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El problema de Büchi para números *p*-ádicos
(Büchi's problem for *p*-adic numbers)

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Introduction

Motivated by a mathematical logic problem, J.R. Büchi proposed the following problem in the early 1970's.

Problem (Büchi's problem)**.** $\mathbf{B}^2(\mathbb{Z})$. *Does there exist a positive integer M such that any sequence of M integer squares, with second difference constant equal to the constant sequence $(2)_n$, is of the form $\big((x+n)^2\big)_n$, where $n = 1, \ldots, M$, for some integer x?*

Büchi's problem is open. However, in 2001, P. Vojta showed that it would have a positive answer if Bombieri's conjecture were true for surfaces.

In [12], Pheidas and Vidaux proposed a generalization of Büchi's problem to any unitary commutative ring and to higher powers.

Definition. *Let $k \geq 0$ be an integer. A sequence of elements of a unitary commutative ring A of characteristic 0 is called a k-*Büchi sequence in *A if the sequence of its k-th powers has k-th difference constant equal to $(k!)_n$. Every sequence whose sequence of k-th powers is of the form $\big((x+n)^k\big)_n$ for some x in A will be referred to as* trivial k-Büchi sequence*.*

Note that a trivial k-Büchi sequence is a Büchi sequence. Büchi's problem is generalized as follows.

Problem. $\mathbf{B}^k(A)$. *Let $k \geq 2$ be an integer and A a unitary commutative ring of characteristic 0. Does there exists an integer M such that every k-Büchi sequence in A of length M is trivial?*

Observe that if $\mathbf{B}^k(A)$ has a positive answer, then for any subring B of A, $\mathbf{B}^k(B)$ has a positive answer. In this thesis, we are interested in those rings for which Büchi's problem has a negative answer in a *non-trivial way* (intuitively, rings with not too many k-powers). For example, if $A = \bar{\mathbb{Q}}$ is the field of algebraic numbers, every sequence of the form

$$\left(x_1, x_2, x_3 = \sqrt{2 + 2x_2^2 - x_1^2}, \ldots, x_M = \sqrt{2 + 2x_{M-1}^2 - x_{M-2}^2}, \ldots\right)$$

is a 2-Büchi sequence (which, in general, is non-trivial). With a similar idea, one sees easily that $\mathbf{B}^k(\bar{\mathbb{Q}})$ has a negative answer for every $k \geq 2$. The sequence

$$(\sqrt[k]{n^k+1})_{n\geq 0}$$

being a non-trivial k-Büchi sequence of infinite length, we see that $\mathbf{B}^k(\bar{\mathbb{Z}}\cap\mathbb{R})$ has a negative answer for every $k \geq 2$. In both examples above, the negative answer to Büchi's problem is due to the existence of an infinite non-trivial Büchi sequence.

In their survey [11] on Büchi's problem, Pasten, Pheidas and Vidaux posed the problem of finding rings for which Büchi's problem had a negative answer without having non-trivial sequences of infinite length. They distinguish two kinds of rings in which Büchi's problem can have a negative answer (in characteristic 0):

- **Type 1**: Rings for which there exists an infinite non-trivial Büchi sequence.
- **Type 2**: Rings for which there exist non-trivial Büchi sequences of any finite length, but there is no infinite one.

In [1], J. Browkin proved that for $k = 2$, the field of p-adic numbers $\mathbb{Q}_p$ is of type 1 and the ring of p-adic integers $\mathbb{Z}_p$ is of type 2. This Thesis is an attempt to generalize Browkin's result to higher powers. Before we state our main results, let us introduce the notion of an Hensley sequence:

Definition (Hensley sequences)**.** *Let $k \geq 0$ be an integer. A sequence (a_n) of elements of a unitary commutative ring A of characteristic 0, whose k-th powers are of the form*

$$(a+n)^k + b_{k-2}n^{k-2} + \cdots + b_1 n + b_0,$$

for some $a, b_{k-2}, \ldots, b_0 \in A$, is called k-Hensley sequence*. If*

$$b_0 = \cdots = b_{k-2} = 0$$

then (a_n) is called a trivial k-Hensley sequence*.*

Problem (Hensley's formulation of Büchis's problem)**. $\mathbf{HF}^k(A)$.** *Let $k \geq 2$ be an integer and A a unitary commutative ring of characteristic 0. Does there exist an integer $M \geq k+1$ such that every k-Hensley sequence in A of length M is a trivial sequence?*