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AN UNFITTED HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD IN SHAPE OPTIMIZATION

POR

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Abstract

Shape optimization seeks to optimize the shape of a region where certain partial differential equation is posed such that a functional of its solution is minimized/maximized. In this thesis we give an introduction to shape optimization through a model problem, introducing the concepts of shape derivative for a function and perturbation of the shape for a functional, we deduce the optimality conditions for the problem, and then we will present a numerical method to seek the solution via a hybridizable discontinuous Galerkin methods on curved domains. Subsequently, we develop a rigorous treatment to analyze the well-posedness of the problems that arise from the optimality conditions, and provide an *a priori* error analysis for each scheme.

Resumen

La optimización de forma busca optimizar la forma de una región en la que se plantea una determinada ecuación diferencial parcial de manera que se minimiza/maximiza un funcional de su solución. En esta tesis damos una introducción a la optimización de forma a través de un problema modelo, introduciendo los conceptos de derivada de la forma para una función y perturbación de la forma para un funcional, deducimos las condiciones de optimalidad para el problema, y luego presentamos un método numérico para buscar la solución a través de métodos de Galerkin discontinuo hibridizables en dominios curvos. Posteriormente, realizamos un tratamiento riguroso para analizar la existencia y unicidad de soluciones de los problemas que surgen de las condiciones de optimalidad, luego hacemos un análisis de error *a priori* para cada esquema.

CHAPTER 1

Introduction

Shape optimization is an important branch of the study of optimal control theory, which was developed extensively in the 1990s and arises from the purpose of minimizing material or energy cost through modification of the design shape. This area has inspired the development of a wide variety of theoretical and purely mathematical tools, and has a large number of applications in science and engineering, such as architecture and civil engineering [5], fluid mechanics [16, 30, 56], modelling of quantum chemistry phenomena [7, 11], electromagnetism or photonics [37, 39], among others research fields. From a mathematical point of view, we can see shape optimization as finding the minimum (whenever it exists) of a cost functional over a set of admissible domains, in many cases this minimization problem is constrained by a partial differential equation (PDE) defined on the target domain.

The first Discontinuous Galerkin (DG) method was developed in 1973 by Reed and Hill in an article for a neutron transport equation corresponding to a linear hyperbolic equation independent of time [47]. Since then, DG methods became one of the most widely used methods for the numerical analysis of PDEs. However, DG methods were criticized for having too many degrees of freedom and for having a complicated computational implementation compared to Continuous Galerkin (CG) methods. These criticisms were resolved after the development of Hybridizable Discontinuous Galerkin (HDG) methods, first for diffusion problems and later presented in a unified framework [20].

During the past and present decade, HDG methods have been extensively developed for different types of equations, for instance, diffusion equations [18,21,22,38], convection-diffusion equations [19, 32, 43], the wave equation [24], Stokes flow [13, 25, 34, 40], Oseen and Brinkman equations [4,14,35], Navier-Stokes equations [15,45,48], linear and nonlinear elasticity [26,44,54], just to name a few.

In recent years, HDG methods have been developed for domains that are not necessarily polygonal/polyhedral, the method we will use for this thesis seeks to approximate the solution in a polygonal subdomain by transferring the data from the curved boundary to the boundary of the polygonal subdomain while maintaining the high order of convergence. This HDG method was introduced within the context of HDG discretizations for linear elliptic equations in [17], subsequently completing its theoretical development in [23]. This method has been used for solving equations, for instance, Stokes flow [52], Oseen equations [53], the Helmholtz equation [10], convection diffusion equations [27], the Grad-Shafranov equation [49, 50], among others.

For the treatment of partial differential equations arising from the shape optimization problem, work has been carried out using a variety of methods, for instance, Finite Element Method (FEM) [29], Cut Finite Element Methods (CutFEM) [8,9], Boundary Element Method (BEM) [6, 42], level-set methods [1, 3], among others. For this manuscript we seek to make a first approach for a development of HDG methods on curved domains for shape optimization problems while maintaining the high order of convergence of these methods in this new context. In turn, proving the existence and uniqueness of these methods and giving a rigorous analysis for the error estimates.

The remainder of this work is organized as follows. In Chapter 2 we give an introduction to shape optimization through our model problem, we will also give some definitions and relevant results to the development of shape optimization problems. In Chapter 3 we present the continuous mixed formulations of the state, adjoint and deformation field equations, which come from the shape optimization problem. Then, in Chapter 4 we construct the computational domain, set the notation associated to the mesh, define the transferring segments and we give some relevant assumptions for the development of the work. In Chapter 5 we introduce the HDG method on curved domains for approximate the solution of the state and adjoint equations with Dirichlet data and for the deformation field equation with Neumann data. We also prove that this schemes are well-posed. Then, in Chapter 6 we carry out the *a priori* error analysis of each scheme. Finally, Chapter 7 concludes by discussing the main contributions of this thesis and possible future directions.

CHAPTER 2

Shape optimization

This chapter presents the main tools used in shape optimization, where, through the model problem, the definitions and results of the shape calculation are developed until the optimality conditions are found. In order to make this manuscript self-contained in Sections 2.2, 2.3, and 2.4 we introduce some concepts of shape differential calculus that can be found in [41].

2.1 The model problem

In this thesis we will consider the following shape optimization model problem:

$$\min_{\Omega\in\mathcal{O}} J(\Omega; y(\Omega)) = \min_{\Omega\in\mathcal{O}} \frac{1}{2}\int_{\Omega}(y(\Omega) - \widetilde{y})^2 \tag{2.1.1}$$

subject to

$$-\nabla\cdot(a\,\nabla y(\Omega)) = f \quad \text{in } \Omega\,, \tag{2.1.2a}$$

$$y(\Omega) \;=\; g \quad \text{on } \Gamma := \partial\Omega\,, \tag{2.1.2b}$$

where:

a) $\mathcal{O}$ is set of admissible subsets Ω of $\mathbb{R}^d$ ($d = 2$ or $d = 3$). i.e., all the domains where the PDE is valid.

b) $\Omega \subset \mathcal{U} \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$), where $\mathcal{U}$ is the called *universe* domain, which we assume to be closed in $\mathbb{R}^d$ and is the domain where all the data of the model problem are defined, for the purpose of the well posedness of the problem.

c) $\widetilde{y} \in H^1(\mathcal{U})$ is given,

d) a is a given positive number,

e) $f \in H^1(\mathcal{U})$ is given,

f) $g \in H^2(\mathcal{U})$ is given,

We seek to determine the domain Ω that minimizes the shape functional J subject to (2.1.2). Thus, we will look for the optimal domain Ω^{opt} contained in $\mathcal{O}$, in other words,

$$\Omega^{opt} \;=\; \arg\min_{\Omega\in\mathcal{O}} J(\Omega)$$

subject to

$$\begin{aligned} -\nabla\cdot(a\,\nabla y(\Omega)) \;&=\; f \quad \text{in } \Omega\,,\\ y(\Omega) \;&=\; g \quad \text{on } \Gamma\,. \end{aligned}$$

Remark. We can note that the structure of our shape optimization problem is similar to the structure of a control problem, for example [57]. If we focus on the functional, J which in our case depends on $\widetilde{y}$ and $y(\Omega)$, but in the case of control problems the functional J depends on y and another unknown. Therefore, we can say that shape optimization problems can be seen as control problems but, instead of having two unknowns governed by PDEs, we have two unknowns where one depends on a PDE and the other one is the domain which also depends on a PDE.

2.2 Introduction to shape differential calculus

In elementary differential calculus, the derivative is an operator that measures the sensitivity of a function with respect to changes of an independent variable, but, what happens if now our independent variable is the domain of the function?. We must consider a new concept of derivative, in the context of changes in the domain, more precisely, an operator will be defined that will measure the sensitivity of a function (later it will be extended to a functional) with respect to changes of the domain. To define a formal shape derivative, first we have to define the concept of a perturbation of the domain.

Definition 2.2.1. Given a bounded domain $\Omega \subset \mathcal{U}$, for any $\varepsilon > 0$ the *deformation map* is defined by

$$\boldsymbol{\Phi}_\varepsilon(\boldsymbol{x}) := \boldsymbol{x} + \varepsilon\, \boldsymbol{V}(\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in \Omega\,, \tag{2.2.1}$$

where $\boldsymbol{V} : \mathcal{U} \to \mathbb{R}^d$ (with d the dimension of $\mathcal{U}$), is a vector field over $\mathcal{U}$ hence, $\varepsilon\,\boldsymbol{V}$ is the displacement of each $\boldsymbol{x}$ in $\mathcal{U}$. Then, with the deformation map, we then define a deformed domain Ω_ε as

$$\Omega_\varepsilon := \boldsymbol{\Phi}_\varepsilon(\Omega) = \{\boldsymbol{\Phi}_\varepsilon(\boldsymbol{x}) : \quad \boldsymbol{x} \in \Omega\}\,.$$

It is easy to notice that $\Omega_0 = \Omega$.

Definition 2.2.2. If $y(\Omega)$ is a function that depends on the domain, we define the *material derivative* of y in the direction $\boldsymbol{V}$ as the following limit,

$$\dot{y}(\Omega; \boldsymbol{V})(\boldsymbol{x}) := \lim_{\varepsilon \to 0} \frac{y(\Omega_\varepsilon)(\boldsymbol{\Phi}_\varepsilon(\boldsymbol{x})) - y(\Omega)(\boldsymbol{x})}{\varepsilon}\,,$$

if it exists, then we have the following identity

$$\dot{y}(\Omega;\boldsymbol{V})(\boldsymbol{x}) \;=\; \frac{\partial}{\partial\varepsilon} y(\Omega_\varepsilon)(\boldsymbol{\Phi}_\varepsilon(\boldsymbol{x}))\Big|_{\varepsilon=0}.$$

The operator defined above obeys the common rules of elementary calculus as chain rule, product rule, etc, [41, 55]. Note that if y does not depend explicitly on the geometry of the domain, we can see y as the restriction of a function defined over the set which contains all the possible domains, that is, a function $\hat{y}$ in $\mathcal{U}$, such that $y(\Omega) = \hat{y}(\mathcal{U})\Big|_\Omega$. Then, if we compute the material derivative of y in the direction $\boldsymbol{V}$ we get

$$\dot{y}(\Omega;\boldsymbol{V}) \;=\; \nabla y(\Omega)\cdot\boldsymbol{V}\,.$$

On the other hand, if a function y depends explicitly on the geometry of the domain we introduce the following definition

Definition 2.2.3. The *shape derivative* of y in the direction $\boldsymbol{V}$ is defined as

$$y'(\Omega;\boldsymbol{V}) \;=\; \dot{y}(\Omega;\boldsymbol{V}) \,-\, \nabla y(\Omega)\cdot\boldsymbol{V} \quad \text{in } \Omega\,. \tag{2.2.2}$$

Remark. It is important to notice that the material derivative does not commute with the standard time and space derivatives but the shape derivative does [55].

Remark. It is important to note that the definitions 2.2.3 and 2.2.2 are the same if instead of Ω we compute the derivatives in a curve, for example the boundary Γ, but is not necessarily the same as restricting derivatives over Ω to Γ.

The next tool that we need to introduce is the concept of shape perturbation of a shape functional, which is defined as follows

Definition 2.2.4. Let $J : \mathcal{O} \to \mathbb{R}$ be a shape functional, $\Omega \in \mathcal{O}$. We define the *shape perturbation* of J at Ω along the direction $\boldsymbol{V}$ as the following limit,

$$\delta\, J(\Omega;\boldsymbol{V}) \;:=\; \lim_{\varepsilon\to 0} \frac{J(\Omega_\varepsilon) \;-\; J(\Omega)}{\varepsilon}\,,$$

if it exists. Then, we have the following identity

$$\delta\, J(\Omega; \boldsymbol{V}) \;=\; \frac{d}{d\varepsilon}\, J(\Omega_\varepsilon)\Big|_{\varepsilon=0}.$$

To provide a more explicit computation of the shape perturbation of a particular functional we present the following characterization

Lemma 2.2.1. *Let $\Omega \subset \mathcal{U}$ be a bounded Lipschitz admissible domain and $\boldsymbol{V}$ a vector field. Let $g = g(\Omega)$ be a function that depends on the domain. Let us assume that the material and shape derivative of y both exist and belong to $L^1(\Omega)$. For the shape functional*

$$J(\Omega) \;=\; \int_\Omega g(\Omega)\,,$$

we have that its shape perturbation is given by

$$\delta\, J(\Omega; \boldsymbol{V}) \;=\; \int_\Omega g'(\Omega; \boldsymbol{V}) + \int_\Gamma g(\Omega)\, \boldsymbol{V} \cdot \boldsymbol{n}\,,$$

where $\boldsymbol{n}$ is the unit outward normal vector of Γ.

Proof. By the definition of shape perturbation of $J(\Omega)$ in the direction $\boldsymbol{V}$ we have

$$\delta\, J(\Omega; \boldsymbol{V}) \;=\; \frac{d}{d\varepsilon} \int_{\Omega_\varepsilon} g(\Omega_\varepsilon)\Big|_{\varepsilon=0}.$$

Using the change of variables theorem and the deformation map (2.2.1)

$$\begin{aligned}
\delta\, J(\Omega; \boldsymbol{V}) \;&=\; \frac{d}{d\varepsilon} \int_\Omega \det(\nabla \boldsymbol{\Phi}_\varepsilon)\, g(\Omega_\varepsilon) \circ \boldsymbol{\Phi}_\varepsilon \Big|_{\varepsilon=0} \\
&= \int_\Omega \frac{d}{d\varepsilon} \left(\det(\nabla \boldsymbol{\Phi}_\varepsilon)\, g(\Omega_\varepsilon) \circ \boldsymbol{\Phi}_\varepsilon\right)\Big|_{\varepsilon=0} \\
&= \int_\Omega \det(\nabla \boldsymbol{\Phi}_\varepsilon)\, \frac{d}{d\varepsilon} \left(g(\Omega_\varepsilon) \circ \boldsymbol{\Phi}_\varepsilon\right)\Big|_{\varepsilon=0} \;+\; \int_\Omega \frac{d}{d\varepsilon}(\det(\nabla \boldsymbol{\Phi}_\varepsilon))\, g(\Omega_\varepsilon) \circ \boldsymbol{\Phi}_\varepsilon\Big|_{\varepsilon=0} \\
&= \int_\Omega \det(\nabla \boldsymbol{\Phi}_\varepsilon)\, \frac{d}{d\varepsilon} \left(g(\Omega_\varepsilon) \circ \boldsymbol{\Phi}_\varepsilon\right)\Big|_{\varepsilon=0} \;+\; \int_\Omega \nabla \cdot \boldsymbol{V}\, g(\Omega_\varepsilon) \circ \boldsymbol{\Phi}_\varepsilon\Big|_{\varepsilon=0} \\
&= \int_\Omega \left(\dot{g}(\Omega; \boldsymbol{V}) \;+\; g(\Omega)\, \nabla \cdot \boldsymbol{V}\right)\,,
\end{aligned}$$

whose the proof of $\frac{d}{d\varepsilon}(\det(\nabla \boldsymbol{\Phi}_\varepsilon)) = \nabla \cdot \boldsymbol{V}$ can be found in [41, Section 11.2.1]. Then, by (2.2.2) and using the divergence theorem, finally we obtain

$$\begin{aligned} \delta J(\Omega; \boldsymbol{V}) &= \int_\Omega \left(g'(\Omega; \boldsymbol{V}) + \nabla g(\Omega) \cdot \boldsymbol{V} + g(\Omega) \nabla \cdot \boldsymbol{V}\right) \\ &= \int_\Omega g'(\Omega; \boldsymbol{V}) + \int_\Omega \nabla \cdot (g(\Omega) \boldsymbol{V}) \\ &= \int_\Omega g'(\Omega; \boldsymbol{V}) + \int_\Gamma g(\Omega)\, \boldsymbol{V} \cdot \boldsymbol{n} \,. \end{aligned}$$

□

We present now a definition that will be followed by a lemma,

Definition 2.2.5. Let κ_1 and κ_2 be the principal curvatures of a surface, then the *total curvature* of the surface is given by

$$\kappa := \kappa_1 + \kappa_2 \,.$$

For more details, see for instance [41, Section 11.2].

Remark. It should be emphasized that κ is not necessarily constant on the surface.

Lemma 2.2.2. *Let $z(\Gamma)$ be so that the material derivative $\dot{z}(\Gamma; \boldsymbol{V})$ and the shape derivative $z'(\Gamma; \boldsymbol{V})$ both exist and belong to $L^1(\Gamma)$. Then we have*

$$\delta J(\Gamma; \boldsymbol{V}) = \int_\Gamma z'(\Gamma; \boldsymbol{V}) + \int_\Gamma \kappa\, z(\Gamma)\, \boldsymbol{V} \cdot \boldsymbol{n} \,. \tag{2.2.3}$$

If $z(\Gamma) = g(\Omega)|_\Gamma$, then

$$\delta J(\Gamma; \boldsymbol{V}) = \int_\Gamma g'(\Omega; \boldsymbol{V})|_\Gamma + \int_\Gamma \left(\frac{\partial g(\Omega)}{\partial \boldsymbol{n}} + \kappa\, g(\Omega) \right) \boldsymbol{V} \cdot \boldsymbol{n} \,, \tag{2.2.4}$$

where κ denotes the total curvature of Γ.

Proof. See [41, Proposition 11.9]. □

2.3 Introduction to shape optimization

In optimal control problems, the goal is to minimize/maximize a functional which depends on a function that is constrained by a PDE. In the context of shape optimization, the shape functional also depends on the domain. In other words we want to determine the domain $\Omega \in \mathcal{O}$ that minimizes/maximize the shape functional. Consider the model problem given in (2.1.1) subject to (2.1.2) with $\widetilde{y}$, $f \in L^2(\mathcal{U})$ and $g \in H^2(\mathcal{U})$. Let us consider $y = y(\Omega)$ to make notation easier. Thus, by Lemma 2.2.1 and the chain rule for the shape derivative, we can compute the shape perturbation of J in the direction $\boldsymbol{V}$ as

$$\begin{aligned} \delta\, J(\Omega; \boldsymbol{V}) &= \frac{1}{2} \int_\Omega \left((y-\widetilde{y})^2\right)' + \frac{1}{2} \int_\Gamma (y-\widetilde{y})^2 \, \boldsymbol{V} \cdot \boldsymbol{n} \\ &= \int_\Omega (y-\widetilde{y})\, y'(\Omega; \boldsymbol{V}) + \frac{1}{2} \int_\Gamma (y-\widetilde{y})^2 \, \boldsymbol{V} \cdot \boldsymbol{n}\,. \end{aligned} \tag{2.3.1}$$

Then, we notice that if A is an open set compactly supported in Ω, by testing (2.1.2) with $\varphi \in H_0^1(A)$, and integrating by parts,

$$\int_A a\, \nabla y \cdot \nabla \varphi = \int_A f \varphi \quad \forall\, \varphi \in H_0^1(A)\,.$$

We emphasize that neither the test function φ nor the datum f depend on the domain Ω. Then, computing the shape derivative on both sides of previous expressions

$$\int_A a\, \nabla y' \cdot \nabla \varphi = 0 \quad \forall\, \varphi \in H_0^1(A)\,.$$

On the other hand, if we test the boundary condition of (2.1.2) by $\phi \in H_0^1(\mathcal{U})$ we get

$$\int_\Gamma (y - g)\, \phi = 0\,.$$

Then, computing the shape derivative of both sides of this expression and using (2.2.4) and recalling the boundary condition (2.1.2b) yields

$$0 = \int_\Gamma y'\, \phi + \int_\Gamma \left(\kappa\, (y - g)\, \phi + \frac{\partial((y - g)\, \phi)}{\partial \boldsymbol{n}} \right) \boldsymbol{V} \cdot \boldsymbol{n} = \int_\Gamma y'\, \phi + \int_\Gamma \frac{\partial (y - g)}{\partial \boldsymbol{n}}\, \phi\, \boldsymbol{V} \cdot \boldsymbol{n}\,.$$

With all of the above identities, we deduce that the shape derivative y' satisfies the following equation

$$\begin{cases} -\nabla\cdot(a\,\nabla y') &=& 0 & \text{in } \Omega\,, \\ y' &=& -\dfrac{\partial(y-g)}{\partial \boldsymbol{n}}\,\boldsymbol{V}\cdot\boldsymbol{n} & \text{on } \Gamma\,. \end{cases} \tag{2.3.2}$$

Therefore, we can obtain δJ in (2.3.1) by solving (2.1.2) and (2.3.2), and look for the optimal domain Ω_{opt} that satisfies

$$\delta\, J(\Omega_{opt};\boldsymbol{V}) \,=\, 0\,.$$

for some admissible direction $\boldsymbol{V}$ (which will be specified later in section 2.4).

Like in optimal control problems, in the context of shape optimization we can also employ an adjoint problem in order to obtain a representation of $\delta\, J$ that does not involve y'. In fact, let us consider z satisfying $-\nabla\cdot(a\,\nabla z) \,=\, y-\tilde{y}$ in a distributional sense in Ω. For any direction $\boldsymbol{V}$, according to (2.3.1)

$$\begin{aligned} \delta\, J(\Omega;\boldsymbol{V}) &= \int_\Omega (y-\tilde{y})\,y'(\Omega;\boldsymbol{V}) + \frac{1}{2}\int_\Gamma (y-\tilde{y})^2\,\boldsymbol{V}\cdot\boldsymbol{n} \\ &= -\int_\Omega \nabla\cdot(a\,\nabla z)\,y'(\Omega;\boldsymbol{V}) + \frac{1}{2}\int_\Gamma (y-\tilde{y})^2\,\boldsymbol{V}\cdot\boldsymbol{n} \\ &= \int_\Omega a\,\nabla z\cdot\nabla y' - \int_\Gamma a\,\partial_{\boldsymbol{n}} z\, y' + \frac{1}{2}\int_\Gamma (y-\tilde{y})^2\,\boldsymbol{V}\cdot\boldsymbol{n} \\ &= -\int_\Omega z\,\nabla\cdot(a\,\nabla y') + \int_\Gamma a\,\partial_{\boldsymbol{n}} y'\, z - \int_\Gamma a\,\partial_{\boldsymbol{n}} z\, y' + \frac{1}{2}\int_\Gamma (y-\tilde{y})^2\,\boldsymbol{V}\cdot\boldsymbol{n}\,. \end{aligned}$$

By (2.3.2) the first term vanishes. This suggests the Dirichlet condition $z \,=\, 0$ on Γ for the adjoint problem, in order to have that

$$\begin{aligned} \delta\, J(\Omega;\boldsymbol{V}) &= -\int_\Gamma a\,\partial_{\boldsymbol{n}} z\, y' + \frac{1}{2}\int_\Gamma (y-\tilde{y})^2\,\boldsymbol{V}\cdot\boldsymbol{n} \\ &= \int_\Gamma a\,\partial_{\boldsymbol{n}} z\,\partial_{\boldsymbol{n}}(y-g)\,\boldsymbol{V}\cdot\boldsymbol{n} + \frac{1}{2}\int_\Gamma (y-\tilde{y})^2\,\boldsymbol{V}\cdot\boldsymbol{n}\,, \end{aligned} \tag{2.3.3}$$

Therefore, in (2.3.3) we obtain an expression for the shape perturbation that does not depend on the shape derivative y', but rather on y and on $z \in H^2(\Omega)$, the solution of the adjoint problem,

$$\begin{cases} -\nabla\cdot(a\,\nabla z) &=& y-\tilde{y} & \text{in } \Omega\,, \\ z &=& 0 & \text{on } \Gamma\,. \end{cases} \tag{2.3.4}$$

Finally, we can deduce that the optimality conditions for (2.1.1) are the state equation (2.1.2), the adjoint equation (2.3.4) and the condition

$$\delta\, J(\Omega;\boldsymbol{V}) \;=\; \int_\Gamma G(\Gamma)\,\boldsymbol{V}\cdot\boldsymbol{n} \;=\; 0\,, \tag{2.3.5}$$

for the deformation field $\boldsymbol{V}$, where

$$G(\Gamma) \;=\; a\,\partial_{\boldsymbol{n}} z\,\partial_{\boldsymbol{n}}(y-g) + \frac{1}{2}\,(g-\tilde{y})^2\,.$$

From now on $G(\Gamma)$ will be called the shape gradient of J.

2.4 An approximation for $\boldsymbol{V}$

Now the question is, how to find a deformation field $\boldsymbol{V}$ satisfying (2.3.5). It is known that shape optimization problems generally do not have a unique solution due that there is no manner to ensure that exists a unique $\boldsymbol{V}$. In fact, non-uniqueness is one of the most common problems that mathematicians have to deal with in this context [36]. Since the deformation field $\boldsymbol{V}$ might not be unique, we look for the one that satisfies

$$\begin{cases} -\Delta\boldsymbol{V} &=& 0 & \text{in } \Omega, \\ \partial_{\boldsymbol{n}}\boldsymbol{V} &=& -G(\Gamma)\,\boldsymbol{n} & \text{on } \Gamma_N, \\ \boldsymbol{V} &=& \boldsymbol{0} & \text{on } \Gamma_D, \end{cases} \tag{2.4.1}$$

where Γ_N is the piece of the boundary that can be deformed and Γ_D is the piece of the boundary that is fixed. We assume that $|\Gamma_N| \neq 0$ and $|\Gamma_D| \neq 0$. On the other hand, the weak formulation for (2.4.1) is given by seeking $\boldsymbol{V}\in[H^1_D(\Omega)]^d$ such that

$$\int_\Omega \nabla\boldsymbol{V} : \nabla\boldsymbol{w} + \int_{\Gamma_N} G(\Gamma)\,\boldsymbol{w}\cdot\boldsymbol{n} \;=\; 0 \quad \forall\,\boldsymbol{w}\in[H^1_D(\Omega)]^d\,,$$

where,

$$[H_D^1(\Omega)]^d := \{\boldsymbol{w} \in [H^1(\Omega)]^d, \quad \boldsymbol{w} = 0 \text{ on } \Gamma_D\}.$$

Here for any tensor fields $\boldsymbol{\phi} = (\phi_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, the tensor inner product is defined as

$$\boldsymbol{\phi} : \boldsymbol{\zeta} = \sum_{i,j=1} \phi_{ij}\, \zeta_{ij}.$$

Thus, by letting $\boldsymbol{w} = \boldsymbol{V}$, it follows from the weak formulation that

$$\delta\, J(\Omega; \boldsymbol{V}) = \int_{\Gamma_N} G(\Gamma)\, \boldsymbol{V} \cdot \boldsymbol{n} = -\int_{\Omega} |\nabla \boldsymbol{V}|^2 < 0. \tag{2.4.2}$$

Thus, this implies that $\boldsymbol{V}$ chosen in this way guarantees a descent direction.

The choice of problem (2.4.1) is not arbitrary. In fact, we can notice that as (2.4.1) satisfies the maximum principle (chapter 2, theorem 4 [31]), the absolute value of $\boldsymbol{V}$ on Γ is grater than the absolute value of $\boldsymbol{V}$ in any point of Ω. Thus, the deformation will be grater in the boundary of the domain. This argument makes finding a $\boldsymbol{V}$ that satisfies (2.4.1) has sense.

Based on the previous analysis we propose a technique for computing an approximation for $\boldsymbol{V}$ based on the gradient descent method: Using what we learned from (2.4.1) and (2.4.2) we can define $\boldsymbol{V}^{(k)}$ as the solution of the following problem

$$\begin{cases} -\Delta \boldsymbol{V}^{(k)} = 0 & \text{in } \Omega^{(k)}, \\ \partial_{\boldsymbol{n}} \boldsymbol{V}^{(k)} = -G(\Gamma^{(k)})\, \boldsymbol{n} & \text{on } \Gamma_N^{(k)}, \\ \boldsymbol{V}^{(k)} = \boldsymbol{0} & \text{on } \Gamma_D^{(k)}. \end{cases} \tag{2.4.3}$$

Then for each step we will update the domain as

$$\Omega^{(k+1)} = (I + \tau_k \boldsymbol{V}^{(k)})(\Omega^{(k)}) = \boldsymbol{\Phi}_{\tau_k}(\Omega^{(k)}) \quad \forall\, k \in \mathbb{N},$$

where τ_k is a step size parameter to be determined and $\boldsymbol{V}^{(k)}$ is the descent direction at step k. Analogously to (2.4.2) we can prove that $\boldsymbol{V}^{(k)}$ is a descent direction for each $k \in \mathbb{N}$, and therefore

$$J(\Omega^{(k+1)}) < J(\Omega^{(k)}) \quad \forall k \in \mathbb{N}.$$

At this point, we are able to present an algorithm (Algorithm 11.1 [41]) to solve the shape optimization problem, as follows

Algorithm 1 Shape Optimization Algorithm

Require: Initial domain $\Omega^{(0)}$ and tolerance parameter tol

$y(0) \leftarrow y(\Omega^{(0)})$ by solving the state equation (2.1.2) in $\Omega^{(0)}$
$z(0) \leftarrow z(\Omega^{(0)})$ by solving the adjoint equation (2.3.4) in $\Omega^{(0)}$
compute $J(\Omega^{(0)})$
compute $G(\Gamma^{(0)})$
compute a deformation field $\boldsymbol{V}^{(0)}$ by solving (2.4.3)
$k \leftarrow 0$
while $|\delta J(\Omega^{(k)}; \boldsymbol{V}^{(k)})/\delta J(\Omega^{(0)}; \boldsymbol{V}^{(0)})| > \text{tol}$ **or** $|J(\Omega^{(k)}) - J(\Omega^{(k-1)})| > \text{tol}$ **do**
  compute the step size parameter τ_k with a line search routine
  $\Omega^{(k+1)} \leftarrow (I + \tau_k \boldsymbol{V}^k)(\Omega^{(k)})$
  $y(k+1) \leftarrow y(\Omega^{(k+1)})$ by solving the state equation (2.1.2) in $\Omega^{(k+1)}$
  $z(k+1) \leftarrow z(\Omega^{(k+1)})$ by solving the adjoint equation (2.3.4) in $\Omega^{(k+1)}$
  compute $J(\Omega^{(k+1)})$
  compute $G(\Gamma^{(k+1)})$
  compute a deformation field $\boldsymbol{V}^{(k+1)}$ by solving (2.4.3)
  $k \leftarrow k+1$
end while

CHAPTER 3

Mixed formulations

In this chapter we will develop the mixed formulation for the state equation (2.1.2), adjoint equation (2.3.4) and the velocity field equation (2.4.1), in a feasible arbitrary domain $\Omega \in \mathcal{O}$. First we will set some notation that will be used from now on. Given a region $D \subset \mathbb{R}^d$, we denote by $(\cdot,\cdot)_D$ and $\langle\cdot,\cdot\rangle_{\partial D}$ the $L^2(D)$ and $L^2(\partial D)$ inner products respectively. The subindex will be dropped whenever the integration domain is clear from the context.

3.1 Mixed formulation for the state equation and the adjoint equation

Recalling the state equation (2.1.2) we have that

$$\begin{cases} -\nabla\cdot(a\,\nabla y) &= f \quad \text{in } \Omega, \\ y &= g \quad \text{on } \Gamma. \end{cases}$$

As we want to introduce a mixed variational formulation, we define the additional unknown $\boldsymbol{p} := -a\nabla y$. Thus we have the following problem

$$\begin{cases} c\,\boldsymbol{p} + \nabla y &= 0 \quad \text{in } \Omega\,, \\ \nabla\cdot\boldsymbol{p} &= f \quad \text{in } \Omega\,, \\ y &= g \quad \text{on } \Gamma\,, \end{cases} \tag{3.1.1}$$

where $c = a^{-1}$. Hence, to obtain the weak formulation, let us multiply by $\boldsymbol{v}_1 \in H(\text{div};\Omega)$ the first equation of (3.1.1) and integrate by parts:

$$\int_\Omega c\,\boldsymbol{p}\cdot\boldsymbol{v}_1 - \int_\Omega y\,\nabla\cdot\boldsymbol{v}_1 + \langle \boldsymbol{v}_1\cdot\boldsymbol{n}, y\rangle_\Gamma = 0\,,$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality between $H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$. On the other hand, multiplying by $w_1 \in H^1(\Omega)$ the second equation of (3.1.1) and integrating by parts, we obtain

$$-\int_\Omega \boldsymbol{p}\cdot\nabla w_1 + \langle \boldsymbol{p}\cdot\boldsymbol{n}, w_1\rangle_\Gamma = \int_\Omega f\,w_1\,.$$

Thus, the weak formulation for the state equation (2.1.2) reads as follows: Find $(\boldsymbol{p}, y) \in H(\text{div};\Omega)\times H^1(\Omega)$, such that

$$(c\,\boldsymbol{p}, \boldsymbol{v}_1)_\Omega - (y, \nabla\cdot\boldsymbol{v}_1)_\Omega = -\langle \boldsymbol{v}_1\cdot\boldsymbol{n}, g\rangle_\Gamma\,, \tag{3.1.2a}$$

$$-(\boldsymbol{p}, \nabla w_1)_\Omega + \langle \boldsymbol{p}\cdot\boldsymbol{n}, w_1\rangle_\Gamma = (f, w)_\Omega\,, \tag{3.1.2b}$$

for all $\boldsymbol{v}_1, w_1 \in H(\text{div};\Omega)\times H^1(\Omega)$.

Now recalling the adjoint equation (2.3.4) we have that

$$\begin{cases} -\nabla\cdot(a\,\nabla z) &= y - \widetilde{y} \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \Gamma. \end{cases}$$

Analogously as in the case of the state equation, we introduce an additional unknown $\boldsymbol{r} :=$

$-\, a\, \nabla z$, thus

$$\left\{ \begin{aligned} c\,\boldsymbol{r} \,+\, \nabla z \;&=\; 0 && \text{in } \Omega\,, \\ \nabla\cdot\boldsymbol{r} \;&=\; y \,-\, \widetilde{y} && \text{in } \Omega\,, \\ z \;&=\; 0 && \text{on } \Gamma\,. \end{aligned} \right. \tag{3.1.3}$$

The first equation of (3.1.3) yields

$$\int_\Omega c\,\boldsymbol{r}\cdot\boldsymbol{v}_2 \,+\, \int_\Omega \nabla z\cdot\boldsymbol{v}_2 \;=\; \int_\Omega c\,\boldsymbol{r}\cdot\boldsymbol{v}_2 \,-\, \int_\Omega z\,\nabla\cdot\boldsymbol{v}_2 \,+\, \langle \boldsymbol{v}_2\cdot\boldsymbol{n}, z\rangle_\Gamma \;=\; 0 \quad \forall\, \boldsymbol{v}_2 \in H(\mathrm{div};\Omega)\,.$$

And for the second equation of (3.1.3) we have that

$$\int_\Omega \nabla\cdot\boldsymbol{r}\, w_2 \;=\; -\int_\Omega \boldsymbol{r}\cdot\nabla w_2 \,+\, \langle \boldsymbol{p}\cdot\boldsymbol{n}, w_2\rangle_\Gamma \;=\; \int_\Omega (y-\widetilde{y})\, w_2 \quad \forall\, w_2 \in H^1(\Omega)\,.$$

Therefore, the weak formulation for the adjoint equation reads: Find $(\boldsymbol{r}, u) \in H(\mathrm{div};\Omega) \times H^1(\Omega)$, such that

$$(c\,\boldsymbol{r}, \boldsymbol{v}_2)_\Omega - (z, \nabla\cdot\boldsymbol{v}_2)_\Omega = 0\,, \tag{3.1.4a}$$

$$-\,(\boldsymbol{r}, \nabla w_2)_\Omega \,+\, \langle \boldsymbol{r}\cdot\boldsymbol{n}, w_2\rangle_\Gamma = (y \,-\, \widetilde{y}, w_2)_\Omega\,, \tag{3.1.4b}$$

for all $(\boldsymbol{v}_2, w_2) \in H(\mathrm{div};\Omega) \times H^1(\Omega)$.

Remark. For the well-posedness analysis of (3.1.2) and (3.1.4), we refer to [33, Section 2.4.1].

3.2 Mixed formulation for the deformation field equation

First of all we need to introduce the following notation: For any tensor fields $\boldsymbol{\phi} = (\phi_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\phi})$ be the divergence operator div acting along the rows of $\boldsymbol{\phi}$.

Recalling the velocity field equation (2.4.1)

$$\begin{cases} -\Delta \boldsymbol{V} &= 0 & \text{in } \Omega\,, \\ \partial_{\boldsymbol{n}} \boldsymbol{V} &= -G(\Gamma)\,\boldsymbol{n} & \text{on } \Gamma_N\,, \\ \boldsymbol{V} &= \boldsymbol{0} & \text{on } \Gamma_D\,, \end{cases}$$

To find a mixed formulation we will introduce a new unknown $\boldsymbol{\sigma} := -\nabla \boldsymbol{V}$, then we have the following mixed formulation of (2.4.1)

$$\begin{cases} \boldsymbol{\sigma} + \nabla \boldsymbol{V} &= 0 & \text{in } \Omega\,, \\ \mathbf{div}(\boldsymbol{\sigma}) &= \boldsymbol{0} & \text{in } \Omega\,, \\ \boldsymbol{\sigma}\,\boldsymbol{n} &= G(\Gamma)\,\boldsymbol{n} & \text{on } \Gamma_N\,, \\ \boldsymbol{V} &= \boldsymbol{0} & \text{on } \Gamma_D\,. \end{cases} \tag{3.2.1}$$

Testing the first equation with $\boldsymbol{\phi} \in [H(\mathbf{div};\Omega)]^{d\times d}$ we obtain

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\phi} + \int_\Omega \nabla \boldsymbol{V} : \boldsymbol{\phi} = 0$$

and by Green's identities we have that

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\phi} - \int_\Omega \boldsymbol{V} \cdot \mathbf{div}(\boldsymbol{\phi}) + \langle \boldsymbol{\phi}\,\boldsymbol{n}, \boldsymbol{V} \rangle_{\Gamma_N} = 0\,,$$

where $\langle \cdot, \cdot \rangle_{\Gamma_N}$ is the duality of $H_{00}^{-1/2}(\Gamma_N)$ and $H_{00}^{1/2}(\Gamma_N)$ (for more details on the definitions of spaces and traces, see for instance [33, Section 2.4.2]). On the other hand, testing the second equation by $\boldsymbol{w} \in [H_0^1(\Omega)]^d$, it follows that

$$\int_\Omega \mathbf{div}(\boldsymbol{\sigma}) \cdot \boldsymbol{w} = 0\,.$$

Thus, by Green's identity

$$-\int_\Omega \boldsymbol{\sigma} : \nabla \boldsymbol{w} + \langle \boldsymbol{\sigma}\boldsymbol{n}, \boldsymbol{w} \rangle_\Gamma = -\int_\Omega \boldsymbol{\sigma} : \nabla \boldsymbol{w} + \langle G(\Gamma)\boldsymbol{n}, \boldsymbol{w} \rangle_{\Gamma_N}\,.$$

Finally, we have that the weak mixed formulation for (2.4.1) is to: Find $(\boldsymbol{\sigma}, \boldsymbol{V}) \in [H(\mathbf{div};\Omega)]^{d\times d} \times [H_0^1(\Omega)]^d$, such that

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\phi} - \int_\Omega \boldsymbol{V}\,\mathbf{div}(\boldsymbol{\phi}) + \langle \boldsymbol{\phi}\,\boldsymbol{n}, \boldsymbol{V}\rangle_{\Gamma_N} &= 0, \\ - \int_\Omega \boldsymbol{\sigma} : \nabla \boldsymbol{w} &= \langle G(\Gamma)\boldsymbol{n}, \boldsymbol{w}\rangle_{\Gamma_N}, \end{aligned}$$

for all $(\boldsymbol{\phi}, \boldsymbol{w}) \in [H(\mathbf{div};\Omega)]^{d\times d} \times [H^1(\Omega)]^d$. The solvability analysis of (3.1.2) and (3.1.4) can be found in [33, Section 2.4.2].

CHAPTER 4

Mesh construction and notation

In this chapter we present the construction of the mesh where the HDG methods will be posed, and the notation that will be needed later for the analysis of the methods.

4.1 Computational domain

For the construction of the computational domain D_h, we will follow the construction done in [23]. For this, we begin by choosing a background polyhedral domain $\mathcal{M} \supset \Omega$. Then, let $\{T_h\}_{h>0}$ be a sequence of triangulations of $\mathcal{M}$ (the triangulations can be composed by triangles or tetrahedrons). We define $\mathcal{T}_h$ as the set of all the elements $K \in T_h$, which are totally included in Ω and also we define $D_h := \left(\bigcup_{K\in\mathcal{T}_h} \overline{K}\right)^{\circ}$ and $\Gamma_h := \partial D_h$. We assume by simplicity that the triangulation does not have hanging nodes. Moreover, we will assume that the family of triangulation are uniformly shape regular, this is, $\exists\, \rho > 0$ such that, for any $K \in \mathcal{T}_h$ and for all $h_K \in (0, 1]$, we have that $\text{diam}(B_K) \geq \rho\, \text{diam}(K)$, where B_K is the biggest ball inscribed in the element K and $h_K = \text{diam}(K)$. From now on we denote h as the maximum of $h_K \in \mathcal{T}_h$.

We can see the Figure 4.1 as an example of the construction of D_h.

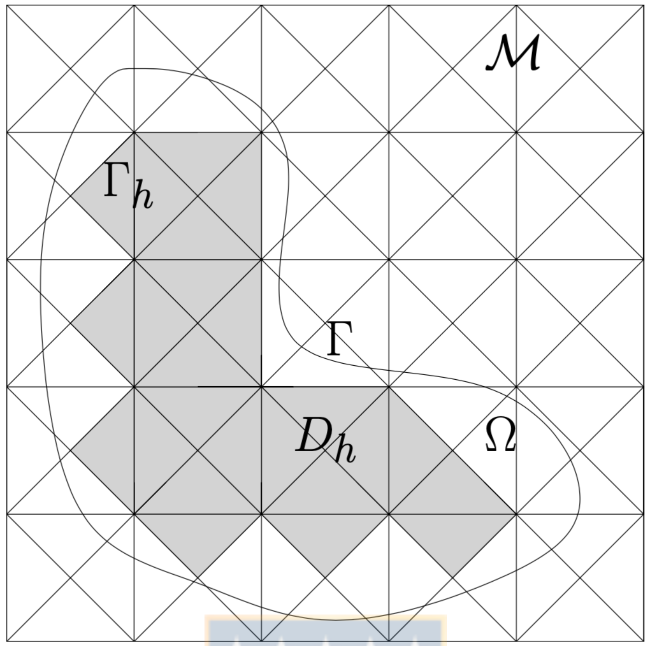


Figure 4.1: Example of a domain Ω, its boundary Γ, a background domain $\mathcal{M}$ and the construction of the polygonal subdomain D_h (gray) [12, Figure 2.1].

On the other hand, we will say that e is an interior edge or face if there are two elements K^+ and K^- in $\mathcal{T}_h$, such that, $e = \partial K^+ \cap \partial K^-$. In the same way, say that e is a boundary edge or face if there is an element $K \in \mathcal{T}_h$ such that $e = \partial K \cap \Gamma_h$.

Let $\mathcal{E}_h$ be the set of edges or faces of the elements of $\mathcal{T}_h$, $\mathcal{E}_h^\circ$ the set of interior edges or faces of $\mathcal{T}_h$ and $\mathcal{E}_h^\partial$ the set of exterior edges or faces of $\mathcal{T}_h$. Thus $\mathcal{E}_h = \mathcal{E}_h^\circ \cup \mathcal{E}_h^\partial$.

We denote by $\boldsymbol{n}$ the outward unit normal of the element $K \in \mathcal{T}_h$. Whenever we want to emphasize that $\boldsymbol{n}$ is the normal to the face e of K, we denote $\boldsymbol{n}_e$. Moreover, for each edge or face e of K, we denote $h_e^\perp$ as the height of the element respect to that edge or face.

For later chapters we need to introduce the notions of averages and interface jumps of functions defined in D_h.

Definition 4.1.1. Let $v : \Omega \to \mathbb{R}$ be a scalar-valued function, such that, $\forall K \in \mathcal{T}_h$, the restriction $v|_K$ of v to the open set K can be defined up to the boundary ∂K. Then, for all

$e \in \mathcal{E}_h^\circ$, the average of v is defined as

$$\{\!\{v\}\!\}_e := \lim_{h\to 0^+} \frac{1}{2} \left(v(\boldsymbol{x} + h\,\boldsymbol{n}) + v(\boldsymbol{x} - h\,\boldsymbol{n})\right) \qquad \text{for } \boldsymbol{x} \in e\,,$$

and the jump of v as

$$[\![v]\!]_e := \lim_{h\to 0^+} \left(v(\boldsymbol{x} + h\,\boldsymbol{n}) - v(\boldsymbol{x} - h\,\boldsymbol{n})\right) \qquad \text{for } \boldsymbol{x} \in e\,.$$

Whenever no confusion can arises, the subscript e can be omitted, and we simply write $\{\!\{v\}\!\}$ and $[\![v]\!]$.

Then, we have that the following lemma, whose proof can be found in [28, Lemma 1.24].

Lemma 4.1.1. *(Characterization of $H(\text{div};\Omega)$). A function $\boldsymbol{\xi} \in H(\text{div};\mathcal{T}_h) \cap [H^1(\mathcal{T}_h)]^d$, where*

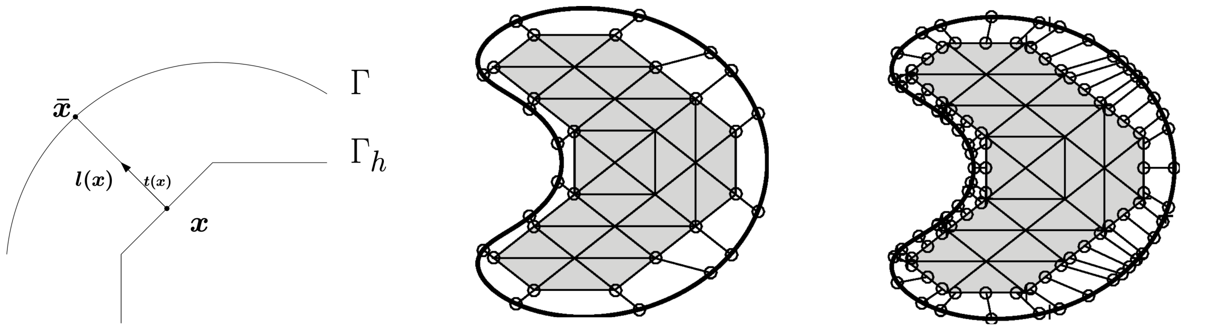
$$H(\text{div};\mathcal{T}_h) := \{\boldsymbol{\xi} \in [L^2(\Omega)]^d \,|\, \forall\, K \in \mathcal{T}_h,\ \boldsymbol{\xi}|_K \in H(\text{div}, K)\}\,,$$

belongs to $H(\text{div};\Omega)$ if and only if

$$[\![\boldsymbol{\xi} \cdot \boldsymbol{n}]\!]_e = \mathbf{0} \quad \forall\, e \in \mathcal{E}_h^\circ\,.$$

4.2 Transfer paths

Following the method developed in [23], we need to describe how to transfer the boundary data from Γ_h to Γ. For this purpose, given a point $\boldsymbol{x} \in \Gamma_h$, we need to choose a specific $\bar{\boldsymbol{x}} \in \Gamma$ to transfer the data from $\boldsymbol{x}$ to $\bar{\boldsymbol{x}}$. Hereinafter the segment joining $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ will be called "transfer path associated to $\boldsymbol{x}$". We will denote by $l(\boldsymbol{x})$ and $\boldsymbol{t}(\boldsymbol{x})$ the length and unit tangent vector, respectively, of the transfer path associated to $\boldsymbol{x}$. See Figure 4.2a for a graphical example of this. This transfer path must satisfy three conditions; $\bar{\boldsymbol{x}}$ and $\boldsymbol{x}$ should be as close as possible, two or more transfer paths must not intersect each other and a transfer path must not intersect the interior of the computational domain D_h. An algorithm for constructing a collection of transfer paths for the two dimensional case was developed in [23]. The three dimensional case can be treated following the sames ideas as in the two dimensional case. Figures 4.2a and 4.2c show examples of transfer paths from the computational domain to a non polygonal domain.



(a) Transfer path associated to $\boldsymbol{x}$. (b) Transfer paths associated to the boundary vertices. (c) Transfer paths associated to the boundary quadrature point.

Figure 4.2: Figure from [12, Figure 2.2].

4.3 Extrapolation regions

Now, we define the complement of the computational domain D_h as $D_h^c := \Omega \backslash \overline{D_h}$. Also, for each $e \in \mathcal{E}_h^\partial$, we denote by K^e as the only element of $\mathcal{T}_h$ that has e as a face. We define

$$\widetilde{K}_{ext}^e := \{\boldsymbol{x} + s\,\boldsymbol{t}(x) \,:\, 0 \leq s \leq l(\boldsymbol{x}),\, \boldsymbol{x} \in e\}\,.$$

We can see an illustration in the Figure 4.3.

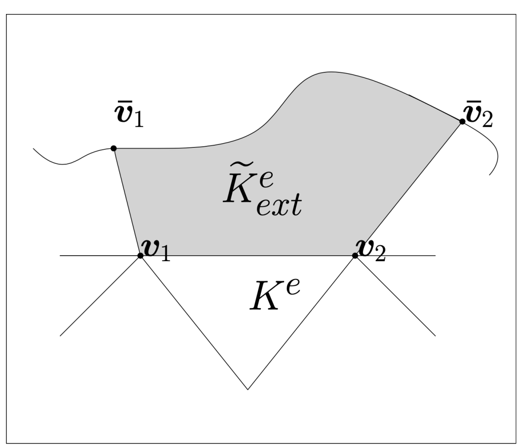


Figure 4.3: Example of $\widetilde{K}_{ext}^e$ [12, Figure 2.3].

4.4 Polynomial spaces and norms

Now, we will define the global polynomial spaces where the HDG method seeks the solution. Let $\mathbb{P}_k(K)$ be the set of polynomials of degree at most k over the element K. Let us define the following spaces

$$\boldsymbol{Z}_h := \left\{ \boldsymbol{v} \in [L^2(\mathcal{T}_h)]^d \,:\, \boldsymbol{v}|_K \in [\mathbb{P}_k(K)]^d \quad \forall\, K \in \mathcal{T}_h \right\}, \tag{4.4.1a}$$

$$W_h := \left\{ w \in L^2(\mathcal{T}_h) \,:\, w|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \right\}, \tag{4.4.1b}$$

$$M_h := \left\{ \mu \in L^2(\mathcal{E}_h) \,:\, \mu|_e \in \mathbb{P}_k(e) \quad \forall\, e \in \mathcal{E}_h \right\}. \tag{4.4.1c}$$

On the other hand, we define the inner products associated to $\mathcal{T}_h$ and $\partial\mathcal{T}_h$ as follows:

$$(u,v)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_K u\, v \quad \text{and} \quad \langle s,t\rangle_{\partial\mathcal{T}_h} := \int_{\partial\mathcal{T}_h} s\, t\,,$$

and the respective induced norms $\|u\|_{\mathcal{T}_h} := (u,u)^{1/2}_{\mathcal{T}_h}$ and $\|s\|_{\partial\mathcal{T}_h} := \langle s,s\rangle^{1/2}_{\partial\mathcal{T}_h}$. For $a > 0$ on $\partial\mathcal{T}_h$ and on Γ_h we define the weighted norms

$$\|v\|_{\partial\mathcal{T}_h,a} := \left(\sum_{K\in\partial\mathcal{T}_h} \langle a\, v, v\rangle_{\partial K} \right)^{1/2} \quad \text{and} \quad \|v\|_{\Gamma,a} := \left(\sum_{e\subset\Gamma_h} \langle a\, v, v\rangle_e \right)^{1/2}.$$

4.5 Parameters and auxiliary constants

We define $H_e^\perp$ as the length of the longest segment connecting e and Γ parallel to the normal direction $\boldsymbol{n}_e$ and completely contained in K_{ext}^e. Recalling the definition of $h_e^\perp$, then we set

$$r_e := \frac{H_e^\perp}{h_e^\perp}\,, \tag{4.5.1}$$

as Figure 4.4 illustrates $H_e^\perp$ and $h_e^\perp$.

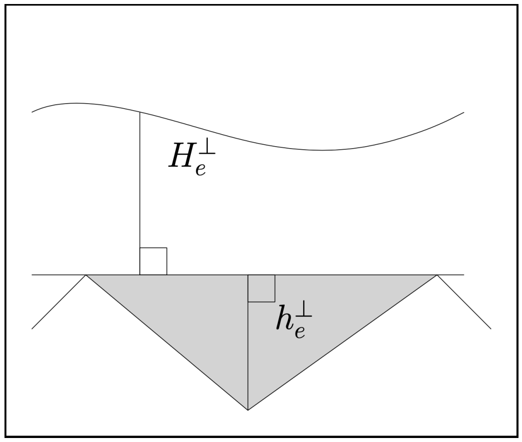


Figure 4.4: Representation of $H_e^\perp$ and $h_e^\perp$ [12, Figure 4.1].

Now, for the analysis in the next chapter, we need to introduce some constants that will be useful, which are:

$$C^e_{ext} := \frac{1}{\sqrt{r_e}} \sup_{\boldsymbol{\zeta}\in[\mathbb{P}_k(K^e)]^d\cdot\boldsymbol{n}_e\setminus\{\mathbf{0}\}} \frac{\|\boldsymbol{E}_h\boldsymbol{\zeta}\|_{K^e_{ext}}}{\|\boldsymbol{\zeta}\|_{K^e}}, \tag{4.5.2a}$$

$$C^e_{inv} := h_e^\perp \sup_{\boldsymbol{\zeta}\in[\mathbb{P}_k(K^e)]^d\cdot\boldsymbol{n}_e\setminus\{\mathbf{0}\}} \frac{\|\partial_{\boldsymbol{n}_e}\boldsymbol{\zeta}\|_{K^e}}{\|\boldsymbol{\zeta}\|_{K^e}}. \tag{4.5.2b}$$

Where, the extrapolation of p from K^e to $\widetilde{K}^e_{ext}$, is denoted by $E_h(p)$, and is defined by $E_h(p)(y) := p|_{K^e}(y), \forall\, y \in \widetilde{K}^e_{ext}$. To simplify notation, from now on we will just write $p(y)$ instead of $E_h(p)(y)$ for $y \in \widetilde{K}^e_{ext}$. The same notation will be used for tensor- and vector-valued polynomial functions defined on K^e.

4.6 Assumptions and previous results

We are in a position to establish our next set of assumptions, which will be used from now on for the analysis in the following chapters. For every face $e \in \mathcal{E}^\partial_h$,

$$\begin{aligned} \max_{\boldsymbol{x}\in\Gamma_h}(1+c)\,\tau\, l(\boldsymbol{x}) &\leq 1/4, & r_e\,(C^e_{ext})^{2/3}\,(C^e_{inv})^{2/3} &\leq 1/2, \\ r_e\,\tau|_e\, h_e^\perp &\leq 1/4, & r_e \leq C \quad C &> 0. \end{aligned} \tag{4.6.1}$$

Since $l(\boldsymbol{x}) \,\leq\, H_e^{\perp} \quad \forall\, \boldsymbol{x} \in e\,,\, e \in \mathcal{E}_h^{\partial}$, the first equation of the first row of the above assumptions implies that $H_e^{\perp} \quad \forall\, e \in \mathcal{E}_h^{\partial}$ has to be controlled by 1/4, the rest of the assumptions suggest the closeness of the curved domain to the computational mesh. For more details see the discussion of [23, Assumptions S].

Above $\tau \,>\, 0$ is a stabilization parameter of the HDG method. On the other hand, for the boundary Γ we will assume that is Lipschitz and there exists $\widetilde{\Gamma} \subset \Gamma$ closed in Γ, such that $|\widetilde{\Gamma}| \,=\, 0$ and $\Gamma \setminus \widetilde{\Gamma}$ is $\mathcal{C}^2$.

Now, we need to state an auxiliary definition that will be used in the subsequent analysis and provide some results proved in [23, Lemma 5.2].

Definition 4.6.1. For any $e \in \mathcal{E}_h^{\partial}$, any point $\boldsymbol{x}$ lying on the face e and any smooth enough function $\boldsymbol{v}$ defined in K_{ext}^e, we set

$$\Lambda^{\boldsymbol{v}}(\boldsymbol{x}) \,:=\, \frac{1}{l(\boldsymbol{x})} \int_0^{l(\boldsymbol{x})} (\boldsymbol{v}(\boldsymbol{x} \,+\, s\, \boldsymbol{n}) \,-\, \boldsymbol{v}(\boldsymbol{x})) \cdot \boldsymbol{n}\, ds\,. \tag{4.6.2}$$

Lemma 4.6.1. *For each $e \in \mathcal{E}_h^{\partial}$, we have that*

$$\|\Lambda^{\boldsymbol{v}}\|_{e,l} \,\leq\, \frac{1}{\sqrt{3}}\, r_e\, \|\partial_{\boldsymbol{n}}(\boldsymbol{v} \cdot \boldsymbol{n})\|_{K_{ext}^e,(h^{\perp})^2}\,.$$

Moreover, if $\boldsymbol{p} \in [\mathbb{P}(K^e)]^d$, we have that

$$\|\Lambda^{\boldsymbol{p}}\|_{\Gamma_h,l} \,\leq\, \frac{1}{\sqrt{3}}\, r_e^{3/2}\, C_{ext}^e\, C_{inv}^e\, \|\boldsymbol{p}\|_{K^e}\,.$$

We define a set of auxiliary constants to be used in subsequent analysis

Definition 4.6.2. We have the following constants

$$\begin{gathered} R \,:=\, \max_{e\in\mathcal{E}_h^{\partial}} r_e\,, \quad R_C \,:=\, \max_{e\in\mathcal{E}_h^{\partial}} r_e\, (C_{ext}^e)^{1/2}\, (C_{inv}^e)^{1/2}\,, \quad R_C' \,:=\, \max_{e\in\mathcal{E}_h^{\partial}} r_e\, (C_{ext}^e\, C_{inv}^e)^2\,, \\ \widetilde{R}_C' \,:=\, \max_{e\in\mathcal{E}_h^{\partial}} \widetilde{r}_e\, (\widetilde{C}_{ext}^e)^2\,, \quad \widehat{C}_{ext} \,:=\, \max_{e\in\mathcal{E}_h^{\partial}} \widetilde{C}_{ext}^e\,, \quad R_\tau \,:=\, \max_{e\in\mathcal{E}_h^{\partial}} r_e\, \tau_e^{1/2}\,, \\ \widetilde{R}_{\tau^{-1}} \,:=\, \max_{e\in\mathcal{E}_h^{\partial}} r_e\, \tau_e^{-1}\,. \end{gathered} \tag{4.6.3a}$$

where,

$$\widetilde{r}_e := \frac{|\widetilde{K}^e_{ext}|}{|K^e|}, \qquad \widetilde{C}^e_{ext} := \frac{1}{\sqrt{\widetilde{r}_e}} \sup_{\boldsymbol{\zeta}\in[\mathbb{P}_k(K^e)]^d\setminus\{\mathbf{0}\}} \frac{\|\boldsymbol{\zeta}\|_{\widetilde{K}^e_{ext}}}{\|\boldsymbol{\zeta}\|_{K^e}}.$$

Finally we state some mesh assumptions. Let C_R, $C_{\boldsymbol{n}}$, δ, β be nonnegative constants independent of h and $\boldsymbol{n}_h$ the outward unit normal of Γ_h, such that

$$R \leq C_R\, h^{\delta}, \tag{4.6.4a}$$

$$\|\boldsymbol{n}_h - \boldsymbol{n}\circ\boldsymbol{\phi}\|_{\infty} \leq C_{\boldsymbol{n}}\, h^{\beta}. \tag{4.6.4b}$$

Where (4.6.4a) gives us the information on how close the computational domain D_h has to be respect of the boundary Γ, i.e. if Ω is polygonal there would be no distance between D_h and Γ, then $C_R = 0$. If Γ_h interpolates Γ, then $C_R > 0$ and $\delta = 1$ and if D_h is completely contained in Ω, i.e. if $\Gamma_h \cap \Gamma = \emptyset$, then $C_R > 0$ and $\delta = 0$. In turn, (4.6.4b) represents the convergence rate of the outward normal vector of Γ_h to Γ. If Ω is polygonal, D_h and Ω are the same set, and then $\boldsymbol{n}_h = \boldsymbol{n}$, then $C_{\boldsymbol{n}} = 0$. If Γ_h interpolates Γ, then $C_{\boldsymbol{n}} > 0$ and $\beta = 1$ and if D_h is completely contained in Ω, then $C_{\boldsymbol{n}} > 0$ and $\beta = 0$.

Finally, to avoid proliferation of constants, we will write $a \lesssim b$ instead of $a \leq C\,b$, where C is a constant independent of h.

CHAPTER 5

The hybridizable discontinuous Galerkin method

In this chapter we present the hybridizable discontinuous Galerkin (HDG) scheme for the state and adjoint equations, the schemes use the technique developed in [17, 23], which allows us to use hybridizable discontinuous Galerkin methods considering solely polyhedral elements to numerically solve problems in domains which are not necessarily polyhedral. In a similar way, within the context of curved domains and inspired by [46], we will furthermore present a scheme for the deformation field equation.

5.1 HDG schemes for state and adjoint equations

Inspired by mixed formulations (3.1.1) and (3.1.3) developed in Chapter 3, and following [23], we have the new mixed formulations for the state and adjoint equations, now posed in the polygonal domain D_h:

$$c\,\boldsymbol{p} + \nabla y = 0 \quad \text{in } D_h\,, \tag{5.1.1a}$$

$$\nabla\cdot\boldsymbol{p} = f \quad \text{in } D_h\,, \tag{5.1.1b}$$

$$y = \varphi_1 \quad \text{on } \Gamma_h\,, \tag{5.1.1c}$$

and

$$c\,\boldsymbol{r} + \nabla z = 0 \quad \text{in } D_h\,, \tag{5.1.2a}$$

$$\nabla\cdot\boldsymbol{r} + y = \widetilde{y} \quad \text{in } D_h\,, \tag{5.1.2b}$$

$$z = \varphi_2 \quad \text{on } \Gamma_h\,, \tag{5.1.2c}$$

where the unknowns φ_1 and φ_2 correspond to the traces of y and z, respectively, on Γ_h. Let $\boldsymbol{x} \in \Gamma_h$ and $\bar{\boldsymbol{x}}$ its corresponding point on Γ. By integrating (5.1.1a) and (5.1.2a) along the transferring segment joining $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$, we obtain

$$\varphi_1(\boldsymbol{x}) := g(\bar{\boldsymbol{x}}) + \int_0^{l(\boldsymbol{x})} c\,\boldsymbol{p}(\boldsymbol{x} + s\,\boldsymbol{t})\cdot\boldsymbol{t}\,ds\,, \tag{5.1.3a}$$

$$\varphi_2(\boldsymbol{x}) := \int_0^{l(\boldsymbol{x})} c\,\boldsymbol{r}(\boldsymbol{x} + s\,\boldsymbol{t})\cdot\boldsymbol{t}\,ds\,. \tag{5.1.3b}$$

The HDG method for the state equation seeks an approximation $(\boldsymbol{p}_h, y_h, \widehat{y}_h)$ of the exact solution $(\boldsymbol{p}, y, y|_{\mathcal{E}_h})$ in the space $\boldsymbol{Z}_h \times W_h \times M_h$ such that

$$(c\,\boldsymbol{p}_h, \boldsymbol{v}_1)_{\mathcal{T}_h} - (y_h, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} + \langle \widehat{y}_h, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0\,, \tag{5.1.4a}$$

$$-(\boldsymbol{p}_h, \nabla w_1)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} = (f, w_1)_{\mathcal{T}_h}\,, \tag{5.1.4b}$$

$$\langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} = 0\,, \tag{5.1.4c}$$

$$\langle \widehat{y}_h, \mu_1\rangle_{\Gamma_h} = \langle \varphi_1^h, \mu_1\rangle_{\Gamma_h}\,, \tag{5.1.4d}$$

for all $(\boldsymbol{v}_1, w_1, \mu_1) \in \boldsymbol{Z}_h \times W_h \times M_h$. In turn, the HDG method for the adjoint equation seeks and approximation $(\boldsymbol{r}_h, z_h, \widehat{z}_h)$ of the exact solution $(\boldsymbol{r}, z, z|_{\mathcal{E}_h})$ in the space $\boldsymbol{Z}_h \times W_h \times M_h$ such that

$$
\begin{aligned}
(c\,\boldsymbol{r}_h, \boldsymbol{v}_2)_{\mathcal{T}_h} \,-\, (z_h, \nabla\cdot\boldsymbol{v}_2)_{\mathcal{T}_h} \,+\, \langle \widehat{z}_h, \boldsymbol{v}_2\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,&=\, 0\,, && (5.1.5a)\\
-\,(\boldsymbol{r}_h, \nabla w_2)_{\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{r}}_h\cdot\boldsymbol{n}, w_2\rangle_{\partial\mathcal{T}_h} \,&=\, (\widetilde{y} - y_h, w_2)_{\mathcal{T}_h}\,, && (5.1.5b)\\
\langle \widehat{\boldsymbol{r}}_h\cdot\boldsymbol{n}, \mu_2\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} \,&=\, 0\,, && (5.1.5c)\\
\langle \widehat{z}_h, \mu_2\rangle_{\Gamma_h} \,&=\, \langle \varphi_2^h, \mu_2\rangle_{\Gamma_h}\,, && (5.1.5d)
\end{aligned}
$$

for all $(\boldsymbol{v}_2, w_2, \mu_2) \in \boldsymbol{Z}_h \times W_h \times M_h$. Moreover, φ_1^h and φ_2^h are the Dirichlet boundary conditions on Γ_h which are defined at all $\boldsymbol{x}$ lying on the face $\mathcal{E}_h^\partial$, by

$$
\begin{aligned}
\varphi_1^h(\boldsymbol{x}) \,&:=\, g(\bar{\boldsymbol{x}}) \,+\, \int_0^{l(\boldsymbol{x})} c\,\boldsymbol{E}_h(\boldsymbol{p}_h)(\boldsymbol{x} \,+\, s\,\boldsymbol{t}(\boldsymbol{x}))\cdot\boldsymbol{t}(\boldsymbol{x})\,ds\,, && (5.1.6a)\\
\varphi_2^h(\boldsymbol{x}) \,&:=\, \int_0^{l(\boldsymbol{x})} c\,\boldsymbol{E}_h(\boldsymbol{r}_h)(\boldsymbol{x} \,+\, s\,\boldsymbol{t}(\boldsymbol{x}))\cdot\boldsymbol{t}(\boldsymbol{x})\,ds\,, && (5.1.6b)
\end{aligned}
$$

where $\boldsymbol{E}_h(\boldsymbol{p}_h)$ and $\boldsymbol{E}_h(\boldsymbol{r}_h)$ are the extrapolation of $\boldsymbol{p}_h$ and $\boldsymbol{r}_h$ respectively. The numerical fluxes $\widehat{\boldsymbol{p}}_h$ and $\widehat{\boldsymbol{r}}_h$ on $\partial\mathcal{T}_h := \{\partial K : \quad K \in \mathcal{T}_h\}$ are defined by

$$
\begin{aligned}
\widehat{\boldsymbol{p}}_h \,&=\, \boldsymbol{p}_h \,+\, \tau\,(y_h \,-\, \widehat{y}_h)\,\boldsymbol{n}\,, && (5.1.7a)\\
\widehat{\boldsymbol{r}}_h \,&=\, \boldsymbol{r}_h \,+\, \tau(z_h \,-\, \widehat{z}_h)\,\boldsymbol{n}\,, && (5.1.7b)
\end{aligned}
$$

with τ being a positive stabilization function, whose maximum value will be denoted by $\bar{\tau}$.

Under the assumptions mentioned for Γ in Chapter 4, we have the following results proven in [23, Lemma 3.1].

Lemma 5.1.1. *We have the following properties:*

(i) $l(\boldsymbol{x}) : \Gamma_h \to \mathbb{R}$ *is measurable.*

(ii) K_{ext} *and* $\widetilde{K}^e_{ext}$ *are measurable sets.*

(iii) $\forall\, g \in H^{1/2}(\Gamma)$, *we have that* $\varphi_1^h : \Gamma_h \to \mathbb{R}$, *is a measurable function.*

(iv) $\varphi_2^h : \Gamma_h \to \mathbb{R}$ *is also a measurable function.*

5.2 Existence and uniqueness of the state and adjoint equations schemes

We will proceed first to prove existence and uniqueness of the HDG scheme (5.1.4) and then of (5.1.5). In the forthcoming analysis, for simplicity we will assume $\boldsymbol{t}(\boldsymbol{x}) = \boldsymbol{n}$, $\forall \boldsymbol{x} \in e$ and $e \in \mathcal{E}_h^\partial$. If this is not true, terms involving $\boldsymbol{t}(\boldsymbol{x}) \cdot \boldsymbol{n}$ would appear in the estimates and the results that we will present still hold true assuming $1 - \boldsymbol{t}(\boldsymbol{x}) \cdot \boldsymbol{n}$ is small enough.

Theorem 5.2.1. *Under the assumptions* (4.6.1) *and if* $\tau > 0$*, there exists a unique solution of the HDG scheme associated to the state equation* (5.1.4)*.*

Proof. Since the linear scheme is finite dimensional, the scheme (5.1.4) will have unique solution if and only if $(\boldsymbol{p}_h, y_h, \widehat{y}_h) = (\boldsymbol{0}, 0, 0)$ when $f = g = 0$. Therefore, suppose that $f = g = 0$, we get

$$
\begin{aligned}
(c\,\boldsymbol{p}_h, \boldsymbol{v}_1)_{\mathcal{T}_h} - (y_h, \nabla \cdot \boldsymbol{v}_1)_{\mathcal{T}_h} + \langle \widehat{y}_h, \boldsymbol{v}_1 \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0\,, \\
-(\boldsymbol{p}_h, \nabla w_1)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n}, w_1 \rangle_{\partial \mathcal{T}_h} &= 0\,.
\end{aligned}
$$

Now, testing with $\boldsymbol{v}_1 = \boldsymbol{p}_h$ and $w_1 = y_h$, we have that that

$$
\begin{aligned}
\left\| c^{1/2}\,\boldsymbol{p}_h \right\|_{\mathcal{T}_h}^2 - (y_h, \nabla \cdot \boldsymbol{p}_h)_{\mathcal{T}_h} + \langle \widehat{y}_h, \boldsymbol{p}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0\,, \\
-(\boldsymbol{p}_h, \nabla y_h)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n}, y_h \rangle_{\partial \mathcal{T}_h} &= 0\,.
\end{aligned}
$$

Then, integrating by parts the second equation

$$
\begin{aligned}
\left\| c^{1/2}\,\boldsymbol{p}_h \right\|_{\mathcal{T}_h}^2 - (y_h, \nabla \cdot \boldsymbol{p}_h)_{\mathcal{T}_h} + \langle \widehat{y}_h, \boldsymbol{p}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0\,, \\
(\nabla \cdot \boldsymbol{p}_h, y_h)_{\mathcal{T}_h} - \langle \boldsymbol{p}_h \cdot \boldsymbol{n}, y_h \rangle_{\partial \mathcal{T}_h} + \langle \widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n}, y_h \rangle_{\partial \mathcal{T}_h} &= 0\,.
\end{aligned}
$$

Adding both equations we obtain

$$
\left\| c^{1/2}\,\boldsymbol{p}_h \right\|_{\mathcal{T}_h}^2 + \langle \widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n} - \boldsymbol{p}_h \cdot \boldsymbol{n}, y_h \rangle_{\partial \mathcal{T}_h} + \langle \widehat{y}_h, \boldsymbol{p}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = 0\,.
$$

Then, by (5.1.7a) we deduce that

$$\left\|c^{1/2}\,\boldsymbol{p}_h\right\|^2_{\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n} - \boldsymbol{p}_h\cdot\boldsymbol{n}, \widehat{y}_h\rangle_{\partial\mathcal{T}_h} \,+\, \langle \tau(y_h-\widehat{y}_h), y_h-\widehat{y}_h\rangle_{\partial\mathcal{T}_h} \,+\, \langle \widehat{y}_h, \boldsymbol{p}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,=\, 0\,.$$

That is,

$$\left\|c^{1/2}\,\boldsymbol{p}_h\right\|^2_{\mathcal{T}_h} \,+\, \left\|\tau^{1/2}(y_h-\widehat{y}_h)\right\|^2_{\partial\mathcal{T}_h} \,+\, \langle \widehat{y}_h, \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,=\, 0\,, \tag{5.2.6}$$

By (5.1.4c) and (5.1.4d), it follows that

$$\langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \widehat{y}_h\rangle_{\partial\mathcal{T}_h} \,=\, \langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \widehat{y}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} \,+\, \langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \widehat{y}_h\rangle_{\Gamma_h} \,=\, \langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \widehat{y}_h\rangle_{\Gamma_h} \,=\, \langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \varphi_1^h\rangle_{\Gamma_h}\,.$$

Moreover, recalling the definition (5.1.6a) let us notice that

$$\begin{aligned}
\varphi_1^h(\boldsymbol{x}) \,&=\, \int_0^{l(\boldsymbol{x})} c\,\boldsymbol{E}_h(\boldsymbol{p}_h)(\boldsymbol{x}\,+\,s\,\boldsymbol{n}_e)\cdot\boldsymbol{n}_e\,ds\\
&=\, \int_0^{l(\boldsymbol{x})} c\,\left[\boldsymbol{E}_h(\boldsymbol{p}_h)(\boldsymbol{x}\,+\,s\,\boldsymbol{n}_e) - \boldsymbol{p}_h(\boldsymbol{x})\right]\cdot\boldsymbol{n}_e\,ds \,+\, \int_0^{l(\boldsymbol{x})} c\,\boldsymbol{p}_h(\boldsymbol{x})\cdot\boldsymbol{n}_e\,ds\\
&=\, \int_0^{l(\boldsymbol{x})} c\,\left[\boldsymbol{E}_h(\boldsymbol{p}_h)(\boldsymbol{x}\,+\,s\,\boldsymbol{n}_e) - \boldsymbol{p}_h(\boldsymbol{x})\right]\cdot\boldsymbol{n}_e\,ds \,+\, c\,\boldsymbol{p}_h(\boldsymbol{x})\cdot\boldsymbol{n}_e\,l(\boldsymbol{x})\,.
\end{aligned}$$

Thus,

$$\boldsymbol{p}_h\cdot\boldsymbol{n}_e \,=\, c^{-1}\,l^{-1}(\boldsymbol{x})\,\varphi_1^h(\boldsymbol{x}) \,-\, \Lambda^{\boldsymbol{p}_h}(\boldsymbol{x})\,,$$

where, we recall (4.6.2)

$$\Lambda^{\boldsymbol{p}_h}(\boldsymbol{x}) \,:=\, l^{-1}(\boldsymbol{x})\,\int_0^{l(\boldsymbol{x})} \left[\boldsymbol{E}_h(\boldsymbol{p}_h)(\boldsymbol{x}\,+\,s\,\boldsymbol{n}_e) \,-\, \boldsymbol{p}_h(\boldsymbol{x})\right]\cdot\boldsymbol{n}_e\,ds \qquad \forall\,\boldsymbol{x}\in e\,,\quad e\in\mathcal{E}_h^{\partial}\,.$$

Then, by (5.1.7a) we have that

$$\begin{aligned}
\langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \varphi_1^h\rangle_{\Gamma_h} \,&=\, \langle \boldsymbol{p}_h\cdot\boldsymbol{n}, \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \tau\,(y_h-\widehat{y}_h), \varphi_1^h\rangle_{\Gamma_h}\\
&=\, \langle c^{-1}\,l^{-1}\,\varphi_1^h, \varphi_1^h\rangle_{\Gamma_h} \,-\, \langle \Lambda^{\boldsymbol{p}_h}, \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \tau\,(y_h-\widehat{y}_h), \varphi_1^h\rangle_{\Gamma_h}\\
&=\, \left\|c^{-1/2}\,l^{-1/2}\,\varphi_1^h\right\|^2_{\Gamma_h} \,-\, \langle c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}, c^{-1/2}\,l^{-1/2}\,\varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \tau\,(y_h-\widehat{y}_h), \varphi_1^h\rangle_{\Gamma_h}\,.
\end{aligned}$$

Hence, we deduce that (5.2.6) yields to

$$\begin{aligned}\left\|c^{1/2}\boldsymbol{p}_h\right\|^2_{\mathcal{T}_h} + \left\|\tau^{1/2}(y_h - \widehat{y}_h)\right\|^2_{\partial\mathcal{T}_h} + \left\|c^{-1/2}\,l^{-1/2}\,\varphi_1^h\right\|^2_{\Gamma_h} &= \langle c^{1/2}\,l^{1/2}\Lambda^{\boldsymbol{p}_h}, c^{-1/2}\,l^{-1/2}\,\varphi_1^h\rangle_{\Gamma_h} \\ &- \langle c^{1/2}\,l^{1/2}\,\tau\,(y_h - \widehat{y}_h), c^{-1/2}\,l^{-1/2}\,\varphi_1^h\rangle_{\Gamma_h}\,.\end{aligned} \tag{5.2.7}$$

Now, let be $\epsilon_1\,,\,\epsilon_2\,>\,0$, at our disposal. Using Young's inequality, we obtain

$$\begin{aligned}\langle c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}, c^{-1/2}\,l^{-1/2}\,\varphi_1^h\rangle_{\Gamma_h} \,-\, \langle c^{1/2}\,l^{1/2}\,\tau\,(y_h - \widehat{y}_h), c^{-1/2}\,l^{-1/2}\,\varphi_1^h\rangle_{\Gamma_h} \,\leq\, \frac{1}{2\epsilon_1}\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}\right\|^2_{\Gamma_h} \\ +\,\frac{\epsilon_1}{2}\left\|c^{-1/2}\,l^{-1/2}\,\varphi_1^h\right\|^2_{\Gamma_h} \,+\, \frac{1}{2\epsilon_2}\left\|c^{1/2}\,l^{1/2}\,\tau\,(y_h - \widehat{y}_h)\right\|^2_{\Gamma_h} \,+\, \frac{\epsilon_2}{2}\left\|c^{-1/2}\,l^{-1/2}\,\varphi_1^h\right\|^2_{\Gamma_h}\,.\end{aligned}$$

Hence, choosing $\epsilon_1 = \epsilon_2 = 1/4$,

$$\begin{aligned}&\langle c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}, c^{-1/2}\,l^{-1/2}\,\varphi_1^h\rangle_{\Gamma_h} \,-\, \langle c^{1/2}\,l^{1/2}\,\tau\,(y_h - \widehat{y}_h), c^{-1/2}\,l^{-1/2}\,\varphi_1^h\rangle_{\Gamma_h} \\ &\quad\leq\, 2\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}\right\|^2_{\Gamma_h} \,+\, \frac{1}{8}\left\|c^{-1/2}\,l^{-1/2}\,\varphi_1^h\right\|^2_{\Gamma_h} \,+\, 2\left\|c^{1/2}\,l^{1/2}\,\tau\,(y_h - \widehat{y}_h)\right\|^2_{\Gamma_h} \\ &\quad+\,\frac{1}{8}\left\|c^{-1/2}\,l^{-1/2}\,\varphi_1^h\right\|^2_{\Gamma_h} \\ &\quad\leq\, 2\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}\right\|^2_{\Gamma_h} \,+\, \frac{1}{4}\left\|c^{-1/2}\,l^{-1/2}\,\varphi_1^h\right\|^2_{\Gamma_h} \,+\, 2\left\|c^{1/2}\,l^{1/2}\,\tau\,(y_h - \widehat{y}_h)\right\|^2_{\Gamma_h}\,.\end{aligned}$$

Using this bound in (5.2.7), we obtain

$$\begin{aligned}&\left\|c^{1/2}\,\boldsymbol{p}_h\right\|^2_{\mathcal{T}_h} \,+\, \left\|\tau^{1/2}\,(y_h - \widehat{y}_h)\right\|^2_{\partial\mathcal{T}_h} \,+\, \frac{3}{4}\left\|c^{-1/2}\,l^{-1/2}\,\varphi_1^h\right\|^2_{\Gamma_h} \\ &\quad\leq\, 2\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}\right\|^2_{\Gamma_h} \,+\, 2\left\|c^{1/2}\,l^{1/2}\,\tau\,(y_h - \widehat{y}_h)\right\|^2_{\Gamma_h} \\ &\quad\leq\, 2\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}\right\|^2_{\Gamma_h} \,+\, 2\,c\,\tau\,\max_{\boldsymbol{x}\in\Gamma_h}\,l(\boldsymbol{x})\,\left\|\tau^{1/2}\,(y_h - \widehat{y}_h)\right\|^2_{\Gamma_h} \\ &\quad\leq\, 2\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}\right\|^2_{\Gamma_h} \,+\, \frac{1}{2}\left\|\tau^{1/2}\,(y_h - \widehat{y}_h)\right\|^2_{\Gamma_h} \\ &\quad\leq\, 2\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}\right\|^2_{\Gamma_h} \,+\, \frac{1}{2}\left\|\tau^{1/2}\,(y_h - \widehat{y}_h)\right\|^2_{\partial\mathcal{T}_h}\,.\end{aligned}$$

Thus, using the second equation of the first row of the assumptions (4.6.1)) it follows

$$\left\|c^{1/2}\,\boldsymbol{p}_h\right\|^2_{\mathcal{T}_h} \,+\, \frac{1}{2}\left\|\tau^{1/2}\,(y_h - \widehat{y}_h)\right\|^2_{\partial\mathcal{T}_h} \,+\, \frac{3}{4}\left\|c^{-1/2}\,l^{-1/2}\,\varphi_1^h\right\|^2_{\Gamma_h} \,\leq\, 2\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}\right\|^2_{\Gamma_h}$$

$$
\begin{aligned}
&\leq 2\,c \sum_{e\subset\Gamma_h} \left\| l^{1/2}\,\Lambda^{\boldsymbol{p}_h} \right\|_e^2 \leq 2\,c \sum_{e\subset\Gamma_h} \frac{1}{3}\, r_e^3\,(C_{ext}^e\, C_{inv}^e)^2\, \|\boldsymbol{p}_h\|_{K^e}^2 \\
&\leq \frac{2c}{3} \max_{e\subset\Gamma_h} r_e^3\,(C_{ext}^e\, C_{inv}^e)^2 \sum_{e\subset\Gamma_h} \|\boldsymbol{p}_h\|_{K^e}^2 \leq \frac{1}{12} \sum_{e\subset\Gamma_h} \left\| c^{1/2}\,\boldsymbol{p}_h \right\|_{K^e}^2 \leq \frac{1}{12} \left\| c^{1/2}\,\boldsymbol{p}_h \right\|_{\mathcal{T}_h}^2 ,
\end{aligned}
$$

then

$$
\frac{11}{12} \left\| c^{1/2}\,\boldsymbol{p}_h \right\|_{\mathcal{T}_h}^2 + \frac{1}{2} \left\| \tau^{1/2}\,(y_h - \widehat{y}_h) \right\|_{\partial\mathcal{T}_h}^2 + \frac{3}{4} \left\| c^{-1/2}\, l^{-1/2}\, \varphi_1^h \right\|_{\Gamma_h}^2 = 0\,.
$$

By this equation, we deduce that $\boldsymbol{p}_h = \mathbf{0}$, $y_h|_{\partial\mathcal{T}_h} = \widehat{y}_h|_{\partial\mathcal{T}_h}$, $\varphi_1^h = 0$, hence, by (5.1.4a) we get

$$
-\,(y_h, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} + \langle \widehat{y}_h, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0 \quad \forall\,\boldsymbol{v}_1 \in \boldsymbol{Z}_h\,,
$$

since $y_h|_{\partial\mathcal{T}_h} = \widehat{y}_h|_{\partial\mathcal{T}_h}$, integrating by parts it follows that

$$
0 = -\,(y_h, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} + \langle y_h, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = (\nabla y_h, \boldsymbol{v}_1)_{\mathcal{T}_h} \quad \forall\,\boldsymbol{v}_1 \in \boldsymbol{Z}_h\,,
$$

Then, setting $\boldsymbol{v}_1 = \nabla y_h$ we have that $\|\nabla y_h\|_{\mathcal{T}_h} = 0$. Thus y_h is a constant in D_h, therefore $y_h = 0$ by (5.1.4d). □

Finally we proved that the HDG scheme for the state equation has a unique solution.

Theorem 5.2.2. *Under the assumptions* (4.6.1) *and if* $\tau > 0$, *there exists a unique solution of the HDG scheme associated to the adjoint equation* (5.1.5).

The proof of the above theorem is analogous to the proof of the Theorem 5.2.1, the difference lies in setting the data $y_h - \widetilde{y} = 0$. Is important to note that y_h can be treated as data, because is the solution of the HDG scheme of the state equation, which does not depend on the adjoint problem.

5.3 HDG scheme for the deformation field equation

Now, we present the HDG scheme for the deformation field equation which is inspired by the work done in [46]. First of all, it should be noted that the technique cannot be the same as in the cases of the state and adjoint equations because we need to deal with a Neumann boundary condition. Then the technique of transferring data from the computational domain to the curved domain will not work in this case. Based on (3.2.1), in the computational domain D_h, this problem can be written as follows:

$$\boldsymbol{\sigma} + \nabla \boldsymbol{V} = 0 \quad \text{in } D_h\,, \tag{5.3.1a}$$

$$\mathbf{div}(\boldsymbol{\sigma}) = 0 \quad \text{in } D_h\,, \tag{5.3.1b}$$

$$\boldsymbol{\sigma}\boldsymbol{n}_h = \boldsymbol{g}_N \quad \text{on } \Gamma_h^N\,, \tag{5.3.1c}$$

$$\boldsymbol{V} = \boldsymbol{g}_D \quad \text{on } \Gamma_h^D\,. \tag{5.3.1d}$$

We assume that the computational boundary Γ_h satisfies $\Gamma_h = \Gamma_h^D \cup \Gamma_h^N$ and $\Gamma_h^D \cap \Gamma_h^N = \emptyset$, where Γ_h^D is the part of Γ_h with Dirichlet datum and Γ_h^N is the part of Γ_h with Neumann datum. On the other hand $\boldsymbol{g}_N$ is defined by

$$\boldsymbol{g}_N := (G(\Gamma)\,\boldsymbol{n}) \circ \boldsymbol{\phi}\,,$$

Hence, recalling the mixed variables $\nabla y = -\,c\,\boldsymbol{p}$, $\nabla z = -\,c\,\boldsymbol{r}$ introduced in the mixed formulations, we can rewrite $G(\Gamma)$ as

$$G(\Gamma) = \boldsymbol{r} \cdot \boldsymbol{n}(c\,\boldsymbol{p} \cdot \boldsymbol{n} + \partial_{\boldsymbol{n}} g) + \frac{1}{2}(g - \widetilde{y})^2\,.$$

In turn, for the definition of $\boldsymbol{\phi}$ we note first the following; let e be a boundary edge with vertices $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, we denote Γ_e as the part of Γ determined by $\bar{\boldsymbol{x}}_1$ and $\bar{\boldsymbol{x}}_2$ as we show in the Figure 5.1:

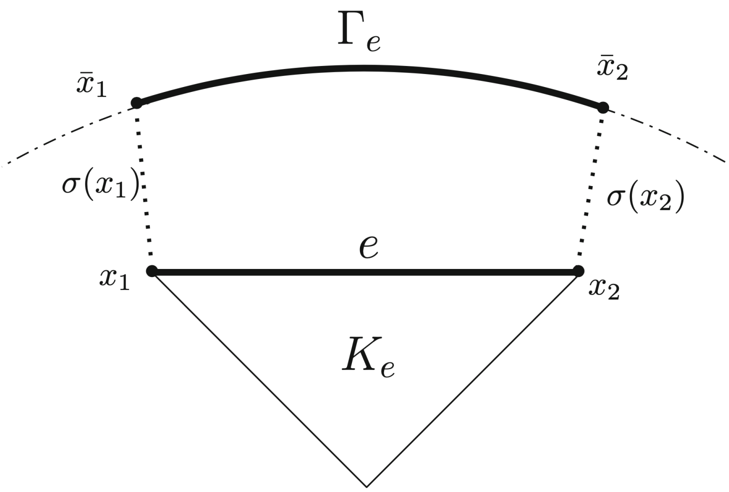


Figure 5.1: Representation of Γ_e.

Then, let $\boldsymbol{\phi} : e \to \Gamma_e$ be a bijection. In addition, we have that $\boldsymbol{g}_D$ is defined by the same technique used for the case of the equations of state and adjoint, that is,

$$\boldsymbol{g}_D(\boldsymbol{x}) := \int_0^{l(\boldsymbol{x})} \boldsymbol{\sigma}(\boldsymbol{x} + s\,\boldsymbol{n})\,\boldsymbol{n}\,ds\,.$$

Before to present the discrete scheme, we define the polynomials spaces, which are a generalization to higher dimensions of $\boldsymbol{Z}_h$, W_h, and M_h. Indeed, they are defined as follows

$$\begin{aligned}
\mathbb{Z}_h &:= \{\boldsymbol{\xi} \in [L^2(\mathcal{T}_h)]^{d\times d} \,:\, \boldsymbol{\xi}|_K \in [\mathbb{P}_k(K)]^{d\times d}, \quad \forall\, K \in \mathcal{T}_h\}\,,\\
\boldsymbol{W}_h &:= \{\boldsymbol{w} \in [L^2(\mathcal{T}_h)]^{d} \,:\, \boldsymbol{w}|_K \in [\mathbb{P}_k(K)]^{d}, \quad \forall\, K \in \mathcal{T}_h\}\,,\\
\boldsymbol{M}_h &:= \{\boldsymbol{\mu} \in [L^2(\mathcal{E}_h)]^{d} \,:\, \boldsymbol{\mu}|_e \in [\mathbb{P}_k(e)]^{d}, \quad \forall\, e \in \mathcal{E}_h\}\,.
\end{aligned}$$

Then, our first attempt of discrete scheme seeks an approximation $(\boldsymbol{\sigma}_h, \boldsymbol{V}_h, \widehat{\boldsymbol{V}}_h) \in \mathbb{Z}_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h$ of the exact solution $(\boldsymbol{\sigma}, \boldsymbol{V}, \boldsymbol{V}|_{\mathcal{E}_h})$, which is given by

$$(\boldsymbol{\sigma}_h, \boldsymbol{\psi})_{\mathcal{T}_h} - (\boldsymbol{V}_h, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{V}}_h, \boldsymbol{\psi}\,\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} = 0\,, \tag{5.3.3a}$$

$$-(\boldsymbol{\sigma}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{\sigma}}_h\,\boldsymbol{n}_h, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} = 0\,, \tag{5.3.3b}$$

$$\langle \widehat{\boldsymbol{\sigma}}_h\,\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} = 0\,, \tag{5.3.3c}$$

$$\langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^N} = \langle \boldsymbol{g}_N^h, \boldsymbol{\mu} \rangle_{\Gamma_h^N}, \tag{5.3.3d}$$

$$\langle \widehat{\boldsymbol{V}}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^D} = \langle \boldsymbol{g}_D^h, \boldsymbol{\mu} \rangle_{\Gamma_h^D}, \tag{5.3.3e}$$

for all $(\boldsymbol{\psi}, \boldsymbol{w}, \boldsymbol{\mu}) \in \mathbb{Z}_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h$, where

$$\widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}_h := \boldsymbol{\sigma}_h \boldsymbol{n}_h + \tau(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h);$$

And $\boldsymbol{g}_N^h$ and $\boldsymbol{g}_D^h$ must by specified. Proceeding analogously to the case of the state and adjoint equations, we define $\boldsymbol{g}_D^h$ as follows

$$\boldsymbol{g}_D(\boldsymbol{x}) \approx \boldsymbol{g}_D^h(\boldsymbol{x}) := \int_0^{l(\boldsymbol{x})} E_h(\boldsymbol{\sigma}_h)(\boldsymbol{x} + s\,\boldsymbol{t}(\boldsymbol{x}))\,\boldsymbol{t}(\boldsymbol{x})\,ds\,.$$

We can note that $\boldsymbol{g}_N^h$ needs to be characterized, for this we define

$$G_h(\Gamma) := \boldsymbol{r}_h \cdot \boldsymbol{n}(c\,\boldsymbol{p}_h \cdot \boldsymbol{n} + \partial_{\boldsymbol{n}} g) + \frac{1}{2}(g - \widetilde{y})^2\,.$$

With the above definition, instead of using equation (5.3.3d), we employ

$$\langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_e = \langle (G_h(\Gamma)\,\boldsymbol{n}) \circ \boldsymbol{\phi}, \boldsymbol{\mu} \rangle_e, \quad \forall e \in \mathcal{E}_h^{\partial}, \quad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h\,, \tag{5.3.4}$$

Therefore, we are able to rewrite the scheme as follows: find $(\boldsymbol{\sigma}_h, \boldsymbol{V}_h, \widehat{\boldsymbol{V}}_h) \in \mathbb{Z}_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h$ such that

$$(\boldsymbol{\sigma}_h, \boldsymbol{\psi})_{\mathcal{T}_h} - (\boldsymbol{V}_h, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{V}}_h, \boldsymbol{\psi}\,\boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = 0\,, \tag{5.3.5a}$$

$$-(\boldsymbol{\sigma}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}_h, \boldsymbol{w} \rangle_{\partial \mathcal{T}_h} = 0\,, \tag{5.3.5b}$$

$$\langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} = 0\,, \tag{5.3.5c}$$

$$\langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^N} = \langle (G_h(\Gamma)\,\boldsymbol{n}) \circ \boldsymbol{\phi}, \boldsymbol{\mu} \rangle_{\Gamma_h^N}\,, \tag{5.3.5d}$$

$$\langle \widehat{\boldsymbol{V}}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^D} = \langle \boldsymbol{g}_D^h, \boldsymbol{\mu} \rangle_{\Gamma_h^D}\,. \tag{5.3.5e}$$

for all $(\boldsymbol{\psi}, \boldsymbol{w}, \boldsymbol{\mu}) \in \mathbb{Z}_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h$, and it also holds

$$\widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}_h := \boldsymbol{\sigma}_h \boldsymbol{n}_h + \tau (\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h) . \tag{5.3.6}$$

Remark. It is ensured that $\boldsymbol{g}_D^h$ is measurable, following a straightforward application of Lemma 5.1.1.

5.4 Existence and uniqueness of the deformation field equation

We will prove the existence and uniqueness with the same strategy as in the case of state and adjoint equations. In fact, we establish the following theorem

Theorem 5.4.1. *Under the assumption (4.6.1), if* $\tau > 0$ *and* $\boldsymbol{t}(\boldsymbol{x}) = \boldsymbol{n} \quad \forall x \in e\,,\ x \in \mathcal{E}_h^\partial$, *there exists a unique solution of the HDG scheme associated to the deformation field equation (5.3.5).*

Proof. As for the case of state and adjoint equations, we will use the Fredholm alternative. Let us start by assuming that $G_h(\Gamma) = 0$, then (5.3.5) becomes.

$$(\boldsymbol{\sigma}_h, \boldsymbol{\psi})_{\mathcal{T}_h} - (\boldsymbol{V}_h, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{V}}_h, \boldsymbol{\psi}\boldsymbol{n}_h \rangle_{\partial\mathcal{T}_h} = 0\,, \tag{5.4.1a}$$

$$-(\boldsymbol{\sigma}_h, \nabla \boldsymbol{w})_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}_h, \boldsymbol{w} \rangle_{\partial\mathcal{T}_h} = 0\,, \tag{5.4.1b}$$

$$\langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\partial\mathcal{T}_h \setminus \Gamma_h} = 0\,, \tag{5.4.1c}$$

$$\langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^N} = 0\,, \tag{5.4.1d}$$

$$\langle \widehat{\boldsymbol{V}}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^D} = \langle \boldsymbol{g}_D^h, \boldsymbol{\mu} \rangle_{\Gamma_h^D} . \tag{5.4.1e}$$

Now setting $\boldsymbol{\psi} = \boldsymbol{\sigma}_h$ and $\boldsymbol{w} = \boldsymbol{V}_h$ in (5.4.1a) and (5.4.1b) respectively we obtain

$$\|\boldsymbol{\sigma}_h\|_{\mathcal{T}_h}^2 - (\boldsymbol{V}_h, \mathbf{div}(\boldsymbol{\sigma}_h))_{\mathcal{T}_h} + \langle \boldsymbol{V}_h, \boldsymbol{\sigma}_h \boldsymbol{n}_h \rangle_{\partial\mathcal{T}_h} = 0\,,$$

$$-\,(\boldsymbol{\sigma}_h, \nabla \boldsymbol{V}_h)_{\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h, \boldsymbol{V}_h \rangle_{\partial \mathcal{T}_h} \,=\, 0\,.$$

Integrating by parts the second equation,

$$\begin{aligned}
\|\boldsymbol{\sigma}_h\|^2_{\mathcal{T}_h} \,-\, (\boldsymbol{V}_h, \mathbf{div}(\boldsymbol{\sigma}_h))_{\mathcal{T}_h} \,+\, \langle \boldsymbol{V}_h, \boldsymbol{\sigma}_h\, \boldsymbol{n}_h \rangle_{\partial \mathcal{T}_h} \,&=\, 0\,,\\
(\mathbf{div}(\boldsymbol{\sigma}_h), \boldsymbol{V}_h)_{\mathcal{T}_h} \,-\, \langle \boldsymbol{\sigma}_h\, \boldsymbol{n}_h, \boldsymbol{V}_h \rangle_{\partial \mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h, \boldsymbol{V}_h \rangle_{\partial \mathcal{T}_h} \,&=\, 0\,.
\end{aligned}$$

Then, adding both equations we get

$$\|\boldsymbol{\sigma}_h\|^2_{\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h - \boldsymbol{\sigma}_h\, \boldsymbol{n}_h, \boldsymbol{V}_h \rangle_{\partial \mathcal{T}_h} \,+\, \langle \boldsymbol{\sigma}_h\, \boldsymbol{n}_h, \boldsymbol{V}_h \rangle_{\partial \mathcal{T}_h} \,=\, 0\,.$$

By (5.3.6) we can rewrite the above equation as

$$\|\boldsymbol{\sigma}_h\|^2_{\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h - \boldsymbol{\sigma}_h\, \boldsymbol{n}_h, \widehat{\boldsymbol{V}}_h \rangle_{\partial \mathcal{T}_h} \,+\, \langle \tau\, (\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h), \boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h \rangle_{\partial \mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{V}}_h, \boldsymbol{\sigma}_h\, \boldsymbol{n}_h \rangle_{\partial \mathcal{T}_h} \,=\, 0\,,$$

which is equivalent to

$$\|\boldsymbol{\sigma}_h\|^2_{\mathcal{T}_h} \,+\, \left\|\tau^{1/2}\, (\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h)\right\|^2_{\partial \mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{V}}_h, \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h \rangle_{\partial \mathcal{T}_h} \,=\, 0\,. \tag{5.4.2}$$

On the other hand, notice that by (5.4.1c), (5.4.1d) and (5.4.1e) we get

$$\begin{aligned}
\langle \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h, \widehat{\boldsymbol{V}}_h \rangle_{\partial \mathcal{T}_h} \,&=\, \langle \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h, \widehat{\boldsymbol{V}}_h \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} \,+\, \langle \widehat{\boldsymbol{\sigma}}_h \boldsymbol{n}_h, \widehat{\boldsymbol{V}}_h \rangle_{\Gamma_h} \,=\, \langle \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h, \widehat{\boldsymbol{V}}_h \rangle_{\Gamma_h}\\
&=\, \langle \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h, \widehat{\boldsymbol{V}}_h \rangle_{\Gamma_h^D} \,+\, \langle \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h, \widehat{\boldsymbol{V}}_h \rangle_{\Gamma_h^N} \,=\, \langle \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h, \boldsymbol{g}_D^h \rangle_{\Gamma_h^D}\,,
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}
\boldsymbol{g}_D^h(\boldsymbol{x}) \,&=\, \int_0^{l(\boldsymbol{x})} \boldsymbol{E}_h(\boldsymbol{\sigma}_h)(\boldsymbol{x} \,+\, s\, \boldsymbol{n}_h)\boldsymbol{n}_h\, ds\\
&=\, \int_0^{l(\boldsymbol{x})} (\boldsymbol{E}_h(\boldsymbol{\sigma}_h)(\boldsymbol{x} \,+\, s\, \boldsymbol{n}_h) \,-\, \boldsymbol{\sigma}_h(\boldsymbol{x}))\, \boldsymbol{n}_h\, ds \,+\, \int_0^{l(\boldsymbol{x})} \boldsymbol{\sigma}_h(\boldsymbol{x})\, \boldsymbol{n}_h\, ds\\
&=\, \int_0^{l(\boldsymbol{x})} (\boldsymbol{E}_h(\boldsymbol{\sigma}_h)(\boldsymbol{x} \,+\, s\, \boldsymbol{n}_h) \,-\, \boldsymbol{\sigma}_h(\boldsymbol{x}))\, \boldsymbol{n}_h\, ds \,+\, \boldsymbol{\sigma}_h(\boldsymbol{x})\, \boldsymbol{n}_h\, l(\boldsymbol{x})\,.
\end{aligned}$$

Thus, $\boldsymbol{\sigma}_h\,\boldsymbol{n}_h \,=\, l^{-1}\,\boldsymbol{g}_D^h \,-\, \Lambda^{\boldsymbol{\sigma}_h}$, and using the above argument and (5.3.6) we have that

$$\begin{aligned}
\langle \widehat{\boldsymbol{\sigma}}_h\,\boldsymbol{n}_h, \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} &= \langle \boldsymbol{\sigma}_h\,\boldsymbol{n}_h, \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} + \langle \tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h), \boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\\
&= \langle l^{-1}\,\boldsymbol{g}_D^h, \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} - \langle \Lambda^{\boldsymbol{\sigma}_h}, \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} + \langle \tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h), \boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\\
&= \left\| l^{-1/2}\,\boldsymbol{g}_D^h \right\|_{\Gamma_h^D}^2 - \langle l^{1/2}\,\Lambda^{\boldsymbol{\sigma}_h}, l^{-1/2}\,\boldsymbol{g}_D^h\rangle_{\Gamma_h^D} + \langle l^{1/2}\,\tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h), l^{-1/2}\,\boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\,.
\end{aligned}$$

Replacing in (5.4.2)

$$\begin{aligned}
\|\boldsymbol{\sigma}_h\|_{\mathcal{T}_h}^2 + \left\|\tau^{1/2}\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h)\right\|_{\partial\mathcal{T}_h}^2 + \left\| l^{-1/2}\,\boldsymbol{g}_D^h \right\|_{\Gamma_h^D}^2 &= \langle l^{1/2}\,\Lambda^{\boldsymbol{\sigma}_h}, l^{-1/2}\,\boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\\
&\quad - \langle l^{1/2}\,\tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h), l^{-1/2}\,\boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\,.
\end{aligned}$$

Now, we use Young's inequality, then, with $\epsilon_1,\,\epsilon_2 \,>\, 0$. At our disposal

$$\begin{aligned}
&\langle l^{1/2}\,\Lambda^{\boldsymbol{\sigma}_h}, l^{-1/2}\,\boldsymbol{g}_D^h\rangle_{\Gamma_h^D} - \langle l^{1/2}\,\tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h), l^{-1/2}\,\boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\\
&\quad\le \frac{1}{2\epsilon_1}\left\| l^{1/2}\,\Lambda^{\boldsymbol{\sigma}_h}\right\|_{\Gamma_h^D}^2 + \frac{\epsilon_1}{2}\left\| l^{-1/2}\,\boldsymbol{g}_D^h\right\|_{\Gamma_h^D}^2 + \frac{1}{2\epsilon_2}\left\| l^{1/2}\,\tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h)\right\|_{\Gamma_h^D}^2 + \frac{\epsilon_2}{2}\left\| l^{-1/2}\,\boldsymbol{g}_D^h\right\|_{\Gamma_h^D}^2\,,
\end{aligned}$$

Setting $\epsilon_1 \,=\, \epsilon_2 \,=\, 1/4$ we obtain

$$\begin{aligned}
&\langle l^{1/2}\,\Lambda^{\boldsymbol{\sigma}_h}, l^{-1/2}\,\boldsymbol{g}_D^h\rangle_{\Gamma_h^D} - \langle l^{1/2}\,\tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h), l^{-1/2}\,\boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\\
&\quad\le 2\left\| l^{1/2}\,\Lambda^{\boldsymbol{\sigma}_h}\right\|_{\Gamma_h^D}^2 + \frac{1}{8}\left\| l^{-1/2}\,\boldsymbol{g}_D^h\right\|_{\Gamma_h^D}^2 + 2\left\| l^{1/2}\,\tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h)\right\|_{\Gamma_h^D}^2 + \frac{1}{8}\left\| l^{-1/2}\,\boldsymbol{g}_D^h\right\|_{\Gamma_h^D}^2\\
&\quad\le 2\left\| l^{1/2}\,\Lambda^{\boldsymbol{\sigma}_h}\right\|_{\Gamma_h^D}^2 + \frac{1}{4}\left\| l^{-1/2}\,\boldsymbol{g}_D^h\right\|_{\Gamma_h^D}^2 + 2\left\| l^{1/2}\,\tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h)\right\|_{\Gamma_h^D}^2\,,
\end{aligned}$$

and replacing the above inequality in (5.4.2), it can be deduced that

$$\begin{aligned}
&\|\boldsymbol{\sigma}_h\|_{\mathcal{T}_h}^2 + \left\|\tau^{1/2}\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h)\right\|_{\partial\mathcal{T}_h}^2 + \left\| l^{-1/2}\,\boldsymbol{g}_D^h \right\|_{\Gamma_h^D}^2\\
&\quad\le 2\left\| l^{1/2}\,\Lambda^{\boldsymbol{\sigma}_h}\right\|_{\Gamma_h^D}^2 + \frac{1}{4}\left\| l^{-1/2}\,\boldsymbol{g}_D^h\right\|_{\Gamma_h^D}^2 + 2\left\| l^{1/2}\,\tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h)\right\|_{\Gamma_h^D}^2\,,
\end{aligned}$$

equivalently,

$$\|\boldsymbol{\sigma}_h\|_{\mathcal{T}_h}^2 + \left\|\tau^{1/2}\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h)\right\|_{\partial\mathcal{T}_h}^2 + \frac{3}{4}\left\| l^{-1/2}\,\boldsymbol{g}_D^h \right\|_{\Gamma_h^D}^2 \le 2\left\| l^{1/2}\,\Lambda^{\boldsymbol{\sigma}_h}\right\|_{\Gamma_h^D}^2 + 2\left\| l^{1/2}\,\tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h)\right\|_{\Gamma_h^D}^2$$

$$\leq 2 \left\| l^{1/2} \, \Lambda^{\boldsymbol{\sigma}_h} \right\|^2_{\Gamma_h^D} + 2\tau \max_{x \in \Gamma_h} l(\boldsymbol{x}) \left\| \tau^{1/2} \, (\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h) \right\|^2_{\Gamma_h^D}$$
$$\leq 2 \left\| l^{1/2} \, \Lambda^{\boldsymbol{\sigma}_h} \right\|^2_{\Gamma_h^D} + \frac{1}{2} \left\| \tau^{1/2} \, (\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h) \right\|^2_{\Gamma_h^D} .$$

Then, from the first equation of the first row of the assumptions (4.6.1), it follows

$$\| \boldsymbol{\sigma}_h \|^2_{\mathcal{T}_h} + \frac{1}{2} \left\| \tau^{1/2} (\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h) \right\|^2_{\partial \mathcal{T}_h} + \frac{3}{4} \left\| l^{-1/2} \boldsymbol{g}_D^h \right\|^2_{\Gamma_h^D} \leq 2 \left\| l^{1/2} \, \Lambda^{\boldsymbol{\sigma}_h} \right\|^2_{\Gamma_h^D} \leq 2 \sum_{e \subset \Gamma_h} \left\| l^{1/2} \, \Lambda^{\boldsymbol{\sigma}_h} \right\|^2_e$$
$$\leq 2 \sum_{e \subset \Gamma_h} \frac{1}{3} r_e^3 \, (C_{ext}^e \, C_{inv}^e)^2 \, \| \boldsymbol{\sigma}_h \|^2_{K^e} ,$$

and therefore,

$$\frac{11}{12} \, \| \boldsymbol{\sigma}_h \|^2_{\mathcal{T}_h} + \frac{1}{2} \left\| \tau^{1/2} \, (\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h) \right\|^2_{\partial \mathcal{T}_h} + \frac{3}{4} \left\| l^{-1/2} \, \boldsymbol{g}_D^h \right\|^2_{\Gamma_D^h} = 0 .$$

By the above equation we deduce that $\boldsymbol{\sigma}_h = \mathbf{0}$ in D_h, $\boldsymbol{V}_h = \widehat{\boldsymbol{V}}_h$ on Γ_h^D and $\boldsymbol{g}_D^h = \mathbf{0}$ on Γ_h^D, hence let us notice the following: recalling (5.4.1a) we have that

$$0 = - (\boldsymbol{V}_h, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{V}}_h, \boldsymbol{\psi} \, \boldsymbol{n}_h \rangle_{\partial \mathcal{T}_h} = (\nabla \boldsymbol{V}_h, \boldsymbol{\psi})_{\mathcal{T}_h} - \langle \boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h, \boldsymbol{\psi} \, \boldsymbol{n}_h \rangle_{\partial \mathcal{T}_h} ,$$

for all $\boldsymbol{\psi} \in \mathbb{V}_h$. Thus, setting $\boldsymbol{\psi} = \nabla \boldsymbol{V}_h$

$$\| \nabla \boldsymbol{V}_h \|^2_{\mathcal{T}_h} = 0.$$

Therefore we have that $\nabla \boldsymbol{V}_h = \mathbf{0}$, i.e. $\boldsymbol{V}_h$ is constant, but $\langle \boldsymbol{V}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^D} = \langle \boldsymbol{g}_D^h, \boldsymbol{\mu} \rangle_{\Gamma_h^D} = 0$ for all $\mu \in \boldsymbol{M}_h$. Hence, $\boldsymbol{V}_h = 0$ in D_h.

□

CHAPTER 6

A priori error estimates

In this chapter we will develop the analysis for the error estimates for each problem, where we will prove under certain assumptions the following estimates for the HDG schemes

$$\|y - y_h\|_\Omega \leq C\, h^{k+1}\,, \qquad \|\boldsymbol{p} - \boldsymbol{p}_h\|_\Omega \leq C\, h^{k+1}\,, \qquad \|z - z_h\|_\Omega \leq C\, h^{k+1}\,,$$

$$\|\boldsymbol{r} - \boldsymbol{r}_h\|_\Omega \leq C\, h^{k+1}\,, \qquad \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_\Omega \leq C\, h^{k}\,,$$

and

$$\|\boldsymbol{V} - \boldsymbol{V}_h\|_\Omega \leq C\left(h + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right) h^k\,,$$

where C is a positive constant independent of h and the values for C_R, δ, $C_{\boldsymbol{n}}$, and β will be specified later in the analysis.

For the error estimates we will use the HDG-projection developed in [21] on the product space $\boldsymbol{Z}_h \times W_h$, which is defined by

$$\Pi_h(\boldsymbol{r}, w) := (\boldsymbol{\Pi_Z r}, \Pi_W w)\,,$$

where $(\boldsymbol{\Pi_Z r}, \Pi_W w)$ satisfies that, for all $K \in \mathcal{T}_h$,

$$(\boldsymbol{\Pi_Z r}, \boldsymbol{s})_K = (\boldsymbol{r}, \boldsymbol{s})_K \qquad \forall\, \boldsymbol{s} \in [\mathbb{P}_{k-1}(K)]^d\,, \tag{6.0.1a}$$

$$(\Pi_W w, v)_K = (w, v)_K \qquad \forall\, v \in \mathbb{P}_{k-1}(K)\,, \tag{6.0.1b}$$

$$\langle \boldsymbol{\Pi_Z r} \cdot \boldsymbol{n} + \tau\, \Pi_W w, \mu\rangle_e = \langle \boldsymbol{r} \cdot \boldsymbol{n} + \tau\, w, \mu\rangle_e \qquad \forall\, \mu \in \mathbb{P}_k(e), \forall\, e \subset \partial K\,, \tag{6.0.1c}$$

This definition can be extended to the case when $(\boldsymbol{\sigma}, \boldsymbol{w}) \in \mathbb{Z}_h \times \boldsymbol{W}_h$, that is, matrix and vector-valued case, where the notation is $\Pi_h(\boldsymbol{r}, w) := (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma}, \boldsymbol{\Pi_W w})$. Furthermore, by the work done in [21], we have the following result.

Theorem 6.0.1. *Suppose $k \geq 0$ and $K \in \mathcal{T}_h$, the projection $(\boldsymbol{\Pi_Z r}, \Pi_W w)$ of $(\boldsymbol{r}, w) \in \boldsymbol{Z}_h \times W_h$ is well-defined. Moreover, there exists a real constant $C > 0$ independent of h_K such that if $\boldsymbol{r} \in \boldsymbol{H}^{k+1}(K)$ and $w \in H^{k+1}(K)$,*

$$\|\boldsymbol{\Pi_V r} - \boldsymbol{r}\|_K \leq C\, h_K^{k+1}\, |\boldsymbol{r}|_{\boldsymbol{H}^{k+1}(K)} + C\, h_K^{k+1}\, \tau_K^*\, |w|_{H^{k+1}(K)}\,, \tag{6.0.2a}$$

$$\|\Pi_W w - w\|_K \leq C\, h_K^{k+1}\, |w|_{H^{k+1}(K)} + C\, \frac{h_K^{k+1}}{\tau_K^{max}}\, |\nabla \cdot \boldsymbol{r}|_{H^k(K)}\,, \tag{6.0.2b}$$

where τ_K^{max} is the maximum value on ∂K and τ_K^* is the second maximum value.

Proof. See [21, Theorem 2.1]. □

On the other hand, for subsequent analysis we need to state the following two lemmas

Lemma 6.0.2. *(Discrete trace inequality) Let K be a element of $\mathcal{T}_h$, where h is its diameter, let $v \in \mathbb{P}_k(K)$, Then $\exists\, C_{tr} > 0$, which depend only of k and mesh regularity, such that*

$$h^{1/2}\, \|v\|_e \leq C_{tr}\, \|v\|_K\,, \tag{6.0.3}$$

where e is a edge or face of K.

Proof. See [28, Lemma 1.52]. □

Lemma 6.0.3. *Let K be an element of $\mathcal{T}_h$, let $0 \leq l \leq k$. We will denote $P_{L^2(K)}v$ the L^2 projection of v in $\mathbb{P}_k(K)$. For $v \in H^{l_v+1}(K)$ exist C_{L^2}, $C_{\partial L^2} > 0$, which do not depend of h, such that:*

$$|v - P_{L^2(K)}v|_{H^m(K)} \leq C_{L^2}\, h^{l+1-m}\, |v|_{H^{l+1}(K)} \quad \forall m \in \{0, \ldots, k\}\,, \tag{6.0.4a}$$

$$\left\| v - P_{L^2(K)}v \right\|_{\partial K} \leq C_{\partial L^2}\, h^{l+1/2}\, |v|_{H^{l+1}(K)}\,. \tag{6.0.4b}$$

Proof. See [28, Lemmas 1.58 and 1.59]. □

Is important to note that the Theorem 6.0.1 and lemmas 6.0.2 and 6.0.3 can be extended to the vector-valued case. In addition, we introduce the error and the projection of the error of the state equation:

$$\boldsymbol{e}^{\boldsymbol{p}} := \boldsymbol{p} - \boldsymbol{p}_h\,, \qquad e^y := y - y_h\,, \qquad e^{\widehat{y}} := y - \widehat{y}_h\,, \qquad \boldsymbol{\varepsilon}_h^{\boldsymbol{p}} := \boldsymbol{\Pi}_{\boldsymbol{Z}}\boldsymbol{p} - \boldsymbol{p}_h\,,$$
$$\varepsilon_h^y := \Pi_W y - y_h\,, \qquad \varepsilon_h^{\widehat{y}} := P_M y - \widehat{y}_h\,, \qquad \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}} := P_M \boldsymbol{p} - \widehat{\boldsymbol{p}}_h\,,$$

in the same way, we define $\boldsymbol{e}^{\boldsymbol{r}}$, e^z, $e^{\widehat{z}}$, $\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}$, ε_h^z, $\varepsilon_h^{\widehat{z}}$ and $\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{r}}}$ for the adjoint equation. Here, P_M denotes the local L^2-projection over M_h. We also define the interpolation errors as follows

$$I_{\boldsymbol{p}} := \boldsymbol{p} - \boldsymbol{\Pi}_{\boldsymbol{Z}}\boldsymbol{p}\,, \qquad I_y := y - \Pi_W y\,, \qquad I_{\boldsymbol{r}} := \boldsymbol{r} - \boldsymbol{\Pi}_{\boldsymbol{Z}}\boldsymbol{r}\,, \qquad I_z := z - \Pi_W z\,.$$

Likewise, we introduce the notation of the error and the projection of the error for the deformation field equation as follows:

$$\underline{\boldsymbol{e}}^{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\,, \qquad \boldsymbol{e}^{\boldsymbol{V}} := \boldsymbol{V} - \boldsymbol{V}_h\,, \qquad \boldsymbol{e}^{\widehat{\boldsymbol{V}}} := \boldsymbol{V} - \widehat{\boldsymbol{V}}_h\,, \qquad \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}} := \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\,,$$
$$\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} := \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{V} - \boldsymbol{V}_h\,, \qquad \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} := P_{\boldsymbol{M}}\boldsymbol{V} - \widehat{\boldsymbol{V}}_h\,, \qquad \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}} := P_{\boldsymbol{M}}\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\,.$$

In turn, let us introduce the notation of the interpolation error as:

$$\underline{I}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma}\,, \qquad I_{\boldsymbol{V}} := \boldsymbol{V} - \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{V}\,.$$

6.1 Error estimates for the state equation

In this chapter we proceed by deriving the error estimates of $\boldsymbol{e^p}$ with an energy argument strategy and then deriving the error estimates of e^y using a duality argument.

6.1.1 Error estimates for $\boldsymbol{e^p}$

The projection of the errors of the state equation satisfies the following equations,

Lemma 6.1.1. *We have that*

$$(c\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \boldsymbol{v}_1)_{\mathcal{T}_h} \;-\; (\varepsilon_h^y, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} \;+\; \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \;=\; -\,(c\,I_{\boldsymbol{p}}, \boldsymbol{v}_1)_{\mathcal{T}_h}\,, \tag{6.1.1a}$$

$$-\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla w_1)_{\mathcal{T}_h} \;+\; \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} \;=\; 0\,, \tag{6.1.1b}$$

$$\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} \;=\; 0\,, \tag{6.1.1c}$$

$$\langle \varepsilon_h^{\widehat{y}}, \mu_1\rangle_{\Gamma_h} \;=\; \langle \varphi_1 - \varphi_1^h, \mu_1\rangle_{\Gamma_h}\,, \tag{6.1.1d}$$

$$\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n} \;=\; \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n} + \tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\,, \tag{6.1.1e}$$

$\forall(\boldsymbol{v}_1, w_1, \mu_1) \in \boldsymbol{Z}_h \times W_h \times M_h$.

Proof. Beginning with (6.1.1a), let $\boldsymbol{v}_1 \in \boldsymbol{Z}_h$, we have that

$$\begin{aligned}
&(c\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \boldsymbol{v}_1)_{\mathcal{T}_h} \;-\; (\varepsilon_h^y, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} \;+\; \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&\quad = \; (c\,(\boldsymbol{\Pi_Z p} - \boldsymbol{p}_h), \boldsymbol{v}_1)_{\mathcal{T}_h} \;-\; (\Pi_W y - y_h, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} \;+\; \langle P_M y - \widehat{y}_h, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&\quad = (c\,\boldsymbol{\Pi_Z p}, \boldsymbol{v}_1)_{\mathcal{T}_h} \;-\; (\Pi_W y, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} \;+\; \langle P_M y, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \;-\; (c\,\boldsymbol{p}_h, \boldsymbol{v}_1)_{\mathcal{T}_h} + (y_h, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} \\
&\quad - \; \langle \widehat{y}_h, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}\,.
\end{aligned}$$

Hence, using (5.1.4a) followed by (6.0.1b), and then integrating by parts,

$$\begin{aligned}
&(c\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \boldsymbol{v}_1)_{\mathcal{T}_h} \;-\; (\varepsilon_h^y, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} \;+\; \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&\quad = \; (c\,\boldsymbol{\Pi_Z p}, \boldsymbol{v}_1)_{\mathcal{T}_h} \;-\; (\Pi_W y, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} \;+\; \langle P_M y, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&\quad = \; (c\,\boldsymbol{\Pi_Z p}, \boldsymbol{v}_1)_{\mathcal{T}_h} \;-\; (y, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} \;+\; \langle P_M y, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}
\end{aligned}$$

$$
\begin{aligned}
&= (c\,\mathbf{\Pi}_{\boldsymbol{Z}}\boldsymbol{p}, \boldsymbol{v}_1)_{\mathcal{T}_h} + (\nabla y, \boldsymbol{v}_1)_{\mathcal{T}_h} - \langle y, \boldsymbol{v}_1 \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle P_M y, \boldsymbol{v}_1 \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&= (c\,\mathbf{\Pi}_{\boldsymbol{Z}}\boldsymbol{p}, \boldsymbol{v}_1)_{\mathcal{T}_h} + (\nabla y, \boldsymbol{v}_1)_{\mathcal{T}_h}\,.
\end{aligned}
$$

Therefore, by (5.1.1a)

$$
\begin{aligned}
(c\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \boldsymbol{v}_1)_{\mathcal{T}_h} - (\varepsilon_h^{y}, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} + \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= (c\,\mathbf{\Pi}_{\boldsymbol{Z}}\boldsymbol{p}, \boldsymbol{v}_1)_{\mathcal{T}_h} - (c\,\boldsymbol{p}, \boldsymbol{v}_1)_{\mathcal{T}_h} \\
&= (c\,(\mathbf{\Pi}_{\boldsymbol{Z}}\boldsymbol{p} - \boldsymbol{p}), \boldsymbol{v}_1)_{\mathcal{T}_h} = -(c\,I_{\boldsymbol{p}}, \boldsymbol{v}_1)\,.
\end{aligned}
$$

Now, following with (6.1.1b), let $w_1 \in W_h$, we have that

$$
\begin{aligned}
-(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla w_1)_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} &= -(\mathbf{\Pi}_{\boldsymbol{Z}}\boldsymbol{p} - \boldsymbol{p}_h, \nabla w_1)_{\mathcal{T}_h} + \langle (P_M\boldsymbol{p} - \widehat{\boldsymbol{p}}_h)\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} \\
&= -(\mathbf{\Pi}_{\boldsymbol{Z}}\boldsymbol{p}, \nabla w_1)_{\mathcal{T}_h} + \langle P_M\boldsymbol{p}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} + (\boldsymbol{p}_h, \nabla w_1)_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h}\,.
\end{aligned}
$$

Thus, by (5.1.4b)

$$
-(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla w_1)_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} = -(\mathbf{\Pi}_{\boldsymbol{Z}}\boldsymbol{p}, \nabla w_1)_{\mathcal{T}_h} + \langle P_M\boldsymbol{p}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} - (f, w_1)_{\mathcal{T}_h}\,.
$$

Then, using (6.0.1a) and integrating by parts

$$
\begin{aligned}
-(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla w_1)_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} &= -(\boldsymbol{p}, \nabla w_1)_{\mathcal{T}_h} + \langle P_M\boldsymbol{p}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} - (f, w_1)_{\mathcal{T}_h} \\
&= (\nabla\cdot\boldsymbol{p}, w_1)_{\mathcal{T}_h} - \langle \boldsymbol{p}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} + \langle P_M\boldsymbol{p}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} - (f, w_1)_{\mathcal{T}_h} \\
&= (\nabla\cdot\boldsymbol{p}, w_1)_{\mathcal{T}_h} - (f, w_1)_{\mathcal{T}_h}\,.
\end{aligned}
$$

Finally, by (5.1.1b)

$$
-(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla w_1)_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} = (f, w_1)_{\mathcal{T}_h} - (f, w_1)_{\mathcal{T}_h} = 0\,.
$$

Now, continuing with (6.1.1c), let $\mu_1 \in M_h$, notice that

$$
\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} = \langle (P_M\boldsymbol{p} - \widehat{\boldsymbol{p}}_h)\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} = \langle P_M\boldsymbol{p}\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} - \langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h}\,.
$$

Thus, by (5.1.4c)

$$\begin{aligned}\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}} \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} &= \langle P_M \boldsymbol{p} \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} = \langle P_M \boldsymbol{p} \cdot \boldsymbol{n} - \boldsymbol{p} \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} + \langle \boldsymbol{p} \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} \\ &= \langle \boldsymbol{p} \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} = 0 \,.\end{aligned}$$

In fact, as $\boldsymbol{p} \in H(\mathrm{div}; \Omega)$, we get

$$\begin{aligned}\langle \boldsymbol{p} \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h} &= \sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{p} \cdot \boldsymbol{n} \, \mu_1 = \sum_{e \in \mathcal{E}_h^\circ} \int_e [\![\boldsymbol{p}]\!]_e \, \{\!\{\mu_1\}\!\}_e + [\![\mu_1]\!]_e \, \{\!\{\boldsymbol{p}\}\!\}_e + \sum_{e \in \mathcal{E}_h^\partial} \int_e \boldsymbol{p} \cdot \boldsymbol{n} \, \mu_1 \\ &= \langle \boldsymbol{p} \cdot \boldsymbol{n}, \mu_1 \rangle_{\Gamma_h} \,.\end{aligned}$$

The above expression was obtained using Lemma 4.1.1. Furthermore, since $\mu_1 \in \mathbb{P}_k(\mathcal{E}_h)$, then $[\![\mu_1]\!]_e = 0$. Now, following with (6.1.1d), recalling (5.1.1c) and (5.1.4d), for any $\mu_1 \in M_h$, let us notice that

$$\begin{aligned}\langle \varepsilon_h^{\widehat{y}}, \mu_1 \rangle_{\Gamma_h} &= \langle P_M y - \widehat{y}_h, \mu_1 \rangle_{\Gamma_h} = \langle P_M y, \mu_1 \rangle_{\Gamma_h} - \langle \widehat{y}_h, \mu_1 \rangle_{\Gamma_h} = \langle y, \mu_1 \rangle_{\Gamma_h} - \langle \varphi_1^h, \mu_1 \rangle_{\Gamma_h} \\ &= \langle \varphi_1, \mu_1 \rangle_{\Gamma_h} - \langle \varphi_1^h, \mu_1 \rangle_{\Gamma_h} = \langle \varphi_1 - \varphi_1^h \rangle_{\Gamma_h} \,.\end{aligned}$$

Finally to prove (6.1.1e), consider any $\mu_1 \in M_h$, then

$$\begin{aligned}\langle \boldsymbol{\varepsilon}_h^{\boldsymbol{p}} \cdot \boldsymbol{n} + \tau \, (\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \mu_1 \rangle_{\partial \mathcal{T}_h} &= \langle (\boldsymbol{\Pi_Z p} - \boldsymbol{p}_h) \cdot \boldsymbol{n} + \tau \, (\Pi_W y - y_h - P_M y + \widehat{y}_h), \mu_1 \rangle_{\partial \mathcal{T}_h} \\ &= \langle \boldsymbol{\Pi_Z p} \cdot \boldsymbol{n} + \tau \Pi_W y, \mu_1 \rangle_{\partial \mathcal{T}_h} - \langle \tau P_M y, \mu_1 \rangle_{\partial \mathcal{T}_h} - \langle \boldsymbol{p}_h \cdot \boldsymbol{n} + \tau (y_h - \widehat{y}_h), \mu_1 \rangle_{\partial \mathcal{T}_h} \,,\end{aligned}$$

finally by (5.1.7a) and (6.0.1c)

$$\begin{aligned}\langle \boldsymbol{\varepsilon}_h^{\boldsymbol{p}} \cdot \boldsymbol{n} + \tau \, (\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \mu_1 \rangle_{\partial \mathcal{T}_h} &= \langle \boldsymbol{p} \cdot \boldsymbol{n} + \tau \, y, \mu_1 \rangle_{\partial \mathcal{T}_h} - \langle \tau P_M y, \mu_1 \rangle_{\partial \mathcal{T}_h} - \langle \widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h} \\ &= \langle \boldsymbol{p} \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h} + \langle \tau (y - P_M y), \mu_1 \rangle_{\partial \mathcal{T}_h} - \langle \widehat{\boldsymbol{p}} \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h} \\ &= \langle \boldsymbol{p} \cdot \boldsymbol{n} - \widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h} \\ &= \langle \boldsymbol{p} \cdot \boldsymbol{n} - P_M \boldsymbol{p} \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h} + \langle P_M \boldsymbol{p} \cdot \boldsymbol{n} - \widehat{p}_h \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h} \\ &= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}} \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h} \,.\end{aligned}$$

□

We will now present a result based on an energy argument.

Lemma 6.1.2. *We have the following identity*

$$\left\|c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\right\|_{\mathcal{T}_h} + \left\|\tau^{1/2}\,(\varepsilon_h^{y} - \varepsilon_h^{\widehat{y}})\right\|_{\partial\mathcal{T}_h}^2 + \langle \boldsymbol{\varepsilon}_h^{\widehat{p}}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} = -\,(c\,\boldsymbol{I}_{\boldsymbol{p}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}})_{\mathcal{T}_h}\,. \tag{6.1.2}$$

Proof. Let $\boldsymbol{v}_1 = \varepsilon_h^{\boldsymbol{p}}$ and $w_1 = \varepsilon_h^{y}$ in (6.1.1a) and (6.1.1b) respectively. Then

$$\begin{aligned}(c\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}})_{\mathcal{T}_h} - (\varepsilon_h^{y}, \nabla\cdot\boldsymbol{\varepsilon}_h^{\boldsymbol{p}})_{\mathcal{T}_h} + \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= -\,(c\,\boldsymbol{I}_{\boldsymbol{p}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}})_{\mathcal{T}_h}\,,\\ -\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla\varepsilon_h^{y})_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varepsilon_h^{y}\rangle_{\partial\mathcal{T}_h} &= 0\,.\end{aligned} \tag{6.1.3}$$

Then, integrating by parts the second equation

$$(\nabla\cdot\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^{y})_{\mathcal{T}_h} - \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n}, \varepsilon_h^{y}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varepsilon_h^{y}\rangle_{\partial\mathcal{T}_h} = 0\,, \tag{6.1.4}$$

and now, adding the first equation of (6.1.3) and (6.1.4),

$$\left\|c^{1/2}\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n} - \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n}, \varepsilon_h^{y}\rangle_{\partial\mathcal{T}_h} + \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = -\,(c\,\boldsymbol{I}_{\boldsymbol{p}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}})_{\mathcal{T}_h}\,.$$

Hence, by (6.1.1e),

$$\left\|c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + \langle \tau\,(\varepsilon_h^{y} - \varepsilon_h^{\widehat{y}}), \varepsilon_h^{y} - \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n} - \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} + \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = -\,(c\,\boldsymbol{I}_{\boldsymbol{p}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}})\,,$$

which is the same as

$$\left\|c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + \left\|\tau^{1/2}\,(\varepsilon_h^{y} - \varepsilon_h^{\widehat{y}})\right\|_{\partial\mathcal{T}_h}^2 + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} = -\,(c\,\boldsymbol{I}_{\boldsymbol{p}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}})\,.$$

On the other hand, notice that using (6.1.1c) and (6.1.1d), we have that

$$\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h} = \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\Gamma_h} = \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varepsilon_h^{\widehat{y}}\rangle_{\Gamma_h} = \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h}\,.$$

Therefore, we deduce that

$$\left\|c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + \left\|\tau^{1/2}\,(\varepsilon_h^{y} - \varepsilon_h^{\widehat{y}})\right\|_{\partial\mathcal{T}_h}^2 + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} = -\,(c\,\boldsymbol{I}_{\boldsymbol{p}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}})_{\mathcal{T}_h}\,.$$

□

Corollary 6.1.1. *It holds*

$$\frac{1}{2}\left\|c^{1/2}\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + \left\|\tau^{1/2}(\varepsilon_h^{y} - \varepsilon_h^{\widehat{y}})\right\|_{\partial\mathcal{T}_h}^2 + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \leq \frac{1}{2}\left\|c^{1/2}\boldsymbol{I}_{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2\,. \tag{6.1.5}$$

Proof. It follows from previous result and Young's inequality. □

Lemma 6.1.3. *We have the following identity*

$$\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n} = c^{-1}\,l^{-1}\,(\varphi_1 - \varphi_1^h) - \Lambda^{\boldsymbol{I}_{\boldsymbol{p}}} - \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}} - \boldsymbol{I}_{\boldsymbol{p}}\cdot\boldsymbol{n} \qquad \forall\,\boldsymbol{x}\in e\,,\quad e\in\mathcal{E}_h^{\partial}\,. \tag{6.1.6}$$

Proof. Let us notice that

$$\begin{aligned}
\varphi_1 - \varphi_1^h &= g(\bar{\boldsymbol{x}}) + \int_0^{l(\boldsymbol{x})} c\,\boldsymbol{p}(\boldsymbol{x}+s\,\boldsymbol{n})\cdot\boldsymbol{n}\,ds - g(\bar{\boldsymbol{x}}) - \int_0^{l(\boldsymbol{x})} c\,\boldsymbol{E}_h(\boldsymbol{p}_h)(\boldsymbol{x}+s\,\boldsymbol{n})\cdot\boldsymbol{n}\,ds\\
&= \int_0^{l(\boldsymbol{x})} c\,(\boldsymbol{p} - \boldsymbol{E}_h(\boldsymbol{p}_h))(\boldsymbol{x}+s\,\boldsymbol{n})\cdot\boldsymbol{n}\,ds\\
&= \int_0^{l(\boldsymbol{x})} c\,(\boldsymbol{p} - \boldsymbol{\Pi}_{\boldsymbol{Z}}\boldsymbol{p} + \boldsymbol{\Pi}_{\boldsymbol{Z}}\boldsymbol{p} - \boldsymbol{E}_h(\boldsymbol{p}_h))(\boldsymbol{x}+s\,\boldsymbol{n})\cdot\boldsymbol{n}\,ds\\
&= \int_0^{l(\boldsymbol{x})} c\,(\boldsymbol{I}_{\boldsymbol{p}} + \boldsymbol{\varepsilon}_h^{\boldsymbol{p}})(\boldsymbol{x}+s\,\boldsymbol{n})\cdot\boldsymbol{n}\,ds\\
&= \int_0^{l(x)} c\,(\boldsymbol{I}_{\boldsymbol{p}}(\boldsymbol{x}+s\,\boldsymbol{n}) - \boldsymbol{I}_{\boldsymbol{p}}(\boldsymbol{x}))\cdot\boldsymbol{n}\,ds + c\,l(\boldsymbol{x})\,\boldsymbol{I}_{\boldsymbol{p}}(\boldsymbol{x})\cdot\boldsymbol{n}\\
&+ \int_0^{l(\boldsymbol{x})} c\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}(\boldsymbol{x}+s\,\boldsymbol{n}) - \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}(\boldsymbol{x}))\cdot\boldsymbol{n}\,ds + c\,l(\boldsymbol{x})\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}(\boldsymbol{x})\cdot\boldsymbol{n}\\
&= c\,l(\boldsymbol{x})\,\left(\Lambda^{\boldsymbol{I}_{\boldsymbol{p}}} + \boldsymbol{I}_{\boldsymbol{p}}\cdot\boldsymbol{n} + \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}} + \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n}\right)(\boldsymbol{x})\,.
\end{aligned}$$

□

The proof of the following lemma can be found in [23, Lemma 5.2].

Lemma 6.1.4. *For each $e \in \mathcal{E}_h^{\partial}$, we have that*

$$\left\|\Lambda^{I_{\boldsymbol{p}}}\right\|_{e,l} \,\leq\, \frac{1}{\sqrt{3}}\, r_e\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{K^e_{ext},(h^{\perp})^2}\,, \tag{6.1.7a}$$

$$\left\|\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}\right\|_{e,l} \,\leq\, \frac{1}{\sqrt{3}}\, r_e^{3/2}\, C^e_{ext}\, C^e_{inv}\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{K^e}\,. \tag{6.1.7b}$$

Now, we establish the following error estimate

Lemma 6.1.5. *Here holds*

$$\left\|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right\| \,\lesssim\, \|I_{\boldsymbol{p}}\|_{\mathcal{T}_h} \,+\, R\,\|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{D_h^c,(h^{\perp})^2} \,+\, R^{1/2}\,\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^{\perp}}\,, \tag{6.1.8}$$

where

$$\left\|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{q}}, \varepsilon_h^u - \varepsilon_h^{\widehat{u}}, \varphi - \varphi^h)\right|\!\right\| \,:=\, \left(\|\boldsymbol{\varepsilon}_h^{\boldsymbol{q}}\|^2_{\mathcal{T}_h,c} + \left\|\varepsilon_h^u - \varepsilon_h^{\widehat{u}}\right\|^2_{\partial\mathcal{T}_h,\tau} + \left\|\varphi - \varphi^h\right\|^2_{\Gamma_h,c^{-1}l^{-1}}\right)^{1/2}.$$

Proof. From (6.1.1e) we can deduce that

$$\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,=\, \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \tau(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \varphi_1 - \varphi_1^h\rangle_{\Gamma_h}\,,$$

and by (6.1.6)

$$\begin{aligned}
\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,&=\, \langle c^{-1}\, l^{-1}\,(\varphi_1 - \varphi_1^h), \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,-\, \langle \Lambda^{I_{\boldsymbol{p}}}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,-\, \langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \\
&\quad -\, \langle I_{\boldsymbol{p}}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \tau(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \\
&=\, \left\|c^{-1/2}\, l^{-1/2}\,(\varphi_1 - \varphi_1^h)\right\|^2_{\Gamma_h} \,-\, \langle \Lambda^{I_{\boldsymbol{p}}}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,-\, \langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \\
&\quad -\, \langle I_{\boldsymbol{p}}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \tau(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \varphi_1 - \varphi_1^h\rangle_{\Gamma_h}\,.
\end{aligned}$$

Hence, replacing this in (6.1.5)

$$\begin{aligned}
\frac{1}{2}\left\|c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\right\|^2_{\mathcal{T}_h} + \left\|\tau^{1/2}(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\right\|^2_{\partial\mathcal{T}_h} + \left\|c^{-1/2}\, l^{-1/2}(\varphi_1 - \varphi_1^h)\right\|_{\Gamma_h} \,&\leq\, \frac{1}{2}\left\|c^{1/2} I_{\boldsymbol{p}}\right\|^2_{\mathcal{T}_h} + \langle \Lambda^{I_{\boldsymbol{p}}}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \\
+ \langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} + \langle I_{\boldsymbol{p}}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} - \langle \tau(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \varphi_1 - \varphi_1^h\rangle_{\Gamma_h}&.
\end{aligned}$$

Let $\epsilon_1,\, \epsilon_2,\, \epsilon_3,\, \epsilon_4 \,>\, 0$, by Cauchy-Schwarz and Young's inequality we get

$$
\begin{aligned}
&\frac{1}{2}\,\left\|c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + \left\|\tau^{1/2}\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\right\|_{\partial\mathcal{T}_h}^2 + \left\|c^{-1/2}\,l^{-1/2}\,(\varphi_1 - \varphi_1^h)\right\|_{\Gamma_h} \\
&\quad\leq \frac{1}{2}\,\left\|c^{1/2}\,I_{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + \frac{1}{2\epsilon_1}\,\left\|c^{1/2}\,l^{1/2}\,\Lambda^{I_{\boldsymbol{p}}}\right\|_{\Gamma_h}^2 + \frac{\epsilon_1}{2}\,\left\|c^{-1/2}\,l^{-1/2}\,(\varphi_1 - \varphi_1^h)\right\|_{\Gamma_h}^2 + \frac{1}{2\epsilon_2}\,\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}\right\|_{\Gamma_h}^2 \\
&\quad+ \frac{\epsilon_2}{2}\,\left\|c^{-1/2}\,l^{-1/2}\,(\varphi_1 - \varphi_1^h)\right\|_{\Gamma_h}^2 + \frac{1}{2\epsilon_3}\,\left\|c^{1/2}\,l^{1/2}\,I_{\boldsymbol{p}}\cdot\boldsymbol{n}\right\|_{\Gamma_h}^2 + \frac{\epsilon_3}{2}\,\left\|c^{-1/2}\,l^{-1/2}\,(\varphi_1 - \varphi_1^h)\right\|_{\Gamma_h}^2 \\
&\quad+ \frac{1}{2\epsilon_4}\,\left\|c^{1/2}\,l^{1/2}\,\tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\right\|_{\Gamma_h}^2 + \frac{\epsilon_4}{2}\,\left\|c^{-1/2}\,l^{-1/2}\,(\varphi_1 - \varphi_1^h)\right\|_{\Gamma_h}^2 \\
&\quad= \frac{1}{2}\,\left\|c^{1/2}\,I_{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + \frac{1}{2\epsilon_1}\,\left\|c^{1/2}\,l^{1/2}\,\Lambda^{I_{\boldsymbol{p}}}\right\|_{\Gamma_h}^2 + \frac{1}{2\epsilon_2}\,\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}\right\|_{\Gamma_h}^2 + \frac{1}{2\epsilon_3}\,\left\|c^{1/2}\,l^{1/2}\,I_{\boldsymbol{p}}\cdot\boldsymbol{n}\right\|_{\Gamma_h}^2 \\
&\quad+ \frac{1}{2\epsilon_4}\,\left\|c^{1/2}\,l^{1/2}\,\tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\right\|_{\Gamma_h}^2 + \frac{1}{2}\,[\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4]\,\left\|c^{-1/2}\,l^{-1/2}\,(\varphi_1 - \varphi_1^h)\right\|_{\Gamma_h}^2 .
\end{aligned}
$$

$\epsilon_1 \,=\, \epsilon_2 \,=\, \epsilon_3 \,=\, \epsilon_4 \,=\, 1/4$, we have that

$$
\begin{aligned}
&\frac{1}{2}\,\left\|c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + \left\|\tau^{1/2}\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\right\|_{\partial\mathcal{T}_h}^2 + \frac{1}{2}\,\left\|c^{-1/2}\,l^{-1/2}\,(\varphi_1 - \varphi_1^h)\right\|_{\Gamma_h} \\
&\qquad\leq \frac{1}{2}\,\left\|c^{1/2}\,I_{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + 2\,\left\|c^{1/2}\,l^{1/2}\,\Lambda^{I_{\boldsymbol{p}}}\right\|_{\Gamma_h}^2 + 2\,\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}\right\|_{\Gamma_h}^2 + 2\,\left\|c^{1/2}\,l^{1/2}\,I_{\boldsymbol{p}}\cdot\boldsymbol{n}\right\|_{\Gamma_h}^2 \\
&\qquad+ 2\,\left\|c^{1/2}\,l^{1/2}\,\tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\right\|_{\Gamma_h}^2 .
\end{aligned}
$$

On the other hand, notice that by the first equation of the first row of the assumptions (4.6.1)

$$
2\,\left\|c^{1/2}\,l^{1/2}\,\tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\right\|_{\Gamma_h}^2 \leq 2\,c\,\tau\,\max_{\boldsymbol{x}\in\Gamma_h} l(\boldsymbol{x})\,\left\|\tau^{1/2}\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\right\|_{\Gamma_h}^2 \leq \frac{1}{2}\,\left\|\tau^{1/2}\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\right\|_{\partial\mathcal{T}_h}^2 .
$$

Thus, we have

$$
\begin{aligned}
&\frac{1}{2}\,\left\|c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + \frac{1}{2}\,\left\|\tau^{1/2}(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\right\|_{\partial\mathcal{T}_h}^2 + \frac{1}{2}\,\left\|c^{-1/2}\,l^{-1/2}\,(\varphi_1 - \varphi_1^h)\right\|_{\Gamma_h} \\
&\qquad\leq \frac{1}{2}\,\left\|c^{1/2}\,I_{\boldsymbol{p}}\right\|_{\mathcal{T}_h}^2 + 2\,\left\|c^{1/2}\,l^{1/2}\,\Lambda^{I_{\boldsymbol{p}}}\right\|_{\Gamma_h}^2 + 2\,\left\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}\right\|_{\Gamma_h}^2 + 2\,\left\|c^{1/2}\,l^{1/2}\,I_{\boldsymbol{p}}\cdot\boldsymbol{n}\right\|_{\Gamma_h}^2 .
\end{aligned}
$$

Also, by (6.1.7)

$$
\begin{aligned}
2\,\left\|c^{1/2}\,l^{1/2}\,\Lambda^{I_{\boldsymbol{p}}}\right\|_{\Gamma_h}^2 &= 2\,c\sum_{e\subset\Gamma_h}\left\|l^{1/2}\,\Lambda^{I_{\boldsymbol{p}}}\right\|_e^2 \leq 2\,c\sum_{e\subset\Gamma_h}\frac{1}{3}\,r_e^2\,\|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{K_{ext}^e,(h^\perp)^2} \\
&\leq \frac{2c}{3}\,\max_{e\subset\Gamma_h} r_e^2\,\|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2} ,
\end{aligned}
$$

and

$$2\,\left\| c^{1/2}\, l^{1/2}\, \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}} \right\|_{\Gamma_h}^2 \;=\; 2\,c \sum_{e\subset\Gamma_h} \left\| l^{1/2}\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}} \right\|_e^2 \;\leq\; 2\,c \sum_{e\subset\Gamma_h} \frac{1}{3}\, r_e^3\, (C_{ext}^e\, C_{inv}^e)^2\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{K^e}^2$$
$$\leq\; \frac{2\,c}{3} \max_{e\subset\Gamma_h} r_e^3\, (C_{ext}^e\, C_{inv}^e)^2\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{\mathcal{T}_h}^2 \;\leq\; \frac{1}{12} \left\| c^{1/2}\, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}} \right\|_{\mathcal{T}_h}^2 .$$

Therefore, we obtain

$$\frac{5}{12} \left\| c^{1/2}\, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}} \right\|_{\mathcal{T}_h}^2 + \frac{1}{2} \left\| \tau^{1/2}\, (\varepsilon_h^y - \varepsilon_h^{\widehat{y}}) \right\|_{\partial\mathcal{T}_h}^2 + \frac{1}{2} \left\| c^{-1/2}\, l^{-1/2}\, (\varphi_1 - \varphi_1^h) \right\|_{\Gamma_h}^2$$
$$\leq \frac{1}{2} \left\| c^{1/2}\, I_{\boldsymbol{p}} \right\|_{\mathcal{T}_h}^2 + \frac{2\,c}{3} \max_{e\subset\Gamma_h} r_e^2\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2}^2 + 2 \left\| c^{1/2}\, l^{1/2}\, I_{\boldsymbol{p}}\cdot\boldsymbol{n} \right\|_{\Gamma_h}^2$$
$$\leq \frac{1}{2} \left\| c^{1/2}\, I_{\boldsymbol{p}} \right\|_{\mathcal{T}_h}^2 + \frac{2\,c\,R^2}{3} \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2}^2 + 2\,R \left\| c^{1/2}\, I_{\boldsymbol{p}}\cdot\boldsymbol{n} \right\|_{\Gamma_h,h^\perp}^2 .$$

□

Now, to control the second term of the right side of (6.1.8), we have the following lemma.

Lemma 6.1.6. *Let e be a edge or face of Γ_h and $C > 0$ independent of h, then*

$$\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} \;\leq\; C\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \;+\; C\, h^{k+1}\, \tau_{\max}\, |y|_{H^{k+1}(\Omega)} . \tag{6.1.9}$$

Proof. Let us notice that using Lemma 6.0.2 and (6.0.4a), we have that for given $e \in \Gamma_h$

$$\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{e,h} \;\leq\; \left\| (\boldsymbol{\Pi_Z p} - \boldsymbol{P}_{L^2(K^e)}\boldsymbol{p})\cdot\boldsymbol{n} \right\|_{e,h} + \left\| (\boldsymbol{p} - \boldsymbol{P}_{L^2(K^e)}\boldsymbol{p})\cdot\boldsymbol{n} \right\|_{e,h}$$
$$\leq \widetilde{C}_1 \left\| \boldsymbol{\Pi_Z p} - \boldsymbol{P}_{L^2(K^e)}\boldsymbol{p} \right\|_{K^e} + \left\| (\boldsymbol{p} - \boldsymbol{P}_{L^2(K^e)}\boldsymbol{p})\cdot\boldsymbol{n} \right\|_{e,h}$$
$$\leq \widetilde{C}_1\, \|\boldsymbol{\Pi_Z p} - \boldsymbol{p}\|_{K^e} + \widehat{C}_2 \left\| \boldsymbol{p} - \boldsymbol{P}_{L^2(K^e)}\boldsymbol{p} \right\|_{K^e}$$
$$\leq \widetilde{C}_1\, h_{K^e}^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(K^e)} + \widetilde{C}_1\, h_{K^e}^{k+1}\, \tau_{K^e}^*\, |y|_{H^{k+1}(K^e)} + \widehat{C}_2\, h_{K^e}^{k+1}\, \tau_{K^e}^*\, |y|_{H^{k+1}(K^e)}$$
$$\leq C\, h_{K^e}^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(K^e)} + C\, h_{K^e}^{k+1}\, \tau_{K^e}^*\, |y|_{H^{k+1}(K^e)} .$$

□

Lemma 6.1.7. *There exists $C > 0$ independent of h, such that*

$$\begin{aligned} \|I_{\boldsymbol{p}}\|_{D_h^c} &\leq \sqrt{\widetilde{R}_C'}\, \|I_{\boldsymbol{p}}\|_{D_h} + C\,(1 + \sqrt{\widetilde{R}_C'})\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}\,, \\ \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}} \cdot \boldsymbol{n})\|_{D_h^c,(h^\perp)^2} &\leq \sqrt{R_C'}\, \|I_{\boldsymbol{p}}\|_{D_h} + C\,(1 + \sqrt{R_C'})\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}\,. \end{aligned}$$

Proof. The proof of this lemma can be found in [23, Lemma 3.8]. □

We are now ready to state the error estimates for $\boldsymbol{p}$.

Lemma 6.1.8. *We have that*

$$\begin{aligned} \|\boldsymbol{p} - \boldsymbol{p}_h\|_{D_h} \lesssim{}& (1 + R\sqrt{R_C'})\, \|I_{\boldsymbol{p}}\|_{D_h} + R^{1/2}\, \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,h^\perp} \\ &+ R\,(1 + \sqrt{R_C'})\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}\,. \end{aligned} \tag{6.1.10}$$

Proof. For this proof we follow the proof of [23, Lemma 3.9]. Using Lemma 6.1.5 and Lemma 6.1.7. Let us note that

$$\begin{aligned} \|\boldsymbol{p} - \boldsymbol{p}_h\|_{D_h} &\leq \|I_{\boldsymbol{p}}\|_{D_h} + \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{D_h} \\ &\lesssim \|I_{\boldsymbol{p}}\| + R^{1/2}\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}} \cdot \boldsymbol{n})\|_{D_h^c,(h^\perp)^2} + R\, \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,h^\perp} \\ &\lesssim \|I_{\boldsymbol{p}}\|_{D_h} + R^{1/2}\, \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,h^\perp} + R\sqrt{R_C'}\, \|I_{\boldsymbol{p}}\|_{D_h} + R\,(1 + \sqrt{R_C'})\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\ &= (1 + R\sqrt{R_C'})\, \|I_{\boldsymbol{p}}\|_{D_h} + R^{1/2}\, \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,h^\perp} + R\,(1 + \sqrt{R_C'})\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}\,. \end{aligned}$$

□

Also we have the following estimates of the error in D_h^c.

Lemma 6.1.9. *There exists a constant $C > 0$ independent of h such that*

$$\begin{aligned} \|\boldsymbol{p} - \boldsymbol{p}_h\|_{D_h^c} &\leq \|I_{\boldsymbol{p}}\|_{D_h^c} + \sqrt{\widetilde{R}_C'}\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{D_h}\,, \\ \|y - y_h\|_{D_h^c} &\leq C\, h\, \|\boldsymbol{p} - \boldsymbol{p}_h\|_{D_h^c}\,. \end{aligned}$$

Proof. The proof of this lemma can be found in [23, Lemma 3.7]. □

Finally, we obtain the following error estimate of $\boldsymbol{p}$ in the entire domain Ω.

Theorem 6.1.10. *We have that*

$$\|\boldsymbol{p}-\boldsymbol{p}_h\|_\Omega \lesssim H_1(R,h)\,\|I_{\boldsymbol{p}}\|_{D_h} + H_2(R,h)\,\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} + H_3(R,h)\,h^{k+1}\,|\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}\,, \tag{6.1.11}$$

where

$$\begin{aligned} H_1(R,h) &:= \left(1 + R + \sqrt{R'_C} + \sqrt{\widetilde{R}'_C} + R\sqrt{\widetilde{R}'_C}\sqrt{R'_C}\right), \\ H_2(R,h) &:= \left(R^{1/2} + R^{1/2}\sqrt{\widetilde{R}'_C}\right), \\ H_3(R,h) &:= \left(1 + \sqrt{\widetilde{R}'_C} + R + R\sqrt{R'_C} + R\sqrt{\widetilde{R}'_C} + R\sqrt{\widetilde{R}'_C}\sqrt{R'_C}\right). \end{aligned}$$

Moreover, we can note that $H_1(R,h) \le C$, $H_3(R,h) \le C$ and also there exists a constant $C>0$ such that $H_2(R,h) \le C\,h^{1/2}$.

Proof. Using Lemma 6.1.8 and Lemma 6.1.9 it follows

$$\begin{aligned} \|\boldsymbol{p}-\boldsymbol{p}_h\|_\Omega &= \|\boldsymbol{p}-\boldsymbol{p}_h\|_{D_h} + \|\boldsymbol{p}-\boldsymbol{p}_h\|_{D_h^c} \\ &\lesssim (1 + R\sqrt{R'_C})\,\|I_{\boldsymbol{p}}\|_{D_h} + R^{1/2}\,\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} + R\,(1+\sqrt{R'_C})\,h^{k+1}\,|\boldsymbol{p}|_{H^{k+1}(\Omega)} \\ &+ \|I_{\boldsymbol{p}}\|_{D_h^c} + \sqrt{\widetilde{R}'_C}\,\|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{\mathcal{T}_h}\,. \end{aligned}$$

Then, by Lemma 6.1.5 and Lemma 6.1.7 it follows

$$\begin{aligned} \|\boldsymbol{p}-\boldsymbol{p}_h\|_\Omega &\lesssim (1 + R\sqrt{R'_C})\,\|I_{\boldsymbol{p}}\|_{D_h} + R^{1/2}\,\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} + R\,(1+\sqrt{R'_C})\,h^{k+1}\,|\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\ &+ \sqrt{\widetilde{R}'_C}\,\|I_{\boldsymbol{p}}\|_{D_h} + (1+\sqrt{\widetilde{R}'_C})\,h^{k+1}\,|\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\ &+ \sqrt{\widetilde{R}'_C}\left(\|I_{\boldsymbol{p}}\|_{D_h} + R\,\|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2} + R^{1/2}\,\|I_{\boldsymbol{p}}\boldsymbol{n}\|_{\Gamma_h,h^\perp}\right) \\ &\lesssim \left(1 + R + \sqrt{R'_C} + \sqrt{\widetilde{R}'_C}\right)\|I_{\boldsymbol{p}}\|_{D_h} + \left(R^{1/2} + R^{1/2}\sqrt{\widetilde{R}'_C}\right)\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} \\ &+ \left(1 + \sqrt{\widetilde{R}'_C} + R + R\sqrt{R'_C}\right)h^{k+1}\,|\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} + R\sqrt{\widetilde{R}'_C}\,\|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2} \\ &\lesssim \left(1 + R + \sqrt{R'_C} + \sqrt{\widetilde{R}'_C}\right)\|I_{\boldsymbol{p}}\|_{D_h} + \left(R^{1/2} + R^{1/2}\sqrt{\widetilde{R}'_C}\right)\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} \end{aligned}$$

$$
\begin{aligned}
&+ \left(1 + \sqrt{\widetilde{R}'_C} + R + R\sqrt{R'_C}\right) h^{k+1} \, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\
&+ R\sqrt{\widetilde{R}'_C}\sqrt{R'_C}\, \|I_{\boldsymbol{p}}\|_{D_h} + \left(R\sqrt{\widetilde{R}'_C} + R\sqrt{\widetilde{R}'_C}\sqrt{R'_C}\right) h^{k+1} \, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\
&= \left(1 + R + \sqrt{R'_C} + \sqrt{\widetilde{R}'_C} + R\sqrt{\widetilde{R}'_C}\sqrt{R'_C}\right) \|I_{\boldsymbol{p}}\|_{D_h} \\
&+ \left(R^{1/2} + R^{1/2}\sqrt{\widetilde{R}'_C}\right) \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h, h^\perp} \\
&+ \left(1 + \sqrt{\widetilde{R}'_C} + R + R\sqrt{R'_C} + R\sqrt{\widetilde{R}'_C} + R\sqrt{\widetilde{R}'_C}\sqrt{R'_C}\right) h^{k+1} \, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \, .
\end{aligned}
$$

□

Corollary 6.1.2. *There exists a constant $C > 0$ independent of h, such that*

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_\Omega \leq C\, h^{k+1} \, .$$

Proof. By a direct application of Theorem 6.1.10 and bearing in mind that $\|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h, h^\perp} \lesssim h^{k+1}$ and $\|I_{\boldsymbol{p}}\| \lesssim h^{k+1}$. □

6.1.2 Error estimates for e^y

For the estimates of e^y we will proceed by following the steps done in [23] for the proof of Lemma 3.4 of that paper. For this purpose we use a dual problem, which is defined as follows: For any given $\eta \in L^2(\Omega)$, the solution $(\boldsymbol{\theta}, \phi)$ of the dual problem:

$$
\begin{aligned}
c\,\boldsymbol{\theta} + \nabla\phi &= 0 \quad \text{in } \Omega \,, && \text{(6.1.12a)} \\
\nabla \cdot \boldsymbol{\theta} &= \eta \quad \text{in } \Omega \,, && \text{(6.1.12b)} \\
\phi &= 0 \quad \text{on } \Gamma \,. && \text{(6.1.12c)}
\end{aligned}
$$

we will assume that ϕ is in $H^2(\Omega)$ and $\boldsymbol{\theta}$ is in $[H^1(\Omega)]^d$. This is true for example when Ω is a convex polygon or when Γ is of boundary $\mathcal{C}^2$. Thus, (6.1.12) satisfies

$$\|\phi\|_{H^2(\Omega)} + \|\boldsymbol{\theta}\|_{[H^1(\Omega)]^d} \leq C\, \|\eta\|_\Omega \,, \tag{6.1.13}$$

where C depends on the domain Ω. Then we can establish the following lemma.

Remark. $\Omega = \Omega_0$ can be chosen arbitrarily, so it can be assumed that all the necessary regularities associated with the domain Ω such as convexity and Lipschitz boundary are satisfied, so that (6.1.13) is satisfied.

Lemma 6.1.11. *We have that*

$$\|\varepsilon_h^y\|_{\mathcal{T}_h}^2 = (c\, I_{\boldsymbol{p}}, \boldsymbol{\Pi_Z \theta})_{\mathcal{T}_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,(\boldsymbol{\theta} - \boldsymbol{\Pi_Z \theta}))_{\mathcal{T}_h} + \mathbb{T}_{y,h}\,, \tag{6.1.14}$$

where

$$\mathbb{T}_{y,h} := \langle \widehat{\varepsilon_h^y}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}\cdot\boldsymbol{n}, \phi\rangle_{\Gamma_h}\,. \tag{6.1.15}$$

Proof. With $\eta = \varepsilon_h^y$ in (6.1.12), notice that

$$\begin{aligned}
\|\varepsilon_h^y\|_{\mathcal{T}_h}^2 &= (\varepsilon_h^y, \nabla\cdot\boldsymbol{\theta})_{\mathcal{T}_h} \\
&= (\varepsilon_h^y, \nabla\cdot\boldsymbol{\theta})_{\mathcal{T}_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,\boldsymbol{\theta})_{\mathcal{T}_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla\phi)_{\mathcal{T}_h} \\
&= (\varepsilon_h^y, \nabla\cdot\boldsymbol{\Pi_Z\theta})_{\mathcal{T}_h} + (\varepsilon_h^y, \nabla\cdot(\boldsymbol{\theta} - \boldsymbol{\Pi_Z\theta}))_{\mathcal{T}_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,\boldsymbol{\Pi_Z\theta})_{\mathcal{T}_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,(\boldsymbol{\theta} - \boldsymbol{\Pi_Z\theta}))_{\mathcal{T}_h} \\
&\quad - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla\Pi_W\phi)_{\mathcal{T}_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla(\phi - \Pi_W\phi))_{\mathcal{T}_h}\,.
\end{aligned}$$

Setting $\boldsymbol{v}_1 = \boldsymbol{\Pi_Z\theta}$ in (6.1.1a) and $w_1 = \Pi_W\phi$ in (6.1.1b), implies that

$$\begin{aligned}
\|\varepsilon_h^y\|_{\mathcal{T}_h}^2 &= (c\, I_{\boldsymbol{p}}, \boldsymbol{\Pi_Z\theta})_{\mathcal{T}_h} + \langle \widehat{\varepsilon_h^y}, \boldsymbol{\Pi_Z\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \widehat{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}\cdot\boldsymbol{n}, \Pi_W\phi\rangle_{\partial\mathcal{T}_h} + (\varepsilon_h^y, \nabla\cdot(\boldsymbol{\theta} - \boldsymbol{\Pi_Z\theta}))_{\mathcal{T}_h} \\
&\quad - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,(\boldsymbol{\theta} - \boldsymbol{\Pi_Z\theta}))_{\mathcal{T}_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla(\phi - \Pi_W\phi))_{\mathcal{T}_h} \\
&= (c\, I_{\boldsymbol{p}}, \boldsymbol{\Pi_Z\theta})_{\mathcal{T}_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,(\boldsymbol{\theta} - \boldsymbol{\Pi_Z\theta}))_{\mathcal{T}_h} + \mathbb{T}_{y,h}\,,
\end{aligned}$$

where, by the moment

$$\mathbb{T}_{y,h} = \langle \widehat{\varepsilon_h^y}, \boldsymbol{\Pi_Z\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \widehat{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}\cdot\boldsymbol{n}, \Pi_W\phi\rangle_{\partial\mathcal{T}_h} + (\varepsilon_h^y, \nabla\cdot(\boldsymbol{\theta} - \boldsymbol{\Pi_Z\theta}))_{\mathcal{T}_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla(\phi - \Pi_W\phi))_{\mathcal{T}_h}\,.$$

Now, integrating by parts and using the HDG-projection, particularly (6.0.1a) and (6.0.1b)

$$\begin{aligned}
\mathbb{T}_{y,h} &= \langle \widehat{\varepsilon^y_h}, \boldsymbol{\Pi_Z}\boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \widehat{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}\cdot\boldsymbol{n}, \Pi_W\phi\rangle_{\partial\mathcal{T}_h} - (\nabla\varepsilon^y_h, \boldsymbol{\theta}-\boldsymbol{\Pi_Z}\boldsymbol{\theta})_{\mathcal{T}_h} + \langle \varepsilon^y_h, (\boldsymbol{\theta}-\boldsymbol{\Pi_Z}\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&+ (\nabla\cdot\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h, \phi-\Pi_W\phi)_{\mathcal{T}_h} - \langle \boldsymbol{\varepsilon}^{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \phi-\Pi_W\phi\rangle_{\partial\mathcal{T}_h} \\
&= \langle \widehat{\varepsilon^y_h}, (\boldsymbol{\Pi_Z}\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \widehat{\varepsilon^y_h}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \widehat{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}\cdot\boldsymbol{n}, \Pi_W\phi-\phi\rangle_{\partial\mathcal{T}_h} - \langle \widehat{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}\cdot\boldsymbol{n}, \phi\rangle_{\partial\mathcal{T}_h} \\
&+ \langle \varepsilon^y_h, (\boldsymbol{\theta}-\boldsymbol{\Pi_Z}\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle \boldsymbol{\varepsilon}^{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \phi-\Pi_W\phi\rangle_{\partial\mathcal{T}_h} \\
&= \langle \widehat{\varepsilon^y_h}-\varepsilon^y_h, (\boldsymbol{\Pi_Z}\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle (\widehat{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}-\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h)\cdot\boldsymbol{n}, \Pi_W\phi-\phi\rangle_{\partial\mathcal{T}_h} + \langle \widehat{\varepsilon^y_h}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&- \langle \widehat{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}\cdot\boldsymbol{n}, \phi\rangle_{\partial\mathcal{T}_h} \\
&= \langle \widehat{\varepsilon^y_h}-\varepsilon^y_h, (\boldsymbol{\Pi_Z}\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} - \langle (\widehat{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}-\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h)\cdot\boldsymbol{n}, \Pi_W\phi-\phi\rangle_{\partial\mathcal{T}_h} + \langle \widehat{\varepsilon^y_h}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\Gamma_h} \\
&- \langle \widehat{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}\cdot\boldsymbol{n}, \phi\rangle_{\Gamma_h}.
\end{aligned}$$

The fact that $\langle \widehat{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}\cdot\boldsymbol{n}, \phi\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} = 0$ is by (6.1.1c) and $\langle \widehat{\varepsilon^y_h}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} = 0$ is by the following argument; as $\widehat{\varepsilon^y_h}$ is single valued in $\mathcal{E}_h$ and $\boldsymbol{\theta}\in H(\mathrm{div};\Omega)$, note that

$$\begin{aligned}
\langle \widehat{\varepsilon^y_h}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \sum_{K\in\mathcal{T}_h}\int_{\partial K}\widehat{\varepsilon^y_h}\,\boldsymbol{\theta}\cdot\boldsymbol{n} \\
&= \sum_{e\in\mathcal{E}^\circ_h}\int_e [\![\widehat{\varepsilon^y_h}]\!]_e\,\{\!\{\boldsymbol{\theta}\}\!\}_e + \{\!\{\widehat{\varepsilon^y_h}\}\!\}_e\,[\![\boldsymbol{\theta}]\!]_e + \sum_{e\in\mathcal{E}^\partial_h}\int_e\widehat{\varepsilon^y_h}\boldsymbol{\theta}\cdot\boldsymbol{n} \\
&= \langle \widehat{\varepsilon^y_h}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\Gamma_h}.
\end{aligned}$$

Then, by (6.1.1e) we have

$$\begin{aligned}
\mathbb{T}_{y,h} &= \langle \widehat{\varepsilon^y_h}-\varepsilon^y_h, (\boldsymbol{\Pi_Z}\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} + \langle \tau\,(\widehat{\varepsilon^y_h}-\varepsilon^y_h), \Pi_W\phi-\phi\rangle_{\partial\mathcal{T}_h} + \langle \widehat{\varepsilon^y_h}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}\cdot\boldsymbol{n}, \phi\rangle_{\Gamma_h} \\
&= \langle \widehat{\varepsilon^y_h}-\varepsilon^y_h, (\boldsymbol{\Pi_Z}\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n} + \tau\,(\Pi_W\phi-\phi)\rangle_{\partial\mathcal{T}_h} + \langle \widehat{\varepsilon^y_h}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}\cdot\boldsymbol{n}, \phi\rangle_{\Gamma_h} \\
&= \langle \widehat{\varepsilon^y_h}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}\cdot\boldsymbol{n}, \phi\rangle_{\Gamma_h},
\end{aligned}$$

where the fact that $\langle \widehat{\varepsilon^y_h}-\varepsilon^y_h, (\boldsymbol{\Pi_Z}\boldsymbol{\theta}-\boldsymbol{\theta})\cdot\boldsymbol{n} + \tau\,(\Pi_W\phi-\phi)\rangle_{\partial\mathcal{T}_h} = 0$ is due to (6.0.1c). □

Now, we will present a lemma that gives us a convenient identity for $\mathbb{T}_{y,h}$.

Lemma 6.1.12. *We have that* $\mathbb{T}_{y,h} := \sum_{i=1}^{7} \mathbb{T}_{y,h}^{i}$*, where*

$$\begin{aligned}
\mathbb{T}_{y,h}^1 &= -\langle c^{-1}\, l^{-1}\,(\varphi_1 - \varphi_1^h), \phi \,+\, c\, l\, \partial_{\boldsymbol{n}}\phi\rangle_{\Gamma_h}\,,\\
\mathbb{T}_{y,h}^2 &= \langle \varphi_1 - \varphi_1^h, \partial_{\boldsymbol{n}}\phi \,-\, P_M \partial_{\boldsymbol{n}}\phi\rangle_{\Gamma_h}\,,\\
\mathbb{T}_{y,h}^3 &= \langle \Lambda^{I_{\boldsymbol{p}}}, \phi\rangle_{\Gamma_h}\,,\\
\mathbb{T}_{y,h}^4 &= \langle I_{\boldsymbol{p}}\cdot\boldsymbol{n}, \phi - P_M\phi\rangle_{\Gamma_h}\,,\\
\mathbb{T}_{y,h}^5 &= -\langle P_M \tau\, I_y, \phi\rangle_{\Gamma_h}\\
\mathbb{T}_{y,h}^6 &= \langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}, \phi\rangle_{\Gamma_h}\,,\\
\mathbb{T}_{y,h}^7 &= -\langle \tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), P_M\phi\rangle_{\Gamma_h}\,.
\end{aligned}$$

Proof. In this proof we proceed analogously to [23, Lemma 5.4]. By (6.1.1e) and (6.1.6), let us notice that

$$\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n} \,=\, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n} \,+\, \tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}) \,=\, c^{-1}\, l^{-1}\,(\varphi_1 - \varphi_1^h) \,-\, \Lambda^{I_{\boldsymbol{p}}} \,-\, \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}} \,-\, I_{\boldsymbol{p}}\cdot\boldsymbol{n} \,+\, \tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\,,$$

and as we proved in (6.1.15), we have that $\mathbb{T}_{y,h} \,=\, \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\Gamma_h} \,-\, \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \phi\rangle_{\Gamma_h}$. Then

$$\mathbb{T}_{y,h} \,=\, \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\Gamma_h} \,-\, \langle c^{-1}\, l^{-1}\,(\varphi_1 - \varphi_1^h) \,-\, \Lambda^{I_{\boldsymbol{p}}} \,-\, \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}} \,-\, I_{\boldsymbol{p}}\cdot\boldsymbol{n} \,+\, \tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \phi\rangle_{\Gamma_h}\,.$$

Hence, by (6.1.1d) and since $\boldsymbol{\theta} \,=\, -\nabla\phi$ in the dual problem, it follows

$$\mathbb{T}_{y,h} \,=\, -\langle \varphi_1 - \varphi_1^h, P_M\partial_{\boldsymbol{n}}\phi\rangle_{\Gamma_h} \,-\, \langle c^{-1}\, l^{-1}\,(\varphi_1 - \varphi_1^h) \,-\, \Lambda^{I_{\boldsymbol{p}}} - \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}} \,-\, I_{\boldsymbol{p}}\cdot\boldsymbol{n} \,+\, \tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \phi\rangle_{\Gamma_h}\,.$$

In turn, by (6.0.1c), we note that

$$\begin{aligned}
\langle I_{\boldsymbol{p}}\cdot\boldsymbol{n}, \phi\rangle_{\Gamma_h} &= \langle I_{\boldsymbol{p}}\cdot\boldsymbol{n}, \phi - P_M\phi\rangle_{\Gamma_h} \,+\, \langle I_{\boldsymbol{p}}\cdot\boldsymbol{n}, P_M\phi\rangle_{\Gamma_h}\\
&= \langle I_{\boldsymbol{p}}\cdot\boldsymbol{n}, \phi - P_M\phi\rangle_{\Gamma_h} \,-\, \langle \tau\, I_y, P_M\phi\rangle_{\Gamma_h}\\
&= \langle I_{\boldsymbol{p}}\cdot\boldsymbol{n}, \phi - P_M\phi\rangle_{\Gamma_h} \,-\, \langle \tau\, P_M I_y, \phi\rangle_{\Gamma_h}\,.
\end{aligned}$$

Therefore, the identity is obtained after a simple rearrangement of terms. □

To continue with our analysis we need the following lemma, that will help us later

Lemma 6.1.13. *If the fourth equation of the second row of the assumptions* (4.6.1) *holds, and if we suppose that* (6.1.13) *holds, then we have*

$$\begin{aligned}
\|\phi - P_M\phi\|_{\Gamma_h,(h^\perp)^{-1}} &\leq C\, h\, \|\eta\|_\Omega\,, \\
\|\partial_{\boldsymbol{n}}\phi - P_M\partial_{\boldsymbol{n}}\phi\|_{\Gamma_h,l} &\leq C\, R\, h\, \|\eta\|_\Omega\,, \\
\|\phi + c\, l\, \partial_{\boldsymbol{n}}\phi\|_{\Gamma_h,l^{-3}} &\leq C\, \|\eta\|_\Omega\,, \\
\|\phi\|_{\Gamma_h,l^{-2}} &\leq C\, \|\eta\|_\Omega\,.
\end{aligned}$$

Proof. See [23, Lemma 5.5]. □

Now, we present the following lemma that gives us another tool we need for error estimates.

Lemma 6.1.14. *Exists $C > 0$ independent of h, such that*

$$\begin{aligned}
|\mathbb{T}_{y,h}| \leq\; & C\,(R + R_\tau + R_C^2\, h^{-1/2})\, h\, \left\|\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\right\|\, \|\eta\|_\Omega \\
& + C\,(R^{3/2}\, h^{1/2}\, \|\partial_{\boldsymbol{n}}(\boldsymbol{I_p}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2} + h\, \|\boldsymbol{I_p}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp})\, \|\eta\|_\Omega \\
& + C\, R_\tau\, h^{1/2}\, \|I_y\|_{\Gamma_h,h^\perp}\, \|\eta\|_\Omega\,.
\end{aligned}$$

where $\left\|\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\right\|$ is defined as in Lemma 6.1.5.

Proof. By Lemma 6.1.12 we can rewrite $\mathbb{T}_{y,h} = \sum_{i=1}^{7} \mathbb{T}_{y,h}^i$, and applying Cauchy-Schwarz inequality

$$\begin{aligned}
|\mathbb{T}_{y,h}^1| &\leq \left\|\varphi_1 - \varphi_1^h\right\|_{\Gamma_h,l}\, \left\|c^{-1}\, l^{-1}\, \phi + c\, \partial_{\boldsymbol{n}}\phi\right\|_{\Gamma_h,l^{-1}}\,, \\
|\mathbb{T}_{y,h}^2| &\leq \left\|\varphi_1 - \varphi_1^h\right\|_{\Gamma_h,l^{-1}}\, \|\partial_{\boldsymbol{n}}\phi - P_M\partial_{\boldsymbol{n}}\phi\|_{\Gamma_h,l}\,, \\
|\mathbb{T}_{y,h}^3| &\leq \left\|\Lambda^{\boldsymbol{I_p}}\right\|_{\Gamma_h,l^2}\, \|\phi\|_{\Gamma_h,l^{-2}}\,, \\
|\mathbb{T}_{y,h}^4| &\leq \|\boldsymbol{I_p}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp}\, \|\phi - P_M\phi\|_{\Gamma_h,(h^\perp)^{-1}}\,, \\
|\mathbb{T}_{y,h}^5| &\leq \|P_M\,\tau\, I_y\|_{\Gamma_h,l^2}\, \|\phi\|_{\Gamma_h,l^{-2}}\,,
\end{aligned}$$

$$|\mathbb{T}^6_{y,h}| \leq \left\|\Lambda^{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}\right\|_{\Gamma_h,l^2} \|\phi\|_{\Gamma_h,l^{-2}} ,$$
$$|\mathbb{T}^7_{y,h}| \leq \left\|\varepsilon^y_h - \varepsilon^{\widehat{y}}_h\right\|_{\Gamma_h,\tau^2 l^2} \|\phi\|_{\Gamma_h,l^{-2}} .$$

Then, by the previous Lemma 6.1.13

$$|\mathbb{T}^1_{y,h}| \leq C \left\|\varphi_1 - \varphi^h_1\right\|_{\Gamma_h,l} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^2_{y,h}| \leq C\, R\, h \left\|\varphi_1 - \varphi^h_1\right\|_{\Gamma_h,l^{-1}} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^3_{y,h}| \leq C \left\|\Lambda^{I_{\boldsymbol{p}}}\right\|_{\Gamma_h,l^2} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^4_{y,h}| \leq C\, h \left\|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\right\|_{\Gamma_h,h^\perp} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^5_{y,h}| \leq C \left\|P_M I_y\right\|_{\Gamma_h,\tau^2 l^2} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^6_{y,h}| \leq C \left\|\Lambda^{\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h}\right\|_{\Gamma_h,l^2} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^7_{y,h}| \leq C \left\|\varepsilon^y_h - \varepsilon^{\widehat{y}}_h\right\|_{\Gamma_h,\tau^2 l^2} \|\eta\|_\Omega .$$

On the other hand, let e be a edge or face of Γ_h, then

$$r_e = \frac{H^\perp_e}{h^\perp_e} \geq \frac{l(\boldsymbol{x})}{h^\perp_e} \Longrightarrow l(\boldsymbol{x}) \leq h^\perp_e\, r_e ,$$

by the above argument and by Lemma 6.1.4, we get

$$|\mathbb{T}^1_{y,h}| \leq C \max_{e\in\mathcal{E}^\partial_h} r_e\, h \left\|\varphi_1 - \varphi^h_1\right\|_{\Gamma_h,l^{-1}} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^2_{y,h}| \leq C\, R\, h \left\|\varphi_1 - \varphi^h_1\right\|_{\Gamma_h,l^{-1}} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^3_{y,h}| \leq C \max_{e\in\mathcal{E}^\partial_h} r_e^{3/2}\, h^{1/2} \left\|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}} \cdot \boldsymbol{n})\right\|_{D^c_h,(h^\perp)^2} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^4_{y,h}| \leq C\, h \left\|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\right\|_{\Gamma_h,h^\perp} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^5_{y,h}| \leq C \max_{e\in\mathcal{E}^\partial_h} \tau_e\, r_e\, h^{1/2} \left\|I_y\right\|_{\Gamma_h,h^\perp} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^6_{y,h}| \leq C \max_{e\in\mathcal{E}^\partial_h} r_e^2\, C^e_{ext}\, C^e_{inv}\, h^{1/2} \left\|\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h\right\|_{\mathcal{T}_h} \|\eta\|_\Omega ,$$
$$|\mathbb{T}^7_{y,h}| \leq C \max_{e\in\mathcal{E}^\partial_h} \tau_e^{1/2}\, r_e\, h \left\|\varepsilon^y_h - \varepsilon^{\widehat{y}}_h\right\|_{\Gamma_h,\tau} \|\eta\|_\Omega .$$

Then, from the definition of $\left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right|\!\right|$ we deduce

$$
\begin{aligned}
|\mathbb{T}_{y,h}^1| &\leq C \max_{e\in\mathcal{E}_h^\partial} r_e\, h \left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right|\!\right| \, \|\eta\|_\Omega \;, \\
|\mathbb{T}_{y,h}^2| &\leq C\, R\, h \left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right|\!\right| \, \|\eta\|_\Omega \;, \\
|\mathbb{T}_{y,h}^3| &\leq C \max_{e\in\mathcal{E}_h^\partial} r_e^{3/2}\, h^{1/2} \, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2} \, \|\eta\|_\Omega \;, \\
|\mathbb{T}_{y,h}^4| &\leq C\, h \, \|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} \, \|\eta\|_\Omega \;, \\
|\mathbb{T}_{y,h}^5| &\leq C \max_{e\in\mathcal{E}_h^\partial} \tau_e\, r_e\, h^{1/2} \, \|I_y\|_{\Gamma_h,h^\perp} \, \|\eta\|_\Omega \;, \\
|\mathbb{T}_{y,h}^6| &\leq C \max_{e\in\mathcal{E}_h^\partial} r_e^2\, C_{ext}^e\, C_{inv}^e\, h^{1/2} \left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right|\!\right| \, \|\eta\|_\Omega \;, \\
|\mathbb{T}_{y,h}^7| &\leq C \max_{e\in\mathcal{E}_h^\partial} \tau_e^{1/2}\, r_e\, h \left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right|\!\right| \, \|\eta\|_\Omega \;.
\end{aligned}
$$

Furthermore, we get

$$
\begin{aligned}
|\mathbb{T}_{y,h}| &\leq C \Big(R\, h \left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right|\!\right| \\
&\quad + R^{3/2}\, h^{1/2} \, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2} \\
&\quad + h \, \|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} \\
&\quad + R_\tau\, h^{1/2} \, \|I_y\|_{\Gamma_h,h^\perp} \\
&\quad + R_C^2\, h^{1/2} \left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right|\!\right| \\
&\quad + R_\tau\, h \left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right|\!\right| \Big) \, \|\eta\|_\Omega \\
&= C \Big(R\, h + R_C^2\, h^{1/2} + R_\tau\, h \Big) \left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right|\!\right| \, \|\eta\|_\Omega \\
&\quad + C \Big(R^{3/2}\, h^{1/2} \, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2} + h \, \|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} \Big) \, \|\eta\|_\Omega \\
&\quad + C\, R_\tau\, h^{1/2} \, \|I_y\|_{\Gamma_h,h^\perp} \, \|\eta\|_\Omega \\
&= C \Big(R + R_\tau^2 + R_C^2\, h^{-1/2} \Big)\, h \left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right|\!\right| \, \|\eta\|_\Omega \\
&\quad + C \Big(R^{3/2}\, h^{1/2} \, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2} + h \, \|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} \Big) \, \|\eta\|_\Omega \\
&\quad + C\, R_\tau\, h^{1/2} \, \|I_y\|_{\Gamma,h^\perp} \, \|\eta\|_\Omega \;.
\end{aligned}
$$

□

The we obtain the following estimate for y in Ω.

Theorem 6.1.15. *We have that*

$$\|y - y_h\|_{\Omega} \leq \widehat{H}_1(R,h)\, \|I_{\boldsymbol{p}}\|_{D_h} + \widehat{H}_2(R,h)\, \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h, h^{\perp}} + \widehat{H}_3(R,h)\, \|I_y\|_{D_h} + \widehat{H}_4(R,h)\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}\,,$$

where

$$\begin{aligned}
\widehat{H}_1(R,h) &:= \Big(h + R\,h + R_\tau\,h + R_C^2\,h^{1/2} + R\sqrt{R'_C}\,h + R^2\sqrt{R'_C}\,h + R\,R_\tau\sqrt{R'_C}\,h \\
&\quad + R\,R_C^2\sqrt{R'_C}\,h^{1/2} + R^{3/2}\sqrt{R'_C}\,h^{1/2} + \sqrt{\widetilde{R}'_C}\,h + R\sqrt{\widetilde{R}_C}\sqrt{R'_C}\,h\Big)\,, \\
\widehat{H}_2(R,h) &:= \Big(R^{1/2}\,h + R^{3/2}\,h + R^{1/2}\,R_\tau\,h + R^{1/2}\,R_C^2\,h^{1/2} + R^{1/2}\sqrt{\widetilde{R}'_C}\,h\Big)\,, \\
\widehat{H}_3(R,h) &:= \Big(1 + R_\tau\,h^{1/2}\Big)\,, \\
\widehat{H}_4(R,h) &:= \Big(h + R\,h + R^2\,h + R\,R_\tau\,h + R\,R_C^2\,h^{1/2} + R^{3/2}\,h^{1/2} + R\sqrt{R'_C}\,h + R^2\sqrt{R'_C}\,h \\
&\quad + R\,R_\tau\sqrt{R'_C}\,h + R\,R_C^2\sqrt{R'_C}\,h^{1/2} + R^{3/2}\sqrt{R'_C}\,h^{1/2} + \sqrt{\widetilde{R}'_C}\,h + R\sqrt{\widetilde{R}'_C}\,h \\
&\quad + R\sqrt{\widetilde{R}'_C}\sqrt{R'_C}\,h\Big)\,.
\end{aligned}$$

Moreover, we can note that there exists a constant $C > 0$ such that $\widehat{H}_i(R,h) \leq C\,h$ for $i = 1,\, 2,\, 4$ and we also have that $\widehat{H}_3(R,h) \leq C$.

Proof. Let us notice that by (6.1.14) and by (6.0.1a), with $\boldsymbol{\theta}_h|_K \in [\mathbb{P}_{k-1}(K)]^d$, for all $K \in \mathcal{T}_h$

$$\begin{aligned}
\|\varepsilon_h^y\|_{D_h}^2 &= (c\,I_{\boldsymbol{p}}, \boldsymbol{\Pi_Z\theta})_{D_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,(\boldsymbol{\theta} - \boldsymbol{\Pi_Z\theta}))_{D_h} + \mathbb{T}_{y,h} \\
&= (c\,I_{\boldsymbol{p}}, \boldsymbol{\Pi_Z\theta} - \boldsymbol{\theta})_{D_h} + (c\,I_{\boldsymbol{p}}, \boldsymbol{\theta})_{D_h} - (\boldsymbol{p} - \boldsymbol{p}_h, c(\boldsymbol{\theta} - \boldsymbol{\Pi_Z\theta}))_{D_h} \\
&\quad - (\boldsymbol{\Pi_Z p} - \boldsymbol{p}, c(\boldsymbol{\theta} - \boldsymbol{\Pi_Z\theta}))_{D_h} + \mathbb{T}_{y,h} \\
&= (c\,I_{\boldsymbol{p}}, \boldsymbol{\theta})_{D_h} - (\boldsymbol{p} - \boldsymbol{p}_h, c(\boldsymbol{\theta} - \Pi_{\boldsymbol{Z}}\boldsymbol{\theta}))_{D_h} + \mathbb{T}_{y,h} \\
&= (c\,I_{\boldsymbol{p}}, \boldsymbol{\theta} - \boldsymbol{\theta}_h)_{D_h} - (\boldsymbol{p} - \boldsymbol{p}_h, c\,(\boldsymbol{\theta} - \Pi_{\boldsymbol{Z}}\boldsymbol{\theta}))_{D_h} + \mathbb{T}_{y,h}\,.
\end{aligned}$$

Then, by Cauchy-Schwarz inequality

$$\|\varepsilon_h^y\|_{D_h}^2 \,\leq\, \|c\, I_{\boldsymbol{p}}\|_{D_h}\, \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{D_h} \,+\, \|\boldsymbol{p} - \boldsymbol{p}_h\|_{D_h}\, \|c\,(\boldsymbol{\theta} - \boldsymbol{\Pi_Z}\boldsymbol{\theta})\|_{D_h} \,+\, |\mathbb{T}_{y,h}|\,.$$

Thus, using (6.0.1) and the elliptic regularity inequality (6.1.13), we get that

$$\begin{aligned}\|\varepsilon_h^y\|_{\mathcal{D}_h}^2 &\leq C\, h \left(\|I_{\boldsymbol{p}}\|_{D_h} \,+\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{D_h} \right) \|\eta\|_{D_h} \,+\, |\mathbb{T}_{y,h}| \\ &\leq C\, h \left(\|I_{\boldsymbol{p}}\| \,+\, \left\|\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\right\| \right) \|\eta\|_{\Omega} + |\mathbb{T}_{y,h}|\,.\end{aligned}$$

Then, by Lemma 6.1.14

$$\begin{aligned}\|\varepsilon_h^y\|_{D_h}^2 &\leq C\, h \left(\|I_{\boldsymbol{p}}\|_{D_h} \,+\, \left\|\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\right\| \right) \|\eta\|_{\Omega} \\ &\quad + C \left(R \,+\, R_\tau \,+ R_C^2\, h^{-1/2} \right) h \left\|\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\right\|\, \|\eta\|_{\Omega} \\ &\quad + C \left(R^{3/2}\, h^{1/2}\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}} \cdot \boldsymbol{n})\|_{D_h^c,(h^\perp)^2} \,+\, h\, \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma,(h^\perp)} \right) \|\eta\|_{\Omega} \\ &\quad + C\, R_\tau\, h^{1/2}\, \|I_y\|_{\Gamma,(h^\perp)}\, \|\eta\|_{\Omega}\,.\end{aligned}$$

Now, setting $\eta = \varepsilon_h^y$ in the dual problem, we get

$$\begin{aligned}\|\varepsilon_h^y\|_{D_h} &\leq C\, h \left(\|I_{\boldsymbol{p}}\|_{D_h} \,+\, \left\|\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\right\| \right) \\ &\quad + C \left(R \,+\, R_\tau \,+\, R_C^2\, h^{-1/2} \right) h \left\|\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\right\| \\ &\quad + C \left(R^{3/2}\, h^{1/2}\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}} \cdot \boldsymbol{n})\|_{D_h^c,(h^\perp)^2} \,+\, h\, \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,h^\perp} \right) \\ &\quad + C\, R_\tau\, h^{1/2}\, \|I_y\|_{\Gamma_h,h^\perp}\,.\end{aligned}$$

By (6.1.8) and Lemma 6.1.7, we notice that

$$\begin{aligned}\|\varepsilon_h^y\|_{D_h} &\lesssim h\, \|I_{\boldsymbol{p}}\|_{D_h} + h \left(\|I_{\boldsymbol{p}}\|_{D_h} \,+\, R\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}} \cdot \boldsymbol{n})\|_{D_h^c,(h^\perp)^2} \,+\, R^{1/2}\, \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,(h^\perp)} \right) \\ &\quad + \left(R \,+\, R_\tau \,+\, R_C^2\, h^{-1/2} \right) h \left(\|I_{\boldsymbol{p}}\|_{D_h} \,+\, R\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}} \cdot \boldsymbol{n})\|_{D_h^c,(h^\perp)^2} \,+\, R^{1/2}\, \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,(h^\perp)} \right) \\ &\quad + R^{3/2}\, h^{1/2}\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}} \cdot \boldsymbol{n})\|_{D_h^c,(h^\perp)} \,+\, R_\tau\, h^{1/2}\, \|I_y\|_{\Gamma,(h^\perp)}\end{aligned}$$

$$
\begin{aligned}
&= \Big(h + R\,h + R_\tau\,h + R_C^2\,h^{1/2}\Big)\,\|I_{\boldsymbol{p}}\|_{D_h} \\
&+ \Big(R\,h + R^2\,h + R\,R_\tau\,h + R\,R_C^2\,h^{1/2} + R^{3/2}\,h^{1/2}\Big)\,\|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)} \\
&+ \Big(R^{1/2}\,h + R^{3/2}\,h + R^{1/2}\,R_\tau\,h + R^{1/2}\,R_C^2\,h^{1/2}\Big)\,\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,(h^\perp)} \\
&+ R_\tau\,h^{1/2}\,\|I_y\|_{\Gamma_h,(h^\perp)} \\
&\lesssim \Big(h + R\,h + R_\tau\,h + R_C^2\,h^{1/2} + R\sqrt{R_C'}\,h + R^2\sqrt{R_C'}\,h + R\,R_\tau\sqrt{R_C'}\,h \\
&+ R\,R_C^2\sqrt{R_C'}\,h^{1/2} + R^{3/2}\sqrt{R_C'}\,h^{1/2}\Big)\,\|I_{\boldsymbol{p}}\|_{D_h} \\
&+ \Big(R^{1/2}\,h + R^{3/2}\,h + R^{1/2}\,R_\tau\,h + R^{1/2}\,R_C^2\,h^{1/2}\Big)\,\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,(h^\perp)} \\
&+ R_\tau\,h^{1/2}\,\|I_y\|_{\Gamma_h,(h^\perp)} \\
&+ \Big(R\,h + R^2\,h + R\,R_\tau\,h + R\,R_C^2\,h^{1/2} + R^{3/2}\,h^{1/2} + R\sqrt{R_C'}\,h + R^2\sqrt{R_C'}\,h \\
&+ R\,R_\tau\sqrt{R_C'}\,h + R\,R_C^2\sqrt{R_C'}\,h^{1/2} + R^{3/2}\sqrt{R_C'}\,h^{1/2}\Big)\,h^{k+1}\,|\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}\,.
\end{aligned}
$$

Then, using discrete trace inequality 6.0.2 with $\|I_y\|_{\Gamma_h,h^\perp}$ we obtain

$$
\begin{aligned}
\|\varepsilon_h^y\|_{D_h} &\lesssim \Big(h + R\,h + R_\tau\,h + R_C^2\,h^{1/2} + R\sqrt{R_C'}\,h + R^2\sqrt{R_C'}\,h + R\,R_\tau\sqrt{R_C'}\,h \\
&+ R\,R_C^2\sqrt{R_C'}\,h^{1/2} + R^{3/2}\sqrt{R_C'}\,h^{1/2}\Big)\,\|I_{\boldsymbol{p}}\|_{D_h} \\
&+ \Big(R^{1/2}\,h + R^{3/2}\,h + R^{1/2}\,R_\tau\,h + R^{1/2}\,R_C^2\,h^{1/2}\Big)\,\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,(h^\perp)} \\
&+ R_\tau\,h^{1/2}\,\|I_y\|_{D_h} \\
&+ \Big(R\,h + R^2\,h + R\,R_\tau\,h + R\,R_C^2\,h^{1/2} + R^{3/2}\,h^{1/2} + R\sqrt{R_C'}\,h + R^2\sqrt{R_C'}\,h \\
&+ R\,R_\tau\sqrt{R_C'}\,h + R\,R_C^2\sqrt{R_C'}\,h^{1/2} + R^{3/2}\sqrt{R_C'}\,h^{1/2}\Big)\,h^{k+1}\,|\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}\,.
\end{aligned}
$$

It follows from the above that

$$
\begin{aligned}
\|y - y_h\|_{D_h} &\le \|I_y\|_{D_h} + \|\varepsilon_h^y\|_{D_h} \\
&\lesssim \Big(h + R\,h + R_\tau\,h + R_C^2\,h^{1/2} + R\sqrt{R_C'}\,h + R^2\sqrt{R_C'}\,h + R\,R_\tau\sqrt{R_C'}\,h \\
&+ R\,R_C^2\sqrt{R_C'}\,h^{1/2} + R^{3/2}\sqrt{R_C'}\,h^{1/2}\Big)\,\|I_{\boldsymbol{p}}\|_{D_h}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(R^{1/2}\, h + R^{3/2}\, h + R^{1/2}\, R_\tau\, h + R^{1/2}\, R_C^2\, h^{1/2}\right) \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,(h^\perp)} \\
&+ \left(1 + R_\tau\, h^{1/2}\right) \|I_y\|_{D_h} \\
&+ \left(R\, h + R^2\, h + R\, R_\tau\, h + R\, R_C^2\, h^{1/2} + R^{3/2}\, h^{1/2} + R\sqrt{R_C'}\, h + R^2 \sqrt{R_C'}\, h \right. \\
&\left. + R\, R_\tau \sqrt{R_C'}\, h + R\, R_C^2 \sqrt{R_C'}\, h^{1/2} + R^{3/2} \sqrt{R_C'}\, h^{1/2}\right) h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}\,.
\end{aligned}
$$

Finally, by Lemmas 6.1.7, 6.1.9 and 6.1.5 we have that

$$
\begin{aligned}
\|y - y_h\|_\Omega &\leq \|y - y_h\|_{D_h} + \|y - y_h\|_{D_h^c} \\
&\lesssim \left(h + R\, h + R_\tau\, h + R_C^2\, h^{1/2} + R\sqrt{R_C'}\, h + R^2 \sqrt{R_C'}\, h + R\, R_\tau \sqrt{R_C'}\, h \right. \\
&\left. + R\, R_C^2 \sqrt{R_C'}\, h^{1/2} + R^{3/2} \sqrt{R_C'}\, h^{1/2}\right) \|I_{\boldsymbol{p}}\|_{D_h} \\
&+ \left(R^{1/2}\, h + R^{3/2}\, h + R^{1/2}\, R_\tau\, h + R^{1/2}\, R_C^2\, h^{1/2}\right) \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,(h^\perp)} \\
&+ \left(1 + R_\tau\, h^{1/2}\right) \|I_y\|_{D_h} \\
&+ \left(R\, h + R^2\, h + R\, R_\tau\, h + R\, R_C^2\, h^{1/2} + R^{3/2}\, h^{1/2} + R\sqrt{R_C'}\, h + R^2 \sqrt{R_C'}\, h \right. \\
&\left. + R\, R_\tau \sqrt{R_C'}\, h + R\, R_C^2 \sqrt{R_C'}\, h^{1/2} + R^{3/2} \sqrt{R_C'}\, h^{1/2}\right) h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\
&+ \|y - y_h\|_{D_h^c} \\
&\lesssim \left(h + R\, h + R_\tau\, h + R_C^2\, h^{1/2} + R\sqrt{R_C'}\, h + R^2 \sqrt{R_C'}\, h + R\, R_\tau \sqrt{R_C'}\, h \right. \\
&\left. + R\, R_C^2 \sqrt{R_C'}\, h^{1/2} + R^{3/2} \sqrt{R_C'}\, h^{1/2}\right) \|I_{\boldsymbol{p}}\|_{D_h} \\
&+ \left(R^{1/2}\, h + R^{3/2}\, h + R^{1/2}\, R_\tau\, h + R^{1/2}\, R_C^2\, h^{1/2}\right) \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,(h^\perp)} \\
&+ \left(1 + R_\tau h^{1/2}\right) \|I_y\|_{D_h} \\
&+ \left(R\, h + R^2\, h + R\, R_\tau\, h + R\, R_C^2\, h^{1/2} + R^{3/2}\, h^{1/2} + R\sqrt{R_C'}\, h + R^2 \sqrt{R_C'}\, h \right. \\
&\left. + R\, R_\tau \sqrt{R_C'}\, h + R\, R_C^2 \sqrt{R_C'}\, h^{1/2} + R^{3/2} \sqrt{R_C'}\, h^{1/2}\right) h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\
&+ h\, \|I_{\boldsymbol{p}}\|_{D_h^c} + \sqrt{\widetilde{R}_C'}\, h\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{D_h} \\
&\lesssim \left(h + R\, h + R_\tau\, h + R_C^2\, h^{1/2} + R\sqrt{R_C'}\, h + R^2 \sqrt{R_C'}\, h + R\, R_\tau \sqrt{R_C'}\, h \right. \\
&\left. + R\, R_C^2 \sqrt{R_C'}\, h^{1/2} + R^{3/2} \sqrt{R_C'}\, h^{1/2}\right) \|I_{\boldsymbol{p}}\|_{D_h}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(R^{1/2}\, h \,+\, R^{3/2}\, h \,+\, R^{1/2}\, R_\tau\, h \,+\, R^{1/2}\, R_C^2\, h^{1/2}\right) \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,(h^\perp)} \\
&+ \left(1 \,+\, R_\tau\, h^{1/2}\right) \|I_y\|_{D_h} \\
&+ \Big(R\, h \,+\, R^2\, h \,+\, R\, R_\tau\, h \,+\, R\, R_C^2\, h^{1/2} \,+\, R^{3/2}\, h^{1/2} \,+\, R\sqrt{R_C'}\, h \,+\, R^2\sqrt{R_C'}\, h \\
&+ R\, R_\tau \sqrt{R_C'}\, h \,+\, R\, R_C^2 \sqrt{R_C'}\, h^{1/2} \,+\, R^{3/2}\sqrt{R_C'}\, h^{1/2}\Big)\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\
&+ \sqrt{\widetilde{R}_C'}\, h\, \|I_{\boldsymbol{p}}\|_{D_h} + h\,(1 \,+\, \sqrt{\widetilde{R}_C'})\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \,+\, \sqrt{\widetilde{R}_C'}\, h\, \left\|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y - \varepsilon_h^{\widehat{y}}, \varphi_1 - \varphi_1^h)\right|\!\right\| \\
&\lesssim \Big(h \,+\, R\, h \,+\, R_\tau\, h \,+\, R_C^2\, h^{1/2} \,+\, R\sqrt{R_C'}\, h \,+\, R^2\sqrt{R_C'}\, h \,+\, R\, R_\tau\sqrt{R_C'}\, h \\
&+ R\, R_C^2\sqrt{R_C'}\, h^{1/2} \,+\, R^{3/2}\sqrt{R_C'}\, h^{1/2}\Big)\, \|I_{\boldsymbol{p}}\|_{D_h} \\
&+ \left(R^{1/2}\, h \,+\, R^{3/2}\, h \,+\, R^{1/2}\, R_\tau\, h \,+\, R^{1/2}\, R_C^2\, h^{1/2}\right) \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,(h^\perp)} \\
&+ \left(1 \,+\, R_\tau\, h^{1/2}\right) \|I_y\|_{D_h} \\
&+ \Big(R\, h \,+\, R^2\, h \,+\, R\, R_\tau\, h \,+\, R\, R_C^2\, h^{1/2} \,+\, R^{3/2}\, h^{1/2} \,+\, R\sqrt{R_C'}\, h \,+\, R^2\sqrt{R_C'}\, h \\
&+ R\, R_\tau\sqrt{R_C'}\, h \,+\, R\, R_C^2\sqrt{R_C'}\, h^{1/2} \,+\, R^{3/2}\sqrt{R_C'}\, h^{1/2}\Big)\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\
&+ \sqrt{\widetilde{R}_C'}\, h\, \|I_{\boldsymbol{p}}\|_{D_h} \,+\, \left(h + \sqrt{\widetilde{R}_C'}\, h\right) h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \,+\, R\sqrt{\widetilde{R}_C'}\, h\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{p}} \cdot \boldsymbol{n})\|_{D_h^c,(h^\perp)^2} \\
&+ R^{1/2}\sqrt{\widetilde{R}_C'}\, h\, \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,h^\perp} \\
&\lesssim \Big(h \,+\, R\, h \,+\, R_\tau\, h \,+\, R_C^2\, h^{1/2} \,+\, R\sqrt{R_C'}\, h \,+\, R^2\sqrt{R_C'}\, h \,+\, R\, R_\tau\sqrt{R_C'}\, h \\
&+ R\, R_C^2\sqrt{R_C'}\, h^{1/2} \,+\, R^{3/2}\sqrt{R_C'}\, h^{1/2}\Big)\, \|I_{\boldsymbol{p}}\|_{D_h} \\
&+ \left(R^{1/2}\, h \,+\, R^{3/2}\, h + R^{1/2}\, R_\tau\, h \,+\, R^{1/2}\, R_C^2\, h^{1/2}\right) \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,(h^\perp)} \\
&+ \left(1 \,+\, R_\tau h^{1/2}\right) \|I_y\|_{D_h} \\
&+ \Big(R\, h \,+\, R^2\, h \,+\, R\, R_\tau\, h \,+\, R\, R_C^2\, h^{1/2} \,+\, R^{3/2}\, h^{1/2} \,+\, R\sqrt{R_C'}\, h \,+\, R^2\sqrt{R_C'}\, h \\
&+ R\, R_\tau\sqrt{R_C'}\, h \,+\, R\, R_C^2\sqrt{R_C'}\, h^{1/2} \,+\, R^{3/2}\sqrt{R_C'}\, h^{1/2}\Big)\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\
&+ \left(\sqrt{\widetilde{R}_C'}\, h \,+\, R\sqrt{\widetilde{R}_C}\sqrt{R_C'}\, h\right) \|I_{\boldsymbol{p}}\|_{D_h} \\
&+ \left(h \,+\, \sqrt{\widetilde{R}_C'}\, h \,+\, R\sqrt{\widetilde{R}_C'}\, h \,+\, R\sqrt{\widetilde{R}_C'}\sqrt{R_C'}\, h\right) h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{h}^{k+1}(\Omega)} \\
&+ R^{1/2}\sqrt{\widetilde{R}_C'}\, h\, \|I_{\boldsymbol{p}} \cdot \boldsymbol{n}\|_{\Gamma_h,h^\perp} \\
&\lesssim \Big(h \,+\, R\, h \,+\, R_\tau\, h \,+\, R_C^2\, h^{1/2} \,+\, R\sqrt{R_C'}\, h \,+\, R^2\sqrt{R_C'}\, h \,+\, R\, R_\tau\sqrt{R_C'}\, h
\end{aligned}
$$

$$+ R\,R_C^2\sqrt{R_C'}\,h^{1/2} + R^{3/2}\sqrt{R_C'}\,h^{1/2} + \sqrt{\widetilde{R}_C'}\,h + R\sqrt{\widetilde{R}_C}\sqrt{R_C'}h\Big)\,\|I_{\boldsymbol{p}}\|_{D_h}$$
$$+\Big(R^{1/2}\,h + R^{3/2}\,h + R^{1/2}\,R_\tau\,h + R^{1/2}\,R_C^2\,h^{1/2} + R^{1/2}\sqrt{\widetilde{R}_C'}\,h\Big)\,\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,(h^\perp)}$$
$$+\Big(1 + R_\tau\,h^{1/2}\Big)\,\|I_y\|_{D_h}$$
$$+\Big(h + R\,h + R^2\,h + R\,R_\tau\,h + R\,R_C^2\,h^{1/2} + R^{3/2}\,h^{1/2} + R\sqrt{R_C'}\,h + R^2\sqrt{R_C'}\,h$$
$$+ R\,R_\tau\sqrt{R_C'}\,h + R\,R_C^2\sqrt{R_C'}\,h^{1/2} + R^{3/2}\sqrt{R_C'}\,h^{1/2} + \sqrt{\widetilde{R}_C'}\,h + R\sqrt{\widetilde{R}_C'}\,h$$
$$+ R\sqrt{\widetilde{R}_C'}\sqrt{R_C'}\,h\Big)\,h^{k+1}\,|\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)}\,.$$

□

Corollary 6.1.3. *There exists a constant $C > 0$ independent of h, such that*

$$\|y - y_h\|_\Omega \le C\,h^{k+1}\,.$$

Proof. By a direct application of Theorem 6.1.15 and bearing in mind that $\|I_{\boldsymbol{p}}\|_{D_h} \lesssim h^{k+1}$, $\|I_{\boldsymbol{p}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} \lesssim h^{k+1}$ and $\|I_y\|_{D_h} \lesssim h^{k+1}$. □

6.2 Error estimates for the adjoint equation

In this section we proceed analogously to the case of the analysis of the state equation. But, here we use the estimates of the state equation in the analysis of this case.

6.2.1 Error estimates for $\boldsymbol{e}^{\boldsymbol{r}}$

The projection of the errors of the adjoint equation satisfies the following equations,

Lemma 6.2.1. *We have that*

$$(c\,\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}, \boldsymbol{v}_2)_{\mathcal{T}_h} - (\varepsilon_h^z, \nabla\cdot\boldsymbol{v}_2)_{\mathcal{T}_h} + \langle\varepsilon_h^{\widehat{y}}, \boldsymbol{v}_2\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = -\,(c\,I_{\boldsymbol{r}}, \boldsymbol{v}_2)_{\mathcal{T}_h}\,, \tag{6.2.1a}$$
$$-\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}, \nabla w_2)_{\mathcal{T}_h} + \langle\widehat{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}}\cdot\boldsymbol{n}, w_2\rangle_{\partial\mathcal{T}_h} = -\,(\varepsilon_h^y, w_2)_{\mathcal{T}_h} - (I_y, w_2)_{\mathcal{T}_h}\,, \tag{6.2.1b}$$
$$\langle\widehat{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}}\cdot\boldsymbol{n}, \mu_2\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} = 0\,, \tag{6.2.1c}$$

$$\langle \varepsilon_h^{\widehat{z}}, \mu_2 \rangle_{\Gamma_h} = \langle \varphi_2 - \varphi_2^h, \mu_2 \rangle_{\Gamma_h} \,, \tag{6.2.1d}$$

$$\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{r}}} \cdot \boldsymbol{n} = \boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \cdot \boldsymbol{n} + \tau \, (\varepsilon_h^z - \varepsilon_h^{\widehat{z}}), \tag{6.2.1e}$$

$\forall \, (\boldsymbol{v}_2, w_2, \mu_2) \in \boldsymbol{Z}_h \times W_h \times M_h$.

Proof. The proof of this lemma is analogous to the proof of Lemma 6.1.1, except by the equation (6.2.1b). Let $w_2 \in W_h$, we have that

$$\begin{aligned} -\,(\boldsymbol{\varepsilon}_h^r, \nabla w_2)_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{r}} \cdot \boldsymbol{n}, w_2 \rangle_{\partial \mathcal{T}_h} &= -\,(\boldsymbol{\Pi}_{\boldsymbol{Z}} \boldsymbol{r} - \boldsymbol{r}_h, \nabla w_2)_{\mathcal{T}_h} + \langle (P_M \boldsymbol{r} - \widehat{\boldsymbol{r}}_h) \cdot \boldsymbol{n}, w_2 \rangle_{\partial \mathcal{T}_h} \\ &= -\,(\boldsymbol{\Pi}_{\boldsymbol{Z}} \boldsymbol{r}, \nabla w_2)_{\mathcal{T}_h} + \langle P_M \boldsymbol{r} \cdot \boldsymbol{n}, w_2 \rangle_{\partial \mathcal{T}_h} + (\boldsymbol{r}_h, \nabla w_2)_{\mathcal{T}_h} \\ &- \langle \widehat{r}_h \cdot \boldsymbol{n}, w_2 \rangle_{\partial \mathcal{T}_h} \,. \end{aligned}$$

Then, by (5.1.5b)

$$-\,(\boldsymbol{\varepsilon}_h^r, \nabla w_2)_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{r}} \cdot \boldsymbol{n}, w_2 \rangle_{\partial \mathcal{T}_h} = -\,(\boldsymbol{\Pi}_{\boldsymbol{Z}} \boldsymbol{r}, \nabla w_2)_{\mathcal{T}_h} + \langle P_M \boldsymbol{r} \cdot \boldsymbol{n}, w_2 \rangle_{\partial \mathcal{T}_h} - (\widetilde{y} - y_h, w_2)_{\mathcal{T}_h} \,.$$

Using (6.0.1a) and integrating by parts

$$\begin{aligned} -\,(\boldsymbol{\varepsilon}_h^r, \nabla w_2)_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{r}} \cdot \boldsymbol{n}, w_2 \rangle_{\partial \mathcal{T}_h} &= -\,(\boldsymbol{r}, \nabla w_2)_{\mathcal{T}_h} + \langle P_M \boldsymbol{r} \cdot \boldsymbol{n}, w_2 \rangle_{\partial \mathcal{T}_h} - (\widetilde{y} - y_h, w_2)_{\mathcal{T}_h} \\ &= (\nabla \cdot \boldsymbol{r}, w_2)_{\mathcal{T}_h} - \langle \boldsymbol{r} \cdot \boldsymbol{n}, w_2 \rangle_{\partial \mathcal{T}_h} + \langle P_M \boldsymbol{r} \cdot \boldsymbol{n}, w_2 \rangle_{\partial \mathcal{T}_h} \\ &- (\widetilde{y} - y_h, w_2)_{\mathcal{T}_h} \,. \end{aligned}$$

Finally, by using (5.1.2a)

$$-\,(\boldsymbol{\varepsilon}_h^r, \nabla w_2)_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{r}} \cdot \boldsymbol{n}, w_2 \rangle_{\partial \mathcal{T}_h} = -\,(y - y_h, w_2)_{\mathcal{T}_h} = -\,(\varepsilon_h^y, w_2)_{\mathcal{T}_h} - (I_y, w_2)_{\mathcal{T}_h} \,.$$

□

We will now present a result based on an energy argument

Lemma 6.2.2.

$$\left\| c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}} \right\|^2_{\mathcal{T}_h} + \left\| \tau^{1/2}\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}) \right\|^2_{\partial\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}} \cdot \boldsymbol{n}, \varphi_1 - \varphi_1^h \rangle_{\Gamma_h} = -\,(c\,I_{\boldsymbol{r}}, \boldsymbol{\varepsilon}_h^r)_{\mathcal{T}_h} - \|\varepsilon_h^y\|^2_{\mathcal{T}_h} - (I_y, \varepsilon_h^y)_{\mathcal{T}_h}\,. \tag{6.2.2}$$

Proof. The proof of this lemma is analogous to the proof of Lemma 6.1.2. □

Corollary 6.2.1. *It holds*

$$\begin{aligned} \frac{1}{2}\left\| c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \right\|^2_{\mathcal{T}_h} + \left\| \tau^{1/2}\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}) \right\|^2_{\partial\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \cdot \boldsymbol{n}, \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} \\ \leq \frac{1}{2}\left\| c^{1/2}\,I_{\boldsymbol{r}} \right\|^2_{\mathcal{T}_h} + \frac{3}{2}\,\|\varepsilon_h^y\|^2_{\mathcal{T}_h} + \frac{1}{2}\,\|I_y\|^2_{\mathcal{T}_h}\,. \end{aligned} \tag{6.2.3}$$

Proof. It follows from the previous result and an application of Young's inequality. □

Lemma 6.2.3. *We have the following identity*

$$\boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \cdot \boldsymbol{n} = c^{-1}\,l^{-1}\,(\varphi_2 - \varphi_2^h) - \Lambda^{I_{\boldsymbol{r}}} - \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}} - I_{\boldsymbol{r}} \cdot \boldsymbol{n} \qquad \forall\, \boldsymbol{x} \in e\,, \quad e \in \mathcal{E}_h^{\partial}\,. \tag{6.2.4}$$

Proof. The proof of this lemma is analogous to the proof of Lemma 6.1.3. □

The proof of the following lemma can be found in [23, Lemma 5.2].

Lemma 6.2.4. *For each* $e \in \mathcal{E}_h^{\partial}$, *we have that*

$$\left\| \Lambda^{I_{\boldsymbol{r}}} \right\|_{e,l} \leq \frac{1}{\sqrt{3}}\,r_e\,\|\partial_{\boldsymbol{n}}(I_{\boldsymbol{r}} \cdot \boldsymbol{n})\|_{K^e_{ext},(h^{\perp})^2}\,, \tag{6.2.5a}$$

$$\left\| \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}} \right\|_{e,l} \leq \frac{1}{\sqrt{3}}\,r_e^{3/2}\,C^e_{ext}\,C^e_{inv}\,\|\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}\|_{K^e}\,. \tag{6.2.5b}$$

Now, we establish the following error estimate for the energy norm.

Lemma 6.2.5. *There holds*

$$\begin{aligned} \left|\!\left|\!\left| (\boldsymbol{\varepsilon}_h^r, \varepsilon_h^z - \varepsilon_h^{\widehat{z}}, \varphi_2 - \varphi_2^h) \right|\!\right|\!\right| \lesssim \|I_{\boldsymbol{r}}\|_{\mathcal{T}_h} + \|I_y\|_{\mathcal{T}_h} + \|\varepsilon_h^y\|_{\mathcal{T}_h} \\ + R\,\|\partial_{\boldsymbol{n}}(I_r \cdot \boldsymbol{n})\|_{D^c_h,(h^{\perp})^2} + R^{1/2}\,\|I_{\boldsymbol{r}} \cdot \boldsymbol{n}\|_{\Gamma_h,h^{\perp}}\,. \end{aligned} \tag{6.2.6}$$

Proof. From (6.2.1e) we can deduce that

$$\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{r}}} \cdot \boldsymbol{n}, \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} = \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \cdot \boldsymbol{n}, \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} + \langle \tau \, (\varepsilon_h^z - \varepsilon_h^{\widehat{z}}), \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} \,,$$

and by (6.2.4)

$$\begin{aligned}\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{r}}} \cdot \boldsymbol{n}, \varphi_1 - \varphi_2^h \rangle_{\Gamma_h} = {} & \left\| c^{-1/2} \, l^{-1/2} \, (\varphi_2 - \varphi_2^h) \right\|_{\Gamma_h}^2 - \langle \Lambda^{I_{\boldsymbol{r}}}, \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} - \langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}}, \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} \\ & - \langle I_{\boldsymbol{r}} \cdot \boldsymbol{n}, \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} + \langle \tau \, (\varepsilon_h^z - \varepsilon_h^{\widehat{z}}), \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} \,.\end{aligned}$$

Hence, replacing this in (6.2.3)

$$\begin{aligned}&\frac{1}{2} \left\| c^{1/2} \, \boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \right\|_{\mathcal{T}_h}^2 + \left\| \tau^{1/2} \, (\varepsilon_h^z - \varepsilon_h^{\widehat{z}}) \right\|_{\partial \mathcal{T}_h}^2 + \left\| c^{-1/2} \, l^{-1/2} \, (\varphi_2 - \varphi_2^h) \right\|_{\Gamma_h}^2 \\ &\quad \leq \frac{1}{2} \left\| c^{1/2} \, I_{\boldsymbol{r}} \right\|_{\mathcal{T}_h}^2 + \frac{3}{2} \, \| \varepsilon_h^y \|_{\mathcal{T}_h}^2 + \frac{1}{2} \, \| I_y \|_{\mathcal{T}_h}^2 + \langle \Lambda^{I_{\boldsymbol{r}}}, \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} + \langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}}, \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} \\ &\quad + \langle I_{\boldsymbol{r}} \cdot \boldsymbol{n}, \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} + \langle \tau \, (\varepsilon_h^z - \varepsilon_h^{\widehat{z}}), \varphi_2 - \varphi_2^h \rangle_{\Gamma_h} \,.\end{aligned}$$

By Cauchy-Schwarz and Young's inequality we get

$$\begin{aligned}&\frac{1}{2} \left\| c^{1/2} \, \boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \right\|_{\mathcal{T}_h}^2 + \left\| \tau^{1/2} \, (\varepsilon_h^z - \varepsilon_h^{\widehat{z}}) \right\|_{\partial \mathcal{T}_h}^2 + \frac{1}{2} \left\| c^{-1/2} \, l^{-1/2} \, (\varphi_2 - \varphi_2^h) \right\|_{\Gamma_h}^2 \\ &\quad \leq \frac{1}{2} \left\| c^{1/2} \, I_{\boldsymbol{r}} \right\|_{\mathcal{T}_h}^2 + \frac{3}{2} \, \| \varepsilon_h^y \|_{\mathcal{T}_h}^2 + \frac{1}{2} \, \| I_y \|_{\mathcal{T}_h}^2 + 2 \left\| c^{1/2} \, l^{1/2} \, \Lambda^{I_{\boldsymbol{r}}} \right\|_{\Gamma_h}^2 + \left\| c^{1/2} \, l^{1/2} \, \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}} \right\|_{\Gamma_h}^2 \\ &\quad + 2 \left\| c^{1/2} \, l^{1/2} \, I_{\boldsymbol{r}} \cdot \boldsymbol{n} \right\|_{\Gamma_h}^2 + 2 \left\| c^{1/2} \, l^{1/2} \, \tau \, (\varepsilon_h^z - \varepsilon_h^{\widehat{z}}) \right\|_{\Gamma_h}^2 \,.\end{aligned}$$

On the other hand, notice that by the first equation of the first row of the assumptions (4.6.1), we deduce

$$2 \left\| c^{1/2} \, l^{1/2} \, \tau \, (\varepsilon_h^z - \varepsilon_h^{\widehat{z}}) \right\|_{\Gamma_h}^2 \leq \frac{1}{2} \left\| \tau^{1/2} \, (\varepsilon_h^z - \varepsilon_h^{\widehat{z}}) \right\|_{\partial \mathcal{T}_h}^2 \,.$$

Thus, we have

$$\begin{aligned}&\frac{1}{2} \left\| c^{1/2} \, \boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \right\|_{\mathcal{T}_h}^2 + \frac{1}{2} \left\| \tau^{1/2} \, (\varepsilon_h^z - \varepsilon_h^{\widehat{z}}) \right\|_{\partial \mathcal{T}_h}^2 + \frac{1}{2} \left\| c^{-1/2} \, l^{-1/2} \, (\varphi_2 - \varphi_2^h) \right\|_{\Gamma_h}^2 \\ &\quad \leq \frac{1}{2} \left\| c^{1/2} \, I_{\boldsymbol{r}} \right\|_{\mathcal{T}_h}^2 + \frac{3}{2} \, \| \varepsilon_h^y \|_{\mathcal{T}_h}^2 + \frac{1}{2} \, \| I_y \|_{\mathcal{T}_h}^2 + 2 \left\| c^{1/2} \, l^{1/2} \, \Lambda^{I_{\boldsymbol{r}}} \right\|_{\Gamma_h}^2 + \left\| c^{1/2} \, l^{1/2} \, \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}} \right\|_{\Gamma_h}^2 \\ &\quad + \left\| c^{1/2} \, l^{1/2} \, I_{\boldsymbol{r}} \cdot \boldsymbol{n} \right\|_{\Gamma_h}^2 \,.\end{aligned}$$

Also, by Lemma 6.2.4

$$2\, \left\| c^{1/2}\, l^{1/2}\, \Lambda^{I_{\boldsymbol{r}}} \right\|_{\Gamma_h}^2 \,\leq\, \frac{2\,c}{3}\, \max_{e\subset\Gamma_h} r_e^2\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{r}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2}\,, \tag{6.2.7}$$

and

$$2\, \left\| c^{1/2}\, l^{1/2}\, \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}} \right\|_{\Gamma_h}^2 \,\leq\, \frac{1}{12}\, \left\| c^{1/2}\, \boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \right\|_{\mathcal{T}_h}^2\,. \tag{6.2.8}$$

Therefore, we obtain

$$\begin{aligned}\frac{5}{12}\, \left\| c^{1/2}\, \boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \right\|_{\mathcal{T}_h}^2 \,+\, \frac{1}{2}\, \left\| \tau^{1/2}\, (\varepsilon_h^z - \varepsilon_h^{\widehat{z}}) \right\|_{\partial\mathcal{T}_h}^2 \,+\, \frac{1}{2}\, \left\| c^{-1/2}\, l^{-1/2}\, (\varphi_2 - \varphi_2^h) \right\|_{\Gamma_h}^2 \,\leq\, \frac{1}{2}\, \left\| c^{1/2}\, I_{\boldsymbol{r}} \right\|_{\mathcal{T}_h}^2 \\ +\, \frac{3}{2}\, \|\varepsilon_h^y\|_{\mathcal{T}_h}^2 \,+\, \frac{1}{2}\, \|I_y\|_{\mathcal{T}_h}^2 \,+\, \frac{2\,c\,R^2}{3}\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{r}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2}^2 \,+\, 2\,R\, \left\| c^{1/2}\, I_{\boldsymbol{r}}\cdot\boldsymbol{n} \right\|_{\Gamma_h,h^\perp}^2\,.\end{aligned}$$

□

Now, to control the fifth term of the right side of (6.2.6), we have the following lemma.

Lemma 6.2.6. *Let e be a edge or face of Γ_h and $C > 0$ independent of h, then*

$$\|I_{\boldsymbol{r}}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} \,\leq\, C\, h^{k+1}\, |\boldsymbol{r}|_{\boldsymbol{H}^{k+1}(\Omega)} \,+\, C\, h^{k+1}\, \tau_{\max} |z|_{H^{k+1}(\Omega)}\,. \tag{6.2.9}$$

Proof. The proof of this lemma is analogous to the proof of Lemma 6.1.6. □

To control the fourth term of the right side of (6.2.6), we have the following lemma

Lemma 6.2.7. *There exists $C > 0$ independent of h, such that*

$$\begin{aligned}\|I_{\boldsymbol{r}}\|_{D_h^c} \,&\leq\, \sqrt{\widetilde{R}'_C}\, \|I_{\boldsymbol{r}}\|_{D_h} \,+\, C\,(1 \,+\, \sqrt{\widetilde{R}'_C})\, h^{k+1}\, |\boldsymbol{r}|_{\boldsymbol{H}^{k+1}(\Omega)}\,, \\ \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{r}}\cdot\boldsymbol{n})\|_{D_h^c,(h^\perp)^2} \,&\leq\, \sqrt{R'_C}\, \|I_{\boldsymbol{r}}\|_{D_h} \,+\, C\,(1 \,+\, \sqrt{R'_C})\, h^{k+1}\, |\boldsymbol{r}|_{\boldsymbol{H}^{k+1}(\Omega)}\,.\end{aligned}$$

Proof. The proof of this lemma can be found in [23, Lemma 3.8]. □

We are now ready to state the error estimates for $\boldsymbol{r}$.

Lemma 6.2.8. *We have that*

$$\|\boldsymbol{r} - \boldsymbol{r}_h\|_{D_h} \lesssim \left(1 + R\sqrt{R'_C}\right) \|I_{\boldsymbol{r}}\|_{D_h} + \|\varepsilon_h^y\|_{\mathcal{T}_h} + \|I_y\|_{\mathcal{T}_h} + R^{1/2} \|I_{\boldsymbol{r}} \cdot \boldsymbol{n}\|_{\Gamma_h, h^\perp} + R\left(1 + \sqrt{R'_C}\right) h^{k+1} |\boldsymbol{r}|_{\boldsymbol{H}^{k+1}(\Omega)} . \tag{6.2.10}$$

Proof. The proof of this lemma is analogous of the proof of Lemma 6.1.8, and additionally adding the terms $\|\varepsilon_h^y\|_{\mathcal{T}_h}$ and $\|I_y\|_{\mathcal{T}_h}$ to the right side of the inequality. □

Also, we have the following estimates of the error in D_h^c

Lemma 6.2.9. *There hold,*

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{D_h^c} \leq \|I_{\boldsymbol{r}}\|_{D_h^c} + \sqrt{\widetilde{R}'_C}\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{D_h} ,$$
$$\|z - z_h\|_{D_h^c} \leq C\, h\, \|\boldsymbol{r} - \boldsymbol{r}_h\|_{D_h^c} .$$

Proof. The proof of this lemma can be found in [23, Lemma 3.7]. □

Finally obtain the following error estimate of $\boldsymbol{r}$ in the entire domain Ω.

Theorem 6.2.10. *We have that*

$$\|\boldsymbol{r} - \boldsymbol{r}_h\|_{\Omega} \lesssim H_1(R,h)\, \|I_{\boldsymbol{r}}\|_{D_h} + H_2(R,h)\, \|I_{\boldsymbol{r}} \cdot \boldsymbol{n}\|_{\Gamma_h, h^\perp} + H_3(R,h)\, h^{k+1}\, |\boldsymbol{r}|_{\boldsymbol{H}^{k+1}(\Omega)} + \|\varepsilon_h^y\|_{D_h} + \|I_y\|_{D_h} ,$$

where $H_i(R,h)$, $i \in \{1,2,3\}$ are defined the same as in Theorem 6.1.10.

Proof. The proof of this theorem is the same that the proof of Theorem 6.1.10, but additionally adding the terms $\|\varepsilon_h^y\|_{D_h}$ and $\|I_y\|_{D_h}$ in the right side of the inequality. □

Corollary 6.2.2. *There exists a constant $C > 0$ independent of h, such that*

$$\|\boldsymbol{r} - \boldsymbol{r}_h\|_{\Omega} \leq C\, h^{k+1} .$$

Proof. By a direct application of Theorem 6.2.10 and bearing in mind that $\|I_{\boldsymbol{r}}\|_{D_h} \lesssim h^{k+1}$, $\|I_{\boldsymbol{r}} \cdot \boldsymbol{n}\|_{\Gamma_h, h^\perp} \lesssim h^{k+1}$, $\|\varepsilon_h^y\|_{D_h} \lesssim h^{k+1}$ and $\|I_y\|_{D_h} \lesssim h^{k+1}$. □

6.2.2 Error estimates for e^z

For the error estimates of e^z we proceed in the same way as the error estimates of e^y. In fact, we use the same dual problem strategy. Thus, we deduce the following lemma.

Lemma 6.2.11. *We have that*

$$\|\varepsilon_h^z\|_{\mathcal{T}_h}^2 = (c\, I_{\boldsymbol{r}}, \boldsymbol{\Pi_Z \theta})_{\mathcal{T}_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}, c\,(\boldsymbol{\theta} - \boldsymbol{\Pi_Z \theta}))_{\mathcal{T}_h} - (e^y, \Pi_W \phi)_{\mathcal{T}_h} + \mathbb{T}_{z,h}\,, \tag{6.2.11}$$

where

$$\mathbb{T}_{z,h} := \langle \widehat{\varepsilon_h^z}, \boldsymbol{\theta}\cdot\boldsymbol{n}\rangle_{\Gamma_h} - \langle \widehat{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}}\cdot\boldsymbol{n}, \phi\rangle_{\Gamma_h}\,. \tag{6.2.12}$$

Proof. The proof of this lemma is analogous to the proof of Lemma 6.1.14. But in the point of the proof when for this case the equation (6.2.1b) is used, here we have to add the term $-(\varepsilon_h^y, \Pi_W\phi)_{\mathcal{T}_h} - (I_y, \Pi_W\phi)_{\mathcal{T}_h} = (e^y, \Pi_W\phi)_{\mathcal{T}_h}$ in the right side of (6.2.11). The rest of the procedure of the proof is the same as in the proof of Lemma 6.1.14. □

Now, we will present a lemma that gives us a convenient identity for $\mathbb{T}_{z,h}$

Lemma 6.2.12. *We have that* $\mathbb{T}_{z,h} := \sum_{i=1}^{7} \mathbb{T}_{z,h}^i$, *where*

$$\begin{aligned}
\mathbb{T}_{z,h}^1 &= -\langle c^{-1}\, l^{-1}\,(\varphi_2 - \varphi_2^h), \phi + c\, l\, \partial_{\boldsymbol{n}}\phi\rangle_{\Gamma_h}\,,\\
\mathbb{T}_{z,h}^2 &= \langle \varphi_2 - \varphi_2^h, \partial_{\boldsymbol{n}}\phi - P_M\partial_{\boldsymbol{n}}\phi\rangle_{\Gamma_h}\,,\\
\mathbb{T}_{z,h}^3 &= \langle \Lambda^{I_{\boldsymbol{r}}}, \phi\rangle_{\Gamma_h}\,,\\
\mathbb{T}_{z,h}^4 &= \langle I_{\boldsymbol{r}}\cdot\boldsymbol{n}, \phi - P_M\phi\rangle_{\Gamma_h}\,,\\
\mathbb{T}_{z,h}^5 &= -\langle P_M\, \tau I_z, \phi\rangle_{\Gamma_h}\,,\\
\mathbb{T}_{z,h}^6 &= \langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}}, \phi\rangle_{\Gamma_h}\,,\\
\mathbb{T}_{z,h}^7 &= -\langle \tau\,(\varepsilon_h^z - \widehat{\varepsilon_h^z}), P_M\phi\rangle_{\Gamma_h}\,.
\end{aligned}$$

Proof. The proof of this Lemma is analogous to the proof of Lemma 6.1.12. □

Now, we present the following estimate for $\mathbb{T}_{z,h}$.

Lemma 6.2.13. *Exists $C > 0$ independent of h, such that*

$$\begin{aligned}|\mathbb{T}_{z,h}| \leq\ & C\left(R + R_\tau + R_C^2\, h^{-1/2}\right) h \left\|\left(\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}, \varepsilon_h^z - \varepsilon_h^{\widehat{z}}, \varphi_2 - \varphi_2^h\right)\right\| \|\eta\|_\Omega \\ & + C\Big(R^{3/2}\, h^{1/2}\, \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{r}} \cdot \boldsymbol{n})\|_{D_h^c,(h^\perp)^2} + h\, \|I_{\boldsymbol{r}} \cdot \boldsymbol{n}\|_{\Gamma_h,h^\perp}\Big) \|\eta\|_\Omega \\ & + C\, R_\tau\, h^{1/2}\, \|I_z\|_{\Gamma_h,h^\perp}\, \|\eta\|_\Omega\,.\end{aligned}$$

Now, we have the following estimate for z in Ω.

Theorem 6.2.14. *We have that*

$$\begin{aligned}\|z - z_h\|_\Omega \leq\ & \widehat{H}_1(R,h)\, \|I_{\boldsymbol{r}}\|_{D_h} + \widehat{H}_2(R,h)\, \|I_{\boldsymbol{r}} \cdot \boldsymbol{n}\|_{\Gamma_h,h^\perp} + \widehat{H}_3(R,h)\, \|I_z\|_{D_h} \\ & + \widehat{H}_4(R,h)\, h^{k+1}\, |\boldsymbol{r}|_{\boldsymbol{H}^{k+1}(\Omega)} + h^{k+1}\,.\end{aligned}$$

Moreover, we can note that there exists a constant $C > 0$ such that $\widehat{H}_i(R,h) \leq C\,h$ for $i = 1,\,2,\,4$ and we also have that $\widehat{H}_3(R,h) \leq C$.

Proof. If we note first from the third term of the right side of (6.2.11) that

$$(e^z, \Pi_W \phi)_{\mathcal{T}_h} \lesssim h^{k+1}\, \|\eta\|_\Omega\,.$$

Then, proceeding analogously to the proof of Theorem 6.1.15 we prove the statement. □

Corollary 6.2.3. *There exists a constant $C > 0$ independent of h, such that*

$$\|z - z_h\|_\Omega \leq C\, h^{k+1}\,.$$

Proof. By a direct application of Theorem 6.2.14 and bearing in mind that $\|I_{\boldsymbol{r}}\|_{D_h} \lesssim h^{k+1}$, $\|I_{\boldsymbol{r}} \cdot \boldsymbol{n}\|_{\Gamma_h,h^\perp} \lesssim h^{k+1}$ and $\|I_z\|_{D_h} \lesssim h^{k+1}$. □

6.3 Error estimates for $\underline{\boldsymbol{e}}^{\boldsymbol{\sigma}}$

We have to mention that in this and in the following section, we use $\boldsymbol{n}_h$ to denote the unitary normal vector for Γ_h and $\boldsymbol{n}$ to denote the unitary normal vector for Γ. To begin with the analysis of the error for the velocity field equation, we start with a energy argument, which is given by the following lemma

Lemma 6.3.1. *We have that the projection of the errors satisfy the following identities*

$$
\begin{aligned}
(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\psi})_{\mathcal{T}_h} \,-\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} \,+\, \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\psi}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} &= -\,(\underline{I}_{\boldsymbol{\sigma}}, \boldsymbol{\psi})_{\mathcal{T}_h}\,, && (6.3.1a)\\
-\,(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \nabla \boldsymbol{w})_{\mathcal{T}_h} \,+\, \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\boldsymbol{n}_h, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} &= 0\,, && (6.3.1b)\\
\langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} &= 0\,, && (6.3.1c)\\
\langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\Gamma_h^N} &= \langle (G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}, \boldsymbol{\mu}\rangle_{\Gamma_h^N}\,, && (6.3.1d)\\
\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\mu}\rangle_{\Gamma_h^D} &= \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \boldsymbol{\mu}\rangle_{\Gamma_h^D}\,, && (6.3.1e)\\
\underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h &= \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\boldsymbol{n}_h + \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\,, && (6.3.1f)
\end{aligned}
$$

for all $(\boldsymbol{\psi}, \boldsymbol{w}, \boldsymbol{\mu}) \in \mathbb{Z}_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h$.

Proof. First we start by proving (6.3.1a), for this let $\boldsymbol{\psi} \in \mathbb{Z}_h$, notice that

$$
\begin{aligned}
&(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\psi})_{\mathcal{T}_h} \,-\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} \,+\, \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\psi}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h}\\
&\quad= (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\psi})_{\mathcal{T}_h} \,-\, (\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{V} - \boldsymbol{V}_h, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} \,+\, \langle P_{\boldsymbol{M}}\boldsymbol{V} - \widehat{\boldsymbol{V}}_h, \boldsymbol{\psi}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h}\\
&\quad= (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma}, \boldsymbol{\psi})_{\mathcal{T}_h} \,-\, (\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{V}, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} \,+\, \langle P_{\boldsymbol{M}}\boldsymbol{V}, \boldsymbol{\psi}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h}\\
&\quad-\,(\boldsymbol{\sigma}_h, \boldsymbol{\psi})_{\mathcal{T}_h} \,+\, (\boldsymbol{V}_h, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} \,-\, \langle \widehat{\boldsymbol{V}}_h, \boldsymbol{\psi}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h}\,,
\end{aligned}
$$

hence, by (5.3.5a) followed by (6.0.1b) and then integrating by parts we obtain

$$\begin{aligned}
(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\psi})_{\mathcal{T}_h} \,-\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} \,+\, \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\psi}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h}
&= (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma}, \boldsymbol{\psi})_{\mathcal{T}_h} \,-\, (\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{V}, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} \,+\, \langle P_{\boldsymbol{M}}\boldsymbol{V}, \boldsymbol{\psi}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} \\
&= (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma}, \boldsymbol{\psi})_{\mathcal{T}_h} \,-\, (\boldsymbol{V}, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} \,+\, \langle P_{\boldsymbol{M}}\boldsymbol{V}, \boldsymbol{\psi}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} \\
&= (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma}, \boldsymbol{\psi})_{\mathcal{T}_h} \,+\, (\nabla\boldsymbol{V}, \boldsymbol{\psi})_{\mathcal{T}_h} \,-\, \langle \boldsymbol{V}, \boldsymbol{\psi}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} \,+\, \langle P_{\boldsymbol{M}}\boldsymbol{V}, \boldsymbol{\psi}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} \\
&= (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma}, \boldsymbol{\psi})_{\mathcal{T}_h} \,+\, (\nabla\boldsymbol{V}, \boldsymbol{\psi})_{\mathcal{T}_h}\,,
\end{aligned}$$

thus, by (5.3.1a) we have that

$$(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\psi})_{\mathcal{T}_h} \,-\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\boldsymbol{\psi}))_{\mathcal{T}_h} \,+\, \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\psi}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} \,=\, (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\psi})_{\mathcal{T}_h} \,=\, -\,(\underline{I}_{\boldsymbol{\sigma}}, \boldsymbol{\psi})_{\mathcal{T}_h}\,.$$

Now following with the proof of (6.3.1b), let $\boldsymbol{w} \in \boldsymbol{W}_h$, then

$$\begin{aligned}
-\,(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \nabla\boldsymbol{w})_{\mathcal{T}_h} \,+\, \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\boldsymbol{n}_h, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}
&= -\,(\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} \,+\, \langle P_{\boldsymbol{M}}\boldsymbol{\sigma}\boldsymbol{n}_h - \widehat{\boldsymbol{\sigma}}_h\boldsymbol{n}_h, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} \\
&= -\,(\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma}, \nabla\boldsymbol{w})_{\mathcal{T}_h} \,+\, \langle P_{\boldsymbol{M}}\boldsymbol{\sigma}\boldsymbol{n}_h, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} \,+\, (\boldsymbol{\sigma}_h, \nabla\boldsymbol{w})_{\mathcal{T}_h} \\
&\quad -\, \langle \widehat{\boldsymbol{\sigma}}_h\boldsymbol{n}_h, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}\,,
\end{aligned}$$

by (5.3.5b), followed by (6.0.1a) and a integration by parts, we have that

$$\begin{aligned}
-\,(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \nabla\boldsymbol{w})_{\mathcal{T}_h} \,+\, \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\boldsymbol{n}_h, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h}
&= -\,(\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma}, \nabla\boldsymbol{w})_{\mathcal{T}_h} \,+\, \langle P_{\boldsymbol{M}}\boldsymbol{\sigma}\boldsymbol{n}_h, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} \\
&= -\,(\boldsymbol{\sigma}, \nabla\boldsymbol{w})_{\mathcal{T}_h} \,+\, \langle P_{\boldsymbol{M}}\boldsymbol{\sigma}\boldsymbol{n}_h, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} \\
&= (\mathbf{div}(\boldsymbol{\sigma}), \boldsymbol{w})_{\mathcal{T}_h} \,-\, \langle \boldsymbol{\sigma}\boldsymbol{n}_h, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} \,+\, \langle P_{\boldsymbol{M}}\boldsymbol{\sigma}\boldsymbol{n}_h, \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} \\
&= 0\,,
\end{aligned}$$

where the first term of the last equation is zero by (5.3.1b). Now, to prove (6.3.1c), let be $\mu \in \boldsymbol{M}_h$, we have that

$$\langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} \,=\, \langle P_{\boldsymbol{M}}\boldsymbol{\sigma}\boldsymbol{n}_h - \widehat{\boldsymbol{\sigma}}_h\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} \,=\, \langle P_{\boldsymbol{M}}\boldsymbol{\sigma}\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} \,-\, \langle \widehat{\boldsymbol{\sigma}}_h\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h}\,,$$

thus, by (5.3.5c)

$$\langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}} \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} = \langle P_{\boldsymbol{M}} \boldsymbol{\sigma} \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} = \langle \boldsymbol{\sigma} \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} .$$

On the other hand, let us note that

$$\begin{aligned} \langle \boldsymbol{\sigma}\, \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h} &= \sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{\sigma}\, \boldsymbol{n}_h \cdot \boldsymbol{\mu} \\ &= \sum_{e \in \mathcal{E}_h^{\circ}} \int_e [\![\boldsymbol{\sigma}]\!]_e \,\{\!\{ \boldsymbol{\mu} \}\!\}_e + \{\!\{ \boldsymbol{\sigma} \}\!\}_e \,[\![\boldsymbol{\mu}]\!] + \sum_{e \in \mathcal{E}_h^{\partial}} \int_e \boldsymbol{\sigma}\, \boldsymbol{n}_h \cdot \boldsymbol{\mu} \\ &= \langle \boldsymbol{\sigma}\, \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\Gamma_h}, \end{aligned}$$

hence, by Lemma 4.1.1 we have that $[\![\boldsymbol{\sigma}]\!]_e = 0$ and also as $\boldsymbol{\mu} \in \mathbb{P}_k(\mathcal{E}_h)$ we have that $[\![\boldsymbol{\mu}]\!]_e = 0$. Therefore $\langle \boldsymbol{\sigma}\, \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \Gamma_h} = 0$. Now, following the proof of (6.3.1d), let $\boldsymbol{\mu} \in \boldsymbol{M}_h$, we have that

$$\langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^N} = \langle P_{\boldsymbol{M}} \boldsymbol{\sigma}\, \boldsymbol{n}_h - \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^N} = \langle P_{\boldsymbol{M}} \boldsymbol{\sigma}\, \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^N} - \langle \widehat{\boldsymbol{\sigma}}_h\, \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^N} ,$$

then, by (5.3.1c) and (5.3.5d),

$$\langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^N} = \langle \boldsymbol{\sigma}\, \boldsymbol{n}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^N} - \langle (G_h \boldsymbol{n}) \circ \boldsymbol{\phi}, \boldsymbol{\mu} \rangle_{\Gamma_h^N} = \langle (G\boldsymbol{n} - G_h \boldsymbol{n}) \circ \boldsymbol{\phi}, \boldsymbol{\mu} \rangle_{\Gamma_h^N} .$$

To prove (6.3.1e), let be $\boldsymbol{\mu} \in \boldsymbol{M}_h$, then

$$\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\mu} \rangle_{\Gamma_h^D} = \langle P_{\boldsymbol{M}} \boldsymbol{V} - \widehat{\boldsymbol{V}}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^D} = \langle P_{\boldsymbol{M}} \boldsymbol{V}, \boldsymbol{\mu} \rangle_{\Gamma_h^D} - \langle \widehat{\boldsymbol{V}}_h, \boldsymbol{\mu} \rangle_{\Gamma_h^D} .$$

Then, by (5.3.5e) and (5.3.1d)

$$\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\mu} \rangle_{\Gamma_h^D} = \langle \boldsymbol{V}, \boldsymbol{\mu} \rangle_{\Gamma_h^D} - \langle \boldsymbol{g}_D^h, \boldsymbol{\mu} \rangle_{\Gamma_h^D} = \langle \boldsymbol{g}_D, \boldsymbol{\mu} \rangle_{\Gamma_h^D} - \langle \boldsymbol{g}_D^h, \boldsymbol{\mu} \rangle_{\Gamma_h^D} = \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \boldsymbol{\mu} \rangle_{\Gamma_h^D} .$$

Finally to prove (6.3.1f), let be $\boldsymbol{\mu} \in \boldsymbol{M}_h$, let us note that

$$\langle \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}} + \tau\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h} = \langle (\boldsymbol{\Pi}_{\mathbb{Z}} \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\, \boldsymbol{n}_h + \tau\, (\boldsymbol{\Pi}_{\boldsymbol{W}} \boldsymbol{V} - \boldsymbol{V}_h - P_{\boldsymbol{M}} \boldsymbol{V} + \widehat{\boldsymbol{V}}_h), \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h}$$

$$= \langle \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma}\,\boldsymbol{n}_h + \tau\,\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{V}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} - \tau\,\langle P_{\boldsymbol{M}}\boldsymbol{V}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h}$$
$$- \langle \boldsymbol{\sigma}_h\,\boldsymbol{n}_h + \tau\,(\boldsymbol{V}_h - \widehat{\boldsymbol{V}}_h), \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h}\,.$$

Then, by (5.3.6) and (6.0.1c), we can deduce that

$$\begin{aligned}
\langle \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}} + \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} &= \langle \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma}\,\boldsymbol{n}_h + \tau\,\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{V}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} - \tau\,\langle P_{\boldsymbol{M}}\boldsymbol{V}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} - \langle \widehat{\boldsymbol{\sigma}}_h\,\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} \\
&= \langle \boldsymbol{\sigma}\,\boldsymbol{n}_h + \tau\,\boldsymbol{V}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} - \tau\,\langle P_{\boldsymbol{M}}\boldsymbol{V}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} - \langle \widehat{\boldsymbol{\sigma}}_h\,\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} \\
&= \langle \boldsymbol{\sigma}\,\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} - \langle \widehat{\boldsymbol{\sigma}}_h\,\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} \\
&= \langle \boldsymbol{\sigma}\,\boldsymbol{n}_h - P_{\boldsymbol{M}}\boldsymbol{\sigma}\,\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} + \langle P_{\boldsymbol{M}}\boldsymbol{\sigma}\,\boldsymbol{n}_h - \widehat{\boldsymbol{\sigma}}_h\,\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h} \\
&= \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h}\,.
\end{aligned}$$

□

To continue with the analysis, we will present the following lemma

Lemma 6.3.2. *We have the following identity*

$$\begin{aligned}
&\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 + \left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h}^2 + \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h\rangle_{\Gamma_h^D} \\
&\quad + \langle (G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\rangle_{\Gamma_h^N} = -(\underline{I}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}})_{\mathcal{T}_h}\,.
\end{aligned} \tag{6.3.2}$$

Proof. let us note that setting $\boldsymbol{\psi} = \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}$ and $\boldsymbol{w} = \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}$ in (6.3.1a) and (6.3.1b) respectively, thus, we have that

$$\begin{aligned}
\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 - (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\underline{\boldsymbol{\varepsilon}}_h^{\sigma}))_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \underline{\boldsymbol{\varepsilon}}_h^{\sigma}\,\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} &= -(\underline{I}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}})_{\mathcal{T}_h}\,, \\
-(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \nabla\boldsymbol{\varepsilon}_h^{\boldsymbol{V}})_{\mathcal{T}_h} + \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\rangle_{\partial\mathcal{T}_h} &= 0\,,
\end{aligned}$$

integrating by parts the second equation,

$$\begin{aligned}
\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 - (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}))_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} &= -(\underline{I}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}})_{\mathcal{T}_h}\,, \\
(\mathbf{div}(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}), \boldsymbol{\varepsilon}_h^{\boldsymbol{V}})_{\mathcal{T}_h} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\rangle_{\partial\mathcal{T}_h} + \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\rangle_{\partial\mathcal{T}_h} &= 0\,,
\end{aligned}$$

then, adding both equations we obtain

$$\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 \,+\, \langle \widehat{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h - \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\, \boldsymbol{n}_h, \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} \rangle_{\partial\mathcal{T}_h} \,+\, \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\, \boldsymbol{n}_h \rangle_{\partial\mathcal{T}_h} \,=\, -\,(\underline{I}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}})_{\mathcal{T}_h}\,,$$

then, by (6.3.1f)

$$\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 + \langle \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \rangle_{\partial\mathcal{T}_h} + \langle \widehat{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h - \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\, \boldsymbol{n}_h, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\, \boldsymbol{n}_h \rangle_{\partial\mathcal{T}_h} = -(\underline{I}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}})_{\mathcal{T}_h},$$

equivalently

$$\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 \,+\, \left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h}^2 \,+\, \langle \widehat{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \rangle_{\partial\mathcal{T}_h} \,=\, -\,(\underline{I}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}})_{\mathcal{T}_h}\,.$$

Furthermore, note that by (6.3.1c), (6.3.1d) and (6.3.1e) we obtain that

$$\begin{aligned}\langle \widehat{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \rangle_{\partial\mathcal{T}_h} \,&=\, \langle \widehat{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} + \langle \widehat{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \rangle_{\Gamma_h^D} \,+\, \langle \widehat{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \rangle_{\Gamma_h^N} \\ &=\, \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \widehat{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h \rangle_{\Gamma_h^D} \,+\, \langle (G\boldsymbol{n} - G_h\boldsymbol{n}) \circ \boldsymbol{\phi}, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \rangle_{\Gamma_h^N}\,.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 \,+\, \left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h}^2 \,+\, \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \widehat{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h \rangle_{\Gamma_h^D} \\ +\langle (G\boldsymbol{n} - G_h\boldsymbol{n}) \circ \boldsymbol{\phi}, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \rangle_{\Gamma_h^N} \,=\, -\,(\underline{I}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}})_{\mathcal{T}_h}\,.\end{aligned}$$

□

Lemma 6.3.3. *We have the following identity for the projection of the error,*

$$\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\, \boldsymbol{n}_h \,=\, l^{-1}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h) \,-\, \Lambda^{\underline{I}_{\boldsymbol{\sigma}}} \,-\, \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \,-\, \underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h\,. \tag{6.3.3}$$

Proof. Let us note that

$$\begin{aligned}\boldsymbol{g}_D - \boldsymbol{g}_D^h \,&=\, \int_0^{l(\boldsymbol{x})} \boldsymbol{\sigma}(\boldsymbol{x} + s\,\boldsymbol{n}_h)\,\boldsymbol{n}_h\,ds \,-\, \int_0^{l(\boldsymbol{x})} \boldsymbol{E}_h(\boldsymbol{\sigma}_h)(\boldsymbol{x} + s\,\boldsymbol{n}_h)\,\boldsymbol{n}_h\,ds \\ &=\, \int_0^{l(\boldsymbol{x})} (\boldsymbol{\sigma} - \boldsymbol{E}_h(\boldsymbol{\sigma}_h))(\boldsymbol{x} + s\,\boldsymbol{n}_h)\,\boldsymbol{n}_h\,ds\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{l(\boldsymbol{x})} (\boldsymbol{\sigma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma} + \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma} - \boldsymbol{E}_h(\boldsymbol{\sigma}_h))(\boldsymbol{x} + s\,\boldsymbol{n}_h)\,\boldsymbol{n}_h\,ds \\
&= \int_0^{l(\boldsymbol{x})} (\underline{I}_{\boldsymbol{\sigma}} + \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}})(\boldsymbol{x} + s\,\boldsymbol{n}_h)\,\boldsymbol{n}_h\,ds \\
&= \int_0^{l(\boldsymbol{x})} (\underline{I}_{\boldsymbol{\sigma}}(\boldsymbol{x} + s\,\boldsymbol{n}_h) - \underline{I}_{\boldsymbol{\sigma}}(\boldsymbol{x}))\,\boldsymbol{n}_h\,ds \,+\, l(\boldsymbol{x})\,\underline{I}_{\boldsymbol{\sigma}}(\boldsymbol{x})\,\boldsymbol{n}_h \\
&+ \int_0^{l(\boldsymbol{x})} (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}(\boldsymbol{x} + s\,\boldsymbol{n}_h) - \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}(\boldsymbol{x}))\,\boldsymbol{n}_h\,ds \,+\, l(\boldsymbol{x})\,\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}(\boldsymbol{x})\,\boldsymbol{n}_h \\
&= l(\boldsymbol{x})\,\left(\Lambda^{\underline{I}_{\boldsymbol{\sigma}}} \,+\, \underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h \,+\, \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \,+\, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\right)(\boldsymbol{x})\,.
\end{aligned}
$$

Therefore the statement had been proved. □

Now we need to state the following lemma to continue with the analysis

Lemma 6.3.4. *For each $e \in \mathcal{E}_h^{\partial}$, we have that*

$$
\left\|\Lambda^{\underline{I}_{\boldsymbol{\sigma}}}\right\|_{e,l} \leq \frac{1}{\sqrt{3}}\,r_e\,\|\partial_{\boldsymbol{n}_h}(\underline{I}_{\boldsymbol{\sigma}}\boldsymbol{n}_h)\|_{K_{ext}^e,(h^{\perp})^2}\,, \tag{6.3.4a}
$$

$$
\left\|\Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\right\|_{e,l} \leq \frac{1}{\sqrt{3}}\,r_e^{3/2}\,C_{ext}^e\,C_{inv}^e\,\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{K^e}\,. \tag{6.3.4b}
$$

Proof. See proof of in [23, Lemma 5.2]. □

Let us now introduce a key lemma for the error analysis,

Lemma 6.3.5. *We have the following inequality*

$$
\begin{aligned}
\left\|\left|(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h)\right|\right\| \lesssim\; &\|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} \,+\, R^{1/2}\,\|\underline{I}_{\boldsymbol{\sigma}}\boldsymbol{n}_h\|_{\Gamma_h^D,h^{\perp}} \,+\, \|(G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|_{\Gamma_h^N} \\
&+ R\,\|\partial_{\boldsymbol{n}_h}(\underline{I}_{\boldsymbol{\sigma}}\boldsymbol{n}_h)\|_{D_h^c,(h^{\perp})^2} \,+\, \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\Gamma_h^N}\,,
\end{aligned} \tag{6.3.5}
$$

where

$$
\left\|\left|(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h)\right|\right\| := \left(\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 \,+\, \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\right\|_{\partial\mathcal{T}_h,\tau}^2 \,+\, \left\|\boldsymbol{g}_D - \boldsymbol{g}_D^h\right\|_{\Gamma_h^D,l^{-1}}^2\right)^{1/2}.
$$

Proof. First we will note the following argument, from (6.3.2) we have that

$$\begin{aligned}\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 \,+\, \left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h}^2 \,+\, \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h\rangle_{\Gamma_h^D} \,+\, \langle (G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\rangle_{\Gamma_h^{\boldsymbol{N}}}\\ \leq\, |(\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}})_{\mathcal{T}_h}| \,\leq\, \frac{1}{2}\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 \,+\, \frac{1}{2}\,\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2\,,\end{aligned}$$

by doing some simple algebraic arrangements, we get

$$\begin{aligned}\frac{1}{2}\,\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 \,+\, \left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h}^2 \,+\, \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h\rangle_{\Gamma_h^D}\\ +\langle (G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\rangle_{\Gamma_h^{\boldsymbol{N}}} \,\leq\, \frac{1}{2}\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2\,.\end{aligned} \tag{6.3.6}$$

On the other hand, let us note from (6.3.1f) that

$$\langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,=\, \langle \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,+\, \langle \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\,,$$

then, by (6.3.3)

$$\begin{aligned}\langle \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,&=\, \langle l^{-1}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h), \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,-\, \langle \Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,-\, \langle \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\\ &\quad-\, \langle \underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,+\, \langle \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\\ &=\, \left\|l^{-1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D}^2 \,-\, \langle \Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,-\, \langle \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\\ &\quad-\, \langle \underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,+\, \langle \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\,.\end{aligned}$$

Then, replacing the obtained above in (6.3.6), we get

$$\begin{aligned}\frac{1}{2}\,\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 \,+\, \left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h}^2 \,+\, \left\|l^{-1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \,+\, \langle (G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\rangle_{\Gamma_h^N}\\ \leq\, \frac{1}{2}\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}^2 \,+\, \langle \Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,+\, \langle \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,+\, \langle \underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\\ -\,\langle \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\,,\end{aligned}$$

equivalently we have that

$$
\begin{aligned}
&\tfrac{1}{2}\,\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} \,+\, \left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|^2_{\partial\mathcal{T}_h} \,+\, \left\|l^{-1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D}\\
&\quad \le\, \frac{1}{2}\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} \,+\, \langle \Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,+\, \langle \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,+\, \langle \underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\\
&\quad -\,\langle \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,-\, \langle (G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}, \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\rangle_{\Gamma_h^N}\\
&\quad \le\, \frac{1}{2}\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} \,+\, \langle \Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,+\, \langle \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,+\, \langle \underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D}\\
&\quad -\,\langle \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \boldsymbol{g}_D - \boldsymbol{g}_D^h\rangle_{\Gamma_h^D} \,+\, \langle \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \tau^{-1}\,((G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi})\rangle_{\Gamma_h^N}\\
&\quad -\,\langle \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, (G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\rangle_{\Gamma_h^N}\,.
\end{aligned}
$$

Then, let be ϵ_1, ϵ_2, ϵ_3, ϵ_4, $\epsilon_5 > 0$, by Cauchy-Schwarz inequality and Young's inequality we have that

$$
\begin{aligned}
&\frac{1}{2}\,\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} \,+\, \left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|^2_{\partial\mathcal{T}_h} \,+\, \left\|l^{-1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D}\\
&\quad \le\, \frac{1}{2}\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} \,+\, \frac{1}{2\,\epsilon_1}\,\left\|l^{1/2}\,\Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}}\right\|^2_{\Gamma_h^D} \,+\, \frac{\epsilon_1}{2}\left\|l^{-1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|^2_{\Gamma_h^D} \,+\, \frac{1}{2\,\epsilon_2}\,\left\|l^{1/2}\,\Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\right\|^2_{\Gamma_h^D}\\
&\quad +\,\frac{\epsilon_2}{2}\,\left\|l^{-1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|^2_{\Gamma_h^D} \,+\, \frac{1}{2\,\epsilon_3}\,\left\|l^{1/2}\,\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\right\|^2_{\Gamma_h^D} \,+\, \frac{\epsilon_3}{2}\,\left\|l^{-1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|^2_{\Gamma_h^D}\\
&\quad +\,\frac{1}{2\,\epsilon_4}\,\left\|l^{1/2}\,\tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|^2_{\Gamma_h^D} \,+\, \frac{\epsilon_4}{2}\,\left\|l^{-1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|^2_{\Gamma_h^D} \,+\, \frac{1}{2\,\epsilon_5}\,\left\|l^{1/2}\,\tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\Gamma_h^N}\\
&\quad +\,\frac{\epsilon_5}{2}\,\left\|\tau^{-1}\,((G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi})\right\|^2_{\Gamma_h^N} \,+\, \frac{1}{2}\,\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|^2_{\Gamma_h^N} \,+\, \frac{1}{2}\,\|(G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|_{\Gamma_h^N}\\
&\quad \le\, \frac{1}{2}\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} \,+\, \frac{1}{2\,\epsilon_1}\,\left\|l^{1/2}\,\Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}}\right\|^2_{\Gamma_h^D} \,+\, \frac{1}{2\,\epsilon_2}\,\left\|l^{1/2}\,\Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\right\|^2_{\Gamma_h^D} \,+\, \frac{1}{2\,\epsilon_3}\,\left\|l^{1/2}\,\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\right\|^2_{\Gamma_h^D}\\
&\quad +\,\frac{1}{2}\left[\frac{1}{\epsilon_4} \,+\, \frac{1}{\epsilon_5}\right]\left\|l^{1/2}\,\tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|^2_{\partial\mathcal{T}_h} \,+\, \frac{1}{2}\,[\epsilon_5 \,+\, 1]\,\|(G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|^2_{\Gamma_h^N}\\
&\quad +\,\frac{1}{2}\,\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|^2_{\Gamma_h^N} \,+\, \frac{1}{2}\,[\epsilon_1 \,+\, \epsilon_2 \,+\, \epsilon_3 \,+\, \epsilon_4]\,\left\|l^{-1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|^2_{\Gamma_h^D}\,,
\end{aligned}
$$

hence, setting $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 1/4$ we get

$$
\begin{aligned}
&\frac{1}{2}\, \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} + \left\| \tau^{1/2} \, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|^2_{\partial \mathcal{T}_h} + \frac{1}{2} \left\| l^{-1/2} \, (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \\
&\quad \leq \frac{1}{2}\, \|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} + 2 \left\| l^{1/2}\, \Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}} \right\|^2_{\Gamma_h^D} + 2 \left\| l^{1/2}\, \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \right\|^2_{\Gamma_h^D} + 2 \left\| l^{1/2}\, \underline{\boldsymbol{I}}_{\boldsymbol{\sigma}} \boldsymbol{n} \right\|^2_{\Gamma_h^D} \\
&\quad + \frac{1}{2} \left[4 + \frac{1}{\epsilon_5} \right] \left\| l^{1/2}\, \tau \, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|^2_{\Gamma_h^D} + \frac{1}{2} \, [\epsilon_5 + 1] \, \| (G\boldsymbol{n} - G_h \boldsymbol{n}) \circ \boldsymbol{\phi} \|^2_{\Gamma_h^N} + \frac{1}{2} \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} \right\|^2_{\Gamma_h^N} .
\end{aligned}
$$

On the other hand, we can note that

$$
\begin{aligned}
\frac{1}{2} \left[4 + \frac{1}{\epsilon_5} \right] \left\| l^{1/2}\, \tau \, (\boldsymbol{\varepsilon}_h^{V} - \boldsymbol{\varepsilon}_h^{\widehat{V}}) \right\|^2_{\Gamma_h^D} &\leq \frac{1}{2} \left[4 + \frac{1}{\epsilon_5} \right] \max_{e \in \mathcal{E}_h^{\partial}} \tau_e \, r_e \, h_e^{\perp} \left\| \tau^{1/2} \, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{V}}}) \right\|^2_{\partial \mathcal{T}_h} \\
&\leq \frac{1}{8} \left[4 + \frac{1}{\epsilon_5} \right] \left\| \tau^{1/2} \, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|^2_{\partial \mathcal{T}_h} ,
\end{aligned}
$$

Thus, setting $\epsilon_5 = 1/2$, we get

$$
\begin{aligned}
&\tfrac{1}{2}\, \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} + \tfrac{1}{4} \left\| \tau^{1/2} \, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|^2_{\partial \mathcal{T}_h} + \tfrac{1}{2} \left\| l^{-1/2} \, (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \\
&\quad \leq \frac{1}{2}\, \|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} + 2 \left\| l^{1/2}\, \Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}} \right\|^2_{\Gamma_h^D} + 2 \left\| l^{1/2}\, \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \right\|^2_{\Gamma_h^D} + 2 \left\| l^{1/2}\, \underline{\boldsymbol{I}}_{\boldsymbol{\sigma}} \boldsymbol{n}_h \right\|^2_{\Gamma_h^D} \\
&\quad + \frac{3}{4} \, \| (G\boldsymbol{n} - G_h \boldsymbol{n}) \circ \boldsymbol{\phi} \|^2_{\Gamma_h^N} + \frac{1}{2} \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} \right\|^2_{\Gamma_h^N} ,
\end{aligned}
$$

Then, by Lemma 6.3.4 and taking into account the assumptions of Chapter 4, we can deduce the following inequalities

$$
\begin{aligned}
2 \left\| l^{1/2}\, \Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}} \right\|^2_{\Gamma_h^D} &= 2 \sum_{e \subset \Gamma_h^D} \left\| l^{1/2} \Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}} \right\|^2_e \leq 2 \sum_{e \subset \Gamma_h} \left\| l^{1/2}\, \Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}} \right\|^2_e \leq 2 \sum_{e \subset \Gamma_h} \frac{1}{3} \, r_e^2 \, \| \partial_{\boldsymbol{n}_h} (\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}} \boldsymbol{n}_h) \|_{K^e_{ext}, (h^{\perp})^2} \\
&\leq \frac{2}{3} \max_{e \subset \Gamma_h} r_e^2 \, \| \partial_{\boldsymbol{n}_h} (\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}} \boldsymbol{n}_h) \|_{D_h^c, (h^{\perp})^2} \leq \frac{2}{3} \, R^2 \, \| \partial_{\boldsymbol{n}} (\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}} \boldsymbol{n}) \|_{D_h^c, (h^{\perp})^2} ,
\end{aligned}
$$

and

$$
\begin{aligned}
2 \left\| l^{1/2}\, \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \right\|^2_{\Gamma_h^D} &= 2 \sum_{e \subset \Gamma_h^D} \left\| l^{1/2}\, \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \right\|^2_e \leq 2 \sum_{e \subset \Gamma_h} \left\| l^{1/2}\, \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \right\|^2_e \leq 2 \sum_{e \subset \Gamma_h} \frac{1}{3} \, r_e^3 \, (C^e_{ext} \, C^e_{inv})^2 \, \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} \\
&\leq \frac{1}{12} \, \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} .
\end{aligned}
$$

Therefore, we obtain the following inequality

$$\begin{aligned}
&\frac{5}{6}\,\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} + \frac{1}{4}\left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|^2_{\partial\mathcal{T}_h} + \frac{1}{2}\left\|l^{-1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \\
&\quad\le \frac{1}{2}\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} + 2\left\|l^{1/2}\,\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\right\|^2_{\Gamma_h^D} + \frac{3}{4}\,\|(G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|^2_{\Gamma_h^N} \\
&\quad + \frac{2}{3}\,R^2\,\|\partial_{\boldsymbol{n}_h}(\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\boldsymbol{n}_h)\|_{D_h^c,(h^\perp)^2} + \frac{1}{2}\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|^2_{\Gamma_h^N} \\
&\quad\le \frac{1}{2}\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} + 2\max_{e\in\mathcal{E}_h^\partial} r_e\,h\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\boldsymbol{n}_h\|^2_{\Gamma_h^D} + \frac{3}{4}\,\|(G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|^2_{\Gamma_h^N} \\
&\quad + \frac{2}{3}\,R^2\,\|\partial_{\boldsymbol{n}_h}(\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\boldsymbol{n}_h)\|_{D_h^c,(h^\perp)^2} + \frac{1}{2}\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|^2_{\Gamma_h^N} \\
&\quad\le \frac{1}{2}\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|^2_{\mathcal{T}_h} + 2\,R\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\boldsymbol{n}_h\|^2_{\Gamma_h^D,h^\perp} + \frac{3}{4}\,\|(G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|^2_{\Gamma_h^N} \\
&\quad + \frac{2}{3}\,R^2\,\|\partial_{\boldsymbol{n}_h}(\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\boldsymbol{n}_h)\|_{D_h^c,(h^\perp)^2} + \frac{1}{2}\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|^2_{\Gamma_h^N}\,.
\end{aligned}$$

□

Now, we see that the first term of the right side of (6.3.5) is controlled by theorem (6.0.1) and the second term is controlled by the following result

Lemma 6.3.6. *Let e be a edge or face of Γ_h, and let be $C > 0$ independent of h, then*

$$\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\|_{\Gamma_h,h^\perp} \le C\,h^{k+1}\,|\boldsymbol{\sigma}|_{\mathbb{H}^{k+1}(\Omega)} + C\,h^{k+1}\,\tau\,|\boldsymbol{V}|_{\boldsymbol{H}^{k+1}(\Omega)}\,. \tag{6.3.7}$$

Proof. The proof of this lemma is analogous to the proof of Lemma 6.1.6. □

On the other hand we can note that to compute the estimates for $\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{D_h}$ we need to state the following lemma to control the term $\|(G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|_{\Gamma_h^N}$.

Lemma 6.3.7. *We have that*

$$\begin{aligned}
\|(G(\Gamma)\boldsymbol{n} - G_h(\Gamma)\boldsymbol{n})\circ\boldsymbol{\phi}\|_{\Gamma_h^N} &\lesssim \|(\boldsymbol{r}-\boldsymbol{r}_h)\cdot\boldsymbol{n}\|_{\Gamma^N}\,\|(\boldsymbol{p}-\boldsymbol{p}_h)\cdot\boldsymbol{n}\|_{\Gamma^N} \\
&+ \|\boldsymbol{r}\|_{\boldsymbol{H}^1(\Omega)}\,\|(\boldsymbol{p}-\boldsymbol{p}_h)\cdot\boldsymbol{n}\|_{\Gamma^N} + \|\boldsymbol{p}\|_{\boldsymbol{H}^1(\Omega)}\,\|(\boldsymbol{r}-\boldsymbol{r}_h)\cdot\boldsymbol{n}\|_{\Gamma^N} + \|(\boldsymbol{r}-\boldsymbol{r}_h)\cdot\boldsymbol{n}\|_{\Gamma^N}\,,
\end{aligned}$$

moreover

$$
\begin{aligned}
\|(\boldsymbol{p}-\boldsymbol{p}_h)\cdot\boldsymbol{n}\|_{\Gamma^N} \lesssim\; & \max_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\, h^{-1/2}\sqrt{\widetilde{R}'_C}\, \left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \varepsilon_h^y-\varepsilon_h^{\widehat{y}}, \varphi_1-\varphi_1^h)\right|\!\right|\!\right| \\
& + \left(\max_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\, h^{-1/2}\sqrt{\widetilde{R}'_C} \,+\, R^{1/2}\, h^{-1}\right) \|I_{\boldsymbol{p}}\|_{D_h} \\
& + \max_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\, h^{-1/2}\left(1 \,+\, \sqrt{\widetilde{R}'_C}\right) h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\
& + \left(1 \,+\, R^{1/2}\right) h^k\, \|\boldsymbol{p}\|_{\boldsymbol{H}^{k+1}(\Omega)}\;,
\end{aligned}
$$

and

$$
\begin{aligned}
\|(\boldsymbol{r}-\boldsymbol{r}_h)\cdot\boldsymbol{n}\|_{\Gamma^N} \lesssim\; & \max_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\, h^{-1/2}\sqrt{\widetilde{R}'_C}\, \left|\!\left|\!\left|(\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}, \varepsilon_h^z-\varepsilon_h^{\widehat{z}}, \varphi_2-\varphi_2^h)\right|\!\right|\!\right| \\
& + \left(\max_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\, h^{-1/2}\sqrt{\widetilde{R}'_C} \,+\, R^{1/2}\, h^{-1}\right) \|I_{\boldsymbol{r}}\|_{D_h} \\
& + \max_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\, h^{-1/2}\left(1 \,+\, \sqrt{\widetilde{R}'_C}\right) h^{k+1}\, |\boldsymbol{r}|_{\boldsymbol{H}^{k+1}(\Omega)} \\
& + \left(1 \,+\, R^{1/2}\right) h^k\, \|\boldsymbol{r}\|_{\boldsymbol{H}^{k+1}(\Omega)}\;.
\end{aligned}
$$

Proof. Let us note the following

$$
\begin{aligned}
G(\Gamma) \,-\, G_h(\Gamma) &= \boldsymbol{r}\cdot\boldsymbol{n}\,(c\,\boldsymbol{p}\cdot\boldsymbol{n}+\partial_{\boldsymbol{n}} g) \,-\, \frac{1}{2}\,(g-\widetilde{y})^2 \,-\, \boldsymbol{r}_h\cdot\boldsymbol{n}\,(c\,\boldsymbol{p}_h\cdot\boldsymbol{n}+\partial_{\boldsymbol{n}} g) \,+\, \frac{1}{2}(g-\widetilde{y})^2 \\
&= c\,(\boldsymbol{p}\cdot\boldsymbol{n}\,\boldsymbol{r}\cdot\boldsymbol{n}-\boldsymbol{p}_h\cdot\boldsymbol{n}\,\boldsymbol{r}_h\cdot\boldsymbol{n}) \,+\, \partial_{\boldsymbol{n}} g\,(\boldsymbol{r}\cdot\boldsymbol{n}-\boldsymbol{r}_h\cdot\boldsymbol{n})\,,
\end{aligned}
$$

hence, we have that

$$
\begin{aligned}
\|(G\boldsymbol{n}-G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|_{\Gamma_h^N} = \|G\boldsymbol{n}-G_h\boldsymbol{n}\|_{\Gamma^N} &\leq \|G_h-G\|_{\Gamma^N} \\
&\lesssim \|\boldsymbol{p}\cdot\boldsymbol{n}\,\boldsymbol{r}\cdot\boldsymbol{n}-\boldsymbol{p}_h\cdot\boldsymbol{n}\,\boldsymbol{r}_h\cdot\boldsymbol{n}\|_{\Gamma^N} \,+\, \|\boldsymbol{r}\cdot\boldsymbol{n}-\boldsymbol{r}_h\cdot\boldsymbol{n}\|_{\Gamma^N} \\
&\lesssim \|\boldsymbol{p}\cdot\boldsymbol{n}\,(\boldsymbol{r}\cdot\boldsymbol{n}-\boldsymbol{r}_h\cdot\boldsymbol{n})\|_{\Gamma^N} \,+\, \|\boldsymbol{r}_h\cdot\boldsymbol{n}\,(\boldsymbol{p}\cdot\boldsymbol{n}-\boldsymbol{p}_h\cdot\boldsymbol{n})\|_{\Gamma^N} \\
&+ \|(\boldsymbol{r}-\boldsymbol{r}_h)\cdot\boldsymbol{n}\|_{\Gamma^N} \\
&= \|\boldsymbol{p}\cdot\boldsymbol{n}\|_{\Gamma^N}\, \|\boldsymbol{r}\cdot\boldsymbol{n}-\boldsymbol{r}_h\cdot\boldsymbol{n}\|_{\Gamma^N} \,+ \|\boldsymbol{r}\cdot\boldsymbol{n}-\boldsymbol{r}_h\cdot\boldsymbol{n}\|_{\Gamma^N}\, \|\boldsymbol{p}\cdot\boldsymbol{n}-\boldsymbol{p}_h\cdot\boldsymbol{n}\|_{\Gamma^N} \\
&+ \|\boldsymbol{r}\cdot\boldsymbol{n}\|_{\Gamma_N}\, \|\boldsymbol{p}\cdot\boldsymbol{n}-\boldsymbol{p}_h\cdot\boldsymbol{n}\|_{\Gamma^N} \,+\, \|\boldsymbol{r}\cdot\boldsymbol{n}-\boldsymbol{r}_h\cdot\boldsymbol{n}\|_{\Gamma^N}\;.
\end{aligned}
$$

Then, by trace inequality we obtain

$$\begin{aligned}\|(G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|_{\Gamma_h^N} \lesssim\; & \|(\boldsymbol{r}-\boldsymbol{r}_h)\cdot\boldsymbol{n}\|_{\Gamma^N}\,\|(\boldsymbol{p}-\boldsymbol{p}_h)\cdot\boldsymbol{n}\|_{\Gamma^N} + \|\boldsymbol{r}\|_{\boldsymbol{H}^1(\Omega)}\,\|(\boldsymbol{p}-\boldsymbol{p}_h)\cdot\boldsymbol{n}\|_{\Gamma^N}\\ & + \|\boldsymbol{p}\|_{\boldsymbol{H}^1(\Omega)}\,\|(\boldsymbol{r}-\boldsymbol{r}_h)\cdot\boldsymbol{n}\|_{\Gamma^N} + \|(\boldsymbol{r}-\boldsymbol{r}_h)\cdot\boldsymbol{n}\|_{\Gamma^N}\,.\end{aligned}$$

On the other hand, we will prove the bound of the statement only for $\|(\boldsymbol{p}-\boldsymbol{p}_h)\cdot\boldsymbol{n}\|_{\Gamma^N}$, since the proof for the bound of $\|(\boldsymbol{r}-\boldsymbol{r}_h)\cdot\boldsymbol{n}\|_{\Gamma^N}$ is analogous. Hence

$$\begin{aligned}\|\boldsymbol{p}\cdot\boldsymbol{n}-\boldsymbol{p}_h\cdot\boldsymbol{n}\|_{\Gamma^N} &\leq \|\boldsymbol{p}\cdot\boldsymbol{n}-\boldsymbol{\Pi_Z p}\cdot\boldsymbol{n}\|_{\Gamma^N} + \|\boldsymbol{\Pi_Z p}\cdot\boldsymbol{n}-\boldsymbol{p}_h\cdot\boldsymbol{n}\|_{\Gamma^N}\\ &\leq \|\boldsymbol{p}-\boldsymbol{\Pi_Z p}\|_{\Gamma^N} + \|\boldsymbol{\Pi_Z p}-\boldsymbol{p}_h\|_{\Gamma^N}\\ &= \|\boldsymbol{p}-\boldsymbol{\Pi_Z p}\|_{\Gamma^N} + \sum_{\Gamma_e\subset\Gamma^N}\|\boldsymbol{\Pi_Z p}-\boldsymbol{p}_h\|_{\Gamma_e}\,,\end{aligned}$$

thus, by [10, Lemma 4], we obtain

$$\begin{aligned}\|\boldsymbol{p}\cdot\boldsymbol{n}-\boldsymbol{p}_h\cdot\boldsymbol{n}\|_{\Gamma^N} &\lesssim \|\boldsymbol{p}-\boldsymbol{\Pi_Z p}\|_{\Gamma^N} + \sum_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\,h_e^{-1/2}\,\|\boldsymbol{\Pi_Z p}-\boldsymbol{p}_h\|_{K_{ext}^e}\\ &\lesssim \|\boldsymbol{p}-\boldsymbol{\Pi_Z p}\|_{\Gamma^N} + \sum_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\,h_e^{-1/2}\,\|\boldsymbol{\Pi_Z p}-\boldsymbol{p}_h\|_{D_h^c}\\ &\lesssim \sum_{\Gamma_e\subset\Gamma^N}\|\boldsymbol{p}-\boldsymbol{\Pi_Z p}\|_{\Gamma_e} + \sum_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\,h_e^{-1/2}\,\|\boldsymbol{\Pi_Z p}-\boldsymbol{p}_h\|_{D_h^c}\\ &\lesssim \sum_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\,h_e^{-1/2}\left(\|\boldsymbol{p}-\boldsymbol{\Pi_Z p}\|_{K_{ext}^e} + h_e^{1/2}\,r_e^{1/2}\,\|\nabla(\boldsymbol{p}-\boldsymbol{\Pi_Z p})\|_{K_{ext}^e}\right)\\ &+ \sum_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\,h_e^{-1/2}\,\|\boldsymbol{\Pi_Z p}-\boldsymbol{p}_h\|_{D_h^c}\\ &\lesssim \sum_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\,h_e^{-1/2}\,\|\boldsymbol{p}-\boldsymbol{\Pi_Z p}\|_{D_h^c} + \|\nabla(\boldsymbol{p}-\boldsymbol{\Pi_Z p})\|_{D_h^c}\\ &+ \sum_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\,h_e^{-1/2}\,\|\boldsymbol{\Pi_Z p}-\boldsymbol{p}_h\|_{D_h^c}\,.\end{aligned}$$

Recalling the definition of $\widetilde{C}_{ext}^e$, hence, we deduce

$$\|\boldsymbol{p}\cdot\boldsymbol{n}-\boldsymbol{p}_h\cdot\boldsymbol{n}\|_{\Gamma^N} \lesssim \sum_{\Gamma_e\subset\Gamma^N} r_e^{-1/2}\,h_e^{-1/2}\,\|\boldsymbol{p}-\boldsymbol{\Pi_Z p}\|_{D_h^c} + \|\nabla(\boldsymbol{p}-\boldsymbol{\Pi_Z p})\|_{D_h^c}$$

$$
\begin{aligned}
&+ \Big(\sum_{\Gamma_e \subset \Gamma^N} r_e^{-1/2} h_e^{-1/2} \Big) \Big(\sum_{e \in \mathcal{E}_h^\partial} \|\boldsymbol{\Pi_Z p} - \boldsymbol{p}_h\|^2_{\widetilde{K}^e_{ext}} \Big)^{1/2} \\
&\lesssim \sum_{\Gamma_e \subset \Gamma^N} r_e^{-1/2}\, h_e^{-1/2}\, \|\boldsymbol{p} - \boldsymbol{\Pi_Z p}\|_{D_h^c} + \|\nabla(\boldsymbol{p} - \boldsymbol{\Pi_Z p})\|_{D_h^c} \\
&+ \Big(\sum_{\Gamma_e \subset \Gamma^N} r_e^{-1/2}\, h_e^{-1/2} \Big) \Big(\sum_{e \in \mathcal{E}_h^\partial} \widetilde{r}_e\, (\widetilde{C}^e_{ext})^2\, \|\boldsymbol{\Pi_Z p} - \boldsymbol{p}_h\|^2_{K^e} \Big)^{1/2} \\
&\lesssim \sum_{\Gamma_e \subset \Gamma^N} r_e^{-1/2}\, h_e^{-1/2}\, \|\boldsymbol{p} - \boldsymbol{\Pi_Z p}\|_{D_h^c} + \|\nabla(\boldsymbol{p} - \boldsymbol{\Pi_Z p})\|_{D_h^c} \\
&+ \sum_{\Gamma_e \subset \Gamma^N} r_e^{-1/2}\, h_e^{-1/2} \sqrt{\widetilde{R}'_C}\, \|\boldsymbol{\Pi_Z p} - \boldsymbol{p}_h\|_{D_h} .
\end{aligned}
$$

Finally, by Lemma 6.1.5 and [10, Lemma 3]

$$
\begin{aligned}
\|\boldsymbol{p}\cdot\boldsymbol{n} - \boldsymbol{p}_h\cdot\boldsymbol{n}\|_{\Gamma^N} \lesssim\ & \max_{\Gamma_e \subset \Gamma^N} r_e^{-1/2}\, h^{-1/2} \sqrt{\widetilde{R}'_C}\, \left\|\!\left|(\boldsymbol{\varepsilon}^{\boldsymbol{p}}_h, \varepsilon^y_h - \varepsilon^{\widehat{y}}_h, \varphi_1 - \varphi_1^h)\right|\!\right\| \\
&+ \Big(\max_{\Gamma_e \subset \Gamma^N} r_e^{-1/2}\, h^{-1/2} \sqrt{\widetilde{R}'_C} + R^{1/2}\, h^{-1} \Big)\, \|I_{\boldsymbol{p}}\|_{D_h} \\
&+ \max_{\Gamma_e \subset \Gamma^N} r_e^{-1/2}\, h^{-1/2} \Big(1 + \sqrt{\widetilde{R}'_C} \Big)\, h^{k+1}\, |\boldsymbol{p}|_{\boldsymbol{H}^{k+1}(\Omega)} \\
&+ \Big(1 + R^{1/2} \Big)\, h^k\, \|\boldsymbol{p}\|_{\boldsymbol{H}^{k+1}(\Omega)} .
\end{aligned}
$$

□

It is easy to see that from the above lemma, we have that $\|(G\boldsymbol{n} - G_h\boldsymbol{n}) \circ \boldsymbol{\phi}\|_{\Gamma^N}$ converges at rate h^k. Moreover, we can note that the fourth term of (6.3.5) is controlled by Lemma 3.8 of [23]. Then, the unique term that is not controlled is $\left\|\boldsymbol{\varepsilon}^{\boldsymbol{V}}_h\right\|_{\Gamma_h^N}$, for this purpose we will to present the following result

Corollary 6.3.1. *$\exists\, C_{tr} > 0$ independent of h, then we have that*

$$
\left\|\boldsymbol{\varepsilon}^{\boldsymbol{V}}_h\right\|_{\Gamma_h^N} \leq C_{tr}\, h^{-1/2} \left\|\boldsymbol{\varepsilon}^{\boldsymbol{V}}_h\right\|_{\mathcal{T}_h} . \tag{6.3.8}
$$

Proof. Let us notice that

$$
\left\|\boldsymbol{\varepsilon}^{\boldsymbol{V}}_h\right\|_{\Gamma_h^N} \leq \left\|\boldsymbol{\varepsilon}^{\boldsymbol{V}}_h\right\|_{\Gamma_h} = \sum_{e \subset \Gamma_h} \left\|\boldsymbol{\varepsilon}^{\boldsymbol{V}}_h\right\|_e ,
$$

then, by discrete trace inequality, $\exists\, C_{tr} > 0$ independent of h, such that

$$\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\Gamma_h^N} \leq C_{tr}\, h^{-1/2} \sum_{K \subset \mathcal{T}_h} \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_K \leq C_{tr}\, h^{-1/2} \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h}.$$

□

With Corollary 6.3.1 we have that the term $\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\Gamma_h^N}$ of the right side of (6.3.5) is bounded by $C_{tr}\, h^{-1/2} \left\|\boldsymbol{\varepsilon}_h^{V}\right\|_{\mathcal{T}_h}$, but to control this last term we have to develop the error estimates for $\boldsymbol{e}^{\boldsymbol{V}}$, this will be done in the next chapter. Once we have performed this analysis, we will have the error analysis done for the deformation field equation.

6.4 Error estimates for $\boldsymbol{e}^{\boldsymbol{V}}$

In this chapter we develop the error estimates for $\boldsymbol{e}^{\boldsymbol{V}}$, for this purpose we will follow the same strategy done for the case of the e^y and e^z estimates, this is, we will use a dual problem to find the estimates. For any given $\boldsymbol{U} \in [L^2(\Omega)]^d$, let be $(\boldsymbol{\gamma}, \boldsymbol{u})$ solution of

$$\boldsymbol{\gamma} + \nabla \boldsymbol{u} = 0 \quad \text{in } \Omega\,, \tag{6.4.1a}$$

$$\mathbf{div}(\boldsymbol{\gamma}) = \boldsymbol{U} \quad \text{in } \Omega\,, \tag{6.4.1b}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma^D\,, \tag{6.4.1c}$$

$$\boldsymbol{\gamma}\,\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma^N\,. \tag{6.4.1d}$$

we will assume that $\boldsymbol{u}$ is in $[H^2(\Omega)]^d$ and $\boldsymbol{\gamma}$ is in $[H^1(\Omega)]^{d\times d}$. This is true for example when Ω is a convex polygon or when Γ is of boundary $\mathcal{C}^2$. Thus, (6.4.1) satisfies

$$\|\boldsymbol{u}\|_{[H^2(\Omega)]^d} + \|\boldsymbol{\gamma}\|_{[H^1(\Omega)]^{d\times d}} \leq C\, \|\boldsymbol{U}\|_\Omega\,, \tag{6.4.2}$$

where $C > 0$ depends on the domain Ω. Then, we have the following identity

Lemma 6.4.1. *We have that*

$$(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{U})_{\mathcal{T}_h} \,=\, (\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} \,-\, (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} \,+\, \mathbb{T}_{\boldsymbol{V},h}\,, \tag{6.4.3}$$

where

$$\mathbb{T}_{\boldsymbol{V},h} \,=\, \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \boldsymbol{\gamma}\, \boldsymbol{n}_h\rangle_{\Gamma_h^D} - \langle (G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}, \boldsymbol{u}\rangle_{\Gamma_h^N} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\, \boldsymbol{n}_h\rangle_{\Gamma_h^N} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h, \boldsymbol{u}\rangle_{\Gamma_h^D}\,. \tag{6.4.4}$$

Proof. Let us note the following

$$\begin{aligned}
(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{U})_{\mathcal{T}_h} \,&=\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\boldsymbol{\gamma}))_{\mathcal{T}_h} \\
&=\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\boldsymbol{\gamma}))_{\mathcal{T}_h} \,-\, (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\gamma})_{\mathcal{T}_h} \,-\, (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \nabla\boldsymbol{u})_{\mathcal{T}_h} \\
&=\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma}))_{\mathcal{T}_h} \,+\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma}))_{\mathcal{T}_h} \,-\, (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} \,-\, (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} \\
&\quad -\, (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \nabla\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u})_{\mathcal{T}_h} \,-\, (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \nabla(\boldsymbol{u} - \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}))_{\mathcal{T}_h}\,,
\end{aligned}$$

setting $\boldsymbol{\psi} \,=\, \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma}$ in (6.3.1a) and $\boldsymbol{w} \,=\, \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}$ in (6.3.1b), which implies that

$$\begin{aligned}
(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{U})_{\mathcal{T}_h} \,&=\, (\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} \,+\, \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma}\, \boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} \,-\, \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h, \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}\rangle_{\partial\mathcal{T}_h} \,+\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma}))_{\mathcal{T}_h} \\
&\quad -\, (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} \,-\, (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \nabla(\boldsymbol{u} - \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}))_{\mathcal{T}_h} \\
&=\, (\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} \,-\, (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\gamma} - \Pi_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} \,+\, \mathbb{T}_{\boldsymbol{V},h}\,,
\end{aligned}$$

where $\mathbb{T}_{\boldsymbol{V},h}$ is defined as

$$\mathbb{T}_{\boldsymbol{V},h} \,:=\, \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma}\, \boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h, \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}\rangle_{\partial\mathcal{T}_h} + (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \mathbf{div}(\boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma}))_{\mathcal{T}_h} - (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \nabla(\boldsymbol{u} - \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}))_{\mathcal{T}_h}\,.$$

Then, integrating by parts and using the HDG-projection, particularly (6.0.1a) and (6.0.1b), we get the following

$$\begin{aligned}
\mathbb{T}_{\boldsymbol{V},h} \,&=\, \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma}\, \boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} \,-\, \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\, \boldsymbol{n}_h, \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}\rangle_{\partial\mathcal{T}_h} \,-\, (\nabla\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\gamma} - \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{\gamma})_{\mathcal{T}_h} \\
&\quad -\, \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, (\boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})\, \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,+\, (\mathbf{div}(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}), \boldsymbol{u} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{u})_{\mathcal{T}_h} \,-\, \langle \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\, \boldsymbol{n}_h, \boldsymbol{u} - \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}\rangle_{\partial\mathcal{T}_h}
\end{aligned}$$

$$
\begin{aligned}
&= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma} - \boldsymbol{\gamma})\,\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u}\rangle_{\partial\mathcal{T}_h} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\partial\mathcal{T}_h} \\
&+ \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, (\boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})\,\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{u} - \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u}\rangle_{\partial\mathcal{T}_h} \\
&= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma} - \boldsymbol{\gamma})\,\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} - \langle (\underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}} - \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}})\,\boldsymbol{n}_h, \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} \\
&- \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\partial\mathcal{T}_h} \\
&= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma} - \boldsymbol{\gamma})\,\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} - \langle (\underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}} - \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}})\,\boldsymbol{n}_h, \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h} \\
&- \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\Gamma_h},
\end{aligned}
$$

the last two terms hold, because by (6.3.1c) we have that $\langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} = 0$, moreover we can note that as $\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}$ is single valued in $\mathcal{E}_h$ and $\boldsymbol{\gamma} \in [H(\mathbf{div};\Omega)]^{d\times d}$, it follows

$$
\begin{aligned}
\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} &= \sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \cdot \boldsymbol{\gamma}\,\boldsymbol{n}_h \\
&= \sum_{e\in\mathcal{E}_h^{\circ}} \int_e [\![\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}]\!]_e\,\{\!\{\boldsymbol{\gamma}\}\!\}_e + \{\!\{\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\}\!\}_e\,[\![\boldsymbol{\gamma}]\!]_e + \sum_{e\in\mathcal{E}_h^{\partial}} \int_e \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \cdot \boldsymbol{\gamma}\,\boldsymbol{n}_h \\
&= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h}.
\end{aligned}
$$

Therefore, $\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} = 0$. In turn, by (6.3.1f) we obtain

$$
\begin{aligned}
\mathbb{T}_{\boldsymbol{V},h} &= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma} - \boldsymbol{\gamma})\,\boldsymbol{n}_h\rangle_{\partial\mathcal{T}_h} + \langle \tau\,(\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}), \boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u}\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h} \\
&- \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\Gamma_h} \\
&= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma} - \boldsymbol{\gamma})\,\boldsymbol{n}_h + \tau\,(\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u})\rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\Gamma_h}.
\end{aligned}
$$

We note by (6.0.1c) that $\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma} - \boldsymbol{\gamma})\,\boldsymbol{n}_h + \tau\,(\boldsymbol{\Pi}_{\boldsymbol{W}}\boldsymbol{u} - \boldsymbol{u})\rangle_{\partial\mathcal{T}_h} = 0$. Finally we obtain by (6.3.1d) and (6.3.1e) that

$$
\begin{aligned}
\mathbb{T}_{\boldsymbol{V},h} &= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\Gamma_h} \\
&= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h^D} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h^N} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\Gamma_h^D} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\Gamma_h^N} \\
&= \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h^D} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h^N} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\Gamma_h^D} - \langle (G\boldsymbol{n} - G_h\boldsymbol{n}) \circ \boldsymbol{\phi}, \boldsymbol{u}\rangle_{\Gamma_h^N}.
\end{aligned}
$$

□

We now establish the following result, which is one of the most important obtained in this chapter. In fact, through this lemma we can deduce the convergence rate of the scheme.

Lemma 6.4.2. *There exists $C > 0$ independent of h such that*

$$\begin{aligned}\Big(1 - H_2^{\boldsymbol{V}}(R,h)\Big)\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} \leq\; & H_1^{\boldsymbol{V}}(R,h)\,\|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}\\ &+ H_3^{\boldsymbol{V}}(R,h)\,\|\partial_{\boldsymbol{n}_h}(\underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h)\|_{D_h^c,(h^\perp)^2}\\ &+ H_4^{\boldsymbol{V}}(R,h)\,\|\underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\|_{\Gamma_h^D,h^\perp}\\ &+ H_5^{\boldsymbol{V}}(R,h)\,\|(G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|_{\Gamma_h^N}\\ &+ H_6^{\boldsymbol{V}}(R,h)\,\|I_{\boldsymbol{V}}\|_{\Gamma_h^D,h^\perp}\end{aligned}$$

where

$$\begin{aligned}H_1^{\boldsymbol{V}}(R,h) &:= C\,\Big(h + R\,h + R\,h\left\|l^{-1/2}(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R^{1/2}h^{1/2} + \widetilde{R}_{\tau^{-1}}^{1/2}h^{1/2}\\ &\quad + \widetilde{R}_{\tau^{-1}}^{1/2}h^{1/2}\left\|l^{-1/2}(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R_C^2\,h^{1/2} + R_\tau\,h\Big)\,,\\ H_2^{\boldsymbol{V}}(R,h) &:= C\,\Big(R^{1/2} + h^{1/2} + R\,h^{1/2} + R\,h^{1/2}\left\|l^{-1/2}(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + \widetilde{R}_{\tau^{-1}}^{1/2}\\ &\quad + \widetilde{R}_{\tau^{-1}}^{1/2}\left\|l^{-1/2}(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R_C^2 + R_\tau\,h^{1/2}\Big)\,,\\ H_3^{\boldsymbol{V}}(R,h) &:= C\,\Big(R^{3/2}h^{1/2} + R\,h + R^2\,h + R^2\,h\left\|l^{-1/2}(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R\,\widetilde{R}_{\tau^{-1}}^{1/2}h^{1/2}\\ &\quad + R\,\widetilde{R}_{\tau^{-1}}^{1/2}h^{1/2}\left\|l^{-1/2}(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R\,R_C^2\,h^{1/2} + R\,R_\tau\,h\Big)\,,\\ H_4^{\boldsymbol{V}}(R,h) &:= C\,\Big(h + R^{1/2}h^{3/2} + R^{3/2}h^{3/2} + R^{3/2}h^{3/2}\left\|l^{-1/2}(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R\,h\\ &\quad + R^{1/2}\,\widetilde{R}_{\tau^{-1}}^{1/2}\,h + R^{1/2}\,\widetilde{R}_{\tau^{-1}}^{1/2}\,h\left\|l^{-1/2}(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R^{1/2}R_C^2\,h + R^{1/2}R_\tau h^{3/2}\Big)\,,\\ H_5^{\boldsymbol{V}}(R,h) &:= C\,\Big(h + R\,h + R\,h\left\|l^{-1/2}(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R^{1/2}h^{1/2} + \widetilde{R}_{\tau^{-1}}^{1/2}h^{1/2}\\ &\quad + \widetilde{R}_{\tau^{-1}}^{1/2}h^{1/2}\left\|l^{-1/2}(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R_C^2\,h^{1/2} + R_\tau\,h\Big)\,,\\ H_6^{\boldsymbol{V}}(R,h) &:= C\,(R_\tau\,h^{1/2})\,.\end{aligned}$$

Proof. First we have to note that from (6.3.1f) and (6.3.3) that

$$\underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h = \underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\,\boldsymbol{n}_h + \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) = l^{-1}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h) - \Lambda^{\underline{I}_{\boldsymbol{\sigma}}} - \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} - \underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h + \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\,,$$

and as we know from (6.4.1) that

$$\mathbb{T}_{\boldsymbol{V},h} = \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \boldsymbol{\gamma}\boldsymbol{n}_h\rangle_{\Gamma_h^D} - \langle (G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}, \boldsymbol{u}\rangle_{\Gamma_h^N} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h^N} - \langle \underline{\boldsymbol{\varepsilon}}_h^{\widehat{\boldsymbol{\sigma}}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\Gamma_h^D},$$

it follows that

$$\begin{aligned}\mathbb{T}_{\boldsymbol{V},h} &= \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h^D} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h^N} - \langle (G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}, \boldsymbol{u}\rangle_{\Gamma_h^N} \\ &\quad - \langle l^{-1}(\boldsymbol{g}_D - \boldsymbol{g}_D^h) - \Lambda^{\underline{I}_{\boldsymbol{\sigma}}} - \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} - \underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h + \tau(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \boldsymbol{u}\rangle_{\Gamma_h^D}.\end{aligned} \tag{6.4.5}$$

On the other hand, keeping in mind that $\boldsymbol{\gamma}\,\boldsymbol{n} = 0$ on Γ^N, let us note that

$$\begin{aligned}\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h^N} &= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h^N} + \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h^N} \\ &= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\rangle_{\Gamma_h^N} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\rangle_{\Gamma_h^N} \\ &\quad + \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\rangle_{\Gamma_h^N} + \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\rangle_{\Gamma_h^N} \\ &= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\rangle_{\Gamma_h^N} + \langle (\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}})\circ\boldsymbol{\phi}^{-1}, \boldsymbol{\gamma}\,\boldsymbol{n}\rangle_{\Gamma^N} \\ &\quad + \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\rangle_{\Gamma_h^N} + \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\circ\boldsymbol{\phi}^{-1}, \boldsymbol{\gamma}\,\boldsymbol{n}\rangle_{\Gamma^N} \\ &= \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\rangle_{\Gamma_h^N} + \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\rangle_{\Gamma_h^N},\end{aligned}$$

thus, replacing the above in (6.4.5) we obtain

$$\begin{aligned}\mathbb{T}_{\boldsymbol{V},h} &= \langle \boldsymbol{g}_D - \boldsymbol{g}_D^h, \boldsymbol{\gamma}\,\boldsymbol{n}_h\rangle_{\Gamma_h^D} + \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\rangle_{\Gamma_h^N} \\ &\quad + \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\rangle_{\Gamma_h^N} \\ &\quad - \langle l^{-1}(\boldsymbol{g}_D - \boldsymbol{g}_D^h) - \Lambda^{\underline{I}_{\boldsymbol{\sigma}}} - \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} - \underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h + \tau(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}), \boldsymbol{u}\rangle_{\Gamma_h^D}.\end{aligned}$$

Then, applying Cauchy-Schwarz inequality we have that

$$\begin{aligned}|\mathbb{T}_{\boldsymbol{V},h}| &\le \left\|\boldsymbol{g}_D - \boldsymbol{g}_D^h\right\|_{\Gamma_h^D} \|\boldsymbol{\gamma}\,\boldsymbol{n}_h\|_{\Gamma_h^D} + \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\right\|_{\Gamma_h^N} \|\boldsymbol{\gamma}\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} \\ &\quad + \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\Gamma_h^N} \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} + \left\|l^{-1}(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\|\Lambda^{\underline{I}_{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\ &\quad + \left\|\Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \langle \underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\Gamma_h^D} + \left\|\tau(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D}.\end{aligned}$$

On the other hand, we have to note from (6.0.1c) that

$$\begin{aligned}
\langle \underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{u}\rangle_{\Gamma_h^D} &= \langle \underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{u} - P_{\boldsymbol{M}}\boldsymbol{u}\rangle_{\Gamma_h^D} + \langle \underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, P_{\boldsymbol{M}}\boldsymbol{u}\rangle_{\Gamma_h^D} \\
&= \langle \underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{u} - P_{\boldsymbol{M}}\boldsymbol{u}\rangle_{\Gamma_h^D} - \langle \tau\, I_{\boldsymbol{V}}, P_{\boldsymbol{M}}\boldsymbol{u}\rangle_{\Gamma_h^D} \\
&= \langle \underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h, \boldsymbol{u} - P_{\boldsymbol{M}}\boldsymbol{u}\rangle_{\Gamma_h^D} - \langle \tau\, P_{\boldsymbol{M}} I_{\boldsymbol{V}}, \boldsymbol{u}\rangle_{\Gamma_h^D} \\
&\leq \|\underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\|_{\Gamma_h^D}\, \|\boldsymbol{u} - P_{\boldsymbol{M}} u\|_{\Gamma_h^D} + \|\tau P_M I_{\boldsymbol{V}}\|_{\Gamma_h^D}\, \|\boldsymbol{u}\|_{\Gamma_h^D}\,.
\end{aligned}$$

Thus, by Lemma 6.4.1 we can deduce the following

$$\begin{aligned}
(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{U})_{\mathcal{T}_h} \leq\ & (\underline{I}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} - (\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} + \left\| l^{1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \left\| l^{-1/2}\boldsymbol{\gamma}\,\boldsymbol{n}_h\right\|_{\Gamma_h^D} \\
&+ \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\right\| \, \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} \\
&+ \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\Gamma_h^N} \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} + \left\| l^{-1}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
&+ \left\| \Lambda^{\underline{I}_{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\| \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \|\underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\|_{\Gamma_h^D} \|\boldsymbol{u} - P_{\boldsymbol{M}}\boldsymbol{u}\|_{\Gamma_h^D} \\
&+ \|\tau P_{\boldsymbol{M}} I_{\boldsymbol{V}}\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\| \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
=\ & (\underline{I}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma} - \boldsymbol{\gamma})_{\mathcal{T}_h} + (\underline{I}_{\boldsymbol{\sigma}}, \boldsymbol{\gamma})_{\mathcal{T}_h} - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} \\
&- (\boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} + \left\| l^{1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \left\| l^{-1/2}\,\boldsymbol{\gamma}\,\boldsymbol{n}_h\right\|_{\Gamma_h^D} \\
&+ \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\right\| \, \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} \\
&+ \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\Gamma_h^N} \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} + \left\| l^{-1}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
&+ \left\| \Lambda^{\underline{I}_{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\| \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \|\underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\|_{\Gamma_h^D} \|\boldsymbol{u} - P_{\boldsymbol{M}}\boldsymbol{u}\|_{\Gamma_h^D} \\
&+ \|\tau\, P_{\boldsymbol{M}} I_{\boldsymbol{V}}\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\| \tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
=\ & (\underline{I}_{\boldsymbol{\sigma}}, \boldsymbol{\gamma})_{\mathcal{T}_h} - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} \\
&+ \left\| l^{1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \left\| l^{-1/2}\,\boldsymbol{\gamma}\,\boldsymbol{n}_h\right\|_{\Gamma_h^D} \\
&+ \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\right\| \, \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} \\
&+ \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\Gamma_h^N} \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} + \left\| l^{-1}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
&+ \left\| \Lambda^{\underline{I}_{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\| \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \|\underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\|_{\Gamma_h^D} \|\boldsymbol{u} - P_{\boldsymbol{M}}\boldsymbol{u}\|_{\Gamma_h^D}
\end{aligned}$$

$$+ \|\tau\, P_{\boldsymbol{M}} I_{\boldsymbol{V}}\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\|\tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} .$$

Then, as $\boldsymbol{\gamma}|_K \in [\mathbb{P}_{k-1}(K)]^{d\times d}$ for all $K \in \mathcal{T}_h$ by HDG-projection, particularly (6.0.1c) and if the elliptic regularity inequality (6.4.2) holds, we have that

$$\begin{aligned}
(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{U})_{\mathcal{T}_h} \leq\; & (\underline{I}_{\boldsymbol{\sigma}}, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)_{\mathcal{T}_h} - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma})_{\mathcal{T}_h} \\
& + \left\|l^{1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \left\|l^{-1/2}\,\boldsymbol{\gamma}\,\boldsymbol{n}_h\right\|_{\Gamma_h^D} \\
& + \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\right\| \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} \\
& + \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\Gamma_h^N} \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} + \left\|l^{-1}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
& + \left\|\Lambda^{\underline{I}_{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\|\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \|\underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\|_{\Gamma_h^D} \|\boldsymbol{u} - P_{\boldsymbol{M}}\boldsymbol{u}\|_{\Gamma_h^D} \\
& + \|\tau P_{\boldsymbol{M}} I_{\boldsymbol{V}}\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\|\tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
\leq\; & \|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{\mathcal{T}_h} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathcal{T}_h} \|\boldsymbol{\gamma} - \boldsymbol{\Pi}_{\mathbb{Z}}\boldsymbol{\gamma}\|_{\mathcal{T}_h} \\
& + \left\|l^{1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \left\|l^{-1/2}\boldsymbol{\gamma}\,\boldsymbol{n}_h\right\|_{\Gamma_h^D} \\
& + \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\right\| \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} \\
& + \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\Gamma_h^N} \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} + \left\|l^{-1}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
& + \left\|\Lambda^{\underline{I}_{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\|\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \|\underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\|_{\Gamma_h^D} \|\boldsymbol{u} - P_{\boldsymbol{M}}\boldsymbol{u}\|_{\Gamma_h^D} \\
& + \|\tau\, P_{\boldsymbol{M}} I_{\boldsymbol{V}}\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\|\tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
\leq\; & C\,h\, \|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} \|\boldsymbol{U}\|_\Omega + C\,h\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathcal{T}_h} \|\boldsymbol{U}\|_\Omega \\
& + \left\|l^{1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \left\|l^{-1/2}\,\boldsymbol{\gamma}\,\boldsymbol{n}_h\right\|_{\Gamma_h^D} \\
& + \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}\right\| \|\boldsymbol{\gamma}\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} \\
& + \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\Gamma_h^N} \|\boldsymbol{\gamma}\,\boldsymbol{n}_h - (\boldsymbol{\gamma}\circ\boldsymbol{\phi})(\boldsymbol{n}\circ\boldsymbol{\phi})\|_{\Gamma_h^N} + \left\|l^{-1}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
& + \left\|\Lambda^{\underline{I}_{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\|\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \|\underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\|_{\Gamma_h^D} \|\boldsymbol{u} - P_{\boldsymbol{M}}\boldsymbol{u}\|_{\Gamma_h^D} \\
& + \|\tau\, P_{\boldsymbol{M}} I_{\boldsymbol{V}}\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\|\tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} ,
\end{aligned}$$

and now by (6.3.1), we obtain

$$
\begin{aligned}
(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{U})_{\mathcal{T}_h} \lesssim\; & h\, \|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}\, \|\boldsymbol{U}\|_\Omega + h\, \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}\, \|\boldsymbol{U}\|_\Omega + \left\| l^{1/2} (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \left\| l^{-1/2} \boldsymbol{\gamma}\, \boldsymbol{n}_h \right\|_{\Gamma_h^D} \\
& + \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}} \right\|_{\Gamma_h^N} \|\boldsymbol{\gamma}\, \boldsymbol{n}_h - (\boldsymbol{\gamma} \circ \boldsymbol{\phi})(\boldsymbol{n} \circ \boldsymbol{\phi})\|_{\Gamma_h^N} \\
& + C_{tr}\, h^{-1/2} \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} \right\|_{\mathcal{T}_h} \|\boldsymbol{\gamma}\, \boldsymbol{n}_h - (\boldsymbol{\gamma} \circ \boldsymbol{\phi})(\boldsymbol{n} \circ \boldsymbol{\phi})\|_{\Gamma_h^N} \\
& + \left\| l^{-1} (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\| \Lambda^{\underline{I}_{\boldsymbol{\sigma}}} \right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\| \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
& + \|\underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h\|_{\Gamma_h^D} \|\boldsymbol{u} - P_{\boldsymbol{M}} \boldsymbol{u}\|_{\Gamma_h^D} + \|\tau\, P_{\boldsymbol{M}} I_{\boldsymbol{V}}\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} + \left\| \tau\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^D} \|\boldsymbol{u}\|_{\Gamma_h^D} \\
\lesssim\; & h\, \|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}\, \|\boldsymbol{U}\|_\Omega + h\, \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}\, \|\boldsymbol{U}\|_\Omega \\
& + \left\| l^{1/2} (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \left\| l^{-1/2} (\boldsymbol{\gamma}\, \boldsymbol{n}_h - (\boldsymbol{\gamma} \circ \boldsymbol{\phi})(\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_{\Gamma_h^D} \\
& + \left\| l^{1/2} (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \left\| l^{-1/2} (l^{-1} \boldsymbol{u} + \partial_{\boldsymbol{n}_h} \boldsymbol{u}) \right\|_{\Gamma_h^D} + \left\| l^{1/2} (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \left\| l^{-3/2} \boldsymbol{u} \right\|_{\Gamma_h^D} \\
& + \left\| l^{1/2} (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^N} \left\| l^{-1/2} (\boldsymbol{\gamma}\, \boldsymbol{n}_h - (\boldsymbol{\gamma} \circ \boldsymbol{\phi})(\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_{\Gamma_h^N} \\
& + h^{-1/2}\, l^{1/2} \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} \right\|_{\mathcal{T}_h} \left\| l^{-1/2} (\boldsymbol{\gamma}\, \boldsymbol{n}_h - (\boldsymbol{\gamma} \circ \boldsymbol{\phi})(\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_{\Gamma_h^N} + \left\| \boldsymbol{g}_D - \boldsymbol{g}_D^h \right\|_{\Gamma_h^D} \left\| l^{-1} \boldsymbol{u} \right\|_{\Gamma_h^D} \\
& + \left\| l\, \Lambda^{\underline{I}_{\boldsymbol{\sigma}}} \right\|_{\Gamma_h^D} \left\| l^{-1} \boldsymbol{u} \right\|_{\Gamma_h^D} + \left\| l\, \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \right\|_{\Gamma_h} \left\| l^{-1} \boldsymbol{u} \right\|_{\Gamma_h^D} \\
& + \left\| \underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h\, h^{1/2} \right\|_{\Gamma_h^D} \left\| h^{-1/2} (\boldsymbol{u} - P_{\boldsymbol{M}} \boldsymbol{u}) \right\|_{\Gamma_h^D} + \|l\, \tau\, P_{\boldsymbol{M}} I_{\boldsymbol{V}}\|_{\Gamma_h^D} \left\| l^{-1} \boldsymbol{u} \right\|_{\Gamma_h^D} \\
& + \left\| l\, \tau\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^D} \left\| l^{-1} \boldsymbol{u} \right\|_{\Gamma_h^D} \\
\lesssim\; & h\, \|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}\, \|\boldsymbol{U}\|_\Omega + h\, \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h}\, \|\boldsymbol{U}\|_\Omega \\
& + \left\| l^{1/2} (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \left\| l^{-1/2} (\boldsymbol{\gamma} - \boldsymbol{\gamma} \circ \boldsymbol{\phi})\, \boldsymbol{n}_h \right\|_{\Gamma_h^D} \\
& + \left\| l^{1/2} (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \left\| l^{-1/2} (\boldsymbol{\gamma} \circ \boldsymbol{\phi})(\boldsymbol{n}_h - (\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_{\Gamma_h^D} \\
& + \left\| l^{1/2} (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \left\| l^{-1/2} (l^{-1} \boldsymbol{u} + \partial_{\boldsymbol{n}_h} \boldsymbol{u}) \right\|_{\Gamma_h^D} + \left\| l^{1/2} (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \left\| l^{-3/2} \boldsymbol{u} \right\|_{\Gamma_h^D} \\
& + \left\| l^{1/2} (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^N} \left\| l^{-1/2} (\boldsymbol{\gamma} - \boldsymbol{\gamma} \circ \boldsymbol{\phi})\, \boldsymbol{n}_h \right\|_{\Gamma_h^N} \\
& + \left\| l^{1/2} (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^N} \left\| l^{-1/2} (\boldsymbol{\gamma} \circ \boldsymbol{\phi})(\boldsymbol{n}_h - (\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_{\Gamma_h^N} \\
& + h^{-1/2}\, l^{1/2} \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} \right\|_{\mathcal{T}_h} \left\| l^{-1/2} (\boldsymbol{\gamma} - \boldsymbol{\gamma} \circ \boldsymbol{\phi})\, \boldsymbol{n}_h \right\|_{\Gamma_h^N} \\
& + h^{-1/2}\, l^{1/2} \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} \right\|_{\mathcal{T}_h} \left\| l^{-1/2} (\boldsymbol{\gamma} \circ \boldsymbol{\phi})(\boldsymbol{n}_h - (\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_{\Gamma_h^N} + \left\| \boldsymbol{g}_D - \boldsymbol{g}_D^h \right\|_{\Gamma_h^D} \left\| l^{-1} \boldsymbol{u} \right\|_{\Gamma_h^D}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left\| l\, \Lambda^{\underline{I}_{\boldsymbol{\sigma}}} \right\|_{\Gamma_h^D} \left\| l^{-1}\, \boldsymbol{u} \right\|_{\Gamma_h^D} + \left\| l\, \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \right\|_{\Gamma_h} \left\| l^{-1}\, \boldsymbol{u} \right\|_{\Gamma_h^D} \\
&+ \left\| \underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h\, h^{1/2} \right\|_{\Gamma_h^D} \left\| h^{-1/2}\, (\boldsymbol{u} - P_{\boldsymbol{M}} \boldsymbol{u}) \right\|_{\Gamma_h^D} + \| l\, \tau\, P_{\boldsymbol{M}} I_{\boldsymbol{V}} \|_{\Gamma_h^D} \left\| l^{-1}\, \boldsymbol{u} \right\|_{\Gamma_h^D} \\
&+ \left\| l\, \tau\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^D} \left\| l^{-1}\, \boldsymbol{u} \right\|_{\Gamma_h^D} .
\end{aligned}
$$

Then, by Lemma 6.1.13, [51, Lemma 2.1] and if the elliptic regularity inequality (6.4.2) holds, we have that

$$
\begin{aligned}
(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}, \boldsymbol{U})_{\mathcal{T}_h} \lesssim\ & h\, \|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} \|\boldsymbol{U}\|_\Omega + h\, \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} \|\boldsymbol{U}\|_\Omega + \left\| l^{1/2}\, (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega \\
&+ \left\| l^{1/2}\, (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \left\| l^{-1/2}\, (\boldsymbol{n}_h - (\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_\infty \|\boldsymbol{\gamma} \circ \boldsymbol{\phi}\|_{\Gamma_h^D} + \left\| \boldsymbol{g}_D - \boldsymbol{g}_D^h \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega \\
&+ \left\| l^{1/2}\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^N} \|\boldsymbol{U}\|_\Omega \\
&+ \left\| l^{1/2}\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^N} \left\| l^{-1/2}\, (\boldsymbol{n}_h - (\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_\infty \|\boldsymbol{\gamma} \circ \boldsymbol{\phi}\|_{\Gamma_h^N} \\
&+ h^{-1/2}\, l^{1/2} \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} \right\|_{\mathcal{T}_h} \|\boldsymbol{U}\|_\Omega + h^{-1/2}\, l^{1/2} \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} \right\|_{\mathcal{T}_h} \left\| l^{-1/2}\, (\boldsymbol{n}_h - (\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_\infty \|\boldsymbol{\gamma} \circ \boldsymbol{\phi}\|_{\Gamma_h^N} \\
&+ \left\| l\, \Lambda^{\underline{I}_{\boldsymbol{\sigma}}} \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega + \left\| l\, \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega + h \left\| \underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h\, h^{1/2} \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega \\
&+ \| l\, \tau\, P_{\boldsymbol{M}} I_{\boldsymbol{V}} \|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega + \left\| l\, \tau\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega \\
\lesssim\ & h\, \|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} \|\boldsymbol{U}\|_\Omega + h\, \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} \|\boldsymbol{U}\|_\Omega + \left\| l^{1/2}\, (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega \\
&+ \left\| l^{1/2}\, (\boldsymbol{g}_D - \boldsymbol{g}_D^h) \right\|_{\Gamma_h^D} \left\| l^{-1/2}\, (\boldsymbol{n}_h - (\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_\infty \|\boldsymbol{U}\|_\Omega + \left\| \boldsymbol{g}_D - \boldsymbol{g}_D^h \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega \\
&+ \left\| l^{1/2}\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^N} \|\boldsymbol{U}\|_\Omega \\
&+ \left\| l^{1/2}\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^N} \left\| l^{-1/2}\, (\boldsymbol{n}_h - (\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_\infty \|\boldsymbol{U}\|_\Omega \\
&+ h^{-1/2}\, l^{1/2} \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} \right\|_{\mathcal{T}_h} \|\boldsymbol{U}\|_\Omega + h^{-1/2}\, l^{1/2} \left\| \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} \right\|_{\mathcal{T}_h} \left\| l^{-1/2}\, (\boldsymbol{n}_h - (\boldsymbol{n} \circ \boldsymbol{\phi})) \right\|_\infty \|\boldsymbol{U}\|_\Omega \\
&+ \left\| l\, \Lambda^{\underline{I}_{\boldsymbol{\sigma}}} \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega + \left\| l\, \Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}} \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega + h \left\| \underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h\, h^{1/2} \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega \\
&+ \| l\, \tau P_{\boldsymbol{M}} I_{\boldsymbol{V}} \|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega + \left\| l\, \tau\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}) \right\|_{\Gamma_h^D} \|\boldsymbol{U}\|_\Omega .
\end{aligned}
$$

Then, setting $\boldsymbol{U} = \boldsymbol{\varepsilon}_h^{\boldsymbol{V}}$, it follows that

$$
\begin{aligned}
\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} \lesssim\; & h\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} + h\,\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} + \left\|l^{1/2}\,\boldsymbol{g}_D - \boldsymbol{g}_D^h\right\|_{\Gamma_h^D} \\
& + \left\|l^{1/2}\,(\boldsymbol{g}_D - \boldsymbol{g}_D^h)\right\|_{\Gamma_h^D} \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_{\infty} + \left\|\boldsymbol{g}_D - \boldsymbol{g}_D^h\right\|_{\Gamma_h^D} + \left\|l^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\Gamma_h^N} \\
& + \left\|l^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\Gamma_h^N} \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_{\infty} + h^{-1/2}\,l^{1/2}\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} \\
& + h^{-1/2}\,l^{1/2}\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_{\infty} + \left\|l\,\Lambda^{\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} + \left\|l\,\Lambda^{\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}}\right\|_{\Gamma_h^D} \\
& + h\left\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\,h^{1/2}\right\|_{\Gamma_h^D} + \|l\,\tau P_{\boldsymbol{M}}\boldsymbol{I}_{\boldsymbol{V}}\|_{\Gamma_h^D} + \left\|l\,\tau\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\Gamma_h^D} \\
\lesssim\; & h\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} + h\,\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} + \max_{e\in\mathcal{E}_h^{\partial}} r_e\,h\left\|\boldsymbol{g}_D - \boldsymbol{g}_D^h\right\|_{\Gamma_h^D,l^{-1}} \\
& + \max_{e\in\mathcal{E}_h^{\partial}} r_e\,h\left\|\boldsymbol{g}_D - \boldsymbol{g}_D^h\right\|_{\Gamma_h^D,l^{-1}} \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_{\infty} + \max_{e\in\mathcal{E}_h^{\partial}} r_e^{1/2}\,h^{1/2}\left\|\boldsymbol{g}_D - \boldsymbol{g}_D^h\right\|_{\Gamma_h^D,l^{-1}} \\
& + \max_{e\in\mathcal{E}_h^{\partial}} r_e^{1/2}\,\tau_e^{-1/2}\,h^{1/2}\left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h} \\
& + \max_{e\in\mathcal{E}_h^{\partial}} r_e^{1/2}\,\tau_e^{-1/2}\,h^{1/2}\left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h} \left\|l^{1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_{\infty} \\
& + \max_{e\in\mathcal{E}_h^{\partial}} r_e^{1/2}\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} + \max_{e\in\mathcal{E}_h^{\partial}} r_e^{1/2}\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_{\infty} \\
& + \max_{e\in\mathcal{E}_h^{\partial}} r_e^{3/2}\,h^{1/2}\,\|\partial_{\boldsymbol{n}_h}(\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\boldsymbol{n}_h)\|_{D_h^c,(h^{\perp})^2} + \max_{e\in\mathcal{E}_h^{\partial}} r_e^2\,C_{ext}^e\,C_{inv}^e\,h^{1/2}\,\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} + h\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\|_{\Gamma_h^D,h^{\perp}} \\
& + \max_{e\in\mathcal{E}_h^{\partial}} \tau_e\,r_e h^{1/2}\,\|\boldsymbol{I}_{\boldsymbol{V}}\|_{\Gamma_h^D,h^{\perp}} + \max_{e\in\mathcal{E}_h^{\partial}} \tau_e^{1/2}\,r_e\,h\left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h}.
\end{aligned}
$$

Hence, from the assumptions (4.6.1), Definition 4.6.3 and keeping in mind the definition of $\left\|\left|(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h)\right|\right\|$ we get

$$
\begin{aligned}
\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\| \lesssim\; & h\,\|\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} + h\,\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} + R\,h\left\|\boldsymbol{g}_D - \boldsymbol{g}_D^h\right\|_{\Gamma_h^D,l^{-1}} \\
& + R\,h\left\|\boldsymbol{g}_D - \boldsymbol{g}_D^h\right\|_{\Gamma_h^D,l^{-1}} \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_{\infty} + R^{1/2}\,h^{1/2}\left\|\boldsymbol{g}_D - \boldsymbol{g}_D^h\right\|_{\Gamma_h^D,l^{-1}} \\
& + \widetilde{R}_{\tau^{-1}}^{1/2} h^{1/2}\left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h} \\
& + \widetilde{R}_{\tau^{-1}}^{1/2}\,h^{1/2}\left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h} \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_{\infty} \\
& + R^{1/2}\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} + R^{1/2}\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_{\infty} + R^{3/2}\,h^{1/2}\,\|\partial_{\boldsymbol{n}_h}(\underline{\boldsymbol{I}}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h)\|_{D_h^c,(h^{\perp})^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ R_C^2\, h^{1/2}\, \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} + h\, \|\underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h\|_{\Gamma_h^D, h^\perp} + R_\tau\, h^{1/2}\, \|I_{\boldsymbol{V}}\|_{\Gamma_h^D, h^\perp} + R_\tau\, h\, \left\|\tau^{1/2}\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}})\right\|_{\partial\mathcal{T}_h} \\
&\lesssim h\, \|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} + R^{1/2}\, \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} + R^{1/2}\, \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} + R^{1/2}\, \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty \\
&+ R^{3/2}\, h^{1/2}\, \|\partial_{\boldsymbol{n}_h}(\underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h)\|_{D_h^c, (h^\perp)^2} + h\, \|\underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h\|_{\Gamma_h^D, h^\perp} + R_\tau\, h^{1/2}\, \|I_{\boldsymbol{V}}\|_{\Gamma_h^D, h^\perp} \\
&+ \Big(h + R\,h + R\,h\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R^{1/2}\, h^{1/2} + \widetilde{R}_{\tau^{-1}}^{1/2}\, h^{1/2} \\
&+ \widetilde{R}_{\tau^{-1}}^{1/2}\, h^{1/2}\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R_C^2\, h^{1/2} + R_\tau\, h\Big) \left\|\left(\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{V}}, \boldsymbol{\varepsilon}_h^{\boldsymbol{V}} - \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{V}}}, \boldsymbol{g}_D - \boldsymbol{g}_D^h\right)\right\| \\
&\lesssim h\, \|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} + R^{1/2}\, \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} + R^{1/2}\, \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} + R^{1/2}\, \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty \\
&+ R^{3/2}\, h^{1/2}\, \|\partial_{\boldsymbol{n}}(\underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n})\|_{D_h^c, (h^\perp)^2} + h\, \|\underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h\|_{\Gamma_h^D, h^\perp} + R_\tau\, h^{1/2}\, \|I_{\boldsymbol{V}}\|_{\Gamma_h^D, h^\perp} \\
&+ \Big(h + R\,h + R\,h\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R^{1/2}\, h^{1/2} + \widetilde{R}_{\tau^{-1}}^{1/2}\, h^{1/2} \\
&+ \widetilde{R}_{\tau^{-1}}^{1/2}\, h^{1/2}\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R_C^2\, h^{1/2} + R_\tau\, h\Big) \Big(\|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} + R^{1/2}\, \|\underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h\|_{\Gamma_h^D} \\
&+ \|(G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|_{\Gamma_h^N} + R\, \|\partial_{\boldsymbol{n}_h}(\underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h)\|_{D_h^c, (h^\perp)^2} + h^{-1/2}\, \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h}\Big) \\
&\lesssim + \Big(h + R\,h + R\,h\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R^{1/2}\, h^{1/2} + \widetilde{R}_{\tau^{-1}}^{1/2}\, h^{1/2} \\
&+ \widetilde{R}_{\tau^{-1}}^{1/2}\, h^{1/2}\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R_C^2\, h^{1/2} + R_\tau\, h\Big)\, \|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} \\
&+ \Big(R^{1/2} + h^{1/2} + R\,h^{1/2} + R\,h^{1/2}\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + \widetilde{R}_{\tau^{-1}}^{1/2} \\
&+ \widetilde{R}_{\tau^{-1}}^{1/2}\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R_C^2 + R_\tau\, h^{1/2}\Big) \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} \\
&+ \Big(R^{3/2}\, h^{1/2} + R\,h + R^2\, h + R^2\, h\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R\,\widetilde{R}_{\tau^{-1}}^{1/2}\, h^{1/2} \\
&+ R\,\widetilde{R}_{\tau^{-1}}^{1/2}\, h^{1/2}\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R\,R_C^2\, h^{1/2} + R\,R_\tau\, h\Big)\, \|\partial_{\boldsymbol{n}_h}(\underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h)\|_{D_h^c, (h^\perp)^2} \\
&+ \Big(h + R^{1/2}\, h^{3/2} + R^{3/2}\, h^{3/2} + R^{3/2}\, h^{3/2}\, \left\|l^{-1}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R\,h + R^{1/2}\, \widetilde{R}_{\tau^{-1}}^{1/2}\, h \\
&+ R^{1/2}\, \widetilde{R}_{\tau^{-1}}^{1/2}\, h\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R^{1/2}\, R_C^2\, h + R^{1/2}\, R_\tau\, h^{3/2}\Big)\, \|\underline{I}_{\boldsymbol{\sigma}}\, \boldsymbol{n}_h\|_{\Gamma_h^D} \\
&+ \Big(h + R\,h + R\,h\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R^{1/2}\, h^{1/2} + \widetilde{R}_{\tau^{-1}}^{1/2}\, h^{1/2} \\
&+ \widetilde{R}_{\tau^{-1}}^{1/2}\, h^{1/2}\, \left\|l^{-1/2}\,(\boldsymbol{n}_h - (\boldsymbol{n}\circ\boldsymbol{\phi}))\right\|_\infty + R_C^2\, h^{1/2} + R_\tau\, h\Big)\, \|(G\boldsymbol{n} - G_h\boldsymbol{n})\circ\boldsymbol{\phi}\|_{\Gamma_h^N} \\
&+ R_\tau\, h^{1/2}\, \|I_{\boldsymbol{V}}\|_{\Gamma_h^D, h^\perp}\,.
\end{aligned}
$$

□

From the above lemma we can state the following corollary which gives us the convergence rate for $\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}$ on D_h,

Corollary 6.4.1. *With the same definition for $H_i^{\boldsymbol{V}}(R,h)\ \forall i \in \{1,\ldots,6\}$ of Lemma 6.4.2. If $0 < \delta \leq 1$ and $\beta > 0$ on the assumptions (4.6.4a) and (4.6.4b) respectively, there exists a positive constant C independent of h such that*

$$\left(1 - H_2^{\boldsymbol{V}}(R,h)\right) \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} \leq C\left(h + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right) h^k\,. \tag{6.4.6}$$

Proof. We can note by Theorem 6.0.1 that $\|\underline{I}_{\boldsymbol{\sigma}}\|_{\mathcal{T}_h} \lesssim h^{k+1}$, by [23, Lemma 3.8] that $\|\partial_{\boldsymbol{n}_h}(\underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h)\|_{D_h^c,(h^\perp)^2} \lesssim h^{k+1}$, by Lemma 6.3.6 that $\|\underline{I}_{\boldsymbol{\sigma}}\,\boldsymbol{n}_h\|_{\Gamma_h^D,h^\perp} \lesssim h^{k+1}$ and by discrete trace inequality followed by Theorem 6.0.1 that $\|I_{\boldsymbol{V}}\|_{\Gamma_h^D,h^\perp} \lesssim h^{k+1}$. Then, by Lemma 6.3.7 we have that

$$\begin{aligned}\left(1 - H_2^{\boldsymbol{V}}(R,h)\right) \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} \lesssim{}& \left(H_1^{\boldsymbol{V}}(R,h) + H_3^{\boldsymbol{V}}(R,h) + H_4^{\boldsymbol{V}}(R,h) + H_6^{\boldsymbol{V}}(R,h)\right) h^{k+1} \\ &+ H_5^{\boldsymbol{V}}(R,h)\, h^k\,.\end{aligned}$$

Keeping in mind the assumptions (4.6.4a) and (4.6.4b), let us to note for each $H_i(R,h)$ the following inequalities

$$\begin{aligned}
H_1^{\boldsymbol{V}}(R,h) &\lesssim \Big(h + C_R\, h^\delta + C_R\, C_{\boldsymbol{n}}\, h^{\delta+1/2+\beta} + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta} + C_R\, h^{2\delta+1/2} \\
&\quad + C_R\, h^{\delta+1}\Big) \\
&\lesssim \left(h + C_R\, h^\delta + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right), \\
H_2^{\boldsymbol{V}}(R,h) &\lesssim \left(C_R\, h^{\delta/2} + h^{1/2} + C_R\, h^{\delta+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta+\beta} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2-1/2+\beta} + C_R\, h^{2\delta}\right) \\
&\lesssim \left(h^{1/2} + C_R\, h^{\delta/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2-1/2+\beta}\right), \\
H_3^{\boldsymbol{V}}(R,h) &\lesssim \Big(C_R\, h^{3\delta/2+1/2} + C_R\, h^{\delta+1} + C_R\, h^{2\delta+1} + C_R\, C_{\boldsymbol{n}}\, h^{2\delta+1/2+\beta} + C_R\, C_{\boldsymbol{n}}\, h^{3\delta/2+\beta} \\
&\quad + C_R\, h^{3\delta+1/2}\Big) \\
&\lesssim \left(C_R\, h^{3\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{3\delta/2+\beta}\right), \\
H_4^{\boldsymbol{V}}(R,h) &\lesssim \Big(h + C_R\, h^{\delta/2+3/2} + C_R\, h^{3\delta/2+3/2} + C_R\, C_{\boldsymbol{n}}\, h^{3\delta/2+1+\beta} + C_R\, h^{\delta+1} \\
&\quad + C_R\, C_{\boldsymbol{n}}\, h^{\delta+1/2+\beta} + C_R\, h^{5\delta/2+1}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(h + C_R\, h^{\delta+1} + C_R\, C_{\boldsymbol{n}}\, h^{\delta+1/2+\beta}\right),\\
H_5^{\boldsymbol{V}}(R,h) &\lesssim \left(h + C_R\, h^{\delta+1} + C_R\, C_{\boldsymbol{n}}\, h^{\delta+1/2+\beta} + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta} + C_R\, h^{2\delta+1/2}\right)\\
&\lesssim \left(h + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right),\\
H_6^{\boldsymbol{V}}(R,h) &\lesssim C_R\, h^{\delta+1/2}\,.
\end{aligned}
$$

It is necessary that $H_2^{\boldsymbol{V}}(R,h)$ tends to zero when h tends to zero so that $(1 - H_2^{\boldsymbol{V}}(R,h))$ be positive an decays slower than the right-hand side of (6.4.6) in order to obtain the estimate for $\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h}$. To ensure the aforementioned, by the inequality found above for $H_2^{\boldsymbol{V}}(R,h)$ we note that for $(1 - H_2^{\boldsymbol{V}}(R,h))$ to be positive for a sufficiently small h, δ must necessarily be greater than zero. On the other hand, from the other inequalities for $H_i(R,h)$, $i = 1,3,4,5,6$ we can deduce

$$
\begin{aligned}
\left(1 - H_2^{\boldsymbol{V}}(R,h)\right)\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{\mathcal{T}_h} &\lesssim \Big(h + C_R\left\{h^{\delta} + h^{3\delta/2+1/2} + h^{\delta+1} + h^{\delta+1/2}\right\}\\
&\quad + C_R\, C_{\boldsymbol{n}}\left\{h^{\delta/2+\beta} + h^{3\delta/2+\beta} + h^{\delta+1/2+\beta}\right\}\Big)\, h^{k+1}\\
&\quad + \left(h + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right) h^{k}\\
&\lesssim \left(h + C_R\, h^{\delta} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right) h^{k+1}\\
&\quad + \left(h + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right) h^{k}\\
&\lesssim \left(h + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right) h^{k}\,.
\end{aligned}
$$

□

With the above Corollary we can now state the estimates for $\boldsymbol{e}^{\boldsymbol{V}}$ on Ω.

Corollary 6.4.2. *Under the same assumptions of Corollary 6.4.1 we have that exists a positive constant C independent of h such that*

$$
\|\boldsymbol{V} - \boldsymbol{V}_h\|_{\Omega} \leq C\left(h + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right) h^{k}\,.
$$

Proof. Let us note that

$$\|\boldsymbol{V} - \boldsymbol{V}_h\|_\Omega \leq \|\boldsymbol{V} - \boldsymbol{V}_h\|_{D_h} + \|\boldsymbol{V} - \boldsymbol{V}_h\|_{D_h^c} \leq \|I_{\boldsymbol{V}}\|_{D_h} + \left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\|_{D_h} + \|\boldsymbol{V} - \boldsymbol{V}_h\|_{D_h^c} \,,$$

then by Theorem 6.0.1, 6.4.1 and Lemma 3.7 of [23] we have that

$$\|\boldsymbol{V} - \boldsymbol{V}_h\|_\Omega \lesssim h^{k+1} + \left(h + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right) h^k + h\, \|\underline{I}_{\boldsymbol{\sigma}}\|_{D_h^c} + h\sqrt{\widehat{R}'_C}\, \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{D_h} \,,$$

By (6.3.5) and due that $\|(G\boldsymbol{n} - G_h\boldsymbol{n}) \circ \boldsymbol{\phi}\|_{\Gamma_h^N}$ is the lowest order term we have that $\|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{D_h} \lesssim h^k$, thus $\|\boldsymbol{V} - \boldsymbol{V}_h\|_{D_h^c} \lesssim h^{k+1}$, therefore

$$\|\boldsymbol{V} - \boldsymbol{V}_h\|_\Omega \lesssim \left(h + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right) h^k \,.$$

□

It is important to note that the estimates for $\underline{\boldsymbol{e}}^{\boldsymbol{\sigma}}$ are not relevant to our purpose due that for the performance of the algorithm 1 we do not need to compute $\boldsymbol{\sigma}_h$. However, in the following corollary we present the estimates of $\underline{\boldsymbol{e}}^{\boldsymbol{\sigma}}$ on Ω.

Corollary 6.4.3. *With the same assumptions that Corollary 6.4.1, we have that, there exists a constant $C > 0$ such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_\Omega \leq C\, h^k \,.$$

Proof. Let us note that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_\Omega = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{D_h} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{D_h^c} \leq \|\underline{I}_{\boldsymbol{\sigma}}\|_{D_h} + \|\underline{\boldsymbol{\varepsilon}}_h^{\boldsymbol{\sigma}}\|_{D_h} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{D_h^c} \,.$$

Then, by Lemma 6.3.5 and Corollary 6.4.1 we deduced that $\left\|\boldsymbol{\varepsilon}_h^{\boldsymbol{V}}\right\| \lesssim h^k$ and also by [23, Lemma 3.7 and 3.8], we obtain that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_\Omega \lesssim (1 + \sqrt{\widetilde{R}'_C})\, \|\underline{I}_{\boldsymbol{\sigma}}\|_{D_h} + (1 + \sqrt{\widetilde{R}'_C})\, h^{k+1}\, |\boldsymbol{\sigma}|_{\mathbb{H}^{k+1}(\Omega)} + h^k \,.$$

□

CHAPTER 7

Conclusions and future work

7.1 Conclusions

In this work, we first developed an introduction to shape optimization problems. Based on our model problem, we find the adjoint equation from the state equation using shape calculus tools to later deduce the optimality conditions.

We proposed (2.4.1) to find the deformation field, where it was later shown that if $\boldsymbol{V}$ satisfies (2.4.1) then $\boldsymbol{V}$ is a descent direction, thus establishing Algorithm 1 to solve the shape optimization problem.

We proposed and analyzed HDG schemes for the state and adjoint equations based on [23], we shown that under certain assumptions (4.6.1) the schemes are well-posed. We proved in Theorem 6.1.10 that under the assumptions (4.6.1), then there exists a constant $C > 0$ such that

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_\Omega + \|\boldsymbol{r} - \boldsymbol{r}_h\|_\Omega \leq C\, h^{k+1}\,.$$

Similarly, in Theorem 6.1.15 it was shown that there exists a constant $C > 0$ such that

$$\|y - y_h\|_\Omega + \|z - z_h\|_\Omega \leq C\, h^{k+1}\,.$$

Thus, we show that, theoretically, the convergence rate with the HDG scheme on curved domains for the state and adjoint equations are optimal.

We also proposed and analyzed a HDG scheme for the deformation field equation on curved domains. This was inspired by [46]. Under the Assumptions 4.6.1 we proved that the scheme is well-posed and in Corollary 6.4.3 we proved that there exists a constant $C > 0$ such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_\Omega \leq C\, h^k\,.$$

On the other hand, in Corollary 6.4.2 we proved that if $0 < \delta \leq 1$ and $\beta > 0$ on the Assumptions 4.6.4b and 4.6.4a respectively, then there exists a constant $C > 0$ such that

$$\|\boldsymbol{V} - \boldsymbol{V}_h\|_\Omega \leq C\left(h + C_R\, h^{\delta/2+1/2} + C_R\, C_{\boldsymbol{n}}\, h^{\delta/2+\beta}\right) h^k\,.$$

To obtain optimal convergence when the polyhedral subdomain D_h is at an order h distance from the boundary of the original domain, i.e. when $H_e^\perp = \mathcal{O}(h)$ for each $e \in \mathcal{E}_h^\partial$ is impossible to ensure because in Corollary 6.4.1 we obtained that for $H_2^{\boldsymbol{V}}(R,h)$ converge, necessarily δ must be greater than zero. Therefore the method have optimal convergence when for each $e \in \mathcal{E}_h^\partial$ $H_e^\perp = \mathcal{O}(h^2)$ and $H_e^\perp = 0$.

7.2 Future work

Based on this work, some avenues for further work that we propose are:

- Computational implementation
- Apply the methodology developed in this work to other problems such as Navier-Stokes, elasticity problems, among others.

- Investigate other ways to characterize $\boldsymbol{V}$. An interesting approach may be to use a level set method such as the strategy presented in [2].

- Deduce a new Neumann data transfer technique on curved domains for the HDG scheme, in order to obtain optimal convergence rates if $\delta = 0$.

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