Universidad de Concepción
Facultad de Ciencias Físicas y Matemáticas
Licenciatura en Matemática

Cox Rings of Anticanonical Surfaces

Anillos de Cox de Superficies Anticanónicas

Tesis para optar al grado de Magíster en Matemática

SOFÍA ALMENDRA PÉREZ GARBAYO
2024

Profesor Guía: Michela Artebani
Departamento de Matemática,
Facultad de Ciencias Físicas y Matemáticas
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Comisión Revisora:
Michela Artebani, Universidad de Concepción
Antonio Laface, Universidad de Concepción
Xavier Roulleau, University of Aix-Marseille
Javier Utreras, Universidad de Concepción

Contents

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Introduction

Given an algebraic, projective, irreducible and normal variety X over $\mathbb{C}$ with finitely generated and free divisor class group $\mathrm{Cl}(X)$, its Cox ring $\mathcal{R}(X)$ (introduced by David Cox in 1995 for the case when X is toric) is the algebra

$$\mathcal{R}(X) = \bigoplus_{D \in K} H^0(X, \mathcal{O}_X(D)),$$

where K is a subgroup of $\mathrm{Div}(X)$ that projects isomorphically onto $\mathrm{Cl}(X)$ through the morphism

$$K \to \mathrm{Cl}(X), \qquad K \mapsto [K] = K + \mathrm{PDiv}(X),$$

and, writing $E \geq 0$ for an effective divisor E,

$$H^0(X, \mathcal{O}_X(D)) := \{f \in \mathbb{C}(X)^* : \mathrm{div}(f) + D \geq 0\} \cup \{0\}.$$

This object turns out to be a graded algebra over $\mathrm{Cl}(X)$ related canonically to the variety and the interaction of divisors over it. It is an important invariant that, when is finitely generated, allows us to realise X as a quotient of a "big" open subset of an affine variety by the action of a torus $(\mathbb{C}^*)^r$, just as happens with complex projective space $\mathbb{P}^n$, and generalizes homogenous coordinate rings. Concretely, we have a diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "i"] \arrow[d, "p_X"] & \operatorname{Spec} \mathcal{R}(X) \\ X & \end{tikzcd}$$

where the complement of $\widehat{X}$ is a codimension ≥ 2 closed subset of $\operatorname{Spec} \mathcal{R}(X)$, i is an embedding and p_X is a GIT quotient by a torus $(\mathbb{C}^*)^r$ associated to $\mathrm{Cl}(X)$ (that acts on $\operatorname{Spec} \mathcal{R}(X)$ precisely because $\mathcal{R}(X)$ is a $\mathrm{Cl}(X)$-graded algebra) ([5, Construction 1.6.3.1]).

There are two major open problems regarding Cox rings in literature. The first one is, given a variety X, to decide whether its Cox ring is finitely generated or not. A variety with finitely generated Cox ring is said to be a Mori dream space, and the classification of these spaces has received a lot of attention over the years. Important examples of Mori dream spaces are toric varieties, shown by Cox in 1995 in [12], and log Fano varieties, shown by Birkar, Cascini, Hacon and McKernan in 2010 in [9].

The second one of these problems is finding a presentation for $\mathcal{R}(X)$, given a variety X. A very important work in this regard is in the case X is a toric variety, developed by Cox in 1995 in [12]. Other notable work in the particular case of X being a surface include, for example, when X is del Pezzo, developed by Batyrev and Popov in 2002 in [7]; when X is a Mori dream K3 surface, where there are works by Artebani, Hausen and Laface in 2010 in [3], by Ottem in 2012 in [26] and by Artebani, Correa Deisler and Laface in 2021 in [1]; and when X is an extremal rational jacobian elliptic surface, developed by Artebani, Garbagnati and Laface in 2015 in [2]. Another class of surfaces that has received major attention are weak del Pezzo surfaces. Derenthal, in his doctoral dissertation [14], developed a method to describe generators of the Cox ring and its relations for many different kinds of weak del Pezzo surfaces, that also appears summarized in his article [15] of 2014. His method is based on those by Batyrev and Popov in [7], and was used by Derenthal to prove Manin's conjecture for certain kinds of cubic and quartic surfaces.

Our interest is set on the second problem. We will work in the context of a type of anticanonical surfaces (studied in general by Harbourne in [18] in 1997) called rational elliptic surfaces (classified by Cossec and Dolgačev by means of [16] in 1966 and [11] in 1989). The main reason to consider these surfaces to find presentations of their Cox rings is the control we have over the cohomology of the space of sections of the Riemann-Roch sheaf $\mathcal{O}_X(D)$ and the base locus of the linear system $|D|$ for a nef divisor D. This is strongly related to the degrees $[D]$ of the generators of the Cox ring of X through the following results:

Corollary 1. *(Corollary 2.3.2) Let X be a smooth complex projective variety, E_1, E_2 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2$ such that $\operatorname{div}_{E_1}(f_1) \cap \operatorname{div}_{E_2}(f_2) = \emptyset$. If $D \in \operatorname{Div}(X)$ is such that $h^1(X, D - E_1 - E_2) = 0$, then there is a surjective morphism*

$$H^0(X, D - E_1) \oplus H^0(X, D - E_2) \to H^0(X, D), \quad (g_1, g_2) \mapsto f_1 g_1 + f_2 g_2.$$

Corollary 2. *(Corollary 2.3.3) Let X be a smooth complex projective variety, E_1, E_2, E_3 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2, 3$ such that $\cap_{i=1}^3 \operatorname{div}_{E_i}(f_i) = \emptyset$. If $D \in \operatorname{Div}(X)$ is such that $h^1(X, D - E_i - E_j) = 0$ for all distinct i, j and $h^2(X, D - E_1 - E_2 - E_3) = 0$, then there is a surjective morphism*

$$\bigoplus_{i=1}^{3} H^0(X, D - E_i) \to H^0(X, D), \quad (g_1, g_2, g_3) \mapsto f_1 g_1 + f_2 g_2 + f_3 g_3.$$

We notice that the surjectivity of the function in each case implies that $[D]$ is not a necessary degree of $\mathcal{R}(X)$; that is, a degree with the property that any minimal set of generators of $\mathcal{R}(X)$ has an element of degree $[D]$. In this case, $[D]$ is not necessary because functions in $H^0(X,D)$ can then be written as a polynomial in homogenous elements of other degrees. In the same vein, another result that proves to be very useful is the following:

Lemma 3. *(Lemma 2.4.1) Let X be either a weak del Pezzo surface with $\rho(X) \geq 3$ or an extremal rational elliptic surface of index $m > 1$. If $[D] \notin \mathrm{BNef}(X)$ and D is not ample, then $[D]$ is not a necessary degree for $R(X)$.*

Which can be proven using a similar technique as in the previous Corollaries, and it helps discard many divisors.

Rational elliptic surfaces can be shown to be Halphen surfaces, that is, surfaces obtained by blowing up the nine (possibly infinitely near) base points of multiplicity m of a pencil of plane curves of degree $3m$; where $m > 0$. The number m is called the index of the surface, and behaviour of divisors on a rational elliptic surface is greatly determined by the index.

In this work, first we characterize the divisors on a rational elliptic surface of index $m > 1$ that are nef and not base point free (the case $m = 1$ is the one considered by Artebani, Garbagnati and Laface in [2]):

Proposition 4. *(Proposition 1.5.8) Let X be a relatively minimal rational elliptic surface of index $m > 1$, F be a nonzero section of $H^0(X, -K_X)$ and D a nef divisor on X that is not base point free. Then,*

- *either $D \sim -aK_X + P$ where P is a (-1)-curve and D has exactly one base point on F,*
- *or $D \sim -aK_X$ for some $a > 0$ non-divisible by m and, writing $a = mp + r$ with $0 \leq r < m$, D has only rF as a fixed component.*

and we use this information to bound the possible necessary degrees of the Cox ring of the surface. We also include a description for the case $m = 1$ (which is just a reinterpretation of a result by Artebani, Garbagnati and Laface in [2]) and for the case of X being weak del Pezzo (which is an improvement of a result by Derenthal in [14]):

Theorem 5. *(Theorem 2.4.9) Let X be a nef anticanonical rational surface such that $\kappa(-K_X) \geq 1$. The necessary degrees D of $\mathcal{R}(X)$ must be*

1. *degrees of negative curves,*
2. *elements of $\mathrm{BNef}(X)$,*
3. *ample classes D with $-K_X \cdot D = 1$ of the form $-\alpha K_X + E$, where $2 \leq \alpha < m$, E is the class of a (-1)-curve and X is an elliptic surface of index $m > 1$.*

Once we have our degrees bounded to a finite number (and a computationally manageable one), we use a Magma program [10] to, in specific cases, find the necessary degrees. This allows us, for example, to conclude that a rational elliptic surface that is 2-Halphen of a jacobian rational elliptic surface of type $\tilde{E}_8$ has the following necessary degrees (Example 3.1.1):

$$\left(\begin{array}{ccc|cccccccccccc}
5 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\
-1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\
-4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -2
\end{array}\right)$$

More generally, we can characterize all possible necessary degrees of Cox rings of rational elliptic surfaces of small index. As an example, we have:

Corollary 6. *(Corollary 2.4.13) Let X be a relatively minimal rational elliptic surface of index 2. Then, the necessary degrees of X must be degrees of negative curves or elements of* $\mathrm{BNef}(X)$.

The thesis is organized as follows:

- In Chapter 1 we recall some basic concepts in the theory of projective varieties, specifically about divisors in Section 1.1 and some specific results on complex surfaces in Section 1.2. We continue specializing these results to many classes of surfaces, including anticanonical surfaces in Section 1.3, weak del Pezzo surfaces in Section 1.4 and elliptic surfaces, our main focus, in Section 1.5. During this Chapter, we prove Propositions 1.4.7 and 1.5.8, that are extremely important to control base points of our divisors.

- In Chapter 2.1.1 we introduce our central object, the Cox ring of a variety, and we give some context about it and some useful properties in 2.2. Afterwards, in Section 2.3, we introduce the main tool for our work: Koszul type sequences, that allow us to show many results in Section 2.4; including 2.4.6, our main result. Then, we show how to continue the process of finding necessary degrees in a computational manner in 2.5.

- In Chapter 3 we give applications of our results, in particular Example 3.1.1.

- In Chapter 4 we give an almost complete description of the Cox rings of weak del Pezzo surfaces of maximum Picard number; the only ones that are left out in Derenthal's result in [14].

- In Chapter 5 we describe the Magma functions and programs [10] that we applied to prove our results. These programs are organized in many libraries and can be found here.

Introducción

Dada una variedad algebraica, proyectiva, irreducible y normal X sobre $\mathbb{C}$ con grupo de clases de divisores $\mathrm{Cl}(X)$ finitamente generado y libre, su anillo de Cox $\mathcal{R}(X)$ (introducido por David Cox en 1995 para el caso particular en que X es una variedad tórica) es el álgebra

$$\mathcal{R}(X) = \bigoplus_{D \in K} H^0(X, \mathcal{O}_X(D))$$

donde K es un subgrupo de $\mathrm{Div}(X)$ que se proyecta isomorfo en $\mathrm{Cl}(X)$ a través del homomorfismo

$$K \to \mathrm{Cl}(X), \qquad K \mapsto [K] = K + \mathrm{PDiv}(X),$$

y, si denotamos por $E \geq 0$ a un divisor E efectivo,

$$H^0(X, \mathcal{O}_X(D)) := \{f \in \mathbb{C}(X)^* : \mathrm{div}(f) + D \geq 0\} \cup \{0\}.$$

Este objeto resulta ser un álgebra graduada sobre $\mathrm{Cl}(X)$, que se relaciona de manera canónica a la variedad y a la interacción de los divisores sobre ella. Es un importante invariante que, cuando resulta ser finitamente generado, permite homologar la construcción del espacio proyectivo complejo $\mathbb{P}^n$, realizando X como el cociente de un abierto "grande" de una variedad afín por la acción de un toro $(\mathbb{C}^*)^r$, y generalizando los anillos de coordenadas homogéneas. Concretamente, tenemos un diagrama

$$\begin{tikzcd} \widehat{X} \arrow[r, "i"] \arrow[d, "p_X"] & \operatorname{Spec} \mathcal{R}(X) \\ X & \end{tikzcd}$$

donde $\widehat{X}$ es un cerrado de $\operatorname{Spec} \mathcal{R}(X)$ cuyo complemento es de codimensión ≥ 2, i es un embedding y p_X es un cociente GIT por un toro $(\mathbb{C}^*)^r$ asociado a $\mathrm{Cl}(X)$ (que actúa sobre $\operatorname{Spec} \mathcal{R}(X)$ precisamente porque $\mathcal{R}(X)$ es un álgebra $\mathrm{Cl}(X)$-graduada) ([5, Construcción 1.6.3.1]).

Existen dos grandes problemas abiertos en relación con los anillos de Cox en la literatura. El primero de ellos es, dada la variedad X, determinar si su anillo de Cox es finitamente generado. Una variedad con anillo de Cox finitamente generada se llama un espacio Mori dream, y la clasificación de estos espacios ha recibido bastante atención a lo largo de los años. Importantes ejemplos de variedades que son Mori dream son las variedades tóricas, demostrado por Cox en 1995 en [12], y las variedades log Fano, demostrado por Birkar, Cascini, Hacon y McKernan en 2010 en [9].

El segundo de estos problemas abiertos es encontrar presentaciones para $\mathcal{R}(X)$ al fijar una variedad X. Un trabajo muy importante en esta dirección fue una descripción para el caso en que X es una variedad tórica, desarrollada por Cox en 1995 en [12]. Otros trabajos notables en el caso particular en que X es una superficie incluyen descripciones, por ejemplo, en el caso de que X sea una superficie de del Pezzo, desarrollada por Batyrev y Popov en 2002 en [7], en el caso que X es una superficie K3 y Mori dream; donde hay trabajos de Artebani, Hausen y Laface en 2010 en [3], de Ottem en 2012 en [26], y de Artebani, Correa Deisler y Laface en 2021 en [1]; y en el caso que X es una superficie racional elíptica extremal, desarrollada por Artebani, Garbagnati y Laface en [2]. Otra clase de superficies que ha recibido atención a lo largo de los años es la de superficies de del Pezzo generalizadas. Derenthal, en su disertación doctoral [14] en 2006, desarrolló un método para describir los generadores del anillo de Cox y sus relaciones para varios tipos diferentes de superficies de del Pezzo generalizadas, que también aparece resumido en su artículo [15] de 2014. Este método se basa en los de Batyrev y Popov de [7], y fue utilizado por Derenthal para probar la Conjetura de Manin para ciertos tipos de superficies cúbicas y cuárticas.

Nuestro interés está en el segundo problema. Trabajamos en el contexto de las superficies anticanónicas (estudiadas en general por Harbourne en [18] en 1997) llamadas superficies racionales elípticas (clasificadas por Cossec y Dolgačev por medio de [16] en 1966 y [11] en 1989). La idea central para considerar estas superficies a la hora de encontrar presentaciones para sus anillos de Cox es el control que se tiene sobre la cohomología del espacio de secciones del haz de Riemann-Roch $\mathcal{O}_X(D)$ y del base locus del sistema lineal $|D|$ para un divisor D nef; estos están fuertemente relacionados con los grados $[D]$ de los generadores del anillo de Cox de X a través de los siguientes resultados:

Corolario 1. *(Corolario 2.3.2) Sea X una variedad proyectiva suave sobre $\mathbb{C}$, E_1, E_2 divisores efectivos sobre X con $E_1 \cap E_2 = \emptyset$ y $s_i \in H^0(X, \mathcal{O}_X(E_i))$ con $\operatorname{div}(s_i) = E_i$ para $i = 1, 2$. Si $D \in \operatorname{Div}(X)$ es tal que $h^1(X, \mathcal{O}_X(D - E_1 - E_2)) = 0$, entonces existe un morfismo sobreyectivo*

$$H^0(X, D - E_1) \oplus H^0(X, D - E_2) \to H^0(X, D), \quad (g_1, g_2) \mapsto f_1 g_1 + f_2 g_2.$$

Corolario 2. *(Corolario 2.3.3) Sea X una variedad proyectiva suave sobre $\mathbb{C}$, E_1, E_2, E_3 divisores efectivos sobre X con $E_1 \cap E_2 \cap E_3 = \emptyset$ y $s_i \in H^0(X, \mathcal{O}_X(E_i))$ con $\operatorname{div}(s_i) = E_i$ para $i = 1, 2, 3$. Si $D \in \operatorname{Div}(X)$ entonces el morfismo*

$$\bigoplus_{i=1}^{3} H^0(X, D - E_i) \to H^0(X, D), \quad (g_1, g_2, g_3) \mapsto f_1 g_1 + f_2 g_2 + f_3 g_3.$$

es sobreyectivo si $h^1(X, \mathcal{O}_X(D-E_i-E_j)) = 0$ *para todo* $i \neq j \in \{1,2,3\}$ *y* $h^2(X, \mathcal{O}_X(D-E_1-E_2-E_3)) = 0$.

Notemos que la sobreyectividad de la función en cada caso implica que $[D]$ no es un grado necesario; esto es, un grado con la propiedad que cualquier conjunto minimal de generadores de $\mathcal{R}(X)$ tiene un elemento de grado $[D]$. En este caso, $[D]$ no es necesario porque las funciones en $H^0(X, D)$ se pueden entonces escribir como un polinomio en elementos homogéneos de otros grados. En la misma dirección, otro resultado que muestra ser muy útil es el siguiente:

Lema 3. *(Lema 2.4.1) Sea* X *o una superficie de del Pezzo generalizada con* $\rho(X) \geq 3$ *o una superficie elíptica relativamente minimal de índice* $m > 1$*. Si* $[D] \notin \mathrm{BNef}(X)$ *y* D *no es amplio, entonces,* $[D]$ *no es un grado necesario de* $\mathcal{R}(X)$.

que se puede mostrar usando una técnica similar a la de los corolarios anteriores, y ayuda a descartar muchos divisores.

Se puede mostrar que las superficies elípticas racionales son superficies de Halphen, esto es, superficies obtenidas explotando los nueve (potencialmente infinitamente cercanos) puntos base de multiplicidad m de un pincel de curvas planas de grado $3m$, donde $m > 0$. El número m se llama el índice de la superficie, y el comportamiento de los divisores en una superficie elíptica racional está determinado en gran medida por su índice.

En este trabajo, primero caracterizamos los divisores de una superficie elíptica racional de índice $m > 1$ que son nef y libres de puntos base (el caso $m = 1$ es el considerado por Artebani, Garbagnati y Laface en [2]):

Proposición 4. *(Proposición 1.5.8) Sea* X *una superficie elíptica racional relativamente minimal de índice* $m > 1$*,* F *una sección no cero de* $H^0(X, -K_X)$ *y* D *un divisor nef en* X *que tiene puntos base. Entonces,*

- *o bien* $D \sim -aK_X + P$ *donde* P *es una* (-1)*-curva y* D *tiene exactamente un punto base en* F*,*
- *o bien* $D \sim -aK_X$ *para algún* $a > 0$ *no divisible por* m *y, escribiendo* $a = mp + r$ *con* $0 \leq r < m$*,* D *tiene solo* rF *como componente fija.*

y utilizamos esta información para acotar los posibles grados necesarios del anillo de Cox de la superficie. También incluimos una descripción para el caso $m = 1$ (que es solo una reinterpretación de un resultado de Artebani, Garbagnati y Laface en [2]) y para el caso en que X es del Pezzo generalizada (que es una mejora de un resultado de Derenthal en [14]):

Teorema 5. *(Teorema 2.4.9) Sea X una superficie nef anticanónica racional con $\kappa(-K_X) \geq 1$. Los grados necesarios de $\mathcal{R}(X)$ deben ser*

- *grados de curvas negativas,*
- *elementos de* $\mathrm{BNef}(X)$*,*
- *clases amplias D con $-K_X \cdot D = 12$ de la forma $-\alpha K_X + E$, donde $2 \leq \alpha < m$ y E la clase de una (-1)-curva si X una superficie elíptica de índice $m > 1$.*

Una vez que los grados están acotados a un número finito (y computacionalmente manejable), ocupamos un programa Magma [10] para, en casos específicos, encontrar los grados necesarios. Esto nos permite, por ejemplo, concluir que los grados necesarios de una superficie elíptica racional que es 2-Halphen de una superficie elíptica racional jacobiana de tipo $\tilde{E}_8$ son (ejemplo 3.1.1):

$$\left(\begin{array}{ccc|cccccccccccc}
5 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\
-1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\
-4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -2
\end{array}\right)$$

Más en general, caracterizamos todos los posibles grados necesarios del anillo de Cox de una superficie elíptica racional de índice pequeño. Como ejemplo, tenemos:

Corolario 6. *(Corolario 2.4.13) Sea X una superficie elíptica racional relativamente minimal Mori dream de índice 2. Entonces, los grados necesarios de $\mathcal{R}(X)$ deben ser grados de curvas negativas o elementos de* $\mathrm{BNef}(X)$*.*

La tesis está ordenada como sigue:

- En el Capítulo 1 recordamos algunos conceptos básicos de la teoría de variedades proyectivas, específicamente de los divisores sobre una en la Sección 1.1 y algunos resultados específicos a superficies complejas en la Sección 1.2. Continuamos especializando estos resultados a varias clases de superficies, que incluyen superficies anticanónicas en la Sección 1.3, superficies de del Pezzo generalizadas en la Sección 1.4, y superficies elípticas, nuestro enfoque principal, en la Sección 1.5. Durante este Capítulo, mostramos las proposiciones 1.4.7 y 1.5.8, que son extremadamente importantes para controlar los puntos base de nuestros divisores.

- En el Capítulo 2.1.1 introducimos el objeto central de estudio, el anillo de Cox de una variedad, y damos un poco de contexto sobre él y sobre algunas propiedades útiles en 2.2. Luego, en la Sección 2.3 introducimos la herramienta central del trabajo: las sucesiones de Koszul, que nos permiten mostrar múltiples resultados en la Sección 2.4; incluyendo el Teorema 2.4.6, nuestro resultado principal. Luego, mostramos cómo continuar el proceso de encontrar grados necesarios de manera computacional en 2.5.

- En el Capítulo 3 damos algunas aplicaciones de nuestros resultados, en particular el Ejemplo 3.1.1.

- En el Capítulo 4 damos una descripción casi completa de los anillos de Cox de las superficies de del Pezzo generalizadas de número de Picard máximo, las únicas que quedan fuera del resultado de Derenthal en [14].

- En el Capítulo 5, describimos las funciones y programas Magma [10] que hemos aplicado para lograr nuestros resultados. Estos programas están organizados en varias librerías, que se pueden encontrar aquí.

1 Projective Varieties

1.1 Divisors

In this section X will always be an algebraic, projective, irreducible and normal variety. We mainly follow [19].

Definition 1.1.1.

- A **prime divisor** of X is an irreducible subvariety of codimension 1 of X.
- The **group of divisors** of X, denoted $\mathrm{Div}(X)$, is the free abelian group generated by all the prime divisors of X. An element of $\mathrm{Div}(X)$ is called a **(Weil) divisor** of X.
- A divisor $\sum_{i=0}^{n} a_i E_i$, where the E_i are prime divisors of X, is **effective** if $a_i \geq 0$ for all i. We write $D \geq 0$ as a shorthand.

Inside $\mathrm{Div}(X)$ we can find divisors that are **principal**, i.e. of the form $\mathrm{div}(f)$, with f a non-zero rational function in X and $\mathrm{div}(f)$ its divisor of zeroes and poles (as in [19, Definition 2.6.1]). Principal divisors form a subgroup of $\mathrm{Div}(X)$, denoted $\mathrm{PDiv}(X)$.

Definition 1.1.2. The **divisor class group** of X is the quotient group

$$\mathrm{Cl}(X) := \mathrm{Div}(X)/\,\mathrm{PDiv}(X).$$

We denote by $[D]$ the class of a Weil divisor D. Two elements in the same class are called **linearly equivalent**.

In the case that X is locally factorial, i.e. all its local rings are factorial, it can be shown [19, Proposition 6.6.11] that every Weil divisor is locally principal, which means that there exists an open covering $\{U_i\}_{i\in I}$ of X such that the divisor $D|_{U_i}$ is principal (in the variety U_i). This is the case, in particular, when X is smooth. A Weil divisor that is locally principal is called a **Cartier divisor**.

Definition 1.1.3. The **Picard group** of X is the set of isomorphism classes of invertible sheaves on X equipped with the tensor product. It is denoted by $\mathrm{Pic}(X)$.

Given a Weil divisor D on X, we can associate to it the sheaf defined by

$$\mathcal{O}_X(D)(U) = \{f \in \mathbb{C}(X)^* : (\mathrm{div}(f) + D)|_U \geq 0)\} \cup \{0\}$$

where $U \subseteq X$ is open and $\mathbb{C}(X)^*$ is the group of non-zero rational functions on X. If D is a Cartier divisor then $\mathcal{O}_X(D)$ is an invertible sheaf [19, Proposition 6.6.13]. This defines a map between the group of Cartier divisors modulo $\mathrm{PDiv}(X)$ and $\mathrm{Pic}(X)$ that turns out to be an isomorphism [19, Proposition 6.6.15]. By the previous remark, if X is locally factorial, then we have an isomorphism $\mathrm{Cl}(X) \simeq \mathrm{Pic}(X)$.

Notation 1.1.4. In what follows, given a sheaf $\mathcal{F}$ on X we will denote by

- $H^i(X, \mathcal{F})$ the **i-th cohomology group** of the sheaf $\mathcal{F}$ and by $h^i(X, \mathcal{F})$ its dimension as a $\mathbb{C}$-vector space. If $D \in \mathrm{Div}(X)$, we will also use the shorthands $H^i(X, D) = H^i(X, \mathcal{O}_X(D))$ and $h^i(X, D) = h^i(X, \mathcal{O}_X(D))$.
- $\chi(\mathcal{F}) = \sum_{i=0}^{\dim(X)} h^i(X, \mathcal{F})$ the **Euler-Poincaré characteristic** of the sheaf $\mathcal{F}$.

Given a Weil divisor D, the map

$$H^0(X, D) \setminus \{0\} \to \mathrm{Div}(X), \quad f \mapsto \mathrm{div}(f) + D$$

defines a bijection between $\mathbb{P}(H^0(X, D))$ (all the nonzero functions in $H^0(X, D)$ modulo multiplying by scalars) and the set of effective Weil divisors that are linearly equivalent to D.

Definition 1.1.5.

- The set of effective Weil divisors linearly equivalent to D is called the **linear system associated to** D and denoted $|D|$.
- The **base locus** of D, denoted $\mathrm{Bs}(D)$, is the set of all points of X that are in the support of every element of $|D|$.
- The linear system $|D|$ is **base point free** if $\mathrm{Bs}(D) = \emptyset$. We will sometimes simply say that D is base point free.
- The **fixed part** of $|D|$ is the greatest divisor L such that $F - L \geq 0$ for every $F \in |D|$. Here, the greatest means that if L' is another divisor with this property, then $L - L' \geq 0$.

If we choose a basis $f_0, ..., f_n$ of $H^0(X, D)$, we can define a map

$$\varphi_{|D|} : X \setminus \mathrm{Bs}(D) \to \mathbb{P}^n, \quad p \mapsto [f_0(p) : ... : f_n(p)]$$

that turns out to be a morphism $\varphi_{|D|} : X \to \mathbb{P}^n$ if $|D|$ is base point free.

Definition 1.1.6. Given a smooth projective variety X over $\mathbb{C}$ one can define the intersection number $D \cdot C$ of a divisor D and a curve C on X (as in [25, Section 1.1.C], or as in [19, Section V.1] for the particular case of surfaces; we also recall this particular case in Theorem 1.2.1). This allows us to define several useful cones:

- The **effective cone** $\mathrm{Eff}(X)$ of X is the cone generated by all classes of effective divisors in $\mathrm{Cl}(X)_{\mathbb{Q}} = \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- A divisor D on X is **nef** if $D \cdot C \geq 0$ for all curves C on X. The **nef cone** $\mathrm{Nef}(X)$ of X is the cone generated by all classes of nef divisors in $\mathrm{Cl}(X)_{\mathbb{Q}}$.
- A divisor D on X is **ample** if there exists a positive integer m such that $\varphi_{|mD|}$ defines an embedding of X in a projective space. The **ample cone** $\mathrm{Ample}(X)$ is the cone generated by all classes of ample divisors in $\mathrm{Cl}(X)_{\mathbb{Q}}$.
- A divisor D on X is **big** if there exists a positive integer m such that the image of the rational map $\varphi_{|mD|}$ has dimension $\dim(X)$.

By the Nakai-Moishezon-Kleiman criterion [25, Theorem 1.2.23] and Kleiman's Theorem [25, Theorem 1.4.23] the nef cone is the closure of the ample cone and the ample cone is clearly contained in the effective cone. Moreover, ample divisors are nef. Thus,

$$\mathrm{Ample}(X) \subseteq \mathrm{Nef}(X) \subseteq \overline{\mathrm{Eff}(X)}.$$

It will be useful to talk about a special generating set of these cones.

Definition 1.1.7. Given a lattice $L \subseteq \mathbb{Z}^d$ and a convex polyhedral cone $C \subseteq L \otimes_{\mathbb{Z}} \mathbb{Q}$, the **Hilbert basis** of C is a minimal generating set of the monoid $C \cap L$.

It can be shown that every convex polyhedral cone has a Hilbert basis, and that it is unique if the cone is **pointed** (that is, that the cone contains no line) [27, Theorem 16.4]. We denote the Hilbert basis of a given cone by preceding the name of the cone by a letter B: for example, the Hilbert basis of the nef cone of a variety X would be written as $\mathrm{BNef}(X)$.

In the case that X is smooth, a very important class in $\mathrm{Cl}(X)$ is the following:

Definition 1.1.8. Given a smooth projective variety X over $\mathbb{C}$ of dimension n, the **canonical sheaf** of X is the invertible sheaf $\omega_X = \wedge^n \Omega_X$, where Ω_X is the sheaf of differentials on X. A **canonical divisor** of X is any Cartier divisor K_X such that $\mathcal{O}_X(K_X) \simeq \omega_X$.

We also recall two classical results about vanishing of cohomology groups.

Theorem 1.1.9. *(Kawamata-Viehweg vanishing Theorem,* [25, Theorem 4.3.1]*) Let X be a smooth projective variety over $\mathbb{C}$ and D a nef and big divisor on X. Then, for all $i > 0$,*

$$H^i(X, K_X + D) = 0.$$

Theorem 1.1.10. *(Serre duality,* [19, Theorem III.7.7]*) Let X be a smooth projective variety of dimension n and D a divisor on X. Then, for every $0 \leq i \leq n$, we have*

$$H^i(X, D) \simeq H^{n-i}(X, K_X - D).$$

1.2 Complex surfaces

For our purposes, a **surface** is always going to be a smooth and projective variety of dimension 2 over the complex numbers. In this section, we give some general facts about surfaces that we will need later. We mainly follow [19, Section V.1], but another classical reference for the subject is [8, Chapter I].

Theorem 1.2.1. [19, Theorem V.1,1] *Let X be a surface. There is a unique pairing*

$$\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}, \quad (C, C') \mapsto (C \cdot C')$$

*such that if C, C' are two smooth curves meeting transversally, $(C \cdot C') = \#(C \cap C')$, is symmetric, additive and depends only on linear equivalence classes. It is called the **intersection form** on X.*

Note that this actually defines a symmetric bilinear form on $\mathrm{Cl}(X) \simeq \mathrm{Pic}(X)$. It can be defined as

$$(C \cdot C') = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-C')) + \chi(\mathcal{O}_X(-C - C'))$$

by [8, Theorem 1.4]. By [8, Definition 1.3] and [8, Lemma 1.6], we actually have that

$$(C \cdot C') = h^0(X, \mathcal{O}_{C \cap C'})$$

whenever C, C' are distinct curves on X, and if C is a smooth curve and C' any divisor on X

$$(C \cdot C') = \deg \mathcal{O}_C(C').$$

We also recall a few useful classical Theorems.

Theorem 1.2.2. *(Riemann-Roch,* [19, Theorem V.1.6]*) Let X be a surface and D a divisor in X. Then,*

$$h^0(D) - h^1(D) + h^2(D) = \chi(O_X) + \frac{1}{2} D \cdot (D - K_X).$$

Observation 1.2.3. When X is **rational**, that is, a surface birational to $\mathbb{P}^2$, we have $\chi(O_X) = 1$. As such, we get

$$h^0(D) - h^1(D) + h^2(D) = 1 + \frac{1}{2} D \cdot (D - K_X).$$

Theorem 1.2.4. *(Genus formula,* [8, Proposition I.15]*) Let X be a surface and C a smooth curve on X. We have*

$$g(C) = 1 + \frac{1}{2} C \cdot (C + K_X),$$

*where $g(C) = h^1(C, \mathcal{O}_C)$ is the **genus** of C.*

Lemma 1.2.5. *(Zariski's Lemma,* [6, Lemma III.8.3] *Let X be a surface, $\pi : X \to \mathbb{P}^1$ be a fibration and $F = \sum n_i C_i$ a fiber. If $D = \sum m_i C_i$ and $D^2 = 0$, then a multiple of D has the form aF for some positive integer a.*

1.3 Anticanonical surfaces

Definition 1.3.1. A **rational anticanonical surface** is a rational surface X whose anticanonical divisor $-K_X$ is effective.

The behaviour of nef divisors in these kinds of surfaces is mainly summarized in the following Theorem.

Theorem 1.3.2. *(Harbourne,* [18, Theorem 3.1]*) Let X be a rational anticanonical surface, F a nef divisor class in X, N the class of the fixed part of $|F|$, $H = F - N$ and D a nonzero section of $H^0(-K_X)$.*

(a) If $-K_X \cdot F \geq 2$, then $h^1(X, F) = 0$ and F is base point free.

(b) If $-K_X \cdot F = 1$, then $h^1(X, F) = 0$. If $N = 0$, F has a unique base point on D. Else, $H = rC$ and $N = N_1 + ... + N_t$, where $C \in K_X^{\perp}$ is a class with $h^1(X, C) = 1$ whose general section is reduced and irreducible and $h^1(X, H) = r$ with $r > 1$ only if $C^2 = 0$, N_i is a smooth rational curve for every i, $N_i^2 = -2$ and $N_i \cdot N_{i+1} = 1$ for $i < t$, $N_t^2 = -1$, $N_i \cdot N_j = 0$ for $j > i + 1$, $C \cdot N_1 = 1$ and $C \cdot N_i = 0$ for $i > 1$.

(c) *If $-K_X \cdot F = 0$, then either $N = 0$ (in which case $F \otimes O_D$ is trivial and either $F^2 > 0$ and $h^1(X, F) = 1$ or $F = rC$ and $h^1(X, F) = r$, where $r > 0$ and C is a class of self-intersection 0 whose general section is reduced and irreducible), or N is a smooth rational curve with $N^2 = -2$ (in which case $h^1(X, F) = 1$, $N \otimes O_D$ is trivial and $H = rC$ where $r > 1$ and C is reduced and irreducible with $C^2 = 0$, $C \cdot N = 1$ and $C \otimes O_D$ is trivial), or $N + K_X$ is effective.*

We will be mainly interested in **nef rational anticanonical surfaces**, that is, rational anticanonical surfaces with $-K_X$ nef (which implies it is effective by the Riemann-Roch Theorem (see Theorem 1.2.2)). Two special types of nef anticanonical surfaces are **weak del Pezzo surfaces** and **elliptic surfaces**, and we will discuss them in the following sections.

Lemma 1.3.3. *Let X be a nef rational anticanonical surface. Then,* $\mathrm{Nef}(X) \subseteq \mathrm{Eff}(X)$.

Proof. By the Riemann-Roch formula (see Theorem 1.2.2), for any nef divisor N we have

$$h^0(N) - h^1(N) = 1 + \frac{1}{2}(N^2 - K_X \cdot N).$$

Since $-K_X$ is effective, $-K_X \cdot N \geq 0$; and as such $N^2 - K_X \cdot N \geq 0$. This implies $h^0(N) - h^1(N) \geq 1$; this is, $h^0(N) \geq 1$. □

1.4 Weak del Pezzo surfaces

Definition 1.4.1.

- A **del Pezzo surface** is a surface X with ample anticanonical divisor $-K_X$.
- A **weak del Pezzo surface** is a surface X whose anticanonical divisor $-K_X$ is nef and big.

Del Pezzo surfaces first appeared in literature in [13], where del Pezzo studied them as blow-ups of the projective plane. Later, in [17], du Val was one of the first mathematicians who studied minimal desingularizations of singular surfaces with ample anticanonical class, which turns to be an equivalent definition of a weak del Pezzo surface [5, Theorem 5.2.1.7]. A pretty useful characterization of del Pezzo surfaces is the following:

Theorem 1.4.2. [15, Theorem 8.1.17] *Up to isomorphism, a del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow up of $\mathbb{P}^2$ in $1 \leq r \leq 8$ points, where in each step no blown up point lies on a (-1)-curve of the surface being blown up.*

This last condition is often referred to as the points being in **general position**. It ensures that the only **negative curves** (that is, curves with negative self intersection) on the surface are (-1)-curves: by the genus formula (see Theorem 1.2.4), given a curve C

$$-2 \leq C^2 + C \cdot K_X$$

and $C \cdot K_X < 0$ because $-K_X$ is ample (by the Nakai-Moishezon-Kleiman criterion [25, Theorem 1.2.23]), thus $C^2 = -1$. In the same vein, a characterization of weak del Pezzo surfaces is the following:

Theorem 1.4.3. [15, Theorem 8.1.15] *Up to isomorphism, a weak del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface $\mathbb{F}_2$ or a blow up of $\mathbb{P}^2$ in $1 \leq r \leq 8$ points, where in each step no blown up point lies on a (-2)-curve of the surface being blown up.*

This last condition is often referred to as the points being in **almost general position**. It ensures that the only negative curves on the surface are (-1)-curves and (-2)-curves: by the genus formula (see Theorem 1.2.4), given a curve C

$$-2 \leq C^2 + C \cdot K_X$$

and $C \cdot K_X \leq 0$ because $-K_X$ is nef. We will often be interested in the behaviour of nef divisors on a weak del Pezzo surface.

Proposition 1.4.4. [5, Theorem 5.2.2.4] *Let D be a nef divisor on a weak del Pezzo surface X. Then either $|D|$ is base point free or $\rho(X) = 9$ holds, $D \sim -K_X$, and $\mathrm{Bs}(D)$ consists of exactly one point.*

Proposition 1.4.5. *Let D be a nef divisor on a weak del Pezzo surface X. Then, $h^i(X, D) = 0$ for all $i > 0$.*

Proof. As $-K_X$ is big and nef, $D - K_X$ is also big and nef. Then, by Kawamata-Viehweg Vanishing (see Theorem 1.1.9), we get the result. □

We can also determine the class of the canonical divisor $-K_X$ by looking at the blow down of the surface:

Proposition 1.4.6. *Let X, Y be two nef rational anticanonical surfaces and $\pi_1, \ldots, \pi_n$ blow ups at one point such that we have a diagram*

$$\begin{tikzcd} X \arrow[r, "\pi_1"] \arrow[rr, bend right, "\pi"'] & \ldots \arrow[r, "\pi_n"] & Y. \end{tikzcd}$$

Then,

$$\pi^*(-K_Y) \sim -K_X + E_1 + ... + E_n$$

where each E_i is a generalized (-1)-curve (that is, either a (-1)-curve or a chain of rational curves whose last component is a (-1)-curve and all others are (-2)-curves) disjoint from every other E_j.

Proof. By induction on n: the case $n = 1$ is obvious. Let's assume it holds for $n-1$, that is, there exists a weak del Pezzo surface X' such that we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi_1"] \arrow[rrr, bend right, "\pi"'] & X' \arrow[r, "\pi_2"] \arrow[rr, bend left, "\pi'"] & ... \arrow[r, "\pi_n"] & Y \end{tikzcd}$$

and $\pi'^*(-K_Y) \sim -K_{X'} + E_2 + ... + E_n$, where each E_i is a generalized (-1)-curve that has intersection number zero with every other E_j. We have two cases:

- If the point being blown up by π_1 is not in any E_i, then

$$\pi'^*(-K_Y) \sim \pi_1^*(-K_{X'} + E_2 + ... + E_n) \sim -K_X + E_1 + E_2 ... + E_n,$$

  where E_1 is the exceptional divisor of π_1 and we denote $\pi_1^*(E_i)$ again by E_i for $i \in \{2, ..., n\}$. As such, the result holds.

- Let $E_2, ..., E_r$ be the generalized (-1)-curves in X' that contain the point being blown up by π_1. As X is nef anticanonical, this point can only be in the (-1)-component of each one of them. Let $E_j = C + C_1 + ... + C_m$, where C is this (-1)-curve and C_i are (-2)-curves. We have

$$\pi_1^*(E_j) \sim \pi_1^*(C + C_1 + ... + C_m) \sim (C' + C_1 + ... + C_m) + E_1,$$

  where E_1 is the exceptional divisor of π_1, we denote $\pi_1^*(C_i)$ again by C_i for $i \in \{1, ..., n\}$ and $C' = C - E_1$ is a (-2)-curve. As such, every E_j is a generalized (-1)-curve; and it is easily checked that they have intersection number zero with each other. As such, the result holds.

□

A natural way in which geometry of weak del Pezzo surfaces influences that of nef rational anticanonical surfaces is the following:

Proposition 1.4.7. *Let X be a nef rational anticanonical surface with polyhedral effective cone, $\rho(X) \geq 3$ and $D \in \mathrm{BNef}(X)$. Then,*

- *either* $D \sim -K_X$,
- *or* D *is pullback of a line in* $\mathbb{P}^2$,
- *or* D *is pullback of the class* $2F + E$ *in* $\mathbb{F}_2$, *where* F *is the class of a fiber and* E *is the unique* (-2)*-curve,*
- *or* D *is a conic bundle (that is, a divisor* D *with* $D \cdot -K_X = 2$ *and* $D^2 = 0$*),*
- *or* $D \sim \pi^*(-K_{X'})$*, where* X' *is a weak del Pezzo surface and* $\pi : X \to X'$ *is a birational morphism.*

Proof. Let X be a nef rational anticanonical surface and $D \in \mathrm{BNef}(X)$. If $D \cdot E \geq 1$ for every (-1)-curve E, then $D + K_X$ is nef because it has non-negative intersection with every negative curve (because in this case negative curves generate $\mathrm{Eff}(X)$ by [4, Proposition 1.1]). It follows that $D = (D + K_X) + (-K_X)$, which implies that $D \sim -K_X$. Also, if $D^2 = 0$, by the genus formula (see Theorem 1.2.4), we have

$$2g - 2 = D^2 + D \cdot K_X = D \cdot K_X \leq 0$$

and so either $D \cdot -K_X = 0$ and then $D \sim -K_X$ by Lemma 1.2.5, or $D \cdot -K_X = 2$ and then D is a conic bundle. As such, we focus on those divisors D with $D^2 > 0$ and such that there exist (-1)-curves E_i with $D \cdot E_i = 0$. In this case, contracting these curves one by one, we have a diagram

$$\begin{array}{ccccccc} X & \longrightarrow & X^1 & \longrightarrow & \cdots & \longrightarrow & X^r \\ \\ [D] & \longmapsto & [D^{(1)}] & \longmapsto & \cdots & \longmapsto & [D^{(r)}] \end{array}$$

where $D^{(r)}$ has positive intersection with every (-1)-curve of X^r. Note that each one of the $D^{(i)}$ is in $\mathrm{BNef}(X^i)$, and since X^r is a weak del Pezzo surface (it is a blow up of $\mathbb{P}^2$ in less than 9 points whose negative curves are only (-1) or (-2)-curves, so we can apply Theorem 1.4.3), then $D^{(r)} \sim -K_{X^r}$ for the same reason as above (weak del Pezzo surfaces are nef rational anticanonical surfaces) unless $\rho(X^r) \leq 2$. In this case, we have four different surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ blown-up at one point and $\mathbb{F}_2$. Checking all divisors in $\mathrm{BNef}(X^r)$ for all of these, we get that $D^{(r)}$ is either a conic bundle (whose pullback is again a conic bundle), $D^{(r)} \sim H$, the class of a line in $\mathbb{P}^2$, or $D^{(r)} \sim 2F + E$ in $\mathbb{F}_2$, where F is the class of a fiber and E is the unique (-2)-curve. This proves our result. $\square$

We finish this section with a lemma on intersection of divisors in a weak del Pezzo surface.

Lemma 1.4.8. *Let X be a weak del Pezzo surface and N_1, N_2, N_3 be nonzero nef divisors on X. Then, there exist nonzero nef divisors N_1', N_2', N_3' on X such that $N_1'+N_2'+N_3' \sim N_1+N_2+N_3$ and $N_1' \cap N_2' \cap N_3' = \emptyset$ unless $\rho(X) = 9$ and $N_1+N_2+N_3 \sim -3K_X$.*

Proof. If one of the N_i is base point free, we can choose $N_i = N_i'$ for each i. Else, by Proposition 1.4.4, $\rho(X) = 9$ and $N_1+N_2+N_3 \sim -3K_X$. □

1.5 Elliptic surfaces

Definition 1.5.1. An **elliptic surface** is a projective surface X over $\mathbb{C}$ that admits an **elliptic fibration** $\pi : X \to \mathbb{P}^1$; that is, a morphism whose general fiber is a smooth curve of genus one.

Elliptic surfaces have been a classic subject of study for many years. We will recall some basic facts about them, mainly following [11, Chapter V].

Definition 1.5.2. An elliptic fibration is called **relatively minimal** if there is no (-1)-curve contained in its fibers. It is called **jacobian** if it admits a section. It is called **extremal** if it is jacobian and it admits only a finite number of sections.

Later, we will be interested in surfaces with non-jacobian relatively minimal elliptic fibrations.

Definition 1.5.3. A **Halphen pencil of index** m is a pencil of plane curves of degree $3m$ with nine (possibly infinitely near) base points of multiplicity m. A **Halphen surface** is the blow up of the nine points of a Halphen pencil.

Many examples of Halphen surfaces with different properties can be found in [30, Section 7], a work that was one of the first giving both constructive and exhaustive descriptions of these. Note that this definition implies that in a Halphen surface X, $-K_X$ is nef and effective and $(-K_X)^2 = 0$. This last property implies that $-K_X$ is not big, by [25, Section 2.2.16].

Proposition 1.5.4. *(*[11, Theorem 5.6.1]*) A Halphen surface is a relatively minimal rational elliptic surface. Conversely, to every rational elliptic surface X can be associated a Halphen pencil such that X is the Halphen surface associated to it. In any case, the fibration is given by the morphism $\varphi_{|-mK_X|}$, where m is the Halphen index of the fibration.*

This means that a lot of the information we will need about the rational elliptic surface we study is codified in the index m and the behaviour of the divisor $-mK_X$. We will now call m the **Halphen index** or the **index** of the surface. We remark that every divisor of the form $-aK_X$ with $a < m$ in one of these surfaces is rigid, and $-mK_X$ is base point free. Along this thesis we will be very interested, specifically, in surfaces of index 2.

Proposition 1.5.5. *Let X be a relatively minimal rational elliptic surface and F a nef divisor class in X.*

- *If $-K_X \cdot F \geq 1$, then $h^1(X, F) = 0$.*
- *If $-K_X \cdot F = 0$, then $F \sim -\alpha K_X$ for some $\alpha > 0$.*

Proof. The first part is straightforward from Theorem 1.3.2. The second part is because if $-K_X \cdot F = 0$, then F is contained in the fibers by Lemma 1.2.5; and the only nef divisor contained in the fibers has to be a multiple of $-K_X$. □

Lemma 1.5.6. *Let X be a rational elliptic surface of index $m > 1$. Then, $h^1(-\alpha K_X) = 0$ if and only if $\alpha < m$.*

Proof. By the Riemann-Roch Theorem (see Theorem 1.2.2) and Serre duality (see Theorem 1.1.10) we have

$$h^0(-\alpha K_X) - h^1(-\alpha K_X) = 1$$

and $h^0(-\alpha K_X) = 1$ if and only if $\alpha < m$ because $-mK_X$ is the first non-rigid multiple of $-K_X$. □

Lemma 1.5.7. *Let X be a rational elliptic surface of index $m > 1$, $\langle f \rangle = H^0(-K_X)$ and F be the curve associated to f, which we assume smooth. For every $a > 0$, writing $a = mp + r$ with $0 \leq r < m$,*

$$H^0(X, -aK_X) \simeq \operatorname{Sym}^p H^0(X, -mK_X) f^r$$

Proof. We have the short exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-(a-1)K_X) \longrightarrow \mathcal{O}_X(-aK_X) \longrightarrow \mathcal{O}_F(-aK_X|_F) \longrightarrow 0$$

and then, taking the associated long exact sequence in cohomology,

$$0 \longrightarrow H^0(X, -(a-1)K_X) \longrightarrow H^0(X, -aK_X) \longrightarrow H^0(F, -aK_X|_F).$$

The divisor $-K_X|_F$ is a m-torsion divisor of degree zero in F, and so if a is not a multiple of m, then $h^0(F, -aK_X|_F) = 0$. Otherwise, $h^0(F, -aK_X|_F) = 1$. This implies that $h^0(X, -aK_X) = p+1$ (it goes up by 1 each time a hits a multiple of m). Then, the result follows because $\operatorname{Sym}^p H^0(X, -mK_X) f^r \subseteq H^0(X, -aK_X)$ and $\dim(\operatorname{Sym}^p H^0(X, -mK_X) f^r) = p+1$. □

Proposition 1.5.8. *Let X be a relatively minimal of index $m > 1$, F be a nonzero section of $H^0(-K_X)$ and D a nef divisor on X that is not base point free. Then,*

- *either $D \sim -aK_X + P$ where P is a (-1)-curve, $a > 0$ and D has exactly one base point in $F \cap P$,*
- *or $D \sim -aK_X$ for some $a > 0$ non-divisible by m and, writing $a = mp + r$ with $0 \leq r < m$, D has only rF as a fixed component.*

Proof. By Theorem 1.3.2, the only nef divisors D that can have base points are those with $D \cdot (-K_X) \leq 1$. We treat both cases:

- If $D \cdot (-K_X) = 1$, by the genus formula (see Theorem 1.2.4), D^2 is odd. Let $D^2 = 2s + 1$ and $R = D + (s+1)K_X$. The divisor R is such that $R^2 = -1$ and $R \cdot (-K_X) = 1$, and is effective by the Riemann-Roch Theorem (see Theorem 1.2.2). Let r be the greatest integer such that $R' = R + rK_X$ is effective. Decomposing R' in reduced and irreducible curves, we get

$$R = -rK_X + P + G_1 + \ldots + G_n$$

where $P \cdot (-K_X) = 1$ and $G_1, \ldots, G_n$ are (-2)-curves (this is because if $G_i \cdot -K_X = 0$, then it has to be contained in the fibers by Lemma 1.2.5; and the only curves properly contained in fibers are (-2)-curves because the fibration is relatively minimal). As R is not nef (it has a negative square), there exists a negative curve C with $R \cdot C < 0$. But

$$R \cdot C = (D + (s+1)K_X) \cdot C = D \cdot C + (s+1)K_X \cdot C,$$

which is non-negative if C is a (-2)-curve. This shows that P is a (-1)-curve. Also,

$$0 > R \cdot P = r - 1 + P \cdot (G_1 + \ldots + G_m)$$

and then, as $P \cdot (G_1 + \ldots + G_m) \geq 0$, we have $r = 0$ and $P \cdot (G_1 + \ldots + G_m) = 0$. Finally,

$$-1 = R^2 = (G_1 + \ldots + G_m)^2 + 2P \cdot (G_1 + \ldots + G_m) - 1$$

and then $(G_1 + \ldots + G_m)^2 = 0$, which implies by Lemma 1.2.5 that $(G_1 + \ldots + G_m) \sim -r'K_X$. As such, $D = -aK_X + P$. The only curves that can be in the base locus of D are F and P: we treat both cases.

  - We note that P is in $\mathrm{Bs}(D)$ if and only if $h^0(-aK_X) = h^0(D) = a+1$ by the Riemann-Roch Theorem (see Theorem 1.2.2) and Theorem 1.3.2. Since $H^0(-aK_X) \simeq \mathrm{Sym}^q(-mK_X)f^s$ where $a = qm+s$ with $0 \leq s < m$ by Lemma 1.5.7, we have $h^0(-aK_X) = q+1$; which is equal to $a+1$ if and only if the index of X is 1. This is a contradiction, and so P is not a fixed component of D.
  - We note that, if F is in $\mathrm{Bs}(D)$, $H = D - F = rC$ for a class $C \in K_X^{\perp}$ by Theorem 1.3.2. Nonetheless, $D - F \cdot K_X = -aK_X + R + F \cdot K_X$=1, and as such F cannot be in $\mathrm{Bs}(D)$.

  From these two points, D has no fixed components. This implies by Theorem 1.3.2 that D has exactly one fixed point on F.

- If $D \cdot (-K_X) = 0$, D is contained in the fibers, so $D = -aK_X + G_1 + ... + G_m$ where each G_i is a (-2)-curve. We have,

$$0 \leq D^2 = (G_1 + ... + G_m)^2 \leq 0,$$

  and then, by Lemma 1.2.5, $(G_1 + ... + G_m) \sim -r'K_X$. As such, $D \sim -a'K_X$. The only curve that can be in the base locus of D is F, so if $a' > 1$ then D has base locus if and only if $h^0(-(a'-1)K_X) = h^0(-a'K_X)$; and so if we write $a' = q'm+s'$ and $a'-1 = q''m+s''$ with $0 \leq s, s' < r$, by Lemma 1.5.7,

$$\dim(\mathrm{Sym}^{q'}(-rK_X)) = \dim(\mathrm{Sym}^{q''}(-rK_X)).$$

  This happens if and only if a' is not a multiple of m, and in this case the curve in $\mathrm{Bs}(D)$ is F with multiplicity s'. Also, if $a' = 1$ then $h^0(-K_X) = 1$ by the hypothesis on m, and so it has F in its base locus. This finishes the proof.

□

We finish this section with a Lemma on intersection of nef divisors.

Lemma 1.5.9. *Let X be a relatively minimal rational elliptic surface of index $m > 1$ and N_1, N_2, N_3 be nef nonzero divisors on X. There exist nef nonzero divisors N_1', N_2', N_3' on X such that $N_1 + N_2 + N_3 \sim N_1' + N_2' + N_3'$ and $N_1' \cap N_2' \cap N_3' = \emptyset$ unless*

- *$N_1 + N_2 + N_3 \sim -3K_X + P + Q$, or*
- *$N_1 + N_2 + N_3 \sim -aK_X + P$, where $a \leq m+1$, or*
- *$N_1 + N_2 + N_3 \sim -aK_X + D$, where $a \leq m$ and D is a nef and base point free divisor, or*
- *$N_1 + N_2 + N_3 \sim -aK_X$, where $a \leq m+1$;*

where P, Q are (-1)-curves and, in the first item, such that $-K_X + P$ and $-K_X + Q$ have a base point in common.

Proof. We study many cases: we always find three divisors that can be taken (inside the same class) to be disjoint, and then we check nefness by calculating intersections with (-1) and (-2) curves. We will repeatedly use the fact that a particular divisor is base point free by Proposition 1.5.5, and Proposition 1.5.8 to characterize nef divisors which are not base point free.

- If no N_i has fixed components and at least one is base point free, we can take $N_i' \sim N_i$.
- If only one of the N_i has a fixed component, and we have another that is base point free, we can take $N_i' \sim N_i$.
- If no N_i has fixed components and none are base point free, then

$$N_1 \sim -\alpha_1 K_X + E_1, \qquad N_2 \sim -\alpha_2 K_X + E_2, \qquad N_3 \sim -\alpha_3 K_X + E_3$$

  where E_i are (-1)-curves. In this case, we reduce to one of the previous two cases by taking

$$N_1' \sim -(\alpha_1 + \alpha_2 + \alpha_3 - 2)K_X, \qquad N_2' \sim -K_X + E_1 + E_2, \qquad N_3' \sim -K_X + E_3.$$

- If only one of the N_i has a fixed component and all others are not base point free, then we can assume

$$N_1 \sim -\alpha_1 K_X, \qquad N_2 \sim -\alpha_2 K_X + E_2, \qquad N_3 \sim -\alpha_3 K_X + E_3$$

  where $\alpha_i > 0$ and E_i are (-1)-curves. We have three subcases:

  - If $\mathrm{Bs}(N_2) \neq \mathrm{Bs}(N_3)$, we can take $N_i \sim N_i'$.
  - If $\mathrm{Bs}(N_2) = \mathrm{Bs}(N_3)$, we can replace N_1, N_2 with $N_1' = N_1 - K_X$ and $N_2' = N_2 + K_X$ as long as $\sum \alpha_i > 3$. Observe that $\mathrm{Bs}(N_2') \neq \mathrm{Bs}(N_3)$ since $(-K_X)_{|F}$ is not trivial. The case $\alpha_1 = \alpha_2 = \alpha_3 = 1$ gives us our first exception.

- If exactly two of the N_i have a fixed component, let's say $N_1 \sim -\alpha_1 K_X$ and $N_2 \sim -\alpha_2 K_X$, we have two subcases:

  - If N_3 is not base point free then $N_3 \sim -\alpha_3 K_X + E$, where E is a (-1)-curve. In this case, we can take

$$N_1' \sim -(\alpha_1 + \alpha_2 + \alpha_3 - m - 1)K_X + E, \qquad N_2' \sim -K_X, \qquad N_3' \sim -mK_X$$

    every time that $\alpha_1 + \alpha_2 + \alpha_3 \geq m + 2$. The case $\alpha_1 + \alpha_2 + \alpha_3 \leq m + 1$ gives us our second exception.

– If N_3 is base point free, then we will consider it our third exception as long as $\alpha_1 + \alpha_2 \leq m$. If $\alpha_1 + \alpha_2 \geq m+1$, we can take

$$N_1' \sim -mK_X, \qquad N_2' \sim (\alpha_1 + \alpha_2 - m)K_X, \qquad N_3' \sim N_3$$

- If all three N_i have fixed components, then $N_i \sim -\alpha_i K_X$ for all i, which gives us our fourth exception as long as $\alpha_1 + \alpha_2 + \alpha_3 \leq m+1$. If $\alpha_1 + \alpha_2 + \alpha_3 \geq m+2$, we can take

$$N_1' \sim -mK_X, \qquad N_2' \sim (\alpha_1 + \alpha_2 + \alpha_3 - m - 1)K_X, \qquad N_3' \sim -K_X$$

□

Observation 1.5.10. The Lemma is also a bit stronger than this: by the proof of it, every time we have a sum $N_1 + N_2 + N_3$ such that at least two of them are base point free, we can find the N_i' as above.

2 Cox Rings

2.1 Preliminaries

Cox rings were first introduced by Cox in [12] for the particular case of a toric variety, and then in general by Hu and Keel in [22] as a way of generalizing homogeneous coordinate rings to a broader class of varieties. For simplicity, we will define the Cox ring of a variety using some extra assumptions that are not required, but that will always be satisfied in our setting. For the more general version, check [5, Construction 1.4.2.1].

Definition 2.1.1. Let X be a normal projective variety over $\mathbb{C}$ with divisor class group $\mathrm{Cl}(X)$ finitely generated and torsion free. We define the **Cox ring** of X as the $\mathbb{C}$-algebra

$$\mathcal{R}(X) = \bigoplus_{D \in K} H^0(X, D),$$

where K is a subgroup of $\mathrm{Div}(X)$ that projects isomorphically onto $\mathrm{Cl}(X)$ via the quotient map $D \mapsto [D]$.

Observation 2.1.2. The Cox ring $\mathcal{R}(X)$ is actually a $\mathrm{Cl}(X)$-graded algebra: given a **homogenous element** $f \in H^0(X, D)$ of **degree** $[D]$ and another homogenous element $f' \in H^0(X, D')$ of degree $[D']$, their product ff' is another homogenous element of degree $[D] + [D'] = [D + D']$ [5, Section 1.4]. It can also be shown that this definition does not depend on the choice of the subgroup K [5, Construction 1.4.1.1].

Definition 2.1.3. Given an effective divisor E in a variety X as in 2.1.1, there is a unique divisor D in K such that $[D] = [E]$, and a unique up to multiplication by scalars $f \in H^0(X, D)$ such that $D = \mathrm{div}(f) + E$. We call f a **defining section** of E.

Cox rings turn out to be important invariants for simplifying construction of the varieties of which they are "sufficiently simple" in the sense of the following definition

Definition 2.1.4. A variety X as in 2.1.1 is called a **Mori dream space** if $\mathcal{R}(X)$ is finitely generated.

Mori dream spaces are important because they always appear in a diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "i"] \arrow[d, "p_X"] & \operatorname{Spec} \mathcal{R}(X) \\ X & \end{tikzcd}$$

where $\widehat{X}$ is an open subset of $\operatorname{Spec} \mathcal{R}(X)$ whose complement has codimension ≥ 2, i is an embedding and p_X is a GIT quotient by an **algebraic torus** $(\mathbb{C}^*)^r$ associated to $\mathrm{Cl}(X)$ (that acts on $\operatorname{Spec} \mathcal{R}(X)$ precisely because $\mathcal{R}(X)$ is a $\mathrm{Cl}(X)$-graded algebra) [5, Construction 1.6.3.1]. As an example, this is the situation of complex projective space $\mathbb{P}^n$:

$$\begin{tikzcd} \mathbb{C}^{n+1} \setminus \{0\} \arrow[r, "i"] \arrow[d, "/\mathbb{C}^*"] & \mathbb{C}^{n+1} \\ \mathbb{P}^n & \end{tikzcd}$$

and more in general of toric varieties. In fact, the latter are the simplest in the theory: their Cox rings are always finitely generated and are polynomial rings [5, Theorem 2.1.3.2].

An important necessary condition for being a Mori dream space is the following.

Proposition 2.1.5. [3, Proposition 2.1] *Let X be a variety as in 2.1.1 and $\{f_i : i \in I\}$ be a homogenous set of generators for $\mathcal{R}(X)$. Then, the monoid of effective divisors of X is generated by $\{\deg(f_i) : i \in I\}$. In particular, if X is a Mori dream space, then $\mathrm{Eff}(X)$ is polyhedral.*

Definition 2.1.6. Let X be a variety as in 2.1.1. A divisor class $[D]$ is a **necessary degree** for the Cox ring of X if any minimal set of generators of $\mathcal{R}(X)$ has an element of degree $[D]$.

We are interested in the necessary degrees for the Cox rings of surfaces. As such, the following Propositions will be useful.

Proposition 2.1.7. *Let X be a variety as in 2.1.1. Every divisor class in $\mathrm{BEff}(X)$ is a necessary degree.*

Proof. Let $[D] \in \mathrm{BEff}(X)$ and $\{f_i : i \in I\}$ be a minimal generating set for $\mathcal{R}(X)$. Since the monoid of effective divisors of X is generated by $\{\deg(f_i) : i \in I\}$ by Proposition 2.1.5, $[D] = \deg(f_i)$ for some i. □

Corollary 2.1.8. *The class of every integral divisor D with $h^0(X, D) = 1$ is a necessary degree.*

2.2 The Cox ring of a surface

When we work with a surface X, we will use the following particular case of Corollary 2.1.8.

Proposition 2.2.1. *Let X be a projective surface. Given any negative curve (that is, C in X, $[C]$ is a necessary degree.*

Proof. This is straightforward from Corollary 2.1.8. □

Negative curves are, of course, not nef. Nonetheless, every other necessary degree has to be nef:

Proposition 2.2.2. *Let X be a projective surface and D be an effective divisor such that $[D]$ is a necessary degree, but D is not a negative curve. Then $[D] \in \mathrm{Nef}(X)$.*

Proof. If $[D]$ is not nef, then there exists a negative curve C such that $D \cdot C < 0$. This implies that C is in the base locus of D, and as such the multiplication map $H^0(X, D - C) \to H^0(X, D)$ by a nonzero element of $H^0(X, C)$ is surjective. This means that $[D]$ is not necessary unless $D \sim C$. □

This way, we only have to worry about which nef divisors are necessary; which simplifies the problem a fair bit and justifies our focus on this type of divisors during Chapter 1.

Classifying which surfaces are Mori dream is an open problem that has received a lot of attention over the years. One approach to this problem considers the **anticanonical Iitaka dimension** of the surface, that is, given a surface X, the number

$$\kappa(-K_X) = \max\{\dim \varphi_{|-nK_X|}(X) : n \in \mathbb{N}\}.$$

Theorem 2.2.3. *(Testa, Várilly-Alvarado, Velasco [28, Theorem 1]) Let X be a rational surface with $\kappa(-K_X) = 2$. Then, X is Mori dream.*

Theorem 2.2.4. *(Artebani, Laface [4, Theorem 5.3]) Let X be a surface with $q(X) = 0$ and $-K_X$ nef. Then, X is Mori dream if and only if one of the following holds:*

- *$\kappa(-K_X) = 2$ and X is a weak del Pezzo surface,*
- *$\kappa(-K_X) = 1$ and X is an elliptic surface with finitely many negative curves,*
- *X is either a $K3$ surface or an Enriques surface with finite automorphism group (which implies $\kappa(-K_X) = 0$).*

Since we are interested in rational elliptic surfaces, we would like to say a bit more than the previous theorem for this case.

Proposition 2.2.5. *(Artebani, Laface* [4, Proposition 5.1] *Let X be a rational elliptic surface. Then, X is Mori dream if and only if any connected component of the set of (-2)-curves of X defines an extended Dynkin diagram of rank r_i with $\sum r_i = 8$.*

This can be further described by classifying all sublattices of E_8 that are generated by roots and are of rank 8. This allows us to talk of the **type** of a rational elliptic surface by referring to the Dynkin diagrams that appear as connected components of the set of (-2)-curves of X. We give all possible types in the following proposition (as well as some more information that will be of use later).

Proposition 2.2.6. [24, Proposition 2.6] *Let X be an relatively minimal rational elliptic surface of index 2. Then, the number of (-1)-curves of X is given in the following table:*

Type	*Number of (-1)-curves*
E_8	3
D_8	$6, 9$
$E_7 + A_1$	$8, 10$
A_8	15
$E_6 + A_2$	18
$A_7 + A_1$	$24, 30$
$D_5 + A_3$	$28, 32$
$2A_4$	45
$A_5 + A_2 + A_1$	$60, 66$
$D_6 + 2A_1$	$26, 28$
$2D_4$	28
$2A_3 + 2A_1$	$108, 112$
$4A_2$	144

In the case of nef rational anticanonical surfaces, the behaviour splits as in the following proposition:

Proposition 2.2.7. *Let X be a nef rational anticanonical surface. Then, one of the following holds:*

- *X is a weak del Pezzo surface (and then $\kappa(-K_X) = 2$), or*
- *X is rational elliptic (and then $\kappa(-K_X) = 1$), or*
- *X is the blow-up of 9 points in $\mathbb{P}^2$ which do not generate a fibration (and then $\kappa(-K_X) = 0$).*

Proof. We know that the first two classes occur, so assume X is not in one of those. We note that, because $-K_X$ is nef, the only negative curves on X are (-1) and (-2) curves by the genus formula (see Theorem 1.2.4). Blowing up $r \leq 8$ points in such a way always gives a weak del Pezzo surface (by Theorem 1.4.3), so X is the blow-up of at least 9 points. Nonetheless, note that a blow-up of 10 or more points of $\mathbb{P}^2$ has an anticanonical divisor of negative square, which implies it is not nef. □

Observation 2.2.8. Surfaces in the third class are never Mori dream because the fact that $-mK_X$ never gives an elliptic fibration implies $h^0(-mK_X) = 1$ for all $m > 0$, and so $-mK_X$ has base points for any $m > 0$ (moreover, it is rigid). This means $-K_X$ is not **semiample**, and a surface with a divisor which is nef but not semiample cannot be Mori dream, by [3, Corollary 2.6]. In this thesis, we will study the cases with $\kappa(-K_X) \geq 1$.

There are a few classes of surfaces where necessary degrees are known. For example,

Theorem 2.2.9. *(Batyrev, Popov* [7, Theorem 3.2]*) If X is a del Pezzo surface with $3 \leq \rho(X) \leq 9$, then the necessary degrees of its Cox ring $\mathcal{R}(X)$ are either degrees of negative curves, or $[-K_X]$ when $\rho(X) = 9$.*

and a generalization of this,

Theorem 2.2.10. *(Derenthal,* [14, Theorem 6.2]*) If X is a weak del Pezzo surface with $3 \leq \rho(X) \leq 8$, the necessary degrees of its Cox ring $\mathcal{R}(X)$ must be degrees of negative curves, $[-K_X]$ or degrees of sections of the form $\pi_E^*(\alpha)$, where $\pi_E^* : X \to Y$ is the contraction of a (-1)-curve E, Y is another weak del Pezzo surface and $\alpha \in \mathcal{R}(Y)$ does not vanish in $\pi_E(E)$ and is not a defining section for a negative curve.*

This result allows Derenthal to calculate all necessary degrees for weak del Pezzo surfaces of Picard number different from 9. This is done using an inductive method, first calculating necessary degrees for surfaces with lower Picard number and then pulling back sections of these degrees. Also, in another direction, we have the following result:

Theorem 2.2.11. *(Artebani, Garbagnati, Laface, [2, Theorem 2.10]) If X is an extremal rational elliptic surface, the necessary degrees of its Cox ring $\mathcal{R}(X)$ must be degrees of negative curves, $[-K_X]$, conic bundles or nef divisors D with $D^2 = 1$ and $D\cdot(-K_X) = 3$.*

This result actually motivates our work on elliptic surfaces, trying to debilitate the hypothesis of jacobianity.

2.3 Koszul type sequences

In [1], Artebani, Correa Deisler and Laface develop methods for showing that certain degrees of the Cox ring of a variety are not necessary. These are based on the following result on Koszul type sequences:

Theorem 2.3.1. [1, Theorem 2.3.1] *Let X be a smooth complex projective variety, $E_1, ..., E_n$ be effective divisors of X and $f_i \in H^0(X, E_i)$ for each $i = 1, ..., n$ such that $\cap_{i=1}^n(\mathrm{div}_{E_i} f_i) = \emptyset$. Let*

$$\mathbb{K}_0 = \mathcal{O}_X, \quad \mathbb{K}_i = \bigoplus_{1\leq j_1<...<j_i\leq n} \mathcal{O}_X(-E_{j_1} - ... - E_{j_i}) \text{ for } i = 1, ..., n$$

Then there is an exact sequence of sheaves:

$$0 \longrightarrow \mathbb{K}_n \xrightarrow{d_n} \mathbb{K}_{n-1} \xrightarrow{d_{n-1}} ... \longrightarrow \mathbb{K}_1 \xrightarrow{d_1} \mathbb{K}_0 \longrightarrow 0$$

where $d_1(u_j) = f_j u_0$ for $j = 1, ..., n$ and

$$d_i(u_{j_1\cdots j_i}) = \sum_{r=1}^{i}(-1)^{r+1} f_{i_r} u_{j_1\cdots j_{r-1}\hat{j}_r j_{r+1}\cdots j_i} \text{ for } i = 2, ..., n$$

where $u_{j_1\cdots j_i}$ is a generator of $\mathcal{O}_X(-E_{j_1} - ... - E_{j_i})$ as a $\mathcal{O}_X$-module.

In the particular case that we have two or three disjoint divisors, we have the following:

Corollary 2.3.2. *Let X be a smooth complex projective variety, E_1, E_2 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2$ such that $\mathrm{div}_{E_1}(f_1) \cap \mathrm{div}_{E_2}(f_2) = \emptyset$. If*

$D \in \mathrm{Div}(X)$ is such that $h^1(X, D - E_1 - E_2) = 0$, then there is a surjective morphism

$$H^0(X, D - E_1) \oplus H^0(X, D - E_2) \to H^0(X, D), \quad (g_1, g_2) \mapsto f_1 g_1 + f_2 g_2.$$

Proof. After tensoring the exact sequence of 2.3.1 with $\mathcal{O}_X(D)$, we get

$$0 \longrightarrow \mathcal{O}_X(D - E_1 - E_2) \longrightarrow \mathcal{O}_X(D - E_1) \oplus \mathcal{O}_X(D - E_2) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

and then the result follows by taking the associated long exact sequence in cohomology. □

Corollary 2.3.3. *Let X be a smooth complex projective variety, E_1, E_2, E_3 be effective divisors of X and $f_i \in H^0(X, E_i)$, $i = 1, 2, 3$ such that $\cap_{i=1}^{3} \mathrm{div}_{E_i}(f_i) = \emptyset$. If $D \in \mathrm{Div}(X)$ is such that $h^1(X, D - E_i - E_j) = 0$ for all distinct i, j and $h^2(X, D - E_1 - E_2 - E_3) = 0$, then there is a surjective morphism*

$$\bigoplus_{i=1}^{3} H^0(X, D - E_i) \to H^0(X, D), \quad (g_1, g_2, g_3) \mapsto f_1 g_1 + f_2 g_2 + f_3 g_3.$$

Proof. After tensoring the exact sequence of 2.3.1 with $\mathcal{O}_X(D)$, we can split it into

$$0 \longrightarrow \mathcal{O}_X(D - E_1 - E_2 - E_3) \xrightarrow{d_3} \bigoplus_{i<j} \mathcal{O}_X(D - E_i - E_j) \xrightarrow{d_2} \mathrm{Im}(d_2) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Im}(d_2) \xrightarrow{i} \bigoplus_{k=1}^{3} \mathcal{O}_X(D - E_k) \xrightarrow{d_1} \mathcal{O}_X(D) \longrightarrow 0$$

where i is the inclusion morphism. Taking the associated long exact sequences in cohomology we get from the first one that $H^1(X, \mathrm{Im}(d_2)) = 0$, and then from the second one we get

$$\ldots \longrightarrow \bigoplus_{k=1}^{3} H^0(X, D - E_k) \xrightarrow{\phi} H^0(X, D) \xrightarrow{\phi'} H^1(X, \mathrm{Im}(d_2)) \longrightarrow \cdots$$

and so ϕ is surjective and the result follows. □

Observation 2.3.4. The importance of Corollaries 2.3.2 and 2.3.3 radicates in that the surjectivity of each morphism implies that $[D]$ is not a necessary degree, since any element of $H^0(X, D)$ can be written as a polynomial of elements in other degrees.

2.4 Theoretical results to calculate Cox rings

In this section, we develop some tools that allow us to calculate the Cox rings of our surfaces of interest. We prove several results first.

Proposition 2.4.1. *Let X be either a weak del Pezzo surface with $\rho(X) \geq 3$ or an extremal rational elliptic surface of index $m > 1$. If $[D] \notin \mathrm{BNef}(X)$ and D is not ample, then $[D]$ is not a necessary degree for $R(X)$.*

Proof. Let E be a smooth rational curve which is orthogonal to D. Consider the exact sequence

$$0 \longrightarrow H^0(X, D-E) \longrightarrow H^0(X, D) \longrightarrow H^0(E, D_{|E}) = H^0(\mathcal{O}_E) \cong \mathbb{C}$$

By Proposition 1.3.2 nef divisors have no smooth rational curves in their fixed locus. This implies that the sequence is also exact on the right. Since $D \notin \mathrm{BNef}(X)$ then $D \sim D_1 + D_2$, where D_1, D_2 are nef. Thus $H^0(D)$ can be generated as a vector space by $s_E H^0(D-E)$ and $s_1 s_2$, where s_E is a defining section of E and $s_i \in H^0(D_i)$ do not vanish along E. Thus in this case D is not necessary to generate $R(X)$. □

Corollary 2.4.2. *Let X be either a weak del Pezzo surface with $\rho(X) \geq 3$ or an extremal rational elliptic surface of index $m > 1$. Then $-\alpha K_X$ is not a necessary degree for $R(X)$ when $\alpha > 1$.*

Proof. If X is either an elliptic surface or a weak del Pezzo surface containing (-2)-curves, this follows from Proposition 2.4.1. If X is a del Pezzo surface this follows from [5, Theorem 5.2.2.1]. □

Proposition 2.4.3. *Let X be an extremal rational elliptic surface of index $m > 1$. Let $D \sim -\alpha K_X + N$, where α is a positive integer and $N \in \mathrm{BNef}(X)$. If $[D]$ is a necessary degree for $R(X)$ then one of the following holds:*

1. *$D \sim -\alpha K_X + E$ with $\alpha < m$, where E is a (-1)-curve which intersects any (-2)-curve;*
2. *$D \sim -2K_X + E + E'$ where $m = 2$ and E, E' are (-1)-curves such that $(E - E')_{|F} \sim (-K_X)_{|F}$ and $E + E'$ intersects any (-2)-curve.*

Proof. If D is not ample, then we are done by Proposition 2.4.1. We restrict to the case where it is ample, or equivalently when N has positive intersection with any (-2)-curve. We consider several cases according to Proposition 1.4.7.

If N is a conic bundle, since there are no (-2)-curves in its orthogonal, its reducible fibers are the union of two (-1)-curves E, E' intersecting at one point, thus $D \sim -\alpha K_X + E + E'$. We will apply Lemma 2.3.3 to shows that $[D]$ is not necessary in several cases. If $\alpha \geq m+1$ Lemma 2.3.3 can be applied with the divisors

$$N_1 = -K_X, \quad N_2 = -K_X + E, \quad N_3 = -mK_X.$$

Observe that $h^1(D-N_i-N_j) = 0$ for all distinct i,j by Proposition 1.3.2, and $h^2(D-N_1-N_2-N_3) = h^0(K_X-D+N_1+N_2+N_3) = 0$ since the last divisor has negative intersection with $-K_X$. Moreover the divisors N_i can be chosen to have empty intersection since N_1, N_2 intersect at one point and N_3 is base point free. If $\alpha \leq m$ Lemma 2.3.3 can be applied with the divisors

$$A_1 = -(\alpha-1)K_X, \quad A_2 = -K_X + E, \quad A_3 = E',$$

or the analogous divisors A_1, A_2', A_3' with the roles of E and E' exchanged, or the divisors

$$B_1 = -(\alpha-2)K_X, \quad B_2 = -2K_X + E, \quad B_3 = E'$$

when $\alpha > 2$. Observe that, if the restrictions of A_2 and A_3 to F are linearly equivalent, then the same is false for A_2', A_3' unless $m = 2$ and for B_2, B_3 if $\alpha > 2$, since $\tau := (-K_X)_{|F}$ is not trivial. In each case one can verify that $h^1(D - N_i - N_j) = 0$ for all distinct i,j by Proposition 1.3.2 and $h^2(0) = 0$. This implies that the only exception is the case $\alpha = m = 2$ and $D \sim -2K_X + E + E'$, where E, E' are (-1)-curves intersecting in one point and $\tau + p \sim p'$, where $p = F \cap E$ and $p' = F \cap E'$.

If $N \sim \pi^*(2F + E)$, where $\pi : X \to \mathbb{F}_2$ is a birational morphism, F is the class of a fiber and E is the class of the (-2)-curve of $\mathbb{F}_2$, then $N \cdot \pi^*(E) = 0$, contradicting our assumption on N.

Now assume $N = \pi^*(H)$, where $\pi : X \to \mathbb{P}^2$ is a birational morphism and H is the class of a line. Let $\pi = b \circ p$ where b is the blow up over one point with exceptional divisor E. We have that $\pi^*(H) = p^*(b^*(H))$, so we consider the three divisors

$$N_1 = p^*(b^*(H) - E) - (\alpha-1)K_X, \quad N_2 = p^*(E), \quad N_3 = -K_X.$$

Notice that N_1 is nef and $-K_X \cdot N_1 = 2$, so it is base point free and $h^1(N_1) = 0$ by Proposition 1.3.2. Moreover $h^1(N_2) = 0$ because N_2 is a (-1)-curve and $h^1(-K_X) = 0$. Observe that we can take N_1, N_2, N_3 to have empty intersection up to linear equivalence since N_2, N_3 intersect at one point and N_1 is base point free. Thus $[D]$ is not necessary by Lemma 2.3.3.

Finally assume that $N \sim \pi^*(-K_Y)$ where $\pi : X \to Y$ is a birational morphism onto a weak del Pezzo surface Y. Since N has positive intersection with any (-2)-curve, then $N \sim -K_X + E_1 + \cdots + E_n$, where the E_i's are disjoint (-1)-curves.

If $n > 2$ consider the divisors

$$N_1 = E_n, \quad N_2 = -\alpha K_X + E_1 + \cdots + E_{n-1}, \quad N_3 = -K_X.$$

Since $-K_X \cdot N_2 \geq 2$, then N_2 is base point free by Proposition 1.3.2. Since E_n and $-K_X$ intersect at one point, then N_1, N_2, N_3 can be chosen to have empty intersection up to linear equivalence. Moreover $h^1(N_1) = h^1(N_2) = h^1(N_3) = h^2(0) = 0$ by the Riemann-Roch Theorem (see Theorem 1.2.2) and Proposition 1.5.5. Thus $[D]$ is not necessary by Lemma 2.3.3.

If $n = 1$, then $D \sim -(\alpha+1)K_X + E_1$. If $\alpha + 1 \geq m$ and $\alpha + 1 \neq 0 \,(\mathrm{mod}\ m)$, then we can apply Lemma 2.3.2 with $N_1 = -rK_X$ and $N_2 = -mqK_X$, where q, r are quotient and remainder of $\alpha + 1$ mod m. Observe that N_1, N_2 are linearly equivalent to disjoint curves, since $-mqK_X$ is base point free. Moreover $h^1(D - N_1 - N_2) = h^1(E_1) = 0$. On the other hand, If $\alpha + 1 > m$ and $\alpha + 1 = 0 \,(\mathrm{mod}\ m)$, i.e. $D \sim -mqK_X + E_1$ with $q > 1$, then we can apply Lemma 2.3.2 with $N_1 = -m(q-1)K_X$ and $N_2 = -mK_X$. If $\alpha + 1 \leq m$ we obtain an exception.

If $n = 2$, then $D \sim -(\alpha+1)K_X + E_1 + E_2$ and we can apply the same arguments of the case when N is a conic bundle. $\square$

Lemma 2.4.4. *Let X be a weak del Pezzo surface. If $D \sim -\alpha K_X + N$, where α is a positive integer and $[N] \in \mathrm{BNef}(X)$, then $[D]$ is not a necessary degree.*

Proof. The proof is similar to the one of the previous Proposition. If $D \sim -\alpha K_X + E$ and $\alpha \geq 3$ we can apply Lemma 2.3.3 with $N_1 \sim -K_X$, $N_2 = -(\alpha-1)K_X$, $N_3 = E$. Moreover if $D \sim -\alpha K_X + E_1 + E_2$, where E_1, E_2 are (-1)-curves, then we can apply Lemma 2.3.3 with $N_1 \sim -K_X$, $N_2 = E_1$, $N_3 = -(\alpha-1)K_X + E_2$.

We now exclude the case when $\rho(X) = 9$ and $D \sim -2K_X + E$, where E is a (-1)-curve passing through the base point of $|-K_X|$ and has positive intersection with any (-2)-curve. Since $D \cdot E = 1$ and $h^1(-2K_X) = 0$ by Proposition 1.3.2 there is an exact sequence

$$0 \longrightarrow H^0(X, -2K_X) \longrightarrow H^0(X, D) \xrightarrow{r} H^0(E, \mathcal{O}_E(1)) \longrightarrow 0.$$

In particular the vector space $H^0(X, -2K_X)f_E$, where f_E is a generator of $H^0(X, E)$, has codimension 2 in $H^0(X, D)$. Let $f \in H^0(X, -K_X + E)$ not vanishing along E. Consider the composite map

$$H^0(X, -K_X) \xrightarrow{m_f} H^0(D) \xrightarrow{r} H^0(E, \mathcal{O}_E(1)),$$

where m_f is the multiplication by f and r the restriction map to E. The image of this map is one, since the image of the restriction map $H^0(X, -K_X) \to H^0(E, -K_{X|E})$ is one. This shows that $H^0(X,D)$ can be generated as a vector space by $H^0(X,D-E)f_E$, $H^0(X,-K_X)f$ and an element not vanishing at the base point of $|-K_X|$.

Since $-K_X\cdot E=1$ and E passes through the base point of $|-K_X|$, then there is a curve R such that $E+R\sim -K_X$. Observe that R is a generalized (-2)-curve since $-K_X\cdot R=0$. Since D is ample, then E intersects any (-2)-curve of X. This implies that all the (-2)-curves of X are components of R and that R is either irreducible or the union of two (-2)-curves intersecting at one point.

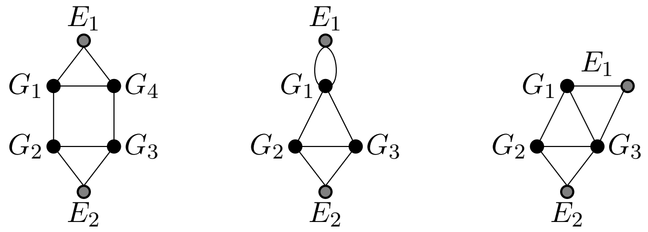
Let E' be a (-1)-curve disjoint from E and consider the divisor $D'=D-E'$. We will show that $D-E'$ is nef. If C is a (-2)-curve, then $D'\cdot C\geq 0$ since $D\cdot C=E\cdot C>0$ and $E'\cdot C\leq 1$ since C is a component of an element in $|-K_X|$. If C is a (-1)-curve then $D'\cdot C=(-2K_X+E-E')\cdot C=2+E\cdot C-E'\cdot C$. Since the intersection of two (-1)-curves of X is at most 3, then $D'\cdot C\geq 0$ unless $E\cdot C=0$ and $E'\cdot C=3$. In this case, after contracting E, the images of E' and C' would be two (-1)-curves on a del Pezzo surface of degree two with intersection number 3, which is not possible.

By Proposition 1.3.2 $D-E'$ is base point free. Thus a non-zero element in $H^0(X,D-E')f_{E'}$, where $f_{E'}$ is a generator of $H^0(X,E')$, is an element of $H^0(X,D)$ not vanishing at the base point of $|-K_X|$. □

Corollary 2.4.5. *Let X be either a weak del Pezzo surface with $\rho(X)\geq 3$ or an extremal rational elliptic surface of index $m>1$. Let $D=D_1+D_2$, with $D_1,D_2\in \mathrm{BNef}(X)$, and such that D is not linearly equivalent to the sum of more than two nef divisors. Then $[D]$ is not a necessary degree for $R(X)$.*

Proof. By Lemma 2.4.1 we can assume D to be ample. Then $D\sim -K_X+(D+K_X)$, where $N:=D+K_X$ is nef and has positive intersection with any (-2)-curve. Moreover $N\in \mathrm{BNef}(X)$. We then have, by Lemma 2.4.3 and Lemma 2.4.4, that $[D]$ is not a necessary degree for $R(X)$ unless $D\sim -2K_X+E_1+E_2$, where E_1,E_2 are (-1)-curves, X is a rational elliptic surface of index $m=2$ and E_1+E_2 intersects all (-2)-curves.

By the classification of singular fibers of extremal rational elliptic surfaces, X can only be of type $2A_3+2A_1$ or $4A_2$ (in every other case, X has a fiber with at least 5 components, and then E_1+E_2 cannot intersect every (-2)-curve). Choosing a fiber with the highest possible amount of components in each type, we have one of the following three cases (up to renaming):



In each of these cases, G_2 and E_1 are disjoint curves. We can then apply Lemma 2.3.2 to these curves: we have that $-2K_X + E_1 + E_2 - G_2 - E_1$ is a generalized (-1)-curve and then $h^1(-2K_X + E_1 + E_2 - G_2 - E_1) = 0$. As such, $[D]$ is not a necessary degree for $R(X)$.

$\square$

All of the above allows us to have the following theorem:

Theorem 2.4.6. *Let X be a relatively minimal rational elliptic surface of index $m > 1$. The necessary degrees D of X must be*

(a) degrees of negative curves,

(b) elements of $\mathrm{BNef}(X)$,

(c) ample classes of the form $D \sim -\alpha K_X + E$, where $2 \leq \alpha < m$ and E is a (-1)-curve,

Proof. Degrees of negative curves are always necessary by Proposition 2.2.1. By Proposition 2.2.2, we only have to focus on nef divisors, so let $D = \sum a_i E_i$, where $[E_i] \in \mathrm{BNef}(X)$, $a_i > 0$ and $\sum a_i \geq 3$. By the hypothesis on D we can choose N_1, N_2, N_3 nef and nonzero such that $D - \sum N_i$ is nef. Also, by Lemma 1.5.9 and Remark 1.5.10, we can take the N_i to be disjoint unless D is of the following types:

1. $D \sim -3K_X + E + E'$, where E, E' are different (-1)-curves such that $-K_X + E$ and $-K_X + E'$ have a common base point,

2. $D \sim -\alpha K_X + E$, where $3 \leq \alpha \leq m + 1$ and E is a (-1)-curve,

3. $D \sim -\alpha K_X + N$, where $2 \leq \alpha \leq m$ and $N \in \mathrm{BNef}(X)$ is base point free,

4. $D \sim -\alpha K_X$, where $3 \leq \alpha$,

Case 1,2,3 are treated in Lemma 2.4.3, and case 4 is treated in Corollary 2.4.2. In the following, we assume that D is not of those types. We observe that $h^2(D - \sum N_i) = h^0(K_X - (D - \sum N_i))$ by Serre duality (see Theorem 1.1.10), which is 0 because $D - \sum N_i - K_X$ is a non-zero nef divisor, which implies effective by Lemma 1.3.3.

Let $A_{ij} = D - N_i - N_j$. By Proposition 1.5.5, $h^1(A_{ij}) = 0$ unless $A_{ij} \sim -\alpha K_X$. So, if $A_{ij} \not\sim -\alpha K_X$, then D is not a necessary degree by Corollary 2.3.3, taking $E_i = N_i$ for each i. If $A_{ij} \sim -\alpha K_X$, then $D \sim N_i + N_j - \alpha K_X$. We observe that $h^1(D + K_X - N_i) = 0$, otherwise,

$$N_j - (\alpha - 1)K_X \sim D + K_X - N_i \sim -sK_X$$

for some positive integer s because the only nef divisors with h^1 nonzero are multiples of $-K_X$ by Propositions 1.5.8 and 1.5.5. As such, $N_j \sim -K_X$ and then we are in one of the exceptions above. This way, we can consider the divisors

$$E_1 = N_i, \qquad E_2 = N_j - (\alpha - 1)K_X, \qquad E_3 = -K_X$$

to apply Corollary 2.3.3, noting that

- $h^2(D - E_1 - E_2 - E_3) = 0$ because $D - E_1 - E_2 - E_3 = 0$,
- $h^1(D - E_1 - E_2) = h^1(-K_X) = 0$,
- $h^1(D - E_1 - E_3) = h^1(N_j - (\alpha - 1)K_X) = 0$, and
- $h^1(D - E_2 - E_3) = h^1(N_i) = 0$ by Propositions 1.5.8 and 1.5.5 because $N_1 \not\sim -K_X$.

As such, D is not a necessary degree. Also, if $D = W_1 + W_2$ is a sum of no more than two elements of $\mathrm{BNef}(X)$, we have shown in Corollary 2.4.5 that D is not necessary. We conclude by noticing we are only left with the exceptions in Lemma 2.4.3.

□

And also an analogue in the weak del Pezzo case:

Theorem 2.4.7. *Let X be a weak del Pezzo surface. The necessary degrees of $\mathcal{R}(X)$ must be degrees of negative curves or elements of* $\mathrm{BNef}(X)$.

Proof. Degrees of negative curves are always necessary by Proposition 2.2.1. Let $D = \sum a_i E_i$, where $[E_i] \in \mathrm{BNef}(X)$, $a_i > 0$ and $\sum a_i \geq 3$. By the hypothesis on D we can choose N_1, N_2, N_3 nef and nonzero such that $D - \sum N_i$ is nef. Also, by Lemma 1.4.8, we can take the N_i to be disjoint unless $D \sim -3K_X$. We discuss the appearance of this degree later, so for now we assume $D \not\sim -3K_X$. Taking $E_i = N_i$, we have

$$h^1(D - E_i - E_j) = h^2(D - E_1 - E_2 - E_3) = 0$$

for each $i \neq j$ because $D - E_i - E_j - K_X$ is a big and nef divisor on X, and so we can apply Kawamata-Viehweg Vanishing (see Theorem 1.1.9) to it; and $D - E_1 - E_2 - E_3$ is an effective divisor on X. As such, D is not a necessary degree. Suppose now $D = W_1 + W_2$, where $W_i \in \mathrm{BNef}(X)$ and D is not a sum of at least three elements of $\mathrm{BNef}(X)$. We have shown in Corollary 2.4.5 that D is not necessary. From these, the only degrees left to treat are the cases where $D \sim -3K_X$ or $D \sim -2K_X$. We have shown that these divisors cannot appear in Corollary 2.4.2, which finishes our proof. $\square$

Observation 2.4.8. Theorem 2.4.7 extends and gives a different proof of Theorem 2.2.10, where Derenthal finds which types of degrees can be necessary to generate the Cox rings of del Pezzo surfaces up to Picard number 8. Derenthal then, in the rest of [14, Section 6], inductively calculates the degrees that are necessary to generate the Cox ring of each specific type of weak del Pezzo surface. Our result then gives a different way to find these degrees without resorting to check those of surfaces with lower Picard number.

Our main Theorem, as a consequence of the above, is the following.

Theorem 2.4.9. *Let X be a nef anticanonical rational surface such that $\kappa(-K_X) \geq 1$. The necessary degrees D of $\mathcal{R}(X)$ must be*

1. *degrees of negative curves,*
2. *elements of* $\mathrm{BNef}(X)$*,*
3. *ample classes D with $-K_X \cdot D = 1$ of the form $-\alpha K_X + E$, where $2 \leq \alpha < m$, E is the class of a (-1)-curve and X is an elliptic surface of index $m > 1$.*

Proof. The weak del Pezzo case follows from Theorem 2.4.7, and the non jacobian elliptic one from Theorem 2.4.6. The jacobian case follows from Theorem 2.2.11. $\square$

We will be very interested in the special case where X is elliptic of small index. As a Corollary of our Lemmas, we can exclude many exceptions:

Lemma 2.4.10. *Let X be a smooth projective surface and let D be an effective divisor without components in its base locus, A, B be two smooth disjoint curves such that $D \cdot A = 0$. Then we have the following diagram:*

$$\begin{tikzcd}
 & & H^0(X, D-B) \arrow[d, "m_B"] & & \\
0 \arrow[r] & H^0(X, D-A) \arrow[r, "m_A"] & H^0(X, D) \arrow[r, "r_A"] & H^0(A, \mathcal{O}_A) \arrow[r, "\simeq"] & \mathbb{C},
\end{tikzcd}$$

where m_A, m_B are multiplications by defining sections of A and B respectively and r_A is the restriction to A. If $h^0(X, D-B) > 0$ and A is not in $Bs(|D-B|)$, then $H^0(X, D)$

is generated by $H^0(X, D-A)$, $H^0(X, D-B)$ and the defining sections of A and B. In particular, D is not necessary to generate $\mathcal{R}(X)$.

Proof. Notice that $D-A$ is effective and r_A is surjective, since otherwise $|D|$ would contain components in its base locus. Since A is not in the base locus of $|D-B|$ then $r_A \circ m_B$ is also surjective. Given $f \in H^0(X, D)$, let $r_A(f) = c \in H^0(A, \mathcal{O}_A)$ and let $g \in H^0(X, D-B)$ such that $r_A \circ m_B(g) = c$, then $f - m_B(g) \in H^0(X, D-A)$, proving the statement. □

Corollary 2.4.11. *Let X be a relatively minimal rational elliptic surface and P a (-1)-curve on X, and suppose X contains a (-2)-curve G which does not intersect P. Then, degrees of the form $-\alpha K_X + P$ are not necessary for any $\alpha \geq 1$. Moreover, if P, Q are two (-1)-curves on X and there exists a (-2)-curve G which does not intersect either one, degrees of the form $-2K_X + P + Q$ are not necessary for any $\alpha \geq 1$.*

Proof. We apply Lemma 2.4.10 with $A = G$ and $B = P$. Notice $(-\alpha K_X + P) \cdot G = 0$, $h^0(-\alpha K_X + P - B) = h^0(-\alpha K_X) > 0$ and G is not in the base locus of $|-\alpha K_X|$ for any α. As such, $-\alpha K_X + P$ is not necessary. For $-\alpha K_X + 2P$ and $-\alpha K_X + P + Q$, the proof is analogous. □

Observation 2.4.12. All the hypotheses of Corollary 2.4.11 are satisfied, for example, anytime X is of index 2 (because it contains a fiber with at least 3 components). Moreover, since X can only have exceptions of the form $-\alpha K_X + E$ when it is of index $m > 2$, the relevant hypothesis of Corollary 2.4.11 is satisfied anytime X has a reducible fiber with at least $m+1$ components.

Corollary 2.4.13. *Let X be a relatively minimal rational elliptic surface of index 2. Then, the necessary degrees of X must be degrees of negative curves or elements of* $\mathrm{BNef}(X)$.

Proof. Straightforward from Theorem 2.4.6 and Remark 2.4.12. □

Proposition 2.4.14. *Let X be a Mori dream relatively minimal rational elliptic surface of index m. Then, $-mK_X$ is not a necessary degree.*

Proof. Since X is Mori dream, Proposition 2.2.5 implies that X has a reducible fiber. This means that we can generate $H^0(-mK_X)$ with an element of $H^0(-K_X)$ and an element defining the reducible fiber of X. □

Following propositions 2.2.5 and 2.2.6, we know all the possible configurations of singular fibers on Mori dream elliptic surfaces. If we consider surfaces of index 3, Remark 2.4.12 turns out to be very strong:

Corollary 2.4.15. *Let X be a Mori dream relatively minimal rational elliptic surface of index* 3*.*

- *If X is not of type $4A_2$, then the necessary degrees of $\mathcal{R}(X)$ must be degrees of negative curves or elements of* $\mathrm{BNef}(X)$*.*
- *If X is of type $4A_2$, then the necessary degrees of $\mathcal{R}(X)$ must be degrees of negative curves, elements of* $\mathrm{BNef}(X)$ *or degrees of the form $-2K_X + E$ where E is a (-1)-curve that intersects every (-2)-curve on X.*

Proof. Follows directly from Theorem 2.4.9, Remark 2.4.12 and Proposition 2.4.14. □

By the same argument as above, and taking into account the multiplicity of components in the fibers, we can produce the following table; which shows the minimal m such that a surface with type on the left column can have exceptions as degrees necessary to generate $\mathcal{R}(X)$.

Type	Minimal m
E_8	30
D_8	14
$E_7 + A_1$	18
A_8	9
$E_6 + A_2$	12
$A_7 + A_1$	8
$D_5 + A_3$	8
$2A_4$	5
$A_5 + A_2 + A_1$	6
$D_6 + 2A_1$	10
$2D_4$	6
$2A_3 + 2A_1$	4
$4A_2$	3

Lemma 2.4.16. *Let X be a relatively minimal rational elliptic surface that contains two disjoint (-2)-curves E_1, E_2 which do not intersect the same negative curve, and such that E_1 and E_2 intersect every other negative curve in no more than one point. If D is an ample divisor on X, then D is not necessary.*

Proof. Notice that the divisor $D - E_1 - E_2$ is nef and $-K_X \cdot (D - E_1 - E_2) = -K_X \cdot D > 0$. This implies that $h^1(D - E_1 - E_2) = 0$ by Proposition 1.5.5, and then the result follows by applying Corollary 2.3.2 to the divisors E_1 and E_2. □

To say if a given degree is necessary, things are more complicated. Nontheless, we can still say something in some cases:

Theorem 2.4.17. *Let X be a smooth projective rational surface such that $-K_X$ is nef. Then the following hold.*

1. *If X is an elliptic surface, $-K_X$ is a necessary degree for $R(X)$ if and only if $m = 1$ and the elliptic fibration has a unique reducible fiber, or $m > 1$ and $F \in |-K_X|$ is irreducible.*
2. *A conic bundle is a necessary degree for $R(X)$ if and only if the associated morphism $\pi : X \to \mathbb{P}^1$ has a unique reducible fiber.*
3. *A class of type $\pi^*(H)$, where $\pi : X \to \mathbb{P}_2$ is a birational morphism, is a necessary degree for $R(X)$ if and only if it contracts all negative curves to one point.*
4. *A class of type $\pi^*(2F + E)$, where $\pi : X \to \mathbb{F}_2$ is a birational morphism, is not a necessary degree for $R(X)$.*
5. *A class of type $\pi^*(-K_Y)$, where $\pi : X \to Y$ is a birational morphism to a weak del Pezzo surface Y and $|-K_X|$ contains irreducible elements, is a necessary degree for $R(X)$ if and only if $m > 1$ and Y is a del Pezzo surface of degree one.*

Proof. The first three items are due to the fact that the class considered is an element of the Hilbert basis of the nef cone. The first item follows from the fact that if X is an extremal elliptic surface, then the only way to write $-K_X$ as a sum of effective divisors is as a sum of (-2)-curves. Thus $-K_X$ is a necessary degree of $R(X)$ if and only if $H^0(-K_X)$ can not be generated by products of defining sections of (-2)-curves. The second item is similar (see, for example, [2, Proposition 2.4]). In the third item, $H^0(\pi^*(H))$ can be generated by defining sections of reducible curves if and only if it blows-up at least two distinct points in $\mathbb{P}^2$. In the fourth item, a class of type $\pi^*(2F+E)$ is not necessary by Lemma 2.4.10 applied to the divisors $B = \pi^*(E)$ and A any (-1)-curve in the exceptional divisor of π. Finally, for the fifth item, we can write $\pi^*(-K_Y) = -K_X+E_1+\cdots+E_n$, where the E_i's are generalized (-1)-curves with $E_i \cdot E_j = 0$ for $i \neq j$. If Y contains a (-2)-curve G, then a class of type $\pi^*(-K_Y)$ is not necessary by Lemma 2.4.10 applied to the divisors $A = \pi^*(G)$ and $B = E_1$, so Y contains no (-2)-curves. We conclude by [7, Theorem 3.2], which states that $-K_Y$ is necessary to generate $R(Y)$ if and only if Y is a del Pezzo surface of degree one. For the converse of this item, observe that if Y is a del Pezzo surface of degree one, then $\pi^*(-K_Y) = -K_X + E$, where E denotes the exceptional divisor over a point p, has a unique reducible fiber since $-K_Y$ has no reducible fibers and there is a unique element of $|-K_Y|$ that passes through p (since $m > 1$). This implies that $\pi^*(-K_Y) = -K_X + E$ is a necessary degree. □

Observation 2.4.18. The hypothesis on the irreducibility of $F \in |-K_X|$ is only needed in the fifth item (we use the fact that G is not in the base locus of $|-K_X|$ to apply Lemma 2.4.10). If we do not assume this and π blows up at least two different points, writing $\pi^*(-K_Y) = -K_X + F_1 \cdots + F_n$, where F_i, F_j are sums of exceptional curves which are disjoint for $i \neq j$, we can apply Lemma 2.4.10 to the divisors $B = F_1$ and $A = F_2$. This implies that a class of type $\pi^*(-K_Y)$ can be necessary only if π is a blow-up over one point.

Observation 2.4.19. If X is a weak del Pezzo surface, we have partial results regarding the necessity of $-K_X$ to generate $R(X)$. For instance: if X is del Pezzo, $-K_X$ is necessary for $R(X)$ if and only if X is of degree one, by [7, Theorem 3.2]; and it is never necessary if X is toric. Observe that by Proposition 2.4.1 $-K_X$ is necessary only if it belongs to the Hilbert basis of the nef cone. If X is *extremal*, that is the lattice generated by classes of (-2)-curves has maximum rank, then by [29, Lemma 3.2] there is an extremal rational elliptic surface that dominates X. By the same arguments of Example 3.3.1, if X is not toric and is dominated by a rational elliptic surface Y whose Cox ring has exactly one relation, then $-K_X$ is necessary for $R(X)$ if and only if $-K_Y$ is necessary for $R(Y)$.

2.5 Computational methods

Given a Mori dream nef rational anticanonical surface X with $\kappa(-K_X) \geq 1$, we construct a list L of all the degrees that are described in Theorem 2.4.6 and we filter it by applying some tests until we arrive at a minimal list of generators. We do this by analysing any $D \in L$ it with the help of a Magma [10] program which can be found in Chapter 5 (which is heavily inspired by that of Artebani, Correa Deisler and Laface in [1, Section 3.3]). Concretely, these are our main steps. We denote by $\mathrm{Neg}(X)$ the set of negative curves of X. Consider the following set:

$$T_1 := \{\{A, B\} : A, B \in \mathrm{Neg}(X) \cup \{-K_X\}, A \cdot B = 0\}$$

We apply the following tests to any $D \in L$: using Test 1, we check whether D has no components in its base locus, $D \cdot A = 0$, $h^0(D - B) > 0$ and A is not contained in $\mathrm{Bs}(|D - B|)$ for some $\{A, B\}$ in T_1. If this holds, by Lemma 2.4.10, D is not necessary and the test returns false. Then, from the list L' of divisors of L that pass the test, we apply Lemma 2.4.16 and Proposition 2.4.17 to decide which ones of them are actually necessary.

3 Examples

3.1 Rational elliptic surfaces of index 2

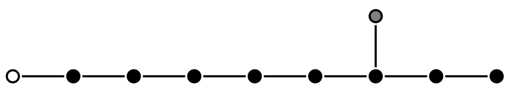
In this section, we give some examples of computation of Cox rings of some rational elliptic surfaces of index 2.

3.1.1 A 2-Halphen rational elliptic surface of type E_8

We will construct, as an example, a 2-Halphen surface. Let us start by considering the pencil of cubics in $\mathbb{P}^2$ given by

$$(x_1^3 + x_0^2 x_2) + t x_2^3 = 0.$$

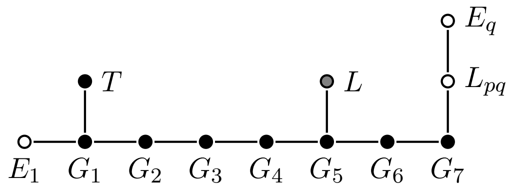
Blowing up the nine infinitely close base points of the pencil, we get a jacobian rational elliptic surface X and a diagram



where the black dots correspond to the (-2)-curves coming from the exceptional divisors of the first 8 blow-ups over the single point p being blown-up; the gray dot corresponds to the (-2)-curve coming from a line passing through p with the same direction as the second blowing up; and the white dot corresponds to the (-1)-curve coming from the final blow-up over p. The black and gray dots form the singular fiber of X, of type II^*. We will take a 2-**Halphen transform** of X as in [5, Proposition 5.1.4.6]: by contracting the (-1)-curve given by the last blow-up over p and blowing up over a point q over a smooth fiber of X such that $q - p$ is of 2-torsion (in the group structure of the elliptic curve), we get a new surface X' as in the following diagram:

$$\begin{tikzcd} X \arrow[dr, "\pi_p"'] & & X' \arrow[dl, "\pi_q"] \\ & Y & \end{tikzcd}$$

As this new surface X' retains the same structure on the singular fiber, we get a diagram



where again the black dots represent (-2)-curves (the seven ones on the bottom are the ones that come from the original exceptional divisors, we call them G_1 through G_7, and the one above is a new one: we call it T), the white dots are (-1)-curves (from the top to the bottom; they are the exceptional divisor over q, we call it E_q, a line from p to q, we call it L_{pq} and the last exceptional divisor over p, we call it E_1) and the gray dot is the original special line through p (we call it L). We know that X' only has three (-1)-curves by Proposition 2.2.6, so we have them all. By writing all these in terms of the basis $\langle H, E_1, ..., E_8, E_q\rangle$ of $\mathrm{Pic}(X')$, where H is a general line and the E_i are the exceptional divisors over p from last to first, we get that

$$G_i = E_{i+1} - E_i, \qquad L = H - E_8 - E_7 - E_6, \qquad L_{pq} = H - E_8 - E_q$$

and, by using the fact that the class of the whole fiber is both $-2K_{X'} \sim 6H - \sum_i^8 2E_i$ and $T + 2G_1 + 3G_2 + 4G_3 + 5G_4 + 6G_5 + 4G_6 + 2G_7 + 4L$ by Kodaira's classification of singular fibers, we get that

$$T \sim 3H - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - 2E_q.$$

As such, we get all the information on the negative curves we need, and we can apply computational methods to find out the necessary degrees of $\mathcal{R}(X')$.

By applying the Magma tests described in Section 2.5, we get that $|L' \setminus \mathrm{Neg}(X)| = 3$ and its elements are:

$$\begin{aligned} C_1 &:= 5H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - 4E_q, \\ C_2 &:= -K_{X'}, \\ C_3 &:= H - E_q. \end{aligned}$$

We study each one of these:

- C_1 is necessary: since $C_1^2 = 1$ and the negative curves orthogonal to it form a connected diagram, it defines a morphism to $\mathbb{P}^2$ that contracts all the negative curves to a single point. This implies that the section of the pullback of a line not passing through this point in $\mathbb{P}^2$ is independent of the other sections of C_1, and as such we need one generator in this degree.

- C_2 is necessary by Lemma 1.5.7.
- C_3 is necessary by Lemma 2.4.17.

As such, the matrix degree of the Cox ring of X' is

$$\left(\begin{array}{ccc|ccccccccccc}
5 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\
-1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & -1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\
-4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -2
\end{array}\right)$$

Observation 3.1.1. By looking at those divisors in $\mathrm{BNef}(X')$ with self intersection one, we get several divisors that represent morphisms to either $\mathbb{P}^2$ (if their intersection with $-K_{X'}$ is 3), or the quadric cone Q (if their intersection with $-K_{X'}$ is 1). Taking those who go down to $\mathbb{P}^2$ and checking for strings of connected negative curves that have intersection 0 with them, we get

- $5H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - 4E_q$: a model of X' as a blow-up in one point, as in [30, Example 7.55],
- $4H - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 - 3E_q$: a model of X' as a blow-up in two points, three times above one and six above the other, as in [30, Example 7.56],
- H: a model of X' as a blow-up in two points, one time above one and eight above the other, as in [30, Example 7.57],
- $2H - E_7 - E_8 - E_q$: a model of X' as a blow-up in two points, two times above one and seven above the other, as in [30, Example 7.58],
- $3H - E_5 - E_6 - E_7 - E_8 - 2E_q$: a model of X' as a blow-up in two points, four times above one and five above the other, as in [30, Example 7.59].

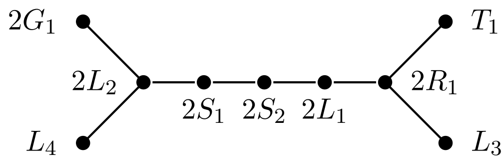
These models end up realising all possible examples of Halphen pencils that give an $\tilde{E}_8$ type fiber, as proven by Zanardini in [30, Theorem 5.15]. By choosing one of these examples and identifying which necessary degrees get blown down to plane curves, we can identify the ideal of relations of $\mathcal{R}(X')$: by [2, Corollary 4.2], it is given by the saturation of the ideal given by the equations of these plane curves. For example, if we choose H as our model, we get that these plane curves are given by the equations

$$x^5 + x^4y + z^3y^2 + x^3yz - xy^3z - y^4z, \qquad x + y, \qquad x,$$
$$z^2x + y^2z - x^3 - x^2z, \qquad y^2z - x^3 - x^2z, \qquad z.$$

We could, with greater computational power, identify a precise description of this ideal. We have been unable to do so in our current situation.

3.1.2 A 2-Halphen rational elliptic surface of type D_8

By [30, Example 7.30], we can compute the Cox ring of a surface of type D_8. We take a smooth cubic C, a flex point p_1 and L_4 its inflection line. Let L_1 be a line through p_1 which is tangent to C at another point p_2, and let L_3 be another line through p_1 which is tangent to C at a different point p_3. Let L_2 be the line joining p_2 and p_3, and p_4 the third intersection point of L_2 and C. The pencil generated by $2L_1 + 2L_2 + L_3 + L_4$ and $2C$ is a 2-Halphen pencil, and we have to blow up three times over p_1, p_2, two times over p_3 and one time over p_4 to get a 2-Halphen rational surface of type D_8. In fact, we get the following (dual) configuration of curves:



We call, from last one to first one, E_1, E_2, E_3 the exceptional divisors over p_1; F_1, F_2, F_3 the exceptional divisors over p_2; G_1, G_2 the exceptional divisors over p_3 and J_1 the exceptional divisor over p_4. By using the basis of $\mathrm{Pic}(X)$ given by $\{H, E_i, F_j, G_k, J_1\}$, we get that the (-2)-curves of X are given by

$$R_i = E_{i+1} - E_i, \qquad S_i = F_{i+1} - F_i, \qquad T_1 = G_2 - G_1,$$
$$L_1 = H - E_3 - F_3 - F_2, \qquad L_2 = H - F_3 - G_2 - J_1,$$
$$L_3 = H - E_3 - G_2 - G_1, \qquad L_2 = H - E_3 - E_2 - E_1.$$

And so we can find all the (-1)-curves on X (for this, we use a Magma program described in Section 5.2) and apply the computational methods we have developed to find the necessary degrees. As such, we get that $|L \setminus \mathrm{Neg}(X)| = 3$ and its elements are

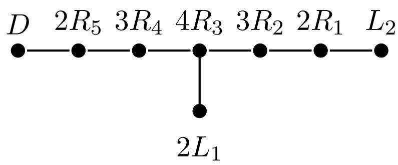
$$C_1 := -K_X,$$
$$C_2 := 4H - 2E_2 - 2E_3 - F_1 - F_2 - F_3 - 2G_2 - J_1,$$
$$C_3 := 2H - F_1 - F_2 - F_3 - J_1.$$

By Lemma 2.4.17, C_2 and C_3 are necessary since they are conic bundles with only one reducible fiber. Moreover, C_1 is necessary by Lemma 1.5.7. As such, the matrix degree of the Cox ring of X is

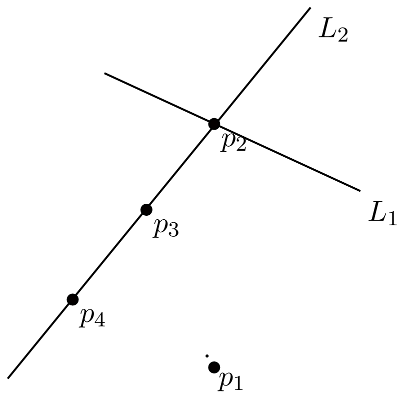
$$\left(\begin{array}{ccc|cccccccccccccccccc}
4 & 2 & -3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 3 & 2 & 3 & 0 & 1 & 4 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\
-2 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -2 & 0 \\
-2 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -2 & 0 \\
-1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 0 \\
-1 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 \\
-2 & -2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & -1 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & -1 & -2 & 0
\end{array}\right).$$

3.1.3 A 2-Halphen rational elliptic surface of type $E_7 + A_1$

Following [30, Example 7.45], we can compute the Cox ring of a surface of type $E_7 + A_1$. We start with a nodal cubic D and call its node p_1. We choose a flex point p_2, denote the respective inflection line by L_1, and we choose a line L_2 through p_2 so that L_2 intersects D at two other points, say p_3 and p_4. Finally, we construct a cubic C through $p_1, ..., p_4$ such that C intersects D with multiplicity 5 at p_2, and intersects L_1 with multiplicity 3 at p_2. The pencil generated by $D + 2L_1 + L2$ and $2C$ is a 2-Halphen pencil, and we have to blow up six times over p_2 and one time over p_1, p_3, p_4 to get a 2-Halphen rational surface of type $E_7 + A_1$. In fact, we get the following (dual) configuration of curves:



We call, from last one to first one, $E_1, E_2, E_3, E_4, E_5, E_6$ the exceptional divisors over p_2; F_1 the exceptional divisor over p_3; G_1 the exceptional divisor over p_4 and H_1 the exceptional divisor over p_1. We need to find the other (-2)-curves that make up the second reducible fiber. Notice that the classes of these must add up to $-2K_X$; moreover, we have the following configuration of points:



Since we need to get a (-2)-curve, the only line that is a candidate is the one through p_2 and p_1. Nonetheless, this cannot be: since this line would have intersection number greater than one with L_1. We switch to conics: it cannot pass through p_2, p_3 and p_4 at the same time, or it would contain L_2. This means we would have to blow it up 3 times over p_2, and then it would have intersection number 3 with L_1. As such, the curves we need are both cubics: and as we are exploding 9 points, it has a node on one of the p_i. We get three options: node on p_1 and not passing through either p_3 or p_4, node on p_3 and not passing through p_4, and node on p_4 and not passing through p_3. A calculation of classes shows that the complement in $-2K_X$ of such a cubic is not irreducible (it contains L_2), and as such we get our two cubics. These are indeed complementary in $-2K_X$. We can then give the classes of all (-2)-curves: by using the basis of $\mathrm{Pic}(X)$ given by $\{H, E_i, F_1, G_1, H_1\}$, we get

$$R_i = E_{i+1} - E_i, \qquad L_1 := H - E_6 - E_5 - E_4, \qquad L_2 := H - E_6 - F_1 - G_1,$$

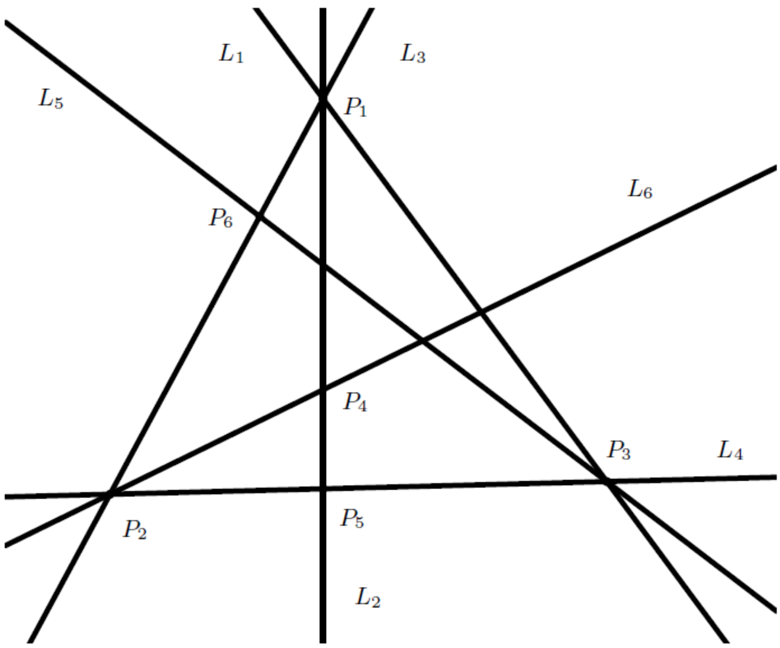
$$D = 3H - E_2 - E_3 - E_4 - E_5 - E_6 - F_1 - G_1 - H_1,$$
$$S_1 = 3H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2G_1 - H_1,$$
$$S_2 = 3H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2F_1 - H_1.$$

And so we can find all the (-1)-curves on X (by following Section 5.2) and apply the computational methods we have developed to find the necessary degrees. As such, we get that $|L \setminus \mathrm{Neg}(X)| = 1$ and the only degree in it is $-K_X$, that is necessary by Lemma 1.5.7. As such, the matrix degree of the Cox ring of X is

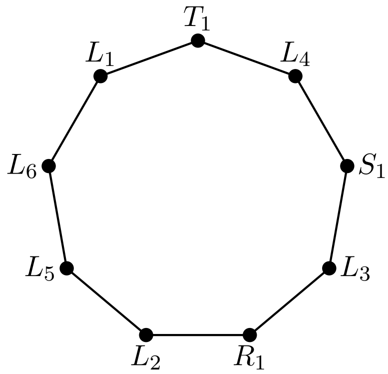
$$\left(\begin{array}{c|cccccccccccccccccc}
-3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 3 & 3 & 1 & 0 & 1 & 3 & 1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & -1 & -1 & 1 & -1 & -2 & -1 & 0 & 0 & 0
\end{array}\right).$$

3.1.4 A 2-Halphen rational elliptic surface of type A_8

By [30, Example 7.13], we can compute the Cox ring of a surface of type A_8. For this, we choose six lines intersecting as in the picture below:



and such that we can choose a smooth cubic C that is tangent with multiplicity 2 to L_1 at p_1, to L_5 at p_3 and to L_6 at p_2; and that also passes through p_4, p_5, p_6. The pencil generated by $L_1 + ... + L_6$ and $2C$ is a 2-Halphen pencil, and we have to blow up two times over p_1, p_2, p_3 and one time over p_4, p_5, p_6 to get a 2-Halphen rational surface of type A_8. In fact, we get the following (dual) configuration of curves:



We call, from last one to first one, E_1, E_2 the exceptional divisors over p_1; F_1, F_2 the exceptional divisors over p_2; G_1, G_2 the exceptional divisors over p_3; H_1 the exceptional divisor over p_4; I_1 the exceptional divisor over p_5 and J_1 the exceptional divisor over p_6. By using the basis of $\mathrm{Pic}(X)$ given by $\{H, E_i, F_j, G_k, H_1, I_1, J_1\}$, we get that the (-2)-curves of X are given by

$$R_1 = E_2 - E_1, \qquad S_1 = F_2 - F_1, \qquad T_1 = G_2 - G_1,$$

$$L_1 = H - E_1 - E_2 - G_2, \qquad L_2 = H - I_1 - H_1 - E_2,$$
$$L_3 = H - E_2 - J_1 - F_2, \qquad L_4 = H - G_2 - F_2 - I_1.$$
$$L_5 = H - J_1 - G_1 - G_2, \qquad L_6 = H - F_1 - F_2 - H_1.$$

And so we can find all the (-1)-curves on X (using Section 5.2 and apply the computational methods we have developed to find the necessary degrees. As such, we get that $|L \setminus \mathrm{Neg}(X)| = 13$ and its elements are (in the basis above)

```
>X1;
[
    [ 3, -1, -1, -1, -1, -1, -1, -1, -1, -1 ],
    [ 19, -6, -6, -2, -8, -7, -7, -9, -4, -5 ],
    [ 9, -3, -3, -2, -2, -2, -3, -5, -1, -4 ],
    [ 10, -3, -3, -1, -4, -4, -4, -5, -2, -2 ],
    [ 11, -4, -4, -2, -2, -3, -3, -1, -6, -5 ],
    [ 7, -1, -1, -2, -2, -1, -2, -3, -3, -4 ],
    [ 6, -2, -2, -3, -3, -2, -2, 0, -1, -1 ],
    [ 13, -2, -2, -4, -4, -3, -3, -5, -6, -7 ],
    [ 9, -1, -4, -3, -3, -4, -4, -3, -2, -1 ],
    [ 17, -2, -8, -6, -6, -7, -7, -5, -4, -3 ],
    [ 5, -2, -2, -1, -1, -1, -1, 0, -3, -2 ],
    [ 13, -4, -4, -6, -6, -5, -5, -1, -2, -3 ],
    [ 17, -6, -6, -4, -4, -5, -5, -9, -2, -7 ]
]
```

We order them from C_1 to C_{13}. Notice that $C_1 \sim -K_X$, which is necessary by Lemma 1.5.7. We can also notice that C_3, C_4, C_6, C_7, C_9 and C_{11} are conic bundles with only one reducible fiber: these are necessary by Lemma 2.4.17. The rest are models to $\mathbb{P}^2$ over a single point: these are also necessary by a similar proof to C_1 in Example 3.1.1. As such, the matrix degree of the Cox ring of X is given by

$$\left(\begin{array}{c|c} A & B \end{array} \right)$$

where A, B are given by

$$A = \begin{pmatrix} 3 & 19 & 9 & 10 & 11 & 7 & 6 & 13 & 9 & 17 & 5 & 13 & 17 \\ -1 & -6 & -3 & -3 & -4 & -1 & -2 & -2 & -1 & -2 & -2 & -4 & -6 \\ -1 & -6 & -3 & -3 & -4 & -1 & -2 & -2 & -4 & -8 & -2 & -4 & -6 \\ -1 & -2 & -2 & -1 & -2 & -2 & -3 & -4 & -3 & -6 & -1 & -6 & -4 \\ -1 & -8 & -2 & -4 & -2 & -2 & -3 & -4 & -3 & -6 & -1 & -6 & -4 \\ -1 & -7 & -2 & -4 & -3 & -1 & -2 & -3 & -4 & -7 & -1 & -5 & -5 \\ -1 & -7 & -3 & -4 & -3 & -2 & -2 & -3 & -4 & -7 & -1 & -5 & -5 \\ -1 & -9 & -5 & -5 & -1 & -3 & 0 & -5 & -3 & -5 & 0 & -1 & -9 \\ -1 & -4 & -1 & -2 & -6 & -3 & -1 & -6 & -2 & -4 & -3 & -2 & -2 \\ -1 & -5 & -4 & -2 & -5 & -4 & -1 & -7 & -1 & -3 & -2 & -3 & -7 \end{pmatrix},$$

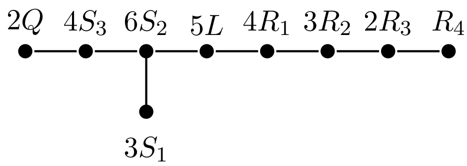
$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 4 & 0 & 0 & 2 & 3 & 1 & 2 & 1 & 1 & 0 & 0 & 4 & 2 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & -2 & 0 & 0 & 0 & -2 & -1 & -1 & 0 & -1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

3.2 Rational elliptic surfaces of index ≥ 3

In this section, we give some examples of computation of Cox rings of some rational elliptic surfaces of index ≥ 3.

3.2.1 A 3-Halphen rational elliptic surface of type E_8

By [21, Example 4.8], we can compute the Cox ring of a 3-Halphen surface of type E_8. In here, Hattori and Zanardini provide an explicit Example of a 3-Halphen pencil of type E_8. Concretely, consider the cubic C given by $z^2y + x(y^2 + xz) = 0$, the conic Q given by $y^2 + xz = 0$ and the line L given by $y = 0$. Then the pencil generated by $2Q + 5L$ and $3C$ is a 3-Halphen pencil. We can calculate classes of divisors using [30, Proposition 4.4] and [30, Lemma 4.5]: these two together imply that we have blow ups over two points $\{p_1, p_2\}$. Moreover, the multiplicity of Q and L in the pencil tells us that we have the following (dual) configuration of curves:



We call, from last one to first one, E_1, E_2, E_3, E_4, E_5 the exceptional divisors over p_1; F_1, F_2, F_3, F_4 the exceptional divisors over p_2; and use Q, L_1 for the classes of the curves Q, L. By using the basis of $\mathrm{Pic}(X)$ given by $\{H, E_i, F_i\}$, we get that the (-2)-curves of X are given by

$$R_i = E_{i+1} - E_i, \qquad S_i = F_{i+1} - Fi$$

$$L_1 = H - E_5 - F_3 - F_4, \qquad Q = 2H - E_1 - E_2 - E_3 - E_4 - E_5 - F_4.$$

And so we can find all the (-1)-curves on X and apply the computational methods we have developed to find the necessary degrees. As such, we get that $|L \setminus \mathrm{Neg}(X)| = 4$ and its elements are

$$\begin{aligned}
C_1 &:= -K_X,\\
C_2 &:= H - F_4,\\
C_3 &:= 2H - F_1 - F_2 - F_3 - F_4,\\
C_4 &:= 7H - 3E_2 - 3E_3 - 3E_4 - 3E_5 - F_1 - 2F_2 - 2F_3 - 2F_4.
\end{aligned}$$

By Lemma 2.4.17, C_2, C_3 and C_4 are necessary. Moreover, C_1 is necessary by Lemma

1.5.7. As such, the matrix degree of the Cox ring of X is

$$\left(\begin{array}{cccc|cccccccccccccc}
1 & 2 & 7 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 4 & 1 & 10 \\
0 & 0 & -3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & -3 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -4 \\
0 & 0 & -3 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & 0 & -4 \\
0 & 0 & -3 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & -1 & -4 \\
0 & 0 & -3 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -2 & -1 & -4 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -3 \\
0 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 \\
0 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -3 \\
-1 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & -1 & 0 & -3
\end{array}\right).$$

3.2.2 A 4-Halphen rational elliptic surface of type E_8

In [24], Laface and Testa give a way to compute the negative curves of a family of m-Halphen rational surfaces for any $m > 0$. Roughly speaking, they show that the ways one can embed the lattice Λ given by intersection of (-2)-curves of a m-Halphen rational elliptic surface in the E_8 lattice are parametrized by the group $\mathrm{Ext}(E_8/\Lambda, C_m)$ [24, Lemma 2.5] and they give a way to, given (m, ξ) with $\xi \in \mathrm{Ext}(E_8/\Lambda, C_m)$, construct a m-Halphen rational elliptic surface with associated lattice Λ [24, Proposition 2.6]. This essentially allows us to skip obtaining a geometric model for the surface.

By using a Magma program for this process (many thanks to Antonio Laface for its implementation) we compute all (-2)-curves of a 4-Halphen rational surface X of type E_8: these and the (-1)-curves of X are

```
Neg(X)=[
    (0, 0, 0, 0, 0, 0, 1, 0, 0, -1),
    (0, 0, 0, 0, 1, 0, 0, 0, -1, 0),
    (0, 0, 0, 1, 0, 0, -1, 0, 0, 0),
    (0, 0, 1, 0, -1, 0, 0, 0, 0, 0),
    (0, 1, -1, 0, 0, 0, 0, 0, 0, 0),
    (1, -1, -1, 0, 0, 0, 0, -1, 0, 0),
    (1, -1, 0, -1, 0, -1, 0, 0, 0, 0),
    (1, 0, 0, -1, 0, 0, -1, -1, 0, 0),
    (2, -1, -1, -1, -1, 0, -1, 0, 0, -1),
    (5, -2, -2, -2, -2, -1, -2, -1, -2, 0),
    (0, 0, 0, 0, 0, 0, 0, 0, 0, 1),
    (7, -3, -3, -2, -3, -2, -2, -1, -3, -1),
    (0, 0, 0, 0, 0, 1, 0, 0, 0, 0),
    (2, -1, -1, 0, -1, -1, 0, 0, -1, 0),
    (2, -1, -1, -1, -1, 0, 0, 0, -1, 0),
    (15, -6, -6, -4, -6, -5, -4, -3, -6, -4),
    (0, 0, 0, 0, 0, 0, 0, 0, 1, 0),
    (0, 0, 0, 0, 0, 0, 0, 1, 0, 0),
    (1, 0, 0, 0, 0, -1, 0, -1, 0, 0)
]
```

Applying the computational methods we have developed to find the necessary degrees, we get that $|L \setminus \mathrm{Neg}(X)| = 4$ and its elements are

```
{
    (12, -5, -5, -3, -5, -4, -3, -1, -5, -3),
    (6, -1, -1, -3, -1, -2, -3, 0, -1, -3),
    (2, 0, 0, -1, 0, -1, -1, 0, 0, -1),
    (3, -1, -1, -1, -1, -1, -1, -1, -1, -1)
}
```

Calling them C_1 through C_4, we have that C_1 and C_3 are necessary by Lemma 2.4.17, C_2 is necessary for being a model to $\mathbb{P}^2$ over one point and C_4 is necessary by Lemma 1.5.7. We have, then, 19 negative curves and 4 extra necessary degrees.

3.2.3 A 5-Halphen rational elliptic surface of type E_8

By using the Magma program described in Example 3.2.2, we compute all (-2)-curves of a 5-Halphen rational surface X of type E_8: these and the (-1)-curves of X are

```
Neg(X)=[
    [ 0, 0, 0, 0, 0, 0, 0, 1, -1, 0 ],
    [ 0, 0, 0, 0, 0, 0, 1, -1, 0, 0 ],
    [ 0, 0, 0, 1, 0, 0, -1, 0, 0, 0 ],
    [ 0, 0, 1, 0, -1, 0, 0, 0, 0, 0 ],
    [ 0, 1, 0, 0, 0, -1, 0, 0, 0, 0 ],
    [ 1, -1, 0, -1, 0, -1, 0, 0, 0, 0 ],
    [ 1, 0, -1, -1, 0, 0, -1, 0, 0, 0 ],
    [ 1, 0, -1, 0, -1, 0, 0, 0, 0, -1 ],
    [ 2, -1, 0, -1, 0, 0, -1, -1, -1, -1 ],
    [ 10, -5, -4, -3, -4, -2, -3, -3, -3, -2 ],
    [ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 ],
    [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 ],
    [ 3, -2, -1, -1, -1, 0, -1, -1, -1, 0 ],
    [ 1, -1, -1, 0, 0, 0, 0, 0, 0, 0 ],
    [ 1, -1, 0, 0, 0, 0, 0, 0, 0, -1 ],
    [ 7, -4, -3, -2, -2, -1, -2, -2, -2, -2 ],
    [ 3, -1, -1, -1, 0, -1, -1, -1, 0, -2 ],
    [ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 ],
    [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 ],
    [ 4, -2, -2, -1, 0, -1, -1, -1, -1, -2 ],
    [ 3, -2, -1, -1, -1, 0, -1, -1, 0, -1 ],
    [ 4, -2, -2, -1, -2, -1, -1, -1, -1, 0 ],
    [ 1, 0, 0, -1, 0, 0, 0, 0, 0, -1 ],
    [ 22, -10, -8, -7, -8, -5, -7, -7, -7, -6 ]
]
```

Applying the computational methods we have developed to find the necessary degrees, we get that $|L \setminus \mathrm{Neg}(X)| = 7$ and its elements are

```
{
    (17, -6, -7, -5, -2, -6, -5, -5, -5, -8),
    (8, -3, -2, -2, -1, -3, -2, -2, -2, -5),
    (11, -4, -4, -3, -1, -4, -3, -3, -3, -6),
    (8, -3, -2, -3, -2, -2, -3, -3, 0, -4),
    (3, -1, -1, -1, -1, -1, -1, -1, -1, -1),
    (22, -11, -8, -7, -8, -4, -7, -7, -6, -6),
    (16, -5, -4, -6, -4, -5, -6, -6, -1, -8)
}
```

Calling them C_1 through C_7, we have that $C_1, \ldots, C_4, C_6$ are necessary by Lemma 2.4.17, C_7 is necessary for being a model to $\mathbb{P}^2$ over one point and C_5 is necessary by Lemma 1.5.7. We have, then, 24 negative curves and 7 extra necessary degrees.

Observation 3.2.1. The reason to consider examples of type E_8 is that the number of negative curves remains small, which helps to make computations feasible; and also because in this case $\mathrm{Ext}(E_8/\Lambda, C_m)$ is trivial, which implies there is always only one possible configuration of necessary degrees (these are only dependent on the classes of negative curves). We also summarize the behaviour of degrees in surfaces of type E_8 with $m \leq 5$ in the following table:

m	Number of necessary degrees	Negative curves	Conic bundles	Models to $\mathbb{P}^2$	$-K_X$
1	13	10	1	1	1
2	15	12	1	1	1
3	18	14	3	0	1
4	23	19	2	1	1
5	31	24	5	1	1

3.3 Weak del Pezzo surfaces

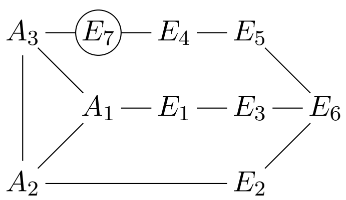
In this section, we give some examples of computations of Cox rings for weak del Pezzo surfaces. We give two different applications: in the first subsection, we recover a known result with our methods; and in the second subsection, we develop a method that allows us to obtain the necessary degrees to generate the Cox ring of all del Pezzo surfaces of Picard number 9.

3.3.1 A weak del Pezzo surface of Picard number 7 that has many different kinds of generators

Let us consider the $\mathbf{E}_6$ cubic surface

$$S = \{(x, y, z, w) : xy^2 + yw^2 + z^3 = 0\} \subseteq \mathbb{P}^3$$

that was studied by Hassett and Tschinkel in [20, Section 3] and is described by Derenthal in [14, Section 5.6] (in the latter, it is called a type xx surface). As Hassett and Tschinkel show, the Cox ring of S has generators in degree $E_1, ..., E_7, A_1, A_2, A_3$; where $E_1, .., E_6$ are (-2)-curves, E_7 is a (-1)-curve and the extended Dynkin diagram of negative curves of S is



By considering the basis of $\mathrm{Pic}(S)$ given by $E_1, ..., E_7$, we have $A_3 = (2,3,4,4,5,6,3) = -K_S$, $A_1 = (0,1,1,2,2,2,2)$ and $A_2 = (1,1,2,3,3,3,3)$. Let us write these divisors in a more manageable form: considering the basis of $\mathrm{Pic}(S)$ given by $H, T_1, ..., T_6$, where H is a general line and

$$T_1 = E_7 + E_4, \quad T_2 = E_7 + E_4 + E_5, \quad T_3 = E_7 + E_4 + E_5 + E_6$$

$$T_4 = E_7 + E_4 + E_5 + E_6 + E_3, \quad T_5 = E_7 + E_4 + E_5 + E_6 + E_3 + E_1, \quad T_6 = E_7$$

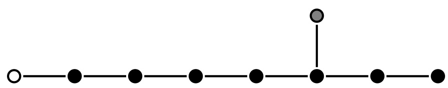
(which are all the (-1)-curves excepting $E_7 + E_4 + E_5 + E_6 + E_2$) we have that

$$A_1 = H - T_5, \qquad A_2 = H$$

and so the necessary degrees of the Cox ring of S include a conic bundle A_1, the pullback of an anticanonical ray A_2 (coming from $\mathbb{P}^2$, in this case) and the anticanonical divisor A_3. This shows that Theorem 2.4.7 is optimal: by Proposition 1.4.7 these are all the different kinds of possible divisors in $\mathrm{BNef}(S)$. Also, by running the Magma program described in 5.4.2, we get that $|L' \setminus \mathrm{Neg}(X)| = 3$: exactly the amount we expected.

3.3.2 A weak del Pezzo surface of Picard number 9

Example 3.3.1. Let us consider a weak del Pezzo surface X with no two disjoint (-1)-curves: since this implies that we can't blow-up more than one point without them being infinitely close, we have a diagram



where the black dots correspond to the (-2)-curves coming from the exceptional divisors of the first 7 exceptional divisors over the single point p being blown-up; the gray dot corresponds to the (-2)-curve coming from a line passing through p with the same direction as the second blowing up; and the white dot corresponds to the (-1)-curve coming from the final blow-up over p. Note that the black and gray dots form a Dynkin diagram of type E_8: this means that, when contracting these curves, we go into a singular surface Y whose minimal desingularization is X and has singularities of type E_8. This is a Gorenstein log del Pezzo surface (see [29, Chapter 1]) and moreover we have $\rho(Y) = 1$ (since we contracted a rank 8 lattice on $\mathrm{Pic}(X)$). By [29, Lemma 3.3], we know that X comes from the contraction of a (-1)-curve on an extremal rational jacobian elliptic surface Z, and, by [29, Table 4.1], Z must have a singular fiber of type II^*II or II^*2I_1. These correspond (in the notation of [2, Table 1]) to the surfaces X_{22} and X_{211}, so let us assume that Z is the surface X_{22}. By the Remark before [2, Example 5.1], we can compute the Cox ring $\mathcal{R}(X)$ by means of putting $T_i = 1$ for any generator T_i of $\mathcal{R}(Z)$

that defines an exceptional divisor of the morphism $Z \to R$. So, by taking the degree matrix of necessary degrees of $\mathcal{R}(X_{22})$, given in [2, Table 4],

$$\left(\begin{array}{cccc|ccccccccc}
1 & 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{array}\right)$$

we notice that the only (-1)-curve present is the last column. So, setting this last variable to 1, we get

$$\left(\begin{array}{cccc|cccccccc}
1 & 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{array}\right)$$

which is the matrix of necessary degrees of our surface $\mathcal{R}(X)$. Notice that we get, ordered by columns,

- The strict transform of the special line passing by p with a special orientation,
- The strict transform of a general line by passing by p,
- The strict transform of a general line not passing by p,
- The anticanonical divisor,
- The (-2)-curves defined by the exceptional divisors of the first blow-ups,
- The (-1)-curve defined by the exceptional divisor of the last blow-up.

This, of course, is in accordance with Theorem 2.4.7. We can also deduce the ideal of relations on $\mathcal{R}(X)$ by the same means: setting $T_{13} = 1$ on [2, Table 4], we get

$$I(X) = \langle T_1T_3^2 + T_2^3T_5^2T_6 - T_4T_8T_9^2T_{10}^3T_{11}^4T_{12}^5\rangle.$$

Similarly, if Z is the surface X_{211}, we get that the necessary degrees of $\mathcal{R}(X)$ are

$$\left(\begin{array}{cccc|cccccccc}
1 & 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{array}\right)$$

and

$$I(X) = \langle T_1T_3^2 + T_2^3T_5^2T_6 - T_4T_8T_9^2T_{10}^3T_{11}^4T_{12}^5 + T_1T_2^2T_5^2T_6^2T_7^2T_8^2T_9^2T_{10}^2T_{11}^2T_{12}^2\rangle.$$

There is nothing special about these two kinds of weak del Pezzo surfaces, as the same procedure can be carried out for any extremal rational jacobian elliptic surface, given that we can choose an appropriate (-1)-curve to contract when there is more than one. Nonetheless, the translation by a section ζ in the Mordell-Weil group of X induces an automorphism of X by [23, Section 3]; and in terms of the Cox rings this permutes the generators associated to the (-1)-curves. Since this can be done for any section, contracting any of them will result in the same surface modulo isomorphism. As such, we need to describe only one of these contractions per each extremal rational jacobian elliptic surface. We describe explicitly the Cox ring for all the possible surfaces (except for three, whose ideal of relations is not known) in Table 4.1, using the descriptions of Artebani, Garbagnati and Laface in [2].

4 Tables

4.1 Cox rings of weak del Pezzo surfaces of Picard number 9

In this table, we give the degree matrix for each possible weak del Pezzo surface of Picard number 9; and ideals of relations where it is known. A surface is denoted X' if it is the blow down of the jacobian elliptic surface X.

Surface	Degree matrix and $I(X)$
X'_{22}	$\left(\begin{array}{cccc\|cccccccc} 1 & 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$ $I(X) = \langle T_1T_3^2 + T_2^3T_5^2T_6 - T_4T_8T_9^2T_{10}^3T_{11}^4T_{12}^5\rangle$
X'_{211}	$\left(\begin{array}{cccc\|cccccccc} 1 & 1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$ $I(X) = \langle T_1T_3^2 + T_2^3T_5^2T_6 - T_4T_8T_9^2T_{10}^3T_{11}^4T_{12}^5 + T_1T_2^2T_5^2T_6^2T_7^2T_8^2T_9^2T_{10}^2T_{11}^2T_{12}^2\rangle$
X'_{411}	$\left(\begin{array}{cccccc\|cccccccc} 1 & 1 & 1 & 2 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $I(X) = \left\langle \begin{array}{c} T_7^2T_8^3T_9^4T_{10}T_{11}^2T_{12}^3T_{13}^4T_{14}^4T_1T_2^4 - T_4T_5 + T_3^2T_6, \\ T_7^2T_8^3T_9^4T_{10}^2T_{14}T_2^3 - T_{10}^2T_{11}T_6 + T_1T_4,\ T_{11}^2T_{12}^3T_{13}^4T_{14}^3T_1^2T_2 - T_7^2T_8T_6 + T_{10}T_5, \\ T_7^4T_8^4T_9^4T_{10}T_{14}T_2^3 - T_{11}^2T_{12}T_5 + T_1T_3^2,\ T_{11}^4T_{12}^4T_{13}^4T_{14}^3T_1T_2 - T_7^2T_8T_4 + T_{10}T_3^2 \end{array} \right\rangle$

Surface	Degree matrix and $I(X)$
X'_{9111}	$\left(\begin{array}{cccccccccc\|cccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -2 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & -1 & -1 & -2 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & -2 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$
X'_{33}	$\left(\begin{array}{cccc\|cccccccc} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$ $I(X) = \langle T_1T_2T_{11}^2T_{12}^3 - T_3T_7T_8^2T_9^3T_{10}^4 - T_4^2T_5\rangle$
X'_{321}	$\left(\begin{array}{cccc\|cccccccc} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$ $I(X) = \langle T_1T_2T_{11}^2T_{12}^3 - T_3T_7T_8^2T_9^3T_{10}^4 - T_4^2T_5 + T_1T_4T_5T_6T_7T_8T_9T_{10}T_{11}T_{12}\rangle$
X'_{8211}	$\left(\begin{array}{ccccccccc\|cccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & -1 & -2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$
X'_{44}	$\left(\begin{array}{cccc\|cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$ $I(X) = \langle T_2T_5T_6^2T_7^3 - T_3T_8T_9^2T_{10}^3 - T_4T_{11}T_{12}^2\rangle$

Surface	Degree matrix and $I(X)$
X'_{431}	$\left(\begin{array}{cccc\|cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$ $I(X) = \langle T_2T_5T_6^2T_7^3 + T_3T_8T_9^2T_{10}^3 - T_4T_{11}T_{12}^2 + T_1T_5T_6T_7T_8T_9T_{10}T_{11}T_{12} \rangle$
X'_{222}	$\left(\begin{array}{cccccc\|cccccccc} 1 & 1 & 1 & 1 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$ $I(X) = \left\langle \begin{array}{c} T_1T_7^2T_8^2T_9^2T_{10} - T_2T_{14}^2 + T_3T_{12}^2T_{13},\ T_1^2T_4^2T_8T_9^2T_{10} + T_2T_5T_{14}^2 - T_3T_6T_{12}^2, \\ T_1T_4^2T_{13} - T_2^2T_3T_{10}T_{11}^2T_{14}^2 + T_6T_7^2T_8,\ T_1T_4^2 - T_2T_3^2T_{10}T_{11}^2T_{12}^2 + T_5T_7^2T_8, \\ T_1T_2T_3T_8T_9^2T_{10}^2T_{11}^2 + T_5T_{13} - T_6 \end{array} \right\rangle$
X'_{141}	$\left(\begin{array}{ccccc\|cccccccc} 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$ $I(X) = \left\langle\ T_1T_4 + T_2T_3T_9T_{11}T_{13} - T_5T_6^2T_7,\ T_1T_6T_7T_8 - T_2T_{10}T_{11}^2 - T_3T_{12}T_{13}^2\ \right\rangle$
X'_{6321}	$\left(\begin{array}{cccccccc\|cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$ $I(X) = \left\langle \begin{array}{c} T_2T_{11}T_{12}T_{16} + T_3T_9T_{10} - T_6T_{14}T_{15},\ T_1T_{11}T_{12}T_{15} + T_3T_{13}T_{14} - T_4T_{10}T_{16}, \\ T_1T_9T_{10}T_{15} + T_2T_{13}T_{14}T_{16} - T_5T_{12},\ T_1T_3T_9T_{15} + T_2T_4T_{16}^2 - T_8T_{12}T_{14}, \\ T_1T_2T_{11}T_{15}T_{16} + T_3T_5 - T_8T_{10}T_{14},\ T_3T_9T_{10}T_{13}T_{14} + T_5T_{11}T_{12}^2 - T_7T_{15}T_{16}, \\ T_1T_9T_{10}T_{11}T_{12} + T_6T_{13}T14^2 - T_7T_{16},\ T_1T_6T_{15}^2 + T_2T_3T_{13}T_{16} - T_8T_{10}T_{12}, \\ T_2T_{11}T_{12}T_{13}T_{14} + T_4T_9T_{10}^2 - T_7T_{15},\ T_1^2T_9T_{11}T_{15}^2 - T_4T_5T_{16} + T_8T_{13}T_{14}^2, \\ T_1T_2T_{11}^2T_{12}^2 + T_3T_7 - T_4T_6T_{10}T_{14},\ T_3^2T_9T_{13} - T_4T_6T_{15}T_{16} + T_8T_{11}T_{12}^2, \\ T_1T_7T_{15}^2 + T_2T_3T_{13}^2T_{14}^2 - T_4T_5T_{10}T_{12},\ T_1T_3T_9^2T_{10}^2 + T_2T_7T_{16}^2 - T_5T_6T_{12}T_{14}, \\ T_2^2T_{11}T_{13}T_{16}^2 - T_5T_6T_{15} + T_8T_9T_{10}^2,\ T_1T_2T_3T_9T_{11}T_{13} + T_4T_5T_6 - T_7T_8 \end{array} \right\rangle$

Surface	Degree matrix and $I(X)$
$X'_{11}(a)$	$\left(\begin{array}{ccccc\|cccccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$ $I(X) = \left\langle \; (a-1)T_2T_8T_9^2 - T_3T_{10}T_{11}^2 + T_5T_{13},\; (a-1)T_1T_6T_7^2 - aT_3T_{10}T_{11}^2 + T_5T_{13} \; \right\rangle$
X'_{5511}	$\left(\begin{array}{cccccc\|cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right)$ $I(X) = \left\langle \begin{array}{c} T_2T_{11}T_{13}T_{14} + T_4T_7^2T_8 - T_5T_9T_{10},\; T_2T_{11}^2T_{12} + T_3T_7T_8 - T_6T_9T_{10}T_{13}, \\ T_1T_9^2T_{10} + T_3T_{13}T_{14} - T_4T_7T_{11}T_{12},\; T_1T_7T_8T_9 - T_5T_{11}T_{12} + T_6T_{13}^2T_{14}, \\ T_1T_2T_9T_{11} + T_3T_5 - T_4T_6T_7T_{13} \end{array} \right\rangle$
X'_{4422}	$\left(\begin{array}{ccccccccccc\|cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$
X'_{3333}	$\left(\begin{array}{cccccccccccc\|cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$ $I(X) = \left\langle \begin{array}{c} (-2\epsilon-1)T_1T_{20} - T_5T_{15}T_{18} + T_6T_{14}T_{17} \\ (\epsilon-1)T_1T_5T_{19}^2 + T_8T_{10}T_{13}T_{16} - T_9T_{11}T_{14}T_{17} \\ T_1T_2T_3 + 1/9(-\epsilon+1)T_4T_9T_{11} + 1/9(\epsilon-1)T_5T_7T_{12} \end{array} \right\rangle^G$

Table 4.1: The Cox rings of weak del Pezzo surfaces of Picard number 9

5 Magma Programs

In this Chapter, we show the programs we used through this thesis along with short descriptions of their uses. They are contained in two different libraries, separated by sections, and can be found here. We also provide the code of each example.

5.1 Generalities

This section describes some functions for general use.

- **L**: The abelian group $\mathbb{Z}^{10}$.
- **M**: The intersection matrix associated to the basis of $\mathrm{Pic}(X)$ given by the pullback of the class of a line and (-1)-curves.
- **K**: The canonical divisor of X in the basis above.
- **qua**: Returns the intersection product of two vectors A and B given the intersection matrix Q.
- **HNef**: Returns the Hilbert basis of the nef cone of X.
- **IsNef**: Checks whether a divisor N is nef by verifying intersections with all negative curves.
- **IsEff**: Checks whether a divisor N is effective.
- **IsNullh1**: For a divisor N on an anticanonical surface X, checks (with a rather coarse test based on Theorems 1.3.2 and Kawamata-Viehweg Vanishing (see Theorem 1.1.9) whether $h^1(X, N)$ is zero.
- **DisjointNef**: Gives the list of all pairs of divisors in $\mathrm{BNef}(X) \cup \{-2K_X\}$ that are disjoint.
- **DisjointNeg**: Gives the list of all pairs of negative curves that are disjoint.
- **DisjointCurves**: Gives the list of all pairs of $\mathrm{Neg}(X) \cup \{-K_X\}$ that are disjoint.

- **Nefification**: Checks whether a divisor A is contained in the base locus of another divisor N.

```
L := ToricLattice(10);
M := DiagonalMatrix([1,-1,-1,-1,-1,-1,-1,-1,-1,-1]);
K := L![-3,1,1,1,1,1,1,1,1,1];
```

```
qua := function(A,B,Q)
  K := CoefficientRing(Q);
  n := Nrows(Q);
return (Matrix(K,1,n,Eltseq(A))*Q*Matrix(K,n,1,Eltseq(B)))[1,1];
end function;
```

```
HNef := function(Neg);
 Eff := Cone([L!v : v in Neg]);
 Nef := Cone([L!Eltseq(v) : v in Rays(Dual(Eff*M))]);
return HilbertBasis(Nef);
end function;
```

```
IsNef := function (N,Neg,M);
if {qua(N,C,M) ge 0 : C in Neg} eq {true} then return true;
else return false;
end if;
end function;
```

```
IsEff:=function(N,Neg,M,H);
if {qua(N,A,M) ge 0 : A in H} eq {true} then return true;
else return false;
end if;
end function;
```

```
IsNullh1 := function(N,Neg,M);
if (IsNef(N,Neg,M) and qua(N,K,M) lt 0) or
(IsNef(L!K+L!N,Neg,M) and qua(N,N,M) gt 0) then return true;
else return false;
end if;
end function;
```

```
DisjointNef := function(Neg,M,H);
return [[A,B] : A,B in Set(H) join {L!N : N in Neg}
join {-2*L!K} | qua(A,B,M) eq 0 and
(qua(A,K,M) lt 0 or qua(B,K,M) lt 0)] cat [[-2*L!K,-2*L!K]];
end function;
```

```
DisjointNeg := function(Neg,M);
return [[A,B] : A,B in Neg | qua(A,B,M) eq 0];
end function;
```

```
DisjointCurves:= function(Neg,M);
return [[A,B] : A,B in {L!N : N in Neg} join {L!-K}
              | qua(A,B,M) eq 0];
end function;
```

```
Nefification := function(A,N,Neg,H);
repeat
if IsEff(N,Neg,M,H) then m,i := Min([qua(N,E,M) : E in Neg]);
C:= Neg[i];
else return false;
end if;
if m lt 0 then N:= L!N - L!Eltseq(C);
else return true;
end if;
until m ge 0 or L!C eq L!A;
if L!C eq L!A then return false;
else return true;
end if;
end function;
```

5.2 Finding (-1)-curves

By [24, Theorem 1], the set of (-1)-curves of a rational elliptic surface is in bijection to the integral points of the Riemann-Roch polyhedron. A program that calculates such polyhedron and then the classes of (-1)-curves is the following (many thanks to Antonio Laface for its implementation):

```
Rv := [L!(Eltseq(Matrix(1,10,Eltseq(r))*M)): r in R];
Kv := L!(Eltseq(Matrix(1,10,Eltseq(K))*M));
P := &meet[HalfspaceToPolyhedron(Rv[i],0) : i in [1..#Rv]]
     meet HyperplaneToPolyhedron(-Kv,1);
pts :=  SetToSequence(Points(CompactPart(P)));
S := [L!p+(qua(p,p,M)+1)/2*K: p in pts];
```

5.3 Tests

This section contains the programs that test our divisors, as explained in 2.5.

```
Test1 := function(N,Neg,M,H);
if L!N eq -L!K or L!N eq -2*L!K then return [];
end if;
S := [Z : Z in DisjointCurves(Neg,M) | qua(L!N,L!Z[1],M) eq 0];
X :=[];
for Z in S do
N2:=L!N-L!Z[2];
if IsEff(N2,Neg,M,H) then
if Nefification(L!Z[1],N2,Neg,H)
then X := X cat [[L!N,L!Z[1],L!Z[2]]];
end if;
end if;
end for;
return X;
end function;
```

5.4 Examples

In this section, we give all programs that we used to compute our examples.

5.4.1 Example 3.1.1

In this subsection, we give the program that returns the 9 divisors that we treat separately in 3.1.1.

- **R**: Gives the list of the classes of (-2)-curves on X.

- **S**: Gives the list of the classes of (-2)-curves on X.
- **NegEx1**: Gives the list of the classes of negative curves on X.
- **Ex:** Gives the list of all exceptions given by Theorem 2.4.6.
- **PND**: Gives the list of the possible necessary divisors of X, excluding negative curves, according to Theorem 2.4.6.
- **X1**: Returns the list of all divisors that do not pass Test 1.

```
G1 := [0,0,0,0,0,0,0,-1,1,0];
G2 := [0,0,0,0,0,0,-1,1,0,0];
G3 := [0,0,0,0,0,-1,1,0,0,0];
G4 := [0,0,0,0,-1,1,0,0,0,0];
G5 := [0,0,0,-1,1,0,0,0,0,0];
G6 := [0,0,-1,1,0,0,0,0,0,0];
G7 := [0,-1,1,0,0,0,0,0,0,0];
T  := [3,0,-1,-1,-1,-1,-1,-1,-1,-2];
Lp := [1,0,0,0,0,0,-1,-1,-1,0];

R:=[G1,G2,G3,G4,G5,G6,G7,T,Lp];

E1 := [0,1,0,0,0,0,0,0,0,0];
Eq := [0,0,0,0,0,0,0,0,0,1];
Lpq:= [1,0,0,0,0,0,0,0,-1,-1];

S:=[E1,Eq,Lpq];
```

```
NegEx1:= S cat R;
H := HNef(NegEx1);
PND:=Seqset(H);
```

```
X:=[];
for A in PND do
if #Test1(A,NegEx1,M,H) ge 1 then X:= X cat [A];
end if;
end for;

X1:=PND diff Set(X);
```

This gives as output

```
> X1;
{
    (3, -1, -1, -1, -1, -1, -1, -1, -1, -1),
    (5, -1, -1, -1, -1, -1, -1, -1, -1, -4),
    (1, 0, 0, 0, 0, 0, 0, 0, 0, -1)
}
```

5.4.2 Example 3.1.2

In this subsection, we give the program that returns the three divisors we found in Example 3.1.2.

- **L**: The abelian group $\mathbb{Z}^{10}$.
- **M**: The intersection matrix associated to the basis of $\mathrm{Pic}(X)$ given by

$$\{H, E_1, E_2, E_3, F_1, F_2, F_3, G_1, G_2, J_1\}.$$

- **K**: The canonical divisor of X in the basis above.
- **R**: Gives the list of the classes of (-2)-curves on X.
- **S**: Gives the list of the classes of (-1)-curves on X.
- **NegEx4**: Gives the list of the classes of negative curves on X.
- **PND**: Gives the list of the possible necessary divisors of X, excluding negative curves (which are the elements of $\mathrm{BNef}(X)$).
- **X1**: Returns the list of all divisors that do not pass Test 1.

```
L := ToricLattice(10);
M := DiagonalMatrix([1,-1,-1,-1,-1,-1,-1,-1,-1,-1]);
K := L![-3,1,1,1,1,1,1,1,1,1];
```

```
R1 := [0,-1,1,0,0,0,0,0,0,0];
R2 := [0,0,-1,1,0,0,0,0,0,0];
S1 := [0,0,0,0,-1,1,0,0,0,0];
S2 := [0,0,0,0,0,-1,1,0,0,0];
T1 := [0,0,0,0,0,0,0,-1,1,0];
L1 := [1,0,0,-1,0,-1,-1,0,0,0];
L2 := [1,0,0,0,0,0,-1,0,-1,-1];
L3 := [1,0,0,-1,0,0,0,-1,-1,0];
L4 := [1,-1,-1,-1,0,0,0,0,0,0];
R:=[R1,R2,S1,S2,T1,L1,L2,L3,L4];

Rv := [L!(Eltseq(Matrix(1,10,Eltseq(r))*M)): r in R];
Kv := L!(Eltseq(Matrix(1,10,Eltseq(K))*M));
P := &meet[HalfspaceToPolyhedron(Rv[i],0) : i in [1..#Rv]]
     meet HyperplaneToPolyhedron(-Kv,1);
pts :=  SetToSequence(Points(CompactPart(P)));
S := [L!p+(qua(p,p,M)+1)/2*K: p in pts];

NegEx4 := [L!r : r in R] cat S;
H := HNef(NegEx4);
PND:=Seqset(H);
```

```
X:=[];
for A in PND do
if #Test1(A,NegEx4,M,H) ge 1 then X:= X cat [A];
end if;
end for;

X1:=PND diff Set(X);
```

This gives as output

```
> X1;
{
    (3, -1, -1, -1, -1, -1, -1, -1, -1, -1),
    (4, 0, -2, -2, -1, -1, -1, 0, -2, -1),
    (2, 0, 0, 0, -1, -1, -1, 0, 0, -1)
}
```

5.4.3 Example 3.1.3

In this subsection, we give the program that returns the three divisors we found in Example 3.1.3.

- **L**: The abelian group $\mathbb{Z}^{10}$.
- **M**: The intersection matrix associated to the basis of $\mathrm{Pic}(X)$ given by

$$\{H, E_1 ..., E_6, F_1, G_1, H_1\}.$$

- **K**: The canonical divisor of X in the basis above.
- **R**: Gives the list of the classes of (-2)-curves on X.
- **S**: Gives the list of the classes of (-1)-curves on X.
- **NegEx4**: Gives the list of the classes of negative curves on X.
- **PND**: Gives the list of the possible necessary divisors of X, excluding negative curves (which are the elements of $\mathrm{BNef}(X)$).
- **X1**: Returns the list of all divisors that do not pass Test 1.

```
L := ToricLattice(10);
M := DiagonalMatrix([1,-1,-1,-1,-1,-1,-1,-1,-1,-1]);
K := L![-3,1,1,1,1,1,1,1,1,1];
```

```
G1 := [0,-1,1,0,0,0,0,0,0,0];
G2 := [0,0,-1,1,0,0,0,0,0,0];
G3 := [0,0,0,-1,1,0,0,0,0,0];
G4 := [0,0,0,0,-1,1,0,0,0,0];
G5 := [0,0,0,0,0,-1,1,0,0,0];
L1 := [1,0,0,0,-1,-1,-1,0,0,0];
L2 := [1,0,0,0,0,0,-1,-1,-1,0];
C1 := [3,-1,-1,-1,-1,-1,-1,0,-2,-1];
C2 := [3,-1,-1,-1,-1,-1,-1,-2,0,-1];
D  := [3,0,-1,-1,-1,-1,-1,-1,-1,-2];
R :=[G1,G2,G3,G4,G5,L1,L2,D,C1,C2];

Rv := [L!(Eltseq(Matrix(1,10,Eltseq(r))*M)): r in R];
Kv := L!(Eltseq(Matrix(1,10,Eltseq(K))*M));
P := &meet[HalfspaceToPolyhedron(Rv[i],0) : i in [1..#Rv]]
     meet HyperplaneToPolyhedron(-Kv,1);
pts :=  SetToSequence(Points(CompactPart(P)));
S := [L!p+(qua(p,p,M)+1)/2*K: p in pts];

NegEx4 := [L!r : r in R] cat S;
H := HNef(NegEx4);
PND:=Seqset(H);
```

```
X:=[];
for A in PND do
if #Test1(A,NegEx4,M,H) ge 1 then X:= X cat [A];
end if;
end for;

X1:=PND diff Set(X);
```

This gives as output

```
> X1;
{
    (3, -1, -1, -1, -1, -1, -1, -1, -1, -1),
}
```

5.4.4 Example 3.1.4

In this subsection, we give the program that returns the three divisors we found in Example 3.1.4.

- **L**: The abelian group $\mathbb{Z}^{10}$.
- **M**: The intersection matrix associated to the basis of $\mathrm{Pic}(X)$ given by

$$\{H, E_1, E_2, F_1, F_2, G_1, G_2, H_1, I_1, J_1\}.$$

- **K**: The canonical divisor of X in the basis above.
- **R**: Gives the list of the classes of (-2)-curves on X.
- **S**: Gives the list of the classes of (-1)-curves on X.
- **NegEx4**: Gives the list of the classes of negative curves on X.
- **PND**: Gives the list of the possible necessary divisors of X, excluding negative curves (which are the elements of $\mathrm{BNef}(X)$).
- **X1**: Returns the list of all divisors that do not pass Test 1.

```
L := ToricLattice(10);
M := DiagonalMatrix([1,-1,-1,-1,-1,-1,-1,-1,-1,-1]);
K := L![-3,1,1,1,1,1,1,1,1,1];
```

```
G1 := [0,-1,1,0,0,0,0,0,0,0];
G2 := [0,0,0,-1,1,0,0,0,0,0];
G3 := [0,0,0,0,0,-1,1,0,0,0];
L1 := [1,-1,-1,0,0,0,-1,0,0,0];
L2 := [1,0,-1,0,0,0,0,-1,-1,0];
L3 := [1,0,-1,0,-1,0,0,0,0,-1];
L4 := [1,0,0,0,-1,0,-1,0,-1,0];
L5 := [1,0,0,0,0,-1,-1,0,0,-1];
L6 := [1,0,0,-1,-1,0,0,-1,0,0];
R  := [G1,G2,G3,L1,L2,L3,L4,L5,L6];

Rv := [L!(Eltseq(Matrix(1,10,Eltseq(r))*M)): r in R];
Kv := L!(Eltseq(Matrix(1,10,Eltseq(K))*M));
P := &meet[HalfspaceToPolyhedron(Rv[i],0) : i in [1..#Rv]]
     meet HyperplaneToPolyhedron(-Kv,1);
pts :=  SetToSequence(Points(CompactPart(P)));
S := [L!p+(qua(p,p,M)+1)/2*K: p in pts];

NegEx4 := [L!r : r in R] cat S;
H := HNef(NegEx4);
PND:=Seqset(H);
```

```
X:=[];
for A in PND do
if #Test1(A,NegEx4,M,H) ge 1 then X:= X cat [A];
end if;
end for;

X1:=PND diff Set(X);
```

This gives as output

```
> X1;
{
    ( 3, -1, -1, -1, -1, -1, -1, -1, -1, -1 ),
    ( 19, -6, -6, -2, -8, -7, -7, -9, -4, -5 ),
    ( 9, -3, -3, -2, -2, -2, -3, -5, -1, -4 ),
    ( 10, -3, -3, -1, -4, -4, -4, -5, -2, -2 ),
    ( 11, -4, -4, -2, -2, -3, -3, -1, -6, -5 ),
    ( 7, -1, -1, -2, -2, -1, -2, -3, -3, -4 ),
    ( 6, -2, -2, -3, -3, -2, -2, 0, -1, -1 ),
    ( 13, -2, -2, -4, -4, -3, -3, -5, -6, -7 ),
    ( 9, -1, -4, -3, -3, -4, -4, -3, -2, -1 ),
    ( 17, -2, -8, -6, -6, -7, -7, -5, -4, -3 ),
    ( 5, -2, -2, -1, -1, -1, -1, 0, -3, -2 ),
    ( 13, -4, -4, -6, -6, -5, -5, -1, -2, -3 ),
    ( 17, -6, -6, -4, -4, -5, -5, -9, -2, -7 )
}
```

5.4.5 Example 3.2.1

In this subsection, we give the program that returns the four divisors we found in Example 3.2.1.

- **L**: The abelian group $\mathbb{Z}^{10}$.
- **M**: The intersection matrix associated to the basis of $\text{Pic}(X)$ given by

$$\{H, E_1, ..., E_5, F_1, .., F_4\}.$$

- **K**: The canonical divisor of X in the basis above
- **R**: Gives the list of the classes of (-2)-curves on X.
- **S**: Gives the list of the classes of (-2)-curves on X.
- **NegEx4**: Gives the list of the classes of negative curves on X.
- **PND**: Gives the list of the possible necessary divisors of X, excluding negative curves (which are the elements of $\text{BNef}(X)$).
- **X1**: Returns the list of all divisors that do not pass Test 1.

```
L := ToricLattice(10);
M := DiagonalMatrix([1,-1,-1,-1,-1,-1,-1,-1,-1,-1]);
K := L![-3,1,1,1,1,1,1,1,1,1];
```

```
G1 := [0,-1,1,0,0,0,0,0,0,0];
G2 := [0,0,-1,1,0,0,0,0,0,0];
G3 := [0,0,0,-1,1,0,0,0,0,0];
G4 := [0,0,0,0,-1,1,0,0,0,0];

H1 := [0,0,0,0,0,0,-1,1,0,0];
H2 := [0,0,0,0,0,0,0,-1,1,0];
H3 := [0,0,0,0,0,0,0,0,-1,1];

L1  := [1,0,0,0,0,-1,0,0,-1,-1];
Q  := [2,-1,-1,-1,-1,-1,0,0,0,-1];

R :=[G1,G2,G3,G4,H1,H2,H3,L1,Q];

NegEx4 := S cat R;
H := HNef(NegEx2);
PND := Set(HNef(NegEx2));
```

```
X:=[];
for A in PND do
if #Test1(A,NegEx4,M,H) ge 1 then X:= X cat [A];
end if;
end for;

X1:=PND diff Set(X);
```

This gives as output

```
> X1;
{
    (2, 0, 0, 0, 0, 0, -1, -1, -1, -1),
    (7, 0, -3, -3, -3, -3, -1, -2, -2, -2),
    (1, 0, 0, 0, 0, 0, 0, 0, 0, -1),
    (3, -1, -1, -1, -1, -1, -1, -1, -1, -1)
}
```

5.4.6 Example 3.3.1

In this subsection, we give the program that returns the three divisors we found in Example 3.3.1.

- **L**: A representation of Pic(X) through an isomorphism with $\mathbb{Z}^7$.
- **M**: The intersection matrix associated to the basis of Pic(X) given by

$$\{H, T_1 ..., T_6\}.$$

- **K**: The canonical divisor of X, represented by the isomorphism above.
- **R**: Gives the list of the classes of (-2)-curves on X.
- **S**: Gives the list of the classes of (-2)-curves on X.
- **NegEx2**: Gives the list of the classes of negative curves on X.
- **PND**: Gives the list of the possible necessary divisors of X, excluding negative curves (which are the elements of BNef(X)).
- **X1**: Returns the list of all divisors that do not pass Test 1.

```
L := ToricLattice(7);
M := DiagonalMatrix([1,-1,-1,-1,-1,-1,-1]);
K := L![-3,1,1,1,1,1,1];
```

```
E1 := [0,0,0,0,-1,1,0];
E2 := [1,0,0,-1,-1,-1,0];
E3 := [0,0,0,-1,1,0,0];
E4 := [0,1,0,0,0,0,-1];
E5 := [0,-1,1,0,0,0,0];
E6 := [0,0,-1,1,0,0,0];
R  := [E1,E2,E3,E4,E5,E6];

E7 := [0,0,0,0,0,0,1];
S  := [E7];

NegEx2 := S cat R;
H := HNef(NegEx2);
PND := Set(HNef(NegEx2));
```

```
X:=[];
for A in PND do
if #Test1(A,NegEx2,M,H) ge 1 then X:= X cat [A];
end if;
end for;

X1:=PND diff Set(X);
```

This gives as output

```
> X1;
{
    (3, -1, -1, -1, -1, -1, -1),
    (1, 0, 0, 0, 0, -1, 0),
    (1, 0, 0, 0, 0, 0, 0)
}
```

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