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Asymptotic Symmetries In Einstein-Scalar Theory in three dimensions

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Chapter 1

Introduction

1.1 Introduction

The identification of infinite-dimensional symmetries at the boundary of spacetime have served as an important tool to unfold the holographic properties of gravity at a quantum level, through the asymptotic symmetry groups that happens to be the global symmetries groups of a lower dimensional, dual field theory. For example, one can determine that the isometry group of Schwarzschild in AdS_4 is $\mathbb{R} \times SO(3)$ [1], however, in the asymptotic region, the asymptotic symmetry group is identified as $SO(3,2)$ corresponding to the algebra of global symmetries found in a conformal field theory in 3 dimensions. In this scenario, one can assert that the conformal field theory (CFT) serves as the dual theory to the gravitational theory in an Anti-de Sitter (AdS) configuration.

Considering all the information above, we encounter a fundamental concept known as the holographic principle [2, 3]. This principle essentially establishes a connection between a gravitational theory in $D+1$ dimensions with a quantum theory living in D dimensions.

The holographic principle has been a rich source of various dualities, with one of the most famous being the AdS/CFT correspondence, proposed by Maldacena [4]. It basically states a way of "translating" phenomena from one theory to another through a relation between correlation functions of the CFT, and the action of the gravitational theory (for more details see [5])

$$\langle O_\Delta(\vec{x}_1)\dots O_\Delta(\vec{x}_n)\rangle_{CFT} = \frac{(-i)^n}{Z_{AdS}[\varphi_0=0]}\frac{\delta^n Z_{AdS}[\varphi_0]}{\delta\varphi_0(\vec{x}_1)\dots\delta\varphi_0(\vec{x}_n)}|_{\varphi_0=0}\,, \tag{1.1.1}$$

where the left side of the equation are correlation functions that represent an observable quantity, and φ_0 acts as a conformal source of the primary operators $O_\Delta(\vec{x})$. On the right hand side, we have functional derivatives of a partition function Z_{AdS}, described completely by the gravitational action in the bulk on-shell

$$Z_{AdS}[\varphi_0] = \int_{\Phi[\varphi_0]} D\Phi e^{iS[\Phi]_{AdS}}\,. \tag{1.1.2}$$

This relations serves as an example of how an AdS space in D dimensions, weakly coupled, is connected to a CFT, strongly coupled, in $D-1$ dimensions, and is part of a “dictionary” that researchers are filling up by working on more complicated gravity theories, usually on asymptotically AdS spacetimes.

Following this information, a natural question arises: Can these holographic ideas be extended to spacetimes with different asymptotic behaviors, such as asymptotically flat spacetimes? By exploring this possibility, researchers hope to gain a deeper understanding of potential solutions and expand the applicability of holographic principles to a broader range of spacetime configurations. Through continued research, it is hoped that a plausible solution and insights into this matter will be obtained.

To start this work, first we are going to review some classical solutions to gravitational theories and emphasizing on their asymptotic behaviors.

1.2 General Relativity

Einstein successfully described a gravitational theory as a geometrical entity that defines how inertial observers navigate through curved spacetimes. The curvature of spacetime may arise as an intrinsic property or due to the presence of massive objects.

To accurately capture these features, a valid mathematical description should be capable of reproducing the system’s dynamics, transform covariantly and remain invariant under diffeomorphisms.

Hilbert found an action principle in 1915 that described perfectly all of the above considerations. The Einstein-Hilbert action is defined as

$$S_{EH} = \frac{1}{16\pi G}\int d^D x\sqrt{-g}(R - 2\Lambda)\,, \tag{1.2.1}$$

and the dynamics of this theory can be obtained from the least action principle, defined above. By varying the action (δS_{EH}) one gets the Einstein field equations[1]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = 0\,, \tag{1.2.2}$$

which is an equation of second order of the metric. This equation helps to compute the components of the metric, and study the geometry of the spacetime. Usually, the differential equations are hard to solve, but one can consider some ansatz that simplify the calculations. The way of applying this is considering some symmetries. For example, for a spherical symmetry that is also stationary, the ansatz is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \tag{1.2.3}$$

where Ω is just a $d-2$ manifold of constant curvature. Then a solution that solves the equation (1.2.2), with a vanishing cosmological constant $\Lambda = 0$ is $f(r) = 1 - \frac{2GM}{r}$. This correspond to the Schwarzschild solution that describe a static and spherically symmetric black hole[2] which has only one conserved quantity, the mass. An important characteristic of this solutions is that they are asymptotically flat, meaning that taking the limit $r \to \infty$, the line element goes to the Minkowski space

$$\lim_{r\to\infty} ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{1.2.4}$$

Now, there is also a rotating solution, where naturally the metric functions now can depend on the angular coordinates and a non diagonal component of the metric is one that represent a "dragging effect" produced by the rotation of the black hole, this solutions are known as the Kerr solutions. An ansatz for this

[1]There is also a boundary term, that for now we are going to set to zero, in order to simplify the analysis.

[2]we can see that it is a black hole, because the presence of a event horizon at $r = r_0 = 2GM$.

black holes are

$$ds^2 = -f(r,\theta)dt^2 + g(r,\theta)dr^2 + h(r,\theta)d\theta^2 + T(r,\theta)d\phi^2 + R(r,\theta)dtd\phi \quad (1.2.5)$$

although is hard to solve the equations, one can find that, in Boyer-Lindquist coordinates, the following solutions are obtained

$$\begin{aligned}
f(r,\theta) =& 1 - \frac{r\, r_s}{\Sigma} \\
g(r,\theta) =& \frac{\Sigma}{\Delta} \\
h(r,\theta) =& \Sigma \\
T(r,\theta) =& (r^2 + a^2 + \frac{r\, r_s}{\Sigma} a^2 sin^2(\theta)) sin^2(\theta) \\
R(r,\theta) =& -2 \frac{r\, r_s\, a sin^2(\theta)}{\Sigma} c
\end{aligned}$$

where $\Sigma = r^2 + a^2 cos^2(\theta)$, $\Delta^2 = r^2 - r\, r_s + a^2$, $a = \frac{J}{Mc}$ and r_s is called the Schwarzschild radius.

As one can see, both solutions, the Schwarzschild and the Kerr solutions belong to a family of asymptotically flat spacetimes, which are known for having an infinite dimensional symmetry group called BMS [6].

1.2.1 $\Lambda \neq 0$

As it is mentioned in the beginning of this section, there is a constant in the action (1.2.2) that carries the information of the curvature of the spacetime without sources. This constant is called the Cosmological constant, which can be positive $\Lambda > 0$, zero $\Lambda = 0$ representing asymptotically flat solutions (explained before) or negative $\Lambda < 0$.

The different geometries dragged by the value of this constant are peculiar, whether shrinks the space ($\Lambda < 0$) or it dilates ($\Lambda > 0$) when one moves through the spacetime. The general solution of the Einstein equations (1.2.2) are described by

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)} + r^2 d\Omega^2 \,, \quad (1.2.6)$$

where the base manifold is a sphere and the parameter m indicates the mass of the black hole, and Λ is a constant that defines whether is a solution in deSitter or

Anti-deSitter spacetime. For $\Lambda > 0$, this solution has a similar characteristic with the Kerr black hole, since it has two horizons, an event horizon and a cosmological horizon. On the opposite, the $\Lambda < 0$ case, describe a black hole with only one event horizon.

The asymptotic symmetry group will be discussed in the subsection (2.3.1) with an example of a asymptotically AdS_3 solution.

1.2.2 Scalar field in the action

In the previous analysis, it has only been examined a free gravitational theory, meaning that no matter has been considered, but of course that one can, in principle, add a matter field to the action and the Einstein equations remains the same, but now with a energy-momentum tensor $T_{\mu\nu}$ that is no longer zero[3].

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.2.7}$$

where now the energy momentum tensor introduced an energetic scale proportional to the amount of matter, and is defined now as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{1.2.8}$$

and can change the form depending on the kind of matter field that one add to the theory.

Although one can consider a big number of matter fields (dilatons, photons, etc..) in this study, our primary focus will center on a scalar field. Where the Coupling is minimal, which denotes that the action being considered is

$$S[g_{\mu\nu}, \Phi] = \int d^3x\sqrt{-g}\left[R - \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi - V(\Phi)\right] . \tag{1.2.9}$$

where now the action depends on two fields, the metric $g_{\mu\nu}$ and the scalar field Φ.

This type of theories has an energy-momentum tensor

$$T_{\mu\nu} = \nabla_\mu\Phi\nabla_\nu\Phi - g_{\mu\nu}\left(\frac{1}{2}\nabla_\rho\Phi\nabla^\rho\Phi - V(\Phi)\right) \tag{1.2.10}$$

[3]From now on, we are going to use natural units, where $c = 1$ and $G = \frac{1}{8\pi}$.

Here the self interacting potential $V(\Phi)$ is arbitrary. The field equations are given by

$$E^{\mu}{}_{\nu} := G^{\mu}{}_{\nu} - \frac{1}{2}\left[\nabla^{\mu}\Phi\nabla_{\nu}\Phi - \delta^{\mu}_{\nu}\left(\frac{1}{2}\nabla_{\alpha}\Phi\nabla^{\alpha}\Phi + V(\Phi)\right)\right]. \tag{1.2.11}$$

If we solve it for the $2+1$ dimension, then the field equations that appears are the same as the ones found by Man and Chan [7, 8]. Where the potential used here is a self interacting potential that has an exponential profile

$$V[\Phi] = -\frac{(2-\alpha)}{\chi^2}\, e^{-\sqrt{2\alpha}\Phi}. \tag{1.2.12}$$

The above type of potentials are of physical interest as they appear in string theory compactifications [9, 10] and also they can be inspired by dimensional reductions where the scalar field represents the size of the extra dimensions [11]. In the present context, they allow us to study the asymptotic behavior of geometries whose Riemann tensor approaches to zero in the region far away from localized sources

$$R^{\alpha\beta}{}_{\gamma\delta} \to 0\,. \tag{1.2.13}$$

Note that this condition does not necessarily imply that the metric resembles Minkowski space in the asymptotic region (see [12] for an example where the leading components of the metric diverge at the boundary). We will refer to configurations satisfying (1.2.13) as asymptotically locally flat spaces. Black hole solutions exhibiting this kind of asymptotics have been found in Einstein-scalar theories in three dimensions with an exponential potential that is unbounded from below in [7, 8, 13, 14].

Chapter 2

Theoretical Basis

2.1 The Model

As it was mentioned before, the objective of this work is to correctly characterize the asymptotic behaviour of asymptotically locally flat solutions that comes from an action principle like (1.2.9). This is archived by potentials like (1.2.12). Then considering static and spherically symmetric ansatz, one can solve the equations of motion, defined in (1.2.11) and get that, considering $\alpha > 0$ and χ a fixed constant with length dimension.

The spherically symmetric solution reads

$$ds^2 = -\left(\frac{r^2}{\chi^2} - Mr^\alpha\right)dt^2 + \left(\frac{r}{L}\right)^{2\alpha}\frac{dr^2}{\frac{r^2}{\chi^2} - Mr^\alpha} + r^2 d\varphi^2\,, \quad \Phi(r) = \sqrt{2\alpha}\log\left(\frac{r}{L}\right)\,, \tag{2.1.1}$$

the solution have two integration constants, M and L. When $M > 0$ the metric has an horizon located at $r = r_0$ with

$$r_0 = (\chi^2 M)^{\frac{1}{2-\alpha}}\,, \tag{2.1.2}$$

Where the case $\alpha = 0$ turn the spacetime into the BTZ black hole, while $\alpha = 1$ is a type of black string found in [15]. The behavior of the metric in the region nearby r_0 is expressed using retarded Eddington-Finkelstein time u and radial coordinate $\rho = r - r_0$,

$$ds^2 = [-2\kappa\rho + O(\rho^2)]du^2 - 2\,[1 + O(\rho)]\,dud\rho + [r_0^2 + O(\rho)]d\varphi^2 \tag{2.1.3}$$

Here we can see that, as it was demonstrated in [16], the surface gravity κ and the horizon area A are captured by the leading order of the metric components g_{uu} and $g_{\varphi\varphi}$ respectively

$$\kappa = \frac{2-\alpha}{2\chi^2} r_0^{1-\alpha} L^\alpha, \quad A = 2\pi r_0 \tag{2.1.4}$$

This means that the geometry describes an event horizon only for $\alpha < 2$ (because otherwise it would mean that gravity at the horizon doesn't pull you in but out). This geometry is an asymptotically locally flat spacetime as the Ricci tensor behaves like $R^\mu{}_\nu = O(r^{-2\alpha})$ i.e., vanishes at the $r \to \infty$ limit. Despite this feature of an asymptotically locally flat space, the conformal boundary shares similarities with AdS_3 spaces. To make this statement much more explicit, let us define the following coordinate transformation

$$r = \frac{\rho^{q+1}}{L^q}\,,\; t = (q+1)^{-1}\tau\,,\; \alpha = \frac{q}{q+1} \tag{2.1.5}$$

Which define the new metric

$$ds^2 = \left(\frac{\rho}{L}\right)^{2q}\left[-\left(\frac{\rho^2}{l^2} - \frac{\nu}{\rho^q}\right)d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{l^2} - \frac{\nu}{\rho^q}} + \rho^2 d\varphi^2\right], \Phi(\rho) = \sqrt{2q(q+1)}\log(\frac{\rho}{L}) \tag{2.1.6}$$

where we have redefined l and ν by $l = (q+1)\chi$ and $\nu = \frac{L^{\alpha+q}}{(q+1)^2}M$. This transformation can be read then as a conformal factor, times a metric that looks like AdS_3, as long as $q > 0$. In fact, we can define a regular metric at infinity by taking $\rho = \frac{1}{\Omega}$. Then at $\Omega = 0$, the boundary metric is defined by

$$\lim_{\Omega\to 0} L^{2q}\Omega^{2(q+1)}ds^2 = -\frac{d\tau^2}{l^2} + d\varphi^2\,, \tag{2.1.7}$$

this correspond to the boundary of the space time, as is defined by Brown and Henneaux in [17], that is related to the cylinder (as it is usual in the anti-deSitter spaces) which means that the entire Virasoro group generates conformal transformations at the boundary.

It is important to notice that the action of bringing the boundary of the spacetime to a finite region is controlled by the constant q, that has beed initially defined as one of the potential's coupling constant, which means that the conformal factor depends on how strong the scalar field is coupled to the theory. In this sense, this is similar to the notion of conformal infinity constructed in [18] for cosmological spacetimes, where q represents a deceleration parameter of the fluid's equation of state. For the sake of simplicity, in the rest of this work we will consider geometries $q = 1$. In the final section, will be comments on generalizations with arbitrary values of q.

2.2 Covariant phase space

In many areas of physics there exist the notion of conservation law, a property that consist on an integral of the dynamical variables whose time evolution is balanced by a spatially driven flux. Mathematically, it is expressed as

$$\partial_t \rho + \partial_i J^i = 0\,, \tag{2.2.1}$$

where J is a density current, and ρ is a charge density. Integrating over a codimension 1 surface, it then follows

$$\frac{d}{dt}\int d^{D-1}x\,\rho = -\int d^{D-1}x\,\partial_i J^i = \int d^{D-2}x\,n_i\,J^i \tag{2.2.2}$$

from where we can read that, in order for the charge[1] $Q = \int d^{D-1}x\rho$ to be conserved there cannot be any flux of current outside the $D-2$ surface. In other words, this means that the normal component of the current $n_i J^i$ vanishes very far from the localized sources.

In 1916, Emmy Noether made a groundbreaking contribution in the field of theoretical physics, by introducing a theorem that establishes a correspondence between each symmetry present in a given theory, defined through a Lagrangian, and its corresponding conserved charge. However, as we delve into the details of this section, we will discover that defining conserved charges becomes more challenging when dealing with symmetries that change continuously through space and time.

Let us first review the above construction for symmetries that are independent of the coordinates, the so-called *Noether's first theorem*. For a Lagrangian L the conserved charges are defined by the zero component of a codimension 1-form J that is determined by the relation

$$J^\mu = B^\mu_X - M^\mu. \tag{2.2.3}$$

Here X is the generator of the symmetry and B^μ_X is the boundary term that arises from the variation of the action once the symmetry is applied. Moreover, M^μ is

[1]In the spirit of the electromagnetic theory for which the conserved quantity is the electric charge

the boundary term that comes from varying the action respect to the generator X, once the equations of motion hold. The current J^μ satisfies an on-shell local conservation law

$$\partial_\mu J^\mu \approx 0. \tag{2.2.4}$$

that is the covariant version of the continuity equation (2.2.1).

An important remark is that J does not have only one representation. Actually, one can obtain an equivalent current by redefined it as

$$J^\mu \to \tilde{J}^\mu = J^\mu + \partial_\nu k^{\mu\nu} + t^\mu\,, \tag{2.2.5}$$

where $k^{\mu\nu}$ is a antisymmetric tensor and t^μ is a function of the equations of motion, which is zero on-shell. Current $\tilde{J}$ and J will result in the same charge, since tensor k and t do not spoil the conservation. In fact, the definition of (2.2.5) assures that the conservation remain even if we consider an ambiguity associated to $k^{\mu\nu}$ and t^μ

$$\partial_\mu J^\mu \approx 0 \to \partial_\mu (J^\mu + \partial_\nu k^{\mu\nu} + t^\mu) \approx\ 0\,. \tag{2.2.6}$$

This construction is still a good way of defining conserved charges, but all of this rumbles when dealing with pure gauge theories, as it is shown below.

2.2.1 First Noether's theorem applied to gauge theories

Gauge theories have the characteristic that, they only have trivial Noether currents, meaning that the only contributions (on-shell) are from the boundary term $k^{\mu\nu}$

$$J^\mu = \partial_\nu k^{\mu\nu} + t^\mu\,, \tag{2.2.7}$$

an important remark about the divergence of $k^{\mu\nu}$ is that it is related to gauge transformations, that for most of the cases are transformations that goes to zero at the boundary. Those kind of transformations are the ones that are called "redundancies" of the theory, and the non-trivial are the ones that do not necessarily vanishes at the boundary.

Then the charge is defined by a form of codimension two k, for example, the easier gauge theory for which we can see this result is the electromagnetic action in four-dimensional flat space

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}\,. \tag{2.2.8}$$

Under the gauge transformation $\delta A_\mu = -\partial_\mu \epsilon(x)$ we have that the terms that originate the Noether current (2.2.3) are

$$M^\mu = 0\,, \tag{2.2.9}$$

$$B^\mu_\epsilon = F^{\mu\nu}\partial_\nu \epsilon(x)\,, \tag{2.2.10}$$

with M^μ is trivial because the variation of the action respect to the symmetry $\delta A_\mu = -\partial_\mu \epsilon$ is identically zero. Then, the Noether current (2.2.7) results to be

$$J^\mu = -\partial_\nu(F^{\mu\nu}\epsilon(x)) + \epsilon(x)\partial_\nu F^{\mu\nu}, \tag{2.2.11}$$

which is just the form of (2.2.5). This may look fine at first sight, but as we mentioned before, one can always add any antisymmetric tensor to the current and it will still be conserved, so there is nothing stopping us to choose k so that we set $J^\mu = 0$. Now, because $\epsilon(x)$ is an arbitrary function, is not clear that the current is conserved, but for a moment, let us think that the parameter is a constant and compute the conserved charge

$$Q = \int (d^2x)_{\mu\nu} F^{\mu\nu}\,. \tag{2.2.12}$$

This give us a glimpse on what we need to obtain when constructing a lower degree conservation law. In what follows, we are going to review the formalism covariant phase space formalism developed by Iyer and Wald to systematically obtain surface integrals like (2.2.12).

2.2.2 A Classical Mechanics example

To see explicitly how to define all of the necessary elements for the formalism, let us visit an example from classical mechanics, which will be helpful to understand the construction of the covariant phase space in field theories. The description made in this section has been based on [19] and also in [20]

For a system defined by a Hamiltonian $\mathcal{H}$ whose canonical coordinates are z^A living in a finite dimensional phase space, the corresponding action is

$$S[z] = \int_{t1}^{t2} dt(\theta_A(z)\dot{z}^A - \mathcal{H}(z)) \,, \tag{2.2.13}$$

where $\dot{z} = \frac{dz}{dt}$ and $\theta_A(z)$ are the components of the Liouville one form $\theta = \theta_A(z)\delta z^A$. One can define a symplectic form ω by taking an exterior derivative

$$\omega = \delta\theta = (\partial_A\theta_B - \partial_B\theta_A)\delta z^A \wedge \delta z^B \,, \tag{2.2.14}$$

this is called a symplectic form because ω is a closed form $\delta\omega = 0$ in the space of fields z^A[2]. There is a theorem postulated by Darboux that states that one can always find a local basis where the symplectic form is diagonalizable, and come back to the usual coordinates (q^i, p_j).

Now, in principle is difficult to identify the presymplectic potential θ from a Hamiltonian action like (2.2.13), then, form a moment let us take the first variation of the action, respect to the canonical coordinates

$$\delta S = \int_{t1}^{t2} dt \left[(\omega_{AB}z^B - \partial_A\mathcal{H})\delta z^A + \frac{d}{dt}(\theta_A\delta z^A)\right] \tag{2.2.15}$$

which, as expected, correspond to the equations of motion plus a boundary term. One of the most important observations of this section is that the presymplectic potential appear as a boundary term once the equations of motion are enforces. Indeed

$$\delta S = [\theta_A\delta z^A]_{t_1}^{t_2} \,, \tag{2.2.16}$$

One can find an explicit expression for the charge associated to the variational principle (2.2.13). In the spirit of the Noether theorem, let us consider a symmetry X, which generate the transformations $\delta_X z^A = X^A(z)$. This is a symmetry since it has to preserves the Hamiltonian action

$$\delta_X[\theta_A(z)\dot{z}^A - \mathcal{H}(z)] = \frac{d}{dt}B_X(z). \tag{2.2.17}$$

[2]It is assummed that ω is non-degenerated. In the cases where there exist zero mode transformations, $\xi^A\omega_{AB} = 0$, it is always assumed (unless explicitly stated) that the directions ξ^A have been moded out from our phase space.

Following Noether's first theorem, a conserved charge must exist for the symmetry spanned by X. Then, by expanding the equation (2.2.17), we end up with the following xpression,

$$\frac{d}{dt}(B_X - \theta_A X^A) = X^A(\omega_{AB}\dot{z}^B - \partial_A \mathcal{H}), \tag{2.2.18}$$

this result is clearly a conservation equation when the system is on-shell, which means that the conserved charge associated to the symmetry X is

$$H_x(z) = B_X - \theta_A X^A + Q_0 \,. \tag{2.2.19}$$

This charge is defined up to a constant Q_0 which holds information of the vacuum energy of the system or in other words, where is the "zero" of the charge. Notice that, considering that the flow of the symmetry X preserves the Hamiltonian, $\mathcal{L}_X \mathcal{H} = X^A \partial_A \mathcal{H} = 0$, then the equation of conservation (2.2.18) reads as $\frac{d}{dt} H_X = X^A \omega_{AB} \dot{z}^B$, or in a covariant way

$$X^A \omega_{AB} \delta z^B = \delta H_X \,. \tag{2.2.20}$$

The above equation corresponds to the definition of a variation of a conserved charge, but not just any variation, an infinitesimal departure in the space of fields consistent with the theory. A generalization of this equation can be written in the language of differential forms as

$$i_X \omega = \delta H_X \,, \tag{2.2.21}$$

where $i_X \omega$ is the interior product that transforms the two form ω into a one form. The charge H_X in these equations is defined as an integrable function, because the one form δH_X is exact, therefore H_X is determined up to a real constant.

2.2.2.1 Charge algebra

In order to understand the way that symmetries are realized through charges, it is important to define Poisson brackets within this formalism. More precisely, the Poisson bracket $\{\cdot, \cdot\}$ for functions H_{X_1} and H_{X_2} canonically conjugated to the symmetries X_1 and X_2 is

$$\{H_{X_1}, H_{X_2}\} = i_{X_2} i_{X_1} \omega \,. \tag{2.2.22}$$

Due to equation (2.2.22) we can express the latter definition as

$$\{H_{X_1}, H_{X_2}\} = i_{X_2} \delta H_{X_1} = \delta_{X_2} H_{X_1} = X_2[H_{X_1}] \,, \tag{2.2.23}$$

where $X_2[H_{X_1}]$ represents the action of the vector X_2 on the charge H_{X_1}. The last equality in (2.2.23) encodes, in particular, the energy conservation that arises in Classical mechanics. Indeed, by using $X_2 = \partial_t$, then the equation (2.2.23) turns into

$$\frac{\partial}{\partial t} H_{X_1} + \{\mathcal{H}, H_{X_1}\} = 0 \,, \tag{2.2.24}$$

where $\mathcal{H}$ represents the energy of the system, the conserved charge associated to time translations. In the case where H_{X_1} is explicitly independent of time, one sees that all the generators commuting with the Hamiltonian are also conserved. This highlights the importance of the algebra of the charges in order to characterize the dynamics of the system.

Now, it is known that the Lie commutator of two symmetries is also a symmetry, which implies that generators form an algebra. The same holds for the corresponding generators of those symmetries when using Poisson brackets. To prove that assertion, we can consider two different ways of expressing an equation. The first relation is obtained after taking an exterior derivative of (2.2.23), yielding

$$\delta\left(\{H_{X_1}, H_{X_2}\}\right) = i_{[X_1, X_2]} \omega \,, \tag{2.2.25}$$

where it is used that $\delta_{X_2} X_1 = -\mathcal{L}_{X_2} X_1$. The second relation comes directly by the definition (2.2.21) that can be expressed as

$$i_{[X_1, X_2]} \omega = \delta H_{[X_1, X_2]} \,. \tag{2.2.26}$$

Thus, taking into account equation (2.2.25) and (2.2.26), one can get as a result that

$$\{H_{X_1}, H_{X_2}\} = H_{[X_1, X_2]} + K_{X_1, X_2} \,, \tag{2.2.27}$$

where K_{X_1, X_2} is a constant in the space of fields that depends explicitly on the vector fields X_1 and X_2. The commutation relation (2.2.27) form what is called an algebraic structure, where K_{X_1,X_2} takes the role of an element of the algebra that is called central extension. Furthermore, it obeys some important rules. Firstly, it is antisymmetric $K_{X_1,X_2} = -K_{X_2,X_1}$. Secondly, in order for (2.2.27) to close an algebra, we must satisfy the Jacobi identity

$$\{H_{X_1}, \{H_{X_2}, H_{X_3}\}\} + \{H_{X_3}, \{H_{X_1}, H_{X_2}\}\} + \{H_{X_2}, \{H_{X_3}, H_{X_1}\}\} \equiv 0\,, \quad (2.2.28)$$

with X_1, X_2 and X_3 three independent vector fields. A direct consequence of the latter is that K_{X_1,X_2} has to obey a 2-cocycle condition

$$K_{X_1,[X_2,X_3]} + cyclic(1,2,3) = 0\,. \quad (2.2.29)$$

This ensures that the algebraic structure is a Lie algebra.

2.2.3 Surfaces charges in field theories

Now, in this section we are going to explore the previous constructed formalism in the case of field theories. We shall see how, even considering a Lagragian written in the second order formulation, the tools previously worked out can be successfully extended to define conservation laws.

Let us begin with the following action principle

$$I(\Phi) = \int d^D x \mathcal{L}(\Phi, \partial\Phi, \dots)\,, \quad (2.2.30)$$

where Φ denotes a set of fields defining the theory. The variation of the action is

$$\delta S = \int d^D x (E_\Phi \delta\Phi + \nabla_\mu \Theta^\mu(\delta\Phi, \Phi))\,,$$

with E_Φ being the classical field equations and Θ gives rise to a boundary term depending on the fields Φ and their first variations. In the language of differential forms this can be written as

$$\delta S = \int (E_\Phi \delta\Phi + d\,(\Theta(\delta\Phi, \Phi))\,, \quad (2.2.31)$$

which readily defines the presymplectic potential $\Theta(\delta\Phi, \Phi)$ after comparing with the on-shell variation found in (2.2.16). In turn, this means that we can define a

Lee-Wald symplectic form [21] as

$$\omega(\Phi;\delta\Phi,\delta\Phi) \equiv \int d^{D-1}x\, \delta\Theta(\Phi,\delta\Phi)\,. \tag{2.2.32}$$

Having defined the symplectic form, one can obtain the variation of the surface charge adapting (2.2.21) to field theories. Before delving into those details, let us define the contraction of ξ on the symplectic form in field space as

$$i_\xi\omega \equiv \omega(\Phi;\delta_\xi\Phi,\delta\Phi) = \int d^{D-1}x \left(\delta_\xi\Theta(\delta\Phi,\Phi) - \delta\Theta(\delta_\xi\Phi,\Phi)\right)\,, \tag{2.2.33}$$

where now the canonical transformation X are understood as local transformations ξ that may represent gauge symmetries or diffeomorphisms. To find the explicit expression for the charges, we first explain an important theorem that that arises in field theory.

2.2.4 Fundamental theorem of the covariant phase space

The fundamental theorem of the covariant phase space formalism states that the integrand of the symplectic form ω contracted with the vector field ξ is an exact differential form $\mathrm{d}k_\xi(\Phi,\delta\Phi)$[3] satisfying the identity

$$\delta_\xi\Theta(\delta\Phi,\Phi) - \delta\Theta(\delta_\xi\Phi,\Phi) = dk_\xi(\Phi,\delta\Phi)\,, \tag{2.2.34}$$

However, it is not guaranteed that k_ξ is an exact differential in the space of fields Φ. In the cases where k_ξ is $\delta-$integrable the conserved charge is represented by a surface term integrated over the phase space

$$Q_\xi = \int_{\mathcal{P}} \int d^{D-2}x\, \delta k_\xi \tag{2.2.35}$$

where the surface charge has a functional integration over the phase space $\mathcal{P}$. Notably, Q_ξ is independent of the path $\mathcal{P}$ taken to perform the integral.

An important feature of the equation (2.2.34) is that k_ξ is conserved if and only if $\omega(\Phi;\delta_\xi\Phi,\delta\Phi)$ is zero, which happens to be the case for exact symmetries, *i.e.*, those vectors $\bar\xi$ preserving the field configurations, $\delta_{\bar\xi}\Phi = 0$. Therefore, for exact symmetries $\bar\xi$, we find a extension of the Noether theorem based on a lower degree

[3] This is a 1-form in fields space but a $D-2$-form in coordinate space.

conservation law

$$dk_{\bar{\xi}} = 0\,. \tag{2.2.36}$$

In the forthcoming section, this condition will be relaxed, extending the concept of symmetry to asymptotic configurations where the non-linear effects of the bulk fields can be neglected.

2.2.5 Some applications

In this section, we work out in detail the expression for $k_{\xi}^{\mu\nu}$ in the cases of Maxwell theory and Einstein gravity.

2.2.5.1 Maxwell theory

Now that a lower degree conservation law has already been explained, its time to come back to the first example of Maxwell theory (2.2.1) and apply the formalism previously explained.

From the first variation of the Maxwell action, the presymplectic form can be defined by looking at the resulting boundary term

$$\delta S = -\int d^4x F^{\mu\nu}\partial_\mu \delta A_\nu\,, \tag{2.2.37}$$

and then rewrite it to see the elements of the current

$$\delta S = \int d^4x \partial_\mu F^{\mu\nu}\delta A_\nu - \int (d^3x)_\mu F^{\mu\nu}\delta A_\nu\,, \tag{2.2.38}$$

from where the first term vanish on-shell and the second term, correspond to the presymplectic form $\Theta^\mu(A,\delta A) = F^{\mu\nu}\delta A_\nu$. Then , taking the exterior derivative (in the phase space), we obtain the symplectic form

$$\omega(A,\delta A,\delta A) \equiv \int (d^3x)_\mu \delta\Theta^\mu(A,\delta A) = -\int (d^3x)_\mu \delta F^{\mu\nu}\delta A_\nu\,, \tag{2.2.39}$$

which is used to compute the surface charge. Then using the equation (2.2.33) and the local symmetries of electromagnetism $\delta_\epsilon A_\mu = \partial_\mu \epsilon$, we get

$$\omega(A,\delta_\epsilon A,\delta A) = -\int (d^3x)_\mu \left[\delta_\epsilon F^{\mu\nu}\delta A_\nu - \delta F^{\mu\nu}\delta_\epsilon A_\nu\right]\,, \tag{2.2.40}$$

where the first term is gauge invariant and thus, it vanishes. The only remaining contribution comes from the second term

$$\int (d^3x)_\mu \left[\delta F^{\mu\nu}\partial_\nu\epsilon\right] = \int (d^3x)_\mu \partial_\nu(\epsilon\delta F^{\mu\nu}) \tag{2.2.41}$$

this defines a variation of a conserved surface charge

$$\omega(A, \delta_\epsilon A, \delta A) = \int (d^2x)_{\mu\nu}\epsilon\delta F^{\mu\nu} \tag{2.2.42}$$

$$=\delta \int (d^2x)_{\mu\nu}\epsilon F^{\mu\nu} \tag{2.2.43}$$

$$=\delta Q_\epsilon\,. \tag{2.2.44}$$

Then, the surface charge has been defined as

$$Q_\epsilon = \int (d^2x)_{\mu\nu}\epsilon F^{\mu\nu} \tag{2.2.45}$$

That is the conserved charge for the exact symmetries $\epsilon =$ const. as it was proposed by the extension of Noether's theorem proposed by the covariant phase space formalism. Generalizations for $\epsilon(x)$ have been studied by [22] in the context of soft theorems.

2.2.5.2 Einstein Gravity

Now that the covariant phase space has demonstrated its applicability to a pure gauge theory, one can apply it directly to a more complicated theory of gravity, where the gauge transformations are now diffeomorphism. The particular case that is going to be studied here is the Einstein-Hilbert action without a cosmological constant[4]

$$S = \frac{1}{16\pi G}\int d^Dx\sqrt{-g}R\,, \tag{2.2.46}$$

from which one need to compute the variation to find explicitly the boundary term

$$\delta S = \frac{1}{16\pi G}\int d^Dx\sqrt{-g}(G_{\mu\nu}\delta g^{\mu\nu} + \nabla_\mu(\nabla^\nu h^\mu_\nu - \nabla^\mu h)) \tag{2.2.47}$$

[4]This term does not add any contribution to the conserved charge, since it does not contain derivatives.

Where we have used that $\delta g_{\mu\nu} = h_{\mu\nu}$. From this term we can identify the presymplectic potential

$$\Theta^\mu(g,\delta g) = \nabla^\nu h^\mu_\nu - \nabla^\mu h\,, \tag{2.2.48}$$

and construct the symplectic form (details of the computation can be found in the appendix)

$$\omega(g,\delta_\xi g,\delta g) = \frac{1}{16\pi G}\int (d^2x)_\nu \nabla_\mu k^{\mu\nu}_{\rm GR} \tag{2.2.49}$$

which is a boundary term, meaning that the conserved charge is indeed a surface integral, and the surface charge is defined as

$$k^{\mu\nu}_{\rm GR} = 2\sqrt{-g}\left[\xi^{[\mu}\nabla_\alpha h^{\nu]\alpha} - \xi^{[\mu}\nabla^{\nu]}h - \frac{1}{2}h\nabla^{[\mu}\xi^{\nu]} - \xi_\alpha\nabla^{[\mu}h^{\nu]\alpha} - h^{\alpha[\mu}\nabla_\alpha\xi^{\nu]}\right] \tag{2.2.50}$$

2.2.5.3 Diffeomorphism invariant scalar matter

Since the interest of this work is to analyze the features of a gravitational theory with a minimally coupled scalar field, it is also necessary to find an expression of the surface charge for a pure scalar theory.

The action for a scalar field that interact with himself is described by

$$S[g,\Phi] = -\int d^Dx\sqrt{-g}\left(\frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi + V[\Phi]\right), \tag{2.2.51}$$

from which it can be seen that the potential does not play any important role in the computation of the charge, because the charge get contributions only from the boundary terms, and the self interacting potentials considered here does not depends on derivatives of the scalar field $\partial\Phi$.

We start by taking the on-shell variation of the action to identify the boundary term

$$\delta S = \int d^Dx\sqrt{-g}\nabla_\mu(-\nabla^\mu\Phi\delta\Phi)\,, \tag{2.2.52}$$

from where the symplectic potential read as the terms inside the derivative $\nabla_\mu\Theta^\mu$

$$\Theta^\mu(\Phi,\delta\Phi) = -\nabla^\mu\Phi\delta\Phi\,, \tag{2.2.53}$$

and using the formula (2.2.33) one can get the equation that defines the surface charge $k^{\mu\nu}$

$$\omega(\Phi, \delta_\xi \Phi, \delta\Phi) = \int (d^{D-1}x)_\mu \nabla_\nu (\xi^\nu \Theta^\mu - \xi^\mu \Theta^\nu) \tag{2.2.54}$$

where we can read off the surface charge

$$k_\Phi = \int (d^{D-2}x)_{\nu\mu} \sqrt{-g} (\xi^\nu \Theta^\mu - \xi^\mu \Theta^\nu) \,. \tag{2.2.55}$$

Now that the gravitational (2.2.50) and the scalar matter (2.2.55) charges have been identified, we can use the sum of both expression to find charges associated to the global symmetries of on-shell solutions to Einstein gravity minimally coupled to a scalar field, as it is defined in [23] (with $\zeta_D = 0$). In D-dimensions, this read

$$\delta Q_\xi = \frac{1}{16\pi G} \int (k^{\mu\nu}_{\rm GR} + k^{\mu\nu}_{\Phi}) \epsilon_{\mu\nu x^1 \dots x^{D-2}} dx^1 \dots dx^{D-2} \,. \tag{2.2.56}$$

Up to this point, the charges have been partially described. We can apply this formalism for the particular solution of the action (2.1.1). This is an static spacetime and thus it has a Killing vector $\xi = \partial_t$. The associated conserved charges can be computed from the (2.2.56). By expressing, the gravitational and scalar contribution to the mass, one finds

$$k^{tr}_{\rm GR} = - 2\frac{r^{2-\alpha}\alpha}{\chi^2} L^{\alpha-1}\delta L + 2\alpha M L^{\alpha-1}\delta L + L^\alpha \delta M \,, \tag{2.2.57}$$

$$k^{tr}_{\Phi} = 2\frac{r^{2-\alpha}\alpha}{\chi^2} L^{\alpha-1}\delta L - 2\alpha M L^{\alpha-1}\delta L \,. \tag{2.2.58}$$

Here one can see that although both sectors have divergent contributions, the total mass is enhanced by the scalar contribution to the energy and the final result becomes finite. This feature has been previously described in configurations with relaxed AdS boundary conditions due to slowly decaying scalar fields [24, 25]. These examples together with the extended asymptotic that will be presented in this work, highlight the importance of the contribution of the matter fields to render a finite charge. Now, the resulting finite charge gives

$$\delta Q(\partial_t) = \int_0^{2\pi} d\varphi \frac{L^\alpha \delta M}{16\pi G} \,, \tag{2.2.59}$$

that as one can notice, it is not integrable in the space of solutions, meaning that a integration condition must be found.

At this point, we must demand an integrability condition between L and M. Here we will use that the scalar field at the horizon must vanish, $\Phi(r_0) = 0$. Let us see whether this condition is allowed by the first law of thermodynamics. Using this, the integration over the one-family parameter M yields

$$Q(\partial_t) = (2-\alpha)\frac{\chi^{\frac{2\alpha}{2-\alpha}} M^{\frac{2}{2-\alpha}}}{16G} \tag{2.2.60}$$

From the equations (2.1.4) one can read the entropy and its temperature

$$S = \frac{2\pi r_0}{4G}\,, \tag{2.2.61}$$

$$T = (2-\alpha)\frac{r_0}{4\pi\chi^2}\,, \tag{2.2.62}$$

where we have used that $r_0 = L$. From the above quantities the first law of thermodynamics can be build up. This allows us to independently compute the conserved charge of this family of solutions.

$$\delta Q^{\text{Thermo}}(\partial_t) = T\delta S = (2-\alpha)\frac{r_0}{8G\chi^2}\delta r_0\,, \tag{2.2.63}$$

which can be easily integrated, giving

$$Q^{\text{Thermo}}(\partial_t) = (2-\alpha)\frac{r_0^2}{16Gl^2}\,, \tag{2.2.64}$$

and because the horizon is related to the mass then

$$Q^{\text{Thermo}}(\partial_t) = (2-\alpha)\frac{\chi^{\frac{2\alpha}{2-\alpha}} M^{\frac{2}{2-\alpha}}}{16G}\,, \tag{2.2.65}$$

since the both equations (2.2.60) and (2.2.65) are the same, then now we can confidently say that the surface charge found in (2.2.56), is indeed the correct form computed from the covariant charges in a gravitational theory with a scalar field minimally coupled.

Now as it is going to be explained in the next section, the notion of conserved charges can be generalized so that we can extend the set of symmetries and conserved charges. This will provide a theory a much richer structure, resulting in new and interesting properties.

2.3 Asymptotic Symmetries

A gravitational theory is described by a action principle that should be invariant under diffeomorphism. This is due to the fact that its main field, the metric $g_{\mu\nu}$ transforms as a tensor under coordinate transformations $x^\mu = x^\mu(x')$

$$g_{\mu'\nu'}(x') = g_{\mu\nu}(x') \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \,, \tag{2.3.1}$$

which infinitesimally corresponds to the Lie derivative $\mathcal{L}_\xi$ acting on the metric

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho} \,, \tag{2.3.2}$$

where ξ is some vector field.

There are some special type of vectors, called Killing vectors, that represent symmetries of the theory and are represented by the equation

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \,. \tag{2.3.3}$$

In gravitational theories coupled to matter fields Φ^i, the idea of isometries applies to them in the same manner, $\mathcal{L}_\xi \Phi^i = 0$.

The notion of symmetry in gravity (and generically in gauge theories) can be modified in order to incorporate a group of configurations respecting certain boundary conditions. In other words, we will relax the exact equation (2.3.3) over the fields, so that we would consider an equation like

$$\mathcal{L}_\xi g_{\mu\nu} = O(r^{-\sigma}) \,, \tag{2.3.4}$$

A sufficient condition to ensure that this new equation is going to behave exactly like (2.3.3) in the asymptotic region is $\sigma > 0$.

Now, as well as the Killing vectors generate conserved charges, the asymptotic symmetries should also generate charges. In order for these symmetries to be physical, they have to lead to charges that are finite in the asymptotic region. Therefore, one has to carefully determine how fields approach to infinity.

The way of constructing asymptotic symmetries can be summarized as

- define the fall-off conditions over the fields.
- find the corresponding asymptotic symmetries that preserve the fall-off.
- compute the contraction of the symplectic form with the asymptotic vector.

After going through these steps, one should end up with

$$\omega(g;\delta_\xi g,\delta g) = O(1/r^\sigma)\,, \tag{2.3.5}$$

where again it is assummed that $\sigma > 0$. Thus, the conservation equation (2.2.36) hold asymptotically, as

$$\lim_{r\to\infty} dk_\xi \to 0\,. \tag{2.3.6}$$

In order to determine the exact value of σ, it is necessary to have a better understanding of the asymptotic symmetries of a theory.

Let us first provide some relevant definition for different kinds of gauge symmetries.

Gauge theories are invariant under spacetime dependent symmetries that are normally regarded as "redundancies of the theory". In setup outlined in this section, these are transformations that change the value of the fields in the bulk but they quickly goes to zero at $r\to\infty$. On the contrary, there is also a set of transformations which do not vanish at infinity, so they actually have a non trivial contribution to the conserved charges, as they have an effect that reaches the boundary of the spacetime. This set of transformations are called large gauge transformations or improper symmetries.

Previously, has been defined in [6, 26, 27, 28, 29, 1, 17, 22] that the asymptotic symmetries in a theory can be found after removing the redundancies of the theory from the allowed gauge transformations. Then, the asymptotic symmetries are defined as

$$\text{Asymptotic Symmetries} = \frac{\text{Allowed gauge transformations}}{\text{Trivial gauge transformations}}\,. \tag{2.3.7}$$

To get a better understanding of these statements, let us revisit the Brown and Henneaux boundary conditions for AdS_3 spaces. This construction will help us to introduce new conserved charges using the covariant phase space.

2.3.1 Brown-Henneaux boundary condition

Brown and Henneaux proposed the proper way of defining the asymptotic behaviour of a locally anti de-Sitter solution [17]. By using the Hamiltonian formalism, they found that the algebra at the boundary is not the typical algebra of asymptotically AdS_3 spaces, instead, they found a much bigger symmetry group, a conformal group described by the Virasoro algebra.

Here, we show the steps that lead to an extension of the asymptotic symmetry group, and the algebra with the central charge of the Virasoro algebra. Let us start with presenting the metric of an AdS_3 spacetime

$$ds^2 = -\left(\frac{r^2}{l^2}+1\right)dt^2 + \left(\frac{r^2}{l^2}+1\right)^{-1}dr^2 + r^2 d\phi^2\,, \tag{2.3.8}$$

where $l = \frac{1}{\sqrt{-\Lambda}}$ is the curvature radius. This spacetime is maximally symmetric, meaning that, it has all the symmetries that a three dimensional spacetime can have[5]. Now, in order to find the boundary conditions let us show a solution with a less amount of symmetries.

A good way of choosing a fall-off is to consider that the new set of solutions contains rotation. This is the case of the BTZ solution [30]

$$ds^2 = -\left(\frac{(r^2-r_+^2)(r^2-r_-^2)}{l^2r^2}\right)dt^2 + \frac{l^2r^2dr^2}{(r^2-r_+^2)(r^2-r_-^2)} + r^2\left(d\phi - \frac{r_+r_-}{lr^2}dt\right)^2$$

From where Brown and Henneaux takes the idea for defining a suitable fall-off condition over the metric, such that this new family of solutions contains the

[5] In this case are 6, since the number of symmetries for a n dimensional maximally symmetric spacetime is given by $\frac{n(n+1)}{2}$

rotating solution as well as the maximally symmetric one.

$$\begin{aligned}
g_{tt} =& -\frac{r^2}{l^2} + O(1),\\
g_{tr} =& O\left(\frac{1}{r^3}\right),\\
g_{t\phi} =& O(1),\\
g_{rr} =& \frac{l^2}{r^2} + O\left(\frac{1}{r^4}\right),\\
g_{r\phi} =& O\left(\frac{1}{r^3}\right),\\
g_{\phi\phi} =& r^2 + O(1).
\end{aligned} \tag{2.3.9}$$

The tr and $r\phi$ components were chosen to decay fast enough to obtain well-defined charges. These boundary conditions turn out to be invariant under the following asymptotic symmetries

$$\xi^t = l\,T(t,\phi) - \frac{l^4}{2r^2}\partial_t R(t,\phi) + O(1/r^4) \tag{2.3.10}$$

$$\xi^r = -r\partial_\phi \Phi(t,\phi) + O(1/r) \tag{2.3.11}$$

$$\xi^\phi = \Phi(t,\phi) + \frac{l^2}{2r^2}\partial_\phi R(t,\phi) + O(1/r^4) \tag{2.3.12}$$

where the functions T, Φ and R are related by the following equations

$$\partial_\phi \Phi = l\,\partial_t T\,, \quad l\,\partial_t \Phi = \partial_\phi T\,, \quad R = -\partial_\phi \Phi = -l\partial_t T\,. \tag{2.3.13}$$

The above equations and the fall-off conditions describe the asymptotic behavior of a broad set of solutions. Furthermore one can compute the conserved charges using the covariant phase space defined in (2.2.50). A straightforward computation shows that they are indeed finite and integrable

$$Q(T,\Phi) = \frac{1}{16\pi G}\int d\phi \left[\frac{T}{l^3}\left(f_{rr} + 2l^2 f_{\phi\phi}\right) + 2f_{t\phi}\Phi\right] \tag{2.3.14}$$

where the functions f_{ij} depends on the boundary coordinates $x^i = (t,\phi)$ and represent the fall-off of the metric components defined in (2.3.9).

Once one has already proven that the fall-offs chosen generated well-defined asymptotic symmetries (producing finite charges), the only thing that it is left to do is fully describe the charge algebra. Then, using the relation (2.2.23) we found that

$$\{Q(T_1,\Phi_1),Q(T_2,\Phi_2)\}=\delta_2 Q(T_1,\Phi_1) \tag{2.3.15}$$

one finds that it needs to know how the relevant components of the metric transform under the asymptotic symmetry group. The way of achieving that is to compare $\delta g_{\mu\nu}$ with $\mathcal{L}_\xi g_{\mu\nu}$, from where one obtains

$$\delta_2 f_{rr} =2l\, f_{rr}\dot{T}_2+\Phi_2 f'_{rr}+l\,T_2\dot{f}_{rr}\,, \tag{2.3.16}$$

$$\delta_2 f_{\phi\phi} =2l f_{t\phi}T'_2-l^2\Phi'''_2+2f_{\phi\phi}\Phi'_2+\Phi_2 f'_{\phi\phi}+lT_2 f'_{t\phi}\,, \tag{2.3.17}$$

$$\delta_2 f_{t\phi} =2f_{t\phi}\Phi'_2+\frac{2}{l}f_{\phi\phi}T'_2-\frac{l}{2}T'''_2+\Phi_2 f'_{t\phi}+\frac{1}{l}T_2 f'_{\phi\phi}\,, \tag{2.3.18}$$

where we have used the asymptotic field equations up to order $O(1/r^3)$ [6]. Also, one can notice from (2.3.16), that f_{rr} transforms into itself, meaning that no other f_{ij} component will appear in the transformation. Without loss of generality one can set f_{rr} to zero. Then, by applying (2.3.15) and transformations (2.3.17) and (2.3.18), the algebra reads

$$\{Q(T_1,\Phi_1),Q(T_2,\Phi_2)\}=Q(T_{[1,2]},\Phi_{[1,2]})+\int d\phi\left[\frac{l}{8\pi G}(\Phi'''_1 T_2-\Phi'''_2 T_1)\right]\,, \tag{2.3.19}$$

where one can identify an important property, which is that the charge algebra realize through the Lie bracket of the generators T and Φ. Now, the symmetries are characterized by the vector $\eta=T\partial_t+\Phi\partial_\phi$ which is a two-dimensional representation of the conformal algebra. Then, computing the Lie bracket $[\eta_1,\eta_2]=\mathcal{L}_{\eta_1}\eta_2$ one can see that produces a new vector whose component now are $T_{[1,2]}\partial_t+\Phi_{[1,2]}\partial_\varphi$, and those components result to be

$$T_{[1,2]} =T_1\Phi'_2-T_2\Phi'_1+\Phi_1 T'_2-\Phi_2 T'_1 \tag{2.3.20}$$

$$\Phi_{[1,2]} =\Phi_1\Phi'_2-\Phi'_1\Phi_2+T_1T'_2-T'_1T_2\,. \tag{2.3.21}$$

Now that the principal term of the algebra has been defined, it is time to analyze the extra term appearing in the algebra. From where one can see that does not

[6]from the field equations one can obtain that $f_{tt}=\frac{1}{l^2}f_{\phi\phi}$, $\dot{f}_{t\phi}=\frac{1}{l^2}f'_{\phi\phi}$ and $\dot{f}_{\phi\phi}=f'_{t\phi}$

depends on any metric function f_{ij} and instead depends only on the generators T and Φ, then can be read as a central extension. Hence the algebra (2.3.19) takes the form of the previously defined in (2.2.27). Therefore, the central extension corresponds to

$$K_{T_1,\Phi_1,T_2,\Phi_2} = \frac{l}{8\pi G}\int d\phi(\Phi_1'''T_2 - \Phi_2'''T_1) \tag{2.3.22}$$

which fullfils the 2-cocycle condition (2.2.29), providing a central extension of the conformal algebra.

Now, we can show that (2.3.19) is in fact the Virasoro that one usually finds in conformal field theory. The process to obtain a more explicit form of the algebra is to consider some properties of the solutions that has been obtained.

From the periodicity condition $T(t,\phi+2\pi) = T(t,\phi)$, $\Phi(t,\phi+2\pi) = \Phi(t,\phi)$ and the equations of the generators (2.3.13) , one can see that the following expansions in terms of the basis

$$T_n^{\pm} = \frac{l}{2}e^{in\left(\frac{t}{l}\pm\phi\right)}, \quad \Phi_n^{\pm} = \frac{1}{2}e^{in\left(\frac{t}{l}\pm\phi\right)}. \tag{2.3.23}$$

Then, one can obtain that the resulting non-trivial commutators of (2.3.19) are

$$i\{Q(T_n^{\pm},\Phi_n^{\pm}),Q(T_m^{\pm},\Phi_m^{\pm})\} = (n-m)Q(T_{n+m}^{\pm},\Phi_{n+m}^{\pm}) + \frac{l}{8G}n^3\delta_{(m+n),0} \tag{2.3.24}$$

As it is showed in [31], the coefficients of the central charge are normalized as $\frac{c}{12}$, meaning that the central charge is

$$c = \frac{3l}{2G}, \tag{2.3.25}$$

which is exactly the central charge of Brown and Henneaux.

This example closes the explanation on how the covariant phase space can be generalized so that we can incorporate an extra set of transformations that preserve asymptotic conditions. Now, in the next following sections, we are going to focus on applying the concepts of asymptotic symmetries, charge algebra and lower degree conservation laws for an Einstein-Scalar theory on three dimensions.

Chapter 3

Asymptotic symmetries in Einstein-Scalar theory on three dimensions

3.1 Asymptotic Symmetries

We would like to study the asymptotic group of a broader set of solutions displaying fall-off conditions incorporating (2.1.6). To make easier the analysis of the metric, let us define,

$$\frac{\rho}{L} = e^{\frac{\Phi}{2}} \tag{3.1.1}$$

which is an useful definition to compare with others similar findings of standard asymptotically flat spacetimes at null infinity [32, 33], from which a Weyl transformation of a version of Bondi-Metzner-Sachs (BMS) gauge [6, 28] can be found. This means that the family of spacetimes that are going to be considered are

$$ds^2 = e^{\Phi} \left[\mathcal{V}(u,\rho,\varphi)du^2 - 2dud\rho + 2\mathcal{U}(u,\rho,\varphi)dud\varphi + \rho^2 d\varphi^2 \right] , \tag{3.1.2}$$

in this gauge choice u is a light-like coordinate, ρ is a radial direction and φ is an angle orthogonal to the null rays $u = cte$, and to introduce more degrees of freedom that can help at the asymptotic behaviour, we consider that the scalar field can now depends on all coordinates $\Phi = \Phi(u, \rho, \varphi)$. An important feature of this gauge is that it let us write the metric as a conformal factor that explicitly depends on the scalar field Φ. Of course that this family of spacetimes has to still be able to describe the asymptotically locally flat spaces that solves the equations of motion of the theory, that's why one has to find symmetries that preserves the gauge conditions known as Bondi gauge fixing [1].

$$g_{\rho\varphi} = g_{\rho\rho} = 0,\ g_{\varphi\varphi} = e^{\Phi}\rho^2 \tag{3.1.3}$$

Therefore, symmetries preserving these conditions must fulfill

$$\mathcal{L}_\xi g_{\rho\rho} = \mathcal{L}_\xi g_{\rho\varphi} = 0,\ \mathcal{L}_\xi g_{\varphi\varphi} = \mathcal{L}_\xi \Phi g_{\varphi\varphi} \tag{3.1.4}$$

these equations lead to a definition of the components of the Killing vector

$$\xi^u = f(u,\rho,\varphi) \tag{3.1.5}$$

$$\xi^\rho = -\rho\partial_\varphi Y(u,\rho,\varphi) + \partial_\varphi^2 f(u,\rho,\varphi) - \frac{1}{\rho}\partial_\varphi f(u,\rho,\varphi)\mathcal{U}(u,\rho,\varphi) \tag{3.1.6}$$

$$\xi^\varphi = Y(u,\rho,\varphi) - \frac{1}{\rho}\partial_\varphi f(u,\rho,\varphi) \tag{3.1.7}$$

Notice that the Killing field includes an explicit dependence on the metric function $\mathcal{U}$, also for the gauge fixing (3.1.4) to be fulfill, the following functions cannot depend on the radial coordinate.

$$\partial_\rho f = \partial_\rho Y = 0 \tag{3.1.8}$$

Then $f = f(u,\varphi)$ and $Y = Y(u,\varphi)$.

Now we are ready to try to find the fall-off conditions [2], that as we can see are physically relevant, because one of the metric functions is showed in the radial component of the Killing ξ^ρ, meaning that how the fall-off conditions are defined, will affect on how the symmetries behaves.

With all this in mind, the symmetries that will be the focus of this work are the

[1]The Bondi gauge fixing is a common gauge fixing for asymptotically flat spaces
[2]sometimes you can find this as boundary conditions on the fields

ones that preserve the following fall-off conditions

$$\mathcal{V} = -\frac{\rho^2}{l^2} + \mathcal{V}_0 + O(\rho^{-1}), \quad \mathcal{U} = \mathcal{U}_0 + O(\rho^{-1}), \quad \Phi = 2\,log\left(\frac{\rho}{l}\right) - 2\theta + O(\rho^{-1}), \tag{3.1.9}$$

Where $\mathcal{V}_0$, $\mathcal{U}_0$ and θ are arbitrary functions of u and φ, while l represent a fixed coupling constant with dimensions of length. Something that will be relevant in the next chapters is that, one can think of l as a constant that represent the AdS radius of the asymptotic region of the conformally related spacetime, whose line element is $e^{-\Phi}ds^2$.

Notice that the fall-off conditions proposed in (3.1.9) will not define conserved charges, according to the definition (2.3.4) previously defined. The reason is that, in this type of solutions, the charges are indeed not conserved, because of radiative terms, coming from the scalar fields $\mathcal{E} = e^{-\theta(u,\varphi)}$, which will be a relevant term interpolating between two asymptotic regions.

Now, by acting with (3.1.7) on the asymptotic expansion previously defined, one can find that the vector components f and Y are restricted by each other as we can see on the following equations.

$$\partial_u Y = \frac{1}{l^2}\partial_\varphi f\,, \quad \partial_u f = \partial_\varphi Y\,, \tag{3.1.10}$$

this is exactly the equations of a conformal Killing vector that is defined in the cylinder [34].

Considering values of the radius such that $l \neq 0$, the functions of the Killing vector comprise a representation of two copies of the Witt algebra. This can be proved by using a modifed version of the Lie bracket [32]

$$[\xi(s_1, g), \xi(s_2, g)] = \mathcal{L}_{\xi(s_1,g)}\xi(s_2, g) - \delta_{\xi(s_1,g)}\xi(s_2, g) + \delta_{\xi(s_2,g)}\xi(s_1, g)\,. \tag{3.1.11}$$

Where $s_i = (f_i, Y_i)$ and the last two terms have been added due to the explicit dependence of the Killing vectors (3.1.7) on the metric $g_{\mu\nu}$. These terms correspond to an infinitesimal change of $\xi(s_2, g)$ due to a diffeomorphism ξ_1. Doing so one gets,

$$[\xi(s_1, g), \xi(s_2, g)] = \xi([s_1, s_2], g)\,. \tag{3.1.12}$$

Where the bracket $[s_1, s_2] = (f_{[1,2]}, Y_{[1,2]})$ is defined as a two-dimensional representation of the conformal algebra with

$$f_{[1,2]} = f_1 Y_2' - f_1' Y_2 + Y_1 f_2' - Y_1' f_2 \tag{3.1.13}$$

$$Y_{[1,2]} = Y_1 Y_2' - Y_1' Y_2 + \frac{1}{l^2}(f_1 f_2' - f_1' f_2)\,. \tag{3.1.14}$$

Some comments regarding the above findings are in order. Firstly, notice that equations (3.1.10) and (3.1.14) are the same as the ones found on the Brown-Henneaux example (2.3.13), (2.3.21), with $Y = T$ and $f = \frac{1}{l}\Phi$. Secondly, it is worth mentioning that the coordinate system chosen in this section, admits a smooth $l \to \infty$ limit of the Killing vectors and asymptotic conditions. By dropping l^{-2} terms in (3.1.10) and (3.1.14), the resulting algebra generates the BMS_3 group [35, 36].

3.2 Phase space and charges

3.2.1 Asymptotic solutions

We now analyze the space of solutions of the Einstein-scalar system (1.2.11) in coordinates (3.1.2) subjected to the boundary conditions (3.1.9). One of the important advantages of the conformal BMS gauge discussed before, is that one can determine the radial dependence of the system without losing the functional form of the self-interacting potential $V(\Phi)$, this property will help to find the asymptotic radial expansion associated to the fields Φ, $\mathcal{U}$ and $\mathcal{V}$. To start solving the system, let us consider the equation $E^u_\rho = 0$ that gives the equation for the scalar field

$$(\partial_\rho \Phi)^2 + 2\,\partial_\rho^2 \Phi = 0 \tag{3.2.1}$$

whose general solution can be written as

$$\Phi = 2\left(log\left[\frac{\rho}{l} + \theta_1(u,\varphi)\right] - \theta(u,\varphi)\right)\,. \tag{3.2.2}$$

And to see the metric functions we first see the fields equations $E^u_u - E^\rho_\rho = E^u_\varphi = E^\varphi_\rho = 0$ that determine the radial dependence of $\mathcal{U}$ which gives

$$\rho(\rho + l\theta_1)\partial_\rho^2 \mathcal{U} - l\theta_1 \partial_\rho \mathcal{U} - 2(\mathcal{U} - \mathcal{U}_0) = 0\,, \quad \mathcal{U}_0 \equiv l(\theta_1 \partial_\varphi \theta - \partial_\varphi \theta_1)\,. \tag{3.2.3}$$

The latter can be solved in a expansion series consistent with the asymptotic behavior at infinity (3.1.9)

$$\mathcal{U} = \mathcal{U}_0 + \frac{\mathcal{U}_1}{\rho} \sum_{n\geq 0} \frac{3}{n+3} \left(\frac{-l\theta_1}{\rho} \right)^n . \tag{3.2.4}$$

where $\mathcal{U}_1 = \mathcal{U}_1(u, \varphi)$ is an arbitrary function of the angles. From $E^\rho_\rho - E^\varphi_\varphi = 0$ and using the above solution we can find

$$\begin{aligned} \rho^2(\rho + l\theta_1)\partial^2_\rho \mathcal{V} - l\theta_1 \rho\, \partial_\rho \mathcal{V} - 2\rho(\mathcal{V} - \mathcal{V}_0) = 2(\mathcal{U} - \mathcal{U}_0)\theta' \\ + 2(\mathcal{U} - \mathcal{U}_0)' + (\rho\theta' + \mathcal{U}_0)\partial_\rho \mathcal{U} + (l\theta_1 + \rho)\partial_\rho \mathcal{U}' . \end{aligned} \tag{3.2.5}$$

with $\mathcal{V}_0 = (\partial_\varphi \theta)^2 - \partial^2_\varphi \theta + l(\theta_1 \partial_u \theta - \partial\theta_1)$. The solution to the latter equation can be again expressed in terms of infinite series around $\rho \to \infty$ as

$$\mathcal{V} = -\frac{\rho^2}{l^2} + \mathcal{V}_0 + \frac{1}{\rho} \sum_{n\geq 0} \frac{1}{\rho^n} \left(\frac{3(-l\theta_1)^n \mathcal{V}_1}{n+3} + \mathcal{F}_n \right) . \tag{3.2.6}$$

Where $\mathcal{V}_1$ is an arbitrary function of u and φ, while $\mathcal{F}_n$ are functions completely determined by $\mathcal{U}_1$, θ_1 and θ. It is not necessary for the present discussion the particular form of the function, in fact the only important conclusion is the fact that $\mathcal{V}_n$ vanishes when $\mathcal{U}_1 = \theta_1 = 0$.

The only remaining equations are $E^\rho_u = E^\rho_\varphi = 0$ and they can be solved up to $O(\rho^{-4})$ obtaining an expression for θ_1,

$$\theta_1 = k e^\theta + l\partial_u \theta\, , \tag{3.2.7}$$

where k, is an integration constant. The sub-leading contributions in the large ρ expansion imply differential equations involving $\mathcal{F}_n$, $\mathcal{U}_1$ and θ. For the purpose of providing the asymptotic solution considered here, it is only necessary to consider the equations arising at ρ^{-4} order. They yield relations for $\mathcal{V}_1$ and $\mathcal{U}_1$

$$\begin{aligned} &2\dot{\mathcal{V}}_1 - 3\mathcal{V}_1\dot{\theta} - \frac{3}{l^2}\left(\mathcal{U}_1' - \mathcal{U}_1\theta'\right) = -2(\mathcal{V}_0 - l\dot{\theta}_1)\Box\theta + 3\theta'\Box\theta' + l\theta_1\Box\dot{\theta}\, , \\ &3\dot{\mathcal{U}}_1 - 3\mathcal{U}_1\dot{\theta} - \mathcal{V}_1' = l\theta_1\left(2\theta'\Box\theta - \Box\theta'\right) . \end{aligned} \tag{3.2.8}$$

Where $\Box\theta = -l^2\partial^2_u\theta + \partial^2_\varphi\theta$ is the two-dimensional d'Alambertian.

Finally, the diagonal components of the field equations $E_u^u = E_\rho^\rho = E_\varphi^\varphi$ determine the self-interacting potential for which one can express a series expansion for large values of the scalar field Φ that gives

$$V(\Phi) = -\frac{6}{l^2}e^{-\Phi} + \frac{4k}{l^2}e^{-\frac{3}{2}\Phi} + O(e^{-\frac{5}{2}\Phi})\,. \tag{3.2.9}$$

What's interesting about this result is that it seems like (2.2.48) for $\alpha = 1$ with subleading corrections and thus, we must regard k as a fixed coupling constant. In the next section exact solutions as well as a potential will be provided.

3.2.2 Surface charges

Now that the asymptotic symmetries has been defined, it is time for the most important part. If the choice of the fall-off on the fields has been correct, then the Killing vectors (3.1.7) can be inserted into the equations (2.2.56) and evaluated in the asymptotic region $\rho \to \infty$ only to find that all source of divergences are zero and the physically relevant terms (the finite part) survives.

It is worthwhile mentioning that the Barnich-Brant formalism [37] yields a different expression for the gravitational contribution, however it coincides with the one presented here when using the conformal Bondi gauge (3.1.2) (see [38] for more details on this point).

For the asymptotic conditions studied here, one can verify that the surface charge of the scalar and gravitational sectors arise linear, quadratic and cubic divergences for a series expansion of $\rho \to \infty$, but thanks to the form of (2.2.56) one can verify that

$$k_{GR}^{tr} = -\frac{4e^{-\theta}f\delta\theta}{l^3}\rho^3 + O(\rho^2) + O(\rho) + Q_{GR} + O(1/\rho) \tag{3.2.10}$$

$$k_{\Phi}^{tr} = \frac{4e^{-\theta}f\delta\theta}{l^3}\rho^3 + O(\rho^2) + O(\rho) + Q_{\Phi} + O(1/\rho) \tag{3.2.11}$$

where Q_{GR} and Q_Φ are the finite contributions of the gravitational and scalar sector. Then, the total conserved charge is the sum of both contributions

$$k_\xi^{tr} = 2f\delta(k)\rho^2 + \left[f\delta\left(e^{-\theta}(\mathcal{V}_0 - \theta_1\dot{\theta} + \dot{\theta}_1 - \theta'^2 + \theta'')\right)\right]\rho \tag{3.2.12}$$

$$+\left[Y\delta\left(e^{-\theta}(\mathcal{U}_0 + \theta_1' - \theta_1\theta')\right)\right]\rho + \delta Q_\xi(\chi)\,, \tag{3.2.13}$$

where now the quadratic divergence is proportional to a variation of a constant $\delta(cons) = 0$, and the linear divergences are just the equation of motion that has been solved in the previous subsection (3.2.1).

A similar behaviour has been observer in relaxed AdS boundary conditions due to a slowly decaying scalar field at infinity [24, 39, 25]. These examples together with the extended asymptotic presented here, highlight the importance of the contribution of the matter fields to render a finite charge.

The explicit form of (2.2.56) is generically non-integrable in the space of fields, but it can be simplified by using a definition $\mathcal{E} \equiv e^{-\theta}$. This leads to the expression

$$\delta Q_\xi(\chi) = \delta Q(s, \chi) + \Theta_s(\chi, \delta\chi) \tag{3.2.14}$$

with $s = (f, Y)$ and

$$Q(s,\chi) = \frac{1}{16\pi G}\oint \left[(k - l\dot{\mathcal{E}})\,(f\mathcal{V}_0 + 2Y\mathcal{U}_0) + \frac{2}{l}f\mathcal{E}\mathcal{V}_1 + \frac{3}{l}Y\mathcal{E}\mathcal{U}_1\right], \tag{3.2.15}$$

$$\Theta_s(\chi,\delta\chi) = \frac{1}{16\pi l G}\oint \left(4f'\mathcal{U}_0\delta\mathcal{E} + 2f\mathcal{U}_0'\delta\mathcal{E} + l^2 f\mathcal{V}_0\delta\dot{\mathcal{E}} + f\mathcal{V}_1\delta\mathcal{E}\right). \tag{3.2.16}$$

Where $Q(s,\chi)$ is the integrable piece, while $\Theta_s(\chi,\delta\chi)$ is the contribution that can not be expressed as a local functional of the fields. This definition is not unique, as we can always add an arbitrary functional N_s and define $Q_s' = Q_s + N_s$ with $\Theta' = \Theta - \delta N_s$, such that (3.2.14) is invariant.

A canonical realization of the asymptotic symmetries needs a definition of an integrable charge Q_ξ. One way of sorting out this issue is by restricting our phase space to configurations satisfying $\delta\Theta(\chi,\delta\chi) = 0$. An immediate consequence of the latter shows that one can not provide a functional relation among the fields $\mathcal{E}$, $\dot{\mathcal{E}}$ and $\mathcal{V}_1$ without breaking the enhanced asymptotic symmetry. Hence, the only option, in order to maintain the amount of symmetries of this configuration is to understand the underlying properties of the integrable charge $Q(s,\chi)$.

3.3 Exact solutions

Even if an asymptotic behaviour has already been defined, one can still study a particular solution that still realize the asymptotic behaviour, and that could be achieve by fixing the arbitrary functions to integration constants. In this section a particular set of solutions will be studied such that the physical properties will be probed by the integrable piece $Q(s,\chi)$. It will also being shown that δQ defines a valid extension of the first law of thermodynamics for an asymptotically locally flat black hole. Furthermore, a second set of solutions that describe radiating spacetimes will be provided, in which $Q(s,\chi)$ samples the dynamical evolution of the system to a very peculiar set of solutions.

3.3.1 Stationary black hole solution

Demanding axial symmetry we find an extension of Mann-Chan solution (2.1.1) admitting rotation. The fields read

$$\mathcal{V} = -\frac{\rho^2}{l^2} + \frac{\mathcal{V}_1}{\rho}, \mathcal{U} = \frac{\alpha l \mathcal{E}^{-3}}{\rho}, \Phi = 2\, log\left(\frac{\rho}{l}\mathcal{E}\right) . \tag{3.3.1}$$

where $\mathcal{V}_1$ and $\mathcal{E}$ are integration constants, as $\mathcal{U} \neq 0$, this is a rotating solution. An interesting fact about this solution is that the rotation cannot disappear with a coordinate transformation. This, for instance, differs from a rotating BTZ black hole, where the angular momentum can be obtained from a boost in the $u - \phi$ plane [40].

On the other hand, $\alpha > 0$ and l are fixed constant defining the self-interacting potential

$$V(\Phi) = -\frac{6}{l^2}e^{-\Phi} + \frac{3\alpha^2}{2l^4}e^{-4\Phi} . \tag{3.3.2}$$

a plot of the potential is presented in (3.3.3). It is easy to see the scalar field potential goes to zero exponentially fast for positive Φ. This yield a vanishing cosmological constant, allowing for the asymptotically flat solution (3.3.1) to arise at large values of the scalar field. The potential also has a global minimum at $\Phi_c = \frac{2}{3} log\left(\frac{\alpha}{l}\right)$. At that point the potential becomes an effective cosmological constant

$$V(\Phi_c) = -\frac{9}{2l^2}\left(\frac{l}{\alpha}\right)^{2/3} \equiv -\frac{2}{L^2} . \tag{3.3.3}$$

Around this critical value the scalar field adquires a mass $m^2 = \frac{8}{L^2}$ and the geometry becomes a locally AdS_3 spacetime. Using the standard Klein-Gordon inner product, the scalar field is normalizable there. In the context of AdS/CFT this means that its sub leading term, the "VEV" is excited around this point. At the end of this section we comment on how this AdS_3 background emerges as a near-horizon limit of (3.3.1) hence this critical points plays a role similar to that of the effective potential of the attractor mechanism.

3.3.1.1 Charges and first law of thermodynamics

The line element described by (3.3.1) represent a rotating black hole, whose event horizon is located at

$$\rho_0^3 = \frac{l^2}{2}\left(\mathcal{V}_1 + \sqrt{\mathcal{V}_1^2 - 4\alpha^2/\mathcal{E}^6}\right) \tag{3.3.4}$$

provided by $\mathcal{V}_1 \neq 2\alpha/\mathcal{E}^3$. This is a null surface generated by a Killing horizon $\chi = -\partial_u + \omega_H \partial_\varphi$. The value of the angular potential ω_H and the surface gravity κ follows from $\chi^\mu \nabla_\mu \chi^\nu = \kappa \chi^\nu$ when evaluated on $\rho = \rho_0$,

$$\kappa = \frac{3}{2}\left(\frac{\rho_0}{l^2} - \frac{l^2\alpha^2}{\mathcal{E}^6\rho_0^5}\right) \quad \omega_H = \frac{l\alpha}{(\mathcal{E}\rho_0)^3}\,. \tag{3.3.5}$$

This solution has global symmetries that luckily has the same asymptotic behaviour as the previous section, which is helpful because the equation (3.2.14) still holds to this situation. This gives as a result that the global charges are; the mass, associated to $f = 1$, and the angular momentum which is conjugated to $Y = 1$. Their values reduce to

$$\oint Q(\partial_u) = \frac{1}{8Gl}\left(2\mathcal{E}\delta\mathcal{V}_1 + 3\delta\mathcal{E}\mathcal{V}_1\right)\,, \quad Q(\partial_\varphi) = \frac{3}{8G}\alpha\mathcal{E}^{-2}\,. \tag{3.3.6}$$

It is worth emphasizing that the first law of thermodynamics still holds in this type of solution regardless of the non-integrability of the mass. Indeed using potentials (3.3.5) one finds that

$$\oint Q(\partial_u) = T_H\delta S + \omega_H \delta Q(\partial_\varphi)\,. \tag{3.3.7}$$

where $T_H = \frac{\kappa}{2\pi}$ is the Hawking temperature and S corresponds to the black hole entropy.

3.3.2 Extremal solutions

Now, another interesting feature of the rotating black hole is what is called "extremal solution", and can be seeing from the equation of the event horizon (3.3.4) that defines two different horizons, an inner horizon and an external horizon. Normally these two horizons are separate it by some "distance" but at a certain value of the constants the two horizon can became one. When this happen we say that the solution is extremal.

For this solution, the extremal value comes when $\mathcal{V}_1 = 2\alpha\mathcal{E}^{-3}$. The scalar field is the same as in (3.3.1), but the line element becomes

$$ds^2 = \frac{\rho^2}{l^2}\mathcal{E}^2\left[-\left(\frac{\rho}{l} - \frac{l\mathcal{V}_1}{2\rho^2}\right)^2 du^2 - 2du\,d\rho + \rho^2\left(d\varphi + \frac{l\mathcal{V}_1}{2\rho^3}du\right)^2\right] \tag{3.3.8}$$

This is interesting because the horizon is given by the value where the scalar potential $V(\Phi)$ reaches its minimum, $\rho_c^3 = l^2\alpha\mathcal{E}^{-3}$ meaning that the extremal solution coincides with the configuration in which the system describe an asymptotically AdS_3 spacetime.

By zooming around this region we will find that the near-horizon description is captured by the Coussaert-Henneaux self-dual solution [41]. To show this, let us make a coordinate transformation that will help to visualize this zoom

$$\rho^3 = \rho_0^3 + \epsilon\delta, \quad u = \frac{U}{\epsilon}, \quad \varphi = \phi - \frac{U}{l\epsilon}\,. \tag{3.3.9}$$

After taking the limit $\epsilon \to 0$ we find a one-parameter solution given by

$$ds^2 = \frac{1}{\rho_c^2}\left(\frac{\alpha}{l^2}\right)^{2/3}\left[-\frac{\Delta^2}{l^2\rho_c^2}dU^2 - \frac{2}{3}dU\,d\Delta + \rho_c^4\left(d\phi - \frac{\Delta}{l\rho_c^3}dU\right)^2\right], \Phi = \frac{2}{3}log\left(\frac{\alpha}{l}\right), \tag{3.3.10}$$

which satisfies $R^\mu_\nu = -\frac{9}{2l^2}\left(\frac{l}{\alpha}\right)\delta^\mu_\nu$ consistent with the minimum of the scalar potential. This is not a surprise, as it is known that the Coussaert-Henneaux geometry arises as a near horizon limit of an extremal BTZ black hole[42].

3.3.3 Dynamical domain-wall

Motivated by the behavior of the previous potential, a peculiar solution, that accommodates the gauge (3.1.2) is

$$V(\Phi) = -\frac{6}{l^2}e^{-\Phi} + \frac{4k}{l^2}e^{-\frac{3}{2}\Phi} \, . \tag{3.3.11}$$

This new self-interacting potential has a minimum, where a AdS_3 space emerge and also describes asymptotically locally flat solutions. This can be seeing from the plot

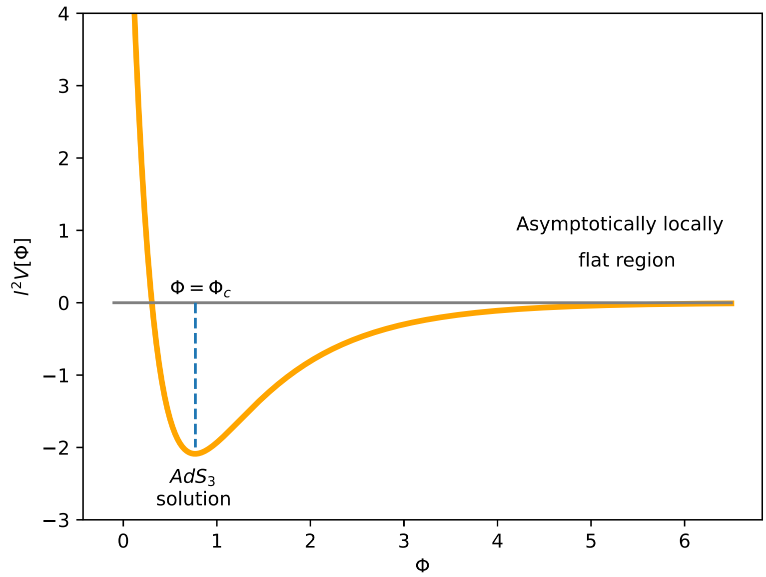

Figure 3.3.1: A schematic representation of the potential $l^2V(\Phi)$ for $k = 10$. At the minimum $\Phi_c = 2\log(k)$ a locally AdS_3 solution arises representing, for instance, a BTZ black hole for the choice $\theta = \mu u$. For large values of Φ, the curvature vanishes signaling the emergence of an asymptotically locally flat geometry. Notice that a very similar profile is also displayed by the scalar potential defined in (3.3.3).

Where now the idea is to show that this self-interacting potential comes from a dynamical configuration that start like an asymptotically locally flat solution and evolves to an AdS_3 solution.

It has been already defined in (3.2.2) that the asymptotic solution of the scalar field, implementing (3.2.7) is

$$\Phi = 2log\left(\frac{\rho}{l}\mathcal{E} - l\dot{\mathcal{E}} + k\right) \, , \tag{3.3.12}$$

Where now the function θ depends on the angles (u,φ) because it satisfies a free boson equation

$$\Box\theta = 0\,. \tag{3.3.13}$$

Another name for this equation which is more common is the wave equation, and it is known that has a solution of a zero mode (on both coordinates) plus in-going and out-going waves

$$\theta(u,\varphi) = \mu u + \sigma\varphi + \theta_+(x^+) + \theta_-(x^-)\,. \tag{3.3.14}$$

where we have explicitly separated the zero mode contributions associated to constants μ and σ from the rest of the oscillators $\theta_\pm$ that are arbitrary functions of $x^\pm = u/l \pm \varphi$. In terms of the free field θ, one completely solves the metric functions where relations (3.2.6) and (3.2.4) reduce to

$$\mathcal{V} = -\frac{\rho^2}{l^2} + l^2\dot\theta^2 + \theta'^2 - 2\theta''\,, \mathcal{U} = l^2(\dot\theta\theta' - \dot\theta')\,. \tag{3.3.15}$$

This means that the phase space of this two-dimensional system is controlled by the coadjoint representation of the Virasoro group [43] in terms of the representatives $(\mathcal{U}_0,\mathcal{V}_0)$. There is an additional restriction, namely that the scalar field Φ must be single-valued on the circle, which in turn implies the condition

$$\theta(u,\varphi+2\pi) = \theta(u,\varphi) \tag{3.3.16}$$

this equation implies that $\sigma = 0$ and that that $\theta_\pm$ are periodically well-defined functions. Furthermore, the momentum density $\mathcal{U}_0$ must have a vanishing zero mode.

The linear dependence on u makes the scalar field becomes a constant at late Bondi time $\Phi(u\to\infty) = 2log(k)$[3], which is the value where the theory describes a spacetime that is locally AdS_3, this has been reported previously in [33]. This produces an effective cosmological constant $\Lambda_{eff} = -\frac{1}{k^2l^2}$.

For simplicity, let us examine in more detail the zero mode solution $\theta = \mu u$, the solution already became dynamical as it was mentioned at the start of this subsection, the point on which one can say that interpolates between an

[3]Without loss of generality, we have set $\mu > 0$. Negative values of μ only revert the behavior of the Φ along u.

asymptotically locally flat solution and a locally AdS_3 can be seeing from the Ricci scalar

$$R = -\frac{2}{l^2}(8e^{-\Phi} - 4e^{-\frac{3}{2}\Phi}k - k^2e^{-2\Phi}) , \tag{3.3.17}$$

where we can see that for $\Phi \to \infty$ the spacetime becomes asymptotically flat. Besides, it becomes constant $R \to 6\Lambda_{\text{eff}}$ when the scalar goes to $2\log(k)$. In these coordinates, there this a curvature singularity when $\Phi \to -\infty$ or $\rho_{\text{sing}} = -l^2\mu - kle^{-\mu u}$. Fortunately, this singularity is hidden behind the null surface $\rho_{\text{null}} = l^2\mu$ for $k > 0$. Therefore, the zero mode geometry of this solution represents a dynamical collapsing spacetime with time-dependent conformal factor that settles down to a non-rotating BTZ black hole at $u \to +\infty$. This final state corresponds to the minimum of the potential.

3.4 Charge algebra

Let us recall the expression for surfaces charges found in section (3.2.2)

$$\not{\delta} Q_\xi(\mathcal{X}) = \delta\mathcal{Q}(s, \mathcal{X}) + \Theta_s(\mathcal{X}, \delta\mathcal{X}) , \tag{3.4.1}$$

which for solutions (3.3.15) reduces to

$$\begin{aligned} \mathcal{Q}(s, \mathcal{X}) &= \frac{1}{16\pi G}\oint (k - l\dot{\mathcal{E}})\,(f\mathcal{V}_0 + 2Y\mathcal{U}_0) , \\ \Theta_s(\mathcal{X}, \delta\mathcal{X}) &= \frac{1}{16\pi lG}\oint \left(4f'\mathcal{U}_0\delta\mathcal{E} + 2f\mathcal{U}_0'\delta\mathcal{E} + l^2 f\mathcal{V}_0\delta\dot{\mathcal{E}}\right) . \end{aligned} \tag{3.4.2}$$

In the standard construction [44, 17, 37], the algebra of the charges is isomorphic to the Lie bracket of the symmetries. This is possible only when δQ_ξ is an exact δ-form. In that case, the Dirac bracket of the charges is given by

$$\{Q_{\xi_1}, Q_{\xi_2}\}^* \equiv \delta_{\xi_2} Q_{\xi_1} \tag{3.4.3}$$

We would still like to have a canonical representation of the symmetries, however we cannot use the standard Dirac bracket due to the presence of $\Theta_s(\mathcal{X}, \delta\mathcal{X})$ in the charges. Fortunately, Barnich and Troessaert (BT) [45] have constructed a bracket for the integrable charges $\mathcal{Q}(s, \mathcal{X})$ that incorporates the non-integrable contribution. The mentioned algebraic structure is given by

$$\{\mathcal{Q}(s_1,\mathcal{X}),\mathcal{Q}(s_2,\mathcal{X})\}_{\mathrm{BT}} \equiv \delta_{s_2}\mathcal{Q}(s_1,\mathcal{X}) + \Theta_{s_2}(\mathcal{X},\delta_{s_1}\mathcal{X})\,, \tag{3.4.4}$$

At this point, we need the action of the asymptotic vectors $s \equiv (f, Y)$ on the leading fields. The relevant field is just θ and its transformation law can be obtained from the preservation of the fall-off at infinity (3.1.9). Doing so, we get

$$\delta_s\theta = f\dot{\theta} + Y\theta' + Y'\,. \tag{3.4.5}$$

The transformation laws of fields $\mathcal{V}_0$ and $\mathcal{U}_0$ can be obtained from the previous relations. They yield

$$\begin{aligned} \delta_s\mathcal{V}_0 &= Y\mathcal{V}_0' + 2Y'\mathcal{V}_0 - 2Y''' + \frac{2}{l^2}(f\mathcal{U}_0' + 2f'\mathcal{U}_0)\,, \\ \delta_s\mathcal{U}_0 &= Y\mathcal{U}_0' + 2Y'\mathcal{U}_0 + \tfrac{1}{2}f\mathcal{V}_0' + f'\mathcal{V}_0 - f'''\,. \end{aligned} \tag{3.4.6}$$

By using these relations we can show that after an straightforward, albeit lengthy application of the BT bracket formula, gives a canonical representation of the asymptotic symmetries. Indeed, by collecting all terms and removing the dependence on $\mathcal{X}$ for simplicity, we find

$$\{\mathcal{Q}(s_1),\mathcal{Q}(s_2)\}_{\mathrm{BT}} = \mathcal{Q}([s_1,s_2]) + \mathcal{K}(s_1,s_2) \tag{3.4.7}$$

where $[s_1, s_2]$ is the modified Lie bracket of two Killing vectors (3.1.14) and $\mathcal{K}(s_1, s_2)$ is <u>field dependent</u> central extension

$$\mathcal{K}(s_1,s_2) = \frac{1}{8\pi G}\oint\Big[(k - l\dot{\mathcal{E}})(Y_1'''f_2 - f_1Y_2''') + \frac{1}{l}\mathcal{E}'''(f_1'f_2 - f_1f_2')\Big]\,. \tag{3.4.8}$$

This term is a new element of the algebra generated by the BT bracket, provided

$$\{\mathcal{K}(s_1,s_2),\mathcal{Q}(s_3)\}_{\mathrm{BT}} \equiv \delta_{s_3}\mathcal{K}(s_1,s_2)\,, \quad \{\mathcal{K}(s_1,s_2),\mathcal{K}(s_3,s_4)\}_{\mathrm{BT}} = 0\,. \tag{3.4.9}$$

It is important to stress that since $\mathcal{K}_{s_1,s_2}$ transforms under the ASG, the Jacobi identity is satisfied by (3.4.7), provided the field dependent central extension satisfies a generalized cocycle condition

$$\delta_{s_3}\mathcal{K}(s_1,s_2) + \mathcal{K}([s_1,s_2],s_3) + \mathrm{cyclic}(1,2,3) = 0\,, \tag{3.4.10}$$

which is indeed satisfied by (3.4.8) (details can be found in appendix A3). Notice that this represents an extension of the cocycle condition defined in (2.2.29), because now we consider that the central extension no longer commute with the charges, as it is defined in (3.4.9).

The field dependent central extension $\mathcal{K}(s_1, s_2)$ has ambiguities associated to the normalization of the integrable part of the charge[45]. In section 3.2.2, we mention that $\mathcal{Q}(s)$ can be shifted by a normalization N_s, so that the total non-integrable charge $\not{\oint} Q_\xi$ remains invariant. At the level of the algebra, we obtained (3.4.7) for the charges $\mathcal{Q}'(s) = \mathcal{Q}(s) + N_s$ and $\mathcal{K}'(s_1, s_2)$ given by

$$\mathcal{K}'(s_1, s_2) = \mathcal{K}(s_1, s_2) + \delta_{s_1} N_{s_2} - \delta_{s_1} N_{s_2} + N_{[s_1,s_2]} \,. \tag{3.4.11}$$

An ambiguity of this sort is considered trivial as the generalized cocycle condition is (3.4.10) immediately satisfied.

3.4.1 Mode expansion

It is enlightening to express the generators of the algebra (3.4.7) in terms of the Fourier modes solutions to (3.1.10)

$$\ell_n^\pm = \tfrac{1}{2} e^{in\left(\frac{u}{l} \pm \varphi\right)} (l\partial_u \pm \partial_\varphi) \,, \quad \mathscr{L}_n^\pm = \mathcal{Q}(\ell_n^\pm) \,. \tag{3.4.12}$$

The algebra (3.4.7) can be explicitly written as

$$i\{\mathscr{L}_n^\pm, \mathscr{L}_m^\pm\}_{\text{BT}} = (n-m)\mathscr{L}_{m+n}^\pm + \mathcal{K}(\ell_n^\pm, \ell_m^\pm) \,, \quad i\{\mathscr{L}_n^+, \mathscr{L}_m^-\}_{\text{BT}} = \mathcal{K}(\ell_n^+, \ell_m^-) \,. \tag{3.4.13}$$

with

$$\begin{aligned} \mathcal{K}(\ell_n^\pm, \ell_m^\pm) &= \frac{kl}{8G} n^3 \delta_{m+n,0} - \frac{l}{16G} e^{i(m+n)\frac{u}{l}} \left[(n^3 - m^3) l\dot{\mathcal{E}}_{m+n} + i(n-m)(m+n)^3 \mathcal{E}_{m+n} \right] \\ \mathcal{K}(\ell_n^+, \ell_m^-) &= \frac{l}{16G} e^{i(m+n)\frac{u}{l}} \left[(n^3 - m^3) l\dot{\mathcal{E}}_{n-m} - i(n-m)^4 \mathcal{E}_{n-m} \right] \,. \end{aligned} \tag{3.4.14}$$

This is an explicitly time-dependent algebra with $\mathcal{E}_n(u) = \frac{1}{2\pi} \oint e^{in\varphi} \mathcal{E}(u, \varphi)$. However, it has an interesting late time behavior since the coefficient $\mathcal{E}_n$ are exponentially decaying for large u. This can be seen by using the periodic properties of θ defined in (3.3.14), implying

$$\mathcal{E}_n(u) = e^{-\left(\mu - i\frac{n}{l}\right)u} \sum_{k=-\infty}^{\infty} \mathcal{E}_k^+ \mathcal{E}_{n+k}^- e^{2i\frac{ku}{l}} \,, \tag{3.4.15}$$

with $\mathcal{E}_n^\pm$ being the Fourier mode of the functions $e^{-\theta_\pm}$. The important conclusion we get from the mode expansion of $\mathcal{E}_n(u)$ is that only the term proportional to k in (3.4.13) survives in the limit $u \to +\infty$. Therefore, we have

$$\mathcal{K}(\ell_n^\pm, \ell_m^\pm) \to \frac{c}{12} m^3 \delta_{m+n,0} \,, \quad \mathcal{K}(\ell_n^+, \ell_m^-) \to 0 \,, \tag{3.4.16}$$

and the time-dependent algebra (3.4.14) becomes two commuting copies of the Virasoro algebra with

$$c = \frac{3kl}{2G} \,, \tag{3.4.17}$$

being the Brown-Henneaux central charge. This asymptotic symmetry algebra emerges for configurations that slightly depart from the minimum of the potential (3.3.11).

Chapter 4

Conclusion

In this work we have analyzed the asymptotic structure of Einstein gravity coupled to a self-interacting scalar field in $2+1$ dimensions. We have focused on the sector in which the self-interaction permits a scalar field with slow fall-off at infinity, compatible with asymptotically locally flat geometries. Remarkably, in spite of the slow fall-off of the scalar and the non-trivial asymptotic geometry, the charges turn out to be finite due to a subtle cancellation of divergences coming from both the gravity and the matter sectors. The potentials here considered are of the same form as those emerging from Kaluza-Klein compactifications of GR coupled to matter fields [11] and in gauged supergravity (see e.g. [46], [47] and [48]). We have proved that for a family of self-interactions, even in the asymptotically locally flat region, two copies of the Witt algebra emerge, since the behavior of the metric at infinity is such that our geometries share the conformal asymptotic geometry with AdS_3. By working on a conformal Bondi gauge, we have shown that the field dependent asymptotic symmetries span two copies of the Witt algebra, which is canonically realized by means of the Barnich-Troessaert bracket [45]. Even more, we were able to find a new infinity family of time dependent solutions. These configurations are governed by a two-dimensional free-boson theory, which dynamically interpolate between the asymptotically locally flat and the locally AdS vacua of the theory, the latter being achieved for late times. The canonical realization of the charges associated to these dynamical solutions is given by two copies of the centrally extended Witt algebra, where the central extension matches the Brown-Hennaux one for late times. This can be interpreted as a realization of the Holographic $c-$theorem, since the spacetime reaches a locally AdS geometry for late times, and therefore the time plays the role of the holographic coordinate leading to a flow of the central charge.

In vacuum, the dynamics of GR is purely given in terms of boundary gravitons, while the inclusion of a scalar field introduces a local degree of freedom which, as we have shown, suitably modify the rich asymptotic structure of GR in vacuum. We have proposed a family of asymptotic behaviors for the metric and the scalar field in the conformal Bondi gauge, which accommodates black holes that were already known in the literature [7, 8] as well as new stationary and even dynamical solutions. We have provided a consistent thermodynamics for the new stationary black holes, which are characterized by two integration constants, and lead to a mass that, in spite of being non-integrable, fulfills the first law of black hole thermodynamics. It would be interesting to study the structure of phase transitions that may emerge within these solutions and the thermal background, or even between the solutions in the AdS sector of the theories we have considered, and the BTZ solution that is achieved for a fixed value of the scalar field, sitting at a negative minimum of the self-interacting potential.

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Appendix A

Appendix

A1 Surface Charge of the Scalar sector

Let us consider an action principle of a free scalar field[1].

$$I[\Phi] = \int d^3x \sqrt{-g} \left(-\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi \right) , \tag{A1.1}$$

although the action (1.2.9) does consider an "arbitrary" potential, it does not contribute to the surface charge. As it was explained in (2.2.16), the only contributions to the charge are the boundary terms.

Now, to find the symplectic potential Θ we take the first variation of (A1.1).

$$\begin{aligned} \delta I &= \int d^3x \delta_\Phi \left[\sqrt{-g} \left(-\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi \right) \right] + \delta_g \left[\sqrt{-g} \left(-\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi \right) \right] \\ &= \int d^3x \left[\sqrt{-g} \left(-\partial_\mu \Phi \partial^\mu \delta \Phi \right) + \delta_g \left[\sqrt{-g} \left(-\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi \right) \right] \right] \\ &= \int d^3x \left[\sqrt{-g} \left(\partial^\mu (-\partial_\mu \Phi \delta \Phi) + \Box \Phi \delta \Phi \right) + T_{\mu\nu} \delta g^{\mu\nu} \right] , \end{aligned} \tag{A1.2}$$

where the energy momentum tensor is defined as

$$T_{\mu\nu} = \sqrt{-g} \left[-\frac{1}{2} \left(\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial_\rho \Phi \partial^\rho \Phi \right) \right] . \tag{A1.3}$$

Now if we analyse these terms on-shell and remember that $\partial \Phi = \nabla \Phi$ then

[1] because Φ is a scalar $\nabla \Phi = \partial \Phi$

$$\delta I = \int d^3x\sqrt{-g}\nabla_\mu(-\nabla^\mu\Phi\delta\Phi) \tag{A1.4}$$
$$= \int (d^2x)_\mu\sqrt{-g}(-\nabla^\mu\Phi\delta\Phi)\,, \tag{A1.5}$$

where the symplectic potential can be read as

$$\Theta^\mu(\Phi,\delta\Phi) = -(\sqrt{-g}\nabla^\mu\Phi\delta\Phi)\,. \tag{A1.6}$$

Now that the symplectic potential has been found, we are ready to compute the symplectic form that has been previously defined as

$$i_\xi\omega = \delta_\xi\Theta(\delta\Phi,\Phi) - \delta\Theta(\delta_\xi\Phi,\Phi)\,, \tag{A1.7}$$

this expression applied to (A1.6) gives

$$\int (d^2x)_\mu\left[\delta_\xi(\sqrt{-g}(-\nabla^\mu\Phi\delta\Phi)) - \delta(\sqrt{-g}(-\nabla^\mu\Phi\delta_\xi\Phi))\right]\,. \tag{A1.8}$$

In order to compute this, we need to remember that a variation with respect of a vector ξ is the same as taking the Lie derivative of the element, with this said, the following is true

$$\delta_\xi\Phi = \xi^\mu\nabla_\mu\Phi \tag{A1.9}$$
$$\delta_\xi\nabla^\mu\Phi = \xi^\nu\nabla_\nu(\nabla^\mu\Phi) - \nabla^\nu\Phi\nabla_\nu\xi^\mu \tag{A1.10}$$
$$\delta_\xi\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta_\xi g_{\mu\nu} = \sqrt{-g}\nabla_\nu\xi^\nu\,, \tag{A1.11}$$

with this definitions, we can start to analyze each term of the symplectic form

$$\begin{aligned}\delta_\xi(\sqrt{-g}(-\nabla^\mu\Phi\delta\Phi)) =& -\delta_\xi(\sqrt{-g})\nabla^\mu\Phi\delta\Phi - \sqrt{-g}\delta_\xi(\nabla^\mu\Phi)\delta\Phi - \sqrt{-g}\nabla^\mu\Phi\delta_\xi\delta\Phi\\ =& -\sqrt{-g}\nabla_\nu\xi^\nu\nabla^\mu\Phi\delta\Phi - \sqrt{-g}(\xi^\nu\nabla_\nu(\nabla^\mu\Phi) - \nabla^\nu\Phi\nabla_\nu\xi^\mu)\delta\Phi\\ & -\sqrt{-g}\nabla^\mu\Phi\xi^\nu\nabla_\nu\delta\Phi\,.\end{aligned}$$

Using the definition (A1.6) and integrating by parts the terms that have derivatives of ξ and we end up with,

$$\begin{aligned}
&= \nabla_\nu(\xi^\nu\Theta^\mu - \xi^\mu\Theta^\nu) + \nabla_\nu\Theta^\nu\xi^\mu - \nabla_\nu\Theta^\mu\xi^\nu - \sqrt{-g}\xi^\nu\nabla_\nu\nabla^\mu\Phi\delta\Phi - \sqrt{-g}\nabla^\mu\Phi\nabla_\nu\delta\Phi\xi^\nu \\
&= \nabla_\nu(\xi^\nu\Theta^\mu - \xi^\mu\Theta^\nu) + \nabla_\nu\Theta^\nu\xi^\mu - \nabla_\nu\Theta^\mu\xi^\nu - \nabla_\nu(\sqrt{-g}\nabla^\mu\Phi\delta\Phi)\xi^\nu \\
&= \nabla_\nu(\xi^\nu\Theta^\mu - \xi^\mu\Theta^\nu) + \nabla_\nu\Theta^\nu\xi^\mu - \nabla_\nu\Theta^\mu\xi^\nu + \nabla_\nu\Theta^\mu\xi^\nu \\
&= \nabla_\nu(\xi^\nu\Theta^\mu - \xi^\mu\Theta^\nu) + \nabla_\nu\Theta^\nu\xi^\mu \,,
\end{aligned}$$

we have a boundary term plus an extra term that we would like to cancel with the other term.

$$\begin{aligned}
\delta(-\sqrt{-g}\nabla^\mu\Phi\xi^\nu\nabla_\nu\Phi) &= -\delta\left(\xi^\nu\sqrt{-g}\left(\nabla^\mu\Phi\nabla_\nu\Phi + \frac{1}{2}(\nabla\Phi)^2\delta^\mu_\nu - \frac{1}{2}(\nabla\Phi)^2\delta^\mu_\nu\right)\right) \\
&= -\delta\left(\xi^\nu\left(T^\mu_{\ \nu} + \frac{1}{2}\sqrt{-g}(\nabla\Phi)^2\delta^\mu_\nu\right)\right) \\
&= -\xi^\nu\delta T^\mu_{\ \nu} - \xi^\nu\delta(-L)\delta^\mu_\nu \,,
\end{aligned}$$

and we know that $\delta L = \nabla_\mu\Theta^\mu$ on-shell, which imply that

$$\delta(\sqrt{-g}\nabla^\mu\Phi\xi^\nu\nabla_\nu\Phi) = \xi^\mu\nabla_\nu\Theta^\nu \,.$$

Now we have all the ingredients to compute the surface charge

$$\begin{aligned}
i_\xi\omega &= \int (d^{D-1}x)_\mu\left[\nabla_\nu(\xi^\nu\Theta^\mu - \xi^\mu\Theta^\nu) + \nabla_\nu\Theta^\nu\xi^\mu - \xi^\mu\nabla_\nu\Theta^\nu\right] \\
&= \int (d^{D-1}x)_\mu\nabla_\nu(\xi^\nu\Theta^\mu - \xi^\mu\Theta^\nu) \\
&= \int (d^{D-2}x)_{\mu\nu}(\xi^\nu\Theta^\mu - \xi^\mu\Theta^\nu) \,,
\end{aligned}$$

here we see that the conserve charge is a surface integral of a term that we call from now on the “surface charge"

$$k^{\mu\nu} = -2\xi^{[\mu}\Theta^{\nu]} \,. \tag{A1.12}$$

A2 Surface charge of the Einstein Hilbert action

To obtain the surface charge of the gravitational sector, let us consider the Einstein-Hilbert action

$$I = \int d^3x\sqrt{-g}R\,, \tag{A2.1}$$

as in this work we focus on theories that describe asymptotically locally flat solutions, we had fixed $\Lambda = 0$.

We are already know at this point that the symplectic potential is the boundary term, but when we take the first variation of (A2.1) we get

$$\begin{aligned}\delta_g I &= \int d^Dx[\delta_g(\sqrt{-g})R + \sqrt{-g}\delta_g R] \\ &= \int d^Dx[G_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta_g(R_{\mu\nu})]\,,\end{aligned}$$

it is not clear the explicit expression of Θ^μ, but we can be sure that the symplectic potential is encoded in the last variation, and in order to see the expression of $\delta_g R_{\mu\nu}$ we first need to take $\delta_g R^\rho{}_{\sigma\mu\nu}$

$$\delta_g R^\rho{}_{\sigma\mu\nu} = \partial_\mu\delta\Gamma^\rho_{\nu\sigma} - \partial_\nu\delta\Gamma^\rho_{\mu\sigma} + \delta\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\sigma} - \delta\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\mu\sigma}\,, \tag{A2.2}$$

where something interesting happen when we study the covariant derivative of the Christoffel symbol

$$\nabla_\mu(\delta\Gamma^\rho_{\nu\sigma}) = \partial_\mu\delta\Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\sigma} - \Gamma^\lambda_{\mu\nu}\delta\Gamma^\rho_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma}\delta\Gamma^\rho_{\nu\lambda}\,, \tag{A2.3}$$

as we can see, the variation of the Riemann (A2.2) it's just the difference between two of the terms define above (A2.3)

$$\delta_g R^\rho{}_{\sigma\mu\nu} = \nabla_\mu(\delta\Gamma^\rho_{\nu\sigma}) - \nabla_\nu(\delta\Gamma^\rho_{\mu\sigma})\,. \tag{A2.4}$$

Now, to see the Ricci tensor we just contract two indices of the Riemman Tensor $\delta R_{\mu\nu} = \delta R^\rho{}_{\mu\rho\nu} = \nabla_\rho(\delta\Gamma^\rho_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\rho_{\mu\rho})$.

As we can see, there is still one step left to do. We need to express the Christoffel symbol in terms of the metric $g_{\mu\nu}$

$$\begin{aligned}
\delta\Gamma^{\rho}_{\mu\nu} &= g^{\rho\lambda}\frac{1}{2}(\partial_\mu\delta g_{\lambda\nu}+\partial_\nu\delta g_{\lambda\mu}-\partial_\lambda\delta g_{\mu\nu})+\delta g^{\rho\lambda}\frac{1}{2}(\partial_\mu g_{\lambda\nu}+\partial_\nu g_{\lambda\mu}-\partial_\lambda g_{\mu\nu})\\
&= g^{\rho\lambda}\frac{1}{2}(\partial_\mu\delta g_{\lambda\nu}+\partial_\nu\delta g_{\lambda\mu}-\partial_\lambda\delta g_{\mu\nu})+\delta^{\rho}_{\sigma}\delta g^{\sigma\lambda}\frac{1}{2}(\partial_\mu g_{\lambda\nu}+\partial_\nu g_{\lambda\mu}-\partial_\lambda g_{\mu\nu})\\
&= g^{\rho\lambda}\frac{1}{2}(\partial_\mu\delta g_{\lambda\nu}+\partial_\nu\delta g_{\lambda\mu}-\partial_\lambda\delta g_{\mu\nu})+g^{\rho\alpha}g_{\alpha\sigma}\delta g^{\sigma\lambda}\frac{1}{2}(\partial_\mu g_{\lambda\nu}+\partial_\nu g_{\lambda\mu}-\partial_\lambda g_{\mu\nu})\\
&= g^{\rho\lambda}\frac{1}{2}(\partial_\mu\delta g_{\lambda\nu}+\partial_\nu\delta g_{\lambda\mu}-\partial_\lambda\delta g_{\mu\nu})-g^{\rho\alpha}g^{\sigma\lambda}\delta g_{\alpha\sigma}\frac{1}{2}(\partial_\mu g_{\lambda\nu}+\partial_\nu g_{\lambda\mu}-\partial_\lambda g_{\mu\nu})\\
&= g^{\rho\lambda}\frac{1}{2}(\partial_\mu\delta g_{\lambda\nu}+\partial_\nu\delta g_{\lambda\mu}-\partial_\lambda\delta g_{\mu\nu})-g^{\rho\lambda}\delta g_{\sigma\lambda}\Gamma^{\sigma}_{\mu\nu}\\
&= g^{\rho\lambda}\frac{1}{2}(\partial_\mu\delta g_{\lambda\nu}+\partial_\nu\delta g_{\lambda\mu}-\partial_\lambda\delta g_{\mu\nu}-2\delta g_{\sigma\lambda}\Gamma^{\sigma}_{\mu\nu})\\
&= g^{\rho\lambda}\frac{1}{2}[\partial_\mu\delta g_{\lambda\nu}-\delta g_{\sigma\lambda}\Gamma^{\sigma}_{\mu\nu}-\delta g_{\sigma\nu}\Gamma^{\sigma}_{\mu\lambda}+\partial_\nu\delta g_{\lambda\mu}-\delta g_{\sigma\lambda}\Gamma^{\sigma}_{\mu\nu}-\delta g_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu}\\
&\qquad -(\partial_\lambda\delta g_{\mu\nu}-\delta g_{\sigma\mu}\Gamma^{\sigma}_{\nu\lambda}-\delta g_{\sigma\nu}\Gamma^{\sigma}_{\mu\lambda})]\\
&= g^{\rho\lambda}\frac{1}{2}(\nabla_\mu\delta g_{\lambda\nu}+\nabla_\nu\delta g_{\lambda\mu}-\nabla_\lambda\delta g_{\mu\nu})
\end{aligned}$$

Then finally the combination $g^{\mu\nu}(\nabla_\rho(\delta\Gamma^{\rho}_{\mu\nu})-\nabla_\nu(\delta\Gamma^{\rho}_{\mu\rho}))$ is

$$\begin{aligned}
&=\nabla_\rho\left[g^{\rho\lambda}\frac{1}{2}(\nabla^\nu\delta g_{\lambda\nu}+\nabla^\mu\delta g_{\lambda\mu}-\nabla_\lambda(g^{\mu\nu}\delta g_{\mu\nu}))\right]-\\
&\quad\nabla_\nu\left[g^{\rho\lambda}\frac{1}{2}(\nabla_\mu(g^{\mu\nu}\delta g_{\lambda\rho})+\nabla_\rho(g^{\mu\nu}\delta g_{\lambda\mu}-\nabla_\lambda(g^{\mu\nu}\delta g_{\mu\rho}))\right]\\
&=\nabla_\rho\left[g^{\rho\lambda}\frac{1}{2}(2\nabla^\nu\delta g_{\lambda\nu}-\nabla_\lambda(g^{\mu\nu}\delta g_{\mu\nu}))\right]-\\
&\quad\nabla_\nu\left[\frac{1}{2}(\nabla_\mu(g^{\rho\lambda}g^{\mu\nu}\delta g_{\lambda\rho})+\nabla^\rho(g^{\mu\nu}\delta g_{\rho\mu})-\nabla^\rho(g^{\mu\nu}\delta g_{\mu\rho}))\right]\\
\nabla_\rho(g^{\mu\nu}\delta\Gamma^{\rho}_{\mu\nu})-\nabla_\nu(g^{\mu\nu}\delta\Gamma^{\rho}_{\mu\rho}) &=\nabla_\rho\left[\nabla^\nu(g^{\rho\lambda}\delta g_{\lambda\nu})-\nabla^\rho(g^{\alpha\beta}\delta g_{\alpha\beta})\right],
\end{aligned}$$

where we have used in the last equation $\rho\leftrightarrow\nu$ conveniently. This defines the symplectic potential $\Theta(g,\delta g)$ which expressing in term of $h_{\mu\nu}=\delta g_{\mu\nu}$ is

$$\Theta^\nu=\sqrt{-g}[\nabla^\mu h^{\nu}{}_{\mu}-\nabla^\nu h] \qquad \text{(A2.5)}$$

then the symplectic form (2.2.33) applied to the Θ^ν defined above is

$$\delta_\xi[\sqrt{-g}[\nabla^\nu(g^{\rho\lambda}\delta g_{\lambda\nu})-\nabla^\rho(g^{\alpha\beta}\delta g_{\alpha\beta})]]-\delta[\sqrt{-g}[\nabla^\nu(g^{\rho\lambda}\delta_\xi g_{\lambda\nu})-\nabla^\rho(g^{\alpha\beta}\delta_\xi g_{\alpha\beta})]],$$

because of the extension of the equations, we are going to analyse each term separately,

$$
\begin{aligned}
\delta_\xi[\sqrt{-g}[\nabla^\mu {h^\nu}_\mu - \nabla^\nu h]] =&\delta_\xi(\sqrt{-g})[\nabla^\mu {h^\nu}_\mu - \nabla^\nu h] + \sqrt{-g}[\delta_\xi(\nabla_\mu h^{\mu\nu}) - \delta_\xi(\nabla^\nu h)]\\
=&\nabla_\alpha \xi^\alpha \Theta^\nu + \sqrt{-g}[\xi^\alpha \nabla_\alpha(\nabla_\mu h^{\mu\nu}) - \nabla_\mu h^{\mu\alpha}\nabla_\alpha \xi^\nu -\\
&(\xi^\alpha \nabla_\alpha \nabla^\nu h - \nabla^\alpha h \nabla_\alpha \xi^\nu)]\\
=&\nabla_\alpha(\xi^\alpha\Theta^\nu) - \xi^\alpha\nabla_\alpha\Theta^\nu + \nabla_\alpha(\sqrt{-g}\xi^\alpha\nabla_\mu h^{\mu\nu}) - \sqrt{-g}\nabla_\alpha\xi^\alpha\nabla_\mu h^{\mu\nu} -\\
&\nabla_\alpha(\sqrt{-g}\nabla_\mu h^{\mu\nu}\xi^\nu) + \sqrt{-g}\nabla_\mu\nabla_\alpha h^{\mu\alpha}\xi^\nu - \nabla_\alpha(\sqrt{-g}\xi^\alpha\nabla^\nu h) +\\
&\sqrt{-g}\nabla_\alpha\xi^\alpha\nabla^\nu h + \nabla_\alpha(\sqrt{-g}\nabla^\alpha h\xi^\nu) - \sqrt{-g}\Box h\xi^\nu\\
=&\nabla_\alpha(\xi^\alpha\Theta^\nu + \sqrt{-g}\xi^\alpha\nabla_\mu h^{\mu\nu} - \sqrt{-g}\xi^\nu\nabla_\mu h^{\mu\alpha} -\\
&\sqrt{-g}\xi^\alpha\nabla^\nu h + \sqrt{-g}\xi^\nu\nabla^\alpha h) - (\xi^\alpha\nabla_\alpha\Theta^\nu + \sqrt{-g}\nabla_\alpha\xi^\alpha\nabla_\mu h^{\mu\nu} -\\
&\sqrt{-g}\nabla_\alpha\xi^\alpha\nabla^\nu h) + \sqrt{-g}\nabla_\alpha\nabla_\mu h^{\mu\nu}\xi^\nu - \sqrt{-g}\Box h\xi^\nu\\
=&\nabla_\alpha(2\xi^\alpha\Theta^\nu - \xi^\nu\Theta^\alpha) - \nabla_\alpha(\xi^\alpha\Theta^\nu) + \xi^\nu\nabla_\alpha\Theta^\alpha\\
=&\nabla_\alpha(\xi^\alpha\Theta^\nu - \xi^\nu\Theta^\alpha) + \xi^\nu\nabla_\alpha\Theta^\alpha\\
=&\nabla_\alpha(\xi^\alpha\Theta^\nu - \xi^\nu\Theta^\alpha) + \xi^\nu\left[\delta L - \sqrt{-g}G_{\alpha\beta}h^{\alpha\beta}\right] ,
\end{aligned}
$$

which is already an antisymmetric term, plus terms that has to cancel out with the next term,

$$
\begin{aligned}
\Theta^\nu(\delta_\xi g) =&\sqrt{-g}\nabla^\mu(g^{\nu\alpha}\delta_\xi g_{\alpha\mu}) - \sqrt{-g}\nabla^\nu(g^{\mu\alpha}\delta_\xi g_{\mu\alpha})\\
=&\sqrt{-g}\nabla^\mu[g^{\nu\alpha}(\nabla_\alpha\xi_\mu + \nabla_\mu\xi_\alpha)] - \sqrt{-g}\nabla^\nu[g^{\mu\alpha}(\nabla_\mu\xi_\alpha + \nabla_\alpha\xi_\mu)]\\
=&\sqrt{-g}\nabla^\mu(\nabla^\nu\xi_\mu + \nabla_\mu\xi^\nu) - \sqrt{-g}\nabla^\nu(\nabla_\mu\xi^\mu + \nabla_\mu\xi^\mu)\\
=&\sqrt{-g}\nabla^\mu\nabla^\nu\xi_\mu + \sqrt{-g}\nabla^\mu\nabla_\mu\xi^\nu - \sqrt{-g}\nabla^\nu\nabla_\mu\xi^\mu - \sqrt{-g}\nabla^\nu\nabla_\mu\xi^\mu\\
=&\sqrt{-g}\nabla_\mu\nabla^\nu\xi^\mu + \sqrt{-g}\nabla_\mu\nabla^\mu\xi^\nu - 2\sqrt{-g}\nabla^\nu\nabla_\mu\xi^\mu\\
=&\sqrt{-g}\nabla_\mu\nabla^\mu\xi^\nu - \sqrt{-g}\nabla_\mu\nabla^\nu\xi^\mu + \sqrt{-g}\nabla_\mu\nabla^\nu\xi^\mu + \sqrt{-g}\nabla_\mu\nabla^\nu\xi^\mu - 2\sqrt{-g}\nabla^\nu\nabla_\mu\xi^\mu\\
=&2\sqrt{-g}\nabla_\mu\nabla^{[\mu}\xi^{\nu]} + 2\sqrt{-g}[\nabla_\mu, \nabla^\nu]\xi^\mu\\
=&2\sqrt{-g}\nabla_\mu\nabla^{[\mu}\xi^{\nu]} + 2\sqrt{-g}R^{\nu\mu}\xi_\mu\\
=&2\sqrt{-g}\nabla_\mu\nabla^{[\mu}\xi^{\nu]} + 2\sqrt{-g}\xi_\mu\left(G^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right)
\end{aligned}
$$

now, we have to take the variation ON-SHELL

$$
\begin{aligned}
\delta\Theta(\delta_\xi g) =&\delta(\sqrt{-g}(\nabla_\mu\nabla^\mu\xi^\nu - \nabla_\mu\nabla^\nu\xi^\mu)) + \sqrt{-g}\xi^\nu\delta L \\
=&\delta(\partial_\mu(\sqrt{-g}(\nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu))) + \sqrt{-g}\xi^\nu\delta L \\
=&\partial_\mu(\delta(\sqrt{-g}(\nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu))) + \sqrt{-g}\xi^\nu\delta L \\
=&\sqrt{-g}\nabla_\mu\left[\frac{1}{2}h\nabla^\mu\xi^\nu - h^{\mu\alpha}\nabla_\alpha\xi^\nu + g^{\mu\rho}\left(\frac{1}{2}g^{\nu\sigma}(\nabla_\rho h_{\sigma\lambda} + \nabla_\lambda h_{\sigma\rho} - \nabla_\sigma h_{\rho\lambda})\xi^\lambda\right)\right] \\
&\sqrt{-g}\nabla_\mu\left[-(\mu\leftrightarrow\nu)\right] + \sqrt{-g}\xi^\nu\delta L \\
=&\sqrt{-g}\nabla_\mu\left[\frac{1}{2}h\nabla^\mu\xi^\nu - h^{\mu\alpha}\nabla_\alpha\xi^\nu + \left(\frac{1}{2}(\nabla^\mu h^\nu{}_\lambda + \nabla_\lambda h^{\nu\mu} - \nabla^\nu h^\mu{}_\lambda)\xi^\lambda\right)\right] \\
&\sqrt{-g}\nabla_\mu\left[-(\mu\leftrightarrow\nu)\right] + \sqrt{-g}\xi^\nu\delta L \\
=&\sqrt{-g}\nabla_\mu\left[\frac{1}{2}h(\nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu) - (h^{\mu\alpha}\nabla_\alpha\xi^\nu - h^{\nu\alpha}\nabla_\alpha\xi^\mu) + \frac{1}{2}(\nabla^\mu h^\nu{}_\lambda + \nabla_\lambda h^{\nu\mu} - \nabla^\nu h^\mu{}_\lambda)\right] \\
&-\sqrt{-g}\nabla_\mu\left[\nabla^\nu h^\mu{}_\lambda - \nabla_\lambda h^{\mu\nu} + \nabla^\mu h^\nu{}_\lambda)\xi^\lambda\right] + \sqrt{-g}\xi^\nu\delta L \\
=&\sqrt{-g}\nabla_\mu\left[\frac{1}{2}h(\nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu) - (h^{\mu\alpha}\nabla_\alpha\xi^\nu - h^{\nu\alpha}\nabla_\alpha\xi^\mu) + (\nabla^\mu h^\nu{}_\lambda - \nabla^\nu h^\mu{}_\lambda)\xi^\lambda\right] \\
&+\sqrt{-g}\xi^\nu\delta L\,.
\end{aligned}
$$

Now, we can finally compute the surface charge

$$
\begin{aligned}
i_\xi\omega =& \int (d^2x)_\nu\nabla_\mu\left[\xi^\mu\Theta^\nu - \xi^\nu\Theta^\mu + \delta L - \delta(\sqrt{-g}\nabla_\mu\nabla^{[\mu}\xi^{\nu]}2) - \delta L\right] \\
=&(dx)_{\nu\mu}2\sqrt{-g}\left[\xi^{[\mu}\nabla_\alpha h^{\nu]\alpha} - \xi^{[\mu}\nabla^{\nu]}h - \frac{1}{2}h\nabla^{[\mu}\xi^{\nu]} - \xi_\alpha\nabla^{[\mu}h^{\nu]\alpha} + h^{\alpha[\mu}\nabla_\alpha\xi^{\nu]}\right]
\end{aligned}
$$

where we can identify the surface charge as the terms next to $(dx)_{\mu\nu}$

$$
k^{\mu\nu} = 2\sqrt{-g}\left[\xi^{[\mu}\nabla_\alpha h^{\nu]\alpha} - \xi^{[\mu}\nabla^{\nu]}h - \frac{1}{2}h\nabla^{[\mu}\xi^{\nu]} - \xi_\alpha\nabla^{[\mu}h^{\nu]\alpha} - h^{\alpha[\mu}\nabla_\alpha\xi^{\nu]}\right]. \tag{A2.6}
$$

This define the surface charge of the gravitational sector without cosmological constant.

A3 Generalized cocycle condition

Here we show that

$$\delta_{s_3}\mathcal{K}(s_1,s_2)+\mathcal{K}([s_1,s_2],s_3)+\mathrm{cyc}(1,2,3)=0 \tag{A3.1}$$

is satisfied for (3.4.13). To do so, we define $\mathcal{K}(s_1,s_2)\equiv\mathcal{A}_{12}+\mathcal{B}_{12}$ such that

$$\mathcal{A}_{12}=\frac{1}{8\pi G}\oint(k-l\dot{\mathcal{E}})(Y_1'''f_2-f_1Y_2''')\quad \mathcal{B}_{12}=\frac{1}{8\pi Gl}\oint\mathcal{E}'''(f_1'f_2-f_1f_2') \tag{A3.2}$$

Then

$$\delta_3\mathcal{A}_{12}+\mathrm{cyc}(1,2,3)=\frac{1}{8\pi \mathrm{G}}\oint\left[\tfrac{1}{\mathrm{l}}\left(\mathrm{f}_3''\mathcal{E}-\mathrm{f}_3'\mathcal{E}'\right)-\mathrm{l}\mathrm{Y}_3\dot{\mathcal{E}}'\right]\left(\mathrm{Y}_1'''\mathrm{f}_2-\mathrm{f}_1\mathrm{Y}_2'''\right)+\mathrm{cyc}(1,2,3) \tag{A3.3}$$

and

$$\begin{aligned}\mathcal{A}_{[1,2]3}+\mathrm{cyc}(1,2,3)&=\frac{1}{8\pi G}\oint(-l\dot{\mathcal{E}})(Y_{[1,2]}'''f_3-f_{[1,2]}Y_3''')+(1,2,3)\\&=\frac{1}{8\pi G}\oint(-l\dot{\mathcal{E}})\Big[Y_3(Y_1'''f_2-f_1Y_2''')+\frac{2}{l^2}f_3(f_1'f_2''-f_1''f_2')\Big]'+\mathrm{cyc}(1,2,3)\,,\end{aligned} \tag{A3.4}$$

thus,

$$\begin{aligned}&\delta_3\mathcal{A}_{12}+\mathcal{A}_{[1,2]3}+\mathrm{cyc}(1,2,3)\\&\quad=\frac{1}{8\pi Gl}\oint\Big[\left(f_3''\mathcal{E}-f_3'\mathcal{E}'\right)\left(Y_1'''f_2-f_1Y_2'''\right)-2\dot{\mathcal{E}}f_3(f_1'f_2'''-f_1'''f_2')\Big]+(1,2,3)\,.\end{aligned} \tag{A3.5}$$

Analogously, for $\mathcal{B}_{12}$ we have

$$\begin{aligned}\delta_3\mathcal{B}_{12}+\mathrm{cyc}(1,2,3)&=\frac{1}{8\pi Gl}\oint\Big[-2f_3\dot{\mathcal{E}}(f_1'''f_2'-f_1'f_2''')\\&\quad-(Y_3'\mathcal{E}''-Y_3'''\mathcal{E}+Y_3\mathcal{E}'''-Y_3''\mathcal{E}')(f_1''f_2-f_1f_2'')\Big]+\mathrm{cyc}(1,2,3)\end{aligned} \tag{A3.6}$$

and

$$\mathcal{B}_{[1,2]3}+\mathrm{cyc}(1,2,3)=\frac{1}{8\pi Gl}\oint\mathcal{E}'''\Big[f_3(f_1'Y_2'-Y_1'f_2'+Y_1f_2''-Y_2f_1'')\Big]+\mathrm{cyc}(1,2,3)\,. \tag{A3.7}$$

that can be used to obtain

$$\delta_3\mathcal{B}_{12}+\mathcal{B}_{[1,2]3}+\text{cyc}(1,2,3) = \frac{1}{8\pi G l}\oint\Big[-2f_3\dot{\mathcal{E}}(f_1'''f_2'-f_1'f_2''')+Y_3'''\mathcal{E}(f_1''f_2-f_1f_2'') \\ +\mathcal{E}'f_3(f_1'Y_2'''-Y_1'''f_2')\Big]+\text{cyc}(1,2,3)\,. \quad \text{(A3.8)}$$

By summing up contributions (A3.5) and (A3.8) together with their cyclic permutations, one gets the desired result (A3.1).