UNIVERSIDAD DE CONCEPCIÓN
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
PROGRAMA DE MAGÍSTER EN MATEMÁTICA - ACADÉMICO

# Análogos del Décimo Problema de Hilbert

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Tesis para ser presentada a la Dirección de Postgrado de la Universidad de Concepción

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CONCEPCIÓN-CHILE
2010

# Introduction

The tenth problem in D. Hilbert's famous list asked the following:

> Devise an algorithm to decide whether a polynomial equation with integer coefficients has an integer solution.

(These equations are called *Diophantine equations.*) In the year 1970, 70 years after Hilbert posed it, Y. Matiyasevich (based on work of M. Davis, J. Robinson and H. Putnam) proved that such an algorithm does not exist [10].

Knowing that the decision problem for integer solutions of Diophantine equations had a negative answer, the problem shifted to smaller classes of equations. For example, it follows from Matiyasevich's negative answer that there exists no algorithm to decide whether a system of second-degree Diophantine equations has integral solutions; while, on the other hand, a result of M. Presburger (1929) implies that an analogous algorithm for systems of linear Diophantine equations exists [15].

Consider all systems of second degree Diophantine equations in where every unknown appears squared in all of its ocurrences. These systems form a subset of all systems of second degree Diophantine equations, and it can be shown that every linear Diophantine equation can be written as such (just because any integer can be written as $x^2+y^2-z^2$ for some integers $x$, $y$ and $z$). Thus, the decision problem for integer solutions to this kind of systems of equations is "in between" the two already known results mentioned in the previous paragraph. This problem is known as the Problem of representation by diagonal quadratic forms, and is currently open.

To study this last problem, let $S \subset \mathbb{Z}$ be the set of all perfect squares. The Problem of representation by diagonal quadratic forms is now equivalent (cf. Proposition 3.3) to

> Determine if there exists an algorithm to decide whether a system of linear Diophantine equations, together with relations of

the form $x_i \in S$, has a solution in $\mathbb{Z}$.

This is the main problem that inspires most of this work, and will be referred to as such in the following.

This thesis is structured as follows: Chapter 1 presents a brief collection of definitions and results from logic needed to correctly state Hilbert's Tenth Problem and its related problems. The results given there can be found in logic textbooks such as [1]. Readers already acquainted with the concepts of *computable function*, *recursive set* and *first order language* may skip this Chapter.

Chapter 2 does not present any original result; it contains instead the general well-known ideas that we might use in our proofs without mentioning them.

In Chapter 3, J. R. Büchi's approach to the main problem is shown. What he did was to conjecture an arithmetical result, today known as *Büchi's $n$ squares problem* (still open; P. Vojta [21] solved it, assuming Bombieri's Conjecture for surfaces). If this problem had a positive answer, he showed that an undecidability answer to the main problem would follow. Afterwards, T. Pheidas and X. Vidaux generalized Büchi's approach to powers other than squares. We will give a new proof of this *reduction* result.

Chapter 4 presents a small variation on the main problem, including relations of the form $x_i x_j \in S$ to the systems (thanks to J. Demeyer for proposing this problem). To approach this new problem, we do a work similar to Büchi's: we formulate an arithmetical problem that, if positively answered, implies an undecidability result to this *a priori* weaker problem. Afterwards, we show the relation between this new arithmetical problem and Büchi's original conjecture.

In Chapter 5, starting from a theorem by Kosovskii for studying systems that include linear Diophantine equations and divisibility relations, we get the following result:

**Theorem.** *For any $k \geq 2$, there exists no algorithm to decide whether there exists a solution in the integers to a system that consists of linear Diophantine equations, relations of the form $x|y$, and one of the following kinds of relations:*

*i) "$x$ is a $k$-th power", or*

*ii) $x = k^y$.*