[theorem]Remarks

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Julia Robinson numbers

Tesis para optar al grado de Magister en Matemática

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Introducción

Luego que se lograra formalizar el concepto de algoritmo en los años treinta, se pudo demostrar que la teoría de primer orden del semi-anillo $\mathbb{N}$ de los números naturales es indecidible: no existe un algoritmo que tome como entrada un enunciado de aritmética y que responda, después de finitas etapas, si o no dicho enunciado es cierto en $\mathbb{N}$. Por otra parte, en 1931, Tarski [3] demostró que la teoría del campo $\mathbb{Q}^{\text{alg}}$ de todos los números algebraicos, asi como la teoría del campo $\mathbb{Q}^{\text{alg}} \cap \mathbb{R}$ de los números algebraicos reales, es decidible. El problema entonces es el siguiente: determinar cuales anillos entre $\mathbb{N}$ y $\mathbb{Q}^{\text{alg}}$ tienen teoría decidible. Un paso importante fue dado por Julia Robinson en 1959: en [4], demuestra que para cualquier campo de números K, $\mathbb{N}$ es definible en el anillo de enteros $\mathcal{O}_K$ de K, y $\mathcal{O}_K$ es definible en K, obteniendo así una definición de $\mathbb{N}$ en K. En particular esto demuestra que la teoría de cualquier campo de números es indecidible, reduciendo el problema al caso de anillos que tienen su campo de fracción de grado infinito sobre $\mathbb{Q}$.

Por un lado, en 1962 J. Robinson [2] demostró que $\mathbb{N}$ es definible en el anillo de enteros $\mathcal{O}_{\mathbb{Q}^{\text{tr}}}$ del campo $\mathbb{Q}^{\text{tr}}$ de todos los números algebraicos totalmente reales (cuyos conjugados son todos números reales), mientras que en 1994 Fried, Haran y Völklein [8] demostraron que la teoría de $\mathbb{Q}^{\text{tr}}$ es decidible. Se conjetura que todo anillo de enteros totalmente reales tiene teoría indecidible. Por otro lado, en ese mismo articulo [2] J. Robinson demostró que $\mathbb{N}$ es definible en el anillo de enteros de $K = \mathbb{Q}(\sqrt{p} \colon p \text{ primo})$, mientras que en 2000 [9] C. Videla probó que $\mathcal{O}_K$ era definible en K, y finalmente en 2020 C. Martínez-Ranero, J. Utreras y C. Videla demostraron que el compositum $\mathbb{Q}^{(2)} = K(\sqrt{-1})$ de todas las extensiones cuadráticas de $\mathbb{Q}$ también tiene teoría indecidible. Este último resultado fue generalizado por C. Springer [10] en el 2020.

Para lograr obtener los resultados ya mencionados, y que también usaremos en este trabajo, J. Robinson demuestra el siguiente teorema [2, Teorema 2]: *Si una familia $\mathcal{F}$ definible de subconjuntos de un anillo R de enteros algebraicos totalmente reales contiene conjuntos finitos arbitrariamente grandes, entonces $\mathbb{N}$ es definible en R.* Luego, para $t \in \mathbb{R}^+ \cup \{+\infty\}$, considera el conjunto

$$R_t = \{x \in R : 0 \ll x \ll t\},$$

donde $a \ll b$ significa que $b - a$ es totalmente positivo (todos sus conjugados son positivos), y el número, que llamamos número de Julia Robinson de R:

$$\text{JR}(R) = \inf\{t \in \mathbb{R}^+ \cup \{+\infty\} \colon R_t \text{ es infinito}\}.$$

De ahí, si $\text{JR}(R)$ es infinito o es un mínimo — en cual caso diremos que R tiene la *propiedad de Julia Robinson* — es facil encontrar una familia $\mathcal{F}$ que permita concluir gracias a su teorema — el punto es que $\ll$ es definible gracias al teorema de los cuatro cuadrados de Siegel.

En este mismo trabajo J. Robinson pregunta si el número JR del anillo de enteros de un campo totalmente real siempre es un mínimo — conocido hoy como el Problema de Julia Robinson. Por otra parte, todos los ejemplos conocidos en ese entonces tenian número JR igual a 4 o a ∞. Motivados por tratar de encontrar anillos que no cumplan con una o la otra de estas dos propiedades del número JR, Vidaux y Videla [5] consideran el siguiente anillo

$$\mathcal{O} = \bigcup_{n \geq 0} \mathbb{Z}[x_n],$$

donde $x_n = \sqrt{\nu + x_{n-1}}$, con x_0 y ν números naturales cumpliendo ciertas condiciones. Para infinitos valores de los parametros x_0 y ν, pueden demostrar que el número JR de $\mathcal{O}$ es un mínimo pero no es 4 ni ∞. Pero demuestran también que para infinitos valores de estos parametros el número JR no es mínimo, pero satisface otra propiedad topologíca que llaman *propiedad de aislación*, la cual definiremos más adelante. En su tesis doctoral [7] M. Castillo completó este trabajo para casi todos los valores de ν y x_0 que faltaban por considerar en [5]. Sin embargo, estos resultados no resuelven *a priori* el problema de J. Robinson porque no se sabe si alguno de estos $\mathcal{O}$ es el anillo de enteros de su campo de fracciones. Cabe mencionar el trabajo de P. Gillibert y G. Ranieri [11] en el cual construyen infinitos anillos, con número JR estrictamente entre 4 e infinito, que son el anillo de enteros de su campo de fracciones. Sin embargo, el número JR de cada uno de estos anillos es un mínimo, dejando también abierta la pregunta de J. Robinson.

El objetivo de esta tesis es, por un lado, obtener nuevos ejemplos de anillos totamente reales indecidibles, y por otro lado contribuir a la pregunta 1.5 de [5] sobre el espectro de los números de J Robinson: ¿cuáles números reales son el número de J. Robinson de un anillo? Para ello consideraremos anillos construidos de manera similar a los de [5], poniendo $x_n = \sqrt{\nu + \lambda x_{n-1}}$, donde $\lambda \geq 1$ es un nuevo parametro. Denotemos por α el límite de (x_n) cuando n va al infinito: $\alpha = \frac{\lambda + \sqrt{\lambda^2 + 4\nu}}{2}$.

En el capítulo 1 se presentarán todos los preliminares de Teoría de Números 1.1, Lógica 1.2, y una presentación más detallada sobre los números de Julia Robinson en 1.3.

En el capítulo 2 comenzaremos con la sección 2.1 estudiando propiedades de la sucesión (x_n). En particular, mostraremos que esta sucesión siempre es monotona. En la sección 2.2 daremos condiciones necesarias y suficientes para que el anillo $\mathcal{O}$ sea totalmente real (lo cual es necesario para poder aplicar las técnicas de Julia Robinson): $\mathcal{O}$ es totalmente real si y solo si, o bien $\nu > x_0^2 - \lambda x_0$ y $\nu \geq 2\lambda^2$, o bien $\nu < x_0^2 - \lambda x_0$ y $\lambda^3 x_0 < \nu^2 - \lambda^2 \nu$. En la sección 2.3 daremos condiciones suficientes para que la torre $(K_n)_{n \geq 0}$, de los campos de fracciones de $\mathcal{O}_n = \mathbb{Z}[x_n]$ sea una 2-torre, es decir, tal que $[K_{n+1} \colon K_n] = 2$ para todo $n \geq 0$ (esto último resulta necesario para aplicar el argumento dado por Vidaux y Videla en [5]). Mas precisamente, mostraremos que la torre crece cuando $\nu + \lambda x_0$ es congruente a 2 o 3 modulo 4, y λ es congruente a 1 o 3 modulo 4.

En el capitulo 3 estudiaremos el caso creciente, dando lugar a nuestro resultado principal (en el teorema siguiente, el caso $\lambda = 1$ es un teorema de Vidaux y Videla):

Teorema 1. *Asumamos $\nu > x_0^2 - \lambda x_0$ y $\nu \geq 2\lambda^2$. Asumamos que para cada $n \geq 1$*

tenemos $[K_{n+1}\colon K_n] = 2$*. Si* $\lambda = 1$ *y* $\nu \neq 3$*, entonces* $\mathcal{O}$ *tiene número de JR igual a* $\lceil\alpha\rceil + \alpha$ *y satisface la propiedad de Julia Robinson. Si* $\lambda \geq 2$*,* $\nu \geq 2\lambda^2 + 2$ *y* $x_0 \geq \dfrac{\lambda}{4}$*, entonces* $\mathcal{O}$ *tiene número de JR igual a* $\lceil\alpha\rceil{+}\alpha$ *y satisface la propiedad de Julia Robinson.*

Este teorema nos da valores nuevos de números de J. Robinson, como por ejemplo para los parametros $\lambda = 3$, $\nu = 20$ y $x_0 = 2$, que dan número de J. Robinson igual a 13.217 aproximadamente, pero con $\lambda = 1$ no se obtiene este número.

Finalmente, en el capítulo 4 presentamos dos nuevos teoremas: el primero de ellos es una directa adaptación de [5, Lemma 3.2, Proposition 3.4 and Proposition 3.5]:

Teorema 2. *Asumamos* $\nu < x_0^2 - \lambda x_0$ *y* $\lambda^3 x_0 < \nu^2 - \lambda^2\nu$*. Asumamos que para cada* $n \geq 1$ *tenemos* $[K_{n+1}\colon K_n] = 2$*. Asumamos que* $\sqrt{\nu - \lambda x_1} \geq 1$ *y* $x_1 < \lfloor\alpha\rfloor + 1$*. El número de Julia Robinson de* $\mathcal{O}$ *es* $\lfloor\alpha\rfloor + \alpha + 1$ *y satisface la propiedad de aislación. Además, existen infinitos anillos* $\mathcal{O}$ *que cumplen lo anterior.*

El segundo teorema resuelve el problema para infinitos valores de los parametros cuando $\lambda = 3$ logrando quitar la hipotesis de $\sqrt{\nu - \lambda x_1} \geq 1$. La misma demostración de este teorema puede ser facilmente adaptada a $\lambda = 2, 4, 5\ldots$, siempre y cuando λ no sea demasiado grande, porque, a pesar que el número de casos a considerar parece disminuir a medida que λ crezca, no pude encontrar un patron que me permita escribir una demostración para λ arbitrario.

Teorema 3. *Asumamos* $\nu < x_0^2 - \lambda x_0$ *y* $\lambda^3 x_0 < \nu^2 - \lambda^2\nu$*. Si* $\lambda = 3$*,* $x_1 < \lfloor\alpha\rfloor + 1$ *y* $\nu \neq 19$*, entonces* $\mathcal{O}$ *también tiene número de JR igual a* $\lfloor\alpha\rfloor + \alpha + 1$ *y satisface la propiedad de aislación. Además, existen infinitos anillos* $\mathcal{O}$ *que cumplen lo anterior.*

Esta tesis es una contribución a dos proyectos lejanos: 1) ¿Cualquier 2-torre sobre $\mathbb{Q}$ debería tener teoría indecidible?; 2) Estudiar la topología del conjunto de números de J. Robinson en el intervalo cerrado $[4, +\infty)$ — por ejemplo, ¿es un conjunto denso?

Introduction

After the algorithm concept was formalized in the 1930's, it was possible to prove that the first-order theory of the semi-ring $\mathbb{N}$ of the natural numbers is undecidable: There is no algorithm that takes an arithmetic statement as input and gives an answer, after finite stages, whether or not said statement is true in $\mathbb{N}$. On the other hand, in 1931, Tarski [3] proved that the field theory $\mathbb{Q}^{\text{alg}}$ of all algebraic numbers, as well as the field theory $\mathbb{Q}^{\text{alg}} \cap \mathbb{R}$ of the real algebraic numbers, is decidable. The problem then is the following: determine which rings between $\mathbb{N}$ and $\mathbb{Q}^{\text{alg}}$ have decidable theory. An important step was taken by Julia Robinson in 1959: In [4], she shows that for any field of numbers K, $\mathbb{N}$ is definable in the ring of integers $\mathcal{O}_K$ of K, and $\mathcal{O}_K$ is definable in K, thus obtaining a definition of $\mathbb{N}$ in K. In particular this shows that the theory of any field of numbers is undecidable, reducing the problem to the case of rings that have their fraction field of infinite degree over $\mathbb{Q}$.

On one hand, in 1962 J. Robinson [2] proved that $\mathbb{N}$ is definable in the ring of integers $\mathcal{O}_{\mathbb{Q}^{\text{tr}}}$ of the $\mathbb{Q}^{\text{tr}}$ field of all totally real algebraic numbers (whose conjugates are all real numbers), while in 1994 Fried, Haran, and Völklein [8] proved that the theory of $\mathbb{Q}^{\text{tr}}$ is decidable. It is conjectured that every ring of totally real integers has an undecidable theory. On the other hand, in that same article [2] J. Robinson proved that $\mathbb{N}$ is definable in the ring of integers $K = \mathbb{Q}(\sqrt{p}\colon p \text{ prime})$, whereas in 2000 [9] C. Videla proved that $\mathcal{O}_K$ was definable in K, and finally in 2020 C. Martínez-Ranero, J. Utreras and C. Videla proved that the compositum $\mathbb{Q}^{(2)} = K(\sqrt{-1})$ of all quadratic extensions of $\mathbb{Q}$ also has undecidable theory. This last result was generalized by C. Springer [10] in 2020.

In order to obtain the results already mentioned, and which we will also use in this work, J. Robinson proves the following theorem [2, Theorem 2]: *If a definable family $\mathcal{F}$ of subsets of a ring R of totally real algebraic integers contains arbitrarily large finite sets, then $\mathbb{N}$ is definable in R.* Then, for $t \in \mathbb{R}^+ \cup \{+\infty\}$, consider the set

$$R_t = \{x \in R : 0 \ll x \ll t\},$$

and the number, which we call the Julia Robinson number of R:

$$\text{JR}(R) = \inf\{t \in \mathbb{R}^+ \cup \{+\infty\}\colon R_t \text{ is infinite}\}.$$

Hence, if $\text{JR}(R)$ is infinite or is a minimum — in which case we say that R has the *Julia Robinson property* — it is easy to find a family $\mathcal{F}$ which allows us to conclude thanks to his theorem — the point is that $\ll$ is definable thanks to Siegel's four squares theorem.

In this same work, J. Robinson asks if the number JR of the ring of integers of a totally real field is always a minimum. On the other hand, all known examples at that time had JR numbers equal to 4 or ∞. Motivated by trying to find rings that do not satisfy one or the other of these two properties of the JR number, Vidaux and Videla [5] consider the following ring

$$\mathcal{O} = \bigcup_{n\geq 0} \mathbb{Z}[x_n],$$

where $x_n = \sqrt{\nu + x_{n-1}}$, with x_0 and ν natural numbers satisfying certain conditions. For infinitely many values of the parameters x_0 and ν, they can prove that the JR number of $\mathcal{O}$ is a minimum but it is not 4 or ∞. But they also show that for infinitely many values of these parameters the number JR is not minimum, but satisfies another topological property called *isolation property*, which we will define later. In her doctoral thesis [7] M. Castillo completed this work for almost all the values of ν and x_0 that remained to be considered in [5]. However, these results do not solve *a priori* J. Robinson's problem because it is not known whether any of these $\mathcal{O}$ is the ring of integers in his field of fractions. It is worth mentioning the work of P. Gillibert and G. Ranieri [11] in which they build infinite rings, with number JR strictly between 4 and infinity, which are the ring of integers of their field of fractions. However, the JR number of each of these rings is a minimum, also leaving J. Robinson's question open.

The objective of this thesis is to obtain new examples of totally real undecidable rings. For this we will consider rings constructed in a similar way to those of [5], putting $x_n = \sqrt{\nu + \lambda x_{n-1}}$, where $\lambda \geq 1$ is a new parameter.

All the preliminaries of Number Theory 1.1, Logic 1.2 will be presented in chapter 1, and a more detailed presentation on numbers by Julia Robinson in **??**.

In chapter 2 we will start with section 2.1 studying properties of the sequence (x_n). In particular, we will show that this sequence is always monotone. In section 2.2 we will give necessary and sufficient conditions for the ring $\mathcal{O}$ to be totally real (which is necessary to be able to apply Julia Robinson's techniques): $\mathcal{O}$ is totally real if and only if either $\nu > x_0^2 - \lambda x_0$ and $\nu \geq 2\lambda^2$ or $\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_0 < \nu^2 - \lambda^2 \nu$. In section 2.3 we will give sufficient conditions for the tower $(K_n)_{n\geq 0}$, of the fraction fields of $\mathcal{O}_n = \mathbb{Z}[x_n]$ is a 2-tower, that is, such that $[K_{n+1} \colon K_n] = 2$ for all $n \geq 0$ (the latter is necessary to apply the argument given by Vidaux and Videla in [5]). More precisely, we will show that the tower grows when $\nu + \lambda x_0$ is congruent to 2 or 3 modulo 4, and λ is congruent to 1 or 3 modulo 4.

In chapter 3 we will study the increasing case, giving rise to our main result (in the following theorem, the case $\lambda = 1$ is a Vidaux and Videla theorem):

Teorema 4. *Let's assume $\nu > x_0^2 - \lambda x_0$ and $\nu \geq 2\lambda^2$. Let's assume that for every $n \geq 1$ we have $[K_{n+1} \colon K_n] = 2$. If $\lambda = 1$ and $\nu \neq 3$, then $\mathcal{O}$ has JR number equal to $\lceil \alpha \rceil + \alpha$ and satisfies the Julia Robinson property. If $\lambda \geq 2$, $\nu \geq 2\lambda^2 + 2$, and $x_0 \geq \dfrac{\lambda}{4}$, then $\mathcal{O}$ has JR number equals $\lceil \alpha \rceil + \alpha$ and satisfies the Julia Robinson property.*

Finally, in chapter 4 we will study the decreasing case. We will solve the problem for infinitely many values of the parameters when $\lambda = 2$, assuming that the tower grows with each step: unfortunately, for this value of λ we do not know if that is true for some values of ν and x_0. However, the same proof as the one presented here can be easily adapted to $\lambda = 3, 4, \ldots$, as long as λ is not too large, because, although the number of cases to consider seems to decrease as λ grows, I couldn't find a pattern that would allow me to write a proof for arbitrary λ.

Teorema 5. *Let's assume* $\nu < x_0^2 - \lambda x_0$ *and* $\lambda^3 x_0 < \nu^2 - \lambda^2 \nu$. *Let's assume that for every* $n \geq 1$ *we have* $[K_{n+1} \colon K_n] = 2$. *If* $\lambda = 1$ *and* $\nu > 3$, *then* $\mathcal{O}$ *has JR number equal to* $\lfloor \alpha \rfloor + \alpha + 1$ *and satisfies the isolation property. If* $\lambda = 2$, $x_1 < \lfloor \alpha \rfloor + 1$ *and* $\nu \neq 9$, *then* $\mathcal{O}$ *also has number of JR equal to* $\lfloor \alpha \rfloor + \alpha + 1$ *and satisfies the isolation property.*

1 Preliminary

1.1 Algebraic Number Theory

Let $A \subseteq B$ be an inclusion of rings. An element $b \in B$ is called integral over A if there is a monic polynomial $f \in A[x]$ such that b is a root of f.

In particular, a complex number which is integral over the field $\mathbb{Q}$ of rational numbers will be called an algebraic number, and if it is also integral over the ring $\mathbb{Z}$ of rational integers, then it will be called an algebraic integer.

Any finite extension of $\mathbb{Q}$ will be called an algebraic number field.

For the rest of the section, K is an algebraic number field. We will denote by $\mathcal{O}_K$ the subset of K consisting of all the elements that are integral over $\mathbb{Z}$.

Definition 1.1.1. Let $a \in K$ be an algebraic number and $f \in \mathbb{Q}[x]$ be the minimal polynomial of a over K. The roots of f are called the conjugates of a. We will say that a is totally real if all the conjugates of a are real numbers. The field K is totally real if all its elements are totally real. We will denote by $\mathbb{Q}^{\mathrm{tr}}$ the field of all the totally real numbers.

Since every separable extension of finite degree is simple, we can write $K = \mathbb{Q}(a)$ for some $a \in K$. So, K is totally real if its generator a is totally real.

Definition 1.1.2. Let K be an algebraic number field and $a \in K$. We will use the notation $\overline{|a|}$ for the largest absolute value of conjugates of a over $\mathbb{Q}$ and we will denote by $\widetilde{a}$ an arbitrary conjugate of a.

Using a result by Kronecker, we have the following proposition about the distribution of totally real algebraic integers.

Proposition 1.1.3 (See [6] for a proof)**.** *The set of totally real algebraic integers in the interval $[0, 4-\varepsilon]$ is finite for every $\varepsilon > 0$.*

Let K be a totally real number field and $x \in K$. If $a, b \in \mathbb{R} \cup \{+\infty\}$, we write $a \ll x \ll b$ if for each conjugate $\widetilde{x}$ of x we have $a < \widetilde{x} < b$.

Using the previous notation, we have that for every $\varepsilon > 0$, there are finitely many totally real algebraic integers that satisfy

$$0 \ll x \ll 4 - \varepsilon.$$

1.2 Logical background

A first order language is a set $\mathcal{L}$ of symbols, where each of its elements is exactly one of the following

1. a constant symbol, or

2. an n-place relation symbol for some $n \geq 1$, or

3. an n-place function symbol, for some $n \geq 1$.

For this section, $\mathcal{L}$ will be a first order language.

A formula is made up of: the equality symbol, variables, logical connectives, quantifiers, constants symbols, relation symbols and function symbols. In a formula, some variables may not be quantified. Those that are not quantified will be called free variables. A formula is a sentence if all its variables appear under the scope of a quantifier.

An interpretation is an assignment of meaning to the symbols (constant, functions and relations) of the language $\mathcal{L}$ with respect to a base set M (domain). Note that a sentence can be true or false depending on the interpretation that is considered.

Definition 1.2.1. An subset A of a ring M is definable if there exists a formula $F(x)$ with exactly one free variable x such that for all element $x \in M$, $x \in A$ if and only if $F(x)$ is true in M (using the interpretation).

The theory of a ring M is the set of all sentences that are true in M. We denote by $\mathrm{Th}(M)$ the theory of M.

Definition 1.2.2. We say that the theory of a ring M is decidable if there exists an algorithm which, given an arbitrary sentence, determines after a finite number of steps whether the sentence belongs or not to $\mathrm{Th}(M)$.

Remark 1.2.3. It is known that $\mathbb{N}$ has an undecidable theory. Hence, any ring in which $\mathbb{N}$ can be defined has an undecidable theory.

Given a number field K, in order to show that it has an undecidable theory, one may divide the problem into two subproblems:

1. Define $\mathbb{N}$ in $\mathcal{O}_K$.

2. Define $\mathcal{O}_K$ in K.

In this thesis, we will focus exclusively on the problem of defining $\mathbb{N}$ in $\mathcal{O}_K$, and only for fields K that are totally real, because in this case there is a technique developed by Julia Robinson as we will show in the next section.

1.3 Julia Robinson's definability criterium

In this section, K is a field of totally real algebraic numbers and R is a subring of $\mathcal{O}_K$.
For $t \in \mathbb{R}^+ \cup \{+\infty\}$, we write:

$$R_t = \{x \in R : 0 \ll x \ll t\}$$

We define the Julia Robinson number of R, or JR number of R, to be

$$\mathrm{JR}(R) = \inf\{t \in \mathbb{R}^+ \cup \{+\infty\} \colon R_t \text{ is infinite}\}.$$

Remark 1.3.1. The JR number of R is greater than or equal to 4 by Proposition 1.1.3.

J. Robinson proved in [2] that a sufficient condition for $\mathbb{N}$ to be definable in $\mathcal{O}_K$ is that $\mathrm{JR}(\mathcal{O}_K)$ is a minimum (if it is $+\infty$ we also consider it as a minimum).
It is possible to generalize this result to any subring R of $\mathcal{O}_K$ (see [7, Theorem 1.2.2]). When it is a minimum, we will say that R satisfies Julia Robinson's property

In [5] X. Vidaux and C. R. Videla defined the following property: the ring R has the *isolation property* if

1. it does not have Julia Robinson's property, and
2. there exists $M > 0$ such that for any $\varepsilon > 0$, if $\varepsilon < M$, then the set

$$R_{\mathrm{JR}(R)+M} \setminus R_{\mathrm{JR}(R)+\varepsilon}$$

is finite.

It is possible (an easy) to adapt the argument given by Julia Robinson to prove that if a ring R has the *isolation property*, then $\mathbb{N}$ is definable in R [5, Definition 1.2] and therefore, R has undecidable theory.

Proposition 1.3.2. *If R has the property of Julia Robinson or the isolation property, then $\mathbb{N}$ is first-order definable in R.*

2 Basic properties of the tower

We define the sequence (x_n) whose general term is $x_n = \sqrt{\nu + \lambda x_{n-1}}$ and

- ν and x_0 are non-negative integers and not zero simultaneously,
- $\lambda > 0$ is a rational integer, and
- $\nu \neq x_0^2 - \lambda x_0$ (in order to avoid $x_1 = x_0$).

We define the following rings and their field of fractions:

$$\begin{array}{ll} \mathcal{O}_0 = \mathbb{Z} & K_0 = \mathbb{Q} \\ \mathcal{O}_n = \mathcal{O}_{n-1}[x_n] & K_n = K_{n-1}[x_n] \\ \mathcal{O} = \bigcup_{n\geq 0} \mathcal{O}_n & K = \bigcup_{n\geq 0} K_n \end{array}$$

Remark 2.0.1. From the construction of the sequence x_n, we have $\mathcal{O}_n = \mathbb{Z}[x_n]$ and $K_n = \mathbb{Q}[x_n]$ for each $n \geq 0$.

2.1 Monotony and Bounds

Several of the lemmas in this section are a straightforward adaptation of the analogous results in [5]. We provide all the proofs for the sake of completeness.

Lemma 2.1.1. [5, Lemma 2.2] *The sequence (x_n) is*

1. *strictly increasing if and only if $\nu > x_0^2 - \lambda x_0$, and*
2. *strictly decreasing if and only if $\nu < x_0^2 - \lambda x_0$.*

Proof. If the sequence (x_n) is strictly increasing, then $x_n > x_{n-1}$ for each $n \geq 1$. In particular, $\nu + \lambda x_0^2 = x_1^2 > x_0^2$. Let us assume $\nu > x_0^2 - \lambda x_0$. We will prove by induction on n that (x_n) is strictly increasing. It is clear for $n = 1$. Assume $n \geq 2$. We have

$$(x_{n+1} + x_n)(x_{n+1} - x_n) = x_{n+1}^2 - x_n^2 = \lambda(x_n - x_{n-1}) > 0$$

by the induction hypothesis. Therefore, we have $x_{n+1} > x_n$ for each $n \geq 1$. The decreasing case is done analogously. $\square$

Lemma 2.1.2. [5, Lemma 2.3] *The sequence (x_n) is convergent with limit*

$$\alpha = \frac{\lambda + \sqrt{\lambda^2 + 4\nu}}{2}.$$

Proof. Suppose that the sequence (x_n) is increasing and suppose that there is an integer $m \geq 1$ such that $2x_m \geq \lambda$. For every $n \geq m+1$, we have

$$\begin{aligned} x_n < \alpha &\iff 2\sqrt{\nu + \lambda x_{n-1}} < \lambda + \sqrt{\lambda^2 + 4\nu} \\ &\iff 4(\nu + \lambda x_{n-1}) < 2\lambda^2 + 4\nu + 2\lambda\sqrt{\lambda^2 + 4\nu} \\ &\iff 2x_{n-1} - \lambda < \sqrt{\lambda^2 + 4\nu} \\ &\iff x_{n-1}^2 < \nu + \lambda x_{n-1} = x_n^2. \end{aligned}$$

If there is no such m, then we have $2x_n < \lambda \leq \alpha$ for every $n \geq 1$. In both cases, the increasing sequence (x_n) is bounded from above, hence it converges. Similarly, if (x_n) is decreasing, then it converges because (x_n) is bounded from below by 0. As the limit is unique, in both cases (increasing and decreasing) the sequence (x_n) converges to α.

□

Lemma 2.1.3. [5, Lemma 2.5] *There exists an integer $n_0 \geq 0$ such that for every $n \geq 0$, we have $n \leq n_0$ if and only if x_n is a rational integer.*

Proof. If $x_n \notin \mathbb{Z}$ for some $n \geq 0$, then $x_n \notin \mathbb{Q}$ since x_n is an algebraic integer. Hence, $\lambda x_n \notin \mathbb{Q}$ for every $\lambda \geq 1$. So, $x_{n+1} = \sqrt{\nu + \lambda x_n} \notin \mathbb{Z}$. Since (x_n) is bounded, the sequence takes finite integer values. We choose n_0 to be the largest index n such that x_n is a rational integer. □

Lemma 2.1.4. [5, Lemma 2.19] *For any real number r, and for any $n \geq 2$ and $a, b \in \mathcal{O}_{n-1}$, if $0 \ll a + bx_n \ll 2r$, then $0 \ll a \ll 2r$ and $\overline{|b|} < \frac{r}{\sqrt{\nu - \lambda x_{n-1}}}$. In particular, for $n = 1$, b must be an integer such that $|b| < \dfrac{r}{x_1}$.*

Proof. Let σ be an embedding of $\mathcal{O}_{n-1}$ in $\mathbb{R}$. We have $(a + bx_n)^\sigma = a^\sigma \pm b^\sigma x_n^\sigma$, hence $0 < a^\sigma \pm b^\sigma x_n^\sigma < 2r$ for every σ. Combining both inequalities, we have $0 < a^\sigma < 2r$ and $|b^\sigma x_n^\sigma| < r$. Since σ is arbitrary, we have $0 \ll a \ll 2r$. For b, there are two cases:

- If $n = 1$, then by definition $b \in \mathbb{Z}$, and hence $|bx_1^\sigma| = |b^\sigma x_1^\sigma| < r$. So $|b| < \frac{r}{x_1}$, because $|x_1^\sigma| = x_1$ for every embedding σ.
- If $n \geq 2$, then we have

  $$|x_n^\sigma| = |(\sqrt{\nu + \lambda x_{n-1}})^\sigma| \geq \sqrt{\nu - \lambda x_{n-1}}.$$

  Therefore, we have
  $$\overline{|b|} < \frac{r}{|x_n^\sigma|} \leq \frac{r}{\sqrt{\nu - \lambda x_{n-1}}}$$
  since σ is arbitrary.

□

2.2 The totally real condition

As we indicated in Section 1.3, Julia Robinson's criterium is only applicable for rings of totally real algebraic integers. In this section we will give sufficient and necessary condition for the ring $\mathcal{O}$ to be totally real.

Lemma 2.2.1. *We have $\nu \geq 2\lambda^2$ if and only if $\nu \geq \lambda\alpha$.*

Proof. Observe that $\nu \geq \lambda\alpha$ if and only if

$$\nu \geq \lambda\left(\frac{\lambda + \sqrt{\lambda^2 + 4\nu}}{2}\right) \geq \frac{\lambda^2}{2},$$

which implies $2\nu \geq \lambda^2$. Therefore, we have

$$\begin{aligned} \nu \geq 2\lambda^2 &\iff 4\nu^2 \geq 8\lambda^2\nu \\ &\iff 4\nu^2 - 4\lambda^2\nu + \lambda^4 \geq \lambda^4 + 4\lambda^2\nu \\ &\iff (2\nu - \lambda^2)^2 \geq \lambda^2(\lambda^2 + 4\nu) \\ &\iff 2\nu - \lambda^2 \geq \lambda\sqrt{\lambda + 4\nu} \\ &\iff \nu \geq \lambda\alpha. \end{aligned}$$

□

Lemma 2.2.2. *If $\mathcal{O}$ is totally real and $\nu > x_0^2 - \lambda x_0$, then $\nu \geq 2\lambda^2$.*

Proof. Since $\mathcal{O}$ is totally real, $x_{n+1} = \sqrt{\nu + \lambda \widetilde{x_n}}$ is a real number for every $n \geq 1$. In particular, $\sqrt{\nu - \lambda x_n}$ will be a real number, so $\nu \geq \lambda x_n$ for every $n \geq 1$. For the sake of contradiction, we suppose $\nu < \lambda\alpha$ (so we can conclude by Lemma 2.2.1). There is some $\varepsilon > 0$ such that $\nu < \lambda\alpha - \lambda\varepsilon$. Since $\nu > x_0^2 - \lambda x_0$, the sequence (x_n) is increasing and converges to α by Lemma 2.1.1. Hence, there is an index $m \in \mathbb{N}$ such that $x_m > \alpha - \varepsilon$. Therefore, we have

$$\nu \geq \lambda x_m > \lambda\alpha - \lambda\varepsilon.$$

□

Lemma 2.2.3. *If $\mathcal{O}$ is totally real, then we have $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$, where n_0 comes from Lemma 2.1.3.*

Proof. We write $n_1 = n_0 + 1$. Since $\mathcal{O}$ is totally real, $x_{n_1+1} = \sqrt{\nu + \lambda \widetilde{x_{n_1}}}$ is a real number. In particular, $\sqrt{\nu - \lambda x_{n_1}}$ will be a real number, which is not zero because λx_{n_1} is an irrational number and ν is a rational integer. So we have $\nu > \lambda x_{n_1} = \lambda\sqrt{\nu + \lambda x_{n_0}}$ if and only if $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$. □

The following theorem gives us a characterization of when our ring $\mathcal{O}$ is totally real and therefore, will allow us to use Julia Robinson' methods explained in Chapter 1.3.

Theorem 2.2.4. *The ring $\mathcal{O}$ is totally real if and only if*

1. *either* $\nu > x_0^2 - \lambda x_0$ *and* $\nu \geq 2\lambda^2$, *or*

2. $\nu < x_0^2 - \lambda x_0$ *and* $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$.

If $\mathcal{O}$ *is totally real, then* $\overline{|x_n|} = x_n$ *for each* $n \geq 0$.

Proof. The implication from left to right is an immediate consequence of Lemma 2.2.2 and Lemma 2.2.3. We show the other implication by induction on n. Let $n_1 = n_0 + 1$. If $n \leq n_0$, then $\mathcal{O}_n = \mathbb{Z}$ which is totally real and hence $\overline{|x_n|} = x_n$. For n_1 we have $x_{n_1} \notin \mathbb{Z}$ and hence its conjugates are of the form $\pm\sqrt{\nu + \lambda x_{n_0}}$. Therefore, $\mathcal{O}_{n_1} = \mathbb{Z}[x_{n_1}]$ is totally real and $\overline{|x_{n_1}|} = x_{n_1}$. Suppose that for some $n \geq n_1$, $\mathcal{O}_n$ is totally real and $\overline{|x_n|} = x_n$. Note that the conjugates of x_{n+1} are of the form $\pm\sqrt{\nu + \lambda \widetilde{x_n}}$. Since $\overline{|x_n|} = x_n$, we have $\overline{|x_{n+1}|} = x_{n+1}$ and it will be enough to prove that $\nu \geq \lambda x_n$ for each $n \geq n_1$. We can separate the proof into cases where the sequence (x_n) is increasing or decreasing:

- If $\nu > x_0^2 - \lambda x_0$ and $\nu \geq 2\lambda^2$, then (x_n) is strictly increasing by Lemma 2.1.1 and hence $\lambda x_n < \lambda\alpha \leq \nu$ by Lemma 2.2.1.

- If $\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_{n_0} < \nu^2 - \lambda^2\nu$, then (x_n) is strictly decreasing by Lemma 2.1.1 and $\lambda x_{n_1} < \nu$. Hence, $\lambda x_n \leq \lambda x_{n_1} < \nu$ for each $n \geq n_1$.

□

From now on we will assume that x_1 is a non-rational integer and the ring $\mathcal{O}$ is totally real.

Lemma 2.2.5. *In the decreasing case, we have* $\nu \geq 3$ *and* $x_0 \geq 3$.

Proof. This is an immediate consequence of the inequalities $\nu < x_0^2 - \lambda x_0$ and $\lambda^3 x_0 < \nu^2 - \lambda^2\nu$, and the fact that λ is at least 1. □

Lemma 2.2.6. *Assume that* (x_n) *is increasing. If* $\nu \geq 2\lambda^2 + 2$, *then* $x_n \geq 2$ *for each* $n \geq 1$.

Proof. Since the sequence (x_n) is increasing, we have

$$x_n \geq x_1 = \sqrt{\nu + \lambda x_0} \geq \sqrt{2\lambda^2 + 2} \geq 2.$$

for each $n \geq 1$. □

Lemma 2.2.7. *We have* $\alpha \geq 2$.

Proof. If (x_n) is decreasing, then by Lemma 2.2.5 we have $\nu \geq 3$, and if (x_n) is increasing, then $\nu \geq 2\lambda^2 \geq 2$. In all cases, we have $\nu \geq 2$. Hence, we have

$$2\alpha = \lambda + \sqrt{\lambda^2 + 4\nu} \geq 4$$

because $\lambda \geq 1$ and $\nu \geq 2$. □

2.3 Conditions for the tower to increase at each step

For the induction arguments to work in the next chapters, we will need the tower (K_n) to increase at each step. In this section, we will provide sufficient conditions for that.

Let $f(t) = \frac{t^2-\nu}{\lambda}$ be a function of the real variable t. We define for each $n \geq 1$

$$P_n = \lambda^{2^n-1} f^{\circ n}(t) - \lambda^{2^n-1} x_0,$$

where $f^{\circ n}$ stands for the composition of f with itself n times.

Lemma 2.3.1. *The polynomial P_n is monic for each $n \geq 1$.*

Proof. We prove it by induction on n. If $n = 1$, then $P_1 = \lambda f(t) - \lambda x_0 = t^2 - \nu - \lambda x_0$ is monic. Suppose that for some $n \geq 2$ the polynomial P_n is monic. We have

$$\begin{aligned}
P_{n+1}(t) &= \lambda^{2^{n+1}-1} f^{\circ(n+1)}(t) - \lambda^{2^{n+1}-1} x_0 \\
&= \lambda^{2^{n+1}-1} \left(\frac{(f^{\circ n}(t))^2 - \nu}{\lambda} \right) - \lambda^{2^{n+1}-1} x_0 \\
&= \lambda^{2^{n+1}-2} (f^{\circ n}(t))^2 - \lambda^{2^{n+1}-2} \nu - \lambda^{2^{n+1}-1} x_0 \\
&= \left(\lambda^{2^n-1} f^{\circ n}(t) \right)^2 - \lambda^{2^{n+1}-2} \nu - \lambda^{2^{n+1}-1} x_0 \\
&= \left(P_n(t) + \lambda^{2^n-1} x_0 \right)^2 - \lambda^{2^{n+1}-2} \nu - \lambda^{2^{n+1}-1} x_0,
\end{aligned}$$

and since P_n is monic by hypothesis, P_{n+1} is monic too. □

Proposition 2.3.2. *If $\nu + \lambda x_0$ is congruent to 2 or 3 module 4 and λ is congruent to 1 or 3 module 4, then for each $n \geq 1$, we have $[K_{n+1} \colon K_n] = 2$.*

Proof. From the definition of f we have $f^{\circ n}(x_n) = x_0$ for each $n \geq 1$. Therefore, x_n is a root of P_n. Also note that, by Lemma 2.3.1, P_n is monic for each $n \geq 1$. Given $a, b \in \mathbb{Z}$, we have

$$P_1(t + a) = (t + a)^2 - \nu - \lambda x_0 = t^2 + 2at + a^2 - (\nu + \lambda x_0), \tag{2.3.1}$$

and

$$\begin{aligned}
P_2(t + b) &= \lambda^3 f^{\circ 2}(t + b) - \lambda^3 x_0 \qquad (2.3.2)\\
&= t^4 + 4bt^3 + 2(3b^2 - \nu)t^2 + 4(b^3 - b\nu)t + (b^4 - 2b^2\nu + \nu^2 - \lambda^2(\nu + \lambda x_0)).
\end{aligned}$$

Also, for each $n \geq 1$, we have

$$
\begin{aligned}
P_{n+2}(t) &= \lambda^{2^{n+2}-1}(f^{\circ(n+2)}(t) - x_0) \\
&= \lambda^{2^{n+2}-1}(f^{\circ 2}(f^{\circ n}(t)) - x_0) \\
&= \lambda^{2^{n+2}-1}\left(f^{\circ 2}\left(\frac{P_n(t)}{\lambda^{2^n-1}} + x_0\right) - x_0\right) \\
&= \lambda^{2^{n+2}-1}\left(\left(\frac{P_2\left(\frac{P_n(t)}{\lambda^{2^n-1}} + x_0\right)}{\lambda^3} + x_0\right) - x_0\right) \\
&= \lambda^{4(2^n-1)}P_2\left(\frac{P_n(t)}{\lambda^{2^n-1}} + x_0\right) \\
&= P_n^4(t) + 4\lambda^{2^n-1}x_0P_n^3(t) + 2\lambda^{2(2^n-1)}(3x_0^2 - \nu)P_n^2(t) + 4\lambda^{3(2^n-1)}(x_0^3 - x_0\nu)P_n(t) \\
&\quad + \lambda^{4(2^n-1)}\left(x_0^4 - 2x_0^2\nu + \nu^2 - \lambda^2(\nu + \lambda x_0)\right).
\end{aligned}
\tag{2.3.3}
$$

We prove by induction on n that the polynomial P_n is irreducible. If $n = 1$, then using Equation (2.3.1) we choose $a = 0$ if $\nu + \lambda x_0$ is congruent to 2 module 4, and $a = 1$ if $\nu + \lambda x_0$ is congruent to 3 module 4. In both cases $P_1(t + a)$ is an Eisenstein polynomial for 2. If $n = 2$, then using Equation (2.3.2), we have that $P_2(t + x_0)$ is an Eisenstein polynomial for 2, because $x_0^4 - 2x_0^2\nu + \nu^2 - \lambda^2(\nu + \lambda x_0)$ is congruent to 2 module 4 when $\nu + \lambda x_0$ is congruent to 2 or 3 module 4 and λ is congruent to 1 or 3 module 4 (we leave the verification to the reader). Note that λ^2 is congruent to 1 module 4 by hypothesis. Therefore, the constant term of $P_{n+2}(t)$, seen as a polynomial in $P_n(t)$, is congruent to 2 modulo 4. So, using Equation (2.3.3), if $P_n(t + c)$ is an Eisenstein polynomial for 2 for some $c \in \mathbb{Z}$, then $P_{n+2}(t + c)$ is an Eisenstein polynomial for 2 too. Thus, we can prove the irreducibility of P_n by induction on n, separating into two cases:

- If n is odd, then $P_n(t + a)$ is an Eisenstein polynomial for 2 (with the respective choice of a).
- If n is even, then $P_n(t + x_0)$ is an Eisenstein polynomial for 2.

□

From now on we will assume that for every $n \geq 1$, we have $[K_{n+1} \colon K_n] = 2$.

3 Increasing Case

Assumption 3.0.1. *For this section, let us assume*

$$\nu \geq 2\lambda^2 + 2 \quad \textit{and} \quad x_0 \geq \frac{\lambda}{4}.$$

Note that x_0 is at least 1.

Definition 3.0.2. For each $n \geq 1$, let k_n be the only rational integer such that

$$\lceil \alpha \rceil - (k_n + 1) < x_n < \lceil \alpha \rceil - k_n.$$

Note that the sequence (k_n) is (non strictly) decreasing, hence the k_n take only finitely many values, and since the sequence (x_n) tends to α, eventually k_n is 0.

We define the following sets:

$$\begin{aligned} X_0 &= \{1, 2, \ldots, 2\lceil \alpha \rceil - 1\}, \\ X_n &= X_0 \cup \{\lceil \alpha \rceil \pm j \pm x_s \colon 0 \leq j \leq k_s \ \text{ and } \ 1 \leq s \leq n\}, \\ X &= \bigcup_{n \geq 0} X_n. \end{aligned}$$

Lemma 3.0.3. *If $\lambda \geq 2$, then $x_1 + x_2 + \lceil x_1 \rceil > 2\lceil \alpha \rceil$.*

Proof. It is enough to prove that we have $x_2 + 2x_1 > 2(\alpha + 1)$. We have

$$\begin{aligned} 2\sqrt{\nu + \lambda x_0} + \sqrt{\nu + \lambda\sqrt{\nu + \lambda x_0}} &\geq \sqrt{4\nu + \lambda^2} + \sqrt{2\lambda^2 + 2 + \lambda\sqrt{2\lambda^2 + 2 + \frac{\lambda^2}{4}}} \\ &\geq \sqrt{4\nu + \lambda^2} + \sqrt{\lambda^2 + 4\lambda + 4} \\ &= 2(\alpha + 1), \end{aligned}$$

where the first inequality is by Assumption 3.0.1. $\square$

Lemma 3.0.4. *Let $n \geq 1$. If $0 \ll a \pm b x_n \ll 2\lceil \alpha \rceil$, with $a, b \in \mathcal{O}_{n-1}$, then $|b| < 2$.*

Proof. Since $\nu \geq 2\lambda^2 + 2$ and $\nu \in \mathbb{N}$, we can write $\nu = 2\lambda^2 + k$, for some $k \geq 2$. Since

$0 < a \pm bx_n < 2\lceil\alpha\rceil$, combining both inequalities we obtain $|b| < \frac{\lceil\alpha\rceil}{x_n}$. So, we have

$$\begin{aligned}|b| &< \frac{\lceil\alpha\rceil}{x_n}\\ &\leq \frac{\alpha+1}{\sqrt{2\lambda^2+k+\lambda x_{n-1}}}\\ &= \frac{\lambda+\sqrt{\lambda^2+4(2\lambda^2+k)}+2}{2\sqrt{2\lambda^2+k+\lambda x_{n-1}}}\\ &\leq \frac{\lambda+2+\sqrt{\lambda^2}+\sqrt{8\lambda^2+4k}}{\sqrt{8\lambda^2+4k}}\\ &\leq 1+\frac{2\lambda+2}{\sqrt{8\lambda^2+4k}}\\ &\leq 1+\frac{2\lambda+2}{\sqrt{8\lambda^2+8}}\\ &\leq 2,\end{aligned}$$

where the last inequality is true because $2\lambda+2 \leq \sqrt{8\lambda^2+8}$ for every $\lambda \geq 1$. □

Lemma 3.0.5. *We have* $\nu - \lambda\alpha > 1$.

Proof. Since $\nu \geq 2\lambda^2+2$ and $\nu \in \mathbb{N}$, we can write $\nu = 2\lambda^2+k$, for some $k \geq 2$. Hence, we have

$$(2\lambda^2+k)-\lambda\left(\frac{\lambda+\sqrt{\lambda^2+4(2\lambda^2+k)}}{2}\right) \geq 1 \iff 3\lambda^2+2k-2 \geq \lambda\sqrt{9\lambda^2+4k}$$
$$\iff 4k^2+12k\lambda^2-8k+9\lambda^4-12\lambda^2+4 \geq 9\lambda^4+4k\lambda^2 \iff 4k^2+(8\lambda^2-8)k+4-12\lambda^2 \geq 0,$$

and since $k \geq 0$, the latter is true for

$$k \geq \frac{8-8\lambda^2+\sqrt{64\lambda^4+64\lambda^2}}{8} = 1-\lambda^2+\sqrt{\lambda^4+\lambda^2}.$$

We consider the continuous function $x \mapsto 1-x^2+\sqrt{x^4+x^2}$. The line $y = \frac{3}{2}$ is an horizontal asymptote for this function, hence we have

$$1-\lambda^2+\sqrt{\lambda^4+\lambda^2} < \frac{3}{2},$$

for every $\lambda \geq 1$. □

Lemma 3.0.6. *Let* $x = a+bx_1 \in \mathcal{O}_1$, *with* $a, b \in \mathbb{Z}$. *If* $0 < a \pm bx_1 < 2\lceil\alpha\rceil$, *then* $x \in X_1$.

Proof. By Lemma 3.0.4, we have $b = \pm 1$ or $b = 0$.

- If $a \leq \lceil\alpha\rceil - (k_1 + 1)$, then $b = 0$. Indeed, if $|b| = 1$, by choosing σ such that $x^\sigma = a - |b|x_1$, we obtain:

  $$a - |b|x_1 \leq \lceil\alpha\rceil - (k_1 + 1) - x_1 \leq 0,$$

  by the definition of k_1, contradicting our hypothesis.

- If $a \geq \lceil\alpha\rceil + (k_1 + 1)$, then $b = 0$. If $|b| = 1$, by choosing σ such that $x^\sigma = a + |b|x_1$, we obtain:

  $$a + |b|x_1 \geq \lceil\alpha\rceil + (k_1 + 1) + x_1 \geq 2\lceil\alpha\rceil,$$

  again contradicting our hypothesis.

Therefore, we have either $|a - \lceil\alpha\rceil| \geq k_1 + 1$ and $b = 0$, or $|a - \lceil\alpha\rceil| < k_1 + 1$ and $|b| \leq 1$. In both cases, we have $x \in X_1$. □

Lemma 3.0.7. *Assume $n > m \geq 1$ and $\lambda \geq 2$.*

1. *We have $\lceil\alpha\rceil \pm j + x_m + x_n \geq 2\lceil\alpha\rceil$ for every $0 \leq j \leq k_m$.*
2. *We have $\lceil\alpha\rceil \pm j - x_m - x_n \leq 0$ for every $0 \leq j \leq k_m$.*

Proof.

1. Note that $\lceil x_1\rceil = \lceil\alpha\rceil - k_1$. By Lemma 3.0.3, and using the fact that (x_n) is increasing, we have

   $$x_m + x_n + \lceil\alpha\rceil - k_1 \geq 2\lceil\alpha\rceil,$$

   for each $n > m \geq 1$. Since $k_1 \geq k_m$ for each $m \geq 1$, we have

   $$x_m + x_n + \lceil\alpha\rceil \pm j \geq 2\lceil\alpha\rceil,$$

   for every $0 \leq j \leq k_m$.

2. For every $0 \leq j \leq k_m$, we have $\lceil\alpha\rceil \pm j - x_m - x_n \leq 0$ if and only if $x_m + x_n + \lceil\alpha\rceil \pm j \geq 2\lceil\alpha\rceil$. So we can conclude by item 1.

□

Lemma 3.0.8. *Assume $\lambda \geq 2$. We have $\lceil x_n\rceil + x_n \geq \lceil\alpha\rceil + 2$ for each $n \geq 1$. In particular, we have $x_n \geq k_n + 2$ for each $n \geq 1$.*

Proof. Since (x_n) is increasing, it is enough to prove that we have $x_1 + \lceil x_1\rceil > \alpha + 3$. If $\lambda = 2$, then we have (recalling that we have $x_0 \geq 1$ and $\nu \geq 10$ by Assumption 3.0.1)

$$x_1 + \lceil x_1\rceil \geq \sqrt{\nu + 2} + \lceil\sqrt{\nu + 2}\rceil > \sqrt{\nu + 1} + \lceil\sqrt{12}\rceil = 3 + \alpha.$$

For $\lambda \geq 3$, we have

$$\begin{aligned} 2x_1 + 2\lceil x_1\rceil &\geq \sqrt{4\nu + \lambda^2} + \sqrt{9\lambda^2 + 8} \\ &> \sqrt{4\nu + \lambda^2} + \lambda + 6 \\ &= 2(\alpha + 3), \end{aligned}$$

where the first inequality is by Assumption 3.0.1, and the second inequality is because $\lambda \geq 3$. In particular, using $\lceil x_n \rceil = \lceil \alpha \rceil - k_n$ for each $n \geq 1$, we have $\lceil x_n \rceil + x_n > \lceil \alpha \rceil + 2$ if and only if $x_n > k_n + 2$. □

Lemma 3.0.9. *If* $0 \ll x \ll 2\lceil \alpha \rceil$, *then* $x \in X$.

Proof. For $\lambda = 1$, this is [5, Lemma 4.9]. For $\lambda \geq 2$, which we now assume, we start as in [5, Lemma 4.9]. We prove by induction on n that if $x \in \mathcal{O}_n$ is such that $0 \ll x \ll 2\lceil \alpha \rceil$, then $x \in X_n$. This is clear for $n = 0$. For $n = 1$, we have $x \in X_1$ by Lemma 3.0.6. Assume $n \geq 2$. Let us fix $x = a + bx_n \in \mathcal{O}_n$ with $a, b \in \mathcal{O}_{n-1}$. By Lemma 2.1.4, we have $0 \ll a \ll 2\lceil \alpha \rceil$, so $a \in X_{n-1}$ by induction hypothesis. Also, by Lemma 2.1.4, we have

$$\overline{|b|} < \frac{\lceil \alpha \rceil}{\sqrt{\nu - \lambda x_{n-1}}} < \frac{\lceil \alpha \rceil}{\sqrt{\nu - \lambda \alpha}} \leq \lceil \alpha \rceil,$$

since $\sqrt{\nu - \lambda \alpha} \geq 1$ by Lemma 3.0.5. Hence, we have $0 \ll \lceil \alpha \rceil + b \ll 2\lceil \alpha \rceil$, and by induction hypothesis we have $\lceil \alpha \rceil + b \in X_{n-1}$. From the definition of X_{n-1}, we have either $b \in \mathbb{Z}$, or $|b| = |j \pm x_s|$ for some $1 \leq s \leq n-1$ and $0 \leq j \leq k_s$. In the first case, we have either $b = 0$ or $b = \pm 1$ by Lemma 3.0.4. In the second case, we have, also by Lemma 3.0.4, either $|j + x_s| < 2$ or $|x_s - j| < 2$. If $|j + x_s| < 2$, then $x_s < 2 - j \leq 2$ and we have a contradiction by Lemma 3.0.8. If $|x_s - j| < 2$, then $x_s < j + 2 \leq k_s + 2$, which is a contradiction, again by Lemma 3.0.8. Therefore, we have $b \in \{-1, 0, 1\}$. For $b = 0$, there is nothing to prove, as we already know that $x = a$ lies in X_{n-1}. Assume $|b| = 1$. We can write $x = a \pm x_n$, and since $a \in X_{n-1}$, we have either $a \in \{1, \ldots, 2\lceil \alpha \rceil - 1\}$, or $a = \lceil \alpha \rceil \pm j \pm x_s$ for some $1 \leq s \leq n-1$ and $0 \leq j \leq k_s$.

- If $a \in \{1, \ldots, \lceil \alpha \rceil - (k_n + 1)\}$, then we can choose an embedding σ such that:

  $$x^\sigma = a - x_n \leq \lceil \alpha \rceil - (k_n + 1) - x_n < 0,$$

  by definition of k_n, which contradicts our hypothesis.

- If $a \in \{\lceil \alpha \rceil + (k_n + 1), \ldots, 2\lceil \alpha \rceil - 1\}$, then again we can choose σ such that

  $$x^\sigma = a + x_n \geq \lceil \alpha \rceil + (k_n + 1) + x_n > 2\lceil \alpha \rceil,$$

  which again contradicts our hypothesis on x.

- If $a = \lceil \alpha \rceil \pm j + x_s$, with $0 \leq j \leq k_s$, then

  $$a + x_n = \lceil \alpha \rceil \pm j + x_s + x_n \geq 2\lceil \alpha \rceil,$$

  by Lemma 3.0.7, a contradiction.

- If $a = \lceil \alpha \rceil \pm j - x_s$, with $0 \leq j \leq k_s$, then

  $$a - x_n = \lceil \alpha \rceil \pm j - x_s - x_n \leq 0,$$

  also by Lemma 3.0.7, again a contradiction.

So, we have $a \in \{\lceil\alpha\rceil - k_n, \dots, \lceil\alpha\rceil + k_n\}$. Therefore, if $|b| = 1$, then x is of the form $\lceil\alpha\rceil \pm j \pm x_n$ where $0 \le j \le k_n$. In any case, we have $x \in X$.

□

Lemma 3.0.10. *Assume $x \in \mathcal{O}$. We have $0 \ll x \ll 2\lceil\alpha\rceil$ if and only if $x \in X$.*

Proof. Thanks to Lemma 3.0.9, we need only to prove the lemma from right to left. Assume $x \in X$. For $x \in X_0$, there is nothing to prove. Assume $x \in X_n$ for some $n \ge 1$, so that $x = \lceil\alpha\rceil \pm j + x_s$ for some s and j such that $1 \le s \le n$ and $0 \le j \le k_s$. By definition of k_s, we have $x_s + k_s < \lceil\alpha\rceil$. Hence, we have

$$\lceil\alpha\rceil \pm j + x_s \le \lceil\alpha\rceil + k_s + x_s < 2\lceil\alpha\rceil,$$

and

$$\lceil\alpha\rceil \pm j - x_s \ge \lceil\alpha\rceil - k_s - x_s > 0.$$

Therefore, we have $0 < x^\sigma < 2\lceil\alpha\rceil$ for every embedding σ of $\mathcal{O}_s$.

□

Proposition 3.0.11. *The JR number of $\mathcal{O}$ is $\lceil\alpha\rceil + \alpha$ and it is a minimum.*

Proof. We have $x_n + \lceil\alpha\rceil < \alpha + \lceil\alpha\rceil$ for infinitely many n, and by Lemma 3.0.10, there are infinitely many $x \in \mathcal{O}$ such that $0 \ll x \ll \lceil\alpha\rceil + \alpha$. Since the sequence (x_n) is increasing and converges to α, for each $\varepsilon > 0$, there are only finitely many n such that $x_n + \lceil\alpha\rceil < \alpha + \lceil\alpha\rceil - \varepsilon$. Moreover, there are infinitely many n such that $k_n = 0$. Hence, there are only finitely many elements of the form $x_n + \lceil\alpha\rceil + j$ where $0 \le j \le k_n$ and $k_n \ge 1$. In particular, only finitely of them satisfy $0 \ll x_n + \lceil\alpha\rceil + j \ll \lceil\alpha\rceil + \alpha$. Therefore, by Lemma 3.0.10, for each $\varepsilon > 0$, there are infinitely many $x \in \mathcal{O}$ such that $0 \ll x \ll \lceil\alpha\rceil + \alpha - \varepsilon$. □

Remark 3.0.12. Note that the hypothesis on x_0, $x_0 \ge \frac{\lambda}{4}$, is only used in the proofs of Lemmas 3.0.8 and 3.0.3. We believe that it is possible to prove the general case (increasing) without this hypothesis.

The following program (written in SageMath 9.2) indicates when one of the two lemmas is not satisfied:

```
def function(n):
    list=[]
    for x_0 in range(n+1): #x_0
        for v in range(n+1): #\nu
            for l in range(n+1): #\lambda
                if (l % 4) in {1,3} and (v+l*x_0 % 4) in {2,3}:
                      if v>x_0**2-l*x_0 and v>=2*l**2+2:
                            x1=sqrt(v+l*x_0)
                            ceil_x1=int(float(x1))+1
                            x2=sqrt(v+l*x_1)
                            alpha=(l+sqrt(l**2+4*v))/2
```

```
                                        ceil_alpha=int(float(alpha))+1
                                        if ceil_x1+x1<=alpha-1 or x1+x2+ceil_x1<=2*ceil_alpha:
                                            append.list((v,x_0,l))
    print(list)
```

With this program we have corroborated that for those $\nu, \lambda, x_0 \in \{0, \ldots, 5000\}$ that satisfy the hypothesis of 2.3.2, 2.2.4 and $\nu \geq 2\lambda^2 + 2$, the conclusions of Lemmas 3.0.8 and 3.0.3 remain true.

4 Decreasing Case

We define the following sets:

$$X_0 = \{1, 2, \dots, 2\lfloor\alpha\rfloor + 1\}$$
$$X_n = X_0 \cup \{\lfloor\alpha\rfloor + 1 \pm x_k : \; 1 \le k \le n\}$$
$$X = \bigcup_{n\ge 0} X_n.$$

The following lemma and theorem are exactly as [5, Lemma 3.2, Proposition 3.4 and Proposition 3.5], changing their hypothesis $\sqrt{\nu - x_1} \ge 1$ by $\sqrt{\nu - \lambda x_1} \ge 1$. For this reason, we will omit the proof.

Lemma 4.0.1. [5, Lemma 3.2] *Assume $\sqrt{\nu - \lambda x_1} \ge 1$ and $x_1 < \lfloor\alpha\rfloor + 1$. For each $n \ge 0$, if $x \in \mathcal{O}_n$ and $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$, then $x \in X_n$.*

Theorem 4.0.2. [5, Proposition 3.4 and 3.5] *Assume $\sqrt{\nu - \lambda x_1} \ge 1$ and $x_1 < \lfloor\alpha\rfloor + 1$. The JR number of $\mathcal{O}$ is $\lfloor\alpha\rfloor + \alpha + 1$ and satisfies the isolation property.*

The following proposition proves that there are infinitely many rings $\mathcal{O}$ for which Theorem 4.0.2 holds.

Proposition 4.0.3. *For any λ congruent to 1 or 3 modulo 4, there are infinitely many distinct values of α corresponding to pairs (ν, x_0) of rational integers such that*

1. $\nu < x_0^2 - \lambda x_0$,
2. $\sqrt{\nu + \lambda x_0}$ *is not a rational integer,*
3. *For every $n \ge 1$, we have $[K_n \colon K_{n-1}]$,*
4. $\lambda^3 x_0 < \nu^2 - \lambda^2 \nu$,
5. $\sqrt{\nu - \lambda x_1} \ge 1$,
6. $\sqrt{\nu + \lambda x_0} < \lfloor\alpha\rfloor + 1$.

Proof. It is enough to show that for every λ there is an duple (ν, x_0) satisfying each of the six conditions. In fact, for any $\lambda \ge 1$ which is congruent to 1 or 3 modulo 4, we choose $\nu = 4\lambda^4$ and $x_0 = 2\lambda^2 + \lambda$. The first 5 conditions are immediate. We have

$$\alpha = \frac{\lambda + \sqrt{\lambda^2 + 4\nu}}{2} = \frac{\lambda + \sqrt{16\lambda^2 + \lambda^2}}{2} = \frac{\lambda + 4\lambda^2 + \varepsilon}{2} = \frac{\lambda - 1}{2} + 2\lambda^2 + \frac{1}{2} + \frac{\varepsilon}{2}$$

for some $0 < \varepsilon < 1$. Since λ is congruent to 1 or 3 modulo 4, we have $\lfloor \alpha \rfloor = 2\lambda^2 + \frac{\lambda-1}{2}$. Therefore, we have

$$(\lfloor \alpha \rfloor + 1)^2 = 4\lambda^4 + 2\lambda^3 + 2\lambda^2 + \left(\frac{\lambda+1}{2}\right)^2 > 4\lambda^4 + 2\lambda^3 + \lambda^2 = \nu + \lambda x_0,$$

so the last condition is satisfied. □

For $\lambda = 1$, M. Castillo [7, Thm. 1] was able to remove the hypothesis $\sqrt{\nu - x_1} \geq 1$ and $x_1 < \lfloor \alpha \rfloor + 1$, and obtain the following theorem:

Theorem 4.0.4. *Assuming $\lambda = 1$ and $\nu > 3$, $\mathcal{O}$ has JR number $\lfloor \alpha \rfloor + \alpha + 1$ and it satisfies the isolation property.*

Now we will present some new results for $\lambda = 3$. The same proof can be easily adapted to the case $\lambda = 2, 4, 5 \ldots$. I could not find the general pattern that would let me write a general proof since for each value of λ there are cases that must be studied independently.

We will prove the following theorem at the end of this section.

Theorem 4.0.5. *If $x_1 < \lfloor \alpha \rfloor + 1$ and $\nu \neq 19$, then $\mathcal{O}$ has JR number $\lfloor \alpha \rfloor + \alpha + 1$ and it satisfies the isolation property.*

Lemma 4.0.6. *If $x_1 < \lfloor \alpha \rfloor + 1$ and $\nu \neq 19$, then $\nu - 3x_2 \geq 1$.*

Proof. Since $x_1 < \lfloor \alpha \rfloor + 1$, we have

$$\nu - 3x_2 > \nu - 3(\lfloor \alpha \rfloor + 1) \geq \nu - 3\alpha - 3.$$

Therefore, it suffices to prove $\nu - 3\alpha - 3 \geq 1$. This is satisfied if and only if $2\nu - 17 \geq 3\sqrt{9 + 4\nu}$, which is true for every $\nu \geq 24$. By Lemma 2.2.5, we have $\nu \geq 3$, so we must analyze the cases when $\nu \in \{3, \ldots, 23\}$. A simple calculation shows that for $\nu \in \{3, \ldots, 18\}$, there is no x_0 that satisfies the inequalities given in Theorem 2.2.4. Hence, $\nu \in \{19, \ldots, 23\}$, and again solving the inequalities given in 2.2.4, we obtain the following cases:

ν	x_0	x_1	x_2	$\nu-3x_1$	$\nu-3x_2$
19	7	$\sqrt{40}$	$\sqrt{19+3\sqrt{40}}$	0.03	0.51
20	7	$\sqrt{41}$	$\sqrt{20+3\sqrt{41}}$	0.79	1.21
	8	$\sqrt{44}$	$\sqrt{20+3\sqrt{44}}$	0.10	1.05
21	7	$\sqrt{42}$	$\sqrt{21+3\sqrt{42}}$	1.56	1.92
	8	$\sqrt{45}$	$\sqrt{21+3\sqrt{45}}$	0.88	1.76
	9	$\sqrt{48}$	$\sqrt{21+3\sqrt{48}}$	0.22	1.61
22	7	$\sqrt{43}$	$\sqrt{22+3\sqrt{43}}$	2.33	2.63
	8	$\sqrt{46}$	$\sqrt{22+3\sqrt{46}}$	1.65	2.48
	9	7	$\sqrt{22+3\sqrt{49}}$	1	2.33
	10	$\sqrt{52}$	$\sqrt{22+3\sqrt{52}}$	0.37	2.18
23	7	$\sqrt{44}$	$\sqrt{23+3\sqrt{44}}$	3.10	3.35
	8	$\sqrt{47}$	$\sqrt{23+3\sqrt{47}}$	2.43	3.20
	9	$\sqrt{50}$	$\sqrt{23+3\sqrt{50}}$	1.79	3.05
	10	$\sqrt{53}$	$\sqrt{23+3\sqrt{53}}$	1.16	2.91
	11	$\sqrt{56}$	$\sqrt{23+3\sqrt{56}}$	0.55	2.78

Table 4.1: Approximate values of $\nu - 3x_1$ and $\nu - 3x_2$ for $\nu \in \{19, \dots, 23\}$.

□

Lemma 4.0.7. *Let $x \in \mathcal{O}_1$ be such that $0 \ll x \ll 2\lfloor\alpha\rfloor+2$. If $(\nu, x_0) \in \{(20,7),(20,8),(21,8),(21,9)\}$, then $x \in X_1$.*

Proof. Let $x = a + bx_1 \in \mathcal{O}_1$, with $a, b \in \mathbb{Z}$, be such that $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$. Note that in all cases we have $\lfloor\alpha\rfloor + 1 = 7$ and $x_1 \geq \sqrt{41}$. Since $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$, by Lemma 2.1.4, we have $a \in \{1, \dots, 13\}$ and

$$|b| < \frac{\lfloor\alpha\rfloor + 1}{x_1} \leq \frac{7}{\sqrt{41}},$$

so we have $b \in \{-1, 0, 1\}$. Finally, using a computer program (I used SageMath 9.2, see below) we can analyze all the cases to see that x is indeed in X_1. □

```
def cases_X1(x_0, nu, l):
    x_1=sqrt(nu+l*x_0)
    floor_alpha=math.floor((l+sqrt(l**2+4*nu))/2)
    for a in srange(1,2*floor_alpha+2,1):
        for b in [-1,0,1]:
            if 0<a+b*x_1<2*floor_alpha+2 and 0<a-b*x_1<2*floor_alpha+2:
                print(a+b*x_1)
cases_X1(7,20,3)
```

```
cases_X1(8,20,3)
cases_X1(8,21,3)
cases_X1(9,21,3)
```

Lemma 4.0.8. *Let $x \in \mathcal{O}_2$ be such that $0 \ll x \ll 2\lfloor\alpha\rfloor+2$. If $(\nu, x_0) \in \{(20,7),(20,8),(21,8),(21,9)\}$, then $x \in X_2$.*

Proof. Let $x = a + bx_2 \in \mathcal{O}_2$, with $a, b \in \mathcal{O}_1$. Note that in all cases we have $x_1 \geq \sqrt{41}$, $x_2 \geq \sqrt{20+3\sqrt{41}}$ and $\lfloor\alpha\rfloor + 1 = 7$. Since $0 \ll a + bx_2 \ll 2\lfloor\alpha\rfloor + 2$, by Lemma 2.1.4 we have $0 \ll a \ll 2\lfloor\alpha\rfloor + 2$. Hence, $a \in \{1, \ldots 13\} \cup \{7 \pm x_1\}$ by Lemma 4.0.7. We will prove that we have $\overline{|b|} < 1.2$. Assume, for the sake of contradiction, that this is not the case. We will see that for whatever choice of a, there is an embedding σ such that x^σ is either negative or larger than 14, contradicting our hypothesis.

- Assume first $a \in \{1,2,3,4,5,6\}$: We choose σ such that $x^\sigma = a - |b|x_2$, so that we have
$$x^\sigma = a - |b|x_2 \leq 6 - x_2 < 0.$$
- Assume $a \in \{8,9,10,11,12,13\}$: We choose σ such that $x^\sigma = a + |b|x_2$, so that we have
$$(a + bx_2)^\sigma = a + |b|x_2 \geq 8 + x_2 > 14.$$
- Assume $a = 7 + x_1$: We choose σ such that $x^\sigma = a + |b|x_2$, so that we have
$$a + \overline{|b|}x_2 \geq 7 + x_1 + x_2 > 14.$$
- Assume $a = 7 - x_1$: We choose σ such that $x^\sigma = a - |b|x_2$, so that we have
$$a - \overline{|b|}x_2 \leq 7 - x_1 - x_2 < 0.$$
- Assume $a = 7$. We choose σ such that $x^\sigma = a - \overline{|b|}x_2$, so that we have
$$a + |b|x_2 \geq 7 + 1.2x_2 \geq 7 + 1.2\sqrt{20+3\sqrt{41}} > 14.$$

We conclude that $\overline{|b|} < 1.2$.
We write $b = b_1 + b_2x_1$, with $b_1, b_2 \in \mathbb{Z}$. We have
$$\overline{|b_1 + b_2x_1|} < 1.2, \tag{4.0.1}$$
so that $|b_1| < 1.2$ and $|b_2| < \frac{1.2}{\sqrt{41}}$. Hence, the only choices for b_1 and b_2 are $(b_1, b_2) \in \{(-1,0),(0,0),(1,0)\}$. Therefore, if $a \in \{1,\ldots,6\} \cup \{8,\ldots,13\} \cup \{7 \pm x_1\}$, then $b = 0$ by the first four cases. Otherwise, if $a = 7$, then we can have either $x = 7 - x_2$, or $x = 7 + x_2$, or $x = 7$. In all the cases we obtain $x \in X_2$. ☐

Lemma 4.0.9. *Assume $x_1 < \lfloor\alpha\rfloor + 1$ and $\nu \neq 19$. For each $n \geq 0$, if $x \in \mathcal{O}_n$ and $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$, then $x \in X_n$.*

Proof. If $\nu - 3x_1 \geq 1$, then we are done by Lemma 4.0.1. Assume $\nu - 3x_1 < 1$. By Lemma 4.0.6, the only cases where $\nu - 3x_1 < 1$ are when

$$(\nu, x_0) \in \{(20, 7), (20, 8), (21, 8), (21, 9), (22, 10), (23, 11)\}$$

(see Table 4.1). However, when $(\nu, x_0) \in \{(22, 10), (23, 11)\}$, a simple calculation shows that $x_1 > \lfloor\alpha\rfloor + 1$, so we may assume $(\nu, x_0) \in \{(20, 7), (20, 8), (21, 8), (21, 9)\}$. We will prove by induction on n that if $x \in \mathcal{O}_n$ is such that $0 \ll x \ll 2\lceil\alpha\rceil + 2$, then $x \in X_n$. It is clear for $n = 0$. For $n = 1$ and $n = 2$ we are done by Lemma 4.0.7 and 4.0.8 respectively. Assume $n \geq 3$. By Lemmas 2.1.4 and 4.0.6 we have

$$|b^\sigma| < \frac{\lfloor\alpha\rfloor + 1}{\sqrt{\nu - 3x_{n-1}}} \leq \frac{\lfloor\alpha\rfloor + 1}{\sqrt{\nu - 3x_2}} \leq \lfloor\alpha\rfloor + 1$$

for every $n \geq 3$. The rest of the proof goes exactly as the proof of [5, Lemma 3.2]. □

Lemma 4.0.10. *Assume $x_1 < \lfloor\alpha\rfloor + 1$ and $\nu \neq 19$. Let $x \in \mathcal{O}$. We have $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$ if and only if $x \in X$.*

Proof. By Lemma 4.0.9, we need only to prove the lemma from right to left. Let $x \in X$. If $x \in X_0$, then there is nothing to prove. Assume $x \in X_n$ for some $n \geq 1$, so that $x = \lfloor\alpha\rfloor + 1 \pm x_k$ for some $1 \leq k \leq n$. Since the sequence (x_n) is decreasing, we have

$$\lfloor\alpha\rfloor + 1 + x_k < 2\lceil\alpha\rceil + 2,$$

and

$$\lfloor\alpha\rfloor + 1 - x_k > 0$$

for every $1 \leq k \leq n$. Therefore, we have $0 \ll x \ll 2\lfloor\alpha\rfloor + 2$. □

Proof Theorem 4.0.5. We will prove that $\lfloor\alpha\rfloor + \alpha + 1$ is the JR number of $\mathcal{O}$ and that it satisfies the isolation property. Since (x_n) is a decreasing sequence and converges to α, for every $\varepsilon > 0$ there exist infinitely many n such that

$$x_n + \lfloor\alpha\rfloor + 1 < \lfloor\alpha\rfloor + \alpha + 1 + \varepsilon.$$

So, by Lemma 4.0.10, for every $\varepsilon > 0$, there exist infinitely many $x \in \mathcal{O}$ such that $0 \ll x \ll \lfloor\alpha\rfloor + \alpha + 1 + \varepsilon$. Also, for each $n \geq 1$, we have $\lfloor\alpha\rfloor + 1 + x_n > \lfloor\alpha\rfloor + 1 + \alpha$. Hence, if $x \in \mathcal{O}$ is such that $0 \ll x \ll \lfloor\alpha\rfloor + \alpha + 1$, by Lemma 4.0.10, then we have $x \in \{1, \ldots, 2\lfloor\alpha\rfloor + 1\}$. Therefore, $\lfloor\alpha\rfloor + \alpha + 1$ is the JR number of $\mathcal{O}$, and it is not a minimum. We now show that it satisfies the isolation property. Let $M = \lfloor\alpha\rfloor + 1 - \alpha$ and $x \in \mathcal{O}$ be such that

$$0 \ll x \ll \mathrm{JR}(\mathcal{O}) + M = 2\lfloor\alpha\rfloor + 2.$$

Again by Lemma 4.0.10, $x \in X$ and so, for every $\varepsilon > 0$, there are finitely many n such that

$$x_n + \lfloor\alpha\rfloor + 1 > \lfloor\alpha\rfloor + \alpha + 1 + \varepsilon.$$

□

As an consequence of Proposition 4.0.3, we obtain:

Corollary 4.0.11. *There are infinitely many distinct values of α corresponding to pairs (ν, x_0) of rational integers such that*

1. $\nu < x_0^2 - 3x_0$,
2. $\sqrt{\nu + 3x_0}$ *is not a rational integer,*
3. $27x_0 < \nu^2 - 9\nu$,
4. *For every* $n \geq 1$, $[K_n : K_{n-1}] = 2$,
5. $x_1 < \lfloor \alpha \rfloor + 1$.

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