UNIVERSIDAD DE CONCEPCIÓN
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
MAGÍSTER EN CIENCIAS MENCIÓN FÍSICA

Holographic renormalization of scalar-tensor theories

Renormalización holográfica de teorías tenso-escalares

Profesor Guía: Dr. Rodrigo Olea UNAB
Departamento de Física
Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción

Tesis para ser presentada a la Dirección de Postgrado de la Universidad de Concepción para optar al grado de Magíster en Física

NICOLÁS ALBERTO CÁCERES DE LA FUENTE
DICIEMBRE DE 2023
CONCEPCIÓN, CHILE

Acknowledgements

I extend my heartfelt gratitude to my family, friends, and teachers, whose unwavering support and guidance made this project a reality.

We gratefully acknowledge support by Anillo Grant ANID/ACT210100 Holography and its applications to High Energy Physics, Quantum Gravity and Condensed Matter Systems.

Resumen

Inspirados por los métodos de renormalización holográfica con contratérminos topológicos en gravedad pura $AAdS4$, extendemos el análisis para el caso con matería, especifícamente, en presencia de campos escalares masivos con dimensiones conformes $\Delta = 2$ y $\Delta = 3$. Estudiamos los casos de acople minimal en la teoría Einstein Klein Gordon y los casos no minimal en la teoría Einstein-dilaton Gauss-Bonnet, finalizando el análisis con un sector de la teoría de Horndeski. A partir de lo anterior somos capaces de renormalizar estas teorías ustilizando contratérminos intrínsecos del borde. Además, exigiendo que las teorías preserven la invariancia conforme somos capaces de fijar los acoples de los casos no minimales.

Abstract

Inspired by the methods of holographic renormalization with topological counterterms in pure AdS_4 gravity, we extend our analysis to the case involving matter, particularly in the presence of massive scalar fields with conformal dimensions $\Delta = 2$ and $\Delta = 3$. We examine both the minimal coupling of the Einstein-Klein-Gordon theory and the non-minimal Einstein-Dilaton-Gauss-Bonnet theory, concluding our analysis with a sector of the Horndeski theory. Based on the foregoing, we are able to renormalize these theories using intrinsic boundary counterterms. Furthermore, by demanding that the theories preserve conformal invariance, we can determine the couplings in the non-minimal cases..

Contents

List of Figures

Chapter 1

Introduction

1.1 Introduction

The coupling of scalar fields with gravity, from the perspective of the AdS/CFT correspondence, has been fundamental in understanding a wide range of phenomena. Among the most significant is high-temperature superconductivity, where the scalar field contains crucial information about the superconducting phase Hartnoll et al. (2008). In addition to its importance in the context of AdS/CMT, it has sparked great interest in the study of entanglement entropy , the $EP = EPR$ correspondence, and the phenomenon of scalarization Caceres et al. (2017)Brown et al. (2016)Doneva and Yazadjiev (2018) .

A crucial aspect of this correspondence is having, from the bulk side, a finite on shell action that allows us to construct the associated generating functional to obtain the correlation functions of the dual theory. In this context, holographic renormalization of tensor-scalar theories plays a crucial role. This work focuses on $D = 3 + 1$ space-time dimensions. This is why considering only gravity coupled to a scalar field, the most general theory leading to second-order equations, ghost-free, and invariant under diffeomorphisms is the Horndeski theory. We will proceed to holographically renormalize this theory.

To do this, we will start with the simplest case of the minimally coupled scalar, which will provide us with general lessons about renormalization in all cases to be studied. Subsequently, we will examine the non-minimal coupling of the scalar to the Gauss-Bonnet density, and finally, we will focus on studying the

Horndeski theory. We will show that, depending on the conformal dimension of the scalar, different counterterms will be needed. Additionally, if one fixes the Gauss-Bonnet density with a particular constant, in all cases, the Einstein-Hilbert lagrangian is automatically renormalized, and all that remains is to address the renormalization of the kinetic and mass contributions of the scalar.

This Thesis is structured as follows: In 2, we provide an overview of the AdS/CFT as well as the standard and topological holographic renormalization procedures. In Chapter 3, we present the holographic renormalizations of the Einstein-Klein-Gordon, Dilaton-Gauss-Bonnet, and Horndeski theories. In 4, we present the conclusions and future work.

Chapter 2

Review

2.1 AdS/CFT correspondence

Global anti-de Sitter space is a maximally symmetric space geometry with negative constant curvature, whose conformal compactification results in a manifold with a conformal boundary. The line element of AdS_{d+1} spacetime can be expressed in Poincaré coordinates as

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = \frac{\ell^2}{z^2}\left(dz^2 + \eta_{ij} dx^i dx^j\right) , \tag{2.1.1}$$

where $x^i = (t, x_1, \ldots, x_{d-1})$ are boundary coordinates, η_{ij} represents the Minkowski metric, z is the coordinate along the additional bulk dimension, and ℓ stands for the AdS radius, which is related to the cosmological constant by $\Lambda = -d(d-1)/2\ell^2$.

In order to overcome the boundary singularity, one can employ a conformal compactification of the spacetime through a Weyl rescaling denoted by $g = z^2\tilde{g}$, where the conformal boundary is located at $z = 0$. As a result, the conformal metric g smoothly extends to the boundary and defines the boundary metric as

$$g_{(0)} = \lim_{z\to 0} z^2 \tilde{g} . \tag{2.1.2}$$

The same procedure can be extended to more general spaces Penrose and Rindler (2011). In particular, asymptotically AdS (AAdS) spacetime are defined in this way, such that the bulk metric is written as

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + \bar{g}_{ij}(z, x^i)dx^i dx^j\right) \ . \tag{2.1.3}$$

By construction, $\bar{g}_{ij}$ has a smooth limit as $z \to 0$, such that it admits the Fefferman-Graham (FG) expansion Fefferman and Graham (1985)

$$\bar{g}_{ij} = g_{(0)ij} + z g_{(1)ij} + z^2 g_{(2)ij} + \dots \ . \tag{2.1.4}$$

The coefficients $g_{(n)ij}$ with $n \neq 0$ can be obtained by solving Einstein's equations order by order. In doing so, the coefficients of the odd powers of z are set to zero. Here, for later convenience, we will choose the radial coordinate as $\rho = z^2$. Then, the metric for AAdS spaces looks like

$$ds^2 = \frac{\ell^2 d\rho^2}{4\rho^2} + \frac{\ell^2}{\rho}\bar{g}_{ij}dx^i dx^j \ , \tag{2.1.5}$$

where the asymptotic expansion of the boundary metric is given by

$$\bar{g}_{ij} = g_{(0)ij} + \rho g_{(2)ij} + \dots + \rho^{d/2}\left(g_{(d)ij} + h_{(d)ij}\log\rho\right) + \dots \ . \tag{2.1.6}$$

The coefficient $h_{(d)ij}$ only appears in even boundary dimensions and the ellipsis denotes higher powers of ρ which correspond to non-normalizable modes.

The near-boundary form of the metric realizes the asymptotically AdS condition on the Riemann tensor

$$R_{\mu\nu\lambda\sigma} = \frac{1}{\ell^2}\left(\tilde{g}_{\mu\lambda}\tilde{g}_{\nu\sigma} - \tilde{g}_{\mu\sigma}\tilde{g}_{\nu\lambda}\right) + \mathcal{O}\left(\rho\right) \ . \tag{2.1.7}$$

The FG expansion unveils the infrared divergences in the gravity action.[1] In order to remove these divergences, one can introduce counterterms, which are the result of solving the Einstein equations to determine the expansion coefficients $g_{(n)ij}$, in terms of $g_{(0)}$. Finally, one inverts the relations to obtain $g_{(n)ij}$ as intrinsic covariant quantities from the point of view of the boundary metric, such that divergences of the action are removed. This procedure is referred to as holographic renormalization and it was proposed

[1] In the AdS/CFT correspondence, the infrared behavior of the gravitational theory is mapped to the physical properties of the gauge theory in the ultraviolet Susskind and Witten (1998).

in Refs. Henningson and Skenderis (1998); de Haro et al. (2001). It is a systematic method for extracting holographic quantities such as correlation functions, Ward identities, and Weyl anomalies (see Skenderis (2002) for a review).

In a similar way that the boundary of AdS defines a conformal class of metrics, a bulk field does not induce a specific one on the boundary. Thus, when imposing boundary conditions in AdS, they have to deal with a conformal class rather than to any specific representative, key ingredient to set the boundary value problem in AdS/CFT duality. For the metric tensor, it is imperative to take a Dirichlet condition on $g_{(0)ij}$ instead of fixing the boundary metric $h_{ij} = \bar{g}_{ij}/\rho$; the latter obviously exhibits a divergent behavior. Therefore, the role of the counterterms is two-fold: (i) they ensure the finiteness of the on-shell action, and (ii) they define a well-posed variational principle for $g_{(0)ij}$.

With a renormalized action, one obtains the holographic stress tensor by performing a variation with respect to the boundary metric. The Gubser-Klebanov-Polyakov-Witten dictionary relates the low-energy limit of String Theory (semiclassical regime of supergravity on an AAdS background) to the generating functional of a gauge theory with conformal symmetry at its boundary. The duality can be stated as

$$\exp\left(iS_{\text{grav}}[\phi^I_{(0)}]\right) = \left\langle \exp\left(i\int_{\mathcal{M}} d^dx\, \phi^I_{(0)}\mathcal{O}_I\right)\right\rangle , \tag{2.1.8}$$

where S_{grav} is a gravitational action with dynamical fields living on an AAdS_{d+1} background, $\phi^I_{(0)}$ is the value of the fields at the conformal boundary, and $\mathcal{O}_I$ is the collection of gauge operators sourced by $\phi^I_{(0)}$. Then, a gravitational action on a $(d+1)$-dimensional AAdS spacetime is related to the quantum effective action of a d-dimensional CFT. Considering a scalar field in an Euclidean AdS gravity, the quantum generating functional of the dual CFT living on a background metric $g_{(0)}$ reads

$$Z_{\text{CFT}}[g_{(0)}, \phi_{(0)}] = \left\langle \exp\left[\int_{\mathcal{M}} d^dx\sqrt{g_{(0)}}\left(\frac{1}{2}g_{(0)ij}T^{ij} + \phi_{(0)}\mathcal{O}\right)\right]\right\rangle , \tag{2.1.9}$$

where T^{ij} is the CFT stress-energy tensor and $\mathcal{O}$ is a scalar operator sourced by $\phi_{(0)}$. Then, one obtains a bulk/boundary relation, that is,

$$\langle T^{ij} \rangle = \frac{2}{\sqrt{-g_{(0)}}} \frac{\delta S_{\rm ren}}{\delta g_{(0)ij}} = \lim_{\rho \to 0} \frac{2}{\sqrt{-\bar{g}}} \frac{\delta S_{\rm ren}}{\delta \bar{g}_{ij}} = \lim_{\rho \to 0} \left(\frac{1}{\rho^{d/2-1}} T^{ij}[h] \right) , \qquad (2.1.10)$$

where $T^{ij}[h]$ is the stress tensor of the renormalized action. This tensor is made of two parts: the first one is the canonical momentum which comes from the addition of the Gibbons-Hawking-York (GHY) term to the Einstein-Hilbert action Brown and York (1993). The second contribution arises from the variation of the counterterms introduced to renormalize the theory. The resulting holographic stress-energy tensor is, indeed, finite and it corresponds to a boundary operator that is dual to the bulk gravitational field. This mapping between the asymptotic behavior of a bulk field and the source of a boundary quantum operator in the dual CFT is a fundamental aspect of the holographic dictionary.

For matter fields coupled to gravity, the procedure is analog. It involves the asymptotic expansion of bulk fields near the conformal boundary. In a general setting, the 1-point function of any field appears as the undetermined, subleading term as described in Ref. Skenderis (2002). The coefficient is determined by the boundary conditions imposed on the field, allowing for a comprehensive understanding of the field's behavior and its connection to the boundary theory. Here, we focus on scalar operators that exhibit nontrivial interactions with the boundary metric.

2.2 Massive scalar field in AdS

We will begin by examining the simplest case of the massive scalar field action in an AdS_{3+1} background and develop the standard holographic renormalization procedure:

$$S_{MS} = -\frac{1}{2} \int_{\mathcal{M}} d^4x \sqrt{-g} (g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + m^2 \phi^2) \qquad (2.2.1)$$

Upon taking variations of this action with respect to the scalar field, we obtain the following expression:

$$\delta_\phi S_{MS} = -\frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}(-2(\Box\phi - m^2\phi)\delta\phi) \tag{2.2.2}$$

$$-\frac{1}{2}\int_{\partial\mathcal{M}} d^3x\sqrt{-g}2n_\mu g^{\mu\nu}\nabla_\nu\phi\delta\phi \tag{2.2.3}$$

By imposing Dirichlet boundary conditions[2] on the scalar field in the holographic sense, we specify that the boundary variations are taken with respect to the boundary value of the scalar. Consequently, the equation of motion becomes:

$$\Box\phi - m^2\phi = 0 \tag{2.2.4}$$

Our objectives now are twofold: first, to investigate the relationship between the asymptotic conditions of the scalar and its mass, and second, to determine the divergences of the on-shell action.
For the first objective, we express the equation of motion using the Fefferman-Graham form for the AdS_{3+1}[3] metric:

$$ds^2 = \frac{\ell^2}{4\rho^2}d\rho^2 + \frac{\ell^2}{\rho}\eta_{ij}dx^i dx^j \tag{2.2.5}$$

Here, $\eta = \text{diag}(-+++)$. In this frame, the equation of motion takes the form:

$$-\frac{2}{\ell^2}\rho\partial_\rho\phi + \frac{4\rho^2}{\ell^2}\partial_{\rho\rho}\phi + \frac{\rho}{\ell^2}\eta^{ij}\partial_{ij}\phi - m^2\phi = 0 \tag{2.2.6}$$

It's worth noting that this equation is invariant under translations $x^\mu \to \overline{x}^\mu = x^\mu + a^\mu$. Consequently, we can decompose the solution into radial and spatial parts, defining $\phi = \psi(x)\overline{\phi}(\rho)$. By inserting this into the equation, we can further separate it by introducing a constant k^2:

$$-\frac{2}{\overline{\phi}}\partial_\rho\overline{\phi} + \frac{4\rho}{\overline{\phi}}\partial_{\rho\rho}\overline{\phi} - \frac{m^2\ell^2}{\rho} = -\frac{1}{\psi}\eta^{ij}\partial_{ij}\psi = k^2 \tag{2.2.7}$$

[2]This is a crucial point for constructing the holographic interpretation of the theories, and it will be further discussed in the following sections.
[3]These coordinates are connected to the standard Poincaré patch coordinates with $\rho = \ell^4/r^2$

Thus, the equation for the spatial part ψ is:

$$\eta^{ij}\partial_{ij}\psi + k^2\psi = 0 \tag{2.2.8}$$

This equation admits plane wave solutions $\psi = e^{ipx}$. Plugging this into the equation yields $p^2 = k^2$. We can now superpose the solutions to obtain:

$$\phi(\rho, x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx}\overline{\phi}_p(\rho) \tag{2.2.9}$$

Substituting $p^2 = k^2$ into the equation for the radial part and omitting the subscript p from every solution, we obtain:

$$4\rho\partial_{\rho\rho}\overline{\phi} - 2\partial_\rho\overline{\phi} - (\frac{m^2\ell^2}{\rho} + p^2)\overline{\phi} = 0 \tag{2.2.10}$$

This can be rewritten as:

$$\rho^2\partial_{\rho\rho}\overline{\phi} - \frac{\rho}{2}\partial_\rho\overline{\phi} - (\frac{m^2\ell^2}{4} + \rho\frac{p^2}{4})\overline{\phi} = 0 \tag{2.2.11}$$

Making the substitution $\rho = z^2$, the equation becomes:

$$z^2\partial_{zz}\overline{\phi} - 2z\partial_z\overline{\phi} - \left(p^2z^2 + m^2\ell^2\right)\overline{\phi} = 0 \tag{2.2.12}$$

Further changing $\overline{\phi} = z^{3/2}\overline{\omega}$, we obtain:

$$z^2\partial_{zz}\overline{\omega} + z\partial_z\overline{\omega} - (p^2z^2 + \frac{9}{4} + m^2\ell^2)\overline{\omega} = 0 \tag{2.2.13}$$

If $p^2 \geq 0$, then this equation coincides with the parametric form of the modified Bessel equation of order ν[4]:

$$x^2y'' + xy' - (\alpha^2x^2 + \nu^2)y = 0 \tag{2.2.14}$$

[4]This occurs for the Euclidean and Lorentzian signature; if $p^2 \leq 0$, then these cases differ, and the solution is in terms of Bessel functions of the first and second kind.

Which has solutions in the form of modified Bessel functions of the first and second type:

$$y = c_1 I_\nu(\alpha x) + c_2 K_\nu(\alpha x) \tag{2.2.15}$$

With α, c_1, and c_2 being real constants. The modified Bessel functions are defined as:

$$I_\nu(x) = i^{-\nu} J_\nu(ix) \tag{2.2.16}$$

with:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} n! \Gamma(n+\nu+1)} x^{2n+\nu} \tag{2.2.17}$$

and:

$$K_v(x) = \frac{\pi}{2 \sin v\pi} \left[I_{-v}(x) - I_v(x) \right] \tag{2.2.18}$$

In this case, $\alpha = |p|$ and $\nu = \sqrt{9/4 + m^2\ell^2}$. So, the solution for $\overline{\omega}$ is:

$$\overline{\omega}(z, p) = A_p K_\nu(|p|z) + B_p I_\nu(|p|z) \tag{2.2.19}$$

Returning to $\overline{\phi}$:

$$\overline{\phi}(z, p) = z^{3/2}(A_p K_\nu(|p|z) + B_p I_\nu(|p|z)) \tag{2.2.20}$$

And in the original variable ρ, we obtain:

$$\overline{\phi}(\rho, p) = A_p \rho^{3/4} K_\nu(|p|\rho^{1/2}) + B_p \rho^{3/4} I_\nu(|p|\rho^{1/2}) \tag{2.2.21}$$

It's worth noting that for ν to be non-complex, we need:

$$\frac{9}{4} + m^2\ell^2 \geq 0 \tag{2.2.22}$$

This is the well-known Breitenlohner-Freedman bound. For the study of the asymptotic behavior of the solutions, we require the series expansions of the modified Bessel functions. The series expansion of the first kind is:

$$I_\nu(x) = i^{-\nu} J_\nu(ix) = i^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} n! \Gamma(n+\nu+1)} x^{2n+\nu} \tag{2.2.23}$$

And the second kind has the following series expansion:

$$K_v(x) = \frac{1}{2}\left(\Gamma(v)\left(\frac{x}{2}\right)^{-v}\left(\sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k}}{(1-v)_k k!}\right) + \Gamma(-v)\left(\frac{x}{2}\right)^{v}\left(\sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k}}{(v+1)_k k!}\right)\right) \tag{2.2.24}$$

This form only works for ν non-integer, and for the case where ν is an integer, this function has the expansion:

For ν integer and non-zero

$$\begin{aligned} K_\nu(x) = &\frac{1}{2}\left(\frac{x}{2}\right)^{-\nu} \sum_{n=0}^{\nu-1} (-1)^n \frac{\Gamma(\nu-n)}{n!}\left(\frac{x}{2}\right)^{2n} \\ &- (-1)^\nu \left(\frac{x}{2}\right)^{\nu} \sum_{n\geq 0}\left[\ln\left(\frac{x}{2}\right) - \frac{\lambda(n+1)+\lambda(\nu+n+1)}{2}\right] \frac{\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(n+1+\nu)}, \end{aligned}$$

where

$$\lambda(1) = -\gamma \quad \lambda(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \quad (n \geq 2),$$

and γ is the Euler constant. The case for $\nu = 0$ is a special case, and its expansion is:

$$K_0(x) = -\sum_{n\geq 0}\left[\ln\left(\frac{x}{2}\right) - \lambda(n+1)\right] \frac{\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(n+1)}.$$

For simplicity in this analysis, we use z that has the same asymptotic behavior as the ρ coordinate. Now we are interested in the asymptotics of the solutions near the boundary $z \to 0$ and in the center $z \to \infty$. We start the analysis near the center $z \to \infty$. In this case, the Modified Bessel functions have the following behavior:

$$I_\nu(z) \sim e^{kz}, \tag{2.2.25}$$

$$K_\nu(z) \sim e^{-kz}. \tag{2.2.26}$$

So only the $K_\nu(z)$ mode makes sense near the center. Near the boundary, this function has the following behavior:

For ν not integer, we have:

$$K_\nu(z) \sim \frac{1}{2}(\Gamma(\nu)(\frac{z}{2})^{-\nu} + \Gamma(-\nu)(\frac{z}{2})^{\nu}). \tag{2.2.27}$$

And for the case integer non-zero:

$$\begin{aligned} K_\nu(z) \sim & \frac{1}{2}(\frac{z}{2})^{-\nu}\left[\Gamma(\nu) - \Gamma(\nu-1)(\frac{z}{2})^2 + \frac{\Gamma(\nu-2)}{2}(\frac{z}{2})^4 + \dots\right] \\ & - (-1)^\nu(\frac{z}{2})^\nu\left[\ln\left(\frac{z}{2}\right)\left(\frac{1}{\Gamma(\nu+1)} + \frac{1}{\Gamma(\nu+2)}(\frac{z}{2})^2 + \frac{1}{2\Gamma(\nu+3)}(\frac{z}{2})^4 + \dots\right)\right. \\ & - \left(\frac{\lambda(1)+\lambda(\nu+1)}{2}\frac{1}{\Gamma(\nu+1)} + \frac{\lambda(2)+\lambda(\nu+2)}{2}\frac{1}{\Gamma(\nu+2)}(\frac{z}{2})^2\right. \\ & \left.\left. + \frac{\lambda(3)+\lambda(\nu+3)}{2}\frac{1}{2\Gamma(\nu+3)}(\frac{z}{2})^4\right) + \dots\right]. \end{aligned} \tag{2.2.28}$$

Finally, for $\nu = 0$, we have:

$$K_0(z) \sim -\left(\ln\left(\frac{z}{2}\right) + \gamma\right). \tag{2.2.29}$$

From this analysis, we can see the relationship between the mass of the scalar and the asymptotic conditions because they depend on ν, which, in turn, depends on the mass. The dimension and the Anti-de Sitter radius are always fixed, so the mass is the only parameter to take into account. Plugging these asymptotic solutions into the full solution ϕ, we will notice that when evaluated in the on-shell action, we are able to read the divergent terms and what kind of counterterms are needed to render the action finite.

We start this second part of the analysis with the case ν real and different from zero. In this case, the solution $\overline{\phi}$ near the boundary is:

$$\overline{\phi}(\rho, p) \sim A_p \frac{2^{\nu-1}}{|p|^{\nu}} \Gamma(\nu) \rho^{\frac{3}{4}-\frac{\nu}{2}} + A_p \frac{|p|^{\nu}}{2^{\nu+1}} \Gamma(-\nu) \rho^{\frac{3}{4}+\frac{\nu}{2}}. \tag{2.2.30}$$

In this way, the full solution near the boundary is:

$$\begin{aligned} \phi(\rho, x) &\sim \rho^{\frac{3}{4}-\frac{\nu}{2}} \int \frac{d^4 p}{(2\pi)^4} e^{ipx} A_p \frac{2^{\nu-1}}{|p|^{\nu}} \Gamma(\nu) \\ &+ \rho^{\frac{3}{4}+\frac{\nu}{2}} \int \frac{d^4 p}{(2\pi)^4} e^{ipx} A_p \frac{|p|^{\nu}}{2^{\nu+1}} \Gamma(-\nu) \\ &= \phi_{(0)}(x) \rho^{\frac{3-\Delta}{2}} + \phi_{(1)}(x) \rho^{\frac{\Delta}{2}} \end{aligned} \tag{2.2.31}$$

Where $\phi_{(0)}$ and $\phi_{(1)}$ are the Fourier terms, and Δ is the conformal dimension[5]:

$$\Delta = \frac{3}{2} + \sqrt{\frac{9}{4} + m^2 \ell^2} \tag{2.2.32}$$

This will be useful because it allows the physical interpretation of the fields $\phi_{(0)}$ and $\phi_{(1)}$ as we will see now. The AdS/CFT correspondence tells us that isometries in AdS_{d+1} are mapped to conformal isometries in the CFT_d. From this, the correspondence between the field in the bulk and the source of the CFT in the boundary has to respect conformal invariance. From the CFT side, we know that the partition function can be written as:

$$Z_{CFT}[J] = \left\langle \exp\left(\int d^d x \mathcal{O} J \right) \right\rangle \tag{2.2.33}$$

A conformal operator $\mathcal{O}(x)$ of scaling dimension Δ has to transform under a rescaling in the following way:

$$\mathcal{O}(x) \to \mathcal{O}(\lambda x) = \lambda^{-\Delta} \mathcal{O}(x) \tag{2.2.34}$$

And the volume element like $d^d(\lambda x) = \lambda^d d^d x$. The integral in the partition function has to be invariant under the scaling so:

[5] A way to derive this relation is assuming the Ansatz $\overline{\phi} \sim \rho^{\Delta}$ (this means that near the boundary there is a dominant power for ρ), putting this into the equation for $\overline{\phi}$ we find the relation $m^2 \ell^2 = \Delta(\Delta - d)$ and his roots are Δ and $d - \Delta$.

$$\int d^d x J(x)\mathcal{O}(x) = \int d^d(\lambda x) J(\lambda x)\mathcal{O}(\lambda x)$$
$$= \int d^d x \lambda^{d-\Delta} J(\lambda x)\mathcal{O}(x) \tag{2.2.35}$$

And we see from this that the source has to transform like:

$$J(x) \to \lambda^{d-\Delta} J(\lambda x) \tag{2.2.36}$$

From the near boundary solution:

$$\phi(\rho, x) \sim \phi_{(0)}(x)\rho^{\frac{3-\Delta}{2}} + \phi_{(1)}(x)\rho^{\frac{\Delta}{2}} \tag{2.2.37}$$

If we consider only the part proportional to $\phi_{(0)}$ and we make a rescaling $x \to \lambda x$, noticing that $\rho \to \lambda^2 \rho$ from dimensional analysis[6]. Then ϕ transforms in the following way:

$$\phi(\rho, x) \to \lambda^{3-\Delta}\phi_{(0)}(\lambda x)\rho^{\frac{3-\Delta}{2}} \tag{2.2.38}$$

And from this, we see that $\phi_{(0)}$ transforms exactly like the source, and we can identify them as the source of the CFT in the boundary of AdS. For the $\phi_{(1)}$ part, we see that it transforms precisely like a VEV, and for this reason, it will have information about the quantum fluctuations of the scalar. This part near the boundary is subleading.

For the $\phi_{(1)}$ part, we observe that it transforms precisely like a vacuum expectation value (VEV), and for this reason, it will carry information about the quantum fluctuations of the scalar. This part near the boundary is subleading.

Now that we know the correct behavior and form of the solution for a good interpretation in the AdS/CFT correspondence[7], let's consider the full expansion of the solution:

[6]In the literature, the Poincaré patch uses the z coordinate as the radial coordinate, the ρ coordinate is related to this with the relation $\rho = z^2$, and as z transforms like λz, then ρ transforms like ρ^2.

[7]In general, for the calculation of correlation functions, one needs to normalize the solution using a cutoff, but our interest lies in relation to the divergences of the on-shell action, so we do not follow this analysis.

$$\phi(\rho,x) = \rho^{\frac{3-\Delta}{2}}(\phi_{(0)}(x) + \rho\phi_{(2)}(x) + \rho^2\phi_{(4)}(x) + \rho^3\phi_{(6)}(x) + \dots) \tag{2.2.39}$$

$$+ \rho^{\frac{\Delta}{2}}(\phi_{(1)}(x) + \dots) \tag{2.2.40}$$

The coefficients of the expansion for the solution near the boundary will be determined algebraically from the equation of motion. And the solution near the center will be determined from the functional derivative of the action on-shell using the GKPW prescription.

We proceed to determine the coefficients near the boundary. Here, the $\phi_{(1)}$ coefficient is subleading and is not determined from the equations of motion. Plugging into the equation of motion, we obtain:

$$(\Delta(\Delta-3) - m^2\ell^2)\overline{\phi} + \rho(2(5-2\Delta)\partial_\rho\overline{\phi} + \eta^{ij}\partial_{ij}\overline{\phi} + 4\rho\partial_{\rho\rho}\overline{\phi}) = 0 \tag{2.2.41}$$

Putting $\rho = 0$, we find the relation:

$$m^2\ell^2 = (\Delta-3)\Delta \tag{2.2.42}$$

Using this and deriving with respect to ρ and putting it equal to zero, we obtain:

$$\phi_{(2)} = \frac{1}{2(2\Delta-5)}\Box\phi_{(0)} \tag{2.2.43}$$

where $\Box = \eta^{ij}\partial_{ij}$. We can continue with this process, but for now, we stop here. Now we know the coefficients of the expansion in function of the source. We use this information to read the divergences of the on-shell action and construct the counterterms that render the action finite.

Considering the action on-shell and by integrating by part and using the equation of motion, it can be written:

$$S_{MS}^{on-shell} = -\frac{1}{2}\int d^3x\sqrt{-h}n_\rho g^{\rho\rho}\phi\partial_\rho\phi \tag{2.2.44}$$

To see the divergences, we put a regulator ϵ which we remove by making it tend to zero:

$$S^{reg}_{MS} = -\int d^3x \frac{\ell^2}{\rho^{3/2}}(\frac{(3-\Delta)}{2}\phi^2_{(0)}\rho^{(3-\Delta)} + (4-\Delta)\phi_{(0)}\phi_{(2)}\rho^{(3-\Delta)+1} + \mathcal{O}(\rho^{(3-\Delta)+2})) \tag{2.2.45}$$

where we use the Fefferman-Graham frame and the asymptotic expansion of ϕ. Of course, the divergences in this will depend on the value of Δ. We now suppose that we are in the values in which the action on shell diverges. To construct the counterterms that remove these divergences, we invert the asymptotic series expansion up to the relevant order:

$$\phi_{(0)} = \overline{\phi} - \frac{\rho}{2(2\Delta-5)}\Box\overline{\phi} + \mathcal{O}(\rho^2) \tag{2.2.46}$$

$$\phi_{(2)} = \frac{1}{2(2\Delta-5)}\Box\overline{\phi} + \mathcal{O}(\rho) \tag{2.2.47}$$

Using this, we can rewrite the regularized on-shell action as a function of intrinsic quantities of the boundary:

$$S^{reg}_{MS} = -\frac{1}{\ell}\int d^3x\sqrt{-h}(\frac{(3-\Delta)}{2}\phi^2 + \frac{\rho}{2(2\Delta-5)}\phi\Box\phi + \mathcal{O}(\rho^2)) \tag{2.2.48}$$

Then the first counterterms are:

$$S^{ct}_{MS} = \frac{1}{\ell}\int d^3x\sqrt{-h}(\frac{(3-\Delta)}{2}\phi^2 + \frac{\ell^2}{2(2\Delta-5)}\phi\Box\phi) \tag{2.2.49}$$

where the box in this case is with respect to the h boundary metric. Then the renormalized action is:

$$S^{ren}_{MS} = -\frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2) \tag{2.2.50}$$

$$+\frac{1}{\ell}\int d^3x\sqrt{-h}(\frac{(3-\Delta)}{2}\phi^2 + \frac{\ell^2}{2(2\Delta-5)}\phi\Box\phi) \tag{2.2.51}$$

And using the GKPW prescription, we can obtain all the CFT observables. The crucial part of this is the counterterm proportional to ϕ^2. This term will be universal for $\Delta \neq 3$ across all the cases we will investigate.
In the following, we are interested in deformation of the CFT. So we study the pure gravity side and then the minimally coupled case, to pass then to the non-minimally coupled case.

2.3 Massive Scalar in AAdS

Now we study the massive scalar for a fixed background metric that is $AAdS_4$. This kind of metric, when no backreaction of the scalar is present, can be expressed as follows:

$$ds^2 = \frac{\ell^2}{4\rho^2}d\rho^2 + \frac{\ell^2}{\rho}\overline{g}_{ij}dx^i dx^j \tag{2.3.1}$$

where, in the dimensions we are working on, the boundary metric $\overline{g}$ admits the following asymptotic expansion:

$$\overline{g}_{ij}(\rho, x) = g_{(0)ij}(x) + \rho g_{(2)ij}(x) + \rho^{3/2} g_{(3)ij}(x) + \rho^2 g_{(4)ij}(x) + \cdots \tag{2.3.2}$$

We will proceed in the same fashion as the previous case to construct the boundary intrinsic counterterms. The equation of motion in this frame is:

$$(-2\rho + 2\frac{\rho^2}{\overline{g}}\partial_\rho \overline{g})\partial_\rho \phi + 4\rho^2 \partial_{\rho\rho}\phi + \rho\left(\frac{\overline{g}^{ij}}{2\overline{g}}\partial_i \overline{g} + \partial_i \overline{g}^{ij}\right)\partial_j \phi + \rho \overline{g}^{ij}\partial_{ij}\phi - m^2\ell^2\phi = 0 \tag{2.3.3}$$

Here, we start by supposing that the scalar admits the following solution with this precise asymptotic expansion:

$$\phi(\rho, x) = \rho^{(3-\Delta)/2}\bar{\phi}(\rho, x) \tag{2.3.4}$$

$$\bar{\phi}(\rho, x) = \phi_{(0)}(x) + \rho\phi_{(2)}(x) + \rho^2\phi_{(4)}(x) + \rho^{\frac{2\Delta-3}{2}}\phi_{(1)}(x) + \cdots \tag{2.3.5}$$

where again the subleading coefficient will not be determined by the equations of motion but, in general, could contribute to the divergences of the action on shell. We need to use the inverse of the boundary metric and the series expansion of the metric determinant:

$$\overline{g}^{ij} = g^{ij}_{(0)} - \rho g^{im}_{(0)} g^{jl}_{(0)} g_{(2)ml} + \rho^2 (g^{mi}_{(0)} g^{lr}_{(0)} g^{jp}_{(0)} g_{(2)ml} g_{(2)rp} - g^{im}_{(0)} g^{jl}_{(0)} g_{(4)ml}) + \mathcal{O}(\rho^3) \tag{2.3.6}$$

$$\overline{g} = g_{(0)}(1 + \rho g_{(2)} + \mathcal{O}(\rho^3)) \tag{2.3.7}$$

From this, we obtain the following form of the equation of motion:

$$((\Delta-3)\Delta-m^2\ell^2)-\overline{\phi}+\rho(2(5-2\Delta)\partial_\rho\overline{\phi}+g^{ij}_{(0)}\partial_{ij}\overline{\phi}+(3-\Delta)g_{(2)}\overline{\phi} \tag{2.3.8}$$

$$+(\partial_i g^{ij}_{(0)}+\frac{1}{2}g^{ij}_{(0)}g^{lm}_{(0)}\partial_i g_{(0)lm})\partial_j\overline{\phi})+\rho^2(2((2g_{(4)}-g_{(2)ij}g^{ij}_{(2)})\overline{\phi} \tag{2.3.9}$$

$$+g_{(2)}\partial_\rho\overline{\phi}+2\partial_{\rho\rho}\overline{\phi})-(\partial_i g^{ij}_{(2)}-\frac{1}{2}(g^{ij}_{(0)}g^{lm}_{(0)}\partial_i g_{(2)lm} \tag{2.3.10}$$

$$-g^{ij}_{(0)}\partial_i g_{(0)lm}g^{lm}_{(2)}-g^{ij}_{(2)}g^{lm}_{(0)}\partial_i g_{(0)lm}))\partial_j\overline{\phi}-g^{ij}_{(2)}\partial_{ij}\overline{\phi})+\mathcal{O}(\rho^3)=0 \tag{2.3.11}$$

where we do not consider terms of cubic order in the metric expansion. Using the same method as before, we find that:

$$m^2\ell^2=(\Delta-3)\Delta \tag{2.3.12}$$

$$\phi_{(2)}=\frac{1}{2(2\Delta-5)}(g^{ij}_{(0)}\partial_{ij}\phi_{(0)}+(3-\Delta)g_{(2)}\phi_{(0)}+(\partial_i g^{ij}_{(0)}+\frac{1}{2}g^{ij}_{(0)}g^{lm}_{(0)}\partial_i g_{(0)lm})\partial_j\phi_{(0)}) \tag{2.3.13}$$

We see that there is no modification to the conformal dimension mass relation, but the second coefficient in the asymptotic expansion of the scalar field is modified. For a constant boundary metric, we recover the previous case. Now we only need to use this information on the on-shell action to construct the counterterms.

Using this, we can rewrite the regularized on-shell action as a function of intrinsic quantities of the boundary:

$$S^{reg}_{MS}=-\frac{1}{\ell}\int d^3x\sqrt{-h}\left(\frac{(3-\Delta)}{2}\phi^2\right) \tag{2.3.14}$$

$$+\frac{\ell}{2(2\Delta-5)}\left(\phi\Box\phi+(3-\Delta)\partial_\rho h\phi+(\partial_i h^{ij}+\frac{1}{2}h^{ij}h^{lm}\partial_i h_{lm})\partial_j\phi\right)+\mathcal{O}(\rho^2)) \tag{2.3.15}$$

where the box in this case is with respect to the h boundary metric. The subleading coefficient of the scalar does not contribute to the divergences of the action on shell. Then the first counterterms are:

$$S_{MS}^{ct} = \frac{1}{\ell}\int d^3x\sqrt{-h}\left(\frac{(3-\Delta)}{2}\phi^2\right) \tag{2.3.16}$$

$$+\frac{\ell^2}{2(2\Delta-5)}\left(\phi\Box\phi + (3-\Delta)\partial_\rho h\phi + (\partial_i h^{ij} + \frac{1}{2}h^{ij}h^{lm}\partial_i h_{lm})\partial_j\phi\right) \tag{2.3.17}$$

Here we notice that the terms of second order are not too covariant to the boundary. When we use the dynamical part of the metric, we will be able to calculate the coefficients of their asymptotic expansion and construct truly covariant counterterms. The renormalized action is:

$$S_{MS}^{ren} = -\frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2\right) \tag{2.3.18}$$

$$+\frac{1}{\ell}\int d^3x\sqrt{-h}\left(\frac{(3-\Delta)}{2}\phi^2\right) \tag{2.3.19}$$

$$+\frac{\ell^2}{2(2\Delta-5)}\left(\phi\Box\phi + (3-\Delta)\partial_\rho h\phi + (\partial_i h^{ij} + \frac{1}{2}h^{ij}h^{lm}\partial_i h_{lm})\partial_j\phi\right) \tag{2.3.20}$$

If we take $\overline{g} = \eta$, we recover the previous case.

2.4 Einstein-Hilbert action

Now, let's turn our attention to the pure gravity case Balasubramanian and Kraus (1999)Skenderis and Solodukhin (2000). We begin by considering the Einstein-Hilbert action in the presence of a negative cosmological constant, along with the Gibbons-Hawking boundary term to establish well-posed Dirichlet boundary conditions:

$$S_{EH} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R-2\Lambda) - \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}K \tag{2.4.1}$$

Here, K represents the extrinsic curvature in a radial foliation in Gaussian normal coordinates (see A2). The variation of the action with respect to the metric is given by:

$$\delta S_{EH} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda)\delta g^{\mu\nu} \tag{2.4.2}$$

$$-\frac{1}{2\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}(K_{ij} - h_{ij}K)\delta h^{ij} \tag{2.4.3}$$

then the equation of motions are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 0 \tag{2.4.4}$$

contracting this equation we find

$$R = 4\Lambda \tag{2.4.5}$$

plugging into the equation of motion we obtain the reduced equation of motion

$$R_{\mu\nu} = g_{\mu\nu} \Lambda \tag{2.4.6}$$

Here we will interested in solutions that are asymptotically AdS and how this solutions make the action on shell diverge. For this kind of solution we use the Fefferman-Graham frame and we use the following asymptotic series expansion:

$$ds^2 = \frac{\ell^2}{4\rho^2} d\rho^2 + \frac{\ell^2}{\rho} \overline{g}_{ij} dx^i dx^j \tag{2.4.7}$$

$$\overline{g}_{ij}(\rho, x) = g_{(0)ij}(x) + \rho g_{(2)ij}(x) + \rho^{3/2} g_{(3)ij}(x) + \rho^2 g_{(4)ij}(x) + \cdots \tag{2.4.8}$$

and $\Lambda = -3/\ell^2$. Plugging it into the equations of motions we can obtain the coefficients of the asymptotic series order by order in function of the source $g_{(0)}$ and again the subleading coefficient $g_{(3)}$ will not be determined by the equation of motion and only define its trace, in this case $tr(g_{(3)}) = 0$. Using the Gauss-Codazzi relations A2 the reduce equation of motion are:

$$\rho(2\overline{g}_{im}\partial_\rho(\overline{g}^{mn}\partial_\rho\overline{g}_{nj}) + \overline{g}^{lo}\partial_\rho\overline{g}_{lo}\partial_\rho\overline{g}_{ij}) - \mathcal{R}_{ij}(\overline{g}) - \partial_\rho\overline{g}_{ij} - \overline{g}_{ij}\overline{g}^{no}\partial_\rho\overline{g}_{no} = 0 \tag{2.4.9}$$

$$2\partial_\rho(\overline{g}^{lm}\partial_\rho\overline{g}_{ml}) + \overline{g}^{lm}\partial_\rho\overline{g}_{mn}\overline{g}^{no}\partial_\rho\overline{g}_{ol} = 0 \tag{2.4.10}$$

$$\delta^{op}_{lj}\nabla_o(\overline{g}^{lm}\partial_\rho\overline{g}_{mp}) = 0 \tag{2.4.11}$$

where $\mathcal{R}(\overline{g})$ is the Ricci tensor of the boundary metric $\overline{g}$. From this we obtain

$$g_{(2)ij} = -(\mathcal{R}_{(0)ij} - \frac{1}{4} g_{(0)ij}\mathcal{R}_{(0)}) \tag{2.4.12}$$

$$= -S_{(0)ij} \tag{2.4.13}$$

where $\mathcal{R}_{(0)}$ is the Ricci tensor of the metric source $g_{(0)}$ and $S_{(0)}$ is the Schouten tensor of the metric $g_{(0)}$. Now we can start to renormalize the action on shell. Expanding the action on shell using the asymptotic expansion and imposing a regulator, we find:

$$S_{EH}^{on-shell} = -\frac{1}{\kappa}\int d^3x\sqrt{-g_{(0)}}\ell^2(\frac{2}{\rho^{3/2}} - \frac{g_{(2)}}{\rho^{1/2}} + \mathcal{O}(\rho)) \tag{2.4.14}$$

Inverting the relations in function of the boundary metric

$$g_{(0)ij} = \overline{g}_{ij} - \rho g_{(2)ij} + \mathcal{O}(\rho^2) \tag{2.4.15}$$

$$\sqrt{-g_{(0)}} = \sqrt{-\overline{g}}(1 - \frac{\rho}{2}g_{(2)} + \mathcal{O}(\rho^2)) \tag{2.4.16}$$

$$g_{(2)} = -\frac{1}{4}\mathcal{R}(\overline{g}) + \mathcal{O}(\rho) \tag{2.4.17}$$

then the reagularized action on shell is

$$S_{EH}^{on-shell} = -\frac{1}{\kappa}\int d^3x\sqrt{-h}(\frac{2}{\ell} + \frac{\ell}{2}\mathcal{R}(h) + \mathcal{O}(\rho)) \tag{2.4.18}$$

then the first counterterms are

$$S_{EH}^{ct} = \frac{1}{2\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}(\frac{4}{\ell} + \ell\mathcal{R}(h)) \tag{2.4.19}$$

this are the Balasubramanian-Krauss counterterms. Finally the renormalized action is

$$S_{EH}^{ren} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - 2\Lambda) - \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}K \tag{2.4.20}$$

$$+ \frac{1}{2\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}(\frac{4}{\ell} + \ell\mathcal{R}(h)) \tag{2.4.21}$$

2.4.1 The Gauss-Bonnet density

The past renormalization procedure is known as the standard renormalization. There is an alternative holographic renormalization, which consists of the addition of topological terms as counterterms,this method was developed by R. Olea Olea (2005)Olea (2007)Miskovic and Olea (2009). In the case of the Einstein-Hilbert action in dimension four, the term that eliminates the divergences of the action on shell is the Gauss-Bonnet term. So, we proceed to

study the origin of this term.
Topological invariants are algebraic objects associated with a manifold that don't change when one makes continuous deformations. Consider, for example, the Platonic solids. [8] Euler in 1758 published a formula that relates, for a

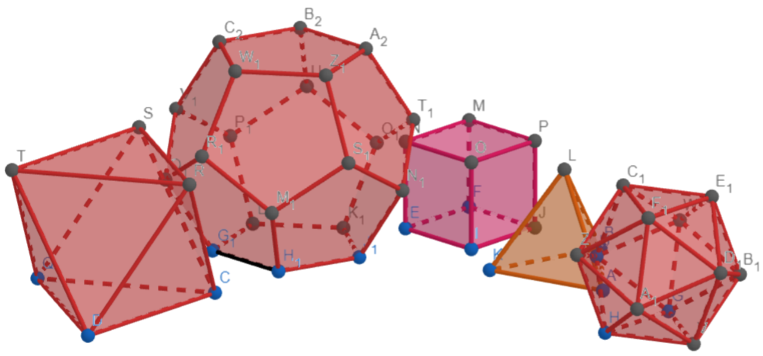

Figure 2.4.1: The five platonic solids

given Platonic solid, the number of vertices, denoted as V, with the number of edges, denoted as E, and the number of faces, denoted as F, with the following formula:

$$\chi = V - E + F = 2 \tag{2.4.22}$$

where χ is know as the Euler characteristic. The reason all Platonic solids have the same Euler characteristic is that they are all homeomorphic to each other, making the Euler characteristic a topological invariant. Then, in 1848, Pierre Ossian Bonnet published a generalization to the case of compact, closed, and oriented two-dimensional manifolds without boundaries, which relates the Gaussian curvature κ to the Euler characteristic of the manifold $\mathcal{M}$ by

$$\int_{\mathcal{M}} \kappa dA = 2\pi\chi(\mathcal{M}) \tag{2.4.23}$$

Finally, a generalization to higher dimensions was made to manifolds with boundaries by Shiing-Shen Chern in 1944. This is known as the Chern-Gauss-Bonnet theorem, and in the important case for us of dimension four, it is also

[8] This image was created with GeoGebra Software, Url: https://www.geogebra.org/

known as the Euler theorem

$$\frac{1}{4}\int_{\mathcal{M}} d^4x\sqrt{-g}\delta^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}R^{\alpha_1\alpha_2}_{\beta_1\beta_2}R^{\alpha_3\alpha_4}_{\beta_3\beta_4} = 32\pi^2\chi(\mathcal{M}) \tag{2.4.24}$$

$$+\int_{\partial\mathcal{M}} d^3x\sqrt{-h}4\delta^{i_1i_2i_3}_{j_1j_2j_3}K^{j_1}_{i_1}(\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2}i_3(h) - \frac{1}{3}K^{j_2}_{i_2}K^{j_3}_{i_3}) \tag{2.4.25}$$

The term on the left is known as the Gauss-Bonnet density and will be used in the following to renormalize the action and also to study its dynamics when coupled non-minimally with a massive scalar field.

2.4.2 Holographic renormalization with topological counterterms

Consider the Einstein-Hilbert action supplemented by the Gauss-Bonnet density

$$S_{EHGB} = S_{EHS} + S_{GB} \tag{2.4.26}$$

$$= \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - 2\Lambda) \tag{2.4.27}$$

$$+ \alpha\int_{\mathcal{M}} d^4x\sqrt{-g}(R^2 - 4R^{\alpha}_{\beta}R^{\beta}_{\alpha} + R^{\alpha\beta}_{\omega\delta}R^{\omega\delta}_{\alpha\beta}) \tag{2.4.28}$$

where α is an arbitrary coupling constant. Lets calculate the variation with respect to the metric of the Gauss-Bonnet part. First note that the Gauss-Bonnet can be writing in a convenient way by using the generalized Kronecker delta:

$$S_{GB} = \alpha\int_{\mathcal{M}} d^4x\sqrt{-g}(R^2 - 4R^{\alpha}_{\beta}R^{\beta}_{\alpha} + R^{\alpha\beta}_{\omega\delta}R^{\omega\delta}_{\alpha\beta}) \tag{2.4.29}$$

$$= \frac{\alpha}{4}\int_{\mathcal{M}} d^4x\sqrt{-g}\delta^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}R^{\alpha_1\alpha_2}_{\beta_1\beta_2}R^{\alpha_3\alpha_4}_{\beta_3\beta_4} \tag{2.4.30}$$

Varying with respect to the metric we have:

$$\delta S_{GB} = \frac{\alpha}{4}\int_{\mathcal{M}} d^4x\sqrt{-g}(2\delta^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} R^{\alpha_1}{}_{\nu\beta_1\beta_2} R^{\alpha_3\alpha_4}_{\beta_3\beta_4} + 4\delta^{\epsilon\eta\beta_3\beta_4}_{\alpha_1\nu\alpha_3\alpha_4} g_{\eta\mu}\nabla^{\alpha_1}\nabla_\epsilon R^{\alpha_3\alpha_4}_{\beta_3\beta_4} \tag{2.4.31}$$

$$-\frac{1}{2}g_{\mu\nu}\delta^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} R^{\alpha_1\alpha_2}_{\beta_1\beta_2} R^{\alpha_3\alpha_4}_{\beta_3\beta_4})\delta g^{\mu\nu} \tag{2.4.32}$$

$$+\frac{\alpha}{4}\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\epsilon(2\delta^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} g^{\alpha_2\gamma}\delta^{\epsilon\eta}_{\beta_1\beta_2}\delta\Gamma^{\alpha_1}_{\eta\gamma} R^{\alpha_3\alpha_4}_{\beta_3\beta_4} \tag{2.4.33}$$

$$+4\delta^{\omega\gamma\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} g^{\alpha_1\epsilon} g^{\alpha_2\eta}\nabla_\omega R^{\alpha_3\alpha_4}_{\beta_3\beta_4}\delta g_{\eta\gamma}) \tag{2.4.34}$$

Expanding the deltas and using the contracted Bianchi identities $\nabla_\mu R^\mu_\nu = \nabla_\nu R/2$ we find:

$$\delta^{\epsilon\eta\beta_3\beta_4}_{\alpha_1\nu\alpha_3\alpha_4} g_{\eta\mu}\nabla^{\alpha_1}\nabla_\epsilon R^{\alpha_3\alpha_4}_{\beta_3\beta_4} = 0 \tag{2.4.35}$$

$$\delta^{\omega\gamma\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} g^{\alpha_1\epsilon} g^{\alpha_2\eta}\nabla_\omega R^{\alpha_3\alpha_4}_{\beta_3\beta_4} = 0 \tag{2.4.36}$$

And from:

$$-\frac{g_{\nu\alpha}}{2}\delta^{\alpha\beta_1\beta_2\beta_3\beta_4}_{\mu\alpha_1\alpha_2\alpha_3\alpha_4} R^{\alpha_1\alpha_2}_{\beta_1\beta_2} R^{\alpha_3\alpha_4}_{\beta_2\beta_3} = 2\delta^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\mu\alpha_3\alpha_4} R^{\alpha_1}{}_{\nu\beta_1\beta_2} R^{\alpha_3\alpha_4}_{\beta_3\beta_4} - \frac{1}{2}g_{\mu\nu}\delta^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} R^{\alpha_1\alpha_2}_{\beta_1\beta_2} R^{\alpha_3\alpha_4}_{\beta_3\beta_4} \tag{2.4.37}$$

notice that the fact that in four dimensions the five index delta saturates the indices and vanishes identically. So we have no contribution to the equations of motions from the Gauss-Bonnet variation and only contribute to the boundary terms.

$$\delta S_{GB} = \frac{\alpha}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\epsilon(\delta^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} g^{\alpha_2\gamma}\delta^{\epsilon\eta}_{\beta_1\beta_2}\delta\Gamma^{\alpha_1}_{\eta\gamma} R^{\alpha_3\alpha_4}_{\beta_3\beta_4}) \tag{2.4.38}$$

Then the total variation is

$$\delta S_{EHGB} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda)\delta g^{\mu\nu} \tag{2.4.39}$$

$$+\frac{1}{8\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}n_\epsilon\delta^{\epsilon\eta\beta_1\beta_2}_{\alpha\nu\alpha_1\alpha_2} g^{\mu\nu}\delta\Gamma^\alpha_{\eta\mu}(\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2} + 8\kappa\alpha R^{\alpha_1\alpha_2}_{\beta_1\beta_2}) \tag{2.4.40}$$

then we have the same equations of motion and the same reduce equations of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 0 \tag{2.4.41}$$

$$R_{\mu\nu} = g_{\mu\nu}\Lambda \tag{2.4.42}$$

again we are interested in $AAdS$ solutions and in have a finite on shell action. Results that the constant coupling α can be fixed by demanding the finiteness of the on shell action for $AAdS$ solution or by render a stationary on shell variation. The both approaches result to be useful in the other cases that we will study. But we start with the finiteness of the on shell action method. For this we make use of the Weyl tensor, that in $d+1$ dimension is:

$$W^{\mu\nu}_{\rho\sigma} = R^{\mu\nu}_{\rho\sigma} - \delta^{\mu\nu}_{\epsilon\eta}\delta^{\epsilon\delta}_{\rho\sigma}S^{\eta}_{\delta} \tag{2.4.43}$$

With S the Schouten tensor:

$$S^{\eta}_{\delta} = \frac{1}{(d-1)}(R^{\eta}_{\delta} - \frac{1}{2d}R\delta^{\eta}_{\delta}) \tag{2.4.44}$$

in this case the on shell Weyl and Schouten tensors is

$$W^{\omega\delta}_{\alpha\beta} = R^{\omega\delta}_{\alpha\beta} + \frac{1}{\ell^2}\delta^{\omega\delta}_{\alpha\beta} \tag{2.4.45}$$

$$S^{\eta}_{\delta} = -\frac{1}{2\ell^2}\delta^{\eta}_{\delta} \tag{2.4.46}$$

then we evaluate the action on shell, that give us

$$S^{on-shell}_{EHGB} = -\int_{\mathcal{M}} d^4x\sqrt{-g}\frac{3}{\kappa\ell^2} + \alpha\int_{\mathcal{M}} d^4x\sqrt{-g}(W^{\omega\delta}_{\alpha\beta}W^{\alpha\beta}_{\omega\delta} + \frac{24}{\ell^4}) \tag{2.4.47}$$

where we made use of this relation on shell

$$R^2 - 4R_{\mu\nu}R^{\mu\nu} = 0 \tag{2.4.48}$$

$$W^{\omega\delta}_{\alpha\beta}W^{\alpha\beta}_{\omega\delta} = R^{\omega\delta}_{\alpha\beta}R^{\alpha\beta}_{\omega\delta} - \frac{24}{\ell^4} \tag{2.4.49}$$

from this we notice that if we chose

$$\alpha = \frac{\ell^2}{8\kappa} \tag{2.4.50}$$

we eliminate the divergences that comes from the Einstein-Hilbert part, and using the asymptotic series of the metric it can be proved that the remaining Weyl square part of the action is finite. So the action

$$S_{EHGB} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R-2\Lambda) \tag{2.4.51}$$
$$+\frac{\ell^2}{8\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R^2 - 4R^{\alpha}_{\beta}R^{\beta}_{\alpha} + R^{\alpha\beta}_{\omega\delta}R^{\omega\delta}_{\alpha\beta}) \tag{2.4.52}$$

is finite on shell. Using the Euler theorem for manifolds with boundary

$$\frac{1}{4}\int_{\mathcal{M}} d^4x\sqrt{-g}\delta^{\beta_1\beta_2\beta_3\beta_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}R^{\alpha_1\alpha_2}_{\beta_1\beta_2}R^{\alpha_3\alpha_4}_{\beta_3\beta_4} = 32\pi^2\chi(\mathcal{M}) \tag{2.4.53}$$
$$+\int_{\partial\mathcal{M}} d^3x\sqrt{-h}4\delta^{i_1i_2i_3}_{j_1j_2j_3}K^{j_1}_{i_1}(\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2}i_3(h) - \frac{1}{3}K^{j_2}_{i_2}K^{j_3}_{i_3}) \tag{2.4.54}$$

we can add and subtract the Gibbons-Hawking term and finally have the renormalized action for Dirichlet boundary conditions

$$S^{ren}_{EHGB} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R-2\Lambda) - \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}K \tag{2.4.55}$$
$$+\frac{\ell^2}{2\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\delta^{i_1i_2i_3}_{j_1j_2j_3}K^{j_1}_{i_1}\left(\frac{1}{2}\mathcal{R}^{j_2j_3}_{i_2i_3}(h) - \frac{1}{3}K^{j_2}_{i_2}K^{j_3}_{i_3} + \frac{1}{\ell^2}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right) \tag{2.4.56}$$

if we use the asymptotic expansion on shell for the counterterm part we notice that up to relevant order we recover the balasubramanian-Krauss counterterm. In the following we show that this method works for eliminate the divergences of the on shell action for the Einstein-Hilbert action minimally coupled to an scalar field.

2.5 Einstein Klein Gordon gravity

Let us examine now the four-dimensional Einstein-AdS gravity minimally coupled to a massive scalar field. The action of such a theory is given by

$$S_{\text{min}} = \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{R+6\ell^{-2}}{\kappa} - (\partial\phi)^2 - m^2\phi^2\right) - \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\,K\,, \tag{2.5.1}$$

where $\mathcal{M}$ is an AAdS$_4$ manifold, κ is the gravitational constant related to the Newton's constant by $\kappa = 8\pi G$, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, ℓ is the AdS radius, m is the mass parameter of the scalar field ϕ, and K is the trace of the extrinsic curvature of the boundary surface, denoted by $\partial\mathcal{M}$. An arbitrary variation of the action gives

$$\begin{aligned}\delta S_{\text{min}} = &\int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{1}{2\kappa}\left(E_{\mu\nu} - \kappa T_{\mu\nu}\right)\delta g^{\mu\nu} + \left(\Box\phi - m^2\phi\right)\delta\phi\right)\\ &-\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left(\pi_{ij}\delta h^{ij} + \pi_\phi\delta\phi\right)\,,\end{aligned} \tag{2.5.2}$$

where, in the bulk, we have defined

$$\begin{aligned}E_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\,,\\ T_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\lambda\phi\nabla_\lambda\phi + m^2\phi^2\right)\,,\end{aligned} \tag{2.5.3}$$

and the boundary variations define the canonical momenta associated with the radial evolution of the metric and the scalar field by

$$\pi_{ij} = \frac{1}{2\kappa}\left(K_{ij} - Kh_{ij}\right) \quad \text{and} \quad \pi_\phi = n^\mu\partial_\mu\phi\,, \tag{2.5.4}$$

respectively. Thus, the field equations for the metric and the scalar field can be read as

$$\begin{aligned}E_{\mu\nu} - \kappa T_{\mu\nu} &= 0\,,\\ \Box\phi - m^2\phi &= 0\,.\end{aligned} \tag{2.5.5}$$

In what follows, we shall consider the FG expansion with no logarithmic modes, which describes a wide class of gravitational setups. In that case, the scalar field can be expanded asymptotically as

$$\phi(\rho,x^i)=\rho^{(3-\Delta)/2}\bar{\phi}=\rho^{(3-\Delta)/2}\left(\phi_{(0)}+\rho\phi_{(2)}+\cdots+\rho^{(2\Delta-3)/2}\phi_{(2\Delta-3)}+\dots\right)\,, \tag{2.5.6}$$

where Δ is a constant to be determined. The holographic renormalization method for the three-dimensional version of Eq.(2.5.1) with $\Delta=2$ was studied in Ref. de Haro et al. (2001). Inserting the asymptotic expansion of the fields into the equation of motion for the metric leads to the following system of equations

$$\begin{aligned}
0 &= \mathcal{R}_{ij}(\bar{g})-\partial_\rho\bar{g}_{ij}-\bar{g}_{ij}\bar{g}^{mn}\partial_\rho\bar{g}_{mn}+\rho\left(2\bar{g}_{im}\partial_\rho\left(\bar{g}^{mn}\partial_\rho\bar{g}_{nj}\right)+\bar{g}^{mn}\partial_\rho\bar{g}_{mn}\partial_\rho\bar{g}_{ij}\right)\\
&\quad+\kappa\rho^{2-\Delta}\left(\tfrac{1}{2}m^2\ell^2\bar{\phi}^2\bar{g}_{ij}+\rho\partial_i\bar{\phi}\partial_j\bar{\phi}\right)\,,\\
0 &= \bar{g}^{mn}\partial_\rho\bar{g}_{ml}\bar{g}^{lp}\partial_\rho\bar{g}_{pn}+2\partial_\rho(\bar{g}^{mn}\partial_\rho\bar{g}_{mn})\\
&\quad+\kappa\rho^{1-\Delta}\left(\left(\tfrac{1}{2}m^2\ell^2+(3-\Delta)\right)\bar{\phi}^2+4\left((3-\Delta)\rho+\rho^2\right)\bar{\phi}\partial_\rho\bar{\phi}\right)\\
0 &= \nabla_i\left(\bar{g}^{jm}\partial_\rho\bar{g}_{jm}\right)-\nabla_j\left(\bar{g}^{jm}\partial_\rho\bar{g}_{mi}\right)+\kappa\rho^{2-\Delta}\left((3-\Delta)\bar{\phi}\partial_i\bar{\phi}+2\rho\partial_\rho\bar{\phi}\partial_i\bar{\phi}\right)\,,
\end{aligned} \tag{2.5.7}$$

where $\mathcal{R}_{ij}(\bar{g})$ is the Ricci tensor of the metric $\bar{g}$. The equation of motion for the scalar field, in turn, can be expressed as

$$\begin{aligned}
0 = &\left(\Delta(\Delta-3)-m^2\ell^2\right)\bar{\phi}+\rho\left(\bar{g}_{ij}\partial_i\partial_j\bar{\phi}-2(5-2\Delta)\partial_\rho\bar{\phi}+(3-\Delta)\partial_\rho(\log\bar{g})\bar{\phi}\right)\\
&+\rho^2\left(2\partial_\rho(\log\bar{g})\partial_\rho\bar{\phi}+\partial_\rho^2\bar{\phi}\right)\,.
\end{aligned} \tag{2.5.8}$$

At zero-th order in the holographic coordinate, this equation gives rise to a relation between the mass and Δ given by

$$m^2\ell^2=\Delta(\Delta-3)\,. \tag{2.5.9}$$

As shown in Refs. Gubser et al. (1998); Witten (1998), this relation corresponds to the one between the mass of a scalar field on an AdS background and the conformal dimension of the dual operator at the boundary. The latter is determined by the rescaling properties of a scalar operator in the CFT and it can be obtained by analyzing the 1-point function of the holographic operator. It is worth noticing that, in a unitary dual theory, the scaling dimension of a scalar operator must be a positive integer, which defines constraints on the allowed values of the mass of the corresponding bulk field. These constraints, combined with the Breitenlohner-Freedman (BF) bound Breitenlohner and Freedman (1982); Klebanov and Witten (1999), further restrict the mass to lie within a given interval, that is,

$$-\left(\frac{d}{2}\right)^2 < m^2\ell^2 < -\left(\frac{d}{2}\right)^2 + 1 \ . \tag{2.5.10}$$

This bound serves to understand the interplay between bulk and boundary physics in AdS/CFT correspondence, matching the stability of the bulk scalar with the unitarity of the dual theory.

In order to determine the coefficients in the expansion (2.5.6), one needs to solve the field equations (2.5.8) order by order in the holographic coordinate. This process, however, cannot be carried out for a generic Δ and it needs to be addressed case by case, as discussed in Ref. de Haro et al. (2001). Looking at the scalar field expansion in Eq. (2.5.8), at first-order in the holographic coordinate, there appears a second-order differential equation for the scalar field which also involves first derivatives of the boundary metric. In turn, in the equations of motion for the metric, the lowest-order term in ρ is proportional to $\rho^{2-\Delta}$. To ensure a consistent asymptotic expansion, one should demand that $\Delta \leq 3$. For instance, when $\Delta = 2$, a self-interaction emerges between the boundary fields at the leading order, producing a backreaction on the geometry of the dual CFT; this represents a critical value of the conformal dimension. Alternatively, for $\Delta = 1$, the kinetic term backreacts on the boundary only at the next-to-leading order. Similarly, when $\Delta = 0$, the bulk scalar field becomes massless and the backreaction mainly comes from the kinetic term of the boundary scalar. These different choices of Δ lead to distinct behaviors and interactions between the bulk and boundary fields, offering valuable insights into the AdS/CFT correspondence.

Let us focus on the case $\Delta = 2$ or, equivalently, $m^2\ell^2 = -2$. This value is, indeed, admissible by the BF bound. Then, we proceed to solve the Einstein equations to determine the first coefficient in the metric expansion in terms of the sources. The coefficient reads

$$g_{(2)ij} = -\mathcal{S}_{ij}(g_{(0)}) - \frac{\kappa}{4}\phi_{(0)}^2 g_{(0)ij} \,, \tag{2.5.11}$$

where

$$S_{ij}(g_{(0)}) := \mathcal{R}_{ij}(g_{(0)}) - \frac{1}{4} g_{(0)ij}\mathcal{R}(g_{(0)}) \,, \tag{2.5.12}$$

is the Schouten tensor of the boundary metric $g_{(0)}$. An arbitrary variation of the on-shell action and using the asymptotic expansion yields

$$\begin{aligned}\delta S_{\text{min}} = \frac{\ell^2}{2\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-g_{(0)}}\rho^{-1/2}\Bigg[&\left(-\frac{2}{\rho}g_{(0)ij} - 3g_{(2)ij} + \mathcal{O}\left(\rho^{1/2}\right)\right)\delta g_{(0)}^{ij} \\ &- \ell^2\left(\phi_{(0)} + \mathcal{O}\left(\rho^{1/2}\right)\right)\delta\phi_{(0)}\Bigg] \,. \end{aligned} \tag{2.5.13}$$

Therefore, in order to preserve a well-posed variational principle, one needs to add only intrinsic boundary terms. Indeed, inverting the series, one finds that the surface terms needed are

$$S_{\text{ct}} + S'_\phi = \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left(\frac{2}{\ell} + \frac{\ell}{2}\mathcal{R}(h)\right) + \frac{1}{\kappa}\int_{\mathcal{M}} d^3x\sqrt{-h}\left(\frac{\phi^2}{2\ell}\right), \tag{2.5.14}$$

which renormalize the gravity sector Balasubramanian and Kraus (1999); de Haro et al. (2001) and the counterterm for a massive scalar field on AdS which cures the divergences coming from the scalar sector Klebanov and Witten (1999).[9] Then, the action

$$S_{\text{min}}^{\text{ren}} = S_{\text{min}} + S_{\text{ct}} + S'_\phi \,, \tag{2.5.15}$$

is finite on shell. Nonetheless, it is possible to add an extra term

$$S'_{\partial\phi} = \frac{\gamma}{2\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\;\phi\, n^\mu\partial_\mu\phi \,, \tag{2.5.16}$$

where n^μ is the outward-pointing unit normal to the boundary. Its coupling γ, by the use of a Legendre transformation, redefines the mass of the scalar field.

[9] The covariant counterterms for the massive scalar including logarithmic modes have been found explicitly in Ref. de Haro et al. (2001), up to second order.

This extra boundary term has been considered in Refs. Lü et al. (2013); Lu et al. (2015) to obtain the correct thermodynamics for a given γ which matches the ADM mass. At first glance, this term may be at odds with a variational principle based on mixed boundary conditions (see Appendix A3). However, within a holographic framework, what is relevant is that the variation of the action is finite and written down in terms of the variation of the holographic sources [10] That is the reason why, one can replace this extrinsic term by another which depends explicitly on the sources and the boundary conditions Anabalon et al. (2016). Then, the counterterm for the scalar field is given by

$$S_{\phi} = \int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left(\frac{\phi^2}{2\ell} + \frac{W\left(\phi_{(0)}\right)}{\ell\phi_{(0)}^3}\phi^3\right) , \tag{2.5.17}$$

where $W\left(\phi_{(0)}\right)$ is determined by the boundary conditions imposed (see Appendix A3). Then, the Euclidean renormalized action for the minimally coupled scalar with $\Delta = 2$ reads

$$\begin{aligned} S_{\text{min}}^{\text{ren}} &= S_{\text{min}} + S_{\text{ct}} + S_{\phi} \\ &= -\frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{g}\left(\frac{R+6\ell^{-2}}{\kappa} - (\partial\phi)^2 - m^2\phi^2\right) \\ &\quad + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{h}\left[K - \frac{2}{\ell} - \frac{\ell}{2}\mathcal{R}(h) - \kappa\left(\frac{\phi^2}{2\ell} + \frac{W\left(\phi_{(0)}\right)}{\ell\phi_{(0)}^3}\phi^3\right)\right] . \end{aligned} \tag{2.5.18}$$

Additionally, the quasi-local stress-energy tensor is given by

$$T_{ij}[h] = -\frac{1}{\kappa}\left[K_{ij} - Kh_{ij} + \frac{2}{\ell}h_{ij} - \ell\left(\mathcal{R}_{ij} - \frac{1}{2}\mathcal{R}h_{ij}\right)\right] - h_{ij}\left(\frac{\phi^2}{2\ell} + \frac{W\left(\phi_{(0)}\right)}{\ell\phi_{(0)}^3}\phi^3\right) , \tag{2.5.19}$$

and it provides a regular holographic stress tensor through Eq. (2.1.10). In the next section, we follow to same procedure to find the covariant counterterms in theories of gravity with a non-minimally coupled scalar field.

[10] A similar discussion, for the metric field, leads to extrinsic counterterms in AdS gravity **?**.

Chapter 3

Case analysis

3.1 Scalar-Gauss-Bonnet gravity

Let us consider scalar-GB gravity (sGB). This theory represents a particular sector of Horndeski gravity and it involves the coupling of the GB invariant with an arbitrary smooth function of the scalar field, building on top of the action in Eq. (2.5.1) while omitting the GHY term. Specifically, the action for sGB gravity is

$$S_{\rm sGB} = \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{R+6\ell^{-2}}{\kappa} - (\partial\phi)^2 - m^2\phi^2 + 2f(\phi)\mathcal{G}\right) , \tag{3.1.1}$$

where the GB term is given by

$$\mathcal{G} \equiv R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 . \tag{3.1.2}$$

The field equations of the theory from arbitrary variations with respect to the metric and the scalar field, which yield

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} &= \kappa\left(T_{\mu\nu} + C_{\mu\nu}\right) , \\ \Box\phi - m^2\phi &= -f'(\phi)\left(R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) , \end{aligned} \tag{3.1.3}$$

respectively, where $T_{\mu\nu}$ has been defined in Eq. (2.5.3) and

$$C^{\mu}_{\nu} = -2\delta^{\mu\alpha\beta\gamma}_{\nu\lambda\sigma\tau}R^{\sigma\tau}_{\beta\gamma}\nabla^{\lambda}\nabla_{\alpha}f(\phi) . \tag{3.1.4}$$

Let us consider a power-series expansion of the scalar coupling function

$$f(\phi) = \sum_{n=0}^{\infty} f_{(n)} \phi^n \, . \tag{3.1.5}$$

Using the same asymptotic expansion as in the minimally coupled scalar [cf. Eq. (2.5.6)], the scalar equation becomes

$$24 f_{(1)} \rho^{(\Delta-3)/2} + \left(48 f_{(2)} + \ell^2 \left(\Delta(\Delta-3) - \ell^2 m^2\right)\right) \bar{\phi} + \mathcal{O}\left(\rho^{(\Delta+1)/2}\right) = 0 \, . \tag{3.1.6}$$

The explicit form of the field equations in this frame turn rather lengthy and we shall not present them here. However, it is worth noticing that the leading-order analysis indicates that $\Delta > 1$ is necessary in order to have a nontrivial scalar source. Additionally, if $\Delta > 3$, there is no backreaction of the scalar field on the boundary metric, giving a trivial scalar source. Consequently, similar to the minimally coupled case, we select $\Delta = 2$ for our analysis. This choice leads to more interesting dynamics, including the interaction between the scalar field and the boundary metric. Notice that, for this value of Δ, the scalar field gives a finite contribution to the on-shell action already at the cubic order. In this case, the relation between Δ and the mass of the scalar field becomes

$$m^2 \ell^2 = \Delta(\Delta-3) + \frac{48}{\ell^2} f_{(2)} = -2 + \frac{48}{\ell^2} f_{(2)} \, . \tag{3.1.7}$$

Then, in order to have a well-defined unitary quantum field theory at the boundary, the BF bound now reads

$$-\left(\frac{d}{2}\right)^2 < \Delta\left(\Delta-3\right) + \frac{48}{\ell^2} f_{(2)} < -\left(\frac{d}{2}\right)^2 + 1 \, . \tag{3.1.8}$$

For $\Delta = 2$ and $D = d + 1 = 4$, this bound is translated into a constraint on the quadratic coupling of the scalar field to the GB, that is,

$$-\frac{\ell^2}{192} < f_{(2)} < \frac{\ell^2}{64} \, . \tag{3.1.9}$$

The asymptotic analysis of Einstein equations requires that linear order in the asymptotic expansion of the scalar vanishes, i.e., $f_{(1)} = 0$. Furthermore, the consistency of higher-order terms in the expansion requires that either $f_{(3)} = 0$ or that the boundary value of the scalar field satisfies $\phi_{(0)}^2 = 0$. In our analysis, we will assume the former. This choice simplifies the equations and allows us to focus on the relevant aspects of the theory. It is worth noticing that $\mathcal{O}(\phi^4)$ contributions in the function $f(\phi)$ are finite when considering the on-shell action. Therefore, they do not play a role in the discussion.

Using the asymptotic expansion to solve the Einstein field equations order by order one finds that

$$\begin{aligned}
g_{(2)ij} &= -\mathcal{S}\left(g_{(0)}\right) - \frac{\kappa}{4}\left(1 - \frac{48}{\ell^2}f_{(2)}\right)\phi_{(0)}^2 g_{(0)ij}\,, \\
\nabla^i g_{(3)ij} &= \frac{2\kappa}{3\ell^2}\left\{\left[32 f_{(2)}\phi_{(1)} - 8\kappa f_{(2)}\left(1 - \frac{48}{\ell^2}f_{(2)}\right)\phi_{(0)}^2\right]\partial_j\phi_{(0)} - \left(\ell^2 - 32 f_{(2)}\right)\phi_{(0)}\partial_j\phi_{(1)}\right\}\,, \\
\mathrm{Tr}\, g_{(3)} &= \frac{4\kappa}{3}\left(1 - \frac{48}{\ell^2}f_{(2)}\right)\left(\frac{4\kappa}{3\ell^2}f_{(2)}\phi_{(0)}^2 - \phi_{(1)}\right)\phi_{(0)}\,,
\end{aligned} \tag{3.1.10}$$

which recovers the coefficient of the minimally coupled scalar theory if $f_{(2)} = 0$. An arbitrary on-shell variation of the action alongside the asymptotic expansion of the fields yield

$$\begin{aligned}
\delta S_{\mathrm{sGB}} = \frac{1}{2\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{-\bar{g}}\frac{1}{\rho^{3/2}}\Big[&\left(\ell^2 - 8\kappa f_{(0)}\right)\bar{g}^{ij}\delta\bar{g}_{ij} + \rho\Big\{\left(16\kappa f_{(0)} - 2\ell^2\right)\bar{g}^{ij}\delta\left(\partial_\rho\bar{g}_{ij}\right) \\
&- 2\ell^2\kappa\bar{\phi}\delta\bar{\phi} - \left(8\kappa f_{(0)}\left(\mathcal{R}(\bar{g})^{ij} - \frac{1}{2}\mathcal{R}(\bar{g})\bar{g}^{ij} + \bar{g}^{ij}\bar{g}^{mn}\partial_\rho\bar{g}_{mn}\right) + 24 f_{(2)}\kappa\bar{\phi}^2\bar{g}^{ij}\right. \\
&\left. + \ell^2\partial_\rho\bar{g}^{ij}\right)\delta\bar{g}_{ij}\Big\} + \mathcal{O}\left(\rho^2\right)\Big]\,.
\end{aligned} \tag{3.1.11}$$

If the zeroth-order coefficient of the scalar function expansion is chosen as

$$f_{(0)} = \frac{\ell^2}{8\kappa}\,, \tag{3.1.12}$$

then, the Einstein-Hilbert sector becomes finite and a well-defined variational principle is achieved in terms of the sources without the need of a GHY term. This coupling coincides with the one obtained in Ref. Olea (2005) for pure Einstein-AdS gravity. In four dimensions, the GB term is purely topological. This means that adding it to gravity action does not introduce modifications to the bulk dynamics, even though it changes the value of the on-shell action and conserved charges in a nontrivial way. Moreover, the Einstein-Hilbert action coupled to the GB term with the coupling (3.1.12) on-shell is a sector of conformal gravity as shown in Ref. Miskovic and Olea (2009). The possibility to embed Einstein gravity in conformal gravity has shown to be useful in renormalizing gravity coupled to conformally coupled scalar fields Anastasiou et al. (2023).

The inclusion of the GB invariant together with the counterterm in Eq. (2.5.17) is sufficient to have a renormalized on-shell action. To see this, let us first define

$$\overline{T}_{\mu\nu} := \left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T_{\mu\nu}\right) + \left(C_{\mu\nu} - \frac{1}{2}g_{\mu\nu}C\right) . \tag{3.1.13}$$

This structure appears in the on-shell value of the Weyl tensor and the GB density. Using the on-shell relation

$$W^{\mu\nu}_{\alpha\beta}W^{\alpha\beta}_{\mu\nu} = R^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}_{\mu\nu} - \frac{24}{\ell^2} + \frac{4\kappa}{\ell^2}\overline{T} + \kappa^2\left(\delta^{\mu\nu}_{\alpha\beta}\delta^{\alpha\lambda}_{\mu\eta}\overline{T}^{\beta}_{\lambda}\overline{T}^{\eta}_{\nu} - 4\overline{T}^{\mu}_{\nu}\overline{T}^{\nu}_{\mu} - \frac{2}{3}\overline{T}^2\right) , \tag{3.1.14}$$

we can write the GB density in terms of the square of the Weyl tensor. Then, the on-shell action can be written as

$$\begin{aligned} S_{\text{sGB}} = \int_{\mathcal{M}} d^4x\sqrt{-g}\Bigg[&-\frac{3}{\kappa\ell^2} + \frac{1}{2}\overline{T} + \left(f(\phi) - \frac{1}{2}\phi f'(\phi)\right)\Bigg(W^{\mu\nu}_{\alpha\beta}W^{\alpha\beta}_{\mu\nu} + \frac{24}{\ell^2} \\ &- \frac{4\kappa}{\ell^2}\overline{T} + \kappa^2\left(\frac{5}{3}\overline{T}^2 - \delta^{\mu\nu}_{\alpha\beta}\delta^{\alpha\lambda}_{\mu\eta}\overline{T}^{\beta}_{\lambda}\overline{T}^{\eta}_{\nu}\right)\Bigg)\Bigg] - \frac{1}{2}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\, n^\mu\phi\partial_\mu\phi \, . \end{aligned} \tag{3.1.15}$$

Notice that

$$f(\phi) - \frac{1}{2}\phi f'(\phi) = f_{(0)} - \frac{3}{2}f_{(3)}\phi^3 - 2f_{(4)}\phi^4 + \cdots = \sum_{n=0}^{\infty}\left(1 - \frac{n}{2}\right) f_{(n)}\phi^n , \tag{3.1.16}$$

does not contain quadratic terms. Then, the scalar-GB coupling cannot be used to remove quadratic self-interaction terms.

Since the Weyl square term is finite for AAdS spacetimes, we can choose the value of Eq. (3.1.12) such that the first two bulk terms cancel. Then, we are left only with quadratic terms in $\overline{T}$. These terms remain finite if the action contains quadratic kinetic terms and/or quadratic self-interacting potentials as in the case in sGB gravity. Hence, the only divergences that cannot be eliminated by the GB density are those associated with the minimally coupled scalar field, that can be renormalized using the counterterm in Eq. (2.5.17). As a result, the renormalized action can be expressed as

$$S_{\rm sGB}^{\rm ren} = S_{\rm sGB} + S_{\phi} \, , \tag{3.1.17}$$

where $f(\phi) = \frac{\ell^2}{8\kappa} + f_{(2)}\phi^2$, even though higher-order terms could be considered as they give finite contributions.

The holographic 1-point for the scalar field depends on the boundary conditions and is controlled $W\left(\phi_{(0)}\right)$. For the holographic stress-energy tensor we obtain

$$\begin{aligned}\langle T_{ij}\rangle =& -\ell^2\left[W\left(\phi_{(0)}\right) - \left(1-\frac{32}{\ell^2}f_{(2)}\right)\phi_{(0)}\phi_{(1)} + \frac{8\kappa}{\ell^2}f_{(2)}\left(1-\frac{48}{\ell^2}f_{(2)}\right)\phi_{(0)}^3\right]g_{(0)ij} \\ &+ \frac{3\ell^2}{2\kappa}g_{(3)ij} \, ,\end{aligned} \tag{3.1.18}$$

whose trace yields

$$\langle T\rangle = -3\ell^2\left[W\left(\phi_{(0)}\right) - \frac{1}{3}\phi_{(0)}\phi_{(1)} - \frac{16\kappa}{3\ell^2}f_{(2)}\left(1-\frac{48}{\ell^2}f_{(2)}\right)\phi_{(0)}^3\right] \, . \tag{3.1.19}$$

Notice that if we consider mixed boundary conditions that respect conformal invariance, i.e., $W\left(\phi_{(0)}\right) = C\phi_{(0)}^3$, with C some constant (see Ref. Henneaux et al. (2007)), one obtains

$$\langle T\rangle = -16\kappa f_{(2)}\left(1-\frac{48}{\ell^2}f_{(2)}\right)\phi_{(0)}^3 \, . \tag{3.1.20}$$

Focusing on the nontrivial quadratic self-interaction, that is $f_{(2)} \neq 0$, we find that conformal invariance of the boundary CFT implies

$$f_{(2)} = \frac{\ell^2}{48} \, , \tag{3.1.21}$$

which is not admissible within the BF bound. Therefore, the theory that consists solely of a scalar field with $\Delta = 2$ and a quadratic self-interacting potential coupled to the GB term does not satisfy the BF bound. Consequently, it leads to a non-unitary dual CFT.

The above results indicate that the addition of the GB term in the bulk is useful to renormalize the bulk theory, in a similar fashion as in the pure AdS gravity case. This invariant, when expressed as a boundary term, can be thought of as an extrinsic counterterm series. It is noteworthy that the boundary contribution of the GB term in four dimensions cannot be seen as a quasilocal stress tensor. However, it does contribute to the holographic stress tensor and renders the variation of the action finite.

If we consider Dirichlet boundary conditions, denoted as $W\left(\phi_{(0)}\right)=0$, the variation of the renormalized action introduces an additional piece arising from the boundary value of the scalar field. As a result, the vacuum expectation value of the boundary scalar operator can be expressed as

$$\langle\mathcal{O}\rangle=\frac{1}{\sqrt{-g_{(0)}}}\frac{\delta S_{\mathrm{H}}^{\mathrm{ren}}}{\delta\phi_{(0)}}=\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{\Delta-d}}\frac{1}{\sqrt{-h}}\frac{\delta S_{\mathrm{H}}^{\mathrm{ren}}}{\delta\phi}\right)=-\phi_{(1)}=-(2\Delta-d)\phi_{(2\Delta-d)}\,, \tag{3.1.22}$$

which matches the holographic 1-point function of the scalar operator dual to a scalar field on an AdS background with Dirichlet boundary conditions.

If we take instead $f_{(3)}\neq 0$ and $\phi_{(0)}^2=0$, we found that asymptotic analysis of the field equations gives

$$\begin{aligned} g_{(2)} &= -\mathcal{S}\left(g_{(0)}\right)_{ij}\,, \\ \nabla^i_{(0)}g_{(3)ij} &= \frac{2\kappa}{3\ell^2}\left[32f_{(2)}\phi_{(1)}\partial_j\phi_{(0)}-\left(\ell^2-32f_{(2)}\right)\phi_{(0)}\partial_j\phi_{(1)}\right]\,, \\ \mathrm{Tr}\; g_{(3)} &= -\frac{4\kappa}{3\ell^2}\left(\ell^2-48f_{(2)}\right)\phi_{(0)}\phi_{(1)}\,. \end{aligned} \tag{3.1.23}$$

Using the same counterterm as in the previous case, given in Eq. (2.5.17), we find that

$$\langle T_{ij}\rangle=\left(\ell^2-32f_{(2)}\right)\phi_{(0)}\phi_{(1)}g_{(0)ij}+\frac{3\kappa}{2\ell^2}g_{(3)ij}\,, \tag{3.1.24}$$

which, independent of the boundary conditions, is traceless. Moreover, considering Dirichlet boundary conditions one gets

$$\langle \mathcal{O} \rangle = -\phi_{(1)} = -(2\Delta - d)\phi_{(2\Delta-d)} \,, \tag{3.1.25}$$

just as in the previous scenario.

Another interesting scenario to explore involves setting $\Delta = 3$. In this case, the scalar field behaves as

$$\phi(\rho, x^i) = \phi_{(0)}(x^i) + \rho\phi_{(2)}(x^i) + \rho^{3/2}\phi_{(3)}(x^i) + \dots \,, \tag{3.1.26}$$

near the boundary. The relation between the mass and Δ becomes

$$m^2\ell^2 = \frac{48}{\ell^2} f_{(2)} \,. \tag{3.1.27}$$

Solving the field equations order by order, we can derive several conditions on the scalar couplings $f_{(n)}$. First, the zeroth-order equations yield $f_{(1)} = 0 = f_{(3)}$, or alternatively, $f_{(1)} = 0$ together with $\phi^2_{(0)} = 0$. Assuming the former condition, the Einstein equations impose $f_{(2)} = 0$. Then, we obtain

$$\begin{aligned} g_{(2)ij} =& -\mathcal{S}(g_{(0)})_{ij} - \frac{\kappa}{4} g_{(0)ij}\partial^m\phi_{(0)}\partial_m\phi_{(0)} + \kappa\partial_i\phi_{(0)}\partial_j\phi_{(0)} \,, \\ \phi_{(2)} =& \frac{1}{2}\Box_{(0)}\phi_{(0)} \,, \\ \text{Tr}\, g_{(3)} =& \; 0 \,, \\ \nabla^i_{(0)} g_{(3)ij} =& \frac{2\kappa}{\ell^2}\phi_{(1)}\partial_j\phi_{(0)} \,. \end{aligned} \tag{3.1.28}$$

Arbitrary variations of the on-shell action make evident that selecting the zeroth-order coupling in Eq. (3.1.12) eliminates the leading-order divergences. However, to address the remaining divergences associated with the scalar field, we need to introduce an appropriate counterterm. We have determined that including the intrinsic counterterm

$$S^{\text{ct}}_{\text{sGB}} = -\frac{\ell}{2}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}h^{ij}\partial_i\phi\partial_j\phi \,, \tag{3.1.29}$$

together with the choice in Eq. (3.1.12), renders the renormalized action

$$S^{\text{ren}}_{\text{sGB}} = S_{\text{sGB}} + S^{\text{ct}}_{\text{sGB}} \,, \tag{3.1.30}$$

finite on shell. This approach allows us to handle and regularize the divergences encountered in the theory. The resulting holographic stress tensor is given by

$$\langle T^{ij} \rangle = \frac{3\ell^2}{2\kappa} g^{ij}_{(3)} \,. \tag{3.1.31}$$

Remarkably, this holographic stress-energy tensor is traceless as it can be seen from Eq. (3.1.28). Additionally, we find a non-zero vacuum expectation value for the boundary scalar operator when we consider Dirichlet boundary conditions

$$\langle \mathcal{O} \rangle = -3\phi_{(3)} = -(2\Delta - d)\phi_{(2\Delta-d)} \,. \tag{3.1.32}$$

This expectation value is akin to the behavior observed for the scalar field in AdS. These results provide insights into the behavior of bulk fields and their dual operators in the context of the AdS/CFT correspondence.

Finally, considering $\phi^2_{(0)} = 0$ while keeping $f_{(3)}$ unconstrained, the holographic data now reads

$$\begin{aligned}
g_{(2)ij} &= -\mathcal{S}(g_{(0)})_{ij} - \frac{\kappa}{4} g_{(0)ij}\partial_m\phi_{(0)}\partial^m\phi_{(0)} \,, \\
\phi_{(2)} &= \frac{1}{2}\left(1 - \frac{72}{\ell^2} f_{(3)}\phi_{(0)}\right)^{-1} \Box_{(0)}\phi_{(0)} \,, \\
\text{Tr g}_{(3)} &= \frac{64\kappa}{\ell^2} f_{(2)}\phi_{(0)}\phi_{(3)} \,, \\
\nabla^i_{(0)} g_{(3)ij} &= \frac{2\kappa}{3\ell^2}\left[\left(3 + \frac{32}{\ell^2} f_{(2)}\right)\phi_{(3)}\partial_j\phi_{(0)} + \frac{32}{\ell^2} f_{(2)}\phi_{(0)}\partial_j\phi_{(3)}\right] \,.
\end{aligned} \tag{3.1.33}$$

In this case, the holographic stress tensor is

$$\langle T_{ij} \rangle = \frac{3\ell^2}{2\kappa} g_{(3)ij} - \frac{32}{\ell^2} f_{(2)}\phi_{(0)}\phi_{(3)} \,, \tag{3.1.34}$$

which is traceless as a consequence of Eq.(3.1.33).

3.2 Kinetic coupling to the Einstein tensor

A particular sector of the Horndeski theory, which has been widely studied, considers the nonminimal coupling of the scalar field to the Einstein tensor. This term belongs to the Horndeski class of gravity theories. In the case under consideration here, a scalar coupling of the GB term is also included

$$S_{\rm H} = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{R-2\Lambda}{2\kappa} - \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}m^2\phi^2 + f(\phi)\mathcal{G} + \frac{\eta}{2}G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi\right)\,, \tag{3.2.1}$$

The field equations can be obtained by demanding arbitrary variations for the metric and scalar field, giving

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa\left(T_{\mu\nu} + C_{\mu\nu} + H_{\mu\nu}\right)\,, \tag{3.2.2}$$

$$\nabla_\mu\left[\left(g^{\mu\nu} - \eta G_{\mu\nu}\right)\nabla_\nu\phi\right] = m^2\phi - f'(\phi)\mathcal{G}\,, \tag{3.2.3}$$

respectively, where $T_{\mu\nu}$ and $C_{\mu\nu}$ have been defined in Eqs. (2.5.3) and (3.1.4), and

$$\begin{aligned} H_{\mu\nu} = \frac{\eta}{4}\Big[&\delta^{\sigma\alpha\beta}_{\mu\lambda\rho}\nabla_\sigma\phi\nabla_\nu\phi R^{\lambda\rho}_{\alpha\beta} + \delta^{\rho\lambda\sigma}_{\alpha\beta\mu}\nabla_\rho\phi\nabla^\alpha\phi R^{\beta}{}_{\nu\lambda\sigma} \\ &+ 2g_{\mu\nu}G_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi + 2\delta^{\alpha\beta\lambda}_{\rho\sigma\nu}g_{\lambda\mu}\nabla^\sigma\phi\nabla_\beta\phi\left(\nabla_\alpha\phi\nabla^\rho\phi\right)\Big]\,, \end{aligned} \tag{3.2.4}$$

is the contribution to the field equations coming from the scalar-kinetic coupling to the Einstein tensor.

For a massless scalar field, the absence of the GB coupling implies that this theory is endowed with a shift symmetry in field space, namely, the field equations remain invariant under $\phi \to \phi + \tilde{\phi}_0$, where $\tilde{\phi}_0$ is a constant. This the typical symmetry exhibited by Galileons Nicolis et al. (2009); Deffayet et al. (2009b,a).[1] In that case, however, there exists a no-hair theorem which prevents from finding black hole solutions with a nontrivial scalar field Hui and Nicolis (2013). Nevertheless, a suitable on-shell condition on the metric can be imposed such that the no-hair theorem is circumvented, allowing for asymptotically locally flat black holes Rinaldi (2012). Later, the same idea was

[1] If the scalar coupling to the GB is linear, the theory still has shift symmetry. However, this possibility was excluded by the dynamics as shown in the previous section.

extended to the case with the cosmological constant Anabalon et al. (2014) and with Maxwell fields Cisterna and Erices (2014). In this section, we consider the coupling between the scalar field and the GB term such that the shift symmetry is broken, to see how the BF bound is modified with respect to the one found in the previous section.

3.2.1 $\Delta = 2$

Performing the asymptotic analysis in the presence of the kinetic coupling to the Einstein tensor, we find that the mass relation is modified according to

$$m^2\ell^2 = \Delta(\Delta-3)\left(1-\frac{3}{\ell^2}\eta\right)+\frac{48}{\ell^2}f_{(2)}\,. \tag{3.2.5}$$

In this case, the BF bound turns out to be

$$-\frac{\ell^2}{192}\left[9-4\Delta(3-\Delta)\left(1-\frac{3}{\ell^2}\eta\right)\right] < f_{(2)} < -\frac{\ell^2}{192}\left[5-4\Delta(\Delta-3)\left(1-\frac{3}{\ell^2}\eta\right)\right]\,, \tag{3.2.6}$$

and, focusing on the case $\Delta = 2$, it becomes

$$-\frac{\ell^2}{192}\left(1+\frac{24}{\ell^2}\eta\right) < f_{(2)} < \frac{\ell^2}{64}\left(1-\frac{8}{\ell^2}\eta\right)\,. \tag{3.2.7}$$

The boundary scalar equations impose that $f_{(1)} = 0$, together with either $f_{(3)} = 0$ or $\phi^2_{(0)} = 0$. We choose $f_{(3)} = 0$ to ensure that the boundary value of the scalar field remains unconstrained, displaying a nontrivial interaction with the boundary metric. Then, solving the equations of motion order by order, the coefficients are found to be

$$\begin{aligned}
g_{(2)ij} &= -\mathcal{S}(g_{(0)})_{ij} - \frac{\kappa}{4}\left(1-\frac{48}{\ell^2}f_{(2)}-\frac{5}{\ell^2}\eta\right)\phi^2_{(0)}g_{(0)ij}\,,\\
\mathrm{Tr}\, g_{(3)} &= \frac{4\kappa}{3}\left[\frac{4\kappa}{3\ell^2}f_{(2)}\left(1-\frac{5}{\ell^2}\eta-\frac{48}{\ell^2}f_{(2)}\right)\phi^3_{(0)} - \left(1-\frac{6}{\ell^2}\eta-\frac{48}{\ell^2}f_{(2)}\right)\phi_{(0)}\phi_{(1)}\right]\,,\\
\nabla^i_{(0)}g_{(3)ij} &= \frac{2\kappa}{3\ell^2}\Bigg\{\left[\left(32f_{(2)}+2\eta\right)\phi_{(1)} - 8\kappa f_{(2)}\left(1-\frac{48}{\ell^2}f_{(2)}-\frac{5}{\ell^2}\eta\right)\phi^2_{(0)}\right]\partial_j\phi_{(0)}\\
&\quad -\left(\ell^2-32f_{(2)}-5\eta\right)\phi_{(0)}\partial_j\phi_{(1)}\Bigg\}\,.
\end{aligned} \tag{3.2.8}$$

Let us consider the variational problem. Using the asymptotic expansion and taking an arbitrary variation of the fields, we find

$$\begin{aligned}\delta S_{\mathrm{H}} = & \int_{\partial\mathcal{M}} d^3x\sqrt{-\bar{g}}\rho^{-\frac{3}{2}}\Big\{ \left(\ell^2 - 8\kappa f_{(0)}\right)\bar{g}^{ij}\left(\delta\bar{g}_{ij} + 2\rho\delta\partial_\rho\bar{g}_{ij}\right) \\ & - 2\rho\Big[\left(\ell^2 - 3\eta\right)\bar{\phi}\delta\bar{\phi} - \Big(4\kappa f_{(0)}\left(\frac{1}{2}\mathcal{R}(\bar{g})\bar{g}^{ij} + \bar{g}^{ij}\bar{g}^{mn}\partial_\rho\bar{g}_{mn} - \mathcal{R}(\bar{g})_{mn}\bar{g}^{im}\bar{g}^{jn}\right) \\ & -2\kappa\left(\eta + 12f_{(2)}\right)\bar{\phi}^2\bar{g}^{ij} + \frac{\ell^2}{2}\bar{g}^{im}\bar{g}^{jn}\partial_\rho\bar{g}_{mn}\Big)\delta\bar{g}_{ij}\Big] + \mathcal{O}(\rho^2)\Big\} \ . \end{aligned} \tag{3.2.9}$$

Notice that, in this case, Eq. (3.1.12) also removes the divergences coming from the gravitational sector. Additionally, we must take into account the counterterm in Eq. (2.5.17) associated with the scalar field. However, this counterterm should involve the kinetic coupling to the Einstein tensor. Then, the renormalized action turns out to be

$$S_{\mathrm{H}}^{\mathrm{ren}} = S_{\mathrm{H}} + \frac{1}{\ell}\left(1 - \frac{3}{\ell^2}\eta\right) S_\phi \ . \tag{3.2.10}$$

Similar to the previous section, we can write the GB in terms of the square of the Weyl tensor and check that the on-shell action in Eq. (3.2.10) is, indeed, finite. For the holographic stress-energy tensor, we find that it is given by

$$\begin{aligned}\langle T_{ij}\rangle = & -\ell^2 g_{(0)ij}\Big[\left(1 - \frac{3}{\ell^2}\eta\right) W\left(\phi_{(0)}\right) - \left(1 - \frac{32}{\ell^2}f_{(2)} - \frac{5}{\ell^2}\eta\right)\phi_{(0)}\phi_{(1)} \\ & + 8\kappa f_{(2)}\left(1 - \frac{5}{\ell^2}\eta - \frac{48}{\ell^2}f_{(2)}\right)\phi_{(0)}^3\Big] + \frac{3\ell^2}{2\kappa}g_{(3)ij} \ . \end{aligned} \tag{3.2.11}$$

Imposing suitable boundary conditions, i.e., $W(\phi) = C\phi_{(0)}^3$, the trace of the latter becomes

$$\langle T\rangle = -16\kappa f_{(2)}\left(1 - \frac{48}{\ell^2}f_{(2)} - \frac{5}{\ell^2}\eta\right)\phi_{(0)}^3 \ . \tag{3.2.12}$$

Therefore, conformal invariance is preserved in the boundary field theory if the trace vanishes. This implies that $f_{(2)} = 0$, or

$$f_{(2)} = \frac{\ell^2}{48}\left(1 - \frac{5}{\ell^2}\eta\right) \ . \tag{3.2.13}$$

The latter condition is admissible by the BF bound if

$$-\frac{5\ell^2}{4} < \eta < -\frac{\ell^2}{4} \; . \tag{3.2.14}$$

Then, η has to be negative and it has to lie within the BF bound defined in the previous equation. Notice that taking $\eta = 0$ is not admissible. This shows how the kinetic coupling to the Einstein tensor can be used to fix the unitarity problem in sGB with $\Delta = 2$. Moreover, if we consider Dirichlet boundary conditions, i.e. $W\left(\phi_{(0)}\right) = 0$, we find an additional contribution to the vacuum expectation value of the boundary scalar coming from the kinetic coupling, namely,

$$\langle \mathcal{O} \rangle = -\left(1 - \frac{3}{\ell^2}\eta\right)\phi_{(1)} = -\left(1 - \frac{3}{\ell^2}\eta\right)(2\Delta - d)\phi_{(2\Delta - d)} \; . \tag{3.2.15}$$

This modifies the result of the minimally coupled scalar field by a factor of $(1 - 3\eta\ell^{-2})$. Moreover, diffeomorphism invariance of $g_{(0)}$ at the boundary implies a holographic Ward identity. While the coefficient in the expansion at a holographic order, $g_{(3)}$, can not be determined, we can determine its trace and divergence throughout the field equations. Then, the Ward identity reads

$$\nabla^i_{(0)}\langle T_{ij} \rangle = \left(1 - \frac{3}{\ell^2}\eta\right)\phi_{(1)}\partial_j\phi_{(0)} = -\langle \mathcal{O} \rangle \partial_j \phi_{(0)} \; , \tag{3.2.16}$$

which matches the result of Ref. de Haro et al. (2001) for the minimally coupled scalar field. This computation of the holographic energy tensor-momentum tensor is not associated to a quasi-local stress tensor on the gravity side. However, the coupling of matter renders its divergence different from zero, as it is related to the flow of momentum out of the boundary Brown and York (1993).

Let us examine Horndeski theory by considering a different possibility. For instance, if we fix $\Delta = 2$ with $\phi^2_{(0)} = 0$ while keeping $f_{(3)}$ arbitrary, we obtain

$$\begin{aligned} g_{(2)ij} &= -\mathcal{S}(g_{(0)})_{ij} , \\ \text{Tr } g_{(3)} &= -\frac{4\kappa}{3}\left(1 - \frac{48}{\ell^2} f_{(2)} - \frac{6}{\ell^2}\eta\right) \phi_{(0)}\phi_{(1)} , \\ \nabla^i_{(0)} g_{(3)ij} &= \frac{2\kappa}{3\ell^2}\left[\left(32 f_{(2)} + 2\eta\right)\phi_{(1)}\partial_j\phi_{(0)} - \left(\ell^2 - 32 f_{(2)} - 5\eta\right)\phi_{(0)}\partial_j\phi_{(1)}\right] . \end{aligned} \tag{3.2.17}$$

The variation of the on-shell action gives the same kind of divergences as in the previous case. Therefore, the renormalized action must be the one in Eq. (3.2.10). The holographic stress tensor in this case becomes

$$\langle T_{ij}\rangle = \ell^2\left(1 - \frac{32}{\ell^2} f_{(2)} - \frac{5}{\ell^2}\eta\right)\phi_{(0)}\phi_{(1)} g_{(0)ij} + \frac{3\ell^2}{2\kappa} g_{(3)ij} , \tag{3.2.18}$$

which is always traceless as $\phi_{(1)}$ is proportional to positive powers of $\phi_{(0)}$ when considering mixed boundary conditions. Therefore, considering $\phi_{(0)}$ to be infinitesimal, one has a well-defined boundary CFT with a continuous $\eta \to 0$ limit. Finally, using Dirichlet boundary conditions we find that the vacuum expectation value and the Ward identity associated to the boundary scalar operator are the same as in the case with $f_{(3)} = 0$ and $\phi_{(0)}$ unfixed.

3.2.2 $\Delta = 3$

Consider now the case $\Delta = 3$. For this choice, the mass simply becomes $m^2\ell^4 = 48 f_{(2)}$ and the scalar equation imposes either $f_{(1)} = 0 = f_{(3)}$ or $f_{(1)} = 0 = \phi^2_{(0)}$. Moreover, the zeroth order of the Einstein equations restricts further the theory with $f_{(2)} = 0$ or $\phi^2_{(0)} = 0$. Focusing first on the case when $\phi_{(0)}$ is arbitrary and solving for the coefficients of the metric, we obtain

$$\begin{aligned} g_{(2)ij} &= -\mathcal{S}(g_{(0)})_{ij} - \frac{\kappa}{4}\left(1 - \frac{3}{\ell^2}\eta\right) g_{(0)ij}\partial_m\phi_{(0)}\partial^m\phi_{(0)} + \kappa\left(1 - \frac{2}{\ell^2}\eta\right)\partial_i\phi_{(0)}\partial_j\phi_{(0)} , \\ \phi_{(2)} &= \frac{1}{2}\Box_{(0)}\phi_{(0)} , \\ \text{Tr } g_{(3)} &= 0 , \\ \nabla^i_{(0)} g_{(3)ij} &= 2\kappa\left(1 - \frac{3}{\ell^2}\eta\right)\phi_{(3)}\partial_j\phi_{(0)} . \end{aligned} \tag{3.2.19}$$

An arbitrary on-shell variation of the action with $f_{(0)} = \ell^2/8\kappa$ yields

$$\begin{aligned}\delta S_{\mathrm{H}} = & -\frac{1}{2}\int_{\partial\mathcal{M}} d^3x\sqrt{-\bar{g}}\rho^{-1/2}\Big[\kappa\eta\left(\bar{g}^{mi}\bar{g}^{nj}\partial_i\bar{\phi}\partial_j\bar{\phi}\delta\bar{g}_{mn} - \bar{g}^{ij}\partial_m\bar{\phi}\partial^m\bar{\phi}\delta\bar{g}_{ij}\right) \\ & + \ell^2\left(\mathcal{R}(\bar{g})^{ij} - \frac{1}{2}\mathcal{R}(\bar{g})\bar{g}^{ij} + \bar{g}^{im}\bar{g}^{jn}\partial_\rho\bar{g}_{mn}\right)\delta\bar{g}_{ij} + 4\kappa\left(\ell^2 - 3\eta\right)\partial_\rho\bar{\phi}\delta\bar{\phi}\Big] \,.\end{aligned} \tag{3.2.20}$$

In order to eliminate the divergencies in this case, we notice that the same counterterm as in the sGB theory renders the theory finite but with a different coupling, that is,

$$S_{\mathrm{H}}^{\mathrm{ct}} = -\frac{\ell}{2}\left(1 - \frac{3}{\ell^2}\eta\right)\int_{\partial\mathcal{M}} d^3x\sqrt{-h}h^{ij}\partial_i\phi\partial_j\phi \,. \tag{3.2.21}$$

Then, the action

$$S_{\mathrm{H}}^{\mathrm{ren}} = S_{\mathrm{H}} + S_{\mathrm{H}}^{\mathrm{ct}} \,, \tag{3.2.22}$$

is on-shell finite. Although the counterterm in Eq. (3.2.21) is necessary, most of the aforementioned solutions have a scalar profile that depends on the radial coordinate only, making the counterterm identically zero. Thus, the particular value of Eq. (3.1.12) is enough to remove the divergences of the theory. Nevertheless, it has been found that the theory admits regular solutions if one considers time-dependent scalar field Babichev and Charmousis (2014), such that the scalar does not inherit the spacetime symmetries, but the stress-tensor does. Moreover, there are interesting solutions of scalar-tensor gravity theories containing scalar fields that depend on both the radial and boundary coordinates such as accelerating black holes Lu and Vazquez-Poritz (2015); Cisterna et al. (2021, 2023); Barrientos and Cisterna (2023), whose holographic properties remain to be fully understood, and instantons de Haro and Petkou (2006); de Haro et al. (2001), that can be used to explore vacuum decay of the boundary conformal theory Papadimitriou (2007). In this case, the holographic stress-energy tensor becomes

$$\langle T_{ij}\rangle = \frac{3\ell^2}{2\kappa}g_{(3)}^{ij} \,, \tag{3.2.23}$$

which is traceless. Moreover, the vacuum expectation value of the boundary source becomes

$$\langle \mathcal{O} \rangle = -3\left(1-\frac{3}{\ell^2}\eta\right)\phi_{(3)} = -\left(1-\frac{3}{\ell^2}\eta\right)(2\Delta-d)\phi_{(2\Delta-d)} \; . \tag{3.2.24}$$

This shows how the couplings of the theory modify the value of the 1-point functions of the dual operators. Moreover, the holographic Ward identity becomes

$$\nabla^i_{(0)}\langle T_{ij}\rangle = 3\left(1-\frac{3}{\ell^2}\eta\right)\phi_{(3)}\partial_j\phi_{(0)} = -\langle \mathcal{O}\rangle \partial_j\phi_{(0)} \; , \tag{3.2.25}$$

just as in the previous cases.

Finally, for an arbitrary $f_{(2)}$ and setting the source of the scalar operator to be infinitesimal, i.e. $\phi^2_{(0)} = 0$, we find that the coefficients can be solved as

$$\begin{aligned}
g_{(2)ij} =&\; -\mathcal{S}_{(0)ij} + \kappa\left(1-\frac{2}{\ell^2}\eta\right)\partial_i\phi_{(0)}\partial_j\phi_{(0)} \\
&+ \frac{\kappa}{\ell^2}\left(24 f_{(2)}\phi_{(0)}\phi_{(2)} - \frac{1}{4}\left(\ell^2-3\eta\right)\partial^m\phi_{(0)}\partial_m\phi_{(0)}\right) g_{(0)ij} \\
\phi_{(2)} =&\; \frac{1}{2}\left(\frac{\ell^2-3\eta}{\ell^2-3\eta-72 f_{(2)}\phi_{(0)}}\right)\Box_{(0)}\phi_{(0)} \\
\mathrm{Tr}\, g_{(3)} =&\; \frac{64\kappa}{\ell^2} f_{(2)}\phi_{(0)}\phi_{(3)} \; , \\
\nabla^i_{(0)} g_{(3)ij} =&\; \frac{2\kappa}{3\ell^2}\left[\left(3+\frac{32}{\ell^2}f_{(2)} - \frac{9}{\ell^2}\eta\right)\phi_{(3)}\partial_j\phi_{(0)} + \frac{32}{\ell^2} f_{(2)}\phi_{(0)}\partial_j\phi_{(3)}\right] \; ,
\end{aligned} \tag{3.2.26}$$

Choosing $f_{(0)}$ as in Eq. (3.1.12) and considering the same counterterm as in Eq. (3.2.21), the on-shell action becomes finite and the variational principle is well-posed. Then, the holographic stress-energy tensor becomes

$$\langle T_{ij}\rangle = -\frac{32}{\ell^2} f_{(2)}\phi_{(0)}\phi_{(3)} g_{(0)ij} + \frac{3\ell^2}{2\kappa} g_{(3)ij} \; , \tag{3.2.27}$$

which is traceless by virtue of Eq. (3.2.26). If we consider Dirichlet boundary conditions, the scalar operator acquires a nontrivial vacuum expectation value given by

$$\langle \mathcal{O} \rangle = -3\left(1-\frac{3}{\ell^2}\eta\right)\phi_{(3)} = -\left(2\Delta - d\right)\left(1-\frac{3}{\ell^2}\eta\right)\phi_{(2\Delta-d)}\ , \qquad (3.2.28)$$

and a Ward identity given by

$$\nabla^i_{(0)}\langle T_{ij}\rangle = 3\left(1-\frac{3}{\ell^2}\eta\right)\phi_{(3)}\partial_j\phi_{(0)} = -\langle \mathcal{O}\rangle \partial_j\phi_{(0)}\ , \qquad (3.2.29)$$

just as in the case with an arbitrary $\phi_{(0)}$.

3.2.3 Minimal Horndeski theory

Finally, we discuss on the particular case $f_{(n)} = 0 = m^2\ , \forall n \in \mathbb{N}_{>0}$ Charmousis et al. (2012b,a), which has received a lot of attention in a holographic context Filios et al. (2019); Kuang and Papantonopoulos (2016); Li and Lu (2018); Liu (2018); Feng and Liu (2019); Li et al. (2019); Jiang et al. (2017) as it contains analytic solutions Arratia et al. (2021); Anabalon et al. (2014); Cisterna and Erices (2014); Cisterna et al. (2015, 2016); Brihaye et al. (2016); Babichev and Charmousis (2014); Babichev et al. (2016); Cisterna et al. (2017); Stetsko (2019); Cisterna et al. (2018). The action in this case is

$$S_{\mathrm{Hmin}} = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{R-2\Lambda}{2\kappa} - \frac{1}{2}\left(\alpha g_{\mu\nu} - \eta G_{\mu\nu}\right)\nabla^\mu\phi\nabla^\nu\phi + f_{(0)}\mathcal{G}\right)\ , \qquad (3.2.30)$$

where we have included an arbitrary constant for the canonical kinetic term; this is achieved by rescaling the scalar field and the metric.[2] Additionally, we have included the GB term as it will serve as a counterterm in the scalar-tensor theories we are interested in. Recall that this sector of Horndeski theory is endowed with a shift symmetry as the scalar field appears only through derivatives in the action.

[2]Consider $\phi \to \alpha^{-1}\phi$ and $g_{\mu\nu} \to \alpha g_{\mu\nu}$, such that $R \to \alpha^{-1}R$. The Einstein tensor is scale-invariant, i.e., $G_{\mu\nu} \to G_{\mu\nu}$ and, rescaling, $\Lambda \to \alpha\Lambda\ ,\ \kappa \to \alpha\kappa$, the minimal Horndeski theory with canonical kinetic term, i.e., $\alpha = 1$, becomes that of Eq. (3.2.30).

The field equations correspond to a subset of Eq. (3.2.2); explicitly, they are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \left(T^{\text{min}}_{\mu\nu} + H_{\mu\nu} \right) \ , \tag{3.2.31}$$

$$\nabla^{\mu} \left[\left(\alpha g_{\mu\nu} - \eta G_{\mu\nu} \right) \nabla^{\nu} \phi \right] = 0 \, , \tag{3.2.32}$$

where $T^{\text{min}}_{\mu\nu} = \frac{\alpha}{2} T_{\mu\nu}$ with the latter defined in Eq. (2.5.3) by setting $m = 0$. Using the FG expansion, we find that the scalar field equation becomes

$$\Delta \left(\Delta - 3 \right) \left(\ell^2 \alpha - 3\eta \right) \bar{\phi} + \mathcal{O} \left(\rho \right) = 0 \, , \tag{3.2.33}$$

which is satisfied either if $\Delta = 3$ or $\eta = \frac{1}{3} \ell^2 \alpha$ to all orders without fixing the conformal weight. If the latter point in the parameter space is assumed, then the action is renormalized simply by the GB term if one fixes $f_{(0)}$ as in Eq. (3.1.12). Nevertheless, this is a critical point of the theory. As shown in Refs. Jiang et al. (2017); Feng and Liu (2019); Li and Lu (2018), the theory admits a solution that is nearly AdS with a nontrivial scalar field whose integration constant appears in the same footing as the cosmological constant. Thus, it is convenient to introduce an effective cosmological constant, say Λ_{eff}, that leads to Λ when ϕ vanishes.

In an Einstein-AdS background, the scalar sector of the minimal Horndeski theory becomes simply a minimally coupled massless scalar, that is,

$$\mathcal{L}_{\phi} = -\frac{1}{2} \left(\alpha - \frac{3}{\ell^2} \eta \right) \left(\partial \phi \right)^2 \ . \tag{3.2.34}$$

Thus, the absence of ghosts implies the inequality

$$\alpha - \frac{3}{\ell^2} \eta \geq 0 \, . \tag{3.2.35}$$

Notice that, if $\alpha \geq 0$, this condition is always fulfilled as long as the parameter η lies within the bound of Eq. (3.2.14). This implies both unitarity and the absence of ghosts. If $\alpha < 0$, on the other hand, there is still a region in the parameter space where the condition (3.2.35) is satisfied. The case $\alpha = 0$ is also allowed by the Eq. (3.2.14). On the other hand, the bound is saturated at the critical point in which the scalar contribution vanishes. These kinds of critical points were also studied in the pure GB gravity Fan et al. (2016) and the black hole solution of Ref. Anabalon et al. (2014) simply becomes the Schwarzschild-AdS black hole with a vanishing scalar field. In this case, the holographic stress-energy tensor is traceless and it equals that of the pure gravity case. Nonetheless, it is possible to obtain an exact global AdS background with a nontrivial scalar. The latter breaks the AdS isometries and it contains a logarithmic mode in the FG expansion. This indicates that the dual theory is scale invariant but not conformally invariant (see Ref. Nakayama (2015) for details).

As logarithmic modes are beyond the scope of this paper, we will move forward and consider $\Delta = 3$ with $\alpha \neq 3\eta\ell^{-2}$. Solving Einstein's equations order by order, we obtain $f_{(1)} = 0 = f_{(3)}$ as before, and the coefficients of the metric expansion are now solved as

$$\begin{aligned} g_{(2)ij} =& -\mathcal{S}_{ij}\left(g_{(0)}\right) - \frac{\kappa}{4\ell^2}\left(\ell^2\alpha - 3\eta\right) g_{(0)ij}\partial_m\phi_{(0)}\partial^m\phi_{(0)} \\ &+ \frac{\kappa}{\ell^2}\left(\ell^2\alpha - 2\eta\right)\partial_i\phi_{(0)}\partial_j\phi_{(0)} \ , \\ \mathrm{Tr}\, g_{(3)} =& \ 0 \ , \\ \nabla^i_{(0)} g_{(3)ij} =& \ \frac{2\kappa}{\ell^2}\left(\alpha - \frac{3}{\ell^2}\eta\right)\phi_{(3)}\partial_j\phi_{(0)} \ . \end{aligned} \tag{3.2.36}$$

Considering an on-shell variation of the action with the GB coupling found in Eq. (3.1.12), we arrive at

$$\begin{aligned} \delta S_{\text{Hmin}} =& \frac{1}{2}\int_{\partial\mathcal{M}} d^3x\sqrt{-g_{(0)}}\left[\rho^{-\frac{1}{2}}\left(\ell^2\alpha - 3\eta\right)\left(\frac{1}{2}g^{ij}_{(0)}\phi^2_{(0)} - \partial^i\phi_{(0)}\partial^j\phi_{(0)}\right)\right. \\ &\left. - \frac{3\ell^2}{2\kappa}g^{ij}_{(3)}\right]\delta g_{(0)ij} \ . \end{aligned} \tag{3.2.37}$$

This is finite if we choose the value in Eq. (3.1.12) but the critical value of η is still needed. Nonetheless, the latter can be rendered arbitrary if we add a suitable counterterm, that is,

$$S_{\text{ct}} = -\frac{1}{2\ell}\left(\ell^2\alpha - 3\eta\right)\int_{\partial\mathcal{M}} d^3x\sqrt{-h}h^{ij}\partial_i\phi\partial_j\phi \ . \tag{3.2.38}$$

With this counterterm, the action is finite. Then, the holographic stress-energy tensor is given by

$$\langle T_{ij} \rangle = \frac{3\ell^2}{2\kappa} g_{(3)ij} \,, \tag{3.2.39}$$

and the vacuum expectation value for the boundary scalar operator, and the holographic Ward identity become

$$\langle \mathcal{O} \rangle = -3 \left(\alpha - \frac{3}{\ell^2}\eta \right) \phi_{(3)} = -\left(\alpha - \frac{3}{\ell^2}\eta \right) (2\Delta - d)\phi_{(2\Delta-d)} \,, \tag{3.2.40}$$

and

$$\nabla^i \langle T_{ij} \rangle = 3 \left(\alpha - \frac{3}{\ell^2}\eta \right) \phi_{(3)}\partial_j\phi_{(0)} = -\langle \mathcal{O} \rangle \partial_j\phi_{(0)} \,, \tag{3.2.41}$$

respectively. Notice that, at the critical point $\alpha = 3\eta\ell^{-2}$, the vacuum expectation value of the scalar field vanishes and the holographic stress-energy tensor becomes covariantly conserved. In this case, the known scalar-field solutions either vanish or do not backreact and the action becomes simply the one of Einstein's gravity. This shows that taking the limit to this critical value is consistent also at the level of 1-point functions.

Chapter 4

Conclusions and future works

4.1 Discussion

In this work, we have considered different sectors of Horndeski theory and analyzed their renormalization in AAdS spaces. To this end, we introduced the GB term nonminimally coupled to an arbitrary function of the scalar field. One of the main results is that the asymptotic analysis of the field equations restricts considerably the form of such a function. In particular, we found that the linear-scalar coupling to the GB term is not allowed in AAdS spaces. Moreover, if the scalar-GB term is the only nonminimal coupling in the bulk, we found that the boundary field theory cannot be unitary and conformal at the same time, since the scalar mass lies outside the BF bound. This provides a holographic argument against this theory.

If the scalar-kinetic coupling to the Einstein tensor is considered, we found that the aforementioned issue can be solved, rendering the boundary field theory self-consistent. We analyzed different possibilities and, in all cases, we obtained the counterterms that render the theory finite, the vacuum expectation value of the scalar operator at the boundary, and the holographic stress-energy tensor. Moreover, we study the holographic Ward identity associated with coordinate transformations at the boundary and identify how the nonminimal coupling of Horndeski's theory produces an anomalous term.

The minimal Horndeski's gravity theory considered in Eq. (3.2.30) encompasses numerous black hole solutions. However, only a limited subset among them possesses exact solutions for the scalar field, as usually only its radial derivative is known, rather than the full analytic expression. Nonetheless, there are a few cases in which this issue can be circumvented. In these situations, the solutions feature either negligible backreaction or modify the geometry as an effective cosmological constant. For instance, in Ref. Anabalon et al. (2014), an analytic solution corresponding to a topological Schwarzschild black hole with a flat transverse section and such effective cosmological constant was found. There exists an analytic solution for the scalar field even though there are new logarithmic divergences near the boundary. In that cases, the additional divergence introduced in the action (3.2.30) can be renormalized by treating it as an effective cosmological constant that takes into account the supplementary contribution of the scalar field. This approach involves selecting the coupling of the GB term as in Eq. (3.1.12) but in terms of an effective cosmological constant. As a result, incorporating the GB term with a distinct coupling is sufficient to regularize the additional divergences in the action, even in the presence of logarithmic modes of the scalar fields.

Interesting questions remain open. In particular, extending this analysis to include the logarithmic modes is certainly very important, since some of the known analytic solutions in the literature are endowed with this asymptotic behavior. We will come back to this point in the future. Additionally, it is well known that in even boundary dimensions, the holographic trace anomaly is related to the logarithmic modes of the metric Henningson and Skenderis (1998). The role of nonminimally coupled scalar fields in the holographic trace anomaly is indeed worth studying, alongside the generalization of the counterterms found here to higher dimensions. On the other hand, the results found here are useful for studying holographic measurements such as entanglement entropy and superconductivity. We will analyze these phenomena for specific analytic solutions in the future.

Appendix A

Supplementary calculations

A1 AdS and AlAdS spacetimes

The Anti-de Sitter spacetime (AdS) is one that is maximally symmetric with constant and negative scalar curvature. In this work, we will focus on dimension $D = 3 + 1$, so the AdS_4 spacetime can be thought of as a spacetime embedded in $\mathbb{R}^{3,2}$, given by the following constraint

$$X_A X^A = -(X_0)^2 + \sum_{i=1}^{3} (X^i)^2 - (X^4)^2 - \ell^2 \tag{A1.1}$$

its associated isometry group is $O(3, 2)$, and being maximally symmetric, it possesses ten Killing generators. Consider now the following set of coordinates $t \in \mathbb{R}, \vec{x} = (x^1, x^2) \in \mathbb{R}^2, r \in \mathbb{R}_+$ given by

$$X^0 = \frac{\ell^2}{2r}\left(1 + \frac{r^2}{\ell^4}(\vec{x}^2 - t^2 + \ell^2)\right) \tag{A1.2}$$

$$X^i = \frac{rx^i}{\ell} \qquad \text{for} \quad i \in \{1, 2\} \tag{A1.3}$$

$$X^3 = \frac{\ell^2}{2r}\left(1 + \frac{r^2}{\ell^4}(\vec{x}^2 - t^2 - \ell^2)\right) \tag{A1.4}$$

$$X^4 = \frac{rt}{\ell}. \tag{A1.5}$$

Given the radial constraint, these coordinates cover half of the spacetime. These local coordinates are known as the Poincaré patch. In these coordinates, the metric takes the form

$$ds^2 = \frac{\ell^2}{r^2}dr^2 + \frac{r^2}{\ell^2}\eta_{ij}dx^i dx^j \tag{A1.6}$$

where $\eta_{ij} = diag(-1, 1, 1)$. If we take $r \to \infty$, we obtain a factor that diverges Ω times a metric

$$ds^2_{r\to\infty} = \Omega ds'^2, \tag{A1.7}$$

the spacetime given by the metric ds'^2 is known as the conformal boundary of AdS_4, and in what follows, we will generalize this notion, where the conformal boundary will have a more general structure, allowing for other types of spacetimes.

Making the coordinate change $\rho = \ell^4/r^2$ [1], we obtain

$$ds^2 = \frac{\ell^2}{4\rho^2}d\rho^2 + \frac{\ell^2}{\rho}\eta_{ij}dx^i dx^j, \tag{A1.8}$$

these are the Fefferman-Graham type coordinates, where the asymptotic boundary is at $\rho \to 0$.
Following Anastasiou et al. (2021), in simple terms, we can say that the Asymptotically locally Anti-de Sitter spacetimes ($AlAdS$) are those spacetimes that are solutions to the Einstein field equations with negative cosmological constant, where the matter tensor is asymptotically subleading concerning the cosmological constant term. These spacetimes admit the metric in the Fefferman-Graham form

$$ds^2 = \frac{\ell^2}{4\rho^2}d\rho^2 + \frac{\ell^2}{\rho}\overline{g}_{ij}dx^i dx^j = \frac{\ell^2}{4\rho^2}d\rho^2 + \frac{\ell^2}{\rho}(g_{(0)ij} + \mathcal{O}(\rho))dx^i dx^j \tag{A1.9}$$

This will be the class of spacetime that we will study in this work.

[1]To use the coordinates z, we simply make the coordinate change to the Poincaré patch $z = \ell^2/r$

A2 Gaussian normal coordinates

Consider the gaussian normal coordinate:

$$ds^2 = N^2(\rho)d\rho^2 + h_{ij}(\rho, x)dx^i dx^j \tag{A2.1}$$

their associated Christoffel symbols are:

$$\Gamma^{\rho}_{\rho\rho} = \frac{\partial_\rho N}{N} \tag{A2.2}$$

$$\Gamma^{\rho}_{ij} = \frac{1}{N}K_{ij} \tag{A2.3}$$

$$\Gamma^{i}_{j\rho} = -NK^i_j \tag{A2.4}$$

$$\Gamma^{i}_{jk}(g) = \Gamma^{i}_{jk}(h) \tag{A2.5}$$

whith $K_{ij} = -\frac{1}{2N}\partial_\rho h_{ij}$ the associated extrinsic curvature. The Gauss-Codazzi relations give us:

$$\begin{aligned}
R^{i\rho}_{jl} &= \frac{1}{N}\left(\nabla_l K^i_j - \nabla_j K^i_l\right) \\
R^{il}_{j\rho} &= N(\nabla^l K^i_j - \nabla^i K^l_j) \\
R^{i\rho}_{j\rho} &= \frac{1}{N}\left(K^i_j\right)' - K^i_n K^n_j \\
R^{ik}_{jl} &= \mathcal{R}^{ik}_{jl}(h) - K^i_j K^k_l + K^i_l K^k_j
\end{aligned} \tag{A2.6}$$

using this relations for the Fefferman-Graham frame:

$$ds^2 = \frac{\ell^2}{4\rho^2}d\rho^2 + \frac{\ell^2}{\rho}\overline{g}_{ij}dx^i dx^j \tag{A2.7}$$

In this case $N(\rho) = \ell/2\rho$ and $h_{ij} = \ell^2\overline{g}_{ij}(\rho, x)/\rho$. Then the Christoffel symbols are

$$\Gamma^{\rho}_{\rho\rho} = -\frac{1}{\rho} \tag{A2.8}$$

$$\Gamma^{\rho}_{ij} = 2(\overline{g}_{ij} - \rho\partial_\rho\overline{g}_{ij}) \tag{A2.9}$$

$$\Gamma^{i}_{j\rho} = -\frac{1}{2\rho}(\delta^i_j - \rho\overline{g}^{ir}\partial_\rho\overline{g}_{rj}) \tag{A2.10}$$

$$\Gamma^{i}_{jk}(h) = \Gamma^{i}_{jk}(\overline{g}) \tag{A2.11}$$

Where the extrinsic curvature is $K_{ij} = \ell(\overline{g}_{ij}/\rho - \partial_\rho \overline{g}_{ij})$. Then the Riemann tensor components are

$$R^{i\rho}_{jl} = \frac{2\rho^2}{\ell^2}\delta^{op}_{jl}\nabla_o(\overline{g}^{im}\partial_\rho\overline{g}_{mp}) \tag{A2.12}$$

$$R^{il}_{j\rho} = \frac{\rho}{2\ell^2}\delta^{il}_{op}\nabla^o(\overline{g}^{pr}\partial_\rho\overline{g}_{rj}) \tag{A2.13}$$

$$R^{i\rho}_{j\rho} = -\frac{1}{\ell^2}(\delta^i_j + \rho^2(2\partial_\rho(\overline{g}^{im}\partial_\rho\overline{g}_{mj}) + \overline{g}^{im}\partial_\rho\overline{g}_{mn}\overline{g}^{no}\partial_\rho\overline{g}_{oj})) \tag{A2.14}$$

$$R^{ik}_{jl} = \frac{1}{\ell^2}(\rho\mathcal{R}^{ik}_{jl}(\overline{g}) - \delta^{mn}_{jl}(\delta^i_m\delta^k_n - \rho\delta^{ik}_{mr}\overline{g}^{ro}\partial_\rho\overline{g}_{on} + \rho^2\overline{g}^{io}\partial_\rho\overline{g}_{om}\overline{g}^{kp}\partial_\rho\overline{g}_{pn})) \tag{A2.15}$$

with this we can calculate the components of the Ricci tensor

$$R^i_j = -\frac{1}{\ell^2}(3\delta^i_j - \rho(\mathcal{R}^i_j(\overline{g}) + \overline{g}^{ir}\partial_\rho\overline{g}_{rj} + \delta^i_j\overline{g}^{mn}\partial_\rho\overline{g}_{mn}) \tag{A2.16}$$

$$+ \rho^2(2\partial_\rho(\overline{g}^{ir}\partial_\rho\overline{g}_{rj}) + \overline{g}^{ir}\partial_\rho\overline{g}_{rj}\overline{g}^{mn}\partial_\rho\overline{g}_{mn})) \tag{A2.17}$$

$$R^\rho_j = \frac{2\rho^2}{\ell^2}\delta^{op}_{lj}\nabla_o(\overline{g}^{lr}\partial_\rho\overline{g}_{rp}) \tag{A2.18}$$

$$R^j_\rho = \frac{\rho}{2\ell^2}\delta^{lj}_{op}\nabla^o(\overline{g}^{pr}\partial_\rho\overline{g}_{rl}) \tag{A2.19}$$

$$R^\rho_\rho = -\frac{1}{\ell^2}(3 + \rho^2(2\partial_\rho(\overline{g}^{lm}\partial_\rho\overline{g}_{ml}) + \overline{g}^{ml}\partial_\rho\overline{g}_{mn}\overline{g}^{no}\partial_\rho\overline{g}_{ol})) \tag{A2.20}$$

and the Ricci scalar is

$$R = -\frac{1}{\ell^2}(12 - \rho(\mathcal{R}(\overline{g}) + 4\overline{g}^{ro}\partial_\rho\overline{g}_\rho) + \rho^2(\delta^{mn}_{ik}\overline{g}^{io}\partial_\rho\overline{g}_{om}\overline{g}^{kp}\partial_\rho\overline{g}_{pn} \tag{A2.21}$$

$$+ 4\partial_\rho(\overline{g}^{lm}\partial_\rho\overline{g}_{ml}) + 2\overline{g}^{lm}\partial_\rho\overline{g}_{mn}\overline{g}^{no}\partial_\rho\overline{g}_{ol})) \tag{A2.22}$$

A3 Scalar field boundary conditions

In order to stress the importance of boundary conditions in the dual field theory, let us analyze the case of a massive scalar field on global AdS as an example. The corresponding action can be expressed as

$$S = \frac{1}{2}\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\left[(\partial\phi)^2 + m^2\phi^2\right] , \tag{A3.1}$$

where $\mathcal{M}$ is an Euclidean AdS_{d+1} background with spacetime coordinates $x^\mu = (z, x^i)$. Its line element in the Poincaré patch is given by

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 + \delta_{ij}dx^i dx^j\right) . \tag{A3.2}$$

An arbitrary variation of the on-shell action with respect to the dynamic field yields

$$\begin{aligned}\delta S &= -\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\left(\nabla^2 - m^2\right)\phi\delta\phi - \int_{\partial\mathcal{M}} d^d x\sqrt{h}\left(n^\mu\nabla_\mu\phi\right)\delta\phi \\ &= -\int_{\partial\mathcal{M}} d^d x\left(\frac{z}{\ell}\right)^{1-d}\partial_z\phi\,\delta\phi .\end{aligned} \tag{A3.3}$$

Depending on the boundary conditions, an extra boundary term must be added such that the on-shell action possesses a minimum. For Dirichlet boundary conditions, the variation of the scalar field vanishes at the boundary, so there is no need for adding such a term. For Neumann boundary conditions, however, the normal derivative of the scalar field is fixed. Thus, one needs to add

$$S_{\mathrm{N}} = \int d^d x\left(\frac{z}{\ell}\right)^{1-d}\phi\partial_z\phi , \tag{A3.4}$$

to the bulk action, such that the variational principle is well-posed. Then, the source of the boundary scalar operator is given by the normal derivative of the field. This corresponds to the radial canonical momentum associated with the scalar field. Additionally, it is possible to impose mixed boundary conditions that involve a relationship between the scalar field and its normal derivative. The latter specifies the behavior of the boundary scalar operator according to

$$\psi := \phi + \lambda n^\mu\partial_\mu\phi , \tag{A3.5}$$

where λ is a non-zero real number. Then, one needs to consider a boundary term of the form

$$S_{\mathrm{M}} = -\frac{1}{2}\int d^d x\,\left(\frac{z}{\ell}\right)^{1-d}\psi\partial_z\phi , \tag{A3.6}$$

and the source is now related to ψ .

On the Euclidean AdS_{d+1} background, the field equation for the scalar field can be written as

$$z^{d+1}\partial_z\left(z^{1-d}\partial_z\phi\right)+z^2\delta^{ij}\partial_i\partial_j\phi=m^2\ell^2\phi\ . \tag{A3.7}$$

Near the boundary, one can check that the solution is,[2]

$$\phi(z,x^i)\sim z^{d-\Delta}\phi_{(0)}(x^i)+z^{\Delta}\phi_{(1)}(x^i)\ , \tag{A3.8}$$

where

$$\Delta=\frac{d}{2}\pm\sqrt{\frac{d^2}{4}+m^2\ell^2}\ . \tag{A3.9}$$

In Ref. Witten (1998), it was shown that Δ corresponds to the conformal weight of a scalar operator of a d-dimensional CFT. This connection implies that for the scaling weight to be a real quantity, the condition $m^2\ell^2>-d^2/4$ must be satisfied. Remarkably, this condition is compatible with the BF bound Breitenlohner and Freedman (1982), suggesting that a scalar field in AdS can possess a negative mass while being stable. The functions $\phi_{(0)}$ and $\phi_{(1)}$ are the two linearly independent solutions of the second-order field equations. The leading-order term can be either singular if $\Delta>d$, trivial if $\Delta<d$, or constant if $\Delta=d$. Then, depending on the mass of the scalar field, the bulk geometry is modified while preserving the asymptotic structure.

As it can be seen from Eq. (A3.9), the conformal weight Δ is a positive real number such that $\Delta>d-\Delta$. Therefore, $\phi_{(0)}$ is associated with non-normalizable modes at the boundary. To identify the source of the boundary scalar operator, one needs to consider

$$\varphi(x^i)=\lim_{z\to 0}z^{\Delta-d}\phi(z,x^i)\ , \tag{A3.10}$$

which is always finite. Boundary conditions fix a function of $\phi_{(0)}$ and $\phi_{(1)}$ at the boundary, that corresponds to the source of the dual scalar operator. The relation between the modes reduces the degrees of freedom in the dual theory by one half. For instance, Dirichlet boundary conditions fix the source to be $\phi_{(0)}$ and the remaining degree of freedom corresponds to the

[2] If $\Delta=d/2$, one needs to consider solutions with logarithmic behavior.

normalizable mode. As shown in Ref. Papadimitriou and Skenderis (2005), the non-normalizable mode $\phi_{(1)}$ does not transform properly under Weyl rescalings. Therefore, from a holographic viewpoint, one needs to consider the renormalized radial momentum $\hat{\pi}_\phi$, i.e., the first regular coefficient in the asymptotic expansion of the canonical momentum π_ϕ. The latter usually differ from $\phi_{(1)}$ by a local functional of $\phi_{(0)}$. Then, for Neumann boundary conditions, one fixes $\hat{\pi}_\phi$ rather than $\phi_{(1)}$. This shows that the leading and sub-leading coefficients in the asymptotic expansion are canonical conjugated variables (see Ref. Papadimitriou (2007) details).

In general, one introduces a boundary term that depends on the scalar field and derivatives thereof, whose explicit form depends on the boundary conditions. As shown before, the field equations fix the relation between derivatives of both the field the boundary term via boundary conditions. The extra boundary contribution to the gravity action implies a modification of the boundary theory. In the case of mixed boundary conditions, this corresponds to modifying the holographic CFT by multi-trace operators if the deformation function is built not only by the fields but also the operators Witten (2001). Then, the vacuum expectation value of the dual operator with conformal dimension $d - \Delta$ corresponds to $\phi_{(0)}$ and its current related to $\phi_{(1)}$. Since the on-shell action is identified with the generating functional of connected correlators of the dual CFT, say Γ, the addition of the new term modifies the dual theory as

$$\Gamma[\phi_{(0)}] \to \Gamma[J] - \int d^d x \sqrt{g_{(0)}} J \phi_{(0)} \ , \tag{A3.11}$$

where J is the current which depends on $\phi_{(0)}$ and $\hat{\pi}_\phi$ and is fixed by boundary condition in the string theory side. Following Ref. Anabalon et al. (2016), we have encoded the deformations of the boundary theory in $W\left(\phi_{(0)}\right)$, which must be fixed such that the variational principle is consistent with the corresponding boundary conditions. Then, the boundary CFT is deformed and the boundary conditions impose Witten (2001)

$$J = \frac{dW\left(\phi_{(0)}\right)}{d\phi_{(0)}} \ . \tag{A3.12}$$

Using Neumann or mixed boundary conditions corresponds to modifying the conformal vacuum at the boundary. In the case of mixed boundary conditions, the multi-trace deformations could break the conformal invariance as they modify the *n*-point functions (see Ref. Minces and Rivelles (2000)). Then, the deformations could be marginal, relevant, or irrelevant depending on the mass of the bulk scalar field. Moreover, multi-trace deformations have been associated to multi-particle states in the dual gravity theory Aharony et al. (2001).

Bibliography

Aharony, O., Berkooz, M., and Silverstein, E. (2001). Multiple trace operators and nonlocal string theories. JHEP, 08:006.

Anabalon, A., Astefanesei, D., Choque, D., and Martinez, C. (2016). Trace Anomaly and Counterterms in Designer Gravity. JHEP, 03:117.

Anabalon, A., Cisterna, A., and Oliva, J. (2014). Asymptotically locally AdS and flat black holes in Horndeski theory. Phys. Rev. D, 89:084050.

Anastasiou, G., Araya, I. J., Busnego-Barrientos, M., Corral, C., and Merino, N. (2023). Conformal renormalization of scalar-tensor theories. Phys. Rev. D, 107(10):104049.

Anastasiou, G., Araya, I. J., and Olea, R. (2021). Einstein Gravity from Conformal Gravity in 6D. JHEP, 01:134.

Arratia, E., Corral, C., Figueroa, J., and Sanhueza, L. (2021). Hairy Taub-NUT/bolt-AdS solutions in Horndeski theory. Phys. Rev. D, 103(6):064068.

Babichev, E. and Charmousis, C. (2014). Dressing a black hole with a time-dependent Galileon. JHEP, 08:106.

Babichev, E., Charmousis, C., and Lehébel, A. (2016). Black holes and stars in Horndeski theory. Class. Quant. Grav., 33(15):154002.

Balasubramanian, V. and Kraus, P. (1999). A Stress tensor for Anti-de Sitter gravity. Commun. Math. Phys., 208:413–428.

Barrientos, J. and Cisterna, A. (2023). Ehlers transformations as a tool for constructing accelerating NUT black holes. Phys. Rev. D, 108(2):024059.

Breitenlohner, P. and Freedman, D. Z. (1982). Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity. Phys. Lett. B, 115:197–201.

Brihaye, Y., Cisterna, A., and Erices, C. (2016). Boson stars in biscalar extensions of Horndeski gravity. Phys. Rev. D, 93(12):124057.

Brown, A. R., Roberts, D. A., Susskind, L., Swingle, B., and Zhao, Y. (2016). Holographic Complexity Equals Bulk Action? Phys. Rev. Lett., 116(19):191301.

Brown, J. D. and York, Jr., J. W. (1993). Quasilocal energy and conserved charges derived from the gravitational action. Phys. Rev. D, 47:1407–1419.

Caceres, E., Mohan, R., and Nguyen, P. H. (2017). On holographic entanglement entropy of Horndeski black holes. JHEP, 10:145.

Charmousis, C., Copeland, E. J., Padilla, A., and Saffin, P. M. (2012a). General second order scalar-tensor theory, self tuning, and the Fab Four. Phys. Rev. Lett., 108:051101.

Charmousis, C., Copeland, E. J., Padilla, A., and Saffin, P. M. (2012b). Self-tuning and the derivation of a class of scalar-tensor theories. Phys. Rev. D, 85:104040.

Cisterna, A., Delsate, T., Ducobu, L., and Rinaldi, M. (2016). Slowly rotating neutron stars in the nonminimal derivative coupling sector of Horndeski gravity. Phys. Rev. D, 93(8):084046.

Cisterna, A., Delsate, T., and Rinaldi, M. (2015). Neutron stars in general second order scalar-tensor theory: The case of nonminimal derivative coupling. Phys. Rev. D, 92(4):044050.

Cisterna, A., Diaz, F., Mann, R. B., and Oliva, J. (2023). Exploring Accelerating Hairy Black Holes in $2+1$ Dimensions: The Asymptotically Locally Anti-de Sitter Class and its Holography.

Cisterna, A. and Erices, C. (2014). Asymptotically locally AdS and flat black holes in the presence of an electric field in the Horndeski scenario. Phys. Rev. D, 89:084038.

Cisterna, A., Fuenzalida, S., Lagos, M., and Oliva, J. (2018). Homogeneous black strings in Einstein–Gauss–Bonnet with Horndeski hair and beyond. Eur. Phys. J. C, 78(11):982.

Cisterna, A., Hassaine, M., Oliva, J., and Rinaldi, M. (2017). Axionic black branes in the k-essence sector of the Horndeski model. Phys. Rev. D, 96(12):124033.

Cisterna, A., Neira-Gallegos, A., Oliva, J., and Rebolledo-Caceres, S. C. (2021). Plebański-Demiański solutions in quadratic gravity with conformally coupled scalar fields. Phys. Rev. D, 103(6):064050.

de Haro, S. and Petkou, A. C. (2006). Instantons and Conformal Holography. JHEP, 12:076.

de Haro, S., Solodukhin, S. N., and Skenderis, K. (2001). Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence. Commun. Math. Phys., 217:595–622.

Deffayet, C., Deser, S., and Esposito-Farese, G. (2009a). Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress-tensors. Phys. Rev. D, 80:064015.

Deffayet, C., Esposito-Farese, G., and Vikman, A. (2009b). Covariant Galileon. Phys. Rev. D, 79:084003.

Doneva, D. D. and Yazadjiev, S. S. (2018). New Gauss-Bonnet Black Holes with Curvature-Induced Scalarization in Extended Scalar-Tensor Theories. Phys. Rev. Lett., 120(13):131103.

Fan, Z.-Y., Chen, B., and Lu, H. (2016). Criticality in Einstein–Gauss–Bonnet gravity: gravity without graviton. Eur. Phys. J. C, 76(10):542.

Fefferman, C. and Graham, C. R. (1985). Conformal invariants. In Élie Cartan et les mathématiques d'aujourd'hui - Lyon, 25-29 juin 1984, number S131 in Astérisque. Société mathématique de France.

Feng, X.-H. and Liu, H.-S. (2019). Holographic Complexity Growth Rate in Horndeski Theory. Eur. Phys. J. C, 79(1):40.

Filios, G., González, P. A., Kuang, X.-M., Papantonopoulos, E., and Vásquez, Y. (2019). Spontaneous Momentum Dissipation and Coexistence of Phases in Holographic Horndeski Theory. Phys. Rev. D, 99(4):046017.

Gubser, S. S., Klebanov, I. R., and Polyakov, A. M. (1998). Gauge theory correlators from noncritical string theory. Phys. Lett. B, 428:105–114.

Hartnoll, S. A., Herzog, C. P., and Horowitz, G. T. (2008). Building a Holographic Superconductor. Phys. Rev. Lett., 101:031601.

Henneaux, M., Martinez, C., Troncoso, R., and Zanelli, J. (2007). Asymptotic behavior and Hamiltonian analysis of anti-de Sitter gravity coupled to scalar fields. Annals Phys., 322:824–848.

Henningson, M. and Skenderis, K. (1998). The Holographic Weyl anomaly. JHEP, 07:023.

Hui, L. and Nicolis, A. (2013). No-Hair Theorem for the Galileon. Phys. Rev. Lett., 110:241104.

Jiang, W.-J., Liu, H.-S., Lu, H., and Pope, C. N. (2017). DC Conductivities with Momentum Dissipation in Horndeski Theories. JHEP, 07:084.

Klebanov, I. R. and Witten, E. (1999). AdS / CFT correspondence and symmetry breaking. Nucl. Phys. B, 556:89–114.

Kuang, X.-M. and Papantonopoulos, E. (2016). Building a Holographic Superconductor with a Scalar Field Coupled Kinematically to Einstein Tensor. JHEP, 08:161.

Li, Y.-Z. and Lu, H. (2018). a-theorem for Horndeski gravity at the critical point. Phys. Rev. D, 97(12):126008.

Li, Y.-Z., Lu, H., and Zhang, H.-Y. (2019). Scale Invariance vs. Conformal Invariance: Holographic Two-Point Functions in Horndeski Gravity. Eur. Phys. J. C, 79(7):592.

Liu, H.-S. (2018). Violation of Thermal Conductivity Bound in Horndeski Theory. Phys. Rev. D, 98(6):061902.

Lü, H., Pang, Y., and Pope, C. N. (2013). AdS Dyonic Black Hole and its Thermodynamics. JHEP, 11:033.

Lu, H., Pope, C. N., and Wen, Q. (2015). Thermodynamics of AdS Black Holes in Einstein-Scalar Gravity. JHEP, 03:165.

Lu, H. and Vazquez-Poritz, J. F. (2015). Dynamic C metrics in gauged supergravities. Phys. Rev. D, 91(6):064004.

Minces, P. and Rivelles, V. O. (2000). Scalar field theory in the AdS / CFT correspondence revisited. Nucl. Phys. B, 572:651–669.

Miskovic, O. and Olea, R. (2009). Topological regularization and self-duality in four-dimensional anti-de Sitter gravity. Phys. Rev. D, 79:124020.

Nakayama, Y. (2015). Scale invariance vs conformal invariance. Phys. Rept., 569:1–93.

Nicolis, A., Rattazzi, R., and Trincherini, E. (2009). The Galileon as a local modification of gravity. Phys. Rev. D, 79:064036.

Olea, R. (2005). Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes. JHEP, 06:023.

Olea, R. (2007). Regularization of odd-dimensional AdS gravity: Kounterterms. JHEP, 04:073.

Papadimitriou, I. (2007). Multi-Trace Deformations in AdS/CFT: Exploring the Vacuum Structure of the Deformed CFT. JHEP, 05:075.

Papadimitriou, I. and Skenderis, K. (2005). AdS / CFT correspondence and geometry. IRMA Lect. Math. Theor. Phys., 8:73–101.

Penrose, R. and Rindler, W. (2011). Spinors and Space-Time. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge, UK.

Rinaldi, M. (2012). Black holes with non-minimal derivative coupling. Phys. Rev. D, 86:084048.

Skenderis, K. (2002). Lecture notes on holographic renormalization. Class. Quant. Grav., 19:5849–5876.

Skenderis, K. and Solodukhin, S. N. (2000). Quantum effective action from the AdS / CFT correspondence. Phys. Lett. B, 472:316–322.

Stetsko, M. M. (2019). Topological black hole in the theory with nonminimal derivative coupling with power-law Maxwell field and its thermodynamics. Phys. Rev. D, 99(4):044028.

Susskind, L. and Witten, E. (1998). The Holographic bound in anti-de Sitter space.

Witten, E. (1998). Anti-de Sitter space and holography. Adv. Theor. Math. Phys., 2:253–291.

Witten, E. (2001). Multitrace operators, boundary conditions, and AdS / CFT correspondence.