Universidad de Concepción

Dirección de Postgrado

Facultad de Ciencias Físicas y Matemáticas - Programa Magíster en Matemática

Minimización del primer valor propio del p-Laplaciano de Dirichlet en ciertos tipos de dominios.

(Minimization of the first eigenvalue of the Dirichlet p-Laplacian in certain classes of domains.)

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CONCEPCIÓN-CHILE

2012

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Introduction.

In this thesis, we will work around the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Δ_p is the p- Laplacian operator, with $1 < p < \infty$, which is a generalization of the Laplacian operator ($p = 2$) and it is defined for a function u in the Sobolev space $W_0^{1,p}(\Omega)$ as

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

More specifically, we will study thoroughly the first eigenvalue $\lambda_1(\Omega)$ of p- Laplacian with Dirichlet condition, which is defined as the minimum of Rayleigh quotient for nonzero functions belonging to $W_0^{1,p}(\Omega)$. i.e.,

$$\lambda_1(\Omega) = \min_{\varphi \in W_0^{1,p}(\Omega), \varphi \neq 0} \frac{\int_\Omega |\nabla \varphi|^p}{\int_\Omega |\varphi|^p}.$$

We note that , λ_1 depends on the domain Ω. We will show the principal properties of $\lambda_1(\Omega)$ and of its eigenfunctions, and later obtain results on the problem of minimization of $\lambda_1(\Omega)$ in certain classes of domains with the same volume or perimeter, similar to a classical problem.

In the first chapter, which corresponds to the preliminaries, we will introduce some basic notions and definitions. We introduce the notion of a distribution, which allows us to define the concept of weak derivative of a function defined in a domain Ω, among other notions. Moreover, in the first chapter we will define the Sobolev space $W^{1,p}(\Omega)$, which is the set of all functions which belong to $L^p(\Omega)$, such that all its weak derivatives of first order also belong to $L^p(\Omega)$. Our aim in this chapter is to recall the definition and to study certain properties of the space $W_0^{1,p}(\Omega)$, which is the domain of the operator p- Laplacian. We also recall some useful notions and results from measure theory and recall the notion of Hausdorff convergence of sets.

In the second chapter, we will describe the principal tools which will allow us to solve the problem of minimization of λ_1 in certain classes of domains. We will introduce the Schwarz symmetrization and the Steiner symmetrization, and discuss their main properties.

In the third chapter, we will describe the principal properties of $\lambda_1(\Omega)$, which we will use for solving the minimization problems previously mentioned. We will see that $\lambda_1(\Omega)$ is a simple eigenvalue, with eigenfunctions which are either strictly positive or strictly negative on Ω and these belong to $C^1(\overline{\Omega})$. Also we will see that $\lambda_1(\Omega)$ is invariant by translation and rotation, and it varies only by a constant if we apply a homothety to the domain Ω, among other properties. Further, we will show that $\lambda_1(\Omega)$ as a function of the domain is continuous with respect to the topology induced by the Hausdorff distance in certain class of domains. Another important tool is the differentiability of λ_1 with respect to domain variations, which will allow us in the last chapter to develop an alternative proof of an analogue of the Faber- Krahn inequality for the p- Laplacian, which says that the domain that minimizes $\lambda_1(\Omega)$ among simply connected bounded domains of the same volume with C^2 boundary is a sphere.

Finally, in the fourth and last chapter we prove the Faber-Krahn inequality for the p- Laplacian and discuss two proofs of the uniqueness of the ball as a minimizer, one being classical and another being an alternative proof. We will see that the classical proof uses Schwarz symmetrization and that the alternative proof uses the fact that if Ω is a minimal domain for λ_1 as a function of the domain, among simply connected bounded domains of the same volume with C^2 boundary , then the eigenfunction corresponding to $\lambda_1(\Omega)$ has the property that its normal derivative is constant on $\partial\Omega$. We will show that the only domain that satisfies this condition is a sphere. As a corollary, we will prove an analogous result for domains with a given surface area. Moreover, we will