_i}(\hat{\boldsymbol{\mu}}_k) + \mathbb{E}(\Delta_{k,i}^*)\frac{\partial f}{\partial z_i^*}(\hat{\boldsymbol{\mu}}_k)\right)\right| \\ & + \left|\mathbb{E}\left(\frac{R_{2,c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k) - R_{2,-c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k)}{2c_k\Delta_{k,l}^*}\middle|\hat{\boldsymbol{\mu}}_k\right)\right|. \end{aligned} \tag{3.53}$$

Because $\Delta_{k,i} = |\Delta_{k,i}|e^{i\phi_{k,i}}$, $\dfrac{\Delta_{k,l}}{\Delta_{k,l}^*} = e^{2i\phi_{k,l}}$. From hypothesis, $\mathbb{E}(\Delta_{[C]ik}) = \mathbb{E}(\Delta_{[C]ik}^*) = 0$, so

$$\begin{aligned}|b_{k,l}(\hat{\boldsymbol{\mu}}_k)| \leq & \left|\mathbb{E}(e^{2i\phi_{kl}})\frac{\partial f}{\partial \boldsymbol{z}_l}(\hat{\boldsymbol{\mu}}_k)\right| \\ & + \mathbb{E}\left(\frac{\left|R_{2,c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k) - R_{2,-c_k\boldsymbol{\Lambda}_k}(\hat{\boldsymbol{\mu}}_k)\right|}{\left|2c_k\Delta_{k,l}^*\right|}\middle|\hat{\boldsymbol{\mu}}_k\right). \end{aligned} \tag{3.54}$$

We assumed $\mathbb{E}(e^{2i\phi_{kl}}) = 0$. From (3.23), we have that

$$\begin{aligned}|b_{k,l}(\hat{\boldsymbol{\mu}}_k)| \leq \mathbb{E}\Bigg(\Bigg|\int_0^1 \frac{1-t}{2c_k\Delta_{k,l}^*}\Big[&(\mathscr{D}^{(2)}_{c_k\boldsymbol{\Lambda}_k}f)(\hat{\boldsymbol{\mu}}_k + tc_k\boldsymbol{\Lambda}_k) \\ &- (\mathscr{D}^{(2)}_{c_k\boldsymbol{\Lambda}_k}f)(\hat{\boldsymbol{\mu}}_k) - (\mathscr{D}^{(2)}_{-c_k\boldsymbol{\Lambda}_k}f)(\hat{\boldsymbol{\mu}}_k - tc_k\boldsymbol{\Lambda}_k) \\ &+ (\mathscr{D}^{(2)}_{-c_k\boldsymbol{\Lambda}_k}f)(\hat{\boldsymbol{\mu}}_k)\Big]\,\mathrm{d}t\Bigg|\,|\hat{\boldsymbol{\mu}}_k\Bigg). \end{aligned} \tag{3.55}$$

We can see that $(\mathscr{D}^{(2)}_{-c_k\mathbf{\Lambda}_k}f)(\hat{\boldsymbol{\mu}}_k) = \sum_{i,j=1}^{2n} c_k^2\Lambda_{k,i}\Lambda_{k,j}\partial_{\mu_i}\partial_{\mu_j}f(\hat{\boldsymbol{\mu}}_k) = (\mathscr{D}^{(2)}_{c_k\mathbf{\Lambda}_k}f)(\hat{\boldsymbol{\mu}}_k)$, and by the same reason, $(\mathscr{D}^{(2)}_{-c_k\mathbf{\Lambda}_k}f)(\hat{\boldsymbol{\mu}}_k - tc_k\mathbf{\Lambda}_k) = (\mathscr{D}^{(2)}_{c_k\mathbf{\Lambda}_k}f)(\hat{\boldsymbol{\mu}}_k - tc_k\mathbf{\Lambda}_k)$. Therefore,

$$|b_{k,l}(\hat{\boldsymbol{\mu}}_k)| \leq \mathbb{E}\left(\left|\int_0^1 \frac{1-t}{2c_k\Delta^*_{k,l}}\left[(\mathscr{D}^{(2)}_{c_k\mathbf{\Lambda}_k}f)(\hat{\boldsymbol{\mu}}_k + tc_k\mathbf{\Lambda}_k)\right.\right.\right.$$
$$\left.\left.\left. -(\mathscr{D}^{(2)}_{c_k\mathbf{\Lambda}_k}f)(\hat{\boldsymbol{\mu}}_k - tc_k\mathbf{\Lambda}_k)\right]\mathrm{d}t\right| \hat{\boldsymbol{\mu}}_k\right). \tag{3.56}$$

Considering $(\mathscr{D}^{(2)}_{c_k\mathbf{\Lambda}_k}f)(\hat{\boldsymbol{\mu}}_k \pm tc_k\mathbf{\Lambda}_k) = \sum_{i,j}^{n} c_k^2\Lambda_{k,i}\Lambda_{k,j}\partial_{\mu_i}\partial_{\mu_j}f(\hat{\boldsymbol{\mu}}_k \pm tc_k\mathbf{\Lambda}_k)$,

$$|b_{k,l}(\hat{\boldsymbol{\mu}}_k)| \leq \mathbb{E}\left(\left|\int_0^1 \frac{1-t}{2c_k\Delta^*_{k,l}}\sum_{i,j}^{2n} c_k^2\Lambda_{k,i}\Lambda_{k,j}\right.\right.$$
$$\left.\left.\cdot\frac{\partial^2}{\partial\mu_i\partial\mu_j}[f(\hat{\boldsymbol{\mu}}_k + tc_k\mathbf{\Lambda}_k) - f(\hat{\boldsymbol{\mu}}_k - tc_k\mathbf{\Lambda}_k)]\mathrm{d}t\right| \hat{\boldsymbol{\mu}}_k\right) \tag{3.57}$$
$$\leq \mathbb{E}\left(\int_0^1 \frac{1-t}{2c_k|\Delta^*_{k,l}|}\sum_{i,j}^{2n} c_k^2|\Lambda_{k,i}||\Lambda_{k,j}|\right.$$
$$\left.\cdot\left[\left|\frac{\partial^2 f}{\partial\mu_i\partial\mu_j}(\hat{\boldsymbol{\mu}}_k + tc_k\mathbf{\Lambda}_k)\right| + \left|\frac{\partial^2 f}{\partial\mu_i\partial\mu_j}(\hat{\boldsymbol{\mu}}_k - tc_k\mathbf{\Lambda}_k)\right|\right]\mathrm{d}t\right| \hat{\boldsymbol{\mu}}_k\right). \tag{3.58}$$

From hypothesis, we have that $|\Delta_{k,i}| \leq \alpha_0$, so $|\Lambda_{k,i}| \leq \alpha_0$, and $|\partial_{\mu_i}\partial_{\mu_j}f(\boldsymbol{\mu})| \leq \alpha_2$, so

$$|b_{k,l}(\hat{\boldsymbol{\mu}}_k)| \leq \mathbb{E}\left(\int_0^1 \frac{1-t}{2c_k|\Delta^*_{k,l}|}\sum_{i,j}^{2n} 2c_k^2\alpha_0^2\alpha_2\mathrm{d}t\right| \hat{\boldsymbol{\mu}}_k\right) \tag{3.59}$$
$$= 4n^2c_k\alpha_0^2\alpha_2\mathbb{E}\left(|\Delta^{-1}_{k,i}|\right| \hat{\boldsymbol{\mu}}_k\right)\int_0^1(1-t)\mathrm{d}t \tag{3.60}$$
$$= 2n^2c_k\alpha_0^2\alpha_2\mathbb{E}\left(|\Delta^{-1}_{k,i}|\right| \hat{\boldsymbol{\mu}}_k\right). \tag{3.61}$$

Because $\Delta_{k,i}$ is independent of $\hat{\boldsymbol{\mu}}_k$, $\mathbb{E}\left(|\Delta^{-1}_{k,i}|\right| \hat{\boldsymbol{\mu}}_k\right) = \mathbb{E}\left(|\Delta^{-1}_{k,i}|\right)$, and from hypothesis, $\mathbb{E}\left(|\Delta^{-1}_{k,i}|\right) \leq \alpha_1$. Then,

$$|b_{k,l}(\hat{\boldsymbol{\mu}}_k)| \leq 2n^2c_k\alpha_0^2\alpha_1\alpha_2. \tag{3.62}$$

Therefore,

$$\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k) = \mathscr{O}(c_k). \tag{3.63}$$

□

3.3 Convergence of the ℂSPSA algorithm

Lemma 1. *Let the conditions of theorem 5 and the following assumptions hold:*

Å1: $a_k, c_k > 0\ \forall k$; $\lim_{k\to\infty}(|a_k| + |c_k|) = 0$; $\sum_{k=1}^{\infty} a_k = \infty$; $\sum_{k=1}^{\infty}(a_k/c_k)^2 < \infty$,

Å2: *For some* $\beta_0, \beta_1, \beta_2 > 0$, $\forall k$, $\mathbb{E}(\varepsilon_{k,\pm}^2) \leq \beta_0$, $\mathbb{E}([f(\hat{\boldsymbol{\mu}}_k \pm c_k \boldsymbol{\Lambda}_k)]^2) \leq \beta_1$, *and* $\mathbb{E}(\Delta_{k,i}^{-2}) \leq \beta_2$,

Å3: $\|\hat{\boldsymbol{\mu}}_k\| < \infty$ *for almost all k,*

Å4: $\boldsymbol{z}(t) = \tilde{\boldsymbol{z}}$ *is an asymptotically stable solution of the differential equation* $\mathrm{d}\boldsymbol{z}(t)/\mathrm{d}t = -\boldsymbol{g}(\boldsymbol{\mu})$,

Å5: *Consider the domain of attraction* $D(\tilde{\boldsymbol{z}}) = \{\boldsymbol{z}_0 : \lim_{t\to\infty} \boldsymbol{z}(t|\boldsymbol{z}_0) = \tilde{\boldsymbol{z}}\}$ *where* $\boldsymbol{z}(t|\boldsymbol{z}_0)$ *denotes the solutions to the differential equation of Å4 based on initial conditions* $\boldsymbol{z}(0) = \boldsymbol{z}_0$. *There exists a closed set* $S \subseteq D(\tilde{\boldsymbol{z}})$ *such that* $\hat{\boldsymbol{z}}_k \in S$ *infinitely often for almost all sample points,*

Then the following are true

S1: $\|\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k)\| < \infty$ *and* $\lim_{k\to\infty} \|\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k)\| = 0$ *almost surely,*

S2: $\lim_{k\to\infty} P\left(\sup_{m\geq k} \left\|\sum_{i=k}^{m} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i)\right\| \geq \eta\right) = 0$ *for all* $\eta > 0$.

Proof. Obtaining S1 is very straightfoward: from theorem 5 and (3.62), we have that

$$\|\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k)\| \leq \sqrt{2n^3 c_k \alpha_0^2 \alpha_1 \alpha_2} < \infty. \tag{3.64}$$

Also, from Å1, $\lim_{k\to\infty} c_k = 0$, so

$$\lim_{k\to\infty} \|\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k)\| = 0. \tag{3.65}$$

To prove S2, we will first show that $\{\sum_{i=k}^{m} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i)\}_{m\geq k}$ is a martingale with respect to $\{\hat{\boldsymbol{\mu}}_m\}$[1]. Indeed,

$$\mathbb{E}\left(\sum_{i=k}^{m+1} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i) - \sum_{i=k}^{m} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i) \middle| \{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{m}\right) = \mathbb{E}(a_{m+1}\boldsymbol{e}_{m+1}(\hat{\boldsymbol{\mu}}_{m+1})|\hat{\boldsymbol{\mu}}_m) \tag{3.66}$$

$$= a_{m+1}[(\mathbb{E}(\hat{\boldsymbol{g}}_{m+1}(\hat{\boldsymbol{\mu}}_{m+1})|\{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{m}) - \mathbb{E}(\mathbb{E}(\hat{\boldsymbol{g}}_{m+1}(\hat{\boldsymbol{\mu}}_{m+1})|\hat{\boldsymbol{\mu}}_{m+1})|\{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{m}))]. \tag{3.67}$$

Because Δ_k is independent of $\{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{k}$,

$$\mathbb{E}(\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k)|\hat{\boldsymbol{\mu}}_k) = \mathbb{E}(\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k)|\{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{k}). \tag{3.68}$$

In particular,

$$\mathbb{E}(\mathbb{E}(\hat{\boldsymbol{g}}_{m+1}(\hat{\boldsymbol{\mu}}_{m+1})|\hat{\boldsymbol{\mu}}_{m+1})|\{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{m})) = \mathbb{E}(\mathbb{E}(\hat{\boldsymbol{g}}_{m+1}(\hat{\boldsymbol{\mu}}_{m+1})|\{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{m+1})|\{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{m})). \tag{3.69}$$

Additionally, because $\{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{m} \subset \{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{m+1}$,

$$\mathbb{E}(\mathbb{E}(\hat{\boldsymbol{g}}_{m+1}(\hat{\boldsymbol{\mu}}_{m+1})|\hat{\boldsymbol{\mu}}_{m+1})|\{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{m})) = \mathbb{E}(\hat{\boldsymbol{g}}_{m+1}(\hat{\boldsymbol{\mu}}_{m+1})|\{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{m+1}). \tag{3.70}$$

Therefore,

$$\mathbb{E}\left(\sum_{i=k}^{m+1} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i) - \sum_{i=k}^{m} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i) \middle| \{\hat{\boldsymbol{\mu}}_i\}_{i=1}^{m}\right) = 0. \tag{3.71}$$

Because $\{\sum_{i=k}^{m} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i)\}$ is a martingale, $|\sum_{i=k}^{m} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i)|$ is a submartingale, thus it satisfies Doob's inequality [39]: for all $\eta > 0$

$$P\left(\sup_{m\geq k}\left\|\sum_{i=k}^{m} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i)\right\| \geq \eta\right) \leq \eta^{-2}\mathbb{E}\left(\left\|\sum_{i=k}^{\infty} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i)\right\|^2\right). \tag{3.72}$$

[1] A *martingale with respect to* $\{Y_i\}$ is a sequence $\{X_i\}$ that satisfies $\mathbb{E}(X_{n+1} - X_n|\{Y_i\}_{i=1}^{n}) = 0$ for every n

Because $\mathbb{E}(\boldsymbol{e}_i^T(\hat{\boldsymbol{\mu}}_i)\boldsymbol{e}_j(\hat{\boldsymbol{\mu}}_j)) = \mathbb{E}(\boldsymbol{e}_i^T(\hat{\boldsymbol{\mu}}_i)\mathbb{E}(\boldsymbol{e}_j(\hat{\boldsymbol{\mu}}_j)|\hat{\boldsymbol{\mu}}_i)) = 0$ when $i < j$ we have

$$P\left(\sup_{m\geq k}\left\|\sum_{i=k}^{m} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i)\right\| \geq \eta\right) \leq \eta^{-2}\sum_{i=k}^{\infty} a_i^2 \mathbb{E}\left(\|\boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i)\|^2\right). \tag{3.73}$$

We can see that

$$\begin{aligned}
\mathbb{E}(|\hat{g}_{k,l}(\hat{\boldsymbol{\mu}}_k)|^2) &= \frac{1}{4}\mathbb{E}[\mathbb{E}(|(c_k\Delta_{k,l}^*)^{-1}(f(\boldsymbol{\mu}_k + c_k\boldsymbol{\Lambda}_k) \\
&\quad - f(\boldsymbol{\mu}_k - c_k\boldsymbol{\Lambda}_k) + \varepsilon_{k,+} - \varepsilon_{k,-})|^2|\boldsymbol{\mu}_k, \boldsymbol{\Lambda}_k)] && (3.74)\\
&= \frac{1}{4}\mathbb{E}[|c_k\Delta_{k,l}^*|^{-2}\mathbb{E}(|f(\boldsymbol{\mu}_k + c_k\boldsymbol{\Lambda}_k) \\
&\quad - f(\boldsymbol{\mu}_k - c_k\boldsymbol{\Lambda}_k) + \varepsilon_{k,+} - \varepsilon_{k,-}|^2|\boldsymbol{\mu}_k, \boldsymbol{\Lambda}_k)] && (3.75)\\
&\leq \frac{1}{2}\mathbb{E}[|c_k\Delta_{k,l}^*|^{-2}\mathbb{E}(|f(\boldsymbol{\mu}_k + c_k\boldsymbol{\Lambda}_k) \\
&\quad - f(\boldsymbol{\mu}_k - c_k\boldsymbol{\Lambda}_k)|^2 + |\varepsilon_{k,+} - \varepsilon_{k,-}|^2|\boldsymbol{\mu}_k, \boldsymbol{\Lambda}_k)]. && (3.76)
\end{aligned}$$

From Å2, $\mathbb{E}(\varepsilon_{k\pm})^2 \leq \beta_0$ and $\mathbb{E}([f(\boldsymbol{\mu}_k \pm c_k\boldsymbol{\Lambda}_k)]^2) \leq \beta_1$, so

$$\mathbb{E}(|\hat{g}_{k,l}(\hat{\boldsymbol{\mu}}_k)|^2) \leq 2\mathbb{E}(|c_k\Delta_{k,l}|^{-2}(\beta_0 + \beta_1)). \tag{3.77}$$

We also assumed that $\mathbb{E}(|\Delta_{k,l}|^{-2}) \leq \beta_2$, thus

$$\mathbb{E}(|\hat{g}_{k,l}(\hat{\boldsymbol{\mu}}_k)|^2) \leq 2(\beta_0 + \beta_1)\beta_2 c_k^{-2}; \tag{3.78}$$

$$\begin{aligned}
\mathbb{E}(\|\boldsymbol{e}_k(\hat{\boldsymbol{\mu}}_k))\|^2) &= \mathbb{E}(\|\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k) - \mathbb{E}(\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k)|\hat{\boldsymbol{\mu}}_k))\|^2) && (3.79)\\
&\leq \mathbb{E}(\|\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k)\| + \|\mathbb{E}(\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k)|\hat{\boldsymbol{\mu}}_k)\|)^2 && (3.80)\\
&\leq 2\mathbb{E}(\|\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k)\|^2) + 2\mathbb{E}(\|\mathbb{E}(\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k)|\hat{\boldsymbol{\mu}}_k)\|^2) && (3.81)\\
&\leq 2\mathbb{E}(\|\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k)\|^2) + 2\mathbb{E}(\|\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k)\|^2) && (3.82)\\
&= 4\mathbb{E}(\|\hat{\boldsymbol{g}}_k(\hat{\boldsymbol{\mu}}_k)\|^2) && (3.83)\\
&= 4\sum_{l=1}^{n}\mathbb{E}(|\hat{g}_{k,l}(\hat{\boldsymbol{\mu}}_k)|^2) && (3.84)\\
&\leq 8n(\beta_0 + \beta_1)\beta_2 c_k^{-2}. && (3.85)
\end{aligned}$$

Substituting in (3.73), we obtain

$$P\left(\sup_{m\geq k}\left\|\sum_{i=k}^{m} a_i \boldsymbol{e}_i(\hat{\boldsymbol{\mu}}_i)\right\| \geq \eta\right) \leq \frac{8n}{\eta^2}(\beta_0+\beta_1)\beta_2 \sum_{i=k}^{\infty}\left(\frac{a_i}{c_i}\right)^2. \tag{3.86}$$

According to Å1, $\sum_{i=0}^{\infty}\left(\frac{a_i}{c_i}\right)^2 < \infty$, so $\lim_{k\to\infty}\sum_{i=k}^{\infty}\left(\frac{a_i}{c_i}\right)^2 = 0$. Therefore, S2 holds.

□

For a pair of sequences $\{x_n\}$ and $\{t_n\}$, with $t_n > 0$ for all n and $\lim_{n\to\infty} t_n = +\infty$, we define the *linear interpolation* $l\cdot i((t_i,x_i),t)$ as a function of $t > 0$ given by

$$l\cdot i((t_i,x_i),t) = \frac{(t-t_n)x_{n+1} + (t_{n+1}-t)x_n}{t_{n+1}-t_n}, \tag{3.87}$$

wtih n such that $t_n \leq t \leq t_{n+1}$. Clearly, $l\cdot i((t_i,x_i),t_n) = t_n$. We also define the *right continuous step interpolation* $s\cdot i((t_i,x_i),t)$ as a function of $t > 0$ given by

$$s\cdot i((t_i,x_i),t) = x_n, \tag{3.88}$$

with n such that that $t_n \leq t < t_{n+1}$.

We can choose the sequence $\{t_n\}$ as

$$t_n = \sum_{i=0}^{n-1} a_i, \tag{3.89}$$

because, by Å1, $a_n > 0$ for all n and $\lim_{n\to\infty}\sum_{i=1}^{n} a_i = \infty$. In this case, $t_1 = 0$. We define

$$\hat{\boldsymbol{z}}^0(t) = l\cdot i((t_i,\hat{\boldsymbol{z}}_i),t), \tag{3.90}$$

$$B^0(t) = l\cdot i\left(\left(t_i, \sum_{j=0}^{i-1} a_j \boldsymbol{b}_j(\hat{\boldsymbol{\mu}}_j)\right),t\right), \tag{3.91}$$

$$M^0(t) = l\cdot i\left(\left(t_i, \sum_{j=0}^{i-1} a_j \boldsymbol{e}_j(\hat{\boldsymbol{\mu}}_j)\right),t\right), \tag{3.92}$$

$$U^0(t) = B^0(t) + M^0(t), \tag{3.93}$$

$$\bar{\boldsymbol{\mu}}^0 = s \cdot i((t_i, \hat{\boldsymbol{\mu}}_i), t). \tag{3.94}$$

It is worth noticing that $\sum_{j=0}^{n-1} a_j \boldsymbol{b}_j(\hat{\boldsymbol{\mu}}_j) = B^0(t_n)$ and $\sum_{j=0}^{n-1} a_j \boldsymbol{e}_j(\hat{\boldsymbol{\mu}}_j) = M^0(t_n)$.

From (3.8), we have that

$$\sum_{i=0}^{n-1} \hat{\boldsymbol{z}}_{i+1} = \sum_{i=0}^{n-1} \hat{\boldsymbol{z}}_i - \sum_{i=0}^{n-1} a_i [\boldsymbol{g}(\hat{\boldsymbol{\mu}}_i) + \hat{\boldsymbol{b}}_i(\hat{\boldsymbol{\mu}}_i) + \hat{\boldsymbol{e}}_i(\hat{\boldsymbol{\mu}}_i)]. \tag{3.95}$$

Because $\hat{\boldsymbol{z}}_n = \hat{\boldsymbol{z}}^0(t_n)$ and $t_1 = 0$,

$$\hat{\boldsymbol{z}}^0(t_n) = \hat{\boldsymbol{z}}^0(0) - \sum_{i=0}^{n-1} a_i \boldsymbol{g}(\hat{\boldsymbol{\mu}}_i) - B^0(t_n) - M^0(t_n), \tag{3.96}$$

or equivalently,

$$\hat{\boldsymbol{z}}^0(t_n) = \hat{\boldsymbol{z}}^0(0) - \sum_{i=0}^{n-1} a_i \boldsymbol{g}(\hat{\boldsymbol{\mu}}_i) - U^0(t_n). \tag{3.97}$$

We can now linearly interpolate the sequence described by both sides of (3.97). For $\hat{\boldsymbol{z}}^0(t_n)$ and $U^0(t_n)$ we already now their interpolations are $\hat{\boldsymbol{z}}^0(t)$ and $U^0(t)$. $\hat{\boldsymbol{z}}^0(0)$ is a constant, and it can be immediately seen that a constant remains the same under interpolation. For the sum, we can see the following: from our choice of t_n it follows that $a_n = t_n - t_{n-1}$, thus $\sum_{i=0}^{n-1} a_i \boldsymbol{g}(\hat{\boldsymbol{\mu}}_i) = \sum_{i=0}^{n-1} (t_i - t_{i-1}) \boldsymbol{g}(\hat{\boldsymbol{\mu}}_i)$. Additionally, because $\bar{\boldsymbol{\mu}}^0(t) = \hat{\boldsymbol{\mu}}_i$ for $t_i \leq t < t_{i+1}$, so

$$\sum_{i=0}^{n-1} (t_i - t_{i-1}) \boldsymbol{g}(\hat{\boldsymbol{\mu}}_i) = \int_0^{t_n} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s)) \mathrm{d}s. \tag{3.98}$$

We can then linearly interpolate the integral as

$$l \cdot i\left(\left(t_i, \int_0^{t_i} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s\right), t\right) = \frac{(t_{n+1}-t)\int_0^{t_n} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s}{t_{n+1}-t_n} + \frac{(t-t_n)\int_0^{t_{n+1}} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s}{t_{n+1}-t_n} \tag{3.99}$$

$$= \frac{t_{n+1}\int_0^{t_n} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s - t_n\int_0^{t_{n+1}} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s}{t_{n+1}-t_n} + \frac{t\int_{t_n}^{t_{n+1}} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s}{t_{n+1}-t_n} \tag{3.100}$$

$$= \frac{(t_{n+1}-t_n)\int_0^{t_n} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s}{t_{n+1}-t_n} + \frac{(t-t_n)\int_{t_n}^{t_{n+1}} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s}{t_{n+1}-t_n} \tag{3.101}$$

$$= \int_0^{t_n} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s + \frac{(t-t_n)(t_{n+1}-t_n)\boldsymbol{g}(\hat{\boldsymbol{\mu}}_n)}{t_{n+1}-t_n} \tag{3.102}$$

$$= \int_0^{t_n} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s + \int_{t_n}^{t} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s \tag{3.103}$$

$$= \int_0^{t} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s. \tag{3.104}$$

Therefore, linearly interpolating both sides of (3.97) gives us

$$\hat{\boldsymbol{z}}^0(t) = \hat{\boldsymbol{z}}^0(0) - \int_0^t \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s - U^0(t). \tag{3.105}$$

Now we will define the following:

$$\hat{\boldsymbol{z}}^n(t) = \begin{cases} \hat{\boldsymbol{z}}^0(t+t_n) & \text{if } t \geq -t_n \\ \hat{\boldsymbol{z}}^0(0) & \text{if } t \leq -t_n \end{cases}, \tag{3.106}$$

$$B^n(t) = \begin{cases} B^0(t+t_n) - B^0(t_n) & \text{if } t \geq -t_n \\ -B^0(t_n) & \text{if } t \leq -t_n \end{cases}, \tag{3.107}$$

$$M^n(t) = \begin{cases} M^0(t+t_n) - M^0(t_n) & \text{if } t \geq -t_n \\ -M^0(t_n) & \text{if } t \leq -t_n \end{cases}, \tag{3.108}$$

$$U^n(t) = B^n(t) + M^n(t). \tag{3.109}$$

We can see that $\hat{\boldsymbol{z}}^n(0) = \hat{\boldsymbol{z}}^0(t_n) = \hat{\boldsymbol{z}}_n$. If we evaluate (3.105) in $t = t_n + t'$ and $t = t_n$, with $t' \geq -t_n$, the difference between the two results is

$$\hat{\boldsymbol{z}}^0(t' + t_n) - \hat{\boldsymbol{z}}^0(t_n) = -\int_{t_n}^{t_n+t'} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(s))\mathrm{d}s - U^n(t'), \tag{3.110}$$

or equivalently, replacing t' with t, and $t \geq -t_n$,

$$\hat{\boldsymbol{z}}^n(t) = \hat{\boldsymbol{z}}^n(0) - \int_0^t \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t_n + s))\mathrm{d}s - U^n(t). \tag{3.111}$$

For $t \geq -t_n$ we have that

$$B^n(t) = \frac{(t + t_n - t_m)B^0(t_{m+1}) + (t_{m+1} - t - t_n)B^0(t_m)}{t_{m+1} - t_m} - B^0(t_n) \tag{3.112}$$

$$= \frac{t_{m+1}B^0(t_m) - t_m B^0(t_{m+1}) + (t + t_n)(B^0(t_{m+1}) - B^0(t_m))}{t_{m+1} - t_m} - B^0(t_n) \tag{3.113}$$

$$= \frac{(t_{m+1} - t_m)B^0(t_m) - t_m a_m \boldsymbol{b}_m(\hat{\boldsymbol{\mu}}_m) + (t + t_n)a_m \boldsymbol{b}_m(\hat{\boldsymbol{\mu}}_m)}{t_{m+1} - t_m} - B^0(t_n) \tag{3.114}$$

$$= \frac{(t + t_n - t_m)(t_{m+1} - t_m)\boldsymbol{b}_m(\hat{\boldsymbol{\mu}}_m)}{t_{m+1} - t_m} - B^0(t_n) + B^0(t_m) \tag{3.115}$$

$$= (t + t_n - t_m)\boldsymbol{b}_m(\hat{\boldsymbol{\mu}}_m) + \sum_{i=n}^{m-1} a_i \boldsymbol{b}_i(\hat{\boldsymbol{\mu}}_i), \tag{3.116}$$

where m is such that $t_m \leq t + t_n \leq t_{m+1}$. The index m can also be defined as $m(n,t) = \max\{k : t_k - t_n \leq t\}$.

Lemma 2. *Let the hypotheses of lemma 1 hold. Then, for all $T > 0$,*

$$\lim_{n\to\infty} \sup_{t \leq T} |U^n(t)| = 0. \tag{3.117}$$

Proof. From (3.116), we have that

$$\|B^n(t)\| \leq \|(t + t_n - t_{m(n,t)})\boldsymbol{b}_{m(n,t)}(\hat{\boldsymbol{\mu}}_{m(n,t)})\| + \left\| \sum_{i=n}^{m(n,t)-1} a_i \boldsymbol{b}_i(\hat{\boldsymbol{\mu}}_i) \right\| \tag{3.118}$$

$$\leq \|(t + t_n - t_{m(n,t)})\boldsymbol{b}_{m(n,t)}(\hat{\boldsymbol{\mu}}_{m(n,t)})\| + \max_{k \geq n} \|\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k)\| \sum_{i=n}^{m(n,t)-1} a_i. \tag{3.119}$$

Because $t_{m(n,t)} - t_n \leq t$, and $t_m = \sum_{i=0}^{m(n,t)-1} a_i$,

$$\|B^n(t)\| \leq t\|\boldsymbol{b}_{m(n,t)}(\hat{\boldsymbol{\mu}}_{m(n,t)})\| + \max_{k\geq n}\|\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k)\|(t_m - t_n) \tag{3.120}$$

$$\leq t\|\boldsymbol{b}_{m(n,t)}(\hat{\boldsymbol{\mu}}_{m(n,t)})\| + t\max_{k\geq n}\|\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k)\|. \tag{3.121}$$

Therefore,

$$\sup_{t\leq T}\|B^n(t)\| \leq T\sup_{t\leq T}\|\boldsymbol{b}_{m(n,t)}(\hat{\boldsymbol{\mu}}_{m(n,t)})\| + T\max_{k\geq n}\|\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k)\|. \tag{3.122}$$

Let $t' \leq T$ be such that $\sup_{t\leq T}\|\boldsymbol{b}_{m(n,t)}(\hat{\boldsymbol{\mu}}_{m(n,t)})\| = \|\boldsymbol{b}_{m(n,t')}(\hat{\boldsymbol{\mu}}_{m(n,t')})\|$ and $k' \geq n$ be such that $\max_{k\geq n}\|\boldsymbol{b}_k(\hat{\boldsymbol{\mu}}_k)\| = \|\boldsymbol{b}_{k'}(\hat{\boldsymbol{\mu}}_{k'})\|$. Then,

$$\sup_{t\leq T}\|B^n(t)\| \leq T(\|\boldsymbol{b}_{m(n,t')}(\hat{\boldsymbol{\mu}}_{m(n,t')})\| + \|\boldsymbol{b}_{k'}(\hat{\boldsymbol{\mu}}_{k'})\|). \tag{3.123}$$

$m(n,t)$ increases as n does, so, because S1 in lemma 1, we have

$$\lim_{n\to\infty}\|\boldsymbol{b}_{m(n,t')}(\hat{\boldsymbol{\mu}}_{m(n,t')})\| = 0. \tag{3.124}$$

Similarly, because $k' \geq n$,

$$\lim_{n\to\infty}\|\boldsymbol{b}_{k'}(\hat{\boldsymbol{\mu}}_{k'})\| = 0. \tag{3.125}$$

Therefore,

$$\lim_{n\to\infty}\sup_{t\leq T}\|B^n(t)\| = 0. \tag{3.126}$$

On the other hand, from S2 in lemma 1, for $\eta > 0$

$$\lim_{k\to\infty} P\left(\sup_{m\geq k}\left\|(M^0(t_{m+1}) - M^0(t_k))-\right\| \geq \eta\right) = 0, \tag{3.127}$$

or equivalently,

$$\lim_{k\to\infty} P\left(\sup_{m\geq k}\left\|M^k(t_{m+1} - t_k)\right\| \geq \eta\right) = 0. \tag{3.128}$$

Therefore, for $T > 0$,

$$\lim_{k\to\infty} \sup_{t\leq T} \left\| M^k(t) \right\| = 0. \tag{3.129}$$

From (3.126) and (3.129), for $T > 0$

$$\lim_{k\to\infty} \sup_{t\leq T} \left(\left\| B^k(t) \right\| + \left\| M^k(t) \right\| \right) = 0. \tag{3.130}$$

And because $\|B^k(t)\| + \|M^k(t)\| \leq \|B^k(t) + M^k(t)\|$,

$$\lim_{k\to\infty} \sup_{t\leq T} \left\| U^k(t) \right\| = 0. \tag{3.131}$$

□

Theorem 6. *Let the hypotheses of lemma 1 hold. Then*

$$\lim_{k\to\infty} \hat{z}_k = \tilde{z} \tag{3.132}$$

for almost all $\omega \in \Omega$.

Proof. This proof was obtained thanks to [40] and follows the spirit of the proof for theorem 2.3.1 in [41]. First, let us see that $\{\hat{z}^n(t)\}_n$ is bounded and equicontinuous[2].

From lemma 2, we can see that $\{U^n(t)\}_n$ is bounded in finite intervals. Also from lemma 2, we can fix $\tau > 0$, so

$$\lim_{k\to\infty} \sup_{t\leq \tau+1} \left\| U^k(t) \right\| = 0. \tag{3.133}$$

Then, for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for $k \geq K$,

$$\sup_{t\leq \tau+1} \left\| U^k(t) \right\| \leq \frac{\varepsilon}{2}. \tag{3.134}$$

[2] A family of functions $\{f_n : X \to Y\}$ is *equicontinuous* if, for every $\varepsilon > 0$ and for each $x \in X$, there is a $\delta > 0$ such that if $d_X(x,y) < \delta$, with $y \in X$ and $d_X(\cdot,\cdot)$ a given distance measure in X, then $d_Y(f_n(x), f_n(y)) < \varepsilon$ for all f_n, with $d_Y(\cdot,\cdot)$ a given distance measure in Y.

In particular, this holds for $t = \tau$. Therefore, for $t \leq \tau + 1$,

$$\left\|U^k(t)\right\| + \left\|U^k(\tau)\right\| \leq \varepsilon. \tag{3.135}$$

And from here we get

$$\left\|U^k(t) - U^k(\tau)\right\| \leq \varepsilon. \tag{3.136}$$

On the other hand, because the $U^k(t)$ are continuous, for every $\varepsilon > 0$ and for each $k < K$ there exists $\delta_k > 0$ such that, if $|t - \tau| < \delta_n$, $\|U^k(t) - U^k(\tau)\| < \varepsilon$. Therefore, fore every ε for each t we can choose $\delta = \min\{\delta_1, ..., \delta_{N-1}, 1\}$ such that if $|t - \tau| < \delta$, then $\|U^k(t) - U^k(\tau)\| < \varepsilon$, that is, the family of $U^k(t)$ is equicontinuous.

From the hypotheses of theorem 5, $f \in \mathscr{C}(\mathfrak{C}^n)$, so $\boldsymbol{g}$ is continuous. Also, from Å3, the $\hat{\boldsymbol{\mu}}_n$ are bounded, while $\bar{\boldsymbol{\mu}}^0(t)$ always has values taken from the set of all $\hat{\boldsymbol{\mu}}_n$, so $\bar{\boldsymbol{\mu}}^0(t)$ is bounded too, and in turn, $\boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t))$ is also bounded. We define $M = \max\{\boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t))\}$. Then,

$$\int_0^t \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t_n + s))\mathrm{d}s \leq \int_0^t M\mathrm{d}s = Mt. \tag{3.137}$$

For every $\varepsilon > 0$, we can see that, for $\delta = \varepsilon/M$, if $|t_1 - t_2| < \delta$, then

$$\begin{aligned}
\varepsilon &> M|t_1 - t_2| && (3.138)\\
&= \left|\int_{t_1}^{t_2} M\mathrm{d}s\right| && (3.139)\\
&\geq \left|\int_{t_1}^{t_2} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t_n + s))\mathrm{d}s\right| && (3.140)\\
&= \left|\int_0^{t_2} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t_n + s))\mathrm{d}s + \int_{t_1}^0 \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t_n + s))\mathrm{d}s\right| && (3.141)\\
&= \left|\int_0^{t_2} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t_n + s))\mathrm{d}s - \int_0^{t_1} \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t_n + s))\mathrm{d}s\right| && (3.142)
\end{aligned}$$

Therefore, $\{\int_0^t \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t_n + s))\mathrm{d}s\}_n$ is bounded and equicontinuous. From (3.111), this means that $\{\hat{\boldsymbol{z}}^n(t)\}_n$ is bounded and equicontinuous too.

Let us define the term $\xi^n(t)$ as

$$\xi^n(t) = \int_0^t \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t_n+s))\mathrm{d}s - \int_0^t \boldsymbol{g}(\hat{\boldsymbol{\mu}}^n(s))\mathrm{d}s. \tag{3.143}$$

Then, we can rewrite (3.111) as

$$\hat{\boldsymbol{z}}^n(t) = \hat{\boldsymbol{z}}^n(0) - \int_0^t \boldsymbol{g}(\hat{\boldsymbol{\mu}}^n(s))\mathrm{d}s + \xi^n(t) - U^n(t). \tag{3.144}$$

We can see that $\bar{\boldsymbol{\mu}}^0(t_n+s)$ is equal to $\bar{\boldsymbol{\mu}}^0(s)$ displaced by t_n to the left, just like $\hat{\boldsymbol{\mu}}^n(s)$ in the interval $(0,t)$ is equal to $\hat{\boldsymbol{\mu}}^0(s)$ displaced by t_n to the left. Also, $\int_0^t \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t_n+s))\mathrm{d}s$ behaves as a left Riemann sum, while $\int_0^t \boldsymbol{g}(\hat{\boldsymbol{\mu}}^n(s))\mathrm{d}s$ corresponds to a trapezoidal Riemann sum. Because $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(t_n - t_{n-1}) = 0$, as n increases the size of the intervals of partition for the Riemann sums decrease and both integrals approach the same value. Therefore,

$$\lim_{n\to\infty} \xi^n(t) = \lim_{n\to\infty} \left(\int_0^t \boldsymbol{g}(\bar{\boldsymbol{\mu}}^0(t_n+s))\mathrm{d}s - \int_0^t \boldsymbol{g}(\hat{\boldsymbol{\mu}}^n(s))\mathrm{d}s \right) = 0. \tag{3.145}$$

According to Ascoli's theorem [42], an infinite, bounded and equicontinuous family of functions has a subsequence that converges to a continuous funtion, with the convergence being uniform in a compact set. Then, we can pick a convergent subsequence of $\{\hat{\boldsymbol{z}}^n(t)\}$ and index it by n_i, with $\boldsymbol{z}(t) = \lim_{n_i\to\infty} \hat{\boldsymbol{z}}^{n_i}(t)$ and, from lemma 2 and (3.145), we have

$$\boldsymbol{z}(t) = \boldsymbol{z}(0) - \int_0^t \boldsymbol{g}(\boldsymbol{\mu}(s))\mathrm{d}s. \tag{3.146}$$

Then, $\boldsymbol{z}(t)$ is a solution of the ODE of Å4.

We can fix $\varepsilon > 0$ such that there exists a $\delta > 0$ that satisfies $\delta < \varepsilon$, $B_\delta(\tilde{\boldsymbol{z}}) \subseteq S$ and, form the stability condition in Å4, for a solution $\boldsymbol{z}(t)$ of the ODE in Å4, if $|\boldsymbol{z}(0) - \tilde{\boldsymbol{z}}| \leq \delta$, then for any $t \geq 0$, $|\boldsymbol{z}(t) - \tilde{z}| < \varepsilon$.

Becuase there are infinite $\hat{z}_k$ in S, there is an infinite set $N_0 \subseteq \mathbb{N}$ such that, for all $i \in N_0$, $\hat{z}^i(t) \in S$ and $\{\hat{z}^i(t)\}_{i\in N_0}$ converges to a function $z(t)$ that is a solution of the ODE in Å4.Because $\hat{z}^i(0) \in S \in D(\tilde{z})$, and S is closed, $z(0) \in S \in D(\tilde{z})$, that is , the domain of attraction of $\tilde{z}$, so

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty}\left(\lim_{i\to\infty, i\in N_0} \hat{z}^i(t)\right) = \tilde{z}. \tag{3.147}$$

Therefore, there is a $T > 0$ such that, for any $t > T$,

$$\left|\lim_{i\to\infty, i\in N_0} \hat{z}^i(t) - \tilde{z}\right| < \frac{\delta}{2}. \tag{3.148}$$

Because $\lim_{i\to\infty} a_i = 0$, we can choose A_{max} such that $a_i = t_i - t_{i-1} < A_{max}$ for every i. Then, for all $t \in [T+1, T+A_{max}+1]$, there exists $J > 0$, such that for $j > J$, with $j \in N_0$,

$$\left|\hat{z}^j(t) - \lim_{i\to\infty, i\in N_0} \hat{z}^i(t)\right| < \frac{\delta}{2}. \tag{3.149}$$

Therefore, adding (3.148) with (3.149), we obtain

$$\left|\hat{z}^j(t) - \tilde{z}\right| = \left|\hat{z}^0(t_j + t) - \tilde{z}\right| < \delta. \tag{3.150}$$

Because the difference between consecutive t_i is at most A_{max}, there is at least one $t \in [T+1, T+A_{max}+1]$ such that there exists a $t_n = t + t_j$. Then

$$\left|\hat{z}^0(t_n) - \tilde{z}\right| = |\hat{z}_n - \tilde{z}| < \delta. \tag{3.151}$$

This is satisfied for every $j > J$, $j \in N_0$, so we have that there are infinitely many $\hat{z}_n$ in $B_\delta(\tilde{z})$.

Now, let us suppose there are infinitely many $\hat{z}_n$ outside $B_\varepsilon(\tilde{z})$. We can find sequences $\{l_i\}_i$ and $\{r_i\}_i$, both in $\mathbb{N}$ such that $l_i < r_i < l_{i+1}$ and $\hat{z}_{l_i} \in B_\delta(\tilde{z})$, $\hat{z}_{r_i} \notin B_\varepsilon(\tilde{z})$ and for every integer n such that $l_i < n < r_i$, $\hat{z}_n \notin B_\delta(\tilde{z})$.

Let us consider the following case: there exists $H \in \mathbb{R}$ such that, for infinitely many i, $r_i - l_i < H$. Then for infinitely many i, $t_{r_i} - t_{l_i} < A_{max}H$. Let us choose a subsequence $\{i_j\}_j$ such that $\lim_{j\to\infty} \hat{z}^{l_{i_j}}(t) = z(t)$ for all $t \in [0, A_{max}H]$, with $z(t)$ a solution of the ODE in Å4. Because all $\hat{z}^{l_{i_j}}(0) = \hat{z}_{l_{i_j}} \in B_\delta(\tilde{z})$, and the subsequence converges in a closed set, $|z(0) - \tilde{z}| \leq \delta$. Therefore, because of the stability condition, $|z(t) - \tilde{z}| < \varepsilon$ for $t \geq 0$. Then, for $t \in [0, A_{max}H]$, there exists $\mu < \varepsilon$ such that $|z(t) - \tilde{z}| \leq \mu$. Because the $\{\hat{z}^{l_{i_j}}(t)\}_j$ converge uniformly to $z(t)$ for $t \in [0, A_{max}H]$, we have that for all ν with $\mu < \nu < \varepsilon$, there exists $J > 0$ such that for every $j > J$, $|\hat{z}^{l_{i_j}}(t) - \tilde{z}| < \nu$ for every $t \in [0, A_{max}H]$. Also, because $t_{r_i} - t_{l_i} < A_{max}H$, in $[0, A_{max}H]$ there must exist $t' \in [0, A_{max}H]$ such that $t' + t_{l_{i_j}} = t_{r_{i_j}}$, so $\hat{z}^{l_{i_j}}(t') = \hat{z}^0(t' + t_{l_{i_j}}) = \hat{z}^0(t_{r_{i_j}}) = \hat{z}_{r_{i_j}}$. Then, $|\hat{z}_{r_{i_j}} - \tilde{z}| < \nu$. However, we have said that $\hat{z}_{r_{i_j}} \notin B_\varepsilon(\tilde{z})$, so $|\hat{z}_{r_{i_j}} - \tilde{z}| \geq \varepsilon$. Therefore, we arrive to a contradiction.

On the other hand, let us consider what happens if there is no such H, that is, the quantity $r_i - l_i$ diverges as i grows. Just like before, we choose a subsequence i_j such that $\lim_{j\to\infty} \hat{z}^{l_{i_j}}(t) = z(t)$ for all $t \geq 0$, with $z(t)$ a solution of the ODE in Å4. Because all $\hat{z}^{l_{i_j}}(0) = \hat{z}_{l_{i_j}} \in B_\delta(\tilde{z})$, $|z(0) - \tilde{z}| \leq \delta$. Therefore, from the asymptotic stability condition, $\lim_{t\to\infty} z(t) = \tilde{z}$, that is, for every $\mu > 0$ there is a $T > 0$ such that, for every $t > T$, $|z(t) - \tilde{z}| < \mu$. In particular, this holds for $\mu = \delta$. Then, in the compact $[T, T + A]$, the $\{\hat{z}^{l_{i_j}}(t)\}_j$ converge uniformly to $z(t)$, so for all $\delta > 0$ there exists $J > 0$ such that for all $j > J$, $|\hat{z}^{l_{i_j}}(t) - \tilde{z}| < \delta$ for all $t \in [T, T + A]$. Because $r_{i_j} - l_{i_j}$ diverges, between $t_{l_{i_j}}$ and $t_{r_{i_j}}$ there is an increasing number of elements of $\{t_i\}$, so almost surely there exists $t \in [T, T + A]$ that satisfies $t' + t_{l_{i_j}} = t_n$ for some t_n. Then, $\hat{z}^{l_{i_j}}(t') = \hat{z}^0(t' + t_n) = \hat{z}^0(t_n) = \hat{z}_n$, and $|\hat{z}_n - \tilde{z}| < \delta$. However, we have said that $\hat{z}_n \notin B_\delta(\tilde{z})$, as $l_{i_j} < n < r_{i_j}$, so $|\hat{z}_n - \tilde{z}| \geq \delta$. Therefore, we arrive to another contradiction.

We can conclude it is impossible for infinitely many $\hat{z}_n$ to be outside $B_\varepsilon(\tilde{z})$. Because this is valid for all $\varepsilon > 0$, we can conclude that $\lim_{n\to\infty} \hat{z}_n = \tilde{z}$.

□

Chapter 4

Tomography of Quantum States using CSPSA

Just like it was possible for SGQT to make use of SPSA to minimize the distance to the state of the system, we are interested a tomographic method using ℂSPSA to minimize the same distance, but this time considering it as a function of complex variables. We will call this method *Complex Self-Guided Quantum Tomography* (ℂSGQT).

4.1 ℂSGQT procedure

Let us consider a quantum system in the state ρ. We are interested in finding the state $\tilde{\sigma}$ such that, for a given distance measure m on the Hilbert space $\mathscr{H}$ associated to the system,

$$m(\rho,\tilde{\sigma}) = \min_{\sigma\in\mathscr{H}} m(\rho,\sigma). \tag{4.1}$$

For pure states $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$, we can consider m to be the infidelity between two pure states, I, defined as

$$I(|\psi\rangle,|\phi\rangle) = 1 - |\langle\psi|\phi\rangle|^2. \tag{4.2}$$

Because the state $|\psi\rangle$ is fixed, we can define the function $f_\psi(\phi) = I(|\psi\rangle,|\phi\rangle)$. The infidelity can be then be obtained experimentally when performing a measurment onto a basis that contains $|\phi\rangle\langle\phi|$ as

$$f_\psi(\phi) = 1 - \frac{n(\phi)}{N}, \tag{4.3}$$

where N is the total numbers of detections made in the experiment, and $n(\phi)$ is the number of detections of $|\phi\rangle$.

We can use ℂSPSA to find $|\tilde{\phi}\rangle \in \mathscr{H}$, $\dim\mathscr{H} = n$, such that

$$f_\psi(\tilde{\phi}) = \min_{|\phi\rangle\in\mathscr{H}} f_\psi(\phi). \tag{4.4}$$

This is achieved using algorithm (3.4),

$$|\hat{\phi}_{k+1}\rangle = \frac{|\hat{\phi}_k\rangle - a_k\hat{\boldsymbol{g}}(\hat{\phi}_k)}{\||\hat{\phi}_k\rangle - a_k\hat{\boldsymbol{g}}(\hat{\phi}_k)\|}, \tag{4.5}$$

where $|\hat{\phi}_k\rangle$ is an estimate of $|\tilde{\phi}\rangle$ at the k-th iteration, the a_k are real parameters and $\hat{\boldsymbol{g}}(\hat{\phi}_k) \in \mathscr{H}$ is an estimate of the gradient of f_ψ at the k-th iteration, whose components are defined as

$$\hat{g}_{k,i}(\hat{\phi}_i) = \frac{f_\psi(\hat{\phi}_k + c_k\boldsymbol{\Delta}_k) + \varepsilon_{k,+} - f_\psi(\hat{\phi}_k - c_k\boldsymbol{\Delta}_k) - \varepsilon_{k,-}}{2c_k\Delta^*_{k,i}}, \tag{4.6}$$

with $i = 1, ..., n$, c_k a real parameter, $\boldsymbol{\Delta}_k \in \mathscr{H}$ a randomly generated vector and $\varepsilon_{k,\pm}$ are noise terms. In (4.5) we normalize the state obtained by the algorithm of $\mathbb{C}$SPSA.

According to the hypotheses of theorem 5, $\mathbb{E}(|\Delta^{-1}_{k,i}|) \leq \alpha_1$, with $\alpha_1 > 0$. Therefore, none of the $\Delta_{k,i}$ can be equal to zero. For $\mathbb{C}$SGQT, we will choose $\Delta_{k,i}$ as a random pick between 1, -1, i or $-i$, all with the same probability. Here lies one of the main differences with SGQT.

In SGQT, it was necessary to represent the quantum state through its real and imaginary parts. Our choice for $\Delta_{k,i}$ is either $+1$ or -1. Because the $\Delta_{k,i}$ also correspond to real and imaginary parts of complex numbers, those numbers are $1+i$, $1-i$, $-1+i$ or $-1-i$, randomly chosen. The values we chose for $\Delta_{k,i}$ in $\mathbb{C}$SGQT are impossible to obtain in SGQT, as that would require that either the real or the imaginary part is zero, which is not permitted by $\mathbb{E}(|\Delta^{-1}_{k,i}|) \leq \alpha_1$.

The values for parameters a_k and c_k are defined, just like in SPSA, as

$$a_k = \frac{A}{(k+1+B)^s}, \tag{4.7}$$

$$c_k = \frac{C}{(k+1)^r}, \tag{4.8}$$

with a, B, C, s and r chosen, as before, as $A = 3$, $B = 0$ and $C = 0.1$, but thise time, $s = 1$ and $r = 1/6$.

4.2 Simulation of ℂSGQT

We performed computational simulations of ℂSGQT in order to study its efficacy. Our program was made following algorithm 1 for obtaining the infidelity inf between the quantum state ψ of the system and the estimate ϕ_0 at each of K iterations

The function FID estimates the fidelity Fid, as it would be measured in a laboratory, between states ψ_1 and ψ_2 using N detections.

The results obtained through this procedure will be shown in the next section.

4.3 Comparison between the perfomances of SGQT and ℂSGQT

Figures 4.1 to 4.4 compare the mean infidelity $\bar{I}$ over 10^4 randomly chosen pairs of qubits $|\psi\rangle$ and $|\hat{\phi}_0\rangle$, by using SGQT (blue, dashed line) and ℂSGQT (red, continuous line) through 100 iterations of the algorithm, simulating an experiment of 10, 100, 1000 and 10^4 total detections, respectively. The shaded regions correspond to the variance around the mean infidelity.

It can be seen that ℂSGQT achieves lower mean infidelities than SGQT. After 100 iterations, the improvement goes from around one order of magnitude to nearly three, increasing as we get more detections per experiment. The variance also decreases as we perform more iterations.

Algorithm 1 $\mathbb{C}$SPSA algorithm

procedure $\mathbb{C}\text{SPSA}(\psi, \phi_0, K, N)$
$A \leftarrow 3$, $s \leftarrow 1$, $B \leftarrow 0$, $C \leftarrow 0.1$, $r \leftarrow 1/6$
$p \leftarrow \dim \psi$
for $k \leftarrow 1, K$ **do**
$a_k \leftarrow \dfrac{A}{(k+B)^s}$
$c_k \leftarrow \dfrac{C}{k^t}$
$\Delta \leftarrow$ a vector of dimension p whose components are randomly chosen from the set $\{+1, -1, +i, -i\}$
$\phi_+ \leftarrow \phi_0 + c_k\Delta$
$\phi_+ \leftarrow \dfrac{\phi_+}{\|\phi_+\|}$
$inf_+ \leftarrow 1 - Fid(\psi, \phi_+, N)$
$\phi_- \leftarrow \phi_0 - c_k\Delta$
$\phi_- \leftarrow \dfrac{\phi_-}{\|\phi_-\|}$
$inf_- \leftarrow 1 - Fid(\psi, \phi_-, N)$
$\hat{g} \leftarrow \dfrac{inf_+ - inf_-}{2c_k}\Delta$
$\phi_0 \leftarrow \phi_0 - a_k\hat{g}$
$\phi_0 \leftarrow \dfrac{\phi_0}{\|\phi_0\|}$
$inf(k) \leftarrow 1 - |\psi \cdot \phi_0|^2$
end for
return inf
end procedure

function FID(ψ_1, ψ_2, N)
for $i \leftarrow 1, N$ **do**
$var \leftarrow$ a random number between 0 and 1
$x(i) \leftarrow \begin{cases} 1 & \text{if } var \leq |\psi_1 \cdot \psi_2|^2 \\ 0 & \text{if } var > |\psi_1 \cdot \psi_2|^2 \end{cases}$
end for
$Fid \leftarrow \dfrac{\sum_{i=1}^{N} x(i)}{N}$
return Fid
end function

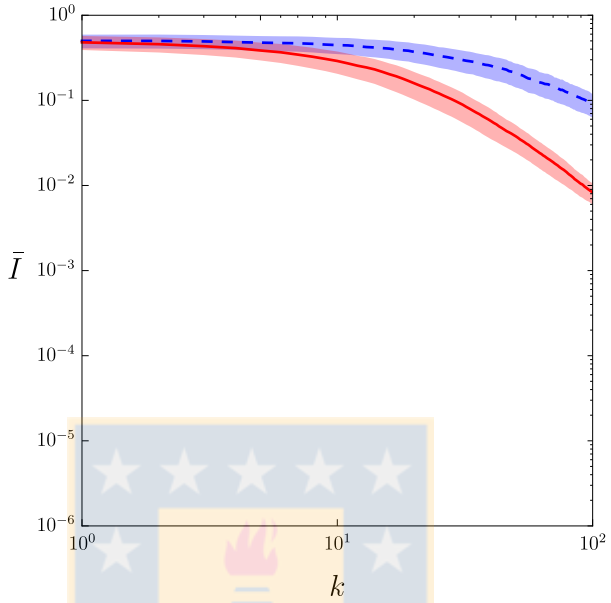

Figure 4.1: Mean infidelity obtained by SGQT (blue) and $\mathbb{C}$SGQT (red) by simulating experiments using 10 detections. Source: Made by the author.

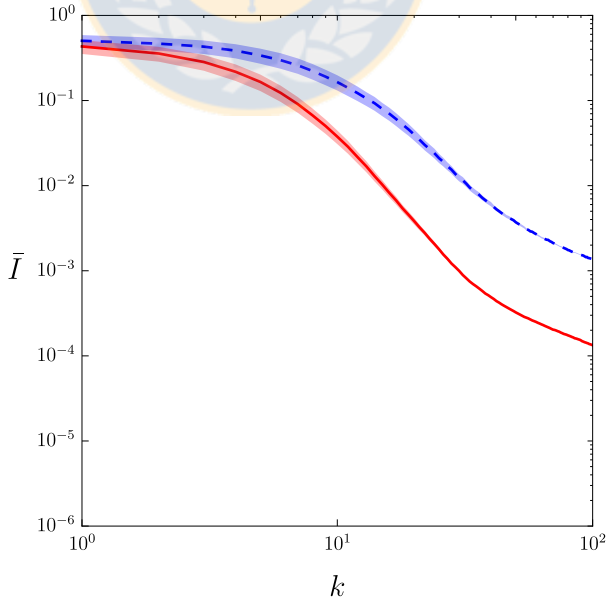

Figure 4.2: Mean infidelity obtained by SGQT (blue) and $\mathbb{C}$SGQT (red) by simulating experiments using 100 detections. Source: Made by the author.

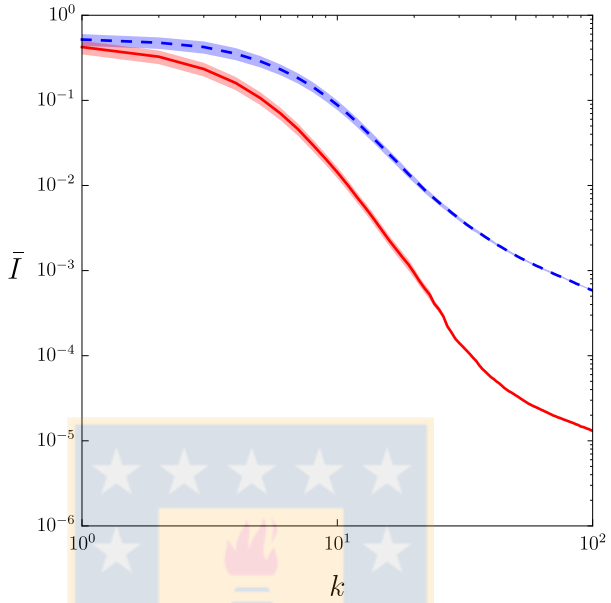

Figure 4.3: Mean infidelity obtained by SGQT (blue) and $\mathbb{C}$SGQT (red) by simulating experiments using 1000 detections. Source: Made by the author.

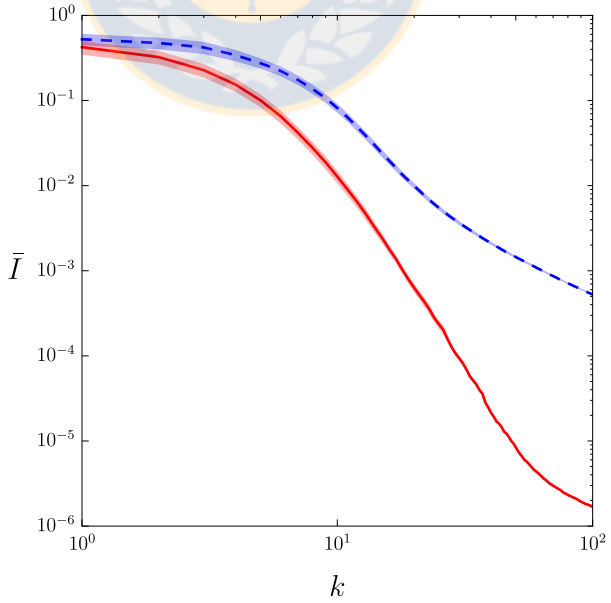

Figure 4.4: Mean infidelity obtained by SGQT (blue) and $\mathbb{C}$SGQT (red) by simulating experiments using 10^4 detections. Source: Made by the author.

Figure 4.5 shows the mean infidelity $\bar{I}$ over 10^4 randomly chosen pair of states $|\psi\rangle$ and $|\hat{\phi}_0\rangle$ versus the dimension d of the Hilbert space of the system, by simulating experiments of 10 (blue), 100 (green) and 1000 (red) detections, using SGQT (dashed lines) and $\mathbb{C}$SGQT (continuous lines).

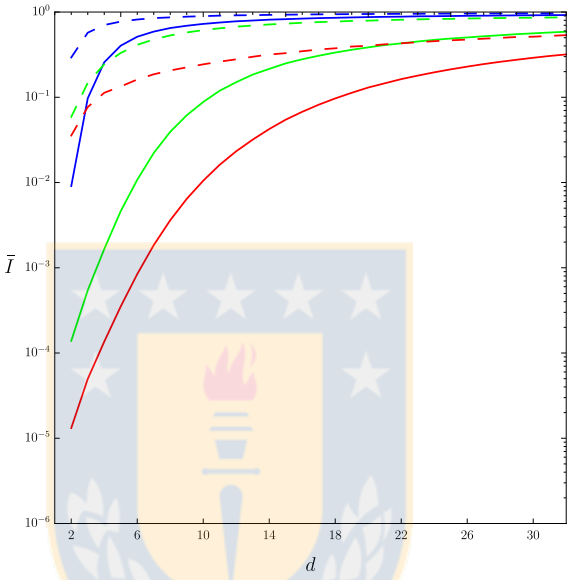

Figure 4.5: Mean infidelity obtained by SGQT (dashed) and $\mathbb{C}$SGQT (continuous) by simulating experiments using 10 (blue), 100 (green) and 1000 (red) detections for states of dimension d. Source: Made by the author.

Even if we increase the dimension, we will obtain a better estimation with $\mathbb{C}$SGQT than with SGQT, in spite of the low performance for high dimensions for both methods.

Figures 4.6 to 4.9 show histograms for the distributions of infidelities after 100 iterations of the algortihm over 10^6 randomly chosen pairs of qubits $|\psi\rangle$ and $|\hat{\phi}_0\rangle$. The blue histograms to the left correspond to the results obtained with SGQT, while the red histograms to the right were done using $\mathbb{C}$SGQT.

These distributions give us mixed results: for 10 detections, $\mathbb{C}$SGQT is less disperse and has a higher frequency of minimum infidelities than SGQT. However, for 100 and 1000

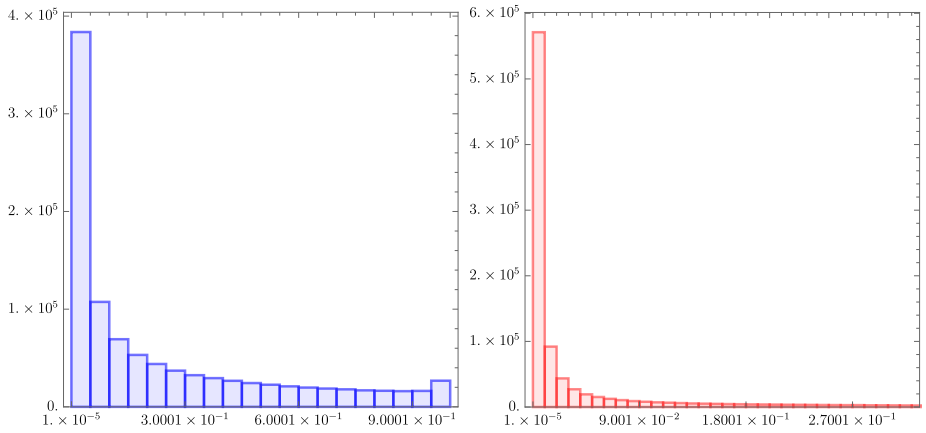

Figure 4.6: Distribution of the infidelities over 10^6 repetitions of the simulation of SGQT (left) and $\mathbb{C}$SGQT (right) for experiments with 10 detections. Source: Made by the author.

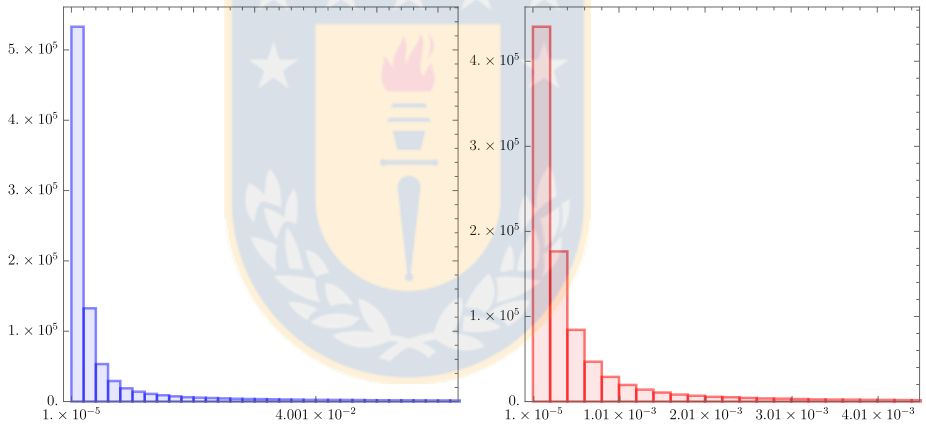

Figure 4.7: Distribution of the infidelities over 10^6 repetitions of the simulation of SGQT (left) and $\mathbb{C}$SGQT (right) for experiments with 100 detections. Source: Made by the author.

detections, the opposite is true, with SGQT showing a smaller deviation than $\mathbb{C}$SGQT. For 10^4 both techniques have a quite similar distribution. It is worth mentioning, though, that in all these cases $\mathbb{C}$SGQT still achieves lower infidelities than SGQT.

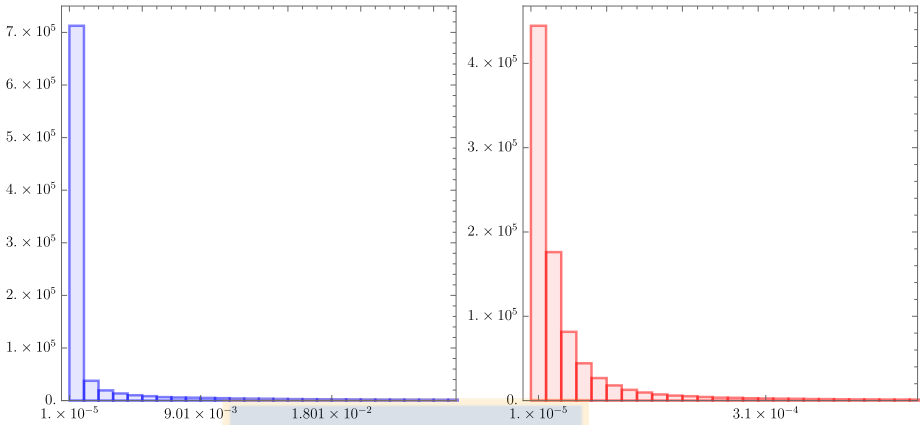

Figure 4.8: Distribution of the infidelities over 10^6 repetitions of the simulation of SGQT (left) and $\mathbb{C}$SGQT (right) for experiments with 1000 detections. Source: Made by the author.

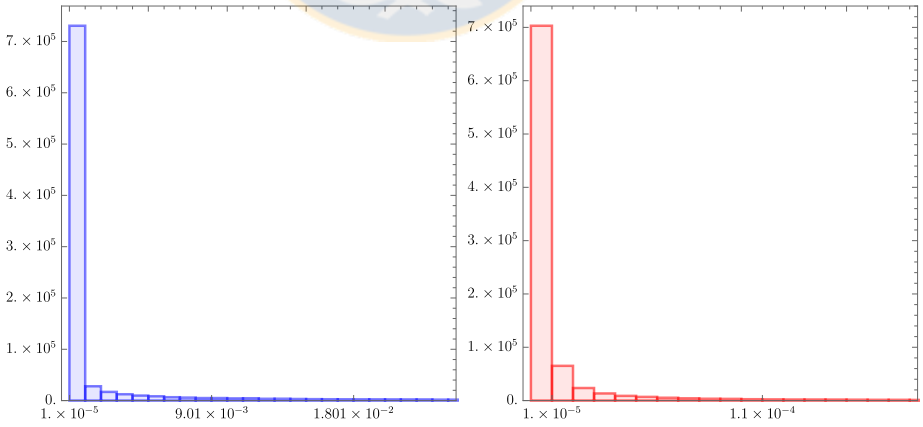

Figure 4.9: Distribution of the infidelities over 10^6 repetitions of the simulation of SGQT (left) and $\mathbb{C}$SGQT (right) for experiments with 10^4 detections. Source: Made by the author.

Chapter 5

Conclusion

We have designed a new method for quantum tomography, that unlike The method proposed by Ferrie, does not need to represent the quantum state i its real and imaginary parts, instead working with it in a complex vector space. Additionally, we have shown that our method performs even better than Ferrie's, which already was an improvement over traditional tomographic techniques.

It should be remembered, however, that our results considered only pure states. For density matrices, it is unknown how to experimentally measure the infidelity between two states. Therefore, another distance measure should be needed for the case of mixed states. It is also sensible to use the $\mathbb{C}$SPSA algorithm in an attempt to find the minimum (and maximum) eigenvalues of a density matrix, providing us with another scheme for tomography.

In fact, $\mathbb{C}$SPSA could be used for other physical problems, not necessarily related with quantum tomography, that involves minimizing or maximizing a function. Examples of this would be minimizing the expectation value of energy to obtain the ground state energy of a quantum system, or even in quantum tomography via postprocessed state estimation, as $\mathbb{C}$SPSA can be used during the maximum likelihood estimation.

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