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Soluciones estacionarias con campo escalar y el efecto de su acoplamiento

Stationary solutions with scalar field and the effect of its coupling

Tesis para optar al grado de Doctor en Ciencias Físicas

por

Cristián Rodrigo Erices Osorio

Profesor Guía : Dr. Cristián Martínez Silva

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Comisión : Dr. Fabrizio Canfora
Dr. Adolfo Cisterna
Dr. Fernando Izaurieta
Dr. Luis Roa

Para mis padres y mi Sofía

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Resumen

Para estudiar los efectos y consecuencias del campo escalar en gravitación, nosotros construimos y analizamos en detalle nuevas soluciones en dos escenarios diferentes.

En el contexto de teorías escalar-tensor, encontramos soluciones de agujero negro tanto asintintóticamente Anti-de Sitter (AdS) como planas para un caso particular de la acción de Horndeski. La solución es dada para dimensión arbitraria, encontrando una nueva clase de agujeros negros esféricamente simétricos y asintóticamente localmente planos cuando la constante cosmológica Λ está presente en el sector cinético no-minimal de la teoría. Cuando la constante cosmológica se anula la solución es asintóticamente plana con una perfecta correspondencia con el espaciotiempo de Minkowski en infinito. En este caso obtenemos una solución que representa un universo con campo eléctrico constante. El campo eléctrico en infinito es sustentado solamente por la constante cosmológica. Anulando la carga eléctrica recuperamos la solución de Schwarzschild. Adicionalmente encontramos un solitón gravitacional no trivial, lo cual permite un análisis termodinámico a través del enfoque de Hawking-Page. Considerando los mismos acoplamientos, es decir, minimal y no-minimal para el campo escalar y la extensión bi-escalar de gravedad de Horndeski, construimos y describimos una solución de estrella bosónica. En esta parte, la estrella bosónica es estudiada para dos casos de especial interés: el caso donde el potencial es dado por un término masivo solamente y el caso auto-interactuante que presenta dos vacíos locales degenerados. Las principales propiedades de la solución son comparadas con las configuraciones estándar construidas con el término cinético minimalmente acoplado.

En la segunda parte de esta tesis, la influencia del campo escalar es investigada en un nivel más simple considerando un campo escalar minimalmente acoplado como campo de materia. Sin embargo, un tipo de solución más compleja es encontrada, la cual corresponde a la solución general en cuatro dimensiones estacionaria cilíndricamente simétrica del sistema de Einstein con campo escalar no masivo y con una constante cosmológica no positiva. La solución tiene dos constantes de integración con información local y adicionalmente dos topológicas. El efecto del campo escalar es explorado usando el esquema de Petrov para la clasificación algebraica de

la solución. Las cargas conservadas asociadas a las simetrías son calculadas usando el método de Regge-Teitelboim. Finalmente, las singularidades de curvatura son removidas cuando la constante cosmológica se anula, encontrando espaciotiempos localmente homogéneos en la presencia de un campo escalar fantasma.

Esta tesis describe el trabajo que fue presentado en las siguientes publicaciones,

- *"Boson Stars in bi-scalar extensions of Horndeski gravity"*,
  Y. Brihaye, A. Cisterna and C. Erices,
  Enviado a Physical Review D.
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- *"Stationary cylindrically symmetric spacetimes with a massless scalar field and a nonpositive cosmological constant"*,
  C. Erices and C. Martínez,
  Phys. Rev. D 92, 044051 (2015),
  arXiv:1504.06321 [gr-qc].

- *"Asymptotically locally AdS and flat black holes in the presence of an electric field in the Horndeski scenario"*,
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Abstract

In order to study the effect and consequences of scalar fields on gravitation, new solutions in two different scenarios are constructed and analyzed in detail.

In the context of scalar-tensor theories, electrically charged asymptotically locally AdS and asymptotically flat black hole solutions are found for a particular case of the Horndeski action. The solution is given for all dimensions and a new class of asymptotically locally flat spherically symmetric black hole is found when the cosmological constant Λ is present in the non-minimal kinetic sector. When the cosmological constant vanishes the black hole is asymptotically flat in perfect matching with Minkowski spacetime at infinity. In this case we get a solution which represents an electric Universe. The electric field at infinity is only supported by Λ. Switching off the electric charge we recover Schwarzschild solution. Additionally a nontrivial gravitational soliton is found, allowing the thermodynamical analysis through the Hawking-Page approach. Considering the same couplings, i.e. minimal and non-minimal for the scalar field and the bi-scalar extension of Horndeski gravity, a boson star solution is constructed and described. In this part the boson star is studied for two cases of special interest: the case where the potential is given by a mass term only and the case of a six order self-interaction that presents two degenerate local vacua. The principal properties of the solution are compared with standard configurations constructed with the usual minimally coupled kinetic term.

In the second part of this thesis, the influence of scalar fields is investigated in a simpler level considering a minimally coupled scalar field as a matter field. Nevertheless, a more complex kind of solution is found and the general four dimensional stationary cylindrically symmetric solution of Einstein-massless scalar field system with a non-positive cosmological constant is presented. The solution possesses two integration constants of local meaning and additionally two topological ones. The effect of the scalar field is explored using the Petrov scheme for the algebraic classification of the solution. Conserved charges associated with the symmetries are computed using the Regge-Teitelboim method. Finally, curvature singularities are removed when the cosmological constant vanishes and locally homogeneous spacetimes are found in the presence of a phantom scalar field.

This thesis describes the work that was presented in the following publications,

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Chapter 1

Introducción

Relatividad General (RG) es una teoría clásica de gravedad basada en sólidos fundamentos matemáticos y físicos. Concuerda con una enorme precisión pruebas observacionales locales tanto en el régimen de gravedad débil como en el fuerte incluyendo experimentos en laboratorios de la ley de fuerza de Newton. RG, no es solamente una teoría física muy exitosa. Es teóricamente muy robusta y como matemáticamente resulta, una teoría métrica única. A pesar del gran progreso que RG ha tenido, aún existen algunas preguntas abiertas en las escalas más bajas y más altas de longitud. Puesto que RG no es una teoría renormalizable, se espera que desviaciones respecto a esta aparezcan a alguna escala entre la de Planck y la escala de longitud más baja a la cual hemos accedido hasta ahora. Es tentador considerar un escenario donde aquellas desviaciones persistan hasta escalas cosmológicas y den cuenta de la Materia Oscura y/o Energía Oscura. Después de todo, nosotros solamente detectamos este componente "oscuro" a través de gravedad. Sin embargo, hay un problema mayor con esta forma de pensar. No hay señales de estas modificaciones en el rango de escalas para las cuales gravedad ha sido exhaustivamente probada. De esta manera, ellas deberían ser relevantes a escalas muy pequeñas, luego de alguna manera desaparecer a escalas intermedias y contener RG, para después reaparecer a escalas mayores. Es difícil imaginar qué puede conducir a tal comportamiento, que en realidad contradice nuestra intuición teórica básica sobre la separación de escalas y las teorías de campo efectivas. Esta es la razón por la cual la comunidad ha puesto más atención a las teorías de gravedad alternativas durante la última década.

En 1971, Lovelock estableció [1] que si uno considera las siguientes cuatro propiedades

(I) Un principio de acción invariante bajo difeomorfismos y con ecuaciones de campo simétricas.

(II) Ecuaciones de campo de segundo orden.

(iii) Espaciotiempo cuadridimensional.

(iv) Solamente la métrica está involucrada en la descripción puramente gravitacional de la teoría.

la acción de Einstein-Hilbert

$$S = \frac{m_{pl}^4}{2} \int d^4x \sqrt{-g}[R - 2\Lambda] \tag{1.1}$$

es la única acción que provee ecuaciones de movimiento de segundo orden para el campo métrico. De este modo, Lovelock nos muestra el camino que podemos seguir para obtener una descripción modificada de la gravitación que contenga RG. Una de las opciones más estudiadas en esta línea, junto a teorías en dimensiones mayores, corresponde a teorías donde nuevos grados de libertad entran en la descripción gravitacional. De hecho, relajando la condición (iv) y dejando que le nuevo grado de libertad sea un campo escalar es que surgen las teorías escalar-tensor.

La teoría escalar-tensor fue concebida originalmente por Jordan, quien comenzó incorporando una variedad curva cuadridimensional en un espaciotiempo plano de cinco dimensiones [2]. Él mostró que una restricción, cuando se formula la geometría proyectiva, puede ser definida por un campo escalar cuadridimensional, lo cual permite describir una "constante" gravitacional dependiente del espaciotiempo, en concordancia con el argumento de Dirac sobre la constante gravitacional que debería ser dependiente del tiempo [3], lo cual obviamente está más allá de lo que puede ser entendido dentro del ámbito de la teoría estándar. Él también discutió sobre la posible conexión de esta teoría con otra teoría en cinco dimensiones, que había sido desarrollada por Kaluza y Klein [4, 5]. La introducción de un campo escalar no-minimalmente acoplado por Jordan, marcó el nacimiento de las teoría escalar-tensor. El esfuerzo de Jordan fue continuado particularmente por Brans y Dicke e implementó el requerimiento de que el principio de equivalencia débil fuera respetado, en contraste con el modelo de Jordan. El prototipo de la gravedad escalar-tensor es la teoría de Brans-Dicke [6] la cual ha sido extensivamente estudiada a lo largo de los años (vea [7, 8, 9] y sus referencias). Nosotros deberíamos notar que en la clase de teorías escalar-tensor caen otras teorías de gravedad modificadas como $f(R)$ o $f(\hat{G})$ las cuales son precisamente teorías escalar-tensor particulares de una manera encubierta [10]. Es más, otras modificaciones de RG tales como bigravedad o teorías de gravedad masiva [11] admiten teorías escalar-tensor como límites particulares, por ejemplo el límite de desacoplamiento para gravedad masiva [12]. Por consiguiente, teorías escalar-tensor son un prototipo consistente de modificación de RG y sus propiedades más importantes son de alguna forma esperables en otras teorías consistentes de gravedad. Un progreso notable fue hecho por Horndeski durante los 70's cuando construyó la teoría escalar-tensor más general con ecuaciones de movimiento de segundo orden para la métrica y el campo escalar [13]. Si bien es cierto que el estudio de teorías escalar-tensor no es un tópico nuevo, actualmente, ha resurgido un gran interés debido al estudio de teorías de Galileón y sus aplicaciones. Fue mostrado que la generalización de los Galileones (originalmente formulados en el espacio plano) a un espaciotiempo curvo, para una parametrización particular de la teoría, se reduce a una parte de la teoría descrita por Horndeski.

En particular, hay un subconjunto de la acción de Horndeski donde la teoría provee un campo escalar con un acoplamiento cinético no-minimal dado por la curvatura. Cuando este acoplamiento es dado por el tensor de Einstein, se ha mostrado que es posible estudiar el proceso de inflación del Universo sin la necesidad de incluir potencial alguno [14]. En este escenario, la teoría exhibe soluciones de agujero negro lo cual incrementó el interés en las teorías de Horndeski. La primera solución de agujero negro fue descubierta por Rinaldi [15] donde el teorema de no-pelo para Galileones [16], que previene la existencia de soluciones de agujero negro asintóticamente planas, es sorteada relajando el comportamiento asintótico, obteniendo una solución asintóticamente AdS. Sin embargo, en este caso, la configuración de campo escalar es imaginaria fuera del horizonte de eventos violando la condición de energía débil. Estos problemas fueron resueltos en [17] donde construyen una solución de agujero negro, asintóticamente localmente tanto AdS como plana, con un campo escalar real fuera del horizonte de eventos y que satisface la condición de energía débil.

Agujeros negros y estrellas compactas son de una importancia significativa en teorías de gravedad alternativas puesto que constituyen pruebas potenciales del régimen de gravedad fuerte. Ya habiendo explorado las soluciones de agujero negro en el escenario de Horndeski y siguiendo la misma línea de trabajo, nos dedicamos al estudio de objetos compactos gravitacionales cuando los acoplamientos minimal y no-minimal del campo escalar son considerados. En este caso, la construcción de estrella de neutrones ha sido primeramente abordada en [18]. Ahí, se muestra que las estrellas de neutrones estáticas y las enanas blancas son sustentadas por este modelo, imponiendo de una manera bastante natural, restricciones de tipo astrofísicas sobre el único parámetro libre que estas soluciones exhiben. Su contraparte de rotación lenta también ha sido estudiada en [19, 20, 21]. Sin embargo, hay soluciones gravitacionales solitónicas conocidas como estrellas bosónicas. Las estrellas de bosones construidas originalmente en [22] son soluciones estacionarias compactas de las ecuaciones de Einstein-Klein-Gordon (EKG) con una configuración de campo escalar complejo. Estas soluciones, que han mostrado la posibilidad de ser estables, representan un balance entre la naturaleza atractiva de la gravedad y el comportamiento dispersivo de los campos escalares, y puede ser pensada como una colección de campos escalares fundamentales estables limitados por la gravedad, donde la carga de Noether representa el número total de partículas bosónicas. Las estrellas bosónicas han sido ampliamente estudiadas durante las últimas tres décadas. Aún más, ha sido mostrado que las propiedades observacionales de las estrellas bosónicas son bastante similares a su contraparte en agujeros negros, siendo propuestas como posibles candidatos para representar objetos súper masivos en el centro de las galaxias y se espera que debido a su dinámica, sean detectadas por observaciones astronómicas [23].

En una perspectiva diferente, los campos escalares han sido considerados como una manera adecuada y representativa para describir fuentes de materia como es sugerido por una abrumadora literatura al respecto. Al mismo tiempo, como es señalado por diversas observaciones astronómicas, las fuentes gravitacionales en nuestro Universo poseen rotación. Con esto en mente, soluciones exactas con rotación en cuarto dimensiones en el contexto de GR describen objetos muy interesantes desde el punto de vista astrofísico, y son particularmente difíciles de encontrar debido a la complejidad de las ecuaciones de campo para ansatz estacionarios. En este marco, la configuración más simple está representada por espaciotiempos de simetría cilíndrica interactuando con un campo escalar minimalmente acoplado.

A pesar que los espaciotiempos cilíndricamente simétricos son bastante conocidos en vacío, soluciones exactas que contienen un campo escalar sin masa como fuente de materia no han sido obtenidas en su forma más general hasta ahora. Solamente soluciones con simetría plana, que son un caso particular de las cilíndricas, han sido reportadas [24, 25].

Para todos estos propósitos previamente mencionados, sería provechoso e interesante explorar y presentar nuestra contribución original en dos aspectos diferentes de la influencia del campo escalar en gravitación. Primero, como un grado de libertad adicional para la interacción gravitacional en el contexto de las teorías de Horndeski a través del estudio de soluciones de agujeros negros y estrellas bosónicas cuando el sector cinético no-minimal es considerado. Segundo, como un campo de materia analizando soluciones estacionarias cilíndricamente simétricas con un campo escalar no masivo en presencia de una constante cosmológica no positiva.

El plan de esta tesis es el siguiente:

En el Capítulo 2 comienza la tesis en inglés con la traducción a dicho idioma de esta introducción.

En el Capítulo 3 se presenta una breve revisión de las teorías escalar-tensor más importantes. Damos algunos detalles sobre el origen y motivación de las teorías escalar-tensor presentando la teoría de Brans-Dicke y su generalización natural. Finalmente tratamos la acción de Horndeski dando un énfasis especial sobre el acoplamiento cinético no-minimal.

El Capítulo 4 contiene una completa descripción de una nueva solución de agujero negro con carga eléctrica en el escenario de Horndeski [26]. Vamos un paso más allá respecto al trabajo de [17] construyendo su contraparte de agujero negro eléctricamente cargado. Hay dos propiedades destacables cuando el acoplamiento minimal se anula y solamente el acoplamiento cinético no-minimal es considerado. Una de las principales características, propio de esta solución, es que la inclusión de carga eléctrica sustenta una configuración asintóticamente plana que asintóticamente coincide con el espaciotiempo de Minkowski cuando la constante cosmológica es nula. Adicionalmente, para constante cosmológica no nula, la solución representa un universo con campo eléctrico constante dado por la constante cosmológica. La solución es también encontrada para dimensiones arbitrarias.

En el Capítulo 5, después de una breve introducción sobre la historia de las estrellas bosónicas, discutimos sobre la naturaleza de ellas dando algunos detalles de cómo sortear el teorema de Derrick así como también un argumento heurístico para estimar la masa crítica para mini-estrellas bosónicas basado en criterios de estabilidad. Luego, presentamos el Lagrangiano considerado y las ecuaciones de EKG. Se dan también expresiones para las características básicas como la masa, radio de la estrella bosónica y la carga de Noether asociada a la simetría $U(1)$. Las consecuencias que el potencial tiene sobre la máxima masa de la estrella bosónica es discutido para potenciales de campo libre y auto-interactuantes de cuarto orden. Finalmente damos algunos comentarios breves sobre la estrella bosónica en el contexto de teorías de gravedad alternativa.

Capítulo 6 es dedicado a las estrellas de bosones en la extensión bi-escalar de la gravedad de Horndeski [27]. El modelo, ansatz para la solución y las ecuaciones de campo son presentadas. Debido a su complejidad, las ecuaciones de campo son resueltas numéricamente definiendo condiciones de borde apropiadas para soluciones regulares asintóticamente planas y un adecuado conjunto de parámetros adimensionales para métodos numéricos. Las soluciones son analizadas para un potencial de campo libre y un auto-interactuante de sexto orden de particular interés, con acoplamiento cinético no-minimal y comparado con las configuraciones minimalmente acopladas.

En el Capítulo 7 damos una reseña sobre soluciones de vacío con simetría cilíndrica y las presentamos para constante cosmológica nula y no nula. Estos espaciotiempos corresponden a la familia de soluciones de Lewis y a la cuerda estática de simetría cilíndrica. Finalmente revisamos la primera solución de este tipo con singularidad de curvatura vestida por un horizonte de eventos encontrada por Lemos [28]. La llamada cuerda negra es transformada a una solución estacionaria realizando una transformación de coordenadas impropia la cual motivará nuestra solución estacionaria en el capítulo siguiente.

El Capítulo 8 muestra la derivación y el estudio de la solución general cilíndricamente simétrica para un campo escalar no masivo minimalmente acoplado en presencia de una constante cosmológica no positiva en un espaciotiempo cuadridimensional [29]. Inesperadamente una subfamilia de la solución no tiene limite estático. Sin embargo, enfocamos el análisis posterior a la clase de soluciones que sí poseen límite estático. Describimos sus propiedades locales y globales así como también damos una interpretación a las constantes de integración. Para esto, calculamos las cargas conservadas asociadas a las simetrías del espaciotiempo relacionándolas con aquellas constantes. El caso para constante cosmológica nula contiene un espaciotiempo regular no trivial de especial interés. Este posee todos sus invariantes escalares constantes (espaciotiempos CSI) y son localmente homogéneos.

Las conclusiones globales de esta tesis son presentadas en el Capítulo 9 con su respectiva traducción al español en Capítulo 10. Posteriormente, el Apéndice A muestra explícitamente las ecuaciones de campo resueltas numéricamente en el Capítulo 6. Finalmente, la biliografía utilizada a lo largo de esta tesis.

Chapter 2

Introduction

General Relativity (GR) is a classical theory of gravity which is based on very solid mathematical and physical foundations. It agrees with overwhelming accuracy local observational tests both for weak and strong gravity including laboratory tests of Newton's force law. GR, is not only a very successful physical theory. It is theoretically very robust and as it turns out mathematically a unique metric theory. Despite the great progress that GR has had, there is still some unanswered questions at lower and higher scales. Since GR is not a renormalizable theory, it is expected that deviations from it will show up at some scale between the Planck scale and the lowest length scale we have currently accessed. It is tempting to consider a scenario where those deviations persist all the way to cosmological scales and account for Dark Matter and/or Dark Energy. After all, we do only detect these dark component through gravity. However, there is a major problem with this way of thinking. There is no sign of these modifications in the range of scales for which gravity has been exhaustively tested. So, they would have to be relevant at very small scales, then somehow switch off at intermediate scales and contain GR, then switch on again at larger scales. It is hard to imagine what can lead to such behavior, which actually contradicts our basic theoretical intuition about separation of scales and effective field theory. This is the reason why the community has paid more attention to alternative theories of gravity during the last decade or so.

It is known due to Lovelock [1] that taking into account a theory that satisfies the following four statements

(i) Action principle invariant under diffeomorphisms and symmetric field equations.

(ii) Second order field equations.

(iii) Four-dimensional spacetime.

(IV) Only the metric field enters in the purely gravitational description of the theory.

then the Einstein-Hilbert action

$$S = \frac{m_{pl}^4}{2} \int d^4x \sqrt{-g}[R - 2\Lambda] \tag{2.1}$$

is the unique action giving equations of motion of second order in the metric field variable. In these lines Lovelock give us the possible paths we can follow in order to obtain a modified description of gravity containing GR. One of the most studied options in this line, along with theories in higher dimensions, corresponds to theories where new degrees of freedom enter in the gravitational description. Indeed, by relaxing the condition (IV) and leaving this new degree of freedom to be a scalar field, scalar-tensor theories arise.

The scalar-tensor theory was conceived originally by Jordan, who started to embed a four-dimensional curved manifold in five-dimensional flat spacetime [2]. He showed that a constraint in formulating projective geometry can be a four-dimensional scalar field, which enables one to describe a spacetime-dependent gravitational "constant", in accordance with Dirac's argument that the gravitational constant should be time-dependent [3], which is obviously beyond what can be understood within the scope of the standard theory. He also discussed the possible connection of his theory with another five-dimensional theory, which had been offered by Kaluza and Klein [4, 5]. The introduction of a non-minimally coupled scalar field by Jordan, marked the birth of the scalar-tensor theory. Jordan's effort was taken over particularly by Brans and Dicke and implemented the requirement that the weak equivalence principle be respected, in contrast to Jordan's model. The prototype of scalar-tensor gravity is Brans-Dicke theory [6] which has been studied extensively throughout the years (see [7, 8, 9] and references within). We should note that in the class of scalar-tensor theories fall also other modified gravity theories like $f(R)$ or $f(\hat{G})$ which are just particular scalar-tensor theories in disguise [10]. Furthermore, other interesting GR modifications such as bigravity or massive gravity theories [11] admit scalar-tensor theories as particular limits, for example the decoupling limit for massive gravity [12]. Hence, scalar-tensor theories are a consistent prototype of GR modification and their important properties are expected in some form, in other consistent gravity theories. A remarkable progress was made by Horndeski during the 70's when he built the most general scalar-tensor theory with equations of motion of second order for both the metric and for the scalar field [13]. While it is true that the study of scalar-tensor theories is not a new topic, currently, great interest resurfaced due to the study of Galileon theories and their applications. It was shown that the generalization of the Galileons (originally formulated in flat space) to a curved background, for a particular parameterization of the theory, reduces to part of the theory before described by Horndeski.

In particular, there is some subset of Horndeski action where the theory provides scalar field with non-minimal kinetic coupling given by the curvature. When this coupling is given by the Einstein tensor, it has been shown that it is possible to study the inflation process of the Universe without need to include any potential term [14]. In this scenario, the theory exhibits black hole solutions regaining the interest in Horndeski theories. The first black hole solution was discovered by Rinaldi [15] where the no-hair theorem for Galileons [16], which prevents the existence of asymptotically flat black hole solutions, is circumvented by relaxing the asymptotic behavior, obtaining an asymptotically AdS solution. However, in this case, the scalar field configuration is imaginary outside the event horizon violating the weak energy condition. Those problems were solved in [17] where they construct an asymptotically locally AdS and flat black hole solution with a real scalar field outside the event horizon and satisfying the weak energy condition.

Black holes and compact stars are of significant importance in alternative theories of gravity as they constitute potential probes of the strong gravity regime. Having explored the black hole solutions in the Horndeski scenario and following the same line of work, we dedicate to the study of compact gravitational objects when the minimal and non-minimal kinetic couplings to the scalar field are considered. In this case, the construction of neutron stars has been tackled first in [18]. There, static neutron stars and white dwarfs are shown to be supported by this model, imposing in a very natural way, astrophysical constraints on the only free parameter that these solutions exhibit. Its slowly rotating counterpart have been also studied in [19, 20, 21]. Nevertheless, there are gravitating solitonic solutions known as boson stars. Boson stars originally constructed in [22] are compact stationary solutions of the Einstein-Klein-Gordon (EKG) equations with a complex scalar field configuration. These solutions, which have shown the possibility to be stable, represent a balance between the attractive nature of gravity and the dispersive behavior of scalar fields, and can be thought as a collection of stable fundamental scalar fields bounded by gravity, where the Noether charge represent the total number of bosonic particles. Boson stars have been widely studied during the last three decades. Furthermore, it has been showed that observational properties of boson star are quite similar to its counterpart in black holes, having proposed as possible candidates to represent super massive objects at the center of galaxies and it is expected due to their dynamics, to be detected by astronomical observations [23].

In a different perspective, scalar fields has been considered as a suitable and representative way to describe matter sources as it is suggested by the overwhelming literature to this respect. At the same time, as it is pointed by several astronomical observations, gravitational sources in our Universe do posses rotation. With this in mind, four dimensional exact rotating solutions in the context of GR describe very interesting object from an astrophysical point of view, and are particularly difficult to find due to the complexity of the field equations for stationary ansatz. In this setting, the simplest configuration are represented by cylindrically symmetric spacetimes interacting with a minimally coupled scalar field.

Despite the static cylindrically symmetric spacetimes are widely known in vacuum, exact solutions containing a massless scalar field as matter source have not been obtained in the most general form until now. Only solutions with plane symmetry, which are a particular case of the cylindrical ones, have been reported [24, 25].

For all the purposes previously mentioned, it would be helpful and interesting to explore and present our original contributions in two different aspects of the influence of the scalar field in gravitation. First, as an additional degree of freedom for the gravitational interaction in the context of Horndeski theories through the study of black hole and boson star solutions when the non-minimal kinetic sector is considered. Second, as a matter field analyzing the stationary cylindrically symmetric solution with a massless scalar field in the presence of a nonpositive cosmological constant.

The plan of this thesis is the following:

In Chapter 3 a short review of the most important scalar-tensor theories is presented. We give some details about the origin and motivation of scalar-tensor theories presenting the Brans-Dicke theory and its natural generalization. Finally we treat the Horndeski action giving special emphasis on the non-minimal kinetic coupling.

Chapter 4 is a complete description of a new electrically charged black hole solution in the Horndeski scenario [26]. We go one step further than the work presented in [17] constructing the electrically charged black hole counterpart. There are two remarkable properties when the minimal coupling to the scalar field is switched off and only the non-minimal kinetic coupling is considered. One of the main features, proper from this solution, is that the inclusion of electric charge supports an asymptotically flat configuration which asymptotically match Minkowski spacetime when the cosmological constant vanishes. In addition, for non zero cosmological constant, the solution represents asymptotically an Electric Universe presenting an asymptotically constant electric field supported by the cosmological constant. The solution is also found for arbitrary dimensions.

In Chapter 5, after a brief introduction of boson star about the precedent history of these kind of solutions, we discuss on the nature of boson star giving some details about how to circumvent Derrick's theorem as well as a heuristic argument to estimate critical mass for mini-boson stars based on a stability criterion. Next, we present the Lagrangian considered and the EKG equations. Expressions for basic features as the mass, radius of the boson star and the Noether charge associated with the $U(1)$ symmetry are provided as well. The consequences that the potential has on the maximum mass of the boson star is discussed for free-field and quartic self-interacting potential. Finally we give some brief remarks on the boson star in the context of alternative theories of gravity.

Chapter 6 is devoted to boson stars in bi-scalar extensions of Horndeski gravity [27]. The model, ansatz for the solution and field equations are presented. Due to its complexity, field equations are solved numerically by defining suitable boundary conditions for regular asymptotically flat solutions and an adequate set of dimensionless parameters for numerical methods. Solutions are analyzed for free-field and six order self-interacting potentials of particular interest, with non-minimally kinetic coupling and compared to the minimally coupled configurations.

In Chapter 7 we give a short overview on the history of cylindrically symmetric vacuum solutions and we present them for vanishing and non vanishing cosmological constant. These spacetimes correspond to the Lewis family of solutions and the static cylindrically symmetric string respectively. Finally we review the first solution of this type with curvature singularity dressed by an event horizon found by Lemos [28]. The so called black string is transformed to the stationary solution by performing an improper coordinate transformation which will motivate our stationary solution in the next chapter.

Chapter 8 displays the derivation and study of the general stationary cylindrically symmetric solution for a minimally coupled massless scalar field in presence of a non-positive cosmological constant in four spacetime dimensions [29]. Surprisingly a subfamily of the solution does not posses static limit. However, we focus the subsequent analysis to the class of solutions that do possesses this limit. We describe its local and global properties as well as provide an interpretation to the integration constants. For this, we compute the conserved charges associated to the symmetries of the spacetime relating them with those constants. The case for vanishing cosmological constant contains a nontrivial regular spacetime of special interest. It possesses all its scalar invariants constant (CSI spacetimes). These are particular regular spacetimes since they are locally homogeneous.

Global conclusions of this thesis are presented in Chapter 9 and its translation to spanish in the next Chapter 10. Then, Appendix A shows explicitly the field equations solved numerically in Chapter 6. Finally, the bibliography employed along this thesis.

Chapter 3

Scalar-tensor theories

Considering alternatives to a theory as successful as General Relativity can be seen as a very radical move. However, from a different perspective it can actually be though of as a very modest approach to the challenges gravity is facing today. Developing a fundamental theory of quantum gravity from first principle and reaching the stage where this theory can make testable predictions has proved to be a very lengthy process. At the same time, it is hard to imagine that we will gain access to experimental data at scales directly relevant to quantum gravity any time soon. Alternative theories of gravity, thought of as effective field theories, are the phenomenological tools that provide the much needs contact between quantum gravity candidates and observations at intermediate and large scales. In this way the so called scalar-tensor theories arises as an alternative modification of General Relativity where a scalar field is an additional degree of freedom that describes the gravitational interaction[1].

3.1. Fundamental properties

The possible scalar-tensor theories could be infinite if we ignore some fundamental properties. In order to get a physically reasonable modification of General Relativity those properties are

1. **Precision tests.** The theory must generate predictions that pass all Solar system, binary pulsar, cosmological and experimental tests carried out to date. This is a threefold requirement in the following aspects:

   a) **General Relativity Limit.** In some limit, such as the weak field one, the theory must predicts the same phenomena than General Relativity

[1]A complete review on modifications of Einstein's theory of gravity and scalar-tensor theories can be found in [30, 31] respectively. This introductory chapter is based on these references.

within experimental precision.

b) **Existence of Known Solutions.** The theory should admit solutions corresponding closely to what we observe, such as (nearly) flat spacetime, (nearly) Newtonian stars, cosmological solutions, etc.

c) **Stability of Solutions.** Solutions of the previous item must be stable to small perturbations on timescales smaller than the age of the Universe.

These properties are not necessarily independent. The existence of weak-field limit usually implies the existence of known solutions. Additionally, the existence of solutions does not necessarily imply stability. In addition to these fundamental properties the modified gravity theory should have some theoretical properties. The most common ones are:

2. **Well-motivated from Fundamental Physics.** One expects that, as a modification of General Relativity, the theory solve some fundamental problem in physics such as late time acceleration or the quantum gravitational description of nature. In other words, there must be some fundamental theory or principle from which the modified theory (effective or not) derives.

3. **Well-posed Initial Value Formulation.** The initial data must uniquely determine the solution to the modified field equations and this must depends continuously on this data.

4. **Strong Field Inconsistency.** The theory must lead to observable deviations from General Relativity in the strong-field regime.

A purely metric description of gravitational interaction which gives second order field equations and is diffeomorphism-invariant restricts terms allowed in the action. In this way, the dynamics of the metric field is left to be governed by Lovelock's action. This is the result of Lovelock's theorem [1] and provide us an action principle which is the natural generalization of General Relativity to higher dimensions. Consequently, in four dimensions this reduces to Einstein-Hilbert action. However, any attempt to modify the action of General Relativity will generically lead to extra degrees of freedom. The simplest way to do this, is not considering a purely metric gravitational description of gravitation but one with an additional degree of freedom as a scalar field. In this way, scalar-tensor theories arise as modified theories of General Relativity when in addition to the metric field the gravitational interaction is described by a scalar field.

3.2. Brans-Dicke theory

Originally, Brans-Dicke proposed the following action [6],

$$S_{BD}[g_{\mu\nu},\varphi,\psi]=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(\varphi R-\frac{\omega_0}{\varphi}\nabla^\mu\varphi\nabla_\mu\varphi-V(\varphi)\right)+S_m[g_{\mu\nu},\psi]\ , \tag{3.1}$$

where ϕ is the scalar field degree of freedom, ω_0 is the so called Brans-Dicke parameter and S_m represents the action for generic matter fields understood to couple minimally to the metric[2]. The field equations derived from this action take the form

$$\begin{aligned}R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{8\varphi G}{\varphi}T_{\mu\nu}+\frac{\omega_0}{\varphi^2}\left(\nabla_\mu\varphi\nabla_\nu\varphi-\frac{1}{2}g_{\mu\nu}\nabla^\alpha\varphi\nabla_\alpha\varphi\right)\\+\frac{1}{\varphi}(\nabla_\mu\varphi\nabla_\nu\varphi-g_{\mu\nu}\Box\varphi)-\frac{V(\varphi)}{2\varphi}g_{\mu\nu}\ ,\end{aligned} \tag{3.2}$$

$$(2\omega_0+3)\Box\varphi=\varphi V'-2V+8\pi G\ T\ , \tag{3.3}$$

with $\Box=\nabla^\alpha\nabla_\alpha$ and a prime stands for differentiation respect to the argument. It is worth to mention that in its original form Brans-Dicke theory did not contain a potential. It is clear that in vacuum, i.e. when $T_{\mu\nu}=0$, the possible solutions for the theory admit a constant scalar field $\varphi=\varphi_0$, provided $\varphi_0V'(\varphi_0)-2V(\varphi_0)=0$. This equation determines an effective cosmological constant $V(\varphi_0)$ and the metric satisfies the Einstein's equations. Therefore, a suitable value for $V(\varphi)$, could predict the same phenomena as General Relativity. As an example, the spacetime around the Sun, could be described by that solution, and the constraints associated with the solar system would by satisfied. In contrast to this, the scalar field could have nontrivial configurations, forcing the metric field to deviates from the solutions given by General Relativity.

As a matter of fact, this is what happen in the case of spherically symmetric solutions. Following with our example, lets consider the Sun and in concrete a potential $V(\varphi)=m^2(\varphi-\varphi_0)^2$. When the newtonian expansion is performed, it is possible to compute the newtonian limit for the metric coefficients. One finds that the standard $1/r$ potential in the perturbations of the metric h_{00} and h_{ij}, gets a Yukawa-like correction. In order to identify new features introduced by the new scalar degree of freedom, it is useful to compute the Eddington parameter

[2]Hereafter, in this thesis we use the "mostly plus signature" and Greek indices stand for indices in the coordinate basis.

$\gamma = h_{ij|i=j}/h_{00}$. It is clear that $\gamma = 1$ recovers General Relativity, which demands ω_0 or m tends to infinity. Additionally, by the equation for the potential, the scalar field φ goes to a constant value φ_0, and the metric $g_{\mu\nu}$ satisfying Einstein equations becomes unique. However, current constraints suggest $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$ [32]. Turns out that, for $m = 0$, this constraint requires a value for ω_0 larger than 4×10^4 making the theory undistinguishable from General Relativity at any scale. Moreover, for $\omega_0 = \mathcal{O}(1)$, the Yukawa correction would be smaller than a few microns, which is lowest scale that the inverse square law has been tested.

Summarizing, the weak gravity constraints are so powerful, that it seems very difficult to satisfy them and have new phenomenology at scales where General Relativity has been recently tested.

3.3. The generalized Brans-Dicke theory

Commonly in literature, scalar-tensor theories are considered as the direct generalization of original Brans-Dicke theory. This refers to promote ω_0 to a general function of the scalar field φ. In this context, the action is

$$S_{st}[g_{\mu\nu}, \varphi, \psi] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(\varphi R - \frac{\omega(\varphi)}{\varphi}\nabla^\mu\varphi\nabla_\mu\varphi - V(\varphi)\right) + S_m[g_{\mu\nu}, \psi] \ . \tag{3.4}$$

After some work, the fields equations are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\varphi G}{\varphi}T_{\mu\nu} + \frac{\omega(\varphi)}{\varphi^2}\left(\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\varphi\nabla_\alpha\varphi\right) \\ + \frac{1}{\varphi}(\nabla_\mu\varphi\nabla_\nu\varphi - g_{\mu\nu}\Box\varphi) - \frac{V(\varphi)}{2\varphi}g_{\mu\nu} \ , \end{aligned} \tag{3.5}$$

$$(2\omega(\varphi) + 3)\Box\varphi = \varphi V' - 2V + 8\pi G\ T - \omega'(\varphi)\nabla^\alpha\varphi\nabla_\alpha\varphi \ , \tag{3.6}$$

It is expected that in the weak limit, scalar-tensor theories of this kind present the same behavior as Brans-Dicke theory. However, the advantage to allow ω be a function of φ is the possibility to get new phenomenology in the strong gravity regime. Up to now, the scalar-tensor theories have been presented with a metric that minimally couples to matter in the so called Jordan frame. In spite of that, it is quite common to write them in the conformal or Einstein frame, where the scalar field is redefined in such a way that couples minimally to gravity and matter. To see this, a conformal transformation is performed $\hat{g}_{\mu\nu} = \varphi g_{\mu\nu}$, supplemented by the scalar field redefinition $4\sqrt{\pi}\varphi d\phi = \sqrt{2\omega(\varphi) + 3}d\varphi$, obtaining the action in the Einstein frame

$$S_{st} = \int d^4x\sqrt{-\hat{g}}\left(\frac{\hat{R}}{16\pi G} - \frac{1}{2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\phi\hat{\nabla}_\nu\phi - U(\phi)\right) + S_m(g_{\mu\nu}, \psi) \ . \tag{3.7}$$

where the potential is $U(\phi) = V(\varphi)/\varphi^2$, $\hat{g}_{\mu\nu}$ is the conformal (or Einstein) frame metric as well as the quantities with a hat are defined with this metric. The corresponding field equations are

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu} \;=\; 8\pi G T^{\phi}_{\mu\nu} + \frac{8\pi G}{\varphi(\phi)} T_{\mu\nu}\ , \hat{\Box}\phi - U'(\phi) = \sqrt{\frac{4\pi G}{(2\omega+3)}} T\ , \tag{3.8}$$

with

$$T^{\phi}_{\mu\nu} = \hat{\nabla}_\mu\phi\hat{\nabla}_\nu\phi - \frac{1}{2}\hat{g}_{\mu\nu}\hat{\nabla}_\alpha\phi\hat{\nabla}^\alpha\phi - U(\phi)\hat{g}_{\mu\nu}\ , \tag{3.9}$$

and the stress-energy tensor and its trace in the Jordan frame given by $T_{\mu\nu}$ and T respectively.

One special thing of the Einstein frame is that as the scalar field ϕ is minimally coupled to $\hat{g}_{\mu\nu}$ calculations can be much simpler, specially in vacuum. Although, computations can be performed in any frame, there are some subtleties about the physical interpretation of the two metrics. The reader can refer to [33] for a complete discussion about this.

3.4. The Horndeski action

So far, up to boundary terms, it has been presented the most general scalar-tensor theory (3.4) quadratic in the derivatives of the scalar field. Nevertheless, this is not the most general one, that can lead to second order field equations. The most general scalar-tensor theory in four dimensional spacetime yielding second order field equations was found by Horndeski [13] and reconsidered recently [34]. It is the analogue to Lovelock theorem in General Relativity but in the context of scalar-tensor theories. A we mention, as a scalar-tensor theory, Horndeski theory possesses a scalar field ϕ an a metric $g_{\mu\nu}$ as the gravitational degrees of freedom of some Lorentzian manifold endowed with a Levi-Civita connection. Let us consider a theory that depends on these degrees of freedom and an arbitrary number of their derivatives

$$\mathcal{L} = \mathcal{L}(g_{\mu\nu},\ g_{\mu\nu,i_1}, \ldots,\ g_{\mu\nu,i_1\ldots i_p},\ \phi,\ \phi_{,i_1}, \ldots,\ \phi_{,i_1\ldots i_q})\ , \tag{3.10}$$

with $p, q \geqslant 2$. As in the previous sections, we consider that matter fields couple only to the metric and not to the scalar field. Therefore the metric and this frame, which is the Jordan frame, are the physical one. In this frame the metric will continue to verify the weak equivalence principle. In simple words, locally the spacetime can be equipped by a normal frame where Christoffel symbols vanish. The Horndeski action can be put in such a way that only second derivatives are involved. Namely,

$$\begin{aligned}\mathcal{L} =&\kappa_1(\phi,\rho)\delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\nabla^\mu\nabla_\alpha\phi R^{\nu\sigma}_{\beta\gamma} - \frac{4}{3}\kappa_{1,\rho}(\phi,\rho)\delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi \\ &+ \kappa_3(\phi,\rho)\delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\nabla_\alpha\phi\nabla^\mu\phi R^{\nu\sigma}_{\beta\gamma} - 4\kappa_{3,\rho}(\phi,\rho)\delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi \\ &+ [F(\phi,\rho)+2W(\phi)]\delta^{\alpha\beta}_{\mu\nu}R^{\mu\nu}_{\alpha\beta} - 4F(\phi,\rho)_{,\rho}\delta^{\alpha\beta}_{\mu\nu}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi \\ &- 3[2F(\phi,\rho)_{,\phi} + 4W(\phi)_{,\phi} + \rho\kappa_8(\phi,\rho)]\nabla_\mu\nabla^\mu\phi + 2\kappa_8\delta^{\alpha\beta}_{\mu\nu}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi + \kappa_9(\phi,\rho)\ ,\end{aligned} \tag{3.11}$$

where $\rho = \nabla_\mu\phi\nabla^\mu\phi$. The Lagrangian (3.11) contains four arbitrary functions $\kappa_i(\phi,\rho)$, $i=\{1,3,8,9\}$ depending on the scalar field and its kinetic term ρ. Additionally, function $F(\phi,\rho)$ fulfills,

$$F_{,\rho} = \kappa_{1,\phi} - \kappa_3 - 2\rho\kappa_{3,\rho} \tag{3.12}$$

and $W(\phi)$ is an arbitrary function of the scalar field. Without loss of generality, it can be set to zero by redefining $F(\phi,\rho)$. Horndeski's theorem [13] states that (3.11) is the unique action -up to total divergence terms- that provides second order field equations for the metric and scalar field, and Bianchi identities. Equations of motion reads,

$$\mathcal{E}^{\mu\nu} = \frac{1}{2}T^{\mu\nu}, \tag{3.13}$$

$$\mathcal{E}_\phi = 0\ , \tag{3.14}$$

with $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g\mu\nu}$ as the energy-momentum tensor. It can be proved that tensor $\mathcal{E}^{\mu\nu}$ is divergenceless. In flat space, a subclass of Horndeski action enjoys of Galilean symmetry, giving rise to Galileon action, which is invariant under $\phi\to\phi+c_\mu x^\mu+c$, where c is a constant and c_μ is a constant one-form. By this reason, these fields are also known as Galileons. However, this symmetry is lost when one tries to generalize Galileon action to curved spacetime [35] (it is local symmetry). Therefore, Horndeski action does not reduce to Galileon action in flat space. In spite of that, the scalar field in the Horndeski scenario is known as Generalised Galileons (GG) [36]. A simpler way to obtain Horndeski Lagrangian is through the general Galileon action

$$\begin{aligned}\mathcal{L}_{GG} =&K(\phi,\rho) - G_3(\phi,\rho)\Box\phi + G_4(\phi,\rho)R + G_{4,\rho}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2] \\ &+ G_5(\phi,\rho)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5,\rho}}{6}[(\nabla^2\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3]\ ,\end{aligned} \tag{3.15}$$

in which is clearly easier to identify other theories like GR, Brans-Dicke, K-essence, etc. as particular subsets of this theory. In [37] it is proved that this theory is equivalent to Horndeski theory when its arbitrary functions K, G_3, G_4 and G_5 are given by

$$
\begin{aligned}
K &= \kappa_9 + \rho \int^{\rho} d\rho' (\kappa_{8,\phi} - 2\kappa_{3,\phi\phi}) && (3.16)\\
G_3 &= 6(F+2W)_{,\phi} + \rho\kappa_8 + 4\rho\kappa_{3,\phi} - \int^{\rho} d\rho' (\kappa_8 - 2\kappa_{3,\phi}) && (3.17)\\
G_4 &= 2(F+2W) + 2\rho\kappa_3 && (3.18)\\
G_5 &= -4\kappa_1 \ . && (3.19)
\end{aligned}
$$

Along this thesis we present two works where a subset of special interest of four dimensional Horndeski Lagrangian is considered. In particular, this action provides cosmological as well as black hole solutions. Namely, by choosing

$$
\begin{aligned}
K(\phi,\rho) &= -\frac{\Lambda}{8\pi G} \ , && (3.20)\\
G_3(\phi,\rho) &= -\frac{\alpha}{2}\phi \ , && (3.21)\\
G_4(\phi,\rho) &= \frac{1}{16\pi G} \ , && (3.22)\\
G_5(\phi,\rho) &= -\frac{\eta}{2}\phi \ , && (3.23)\\
& && (3.24)
\end{aligned}
$$

the theory exhibits a scalar degree of freedom coupled to gravitation through a non-minimal kinetic term that contains the Einstein tensor in the presence of a cosmological constant. The minimal coupling is mediated by the parameter α while the non-minimal coupling is provided through the factor η. Thus, the action principle is given by

$$
I[g_{\mu\nu},\phi] = \int \sqrt{-g} d^n x \ \left[\kappa \left(R - 2\Lambda \right) - \frac{1}{2} \left(\alpha g_{\mu\nu} - \eta G_{\mu\nu} \right) \nabla^{\mu}\phi \nabla^{\nu}\phi \right] \ . \tag{3.25}
$$

where $\kappa := \frac{1}{16\pi G}$. The variation of the action (3.25) with respect to the metric tensor and the scalar field

$$
G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{\alpha}{2\kappa} T^{(1)}_{\mu\nu} + \frac{\eta}{2\kappa} T^{(2)}_{\mu\nu} \ , \tag{3.26}
$$

$$
\nabla_{\mu} \left[\left(\alpha g^{\mu\nu} - \eta G^{\mu\nu} \right) \nabla_{\nu}\phi \right] = 0 \ , \tag{3.27}
$$

respectively. Here we have defined[3]

$$
\begin{aligned}
T^{(1)}_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\lambda\phi\nabla^\lambda\phi \ , \\
T^{(2)}_{\mu\nu} &= \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi R - 2\nabla_\lambda\phi\nabla_{(\mu}\phi R^\lambda_{\nu)} - \nabla^\lambda\phi\nabla^\rho\phi R_{\mu\lambda\nu\rho} \\
&\quad -(\nabla_\mu\nabla^\lambda\phi)(\nabla_\nu\nabla_\lambda\phi) + (\nabla_\mu\nabla_\nu\phi)\Box\phi + \frac{1}{2}G_{\mu\nu}(\nabla\phi)^2 \\
&\quad -g_{\mu\nu}\left[-\frac{1}{2}(\nabla^\lambda\nabla^\rho\phi)(\nabla_\lambda\nabla_\rho\phi) + \frac{1}{2}(\Box\phi)^2 - \nabla_\lambda\phi\nabla_\rho\phi R^{\lambda\rho}\right] \ .
\end{aligned}
$$

The non-minimal derivative coupling are an interesting source of new cosmological dynamics. As we mentioned, this theory in particular can explain and describe the accelerated expansion of the Universe without the use of any fine-tuned potential [14]. This work motivated many subsequent researches about inflationary cosmology and late-time cosmology [38, 39, 40]. The research of exact solutions is rather recent. In [16] the authors present the no-hair theorem for Galileon gravity, which prevents the existence of asymptotically flat black holes endowed by a nontrivial regular scalar field configuration. Undoubtedly, the strategy in the quest of black solutions is circumvent such a no-hair theorem by relaxing some of its hypothesis. In this way, Rinaldi [15] found the first black hole solution by allowing an asymptotically AdS behavior with a cosmological constant given in terms of the non-minimal coupling factor. However, such black hole solution does not satisfy the weak energy condition and the scalar field is imaginary in the domain of outer communications. A later work presented in [17] found the way to solve this problem by adding a cosmological constant as an additional free parameter. This allowed new remarkable solutions. Namely, asymptotically locally flat black hole and the first gravitational soliton found in the theory opening the possibility of regularizing the Euclidean action and to describe the thermodynamics of the system using the Hawking-Page approach [41]. In this way, phase transitions are found between the soliton and the large black hole for specific value of the parameters. In this thesis our original contribution relies on the same line of research as an extension of work [17] where we include a Maxwell field. The inclusion of electric field confers unique properties compared with the aforementioned ones. To be specific, switching off the minimal coupling we obtain an asymptotically flat black hole, i.e. it matches perfectly with Minkowski spacetime at infinity and recover Schwarzschild solution when electric charge vanishes. This work will be presented in detail in the next chapter. Additionally, a gravitational soliton is also found. Other exact solutions are time-dependent galileons, where the scalar field is allowed to depend linearly on time [34] and many other were found subsequently [42, 43, 44].

[3] We use a normalized symmetrization $A_{(\mu\nu)} := \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$.

Chapter 4

Asymptotically locally AdS and flat black holes in the presence of an electric field in the Horndeski scenario

A great interest has been generated by spacetimes which are asymptotically of constant curvature, particularly asymptotically AdS spacetimes. This interest is largely motivated by the AdS/CFT correspondence [45] which relates the observables in a gauged supergravity theory with those of a conformal field theory in one dimension less. In this way, black hole solutions with a negative cosmological constant are important because in principle they could provide the possibility of studying the phase diagram of a CFT theory. Therefore, it seems natural to study the case where a negative cosmological constant is present. This was done in [17], where a real scalar field outside the horizon was found and where the positivity of the energy density is given by this reality condition. Recently in reference [34] it has been shown that allowing the scalar to depend on time permits to construct a black hole solution in which the scalar field is analytic at the future or at the past horizon. In a similar context exact solutions were found in [42].

In this chapter we present asymptotically locally AdS and asymptotically flat black hole solutions for a particular case of the Horndeski action. The action contains the Einstein-Hilbert term with a cosmological constant, a real scalar field with a non-minimal kinetic coupling given by the Einstein tensor, the minimal kinetic coupling and the Maxwell term. There is no scalar potential. The solution has two integration constants related with the mass and the electric charge. The solution is given for all dimensions. A new class of asymptotically locally flat spherically symmetric black holes is found when the minimal kinetic coupling vanishes and the cosmological constant is present. In this case we get a solution which represents an electric Universe. The electric field at infinity is only supported by Λ. When the cosmological constant vanishes the black hole is asymptotically flat.

The outline of this chapter is as follows: Section 4.1 presents the field equations and the ansatz employed to solve them. In Section 4.2 the four-dimensional solution is given for arbitrary K, and the energy density is computed. In Section 4.3, the spherically symmetric solution is described in detail and the constraints on the coupling parameters are described in order to obtain a real scalar field and positive energy density. We comment as well on some of the thermodynamical properties of the solution. In Section 4.4, the solution in arbitrary dimension n is given. Finally in Section 4.5 the solution in the special case when $\alpha = 0$ is analyzed.

4.1. Field equations

The aim of this work is to continue in this line and generalize the results in reference [17] by adding a Maxwell term given by a spherically symmetric gauge field $A = A_0(r)dt$. As was emphasized in 3.4 we consider the non-minimal kinetic sector of Horndeski theory. In the work presented in this chapter we shall focus on the study of black hole solutions and their properties that emerge from this theory. The action principle is given by

$$I[g_{\mu\nu}, \phi] = \int \sqrt{-g} d^n x \left[\kappa \left(R - 2\Lambda \right) - \frac{1}{2} \left(\alpha g_{\mu\nu} - \eta G_{\mu\nu} \right) \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] . \tag{4.1}$$

The strength of the non-minimal kinetic coupling is controlled by η. Here $\kappa := \frac{1}{16\pi G}$. The possible values of the dimensionful parameters α and η will be determined below requiring the positivity of the energy density of the matter field. The variation of the action (8.1) with respect to the metric tensor, the scalar field and the gauge field yields

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{\alpha}{2\kappa} T^{(1)}_{\mu\nu} + \frac{\eta}{2\kappa} T^{(2)}_{\mu\nu} + \frac{1}{2\kappa} T^{em}_{\mu\nu} \ , \tag{4.2}$$

$$\nabla_\mu \left[\left(\alpha g^{\mu\nu} - \eta G^{\mu\nu} \right) \nabla_\nu \phi \right] = 0 \ , \tag{4.3}$$

$$\nabla_\mu F^{\mu\nu} = 0 \ , \tag{4.4}$$

respectively. Here we have defined[1]

$$
\begin{aligned}
T^{(1)}_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\lambda\phi\nabla^\lambda\phi \ , \\
T^{(2)}_{\mu\nu} &= \frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi R - 2\nabla_\lambda\phi\nabla_{(\mu}\phi R^\lambda_{\nu)} - \nabla^\lambda\phi\nabla^\rho\phi R_{\mu\lambda\nu\rho} \\
&\quad -(\nabla_\mu\nabla^\lambda\phi)(\nabla_\nu\nabla_\lambda\phi) + (\nabla_\mu\nabla_\nu\phi)\Box\phi + \frac{1}{2}G_{\mu\nu}(\nabla\phi)^2 \\
&\quad -g_{\mu\nu}\left[-\frac{1}{2}(\nabla^\lambda\nabla^\rho\phi)(\nabla_\lambda\nabla_\rho\phi) + \frac{1}{2}(\Box\phi)^2 - \nabla_\lambda\phi\nabla_\rho\phi R^{\lambda\rho}\right] \ , \\
T^{em}_{\mu\nu} &= F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4}g_{\mu\nu}F^2 \ .
\end{aligned}
$$

We will consider the family of spacetimes

$$ds^2 = -F(r)dt^2 + G(r)dr^2 + r^2 d\Sigma_K^2 \ , \tag{4.5}$$

where $d\Sigma_K$ is the line element of a closed, $(n-2)$-dimensional Euclidean space of constant curvature $K = 0, \pm 1$. The metric (6.2) corresponds to the most general static spacetime compatible with the possible local isometries of Σ_K acting on a spacelike section. For $K = 1$, the space Σ_K is locally a sphere, for $K = 0$ it is locally flat, while for $K = -1$ it locally reduces to the hyperbolic space. Hereafter we will consider a static and isotropic scalar field, i.e. $\phi = \phi(r)$.

4.2. Four dimensional solution

Using the ansatz (6.2) the equation of motion for the scalar field (4.3) admits a first integral, which implies the equation

$$r\frac{F'(r)}{F(r)} = \left[K + \frac{\alpha}{\eta}r^2 - \frac{C_0}{\eta}\frac{G(r)}{\psi(r)\sqrt{F(r)G(r)}}\right]G(r) - 1 \ , \tag{4.6}$$

where C_0 is an integration constant, $\psi(r) := \phi'(r)$, and $(')$ stands for derivation with respect to r. As it was done in reference [15], and then in [17] we (arbitrarily[2]) set $C_0 = 0$, which allows to find a simple relation between the metric functions $F(r)$ and $G(r)$

$$G(r) = \frac{\eta}{F(r)}\left(\frac{rF'(r) + F(r)}{r^2\alpha + \eta K}\right) \ . \tag{4.7}$$

[1] We use a normalized symmetrization $A_{(\mu\nu)} := \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$.

[2] After the publication of this result, in [46] was proved that this is the unique choice compatible with black hole solutions.

The Maxwell equation admits a first integral as well, providing the following relation

$$G(r) = \frac{r^4}{q^2 F(r)} (A_0'(r))^2 \ , \tag{4.8}$$

where $\frac{1}{q^2}$ is an integration constant. These two last equations allow us to find an expression for the first radial derivative of the electric potential

$$(A_0'(r))^2 = \frac{q^2 \eta}{r^4} \left(\frac{rF'(r) + F(r)}{r^2\alpha + \eta K} \right) \ . \tag{4.9}$$

In this way, equations (4.7) and (4.9) together with the tt and rr components of (4.2), provide a consistent system which for $K = \pm 1$ and $\eta\Lambda \neq \alpha$, has the following solution

$$\begin{aligned}
F(r) &= \frac{r^2}{l^2} + \frac{K}{\alpha}\sqrt{\alpha\eta K} \left(\frac{\alpha + \Lambda\eta + \frac{\alpha^2}{4\eta\kappa K} q^2}{\alpha - \Lambda\eta} \right)^2 \frac{\arctan\left(\frac{\sqrt{\alpha\eta K}}{\eta K} r \right) - \mu}{r} \\
&\quad + \frac{\alpha^2}{\kappa(\alpha - \Lambda\eta)^2} \frac{q^2}{r^2} + \frac{\alpha^3}{16\eta\kappa^2 K^2 (\alpha - \Lambda\eta)^2} \frac{q^4}{r^2} - \frac{\alpha^2}{48\kappa^2 K (\alpha - \Lambda\eta)^2} \frac{q^4}{r^4} + \frac{3\alpha + \Lambda\eta}{\alpha - \Lambda\eta} K \ , \\
G(r) &= \frac{1}{16} \frac{\alpha^2 (4\kappa\,(\alpha - \eta\Lambda)\, r^4 + 8\eta\kappa K r^2 - \eta q^2)^2}{r^4 \kappa^2 (\alpha - \eta\Lambda)^2 (\alpha r^2 + \eta K)^2 F(r)} \ , \\
\psi^2(r) &= -\frac{1}{32} \frac{\alpha^2 (4\kappa(\alpha + \eta\Lambda) r^4 + \eta q^2)(4\kappa\,(\alpha - \eta\Lambda)\, r^4 + 8\eta\kappa K r^2 - \eta q^2)^2}{r^6 \eta\kappa^2 (\alpha - \eta\Lambda)^2 (\alpha r^2 + \eta K)^3 F(r)} \ , \\
A_0(r) &= \frac{1}{4} \frac{q\sqrt{\alpha}}{\eta^{\frac{3}{2}} K^{\frac{5}{2}} \kappa} \left(\frac{4\beta\kappa K^2 (\alpha + \eta\Lambda) + \alpha^2 q}{(\alpha - \eta\Lambda)} \right) \arctan\left(\frac{\sqrt{\alpha\eta K}}{\eta K} r \right) \\
&\quad + \alpha \left(\frac{8\eta\kappa K^2 + \alpha q}{4\eta\kappa K^2 (\alpha - \eta\Lambda)} \right) \frac{q}{r} - \frac{\alpha}{12\kappa K (\alpha - \eta\Lambda)} \frac{q^3}{r} \ .
\end{aligned}$$

Here we have defined the effective (A)dS radius l by $l^{-2} := \frac{\alpha}{3\eta}$ and μ is an integration constant related with the mass. In the case of a locally flat transverse section ($K = 0$) the system integrates in a different manner and the solution takes the form

$$F(r) = \frac{r^2}{l^2} - \frac{\mu}{r} + \frac{\alpha}{2\kappa(\alpha-\eta\Lambda)}\frac{q^2}{r^2} + \frac{\alpha\eta}{80\kappa^2(\alpha-\eta\Lambda)^2}\frac{q^4}{r^6} \ ,$$
$$G(r) = \frac{1}{16}\frac{\left(4\kappa(\alpha-\eta\Lambda)r^4 - \eta q^2\right)^2}{\kappa^2(\alpha-\Lambda\eta)r^8 F(r)} \ ,$$
$$\psi(r)^2 = -\frac{1}{32}\frac{\left(4\kappa(\alpha+\eta\Lambda)r^4+\eta q^2\right)\left(4\kappa(\alpha-\eta\Lambda)r^4+\eta q^2\right)^2}{\alpha\eta r^{12}\kappa^2(\alpha-\Lambda\eta)^2 F(r)} \ ,$$
$$A_0(r) = -\left(\frac{20\kappa(\alpha-\eta\Lambda)r^4 - \eta q^2}{20\kappa(\alpha-\eta\Lambda)r^5}\right) q \ .$$

In the case when we set $q \to 0$ we recover the result obtained in [17] for the cases $K = \pm 1$ as well as for the case $K = 0$. The later case reduces to topological Schwarzschild solution with locally flat horizon [28].

It can be seen that this solution is asymptotically locally dS or AdS for $\alpha/\eta < 0$ or $\alpha/\eta > 0$, respectively, since when $r \to \infty$ the components of the Riemann tensor go to

$$R^{ab}{}_{cd} \underset{r\to\infty}{=} -\frac{\alpha}{3\eta}\delta^{ab}_{cd} := -\frac{1}{l^2}\delta^{ab}_{cd} \ ,$$

justifying our previous definition of the effective (A)dS radius. The asymptotic expansion ($r \to \infty$) of the metric functions and of the gauge field reads

$$g_{tt} \underset{r\to\infty}{=} \frac{r^2}{l^2} + \frac{3\alpha+\eta\Lambda}{\alpha-\eta\Lambda}K + \frac{K}{2\alpha}\sqrt{\alpha\eta K}\left(\frac{(\alpha+\eta\Lambda)+\frac{\alpha^2 q^2}{4\eta\kappa K^2}}{\alpha-\eta\Lambda}\right)^2 \frac{\pi\sigma - 2\mu}{r} + O\left(r^{-2}\right) \ ,$$
$$g^{rr} \underset{r\to\infty}{=} \frac{r^2}{l^2} + \frac{7\alpha+\eta\Lambda}{3(\alpha-\eta\Lambda)}K + \frac{K}{2\alpha}\sqrt{\alpha\eta K}\left(\frac{(\alpha+\eta\Lambda)+\frac{\alpha^2 q^2}{4\eta\kappa K^2}}{\alpha-\eta\Lambda}\right)^2 \frac{\pi\sigma - 2\mu}{r} + O\left(r^{-2}\right) \ ,$$
$$A_0(r) \underset{r\to\infty}{=} a_0 - \frac{q}{r} + O(r^{-2}) \ ,$$

where σ is the sign of ηK and a_0 is a constant. From here it is possible to see that our electric potential reproduces the Coulomb potential at infinity. There is a curvature singularity at $r = 0$ since for example the Ricci scalar diverges as

$$R \underset{r\to 0}{=} \frac{4K}{r^2} + O(1) \ . \tag{4.10}$$

If $\rho(r)$ is the energy density, then the total energy $\mathcal{E}$ is given by

$$\mathcal{E} = V(\Sigma)\int dr \rho(r) \ , \tag{4.11}$$

where $V(\Sigma)$ stands for the volume of Σ. Therefore

$$\rho(r) := r^2 \sqrt{G(r)} F(r)^{-1} T_{tt} \ . \tag{4.12}$$

Now, the tt component of the energy momentum tensor reads

$$T_{tt} = -\frac{(\alpha + \Lambda\eta)}{\eta\kappa^2} F(r) \left[1 - H(r)F(r)\right] \ , \tag{4.13}$$

where $H(r)$ is the given by the expression

$$H(r) = \frac{64\eta^2 r^2 (\alpha - \Lambda\eta)^2 (r^2\alpha + \eta K)}{\alpha^2\kappa^2(\alpha + \Lambda\eta)} \left(\frac{q^2\kappa(2r^2\alpha + \eta K) - 4K(\alpha + \Lambda\eta) r^4}{4(\alpha - \Lambda\eta) r^4 + 8r^2\eta K - \eta\kappa q^2} \right) \ .$$

If we take the limit $q \to 0$ we recover the T_{tt} component of the uncharged case.

4.3. Spherically symmetric case

Now we study the particular case with a spherically symmetric transverse section $K = 1$. The solution for the metric components and for the square of the derivative of the scalar field reduces to

$$\begin{aligned}
F(r) &= \frac{r^2}{l^2} + \frac{1}{\alpha}\sqrt{\alpha\eta} \left(\frac{\alpha + \Lambda\eta + \frac{\alpha^2}{4\eta\kappa} q^2}{\alpha - \Lambda\eta} \right)^2 \frac{\arctan\left(\frac{\sqrt{\alpha\eta}}{\eta} r\right) - \mu}{r} \\
&\quad + \frac{\alpha^2}{\kappa(\alpha - \Lambda\eta)^2} \frac{q^2}{r^2} + \frac{\alpha^3}{16\eta\kappa^2(\alpha - \Lambda\eta)^2} \frac{q^4}{r^2} - \frac{\alpha^2}{48\kappa^2(\alpha - \Lambda\eta)^2} \frac{q^4}{r^4} + \frac{3\alpha + \Lambda\eta}{\alpha - \Lambda\eta} \ , \\
G(r) &= \frac{1}{16} \frac{\alpha^2 (4\kappa(\alpha - \eta\Lambda) r^4 + 8\eta\kappa r^2 - \eta q^2)^2}{r^4\kappa^2(\alpha - \eta\Lambda)^2(\alpha r^2 + \eta)^2 F(r)} \ , \\
\psi^2(r) &= -\frac{1}{32} \frac{\alpha^2 (4\kappa(\alpha + \eta\Lambda) r^4 + \eta q^2)(4\kappa(\alpha - \eta\Lambda) r^4 + 8\eta\kappa r^2 - \eta q^2)^2}{r^6\eta\kappa^2(\alpha - \eta\Lambda)^2(\alpha r^2 + \eta)^3 F(r)} \ , \\
A_0(r) &= \frac{1}{4} \frac{q\sqrt{\alpha}}{\eta^{\frac{3}{2}}\kappa} \left(\frac{4\beta\kappa^2(\alpha + \eta\Lambda) + \alpha^2 q}{(\alpha - \eta\Lambda)} \right) \arctan\left(\frac{\sqrt{\alpha\eta}}{\eta} r \right) \\
&\quad + \alpha \left(\frac{8\eta\kappa + \alpha q}{4\eta\kappa(\alpha - \eta\Lambda)} \right) \frac{q}{r} - \frac{\alpha}{12\kappa(\alpha - \eta\Lambda)} \frac{q^3}{r} \ .
\end{aligned}$$

In order to analyze the proper features of a black hole in our solution we need to analyze the lapse function $F(r)$. As we approach the origin, the lapse function goes to minus infinity. On the other hand, as we go to infinity along coordinate r, $F(r)$ tends to plus infinity. Therefore, it is clear that this function being continuous has at least one zero. We can prove that this function has more than one zero. Since we know the existence of at least one zero r_H, we can parametrize the function with r_H as parameter. From the equation $F(r_H) = 0$ we get $\mu \equiv \mu(r_H)$ which can be used to express the lapse function as $F(r, \mu(r_H))$. To prove the existence of the second event horizon, we can do the same as before but with the electric charge. We propose the existence of r_h, then $F(r_h) = 0$, and using this we get $q^2 \equiv q^2(r_h, r_H)$. It is possible to find two roots for $F(r_h) = 0$ or in other words, two suitable values of q^2 for a possible r_h. This values in some cases are both negatives, both positive or one positive and the second negative, but at least the existence of one positive root is enough to prove the existence of r_h. As we said, due to the shape of the lapse function near the origin and at infinity, the existence of two zeros of the function implies the existence of a third zero for some range of parameters. Therefore $F(r)$ can have just one zero, two zeros[3] or three zeros. Each of these cases exist for a specific set of values of the coupling and cosmological constants. From hereafter and for simplicity, we will focus in the case when the lapse function has just one zero.

Reality condition of the lapse function requires $\alpha\eta > 0$. Therefore $l^{-2} := \frac{\alpha}{3\eta}$ is positively defined and the spacetime is asymptotically AdS. As it was noted in the uncharged case [17] without loss of generality it is possible to choose both parameters positive, since the solution with both α and η negative is equivalent to the former by changing $\mu \to -\mu$.

In order to obtain a real scalar field in the domain of outer communications and satisfy the positivity of the energy, we need to impose some constraints in our parameters. In fact, the value of the cosmological constant is restricted to be

$$\Lambda < -\frac{q^2}{4r_H^4\kappa} - \frac{\alpha}{\eta} \ . \tag{4.14}$$

It is important to note that we cannot switch off the scalar field. This implies that our solution is not continuously connected with the maximally symmetric background. Despite of this, setting $\mu = 0$ and $q = 0$ we observe that the spacetime is regular, actually is the only regular spacetime that can be found within this family. Such a case describes an asymptotically AdS gravitational soliton. Close to $r = 0$ and after a proper rescaling on the time coordinate the spacetime metric takes the following form

$$ds^2_{soliton} = -\left(1 - \frac{\Lambda}{3}r^2 + O(r^4)\right)dt^2 + \left(1 - \frac{3\alpha + 2\Lambda\eta}{3\eta}r^2 + O(r^4)\right)dr^2 + r^2 d\Omega^2 \ . \tag{4.15}$$

[3]This case is an special case in the sense that contains a zero which is a local minimum. When that local minimum is the outer horizon this corresponds to an extremal black hole.

The thermal version of this spacetime can be used as the background metric for obtain a regularized euclidean action which could be used to obtain the thermodynamical properties of the black holes in the Hawking-Page approach.

4.4. n-dimensional case

In this section we analyze the $n-$dimensional solution to the action principle defined by (8.1). For doing this, we take the variation of our Lagrangian with respect to all the functions involved $F(r)$, $G(r)$, $\phi(r)$ and $A_0(r)$. This procedure gives us the equations of motion of the system.

Therefore, following the same strategy than in four dimensions, the equation of motion for the scalar field admits a first integral. Setting to zero the integration constant of this equation we obtain a relation between the metric coefficients, but now in arbitrary dimension

$$G_n(r) = \frac{\eta(n-2)}{F_n(r)} \left(\frac{F_n'(r)r + F_n(r)(n-3)}{2r^2\alpha + \eta K(n-2)(n-3)} \right) \ . \tag{4.16}$$

The equation coming from the variation with respect to the electric field gives us the following relation

$$\left(A_{0_n}'(r)\right)^2 = q^2 F_n(r) G_n(r) r^{(4-2n)} \ .$$

In the same spirit, and using the last result, it is possible to obtain a relation for $\psi(r)^2$. Then

$$\psi_n(r)^2 = -\frac{1}{2}(n-2)\left(\frac{\Xi_n^1 + \Xi_n^2}{\Xi_n^3}\right) \ ,$$

where we have defined

$$\begin{aligned}
\Xi_n^1 &= (n-3)^2(4\kappa\Lambda\eta r^2 + 4\kappa r^2\alpha + q^2 r^{(-2n+6)}\eta)F_n(r)^2 \\
&\quad + 2(n-3)(q^2 r^{(-2n+7)}\eta + 4\kappa\Lambda\eta r^3 + 4\alpha r^3\kappa)F_n'(r)F_n(r) \ , \\
\Xi_n^2 &= (4\kappa\Lambda\eta r^4 + q^2 r^{(-2n+8)}\eta + 4\alpha r^4\kappa)F_n'(r)^2 \ , \\
\Xi_n^3 &= F_n(r)(2r^2\alpha\eta K n^2 - 5\eta K n + 6\eta K)^2((n-3)F_n(r) + F_n'(r)r) \ .
\end{aligned}$$

Using these expressions and the equation resulting from the variation with respect to the function $F_n(r)$, we can obtain a relation which allows to obtain the explicit form of $F_n(r)$ for an arbitrary value of the dimension n, and in this way, the complete solution to our system. We checked the result from $n = 4$ to $n = 10$.

4.5. Asymptotically locally flat black holes with charge supported by the Einstein-kinetic coupling

In this section we will study the particular case where the scalar field is coupled to the background only with the Einstein tensor. It is possible to do this by setting $\alpha = 0$. Under the presence of an electric field, we obtain asymptotically locally flat black hole solutions in the case where the cosmological constant is present. Therefore, the action principle is given by

$$I[g_{\mu\nu}, \phi] = \int \sqrt{-g} d^4x \left[\kappa \left(R - 2\Lambda \right) + \frac{\eta}{2} G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] . \tag{4.17}$$

Following the same procedure (with $\alpha \neq 0$ and $K = 1$)[4] we obtain

$$ds^2 = -F(r) dt^2 + \frac{15[4\kappa r^2(2 - \Lambda r^2) - q^2]^2}{r^4} \frac{dr^2}{F(r)} + r^2 d\Omega^2 , \tag{4.18}$$

where

$$\begin{aligned} F(r) &= 48\kappa^2 \Lambda^2 r^4 - 320\kappa^2 \Lambda r^2 + 120\kappa(8\kappa + \Lambda q^2) - \frac{\mu}{r} + 240\kappa \frac{q^2}{r^2} - 5\frac{q^4}{r^4} , \\ \psi(r)^2 &= -\frac{15}{2} \frac{(4\kappa\Lambda r^4 + q^2)(4\kappa r^2(2 - \Lambda r^2) - q^2)^2}{r^6 \eta} \frac{1}{F(r)} , \\ A_0(r) &= \sqrt{15} \left(\frac{q^3}{3r^3} - 8\kappa \frac{q}{r} - 4\kappa\Lambda r q \right) . \end{aligned}$$

This solution shows the following features:

- The solution is asymptotically locally flat, namely we have

$$\lim_{r \to \infty} R^{\mu\nu}{}_{\lambda\rho} \to 0 .$$

- For a non degenerated horizon $r = r_H$ we have $F(r_H) = 0$, then the scalar field vanishes at the horizon and is not analytic there.

- In order to obtain a real scalar field outside of the horizon we can impose two different conditions:

[4] In the case where $K = 0$, the system integrate in a different manner. In fact, Λ and q have to vanish in order to fulfil the field equations. Then, we obtain the same degenerated system found in [17].

1. $\Lambda > 0$ and $\eta < 0$ or
2. $\Lambda < -\frac{q^2}{4\kappa r_H^4}$ and $\eta > 0$.

- For any value of the integration constant μ we have the curvature singularities

$$r_0 = 0 \ ,$$
$$r_{1,2} = \frac{\sqrt{2\kappa\Lambda(2\kappa \pm \sqrt{4\kappa^2 - \kappa\Lambda q^2})}}{2\kappa\Lambda} \ .$$

  Then for $\Lambda < 0$ the only singularity is located at the origin of coordinates. If the cosmological constant is positive, in order to rule out the existence of singularities different than $r = 0$, we need to impose the following constraint on the value of Λ

$$\Lambda > \frac{4\kappa}{q^2} \ . \tag{4.19}$$

- We point out that in the limit $r \to \infty$ our electric potential represents a constant electric field at that point supported by the cosmological constant, and in this way we obtain an asymptotically electric Universe.

- Finally the limit $q \to 0$ we recover the results obtained in [17].

Let us put $\Lambda = 0$, then the solution takes the form

$$ds^2 = -F(r)dt^2 + \frac{3(8\kappa r^2 - q^2)^2}{r^4}\frac{dr^2}{F(r)} + r^2 d\Omega^2 \ ,$$

where

$$F(r) = 192\kappa^2 - \frac{\mu}{r} + 48\kappa\frac{q^2}{r^2} - \frac{q^4}{r^4} \ ,$$
$$\psi(r)^2 = -\frac{15}{2}\frac{(8\kappa r^2 - q^2)^2}{r^6\eta}\frac{q^2}{F(r)} \ ,$$
$$A_0(r) = \sqrt{15}\left(\frac{q^3}{3r^3} - 8\kappa\frac{q}{r}\right) \ .$$

In this case we have:

- The solution is asymptotically flat

$$ds^2 = -\left(1 - \frac{\mu}{r} + O(r^{-2})\right)dt^2 + \left(1 + \frac{\mu}{r} + O(r^{-2})\right)dr^2 + r^2 d\Omega^2 \ ,$$

  which is reasonable because when we have $\Lambda = 0$, the electric field at infinity vanishes.

- For a non degenerated horizon $r = r_H$ we have $F(r_H) = 0$, then the scalar field vanish at the horizon, as in the previous cases, is not analytic there.
- In order to obtain a real scalar field outside of the horizon we impose

$$\eta < 0 \ .$$

- For any value of the integration constant μ we have the curvature singularities

$$r_0 = 0 \ ,$$
$$r_1 = \sqrt{\frac{1}{8\kappa}}|q| \ .$$

- The electric field goes to zero at infinity.
- Taking the limit when $q \to 0$ we obtain a trivial scalar field and then we recover the Schwarzschild solution.

4.6. Ending remarks

In this work a particular sector of the Horndeski theory was considered where the gravity part is given by the Einstein-Hilbert term, and where the matter source is represented by a scalar field which has a non-minimal kinetic coupling constructed with the Einstein tensor. The main novelty of this work is the inclusion of the Maxwell field. We found exact solutions to this system for a spherically symmetric and topological horizons in all dimensions. The solution gives a new class of asymptotically locally AdS and asymptotically locally flat black hole solutions.

These solutions are obtained using two important observations. The first one, is the fact that the equation of motion for the scalar field admits a first integral, which after setting the integration constant to zero (arbitrarily) gives a simple relation between the two metric functions. The second one, is that the Maxwell equations are easily integrated for our ansatz and symmetry conditions, given a simple relation between the electric potential term and the metric functions. Mixing these two results we obtain a complete description of the system, obtaining in that way the exact solution for the topological case in $n \geq 4$ dimensions.

We observe and point out that in the case of the asymptotically locally AdS solution, the cosmological constant at infinity is not given by the cosmological Λ term in the action but rather in terms of the coupling constants α and η that appear in the kinetic coefficients of the field. The electric field is well behaved and goes to the Coulombic one at infinity.

The solutions are not continuously connected with the maximally symmetric AdS or flat backgrounds since the scalar field cannot be turned off. Nevertheless, since our family of metrics contains a further integration constant, it is possible to show that within such a family there is a unique regular spacetime. Such spacetime is a gravitational soliton and it is useful in the four dimensional spherically symmetric case to define a regularized Euclidean action and to explore the thermodynamics of the black hole solution. A similar situation occurs with the AdS soliton, which can be considered as the background for the planar AdS black holes, as well as in gravity in 2+1 with scalar fields, where the gravitational solitons are the right backgrounds to give a microscopic description of the black hole entropies [47, 48, 49].

In the particular case when the scalar field is only coupled to the metric through the Einstein tensor, namely, $\alpha = 0$ we obtain an asymptotically locally flat black hole solution. When $\Lambda \neq 0$ this solution presents some interesting properties. The solution exist in both cases, where the cosmological constant is positive and when is negative, given a real scalar field configuration depending on constraints imposed on the electric charge and on the coupling constant η. In any of these cases we obtain a constant electric field at infinity, representing in this way our solution a electric Universe. This constant electric field at infinity is just supported by the cosmological constant.

In the case where $\Lambda = 0$ we obtain a real scalar configuration just in case where the coupling constant is negative. The solution is asymptotically flat and the electric field vanishes at infinity when $\Lambda = 0$. If we switch off the electric field setting $q = 0$, we get a trivial scalar field and then we recover the Schwarzschild solution.

It is important to note that Horndeski theory offers the possibility of exploring its solutions in many different ways. In another context, using the same action principle, but without the Maxwell term an asymptotically Lifshitz solution was recently found in [43]. Moreover, even if it is not possible to obtain an analytic solution to the most general case of the Horndeski theory for the general static black hole solution, it would be interesting to study the cases where the non-minimal coupling is given by more general tensors than the Einstein one, namely the Lovelock tensors.

Chapter 5

Boson Stars

In the same way as the scalar fields have been used to tackle cosmological problems, these fields can be employed to describe astrophysical objects composed of massive scalar particles (or bosons). In this context, it is possible to find configurations which are held together simply by gravitational force. These configurations were dubbed as Boson Stars (BSs) and were found theoretically almost 50 years ago by Kaup [22] and by Ruffini and Bonazzola [50]. Astrophysical objects of fermionic nature like neutron stars and white dwarfs, are prevented from collapse due to the Pauli exclusion principle. Instead, as bosons, scalar particles experience Bose-Einstein condensation, i.e. all the bosons may occupy the same ground state. However, by the Heisenberg uncertainty principle, the scalar particles can not be localized within the Compton wavelength, preventing the gravitational collapse of the boson star into a black hole. This is the same mechanism that confers stability to atoms [51]. For this reason, boson stars are also called "gravitational atoms".

The study of boson stars got stuck until its rediscovery during the 1980's when scalar fields acquired much more attention since its discovery [52, 53, 54, 55]. During these years, boson star configurations with self-coupling additionally to a mass term were found [56]. The study of boson star including a scalar field non-minimally coupled to gravity were also studied [57] and consequently the electrically charged version [58]. Configurations mixing bosonic and fermionic particles were also constructed [59, 60].

It is of particular interest the study of mechanisms that allow the formation of boson star. To this respect, two possibilities are apparently possible: gravitational collapse as occurs with galaxies and clusters [61] and phase transitions. The latter was investigated for non-topological solitons [62], however a large object as a boson star is very unlikely to form by this mechanism and gravitational collapse is more interesting in this context.

Other regular solutions related to boson stars have been studied. For example, nontopological soliton stars [55, 63, 64], which are nontopological solitons with gravitational interaction. The Q-star [65, 66], which is the gravitating version of a Q-ball [67]. These solutions consider additional fields which localize the scalar initially, and then gravity is incorporated. Even so, the simplicity of boson star makes them a more appealing and interesting object. For this reason different kind of boson stars has been explored in the literature. In this way, it is possible to find boson stars with electric charge [58], rotation [68, 69] and even with fermionic component [59], each one with different kind of potentials for the scalar field.

This chapter is devoted to present a brief review of boson stars and is organized as follows[1]. Section 5.1 is dedicated to give an introduction to boson stars as well as to provide an insight about the nature of this objects and a heuristic estimation of the maximum mass attainable by the simplest configuration. In Section 5.2 we present the Lagrangian, evolution equations and conserved quantities. In Section 5.3 the mini-boson star is presented, which are the simplest spherically symmetric configurations with only massive term in the scalar potential. Boson stars with self coupled scalar field are also considered with simple potentials that retain the global gauge symmetry $U(1)$ in Section 5.4. The form in which gravitational interaction is described can be modified too, namely, in the Newtonian gravity context, scalar-tensor theories or even with no gravity at all as in the case of Q-balls.This chapter gives the basic foundations to understand our main result in this stage of the thesis, presented in the next chapter where we give a complete analysis of boson stars in bi-scalar extensions of Horndeski gravity.

5.1. Nature of boson star

Boson stars are described by a complex scalar field coupled to gravity. A complex scalar field $\phi(t, \mathbf{r})$ can be decomposed into two real scalar fields ϕ_R and ϕ_I

$$\phi(t, \mathbf{r}) \equiv \phi_r(t, \mathbf{r}) + i\phi_I(t, \mathbf{r}) \ . \tag{5.1}$$

The energy of this field -determined by its stress-energy tensor- gravitates holding the star together. On the other hand, to know what prevents the gravitational collapse one must to consider that the scalar field obeys a Klein-Gordon wave equation which confers a dispersive behavior to the fields. To this respect, Kaup found the energy eigenstates for semi-classical, complex scalar field, concluding that gravitational collapse was not inevitable [22]. This work was continued by Ruffini and Bonazzola [50], quantizing a real scalar field and finding the same equations.

None of the arguments previously exposed ensure a balance between the dispersion and gravitational attraction of the scalar field. Indeed, Derrick's theorem

[1]The reader may refer to [70, 71, 72] for detailed reviews on boson stars.

proves that in three spatial dimensional flat space there are no stable configurations which are regular, static, nontopological scalar field solutions [73]. It is possible to circumvent this constraint by considering a harmonic ansatz for the complex scalar field

$$\phi(t,\mathbf{r}) = \phi_0 e^{i\omega t} \ . \tag{5.2}$$

and turns the configuration stable. Although the scalar field is no longer static, the spacetime remains static. This can be seen by computing the stress-energy tensor which is independent of the time coordinate. Therefore, the star itself is a stationary, soliton-like solution.

A good estimation of the maximum mass of a boson star can be made considering, on the one hand, the Heisenberg uncertainty principle

$$\Delta p \Delta x \geqslant \hbar \ , \tag{5.3}$$

assuming m as the mass of the constituent particle and a star confined to a radius $\Delta x = R$. We determine its minimum possible value when the maximum momentum for a relativistic boson star $\Delta p = mc$ is reached and the uncertainty bound is saturated. Namely

$$R = \frac{\hbar}{mc} \equiv \lambda_C \ , \tag{5.4}$$

which is precisely its Compton wavelength λ_C. On the other hand to prevent gravitational collapse of the boson star, it requires R to get a minimum value given by its Schwarzschild radius $R_S \equiv 2GM/c^2$. This means that $R = R_S$ determines the maximum value M_{max} or[2]

$$\frac{2GM_{max}}{c^2} = \frac{\hbar}{mc} \ , \tag{5.5}$$

giving

$$M_{max} = \frac{1}{2}\frac{\hbar c}{Gm}. \tag{5.6}$$

The estimation is $M_{max} = 0.5 M_{Planck}^2/m$ where $M_{Planck} \equiv \sqrt{\hbar c/G}$ is the Planck mass . This inverse relationship is modified by self-interaction terms and the strength of the coupling m. Thus, depending on those conditions, the mass and size of the boson stars can vary from atomic to astrophysical scales.

More than just an analogy, boson stars can serve as a very useful model of a compact star, having certain advantages over a fluid neutron star model: (i) the

[2]In some literature the criterion to avoid an instability against complete gravitational collapse requires a star's radius bigger than the last stable Kepler orbit $3R_S$. Furthermore, some numerical calculation shows that $M_{max} = 0.633 M_{Planck}^2/m$.

equations governing its dynamics avoid developing discontinuities, in particular there is no sharp stellar surface, (ii) there is no concern about resolving turbulence, and (iii) one avoids uncertainties in the equation of state.

5.2. Lagrangian, field equations and conserved charges

The equations governing boson star are the Einstein equations for the geometric description and the Klein-Gordon equation determining the scalar field dynamics. Hereafter, we will refer to this system as the Einstein-Klein-Gordon (EKG) equations.

The action to be considered is as follows,

$$S = \int \sqrt{-g} d^4x \left(\frac{R}{16\pi G} + \mathcal{L}_m \right) \tag{5.7}$$

where R is the Ricci scalar given by the metric $g_{\mu\nu}$. The matter Lagrangian is given by

$$\mathcal{L}_m = -\frac{1}{2} \left[g^{\mu\nu} \nabla_\mu \bar{\phi} \nabla_\nu \phi + V(|\phi|^2) \right] \tag{5.8}$$

where we have denoted to the complex conjugate of the field as $\bar{\phi}$ and $V(|\phi|^2)$ as a potential depending only on the magnitude of the scalar field, which enjoys of $U(1)$ invariance of the field in the complex plane.

The Einstein field equations for this system turn out to be

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{5.9}$$

with $T_{\mu\nu} = \frac{1}{2} \left[\nabla_\mu \bar{\phi} \nabla_\nu \phi + \nabla_\mu \phi \nabla_\nu \bar{\phi} \right] - \frac{1}{2} g_{\mu\nu} \left[g^{\alpha\beta} \nabla_\alpha \bar{\phi} \nabla_\beta \phi + V(|\phi|^2) \right]$.

On the other hand Klein-Gordon (KG) equation is

$$\Box \phi = \phi \frac{dV}{d|\phi|^2} \ , \tag{5.10}$$

which is equivalent to the equation of motion for $\bar{\phi}$.

According to Noether's theorem, the global symmetry of the KG Lagrangian under the group $U(1)$ implies a conserved current

$$J_\mu = \frac{i}{2} (\bar{\phi} \nabla_\mu \phi - \phi \nabla_\mu \bar{\phi}), \tag{5.11}$$

satisfying the conservation law

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} J_\nu) = 0 \ . \tag{5.12}$$

Therefore, the conserved Noether current, given by the spatial integral of the time component of this current is

$$N = \int g^{0\mu} J_\mu \sqrt{-g} d^3x \ , \tag{5.13}$$

which is interpreted as the total number of bosonic particles.

Other important quantities that describe the boson star is the mass and the notion of radius. The mass is given by the expression

$$M = 4\pi \int_0^\infty \rho r^2 dr \tag{5.14}$$

where ρ is the mass-energy density of the scalar field. Equivalently, outside the star the solution can be matched with a Schwarzschild solution obtaining,

$$M = \lim_{r\to\infty} \frac{r}{2} \left[1 - \frac{1}{A(r)} \right] \ . \tag{5.15}$$

We require suitable boundary conditions for our desired boson star configurations. The appropriate set of boundary conditions requires that our solution must be

- non-singular,
- asymptotically flat,
- of finite mass.

The radius of the boson star has many definitions in the literature. The cause of this, is that the scalar field does not reach zero until radial infinity, therefore there is nonzero probability to find the presence of the scalar field at any radius. A definition by Gleiser [74] defines de radius R as

$$R \equiv \frac{4\pi}{M} \int_0^\infty \rho r^3 dr \ , \tag{5.16}$$

which is nothing else than the mean value of the radius with respect to the mass function integral given above. Other alternative definitions as the distance enclosing

95% of the total mass or the maximum of the radial metric function, gives similar results confirming a radius comparable to the Compton wavelength $1/m$. This is expected as all the bosons are in their ground state.

Finally and non less important the binding energy E_B of the star which is a measure of the difference in mass between the gravitationally bound configuration and the same number of particles dispersed to infinity. The expression for this quantity is defined as

$$E_B = Nm - M \ . \tag{5.17}$$

Positive binding energy configurations are unable to disperse completely to infinity, and those configurations are the most likely to be stable. In contrast, negative binding energy makes the entire star unstable, so the star disperse to infinity radiating off some particles, leaving a less centrally condensed stable configuration.

5.3. Mini-boson stars

Mini-boson stars are the simplest boson star solutions. These configurations are supported by a free-field potential for Lagrangian (5.7) which consist only in a massive term

$$V(|\phi|^2) = m^2|\phi|^2 \ , \tag{5.18}$$

with m as a parameter that can be recognized as the bare mass of the field theory.

As it was proved in [55], real scalar fields are not compatible with a regular and stationary gravitational field. Even so, one can assume a harmonic ansatz for the scalar field

$$\phi(t, \mathbf{r}) = \phi_0(\mathbf{r})e^{\mathrm{i}\omega t} \ , \tag{5.19}$$

and obtain time-independent equation of motion. Here ϕ_0 is a real scalar field which is the profile of the boson star and ω is the angular frequency of the phase of the complex scalar field. The variable ω is determined via an eigenvalue equation and is crucial for obtaining configurations of finite mass. For simplicity, let us consider spherically symmetric and static ansatz for the spacetime according to the minimal energy configuration requirement. Then

$$ds^2 = -B(r)^2dt^2 + A(r)^2dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\psi^2 \ , \tag{5.20}$$

where the real metric functions are A and B. The structure equations follow from the Einstein equations and the scalar wave equations for each of the two components. It is convenient also to include the equation of energy conservation. The Bianchi identities enforce energy conservation from the Einstein equations alone, thus implying that there are redundancies between these equations. This allows us to choose any set of independent ones. There are three nontrivial Einstein equations coming from the G_{tt}, G_{rr} and $G_{\theta\theta}$ components of the Einstein tensor. Once the scalar field has been reduced to one nonzero component as described above, the scalar wave equation and the energy conservation equation $\nabla_\nu T^\nu_\mu = 0$ (which here has the only one nontrivial component) give the same equation. Furthermore, the Bianchi identities then imply that this equation is a consequence of the Einstein equations. The most convenient set of independent equations to take are the G_{tt} and G_{rr} Einstein equations together with the scalar wave equation, so the EKG system can be written after a little manipulation as

$$\left(\frac{r}{A}\right)' = 1 - 8\pi G r^2 \left[\left(m^2 + \frac{\omega^2}{B}\right)\phi^2 + \frac{\phi'^2}{A}\right] , \tag{5.21}$$

$$\frac{B'}{ABr} - \frac{1}{r^2}\left(1 - \frac{1}{A}\right) = -8\pi G \left[\left(m^2 - \frac{\omega^2}{B}\right)\phi^2 - \frac{\phi'^2}{A}\right] , \tag{5.22}$$

$$\phi'' + \left[\frac{2}{r} + \frac{1}{2}\left(\frac{B'}{B} - \frac{A'}{A}\right)\right] - A\left(m^2 - \frac{\omega^2}{B}\right)\phi = 0 , \tag{5.23}$$

with prime as differentiation respect to r. For the numerical methods, it is more practical, to express the equations using dimensionless variables

$$x = mr , \qquad \sigma(x) = \sqrt{8\pi G}\phi(x/m) , \qquad \Omega = \omega/m . \tag{5.24}$$

Now, prime denotes differentiation respect to new variable x and the redefinition of the scalar field is given by σ. Finally we get the EKG system as follows

$$A' = xA^2\left[\left(\frac{\Omega^2}{B} + 1\right)\sigma^2 + \frac{\sigma'^2}{A}\right] - \frac{A}{x}(A-1) , \tag{5.25}$$

$$B' = xAB\left[\left(\frac{\Omega^2}{B} - 1\right)\sigma^2 + \frac{\sigma'^2}{A}\right] + \frac{B}{x}(A-1) , \tag{5.26}$$

$$\sigma'' = -\left(\frac{2}{x} + \frac{B'}{2B} - \frac{A'}{2A}\right)\sigma' - A\sigma\left(\frac{\Omega^2}{B} - 1\right) . \tag{5.27}$$

The use of dimensionless variables makes equations of motion explicitly non dependent on the boson mass m obtaining configurations with no dependence on this quantity. In order to obtain a physical and reasonable solution of this system, we have to impose suitable boundary conditions,

$$\sigma(0) = \sigma_0 \ , \tag{5.28}$$
$$\sigma'(0) = 0 \ , \tag{5.29}$$
$$A(0) = 1 \ , \tag{5.30}$$
$$B(0) = B_0 \ , \tag{5.31}$$
$$\lim_{x\to\infty} \sigma(x) = 0 \ , \tag{5.32}$$
$$\lim_{x\to\infty} B(x) = \lim_{x\to\infty} \frac{1}{A(x)} \ , \tag{5.33}$$

which guarantee regularity at the origin and asymptotic flatness and therefore being consistent with the requirements given in the previous Section 5.2.

Once the value for σ_0 is fixed, we need to adjust the eigenvalue $\omega^{(n)}$ in order to find solutions matching the asymptotic behavior (5.32) and (5.33). The system can be solved as a shooting problem by integrating from the origin $r = 0$ towards the boundary $r = r_{out}$. The configuration without nodes $\omega^{(0)}$ is the ground state, while all those with any nodes are excited states. As the number of nodes increases, the distribution of the mass as a function of the radius becomes more homogeneous.

As the amplitude σ_0 increases, the stable configuration has a larger mass while its effective radius decreases. This trend indicates that the compactness of the boson star increases. However, there is a point in which the mass decreases while the central amplitude increases. So, in the ground state there is a maximum allowed mass for a boson star, which is numerically found with the value $M_{max} = 0.633 M_{Planck}^2/m$ in very good agreement with the heuristic arguments provided in Section 5.1.

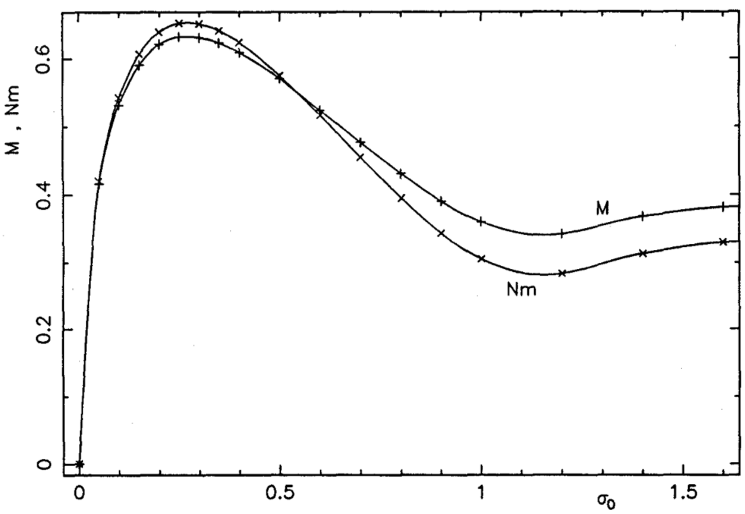

Figure 5.1: Mass M and particle number Nm respect to the central value of scalar field σ_0 for a ground state mini-boson star configuration. The first peak shows the maximum mass attainable for a boson star $M_{max} = 0.633 M^2_{Planck}/m$.

This can be seen in Fig. 5.1, as well as the particle number Nm respect to σ_0. Positive values for the binding energy E_B are presented even after the maximum mass, until a value for $\sigma_0 \approx 0.5$. This means that our maximum mass configuration is stable, and even more, that for $\sigma_0 \gtrsim 0.5$ is not possible to find stable configurations.

This behavior is the same exhibited by neutron stars, where static configurations are a function dependent only on the central density and they present a similar shape.

5.4. Boson stars in presence of self-interacting potentials

As we mentioned in 5.3, the first boson star solution was constructed with a free-field potential, without any kind of self-interaction, and with a maximum mass going like $M_{max} \approx M^2_{Planck}/m$. This mass is much smaller than the Chrandasekhar mass $M_{Ch} \approx M^3_{Planck}/m^2$ obtained for fermionic stars. For this reason those configurations are known as mini-boson stars. The main motivation for introducing self-interaction terms in the potential, is to reach astrophysical masses comparable to the Chandrasekhar mass since these terms provide an extra pressure against gravitational collapse.

The action (5.7) for these configurations contains a potential given by

$$V(|\phi|^2) = m^2|\phi|^2 - \frac{\lambda}{2}|\phi|^4 \ , \tag{5.34}$$

which retains global phase invariance and particle number conservation. In this part we restrict ourselves to positive λ, although a negative one is not excluded *a priori*. Indeed, in the next chapter we will consider that case where the system exhibits two degenerate local vacua but in the context of Horndeski Lagrangians where we explore the consequences of the non-minimal kinetic term. Let us check how important is the coupling constant λ. The values of central density for the most massive boson star is approximately $\phi_0 \sim M_{Planck}/20$. Thus, the ratio of the self-coupling term to the mass term at the centre of the star is

$$\frac{\lambda\phi_0^4/2}{m^2\phi_0^2} \sim \frac{\lambda}{1000}\frac{M_{Planck}^2}{m^2} \ , \tag{5.35}$$

which determines that the self-coupling term is important for,

$$\lambda > 1000\frac{m^2}{M_{Planck}^2} \ . \tag{5.36}$$

Even for a boson with mass equivalent to that of the neutron, the self-coupling terms only needs $\lambda > 10^{-35}$. This tell us, that the case $\lambda = 0$ examined previously is unlikely to be of astrophysical relevance.

The additional parameter λ increases the size of the space of configurations. As in the previous Section 5.3 it is convenient to redefine it by a dimensionless parameter,

$$\tilde{\Lambda} = \frac{\lambda M_{Planck}^2}{4\pi m^2} \ . \tag{5.37}$$

Any value of $\tilde{\Lambda}$ will label a family of equilibrium solutions of different central densities.

The equations of motion are quite similar to equations (5.25)-(5.27) but with an additional term coming from the self-interacting part of the potential,

$$A' = xA^2\left[\left(\frac{\Omega^2}{B}+1\right)\sigma^2 + \frac{\tilde{\Lambda}}{2}\sigma^4 + \frac{\sigma'^2}{A}\right] - \frac{A}{x}(A-1) \ , \tag{5.38}$$

$$B' = xAB\left[\left(\frac{\Omega^2}{B}-1\right)\sigma^2 - \frac{\tilde{\Lambda}}{2}\sigma^4 + \frac{\sigma'^2}{A}\right] + \frac{B}{x}(A-1) \ , \tag{5.39}$$

$$\sigma'' = -\left(\frac{2}{x}+\frac{B'}{2B}-\frac{A'}{2A}\right)\sigma' - A\left[\left(\frac{\Omega^2}{B}-1\right)\sigma - \tilde{\Lambda}\sigma^3\right] \ . \tag{5.40}$$

Large values for $\tilde{\Lambda}$ become a problem at the time of solving these equations numerically since the terms involved in the system differ by many orders of magnitude. In spite of that, Colpi, Shapiro and Wasserman [56] were able to construct a method to get numerical solutions with arbitrarily accuracy in the large $\tilde{\Lambda}$ limit. Let us redefine the radial metric function

$$A(x) = \left(1 - \frac{\mathcal{M}(x)}{x}\right)^{-1} , \tag{5.41}$$

and then we can rewrite (5.38) as

$$\mathcal{M}'(x) = \frac{1}{2}x^2 \left[\left(\frac{\Omega^2}{B} + 1\right)\sigma^2 + \frac{\tilde{\Lambda}}{2}\sigma^4 + \frac{\sigma'^2}{A}\right] . \tag{5.42}$$

As we mentioned above, the matching with the Schwarzschild solution at large values of the radial coordinates makes possible a definition for the boson star given by $M = \mathcal{M}(\infty)$. In the same way as in the case of mini-boson star, we redefine our variables in terms of dimensionless ones

$$\sigma^* = \tilde{\Lambda}^{1/2}\sigma , \qquad x^* = \tilde{\Lambda}^{1/2}x , \qquad \mathcal{M}^* = \tilde{\Lambda}^{1/2}\mathcal{M} . \tag{5.43}$$

Then, the large $\tilde{\Lambda}$ approximation neglects terms of order $1/\tilde{\Lambda}$ and higher [56] yielding to an algebraic equation for the scalar field (5.40),

$$\sigma^* = \left(\frac{\Omega^2}{2} - 1\right)^{1/2} . \tag{5.44}$$

This algebraic relation can be substituted into the other equations of motion which are consistently restricted to the same approximation. Thus, we left with the remaining equations as

$$\mathcal{M}^{*\prime} = \frac{1}{4}x^{*2}\left(3\frac{\Omega^2}{B} + 1\right)\left(\frac{\Omega^2}{B} - 1\right)^2 , \tag{5.45}$$

$$\frac{B'}{ABx^*} - \frac{1}{x^{*2}}\left(1 - \frac{1}{A}\right) = \frac{1}{2}\left(\frac{\Omega^2}{B} - 1\right)^2 . \tag{5.46}$$

By using this approximation, numerical solutions become functions of a new free parameter $\Omega/B(0)$. The maximum mass is shown to be [74]

$$M_{max} \sim 0.22\tilde{\Lambda}^{1/2}\frac{M_{Planck}^2}{m} = 0.06\lambda^{1/2}\frac{M_{Planck}^3}{m^2} . \tag{5.47}$$

Boson star masses with self-interacting coupling are much greater than the masses achieved by mini-boson stars as it can be seen in Fig. 5.2, suggesting a different range of possible masses from the uncoupled case discussed above. Indeed, for a boson mass of the order of a neutron mass and $\lambda \sim 1$ the boson star mass is comparable with a neutron star mass. For less heavier particles, boson star configurations can reach a huge mass. Because of their astrophysical relevance to this respect, in the literature the name "boson star" is reserved to this configuration, making an explicit distinction with the smallest mass configuration described in the previous Section 5.3 which is called "mini-boson star".

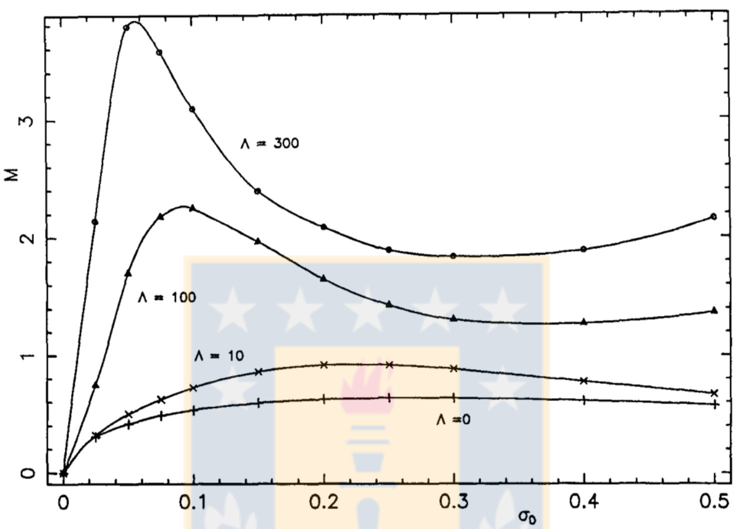

Figure 5.2: Mass M respect to the central value of scalar field σ_0 for a ground state boson star configuration. It can be seen that the maximum mass increases as $\tilde{\Lambda}$ is increased.

5.5. Alternative theories of gravity

Instead of modifying the scalar field potential, one can explore boson star in the context of alternative theories of gravity. It has been found that scalar-tensor theories exhibit spontaneous scalarization in which the scalar field, as new degree of freedom for the gravitational interaction, undergoes a transition to a nontrivial configuration, in the same way as happens with ferromagnetism in neutron stars [75]. Such effect is also found in the context of boson star evolution [76].

Some examples of boson star in scalar-tensor extensions in conformal gravity are in [77, 78]. Alternative theories of gravity have rise as a explanation to the apparent existence of dark matter without employing an unknown dark matter component. The most well known case is the Modified Newtonian Dynamics (MOND) in which gravity suffers modifications only at large distances (for a review reader may refer [79]). The boson star has been studied in the frame of Tensor-Vector Scalar theories (TeVeS), as a generalization of MOND [80]. In particular, their evolutions for boson stars develop caustic singularities, which has motivated several modifications of the theory in order to avoid such problems.

In the next chapter we will explore boson stars in bi-scalar extensions of Horndeski gravity. This is the most general ghost-free scalar-tensor theory with non-minimal coupling. We will analyze the influence and consequences on stability and critical masses of the new non-minimal coupling constant by comparing to the mini-boson star and Q-ball cases as well as a detailed description for two different potential of special interest.

Chapter 6

Boson Stars in bi-scalar extensions of Horndeski gravity

This chapter is concerned with the construction and analysis of boson stars in the context of non-minimal derivative coupling theories. In particular we embed our model in the bi-scalar extension of Horndeski gravity, considering a scalar field theory displaying a non-minimally coupled kinetic term given by the Einstein tensor. We focus on the case where the potential is given by a mass term only, and when a six order self-interaction is included. In the latter case we consider specific couplings in the self-interacting terms in such a way that our self-interaction is given by a positive definite potential presenting two degenerate local vacua. We show how solutions can be obtained and we compare its principal properties with standard configurations constructed with the usual minimally coupled kinetic term.

This chapter is organized as follows: Section 6.1 is devoted to present our model and the general setting in which we will study boson star configurations. In Section 6.2 we will construct BS's considering no self-interaction, this means considering only a mass term, and we will compare them with the results obtained for the standard case of minimally coupled scalar field theories. Section 6.3 considers the inclusion of self-interaction, in particular the sixth order potential with nontrivial vacuum manifold. Finally we conclude in Section 6.4.

6.1. General setting

6.1.1. The model

In the following we will extend (3.25) to contain a complex scalar field. The action then reads

$$S=\int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N}\right)-\int d^4x\sqrt{-g}[(\alpha g^{\mu\nu}-\eta G^{\mu\nu})\nabla_\mu\Phi\nabla_\nu\Phi^*+U(|\Phi|)] \quad (6.1)$$

where Φ denotes a complex scalar field. We work in the "mostly plus" signature. As we mention above α and η are the dimensionful parameters controlling the standard and nonminimal couplings. The potential $U(|\Phi|)$ contains the mass term m and, eventually, a self-interaction to be specified below.

To embed this model in the context of the STT we are considering here, it is necessary to go beyond the original Horndeski theory and to consider its extension. Indeed, as we know, a system composed by a complex scalar field can be treated as a system composed by two real scalar fields. Extensions of Galileon theory or Horndeski gravity for the case in which two scalar fields degrees of freedom are considered have already been constructed in [81, 82, 83, 84, 85]. We observe that in the bi-scalar extension also appears the nonminimal kinetic sector described above in (3.15) and that our model can be supported by that kind of Lagrangians. Construction of relativistic stars on these kind of models have been considered in [19]. We point out that these kind of theories have been recently considered in cosmology [86, 87, 88] where the authors have studied theories beyond Horndeski (higher order terms) imposing conformal invariance, thus arriving to a healthy ghost free bi-scalar tensor theory.

6.1.2. The Ansatz

Due to the complexity of the equations, we will limit ourselves to stationary non-spinning solutions. For this purpose we use a spherically symmetric ansatz for the metric and specify the radial variable through the isotropic coordinates

$$ds^2=-F(r)dt^2+\frac{G(r)}{F(r)}\left[dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2\right] \ . \quad (6.2)$$

The scalar field will be given by

$$\Phi=\Phi_0\phi(r)e^{i\tilde{\omega}t} \quad (6.3)$$

where the constant Φ_0 supports the dimension of the scalar field and the frequency $\tilde{\omega}$ encodes the harmonic dependence of the solution. The harmonic ansatz is used in order to circumvent Derrick's theorem [73], which states that time-independent localized solutions of nonlinear wave equations in spacetime with three or more space dimensions are unstable. For this precise form of the scalar field we obtain that the contribution of the scalar field in the equations of motion remains static, even if the scalar field degree of freedom is no longer static, not sharing in this way the same symmetries than the spacetime. The coupled system of non-linear equations then reads

$$\begin{aligned} A_{11}F'' + A_{12}G'' + A_{13}\phi'' &= K_1(F, F', G, G', \phi, \phi', \tilde{\omega}) \\ A_{21}F'' + A_{22}G'' + A_{23}\phi'' &= K_2(F, F', G, G', \phi, \phi', \tilde{\omega}) \\ A_{31}F'' + A_{32}G'' + A_{33}\phi'' &= K_3(F, F', G, G', \phi, \phi', \tilde{\omega}) \end{aligned} \tag{6.4}$$

where the prime denotes the derivative with respect to r. K_a are polynomials given in term of the metric functions, the scalar field and their first derivatives respectively. The coefficients A_{ab} depends on the fields in the same way than the polynomials K_a. They are given in the Appendix 8.1.

In the case of a minimal coupling, i.e. for $\eta = 0$, the matrix A is diagonal and positive definite. Nevertheless, for $\eta \neq 0$, this matrix becomes non-diagonal and the determinant $|\text{det}A(r)|$ plays a fundamental role in the existence of solutions. When this determinant presents zeros the corresponding system is singular and no regular solution can be found. We will see that this affects significantly the pattern of solutions.

6.1.3. Boundary conditions

For the construction of BSs, the system has to be solved with the following boundary conditions:

$$\begin{aligned} &F(0) = 1\ ,\ G(0) = 1\ ,\ \phi(0) = \phi_0\ ,\ F'(0) = 0\ ,\ G'(0) = 0\ ,\ \phi'(0) = 0\ , \\ &F(\infty) = 1\ ,\ G(\infty) = 1\ ,\ \phi(\infty) = 0\ . \end{aligned} \tag{6.5}$$

Here ϕ_0 represents the central value of the scalar field. On the one hand, the conditions at $r = 0$ are necessary for soliton solutions to be regular at the origin. On the other hand the conditions at $r = \infty$ ensure localized and asymptotically flat solutions. To find solutions respecting these conditions on $r = 0$ and $r = \infty$, the eigenvalue $\tilde{\omega}$ has to be fine tuned for a given central value, ϕ_0, of the scalar function ϕ. This leads in general to a relation of the form $\omega(\phi_0)$. In principle, the equations can be solved by a shooting technique; we used instead the routine Colsys [89, 90] based on the Newton-Raphson algorithm.

6.1.4. Rescaling

For the numerical study of our system (6.4), it is convenient to perform suitable rescalings of the parameters leading to dimensionless quantities. For this purpose, we define the dimensionless variable x and parameters κ, ξ, ω by mean of

$$x = mr \quad , \quad \kappa = 8\pi G_N \Phi_0^2 \quad , \quad \xi = \frac{\eta}{m^2} \quad , \quad \tilde{\omega} = m\omega \ . \tag{6.6}$$

where m denotes the mass of the scalar field. One reason for including the parameter α is to allow -if they would exist- exotic solutions corresponding to $\alpha = 0$ and $\eta = 1$. Since we failed to construct such solutions in the model under consideration, we set, without loosing generality, $\alpha = 1$ throughout the paper.

6.1.5. Physical Quantities

The solutions can be characterized by several quantities. The global symmetry of the action under phase change of the scalar field leads to a conserved current j^μ and conserved charge Q :

$$j^\mu = -i(\Phi^* \partial^\mu \Phi - (\partial^\mu \Phi^*)\Phi) \ \ , \ \ Q_{phys} = -\int j^0 \sqrt{-g} d^3 r \tag{6.7}$$

With the ansatz and rescaling used above, the conserved charge is computed as follow

$$Q_{phys} = 8\pi \frac{\Phi_0^2}{m^2} \int_0^\infty \frac{\sqrt{G^3}}{F^2} x^2 \tilde{\omega} \phi^2 dx \quad , \quad \frac{\Phi_0^2}{m^2} = \kappa \frac{M_{Pl}^2}{m^2} \ , \tag{6.8}$$

where $8\pi G_N \equiv M_{Pl}^{-2}$ is the Planck mass; the quantity Q is interpreted as the number of bosonic particles. The solution is also characterized by the mass M, it can be read out of the asymptotic decay of the metric function F

$$F(r) = 1 - \frac{2G_N M_{phys}}{r} + o(\frac{1}{r^2}) \ = \ 1 - \frac{2mM_{phys}}{8\pi M_{Pl}^2} \frac{1}{x} + o(\frac{1}{x^2}). \tag{6.9}$$

The quantities Q and M reported on the figures will be related to the physical quantities according to

$$Q_{phys} = \kappa \frac{M_{Pl}^2}{m^2} Q \quad , \quad M_{phys} = \kappa \frac{M_{Pl}^2}{m} M \tag{6.10}$$

The boson star can also be characterized by a radius. There are many ways to define such a parameter since the scalar field does not strictly vanish, along many authors (see namely [91]) we define the dimensionless radius R of the boson star as

$$\frac{R}{m} = \frac{1}{Q_{phys}} \int r \ j^0 \sqrt{-g} d^3 r \ \ . \tag{6.11}$$

We will find R of order one, as a consequence, a mass m for the boson field of order one MeV would corresponds to R_{phys} of order 200 Fermi. The ratio M/mQ provides some informations about the stability of the soliton. The condition $M < mQ$ is necessary for the soliton to be stable; indeed if $M > mQ$ the mass of the full soliton exceeds the mass of Q scalar field quanta and no binding energy is left to stabilize the lump. In the discussion of the solutions we will refer to this argument only; the full study of the stability is out of the scope of this paper.

6.1.6. The potentials

BS solutions minimally coupled to Einstein gravity with no self-interaction (mass term only) have been studied in great detail in [91]. As we pointed out, the non-gravitating counterpart of BSs, Q-balls, do not exist. Indeed, to obtain the later configurations is necessary to consider self-interaction with at least six order powers of the scalar field (see [92]). Motivated by this, we will also investigate BSs in the context of this kind of potentials for our non minimally coupled model. The potential reads

$$V = \lambda_3|\Phi|^6 - \lambda_2|\Phi|^4 + \lambda_1|\Phi|^2 \quad , \quad \lambda_1 \equiv m^2. \tag{6.12}$$

Because of the numerous parameters, we will put the emphasis on the following two cases

- $\lambda_2 = 0$, $\lambda_3 = 0$ which corresponds to a mass term only. We will examine the influence of the nonminimal coupling on the spectrum of the solutions.
- $\lambda_2/\lambda_1 = 2\lambda_3/\lambda_1 = 2$. This corresponds to a positive definite potential presenting two degenerate local minima at $\phi = 0$ and $\phi = 1$. We will denote it V_6.

6.2. Boson Stars with the mass potential

As pointed out already, the occurrence of nodes of the quantity $|\det A(r,\xi)|$ play a role in the construction of the solutions. For all parameters that we have explored, the minimum of this determinant is always located at the origin (i.e. $x = 0$). Therefore we find it convenient to define

$$\Delta(\xi) = \frac{\det A(0,\xi)}{\det A(0,0)} \tag{6.13}$$

as a control parameter. The set of numerical routines employed lead to reliable solutions as long as $\Delta > 10^{-6}$.

6.2.1. Mini Boson Stars with $\xi = 0$

In this section we comment on some properties of BSs when the nonminimal coupling is absent. For more details please see [70]. In this case, the constant κ can be rescaled in the scalar field and the mass m of the scalar field can be rescaled in the radial variable. We can therefore set $\kappa = 1$, $m = 1$ without loosing generality. BSs are then essentially characterized by the value $\phi(0) \equiv \phi_0$ of the soliton at the center. In particular, the numerical integration determines the frequency of the scalar field ω as a function of ϕ_0. In this paper, we discuss only the fundamental solutions where the function $\phi(r)$ has no nodes, a series of excited solutions presenting zeros of $\phi(r)$ exists as well. In the limit $\phi_0 \to 0$, the vacuum solution is approached ($M = Q = 0$) and this corresponds to $\omega \to 1$. Increasing gradually the parameter ϕ_0, it turn out that the frequency ω first reaches a minimal value $\omega_m \approx 0.7677$ and then oscillates around an asymptotic mean value $\omega_a \sim 0.8425$ (see top part of Fig. 6.1). In spite of the fact the the frequency ω does not characterize the solutions uniquely, it is common to display the mass M and the charge Q as functions of this parameter. Due to the oscillations, these plots currently present the form of spirals as seen in the bottom part of Fig. 6.1.
The three symbols -bullet, triangle and square- symbolise the special values where the charge Q reaches its absolute maximum and minimum (Q_{max}, Q_{min} and Q_c where $M = mQ_c$; these values will play a role in the discussion of stability. Completing Fig.6.1, we show on the top panel of Fig.6.2 the dependance of the mass M and of the charge Q as functions of the central value of the scalar field ϕ_0. The three exeptional values Q_{max}, Q_c, Q_{min} refer to the minimal case $\xi = 0$ and correspons respectively to $\phi_0 \sim 0.6, 1.37, 2.6$. On the bottom panel of the figure, the dependance of Q, M on the radius R are reported. Referring to the argument of stability invoked above, it turns out that the condition $M/Q < 1$ is fulfilled only for the small values of ϕ_0, typically for $\phi_0 \leq 1.25$. The value $\phi_0 = 1.37$ corresponds to $M = Q = Q_c \approx 67$, as seen on Fig. 6.3. The plot of the ratio M/Q as function of Q reveals the occurrence of at least three branches joining at spikes. For later convenience let us call the branch connected to the vacuum (i.e. with $M = Q = 0$) the main branch and the other branches as the second, third branch and so on. The spike connecting the main and the second branches corresponds to the maximal value of the charge, say $Q = Q_{max}$. We find it for $Q_{max} \approx 82$, $M \approx 79.5$, $\omega \sim 0.85$, $\phi_0 = 0.6$; it belongs in the domain of classical stability. The second spike connects the second and third branches and corresponds to a local minimum of Q, say $Q = Q_{min}$. We find $Q_{min} \approx 36.00$, $M \approx 43.0$. This second spike belongs to a region where the solutions are unstable. On the second branch, only the BSs corresponding to $Q_c \leq Q \leq Q_{max}$ are classically stable.

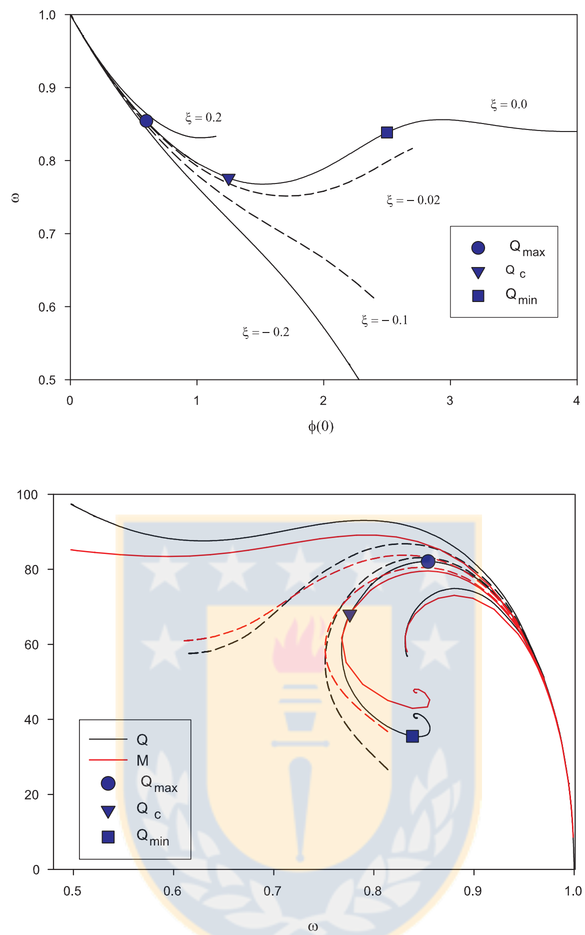

Figure 6.1: Top: The frequency ω as function of $\phi(0)$ for BSs without self-interacting scalar fields and for different values of ξ. Bottom: The mass and charge as functions of ω for the same values of ξ. The three symbols (bullet,etc...) show up three critical values of Q on the the $\xi = 0$ line.

6.2.2. Mini Boson Stars with $\xi \neq 0$

We now discuss how the spectrum of the BSs is affected by the inclusion of the non minimal coupling, i.e. for $\xi \neq 0$. The classical equations now depend on two non trivial parameters ω and ξ. As expected by a continuity argument, integrating the field equations for a fixed value of ϕ_0, the minimally coupled BSs (i.e. with $\xi = 0$) can be continuously deformed by increasing (or decreasing) gradually the coupling parameter ξ.

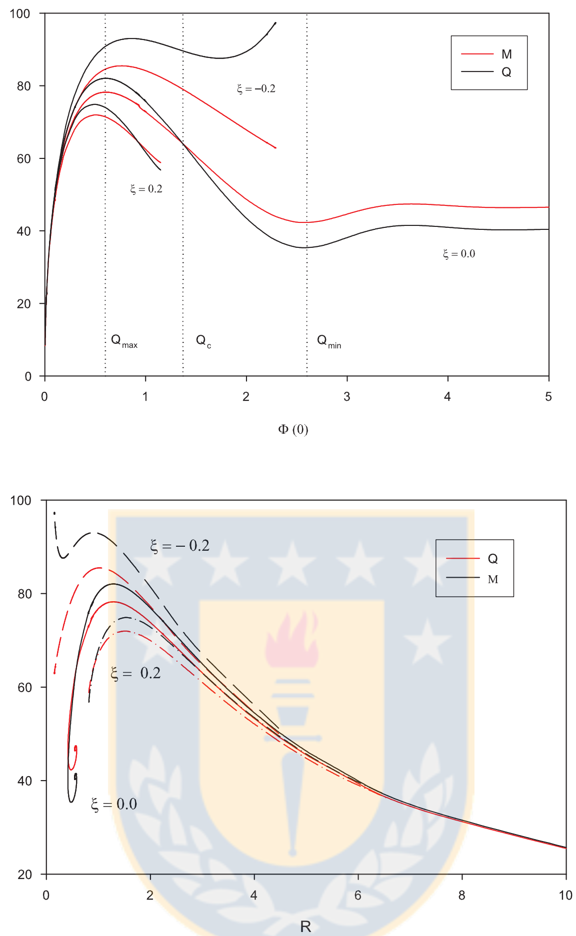

Figure 6.2: Top: The mass and charge as functions of $\phi(0)$ for $\xi = 0$ and $\xi = \pm 0.2$. Bottom: The mass and charge as functions of the radius R for the same values of ξ.

Let us first discuss the case $\xi \neq 0$. Similarly to the case $\xi = 0$, a branch of BSs can then be constructed by increasing the parameter $\phi(0)$. This leads to families of solutions characterized by the frequency ω, the charge Q and the mass M. In Fig. 6.1 we present some data corresponding to different values of ξ together with the case $\xi = 0$. For $\xi \neq 0$, the curves stop at some critical values of $\phi(0)$; the numerical integration indeed becomes problematic at some stage for high values of $\phi(0)$. Our numerical results suggest that the critical phenomena limiting the

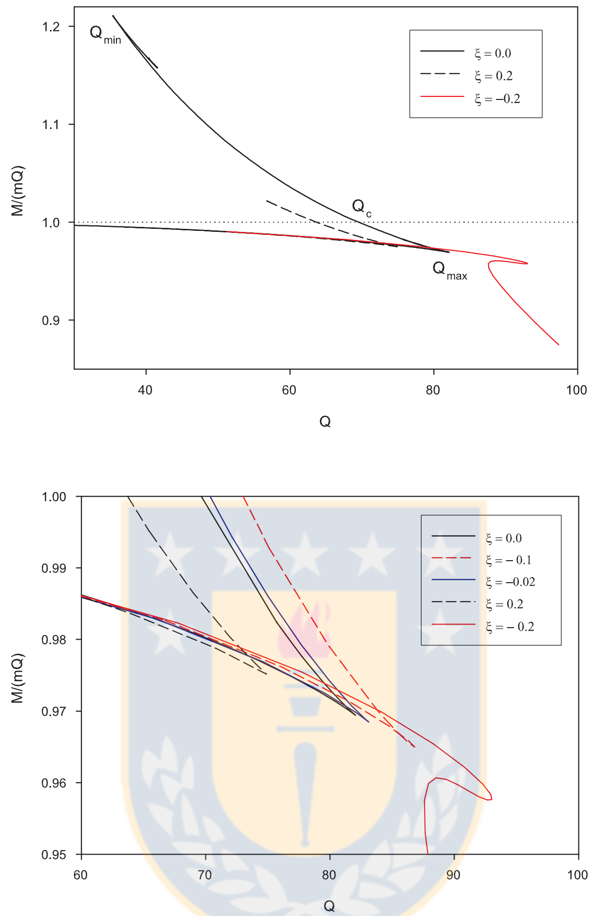

Figure 6.3: The ratio M/Q as a function of Q for several values of ξ.

solutions for positive and negative values of ξ have different origins :

- For **positive** values of ξ the solutions cannot be constructed for large values of ϕ_0 because the determinant Δ approaches zero at a critical value of the parameter ϕ_0, say $\phi_0 = \phi_{0,max}$. For example for $\xi = 0.2$, we find $\phi_{0,max} \approx 1.15$.

- For **negative** values of ξ, the situation is different : Δ decreases monotonically but not reaching zero while ϕ_0 increases.

One of the main effect of the non minimal coupling is then to limit the possible values of the central value ϕ_0 of the boson field. In particular setting $|\xi| > 0$ has the tendency to 'unwind' the spiral curves $M(\omega)$ as seen in Fig. 6.1. Qualitatively, this resemble the effects of the Gauss-Bonnet interaction in the pattern of higher dimensional boson stars in Einstein-Gauss-Bonnet Gravity. These solutions have been studied in [93] where it was shown that the origin of the critical phenomenon is related to the occurrence of a singularity of the metric at the origin. In the present case, the geometry remain regular in the critical limit, instead the system of equations becomes singular when the determinant Δ approaches to zero.

Before reexamining this phenomenon with a different point of view, let us discuss the effects of the non-minimal coupling on the classical stability of the BSs. For small values of $|\xi|$, the plot of the ratio M/Q as a function of Q generally presents two branches joining in a spike at, say $Q = Q_{max}$ (see Fig. 6.3). The main branch is stable all long. On the other hand a piece of the second branch is stable for $Q_c \leq Q \leq Q_{max}$ where we define Q_c as the value of the charge where $M/Q = 1$. For $Q \leq Q_c$, the solutions of the second branch are unstable. Both values Q_c, Q_{max} increase while ξ decreases. This scenario holds true for small enough values of $|\xi|$. Interestingly, for $\xi < -0.15$ the pattern changes: both the main and second branches are classically stable. Hence, negative values of the non-minimal coupling have the tendency to enhance the stability of the solutions.

To complete the discussion we study how solutions corresponding to a particular central value ϕ_0 are affected by the non minimal coupling. The results are the object of Figs. 6.4. The ratio $M/(mQ)$ is reported as a function of ξ for three values of ϕ_0 on the top panel: it shows that the ratio increases monotonically with ξ. As noticed already, the lump is more bounded. for negative values of ξ The critical phenomenon limiting the BSs for $|\xi| \neq 0$ is revealed on the bottom side of Fig 6.4. We see that the determinant $\Delta(\xi)$ suddenly approaches to zero for a *positive* critical value of ξ (this value, depends of course on ϕ_0). In contrast, for $\xi < 0$, the value Δ regularly decreases to zero, although not reaching $\Delta = 0$, while decreasing ξ. The numerical difficulties occur typically when Δ becomes of the order of the tolerance imposed for the numerical integrator. We manage to construct robust solutions up to $\Delta \sim 10^{-8}$. For large values of ϕ_0 (typically $\phi_0 \geq 2$) the following features, illustrated by the red curve in Fig. 6.4, should be stressed

- the value Δ becomes very sensitive to ξ
- the interval of ξ where the solutions exist decreases.

These constitute the sources of the numerical difficulties.

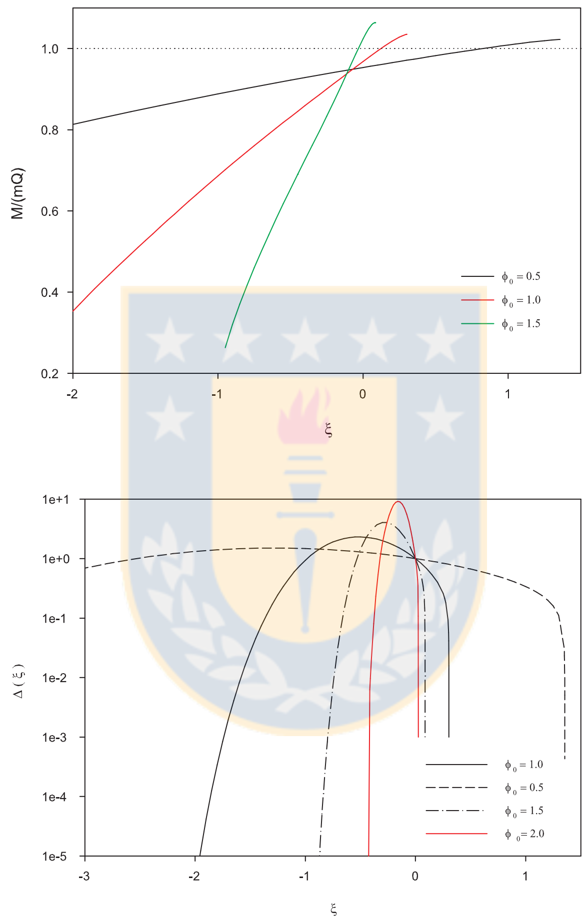

Figure 6.4: Top: Ratio $M/(mQ)$ as functions of ξ for different values of ϕ_0. Bottom: Discriminant Δ as function of ξ for several values of ϕ_0.

6.3. Self-interacting solutions

6.3.1. $\xi = 0$ case

We now discuss the effects of the self-interaction of the scalar field on the solutions. As stated above, we choose the particular potential

$$V_6(|\Phi|) = m^2 |\Phi|^2 (|\Phi|^2 - 1)^2 \tag{6.14}$$

which possesses a non trivial vacuum manifold : $|\Phi| = 0$ and $|\Phi| = 1$. Many properties of BSs in this potential (including also the effect of an electric charge) have been discussed in [94]. Perhaps one of the main property is that the BSs can be continued to the non gravitating limit $\kappa = 0$, constituting a family of Q-ball solutions labelled by ω. The self-interaction due to the potential confers very specific features to the Q-balls, some of which are shown on Fig. 6.5 (dashed lines) :

- The solutions exist up to a maximal value of $\phi(0)$.
- The solutions exist for arbitrarily small values of ω. The limit $\omega \to 0$ corresponds to $\phi(0) \to 1$; the profile of the scalar field approaches a step function with $\phi(r) \sim 1$ for $r < R$ and with $\phi(r) \sim 0$ for $r > R$, so that the boson field is essentially concentrated in a sphere of radius R. This corresponds to the so-called "thin-wall limit"; the mass, the charge and the radius R diverge while ω approaches zero.
- In the limit $\phi(0) \to 0$ the matter field approaches uniformly the vacuum configuration $\phi(r) = 0$ although the mass and the charge remain finite, forming a "mass gap". This is denoted by Y on the bottom side of Fig.6.5.

The coupling to gravity has the effect to regularize the Q-balls configurations. This is shown in Fig. 6.5 where the data corresponding to $\kappa = 0.1$ (we set $\xi = 0$ in this section) is reported by means of the solid lines. In contrast with Q-balls, the following features hold :

- BSs exist for large values of $\phi(0)$. The mass and charge remain finite and bounded.
- There is a minimal value of ω. The minimal value depend on the constant κ.
- In the limit $\phi(0) \to 0$ the matter field approaches uniformly the vacuum configuration $\phi(r) = 0$. The mass and the charge converge to zero.

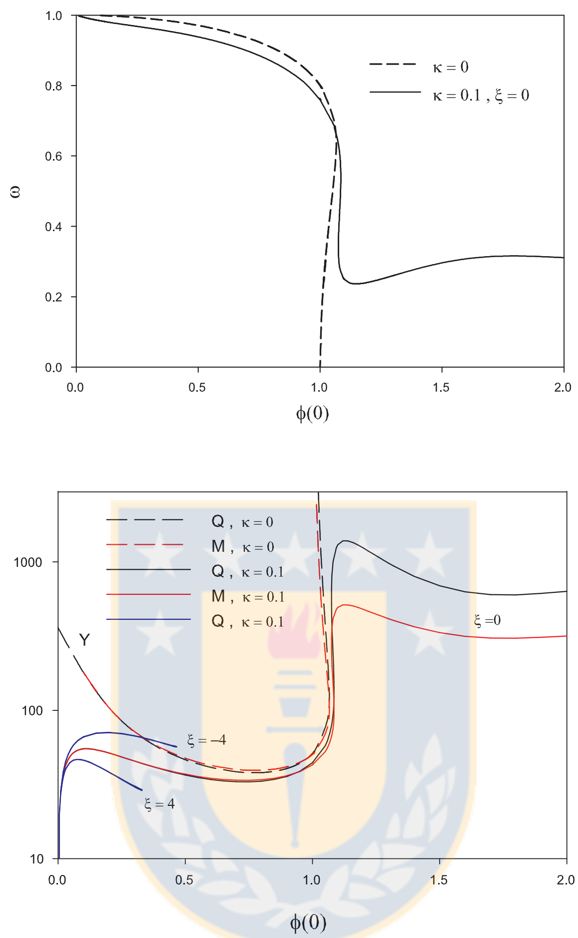

Figure 6.5: Top: The dependence of ω on ϕ_0 for Q-balls (dashed) and BSs (solid). Bottom: The mass, charge dependance of $\phi(0)$.

The classical stability of self-interacting Q-balls and (minimal-coupled) BSs can be read from the M/Q plot provided in Fig.6.6. The curve corresponding to BSs is the black-solid line. It shows the occurrence of three branches joining in two spikes (labelled A and B in the figure) and forming a curve with the shape of a butterfly. The main branch, connected to the vacuum and terminating at A, corresponds to a set of stable solutions. The intermediate branch $A - B$ is essentially unstable (only on a small fraction of it the solutions are stable). The third branch terminating at B is stable in its part corresponding to large values of $\phi(0)$.

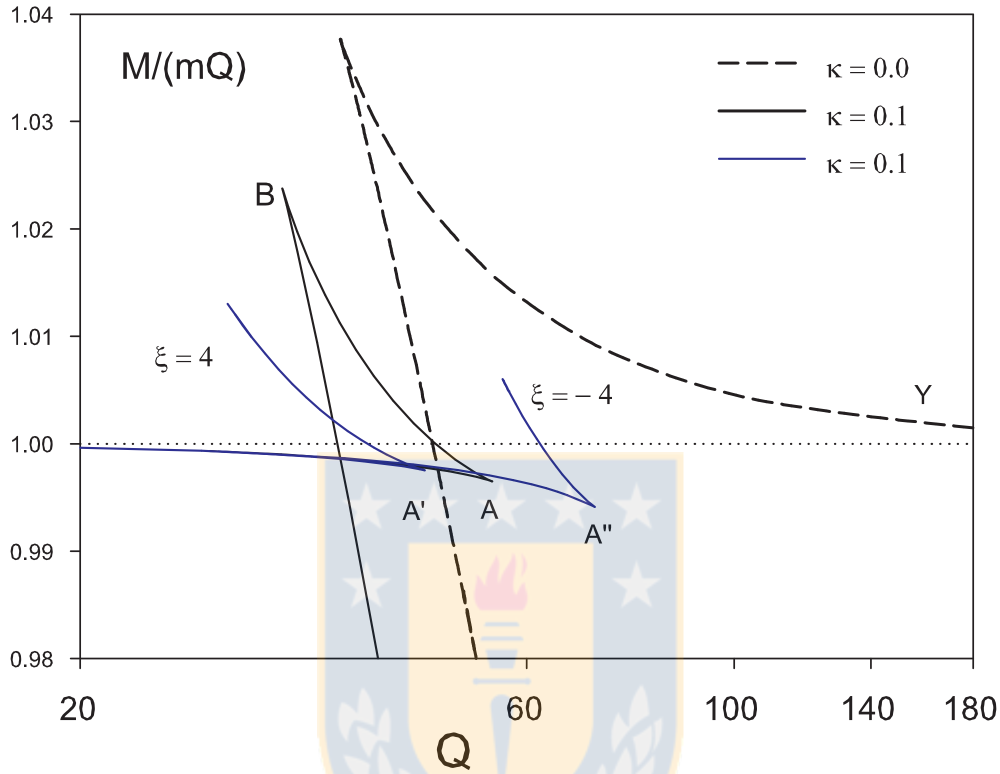

Figure 6.6: The ratio $M/(mQ)$ as function of Q for different self-interacting solutions.

6.3.2. $\xi \neq 0$ case

We now discuss the influence of the non-minimal coupling to the solutions. For definiteness we set $\kappa = 0.1$ in our numerical construction. Following the same lines as in the previous section, we analyzed the deformation of the BSs for $\xi \neq 0$. As expected, it turns out that the non-minimal coupling reduces considerably the domain of existence of the BSs : for both signs of ξ the solutions exist only for small enough values of ϕ_0. In particular the quantity $\Delta(\xi)$ approaches zero while increasing the value of ϕ_0 of the solution; leading to a maximal value, say $\phi_{0,max}$. The precise determination of $\phi_{0,max}(\xi)$ is beyond the scope of this paper but our numerical results demonstrate that it is monotonically decreasing while $|\xi|$ increases. The data corresponding to $\xi = \pm 4$ is shown in Fig. 6.5 (the blue lines for the charge

Q, the curve for the corresponding mass is very close and not reported).

Remembering that the self-interaction allows for solutions to exist in the absence of normal gravity, the question arises naturally whether solitons could interact with gravity through the non-minimal derivative term only, i.e. with $\alpha = 0, \xi = 1$. We therefore put some emphasis on solutions with $\alpha/|\xi| \ll 1$. Our numerical results strongly suggest that the standard BSs do not survive in the limit $\alpha/|\xi| \to 0$; their domain being too restricted by the condition $\Delta(\xi) > 0$. It is possible, however, that new types of solitons exist on a domain of the parameter space connected to $\alpha = 0, \xi = 1$. This would constitute a bosonic counterpart of the neutron stars obtained in [18]. So far, we failed to construct such solutions numerically.

Let us finally comment on the way how the stability pattern is affected for $\xi \neq 0$. Due to the reduction of the domain of the solutions, the "butterfly" curve occurring for $\xi = 0$ is progressively reduced as well. For the cases $\xi = 4$ and $\xi = -4$, chosen for Fig. 6.6, only two of the three branches remain; They are joining at points A' and A'' respectively. The solutions on the branch joining to the vacuum are stable, irrespectively of the sign of ξ. Negative values of ξ allow for stable solutions with higher values of the charge Q and of the energy binding $M/Q - 1$.

6.4. Final remarks

In this work we have constructed BS configurations for STT possessing a non-standard kinetic term coupled through the Einstein tensor. This particular coupling is contained in the most general STT with second order equations of motion for a single new scalar degree of freedom, the so called Horndeski theory.
Due to the fact that we are dealing with a complex scalar field, instead of Hordenski gravity, our model is embedded in its bi-scalar extension, namely, in the context of the most general STT with second order equations of motion, constructed with a single massless metric tensor and with two real scalar field degrees of freedom. In this scenario BSs are supported by new degrees of freedom and not by external matter sources. It is important to stress that along with the new degrees of freedom also external matter fields may be included. In the context of STT, in [76, 95] the authors have tackled this problem showing that the phenomenon of spontaneous scalarization originally predicted for neutron stars, can also occur for BSs. Moreover, we are not considering here any kind of interaction between external fields and new scalar degrees of freedom.
We have analyzed the existence of mini-BSs configurations (where only a mass term is considered) and of self-interacting BSs where the self interaction possesses a six-order potential which can be written, for specific values of the involved couplings, as a positive definite potential presenting two degenerate local vacua. In both cases

we have shown that the determinant of our system of equations (6.4) plays a fundamental role in the pattern of solutions. Indeed, when this determinant approach to zero no solutions can be obtained. In practice we have seen that the pole of this determinant is always located at the origin, luring as to define the control parameter Δ (6.13) in order to look for non-singular solutions.
Mini-BS solutions exist for both, positive and negative values of the nonminimal rescaled parameter ξ. For $\xi > 0$ the solutions cannot be obtained when the central value ϕ_0 exceed some maximal value $\phi_0 = \phi_{0,max}$, for which our function Δ goes to zero. On the other hand, the $\xi < 0$ case, is different. Δ shows a monotonically decreasing behavior when increasing ϕ_0 not reaching the conflictual point $\Delta = 0$, nevertheless complications arise when this function approaches to values of the same order than the tolerance imposed by the numerical integrator. The ξ negative branch also shows a tendency to enhance the stability of the solutions.
For the self-interacting solutions considered here, the situation is similar to the mass term case. The pattern of solutions is harshly constrained and exists for a limited branch of values of ϕ_0, later that indeed depend on ξ. For this particular case we have also investigated the existence of configurations supported only by the presence of the nonminimal kinetic coupling, this means, for the $\alpha/|\xi| \to 0$ case. Our results suggest that BSs do not survive in this case. It would be interesting to circumvent this problem in oder to construct the bosonic counterpart of the neutron stars constructed in [18], and make qualitative comparisons. We leave this for future work.

Chapter 7

Cylindrically symmetric spacetimes

In vacuum, the static cylindrically symmetric spacetimes, in absence of a cosmological constant, was found by Levi-Civita [96] just few years after the emerging of General Relativity. However, the inclusion of a nonzero cosmological constant was only achieved almost 70 years later by Linet [97] and Tian [98]. More recently, some geometrical properties of these spacetimes, such as the presence of conical singularities, were reviewed in [99, 100]. The stationary cylindrically symmetric vacuum solution was discovered independently by Lanczos [101] and Lewis [102]. The general solution contains a number of integration constant, whose physical interpretation has been studied in [103, 104]. In vacuum, the cylindrical stationary spacetime with a nonvanishing cosmological constant was derived in [105] and [106]. The interpretation of the integration constants was clarified in [107], where it was proved that three of them are indeed essential parameters. Two integration constant have a topological origin [108], and a third one characterizes the local gravitational field.

The physical interpretation of this kind of solutions are far from being trivial. As an example, the identification of points associated with the periodic coordinate represents a problem in this context, since without such identifications, spacetimes with both temporal and spatial symmetries may be alternatively interpreted as being plane symmetric. An additional problem appears when singularities, normally located at the axis, depend on the direction from which they are approached. The aim of this chapter is to present this kind of solutions and to describe their physical properties.

In this chapter we will give a short review of the main cylindrically symmetric vacuum solutions. This will be useful in order to gain familiarity with this kind of solutions which will be generalized in the next chapter when there is gravitational interaction with a matter field and in presence of a cosmological constant.

Let us consider this family of spacetimes but with an additional Killing vector ∂_z, so that components of the metric depends on ρ only. With the periodicity of ϕ we can interpret this metric as a stationary spacetime provided with cylindrical symmetry. It is convenient to perform the substitution $\gamma = \mu + \frac{1}{2}\log f$. Then,

$$ds^2 = -f(dt + Ad\phi)^2 + e^{2\mu}(d\rho^2 + dz^2) + \rho^2 f^{-1} d\phi^2 \ . \tag{7.1}$$

This substitution makes easier analytical extension of coordinates in regions where f becomes negative.

7.1. The Lewis family of vacuum solutions

Lanczos [101] and Lewis [102] found independently the general family of exact stationary cylindrically symmetric vacuum solutions using the WLP form of the metric. It turns out to be,

$$\begin{aligned} f &= \rho\left(a\rho^{-n} - \frac{c^2}{n^2 a}\rho^n\right) \\ e^{2\mu} &= k^2 \rho^{(n^2-1)/2} \\ A &= \frac{c}{na}\rho^{n+1} f^{-1} + b \end{aligned} \tag{7.2}$$

where n, a, b, c and k are five constant parameters and one expects to gauge away some of them; for example, fixing $k = 1$ by using a coordinate rescaling. This line element is known as the *Lewis metric.* There are two classes of spacetimes depending on whether n is real or imaginary. In the case n real the other parameters must be real. However, when n is imaginary, the remaining parameters have to be chosen so that the metric is real.

Forcing to have a regular axis at $\rho = 0$ implies only the flat case $n = 1$ as possible spacetime as was proved by Davis and Caplan [109]. In some cases, it could be considered to exclude this infinite axis from the spacetime assuming that we are describing the gravitational interaction with a source located there. The general interpretation of these parameters and the constraint relating them is still not clear at all.

However in the next chapter we will find a natural extension of this solution, but including a minimally coupled scalar field as a matter source. To find the solution it is convenient to use other coordinate system where the radial coordinate is the radial proper distance ($g_{\rho\rho} = 1$), and as expected many of the integration constants can be gauged away by using a proper interpretation of the global and local properties as it will become clear. Additionally we will compute the conserved charges in order to have a deeper insight about this interpretation by relating them with these quantities.

7.1.1. The Weyl class of Lewis metrics

In the case of n real, we say that Lewis metric belongs to the Weyl class. In this case a, b, c and k must be real. It is possible to diagonalize the metric performing a coordinate transformation. Namely, when $a > 0$,

$$t = \frac{1}{\sqrt{a}}\left(1 - \frac{bc}{n}\right)\tau - b\sqrt{a}\tilde{\phi} \ , \qquad \phi = \frac{c}{n\sqrt{a}}\tau + \sqrt{a}\tilde{\phi} \ , \tag{7.3}$$

put the Lewis metric in the form,

$$ds^2 = -\rho^{1-n}d\tau^2 + k^2\rho^{(n^2-1)/2}(d\rho^2 + dz^2) + \rho^{n+1}d\tilde{\phi}^2 \ , \tag{7.4}$$

which is nothing else than the static Levi-Civita solution with $n = 1-4\sigma$ considering $a = 1$, $b = 0$ and $c = 0$. In the Newtonian limit σ can be identified as the mass per unit length of an infinite line source located on the axis. If $a < 0$, the transformation,

$$t = b\sqrt{-a}\tau + \frac{1}{\sqrt{-a}}\left(1 - \frac{bc}{n}\right)\tilde{\phi} \qquad \phi = -\sqrt{-a}\tau + \frac{c}{n\sqrt{-a}}\tilde{\phi} \tag{7.5}$$

takes the Lewis metric to the form,

$$ds^2 = -\rho^{1+n}d\tau^2 + k^2\rho^{(n^2-1)/2}(d\rho^2 + dz^2) + \rho^{1-n}d\tilde{\phi}^2 \tag{7.6}$$

which in any case is the known Levi-Civita solution with $n = 4\sigma - 1$. Therefore, we can conclude that this family of spacetime is locally isomorphic to the static Levi-Civita solutions. Levi-Civita solution corresponds to a static and cyllindrically symmetric form of the Weyl metric. Consequently, (7.4) and (7.6) are referred as the locally static Weyl class of Lewis solution. However, since ϕ is a periodic coordinate, the time coordinate τ is periodic unless $b = 0$. When $b \neq 0$ spacetime contains closed timelike curves, this implies that these spacetimes are topologically different from the Levi-Civita solution [103].

It is worth to mention that f function possesses a zero at $\rho_1 = |\frac{na}{c}|^{1/n}$ when $na > 0$. The opposite occurs for $na < 0$. When $f < 0$ the time coordinate b becomes spacelike. However this is not a Killing horizon since the metric can always be transformed to Levi-Civita form for al values of ρ, thus it does not have a local significance. It comes from the rotational part of the coordinates resembling much the same the properties of an ergosphere like in Kerr spacetime.

Additionally the angular metric function also possesses a zero. This means that there is a region where ϕ is a timelike coordinate. In fact, closed timelike curves occur asymptotically when $na < 0$ or near the axis when $na > 0$.

Excluding the flat case, these solutions contain a curvature singularity at $\rho = 0$. These spacetimes are appropriate as exterior matched solutions to infinite cylindrical sources. Like the Levi-Civita solution, they are asymptotically locally flat and not asymptotic to a flat cylindrically symmetric spacetime if $n \neq 1$ with $a > 0$.

7.1.2. The Lewis class of Lewis metrics

When the parameter n is purely imaginary, the functions (7.2) represent a different class of spacetimes. The reality condition for the metric requires a and b have to be complex. Then, these parameters can be expressed in terms of the real parameter $\tilde{n}$, a_1, b_1, a_2 and b_2 where $a_1 b_2 - a_2 b_1 = 1$, and

$$\begin{aligned} n &= \mathrm{i}\tilde{\mathrm{n}} \ , & a &= \frac{1}{2}(a_1^2 - b_1^2) + \mathrm{i a_1 b_1} \ , \\ b &= \frac{a_1 a_2 + b_1 b_2 + \mathrm{i}}{a_1^2 + b_1^2} \ , & c &= \frac{1}{2}\tilde{n}(a_1^2 + b_1^2) \ , \end{aligned}$$

this allows us to rewrite the metric functions (7.2) as,

$$\begin{aligned} f &= \rho[(a_1^2 - b_1^2)\cos(\tilde{n}\log\rho) + 2a_1 b_1 \sin(\tilde{n}\log\rho)] \ , \\ e^{2\mu} &= k^2 \rho^{-(\tilde{n}^2+1)/2} \ , \\ A &= \rho f^{-1}[(a_1 a_2 - b_1 b_2)\cos(\tilde{n}\log\rho) + (a_1 b_2 + a_2 b_1)\sin(\tilde{n}\log\rho)] \ . \end{aligned} \tag{7.7}$$

In [104] it was shown that these spacetimes which belong to Lewis class are not locally isomorphic to the Levi-Civita solutions. There is no a suitable coordinate transformation that diagonalize the metric and even more, they do not contain any locally flat spacetime. They are asymptotically locally when $\tilde{n}^2 < 3$ and in the opposite case a curvature singularity is located at $\rho = \infty$ but at a finite proper radial distance from the axis.

7.2. Static, cylindrically symmetric strings with cosmological constant

In this section we present the generic four dimensional static solution with cylindrical symmetry and cosmological constant in General Relativity found by Linet in [97]. The action for this system gives us the field equations

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \ . \tag{7.8}$$

To find static, cylindrically symmetric strings with cosmological constant it is simpler to use the following ansatz for the metric

$$ds^2 = -g_0(r)dt^2 + g_1(r)l^2d\phi^2 + g_2(r)dz^2 + dr^2. \tag{7.9}$$

where the coordinates system (t, r, z, ϕ) is considered with proper radial distance as coordinate r with $r \geqslant 0$ and compact coordinate ϕ with $0 \leqslant \phi < 2\pi$. Then, hypersurfaces $\phi = 0$ and $\phi = 2\pi$ are identified. The square root of the determinant of metric (7.9) is conveniently defined as

$$u = \sqrt{g_0 g_1 g_2} \ . \tag{7.10}$$

This is useful since allows us to express the field equations in a very compact and simple form

$$\left(\left(\frac{u}{g_i}\right) g_i'\right)' = 0, \quad i = 0, 1, 2, \tag{7.11}$$

$$\frac{g_0' g_1'}{g_0 g_1} + \frac{g_1' g_2'}{g_1 g_2} + \frac{g_2' g_3'}{g_2 g_3} + 4\Lambda = 0 \tag{7.12}$$

where a prime means differentiation wit respect to r. Using (7.11) it is obtained an equation for u

$$u'' + 3\Lambda u = 0 \ , \tag{7.13}$$

which can be recast as follows

$$u'^2 = -3\Lambda u^2 + K^2 \ , \tag{7.14}$$

with K a positive integration constant. Plugging this into (7.12) we get

$$\frac{g_i'}{g_i} = \frac{KK_i}{u} + \frac{2u'}{3u} \ , \quad i = 0, 1, 2 \ , \tag{7.15}$$

where K_i are three integration constants. They are algebraically related in order to fulfill the field equations

$$K_0 + K_1 + K_2 = 0 \ , \tag{7.16}$$

$$K_0K_1 + K_1K_2 + K_2K_0 = -\frac{4}{3} \ , \tag{7.17}$$

which tell us that there is only one independent parameter. Solving equation (7.14) and fixing the axis at $r = 0$ we obtain

$$u(r) = \frac{K}{\sqrt{3\Lambda}} \sin\left[\sqrt{3\Lambda} r\right] \ , \qquad \Lambda > 0 \ , \tag{7.18}$$

$$u(r) = \frac{K}{\sqrt{-3\Lambda}} \sinh\left[\sqrt{-3\Lambda} r\right] \ , \quad \Lambda < 0 \ . \tag{7.19}$$

From this and using equation (7.15) the solution for the metric functions g_i is

$$g_i(r) = g_i^0 \left\{ \tan\left[\frac{\sqrt{3\Lambda}}{2} r\right]\right\}^{K_i} \sin^{2/3}[\sqrt{3\Lambda} r] \ , \qquad \Lambda > 0 \ , \tag{7.20}$$

$$g_i(r) = g_i^0 \left\{ \tanh\left[\frac{\sqrt{-3\Lambda}}{2} r\right]\right\}^{K_i} \sinh^{2/3}[\sqrt{-3\Lambda} r] \ , \quad \Lambda < 0 \ , \tag{7.21}$$

where g_i^0 are three integration constant that satisfy $g_0^0 g_1^0 g_2^0 = K^2/|3\Lambda|$ for consistency with relation (7.10). It is clear that integration constant g_0^0 and g_2^0 can always be gauged away by a coordinate rescaling. This is not the case for g_1^0 as it is related with an angular defect since ϕ is a compact coordinate with period 2π. In fact, as was pointed out in [97] the geometry near the axis corresponds to a cosmic string. When $K_0 = K_2 = -2/3$ and $K_1 = 4/3$ and

$$g_0^0 = g_2^0 = 2^{-2/3} \ , \qquad g_1^0 = 2^{4/3} K^2/|3\Lambda| \ , \tag{7.22}$$

the geometry near the origin $r = 0$ is

$$ds^2 = -dt^2 + K^2 d\phi^2 + dz^2 + dr^2. \tag{7.23}$$

This near-origin form induces a stress-energy momentum

$$T_t^t = T_z^z = \frac{1-K}{4G} \frac{\delta(r)}{\sqrt{-\gamma}} \ , \qquad T_r^r = T_\phi^\phi = 0 \ , \tag{7.24}$$

where γ is the induced metric on the 2-hypersurface $t = const$ and $z = const$. This stress-energy tensor is characteristic of a static, cylindrically symmetric string of mass per unit length M given by

$$M = \frac{1-K}{4G} \ , \tag{7.25}$$

where $K \neq 1$. We see clearly that the integration constant g_1^0 confers the mass to the cosmic string. Imposing regularity on the origin implies $K = 1$ and flat spacetime geometry at the origin.

7.3. The black string

A solution of special interest was found in [28]. This is a four dimensional black hole solution of Einstein's field equations with cylindrical symmetry in the presence of a cosmological constant. Due to its geometry is commonly referred in the literature as "black string". In a coordinate system (t, r, ϕ, z) this solution is

$$ds^2 = -\left(\alpha^2 r^2 - \frac{b}{\alpha r}\right) dt^2 + \frac{dr^2}{\alpha^2 r^2 - \frac{b}{\alpha r}} + r^2 d\phi^2 + \alpha^2 r^2 dz^2 \ , \tag{7.26}$$

with $-\infty < t < \infty$, $0 \leqslant r < \infty$, $0 \leqslant \phi < 2\pi$, $-\infty < z < \infty$. Here r is the radial circumferencial coordinate, $\alpha^2 \equiv -\Lambda/3 > 0$ and b is an integration constant directly related with the mass per unit length of the black string. The Kretschmann scalar given by $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 24\alpha^4(1 + \frac{b^2}{\alpha^6 r^6})$ determines a curvature singularity at the origin $r = 0$ which is dressed by the event horizon of the black string located at $r_H = b^{1/3}/\alpha$. The spacetime can be provided of angular momentum by performing coordinate transformations on the $t - \phi$ plane

$$t = \lambda\tau - \frac{\omega}{\alpha^2}\theta \ , \qquad\qquad \phi = \lambda\theta - \omega\tau \ , \tag{7.27}$$

where ω and λ are constant parameters of the transformations. Once we apply them on (7.26) we obtain

$$\begin{aligned} ds^2 = &- \left[\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right) \alpha^2 r^2 - \frac{b\lambda^2}{\alpha r} \right] d\tau^2 + \frac{dr^2}{\alpha^2 r^2 - \frac{b}{\alpha r}} - \frac{\omega b}{\alpha^3 r} 2 d\theta d\tau \\ &+ \left[\left(\lambda^2 - \frac{\omega^2}{\alpha^2}\right) r^2 + \frac{\omega^2 b}{\alpha^5 r} \right] d\theta^2 + \alpha^2 r^2 dz^2 \ , \end{aligned} \tag{7.28}$$

with $-\infty < \tau < \infty$, $0 \leqslant r < \infty$, $0 \leqslant \theta < 2\pi$, $-\infty < z < \infty$. The transformation (7.27) is not a proper coordinate transformation, since it changes the topology of the original spacetime. In other words it converts an exact 1-form into a closed but not exact 1-form [110]. Therefore, although solutions (7.26) and (7.28) are locally equivalent, they are globally different. This is the reason why (7.28) is a new metric, being the stationary version of the original static one.

As we mention in Section 7.2, Linet found the generic cylindrically symmetric solutions in the presence of cosmological constant for General Relativity. Unfortunately, the coordinate system he employed makes quite difficult to identify the black string solution. However, there is an essential clue mentioned by Linet that makes possible to obtain the black string solution presented here from Linet's solution presented in the previous section. In his work he mention that the Weyl tensor is regular when K_i integration constants from (7.16) takes any circular permutation of values $-\frac{2}{3}$, $-\frac{2}{3}$ and $\frac{4}{3}$. The fact is that all the scalar invariants are regular with this condition, which means that there is no curvature singularity in the range of coordinate considered. Thus, the coordinate r from (7.9) can be analytically extended and by performing a suitable radial coordinate transformation it possible to obtain the black string solution (7.26) starting from metric (7.9) with (7.21). We will show this explicitly (see relation (8.87)) in Section 8.3 of the next chapter, since some comments about the role of parameter b are needed previously. There we show that the origin in the Linet side is mapped to the event horizon. Therefore, the cosmic string described by (7.23) is nothing but the regularized near horizon geometry of the black string up to a double Wick rotation.

Chapter 8

Stationary cylindrically symmetric spacetimes with a massless scalar field and a nonpositive cosmological constant

Cylindrically symmetric spacetimes are widely known in vacuum, however exact solutions containing a massless scalar field as matter source in presence of a cosmological constant have received almost null attention until now. Previously, solutions with plane symmetry, which are a particular case of the cylindrical ones, have been reported [24, 25] and other particular solutions in [111, 112][1]. The main efforts on this subject can be found in [113, 114] and [115] for the static and stationary cases, respectively. In these articles the existence of soliton and wormhole solutions in the presence of an arbitrary self-interaction potential for the scalar field were analyzed, providing also a useful method for obtaining general cylindrically symmetric solutions.

In this chapter, the general stationary cylindrically symmetric solution of Einstein-massless scalar field system with a nonpositive cosmological constant Λ is found, and its geometrical properties are studied. Our aim is to determine the implications of a massless scalar field in a cylindrically symmetric system. Due to the high interest in exact solutions whose asymptotic behavior approaches the anti-de Sitter spacetime, we include in the analysis a negative cosmological constant. In fact, the solutions presented here, for $\Lambda < 0$, have that asymptotic behavior. Moreover, we study the effect of a massless scalar field in the case of a vanishing cosmological constant,

[1]Unfortunately, along these two articles there are inconsistencies in the signs of the cosmological constant and the kinetic term of the scalar field. Additionally, the solution provided there is not general.

i.e., we explore the backreaction generated by the scalar field in the well-known Lanczos-Lewis and Levi-Civita spacetimes.

As is expected, in the absence of suitable potentials and non-minimal couplings for the scalar field, the no-hair theorem rules out solutions having event horizons, and this is precisely our case. We are just considering a massless scalar field with a constant potential (zero or negative). Thus, in general, the solutions presented here contain naked singularities, which however could have some physical interest [116].

We find that the general stationary cylindrically symmetric solution contains two different classes. In the first of them, the stationary spacetimes become static by adjusting smoothly the integration constants related with the rotation. For the second class this process is not possible, and in consequence, such a class of solutions does not have a static limit. In this sense, this class has an unclear physical relevance. The first section of this chapter is devoted to present all the solutions, however, along the subsequent sections we will focus our analysis on the first class of solutions.

The chapter is organized as follows. Section 8.1 presents the action and offers a very detailed derivation of the general solution. The key point is to reduce the field equations to a very simple uncoupled system of differential equations, which allows us to find (i) all the solutions and (ii) figure out how they split in two classes: the one containing the static solution, and the other one lacking a static limit. In Section 8.2, the ansatz is established and the general solution with static limit is presented as a linear combination of three functions, according to the cosmological constant. In Section 8.3, the local properties of the solutions are studied using the Newman-Penrose (NP) formalism, where the Weyl-NP scalars allow to obtain the Petrov classification of these spacetimes. It is shown that a parameter included through the scalar field enlarges the family of spacetimes with respect to the vacuum ones. Afterwards, following [108] and inspired in [28], the stationary spacetime is obtained from the static one by means of a topological construction. These formalisms allow us to identify the four essential parameters of the general solution. One of them is the amplitude of the scalar field, which in conjunction with a second one describe the strength of the gravitational field. The remaining parameters have a topological origin and are just globally defined, because they cannot be removed by a proper coordinate transformation. Moreover, the mass and angular momentum are computed by using the Regge-Teitelboim method [117]. These conserved charges illustrate the physical meaning of the essential parameters. The case of a vanishing cosmological constant is considered in Section 8.4. We note that it is necessary to integrate the field equations from scratch, because a special class of solutions is not available by just taking the limit $\Lambda \to 0$ in the solutions presented in Section 8.2. We found that these spacetimes have all their scalar invariants constant, and are supported by a phantom scalar field. The last section contains some concluding remarks.

8.1. Solving the field equations

In this section we present a complete and detailed derivation of the general solution for a massless stationary cylindrically symmetric scalar field in presence of a non-positive cosmological constant in four spacetime dimensions. The methodology used in this derivation is based on that proposed in [115][2].

We consider the Einstein-Hilbert action with a massless scalar field and a cosmological constant Λ,

$$I = \int d^4x\sqrt{-g}\left[\frac{R-2\Lambda}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\right], \tag{8.1}$$

where $\kappa = 8\pi G$ is the gravitational constant. The stress-energy tensor turns out to be

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi, \tag{8.2}$$

and the field equations are given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad \Box\Phi = 0. \tag{8.3}$$

Let us consider the general stationary cylindrically symmetric metric

$$ds^2 = e^{2\alpha}dr^2 + e^{2\mu}dz^2 + e^{2\beta}d\phi^2 - e^{2\gamma}(dt - Se^{-2\gamma}\,d\phi)^2, \tag{8.4}$$

where $\alpha, \beta, \gamma, \mu, S$ are functions of the radial coordinate r. The nonvanishing components of the Ricci tensor are

$$R^r{}_r = \bar{R}^r{}_r + 2\omega^2, \tag{8.5}$$

$$R^z{}_z = \bar{R}^z{}_z, \tag{8.6}$$

$$R^\phi{}_\phi = \bar{R}^\phi{}_\phi + 2\omega^2 + WSe^{-2\gamma}, \tag{8.7}$$

$$R^t{}_t = \bar{R}^t{}_t - 2\omega^2 - WSe^{-2\gamma}, \tag{8.8}$$

$$R^\phi{}_t = -W, \tag{8.9}$$

$$R^t{}_\phi = e^{-2\gamma}\left[S(\bar{R}^\phi{}_\phi - \bar{R}^t{}_t + 4\omega^2) + W(e^{2\beta} + S^2e^{-2\gamma})\right]. \tag{8.10}$$

The auxiliary functions ω and W appearing above are defined as

$$\omega \equiv \frac{1}{2}e^{\gamma-\beta-\alpha}(Se^{-2\gamma})', \tag{8.11}$$

$$W \equiv e^{-\alpha-\beta-\gamma-\mu}(\omega e^{2\gamma+\mu})', \tag{8.12}$$

[2] Our results contain the static solution with $\Lambda < 0$ in [114], and the stationary solution with $\Lambda = 0$ in [115].

and the barred symbols denote the Ricci tensor components for the static metric obtained from (8.4) by setting $S = 0$, which are

$$\begin{aligned}
-e^{2\alpha}\bar{R}^r{}_r &= \beta'' + \gamma'' + \mu'' + \beta'^2 + \gamma'^2 + \mu'^2 - \alpha'(\beta' + \gamma' + \mu'), && (8.13)\\
-e^{2\alpha}\bar{R}^z{}_z &= \mu'' + \mu'(\beta' + \gamma' + \mu' - \alpha'), && (8.14)\\
-e^{2\alpha}\bar{R}^\phi{}_\phi &= \beta'' + \beta'(\beta' + \gamma' + \mu' - \alpha'), && (8.15)\\
-e^{2\alpha}\bar{R}^t{}_t &= \gamma'' + \gamma'(\beta' + \gamma' + \mu' - \alpha'). && (8.16)
\end{aligned}$$

The field equations can be expressed as

$$R^\mu{}_\nu = \Lambda\delta^\mu{}_\nu + \kappa\partial^\mu\Phi\partial_\nu\Phi \equiv \tau^\mu{}_\nu, \tag{8.17}$$

where $\tau^\mu{}_\nu$ is reduced to $\mathrm{diag}(\Lambda + \kappa e^{-2\alpha}\Phi'^2, \Lambda, \Lambda, \Lambda)$ for the metric (8.4) and a scalar field depending only on r.

Since $\tau^\phi{}_t = 0$, Eq. (8.9) implies $W = 0$. Then, from (8.12) we obtain

$$\omega = \omega_0 e^{-2\gamma-\mu}, \tag{8.18}$$

where ω_0 is an integration constant.

The definition (8.11) yields $(Se^{-2\gamma})' = 2\omega_0 e^{\beta+\alpha-3\gamma-\mu}$, so that

$$S = e^{2\gamma}\left(S_0 + 2\omega_0\int e^{\alpha+\beta-3\gamma-\mu}\,dr\right), \tag{8.19}$$

where S_0 is a second integration constant (hereafter, all the quantities with subscripts 0, 1 or 2 denote integration constants). Then, the the field equations are reduced to

$$\begin{aligned}
\bar{R}^r{}_r &= \Lambda + \kappa e^{-2\alpha}\Phi'^2 - 2\omega^2, && (8.20)\\
\bar{R}^z{}_z &= \Lambda, && (8.21)\\
\bar{R}^\phi{}_\phi &= \Lambda - 2\omega^2, && (8.22)\\
\bar{R}^t{}_t &= \Lambda + 2\omega^2. && (8.23)
\end{aligned}$$

The components ${}^\phi{}_t$ and ${}^t{}_\phi$ of the field equations are satisfied by virtue of $W = 0$ and Eqs. (8.22, 8.23). The equation for the scalar field,

$$\Phi'' + (\beta' + \gamma' + \mu' - \alpha')\Phi' = 0, \tag{8.24}$$

admits a first integral given by

$$\Phi' = \frac{P_0}{\sqrt{2\kappa}}e^{\alpha-\beta-\gamma-\mu}. \tag{8.25}$$

We choose the gauge $\alpha = 0$. Introducing the functions U, V, σ as follows,

$$\mu = \log U - \sigma, \quad \beta = \frac{1}{2}(\sigma - \log V), \quad \gamma = \frac{1}{2}(\sigma + \log V), \tag{8.26}$$

an equivalent system of equations is obtained,

$$\bar{R}^z{}_z + \bar{R}^\phi{}_\phi + \bar{R}^t{}_t = -\frac{U''}{U} = 3\Lambda, \tag{8.27}$$

$$\bar{R}^\phi{}_\phi + \bar{R}^t{}_t = -\sigma'' - \frac{U'}{U}\sigma' = 2\Lambda, \tag{8.28}$$

$$\bar{R}^\phi{}_\phi - \bar{R}^t{}_t = \frac{V''}{V} - \frac{V'^2}{V^2} + \frac{U'V'}{UV} = -\frac{4\omega_0^2}{U^2V^2}, \tag{8.29}$$

$$\bar{R}^r{}_r = -\frac{3}{2}\sigma'^2 - \frac{U''}{U} + \frac{2\sigma'U'}{U} - \frac{V'^2}{2V^2} = \Lambda + \frac{P_0^2}{2U^2} - \frac{2\omega_0^2}{U^2V^2}. \tag{8.30}$$

In terms of these functions, the scalar field and the metric reads

$$\Phi = \Phi_0 + \frac{P_0}{\sqrt{2\kappa}}\int \frac{dr}{U}, \tag{8.31}$$

and

$$ds^2 = dr^2 + U^2 e^{-2\sigma} dz^2 + \frac{e^\sigma}{V} d\phi^2 - V e^\sigma (dt - \frac{S}{Ve^\sigma} d\phi)^2, \tag{8.32}$$

respectively, with

$$S = Ve^\sigma \left(S_0 + 2\omega_0 \int \frac{dr}{UV^2} \right). \tag{8.33}$$

Note that $U^2V'V \times$ Eq. (8.29) $= -(U^2 \times \text{Eq. } (8.30))'$, so that in the general case $V' \neq 0$ it is enough to consider just Eq. (8.30) because it implies (8.29). In the special case $V' = 0$, Eq. (8.29) yields $\omega_0 = 0$ and (8.30) becomes

$$-\frac{3}{2}\sigma'^2 + \frac{2\sigma'U'}{U} = -2\Lambda + \frac{P_0^2}{2U^2}. \tag{8.34}$$

Equations (8.28) and (8.27) yield

$$\sigma = \begin{cases} \sigma_0 + \sigma_1 \displaystyle\int \frac{dr}{U} & : \Lambda = 0, \\ \sigma_0 + \log U^{2/3} + \sigma_1 \displaystyle\int \frac{dr}{U} & : \Lambda \neq 0. \end{cases} \tag{8.35}$$

Replacing (8.35) and (8.27) in (8.30) we get

$$U^2V'^2 - aV^2 - 4\omega_0^2 = 0, \tag{8.36}$$

where $a = 4\sigma_1 U' - 3\sigma_1^2 - P_0^2$ for $\Lambda = 0$, and $a = 4U'^2/3 + 4\Lambda U^2 - P_0^2 - 3\sigma_1^2$ otherwise. Equation(8.27) implies that U' and $4U'^2/3 + 4\Lambda U^2$ are constants for $\Lambda = 0$ and $\Lambda \neq 0$, respectively. Therefore, in both cases a is a constant.

The change of variable

$$x = \int \frac{dr}{U}, \tag{8.37}$$

transforms Eq. (8.36) into

$$\left(\frac{dV}{dx}\right)^2 - aV^2 - 4\omega_0^2 = 0, \tag{8.38}$$

which can be integrated by quadrature yielding

$$V = \begin{cases} e^{\sqrt{a}(x-x_0)} - \dfrac{\omega_0^2}{a} e^{-\sqrt{a}(x-x_0)}, & \text{if} \quad a > 0, \qquad (8.39\text{a}) \\ 2\omega_0\, x + V_0, & \text{if} \quad a = 0, \qquad (8.39\text{b}) \\ \dfrac{2\omega_0}{\sqrt{-a}} \sin\left[\sqrt{-a}(x - x_0)\right], & \text{if} \quad a < 0. \qquad (8.39\text{c}) \end{cases}$$

The integral appearing in Eq. (8.33) is equivalent to $\int dx V^{-2}$. Then, from (8.39) we obtain

$$\int \frac{dx}{V^2} = \begin{cases} -\dfrac{\sqrt{a}}{2} \dfrac{e^{-\sqrt{a}(x-x_0)}}{a e^{\sqrt{a}(x-x_0)} - \omega_0^2 e^{-\sqrt{a}(x-x_0)}} & : a > 0, \\ -\dfrac{1}{2\omega_0(2\omega_0\, x + V_0)} & : a = 0,\ \omega_0 \neq 0, \\ \dfrac{x}{V_0^2} & : a = 0,\ \omega_0 = 0, \\ -\dfrac{\sqrt{-a}}{4\omega_0^2 \tan\left[\sqrt{-a}(x - x_0)\right]} & : a < 0. \end{cases} \tag{8.40}$$

An important consequence can be derived from Eq. (8.36) (or equivalently from (8.38)). For $a < 0$ there are no real nonvanishing solutions for this equation if $\omega_0 = 0$. This means that all the real solutions are stationary, but they do not contain a static limit. As opposite, the solutions in the case $a > 0$ can be reduced to static ones. The case $a = 0$ has two different branches. The first one, defined by the conditions $V' \neq 0, \omega_0 \neq 0$, provides stationary solutions that fail in containing a static limit. The second branch, $V' = 0, \omega_0 = 0$, corresponds to the special case mentioned before. In fact, Eq. (8.34) implies $a = 0$ regardless the value of the cosmological constant. This special branch contains solutions with static limit.

In what follows, we explicitly show all the possible solutions, which will classify according the value of a.

8.1.1. General solution with $\Lambda = 0$

From (8.27) we obtain

$$U = u_0 + u_1 r \ , \tag{8.41}$$

and from (8.35),

$$\sigma = \begin{cases} \sigma_0 + \dfrac{\sigma_1}{u_1}\log(u_0 + u_1 r) & \text{if} \quad u_1 \neq 0, & (8.42a) \\ \sigma_0 + \dfrac{\sigma_1}{u_0} r & \text{if} \quad u_1 = 0. & (8.42b) \end{cases}$$

Moreover, from (8.37) we get

$$x = \begin{cases} \dfrac{\log(u_0 + u_1 r)}{u_1} & \text{if} \quad u_1 \neq 0, & (8.43a) \\ \dfrac{r}{u_0} & \text{if} \quad u_1 = 0. & (8.43b) \end{cases}$$

and the constant a is given by

$$a = 4\sigma_1 u_1 - 3\sigma_1^2 - P_0^2. \tag{8.44}$$

8.1.1.1. Type A solutions: $a > 0$

A necessary condition for $a > 0$ is $\sigma_1 u_1 > 0$. Then, from (8.39a) and (8.43a) we obtain

$$V = V_0 (u_0 + u_1 r)^{\frac{\sqrt{a}}{u_1}} - \frac{\omega_0^2}{a V_0} (u_0 + u_1 r)^{-\frac{\sqrt{a}}{u_1}}, \tag{8.45}$$

where the constant V_0 is a redefinition of $e^{-\sqrt{a} x_0}$.

After some algebraic manipulations we can express the general solution in a manner that will be presented in Section 8.4.1. The functions g_0, g_1 and g_2 are given by

$$g_i = (r + \bar{u}_0)^{K_i + \frac{2}{3}} \ , \tag{8.46}$$

where

$$K_0 = \frac{\sigma_1 + \sqrt{a}}{u_1} - \frac{2}{3}, \ K_1 = \frac{\sigma_1 - \sqrt{a}}{u_1} - \frac{2}{3}, \ K_2 = \frac{4}{3} - \frac{2\sigma_1}{u_1}, \tag{8.47}$$

and $\bar{u}_0 = u_0/u_1$. The constants a_0, a_1, b_0, b_1, c_0 and α are given by

$$a_0 = e^{\sigma_0} V_0 u_1^{\frac{\sigma_1 + \sqrt{a}}{u_1}}, \quad b_0 = S_0^2 a_0 \tag{8.48}$$

$$a_1 = \frac{e^{\sigma_0} \omega_0^2}{a V_0} u_1^{\frac{\sigma_1 - \sqrt{a}}{u_1}}, \quad b_1 = \frac{e^{\sigma_0}}{a V_0} \left(1 + \frac{\omega_0 S_0}{\sqrt{a}}\right) u_1^{\frac{\sigma_1 - \sqrt{a}}{u_1}} \tag{8.49}$$

$$c_0 = e^{-2\sigma_0} u_1^{2 - \frac{2\sigma_1}{u_1}}, \quad \alpha = \frac{P_0^2}{u_1^2}. \tag{8.50}$$

It is possible to map the condition $a > 0$ to an equivalent one in terms of K_2 and α,

$$(K_2)^2 < \frac{4}{3}\left(\frac{4}{3} - \alpha\right). \tag{8.51}$$

8.1.1.2. Type B solutions: $a = 0$

As before, this case requires $\sigma_1 u_1 > 0$. From (8.39b) and (8.43a) we get

$$V = V_0 + \frac{2\omega_0}{u_1}\log(u_0 + u_1 r). \tag{8.52}$$

In the general case $\omega_0 \neq 0$, the function V is not a constant and the metric has no a static limit. The special case $V = V_0$ appears provided $\omega_0 = 0$ and corresponding metric can be obtained from an improper gauge transformation in the $t - \phi$ plane. In this case, the functions g_0, g_1 and g_2 are given by (8.46), where

$$K_0 = K_1 = \frac{\sigma_1}{u_1} - \frac{2}{3}, \quad K_2 = \frac{4}{3} - \frac{2\sigma_1}{u_1}. \tag{8.53}$$

The constants a_0, a_1, b_0, b_1, c_0 and α are given by

$$a_0 = e^{\sigma_0} V_0 u_1^{\frac{\sigma_1}{u_1}}, \quad b_0 = S_0^2 a_0 \tag{8.54}$$

$$a_1 = 0, \quad b_1 = \frac{e^{\sigma_0}}{V_0} u_1^{\frac{\sigma_1}{u_1}} \tag{8.55}$$

$$c_0 = e^{-2\sigma_0} u_1^{2-\frac{2\sigma_1}{u_1}}, \quad \alpha = \frac{P_0^2}{u_1^2}. \tag{8.56}$$

In terms of K_2 and α, the condition $a = 0$ becomes

$$(K_2)^2 = \frac{4}{3}\left(\frac{4}{3} - \alpha\right). \tag{8.57}$$

8.1.1.3. Type C solutions: $a < 0$

From (8.39c) and (8.43a)-(8.43b), we get

$$V = \begin{cases} \dfrac{2\omega_0}{\sqrt{-a}} \sin\left[\sqrt{-a}\left(\dfrac{\log(u_0 + u_1 r)}{u_1} - x_0\right)\right] & : u_1 \neq 0 \\[2ex] \dfrac{2\omega_0}{\sqrt{-a}} \sin\left[\sqrt{-a}\left(\dfrac{r}{u_0} - x_0\right)\right] & : u_1 = 0 \end{cases} \tag{8.58}$$

Since that V has no a definitive sign, the norm of the Killing vectors ∂_t and ∂_ϕ does not maintain a fixed sign. This type of solutions has no a static limit.

8.1.2. General solution with $\Lambda = -3l^{-2} < 0$

Let us consider a negative cosmological constant $\Lambda = -3l^{-2}$. The Eq. (8.27) is easily solved. It gives

$$U = c_1 e^{3r/l} - c_2 e^{-3r/l}, \tag{8.59}$$

and Eq. (8.35) provides σ as

$$\sigma = \sigma_0 + \log U^{2/3} + \sigma_1 \int \frac{dr}{U}, \tag{8.60}$$

where

$$x = \int \frac{dr}{U} = \begin{cases} \dfrac{l}{6\sqrt{c_1 c_2}} \log\left(\dfrac{e^{3r/l} - \sqrt{c_2/c_1}}{e^{3r/l} + \sqrt{c_2/c_1}}\right) & : c_1 c_2 > 0, \\ -\dfrac{l e^{-3r/l}}{3c_1} & : c_2 = 0, \\ -\dfrac{l e^{3r/l}}{3c_2} & : c_1 = 0, \\ -\dfrac{l}{3\sqrt{-c_1 c_2}} \arctan\left(\sqrt{-c_1/c_2}\, e^{3r/l}\right) & : c_1 c_2 < 0. \end{cases} \tag{8.61}$$

In this case, the constant a becomes

$$a = 48 c_1 c_2 l^{-2} - 3\sigma_1^2 - P_0^2. \tag{8.62}$$

8.1.2.1. Type A solutions: $a > 0$

The case $a > 0$ requires the necessary condition $c_1 c_2 > 0$. From (8.39a) and the first line in (8.61)

$$V = V_0 \left(\frac{e^{3r/l} - \sqrt{c_2/c_1}}{e^{3r/l} + \sqrt{c_2/c_1}}\right)^{\frac{l\sqrt{a}}{6\sqrt{c_1 c_2}}} - \frac{\omega_0^2}{a V_0} \left(\frac{e^{3r/l} - \sqrt{c_2/c_1}}{e^{3r/l} + \sqrt{c_2/c_1}}\right)^{-\frac{l\sqrt{a}}{6\sqrt{c_1 c_2}}}. \tag{8.63}$$

In the same way as for the case $\Lambda = 0$, algebraic manipulations allow us to express the general solution in a convenient manner that will be used in the next section as a linear combination of functions g_0, g_1 and g_2 given by

$$g_i = \left(e^{3r/l} - b^2 e^{-3r/l}\right)^{2/3} \left(\frac{e^{3r/l} - b}{e^{3r/l} + b}\right)^{K_i}, \tag{8.64}$$

where

$$K_0 = \frac{(\sigma_1 + \sqrt{a})l}{6\sqrt{c_1 c_2}},\ K_1 = \frac{(\sigma_1 - \sqrt{a})l}{6\sqrt{c_1 c_2}},\ K_2 = \frac{-2\sigma_1 l}{6\sqrt{c_1 c_2}}, \tag{8.65}$$

and $b = \sqrt{c_2/c_1}$. The constants a_0, a_1, b_0, b_1, c_0 and α are given by

$$a_0 = e^{\sigma_0} V_0 c_1^{2/3}, \quad b_0 = S_0^2 a_0, \tag{8.66}$$

$$a_1 = \frac{e^{\sigma_0}\omega_0^2}{aV_0} c_1^{2/3}, \quad b_1 = \frac{e^{\sigma_0}}{aV_0}\left(1 + \frac{\omega_0 S_0}{\sqrt{a}}\right) c_1^{2/3}, \tag{8.67}$$

$$c_0 = e^{-2\sigma_0} c_1^{2/3}, \quad \alpha = \frac{P_0^2 l^2}{36 c_1 c_2}. \tag{8.68}$$

In terms of K_2 and α the condition $a > 0$ reads

$$(K_2)^2 < \frac{4}{3}\left(\frac{4}{3} - \alpha\right). \tag{8.69}$$

8.1.2.2. Type B solutions: $a = 0$

For $a = 0$, the condition $c_1 c_2 > 0$ is also necessary. From (8.39b) and the first line in (8.61) we get

$$V = V_0 + \frac{\omega_0 l}{3\sqrt{c_1 c_2}} \log\left(\frac{e^{3r/l} - \sqrt{c_2/c_1}}{e^{3r/l} + \sqrt{c_2/c_1}}\right). \tag{8.70}$$

The special case $V = V_0$ appears provided $\omega_0 = 0$ and corresponding metric can be obtained from an improper gauge transformation in the $t - \phi$ plane. In this case, the functions g_0, g_1 and g_2 are given by (8.64), where

$$K_0 = K_1 = \frac{\sigma_1 l}{6\sqrt{c_1 c_2}}, \quad K_2 = \frac{-2\sigma_1 l}{6\sqrt{c_1 c_2}}, \tag{8.71}$$

The constants a_0, a_1, b_0, b_1, c_0 and α are given by

$$a_0 = e^{\sigma_0} V_0 c_1^{2/3}, \quad b_0 = S_0^2 a_0 \tag{8.72}$$

$$a_1 = 0, \quad b_1 = \frac{e^{\sigma_0}}{V_0} c_1^{2/3} \tag{8.73}$$

$$c_0 = e^{-2\sigma_0} c_1^{2/3}, \quad \alpha = \frac{P_0^2 l^2}{36 c_1 c_2}. \tag{8.74}$$

In terms of K_2 and α, the condition $a = 0$ becomes

$$(K_2)^2 = \frac{4}{3}\left(\frac{4}{3} - \alpha\right). \tag{8.75}$$

8.1.2.3. Class C solutions: $a < 0$

In this case V is given by (8.39c), where x is provided by (8.61) according to the constants c_1 and c_2 appearing in the definition of U in Eq. (8.59). Analogously to the case of a vanishing cosmological constant, the Killing vectors ∂_t and ∂_ϕ have not a norm with definite sign, and the solutions do not contain a static limit.

8.2. General stationary cylindrically symmetric solutions with static limit

The general stationary, cylindrically symmetric[3] configuration can be described by the line element

$$ds^2 = g_{tt}(r)dt^2 + g_{\phi\phi}(r)d\phi^2 + g_{zz}(r)dz^2 + 2g_{t\phi}(r)dtd\phi + dr^2, \tag{8.76}$$

where the coordinates range as $t \in (-\infty, \infty)$, $r \in [0, \infty)$, $z \in (-\infty, \infty)$ and $\phi \in [0, 2\pi)$, and a scalar field depending just on the radial coordinate, $\Phi = \Phi(r)$.

As we showed in the previous section, the general solution with static limit (8.76) of the field equations (8.3) can be written as a linear combination of three functions

$$\begin{aligned} g_{tt}(r) &= a_1 g_1(r) - a_0 g_0(r), \\ g_{\phi\phi}(r) &= b_1 g_1(r) - b_0 g_0(r), \\ g_{t\phi}(r) &= \sqrt{a_0 b_0} g_0(r) - \sqrt{a_1 b_1} g_1(r), \\ g_{zz}(r) &= c_0 g_2(r). \end{aligned} \tag{8.77}$$

where for a negative cosmological constant $\Lambda = -3l^{-2}$,

$$g_i(r) = \left(\frac{e^{3r/l} - b}{e^{3r/l} + b}\right)^{K_i} \left(e^{3r/l} - b^2 e^{-3r/l}\right)^{2/3}, i = \{0, 1, 2\}, \tag{8.78}$$

and the scalar field is given by

$$\Phi(r) = \Phi_0 + \frac{1}{2}\sqrt{\frac{\alpha}{2\kappa}} \log\left(\frac{e^{3r/l} - b}{e^{3r/l} + b}\right)^2. \tag{8.79}$$

For $\Lambda = 0$, the functions are

$$g_i(r) = r^{2/3 + K_i}, \quad i = \{0, 1, 2\}, \tag{8.80}$$

[3] In order to include spacetimes lacking of a regular axis, we are adopting the less restrictive definition of cylindrical symmetry given in [107].

and the scalar field is

$$\Phi = \Phi_0 + \sqrt{\frac{\alpha}{2\kappa}} \log(r) \ , \tag{8.81}$$

where the origin has been chosen at $r = 0$. Here K_i, a_0, a_1, b, b_0, b_1, c_0, α and Φ_0 are integration constants. The constants K_i are not independent, since they verify the algebraic relations

$$K_0 + K_1 + K_2 \;=\; 0, \tag{8.82}$$

$$K_0K_1 + K_1K_2 + K_2K_0 \;=\; -\frac{4}{3} + \alpha. \tag{8.83}$$

In order to ensure a real metric and scalar field, the previous algebraic relations fix bounds for the constants. The constant α runs in the interval $0 \leq \alpha \leq 4/3$, and the constants $|K_i|$ are bounded from above by $\frac{2}{3}\sqrt{4-3\alpha}$, $\frac{1}{3}\sqrt{4-3\alpha}$, and $\frac{1}{3}\sqrt{4-3\alpha}$ in any order.

Note that the presence of the scalar field is encoded in the additional integration constant α in (8.83). In absence of the scalar field, the stationary solutions presented in [105], and the static ones in [97, 98], are recovered.

The constant c_0 can be absorbed by rescaling the noncompact coordinate z, and only one of the constants a_0, a_1, b_0, b_1 is essential, as it will become clear in the next section.

8.3. Analysis of the solution with $\Lambda < 0$

In order to get insight about the parameter b, it is convenient to start with static metric

$$ds^2 = -g_0(r)dt^2 + g_1(r)l^2d\phi^2 + g_2(r)dz^2 + dr^2. \tag{8.84}$$

The constant b determines the location of the axis of symmetry at $r_0 = l/3 \log |b|$, and it can be removed from the scalar field by a shift of the radial coordinate $r \to r + r_0$. With this shift, b just appears as a multiplicative factor $b^{2/3}$ in g_i, and consequently, the invariants do not depend on b. In other words, b could be removed from the solution by rescaling the coordinates t, z, ϕ. However, ϕ is a compact coordinate and global properties will be modified with this rescaling. In fact, the metric with the shifted radial coordinate reduces in absence of the scalar field to that shown in [100], where a conicity parameter equivalent to $b^{-1/3}$ is explicitly exhibited. In summary, b has no relevance for the local properties, but it is a topological parameter that contributes to the mass of the solution (see Subsection 8.3.4). On the contrary, note that for $\Lambda = 0$ a shift of the radial coordinate does not have any local or global implication.

The general solution previously considered for the vacuum case do not contain a locally anti-de Sitter (AdS) spacetime [99]. Indeed, the locally AdS solution appears as a special branch disconnected from the general one [100]. The advantage of our general static solution is that it is smoothly connected to a locally AdS spacetime, and in fact, this is achieved just doing $b = 0$ in (8.84). Explicitly, we obtain

$$ds^2 = dr^2 + e^{2r/l}\left(-dt^2 + l^2 d\phi^2 + dz^2\right), \tag{8.85}$$

which becomes the background required for computing the conserved charges in Subsection 8.3.4.

8.3.1. Local properties

In order to obtain a deeper insight into the geometrical properties of the solution, we make use of an invariant characterization of the spacetimes. Spacetimes are usually classified according to the Petrov classification of their Weyl invariants. Note that for analyzing the local properties it is enough to consider the static solutions because, as it will be shown in the next subsection, the stationary solutions can be obtained from a topological construction, and therefore they are locally equivalent. The general solution presented above, (8.84), is of type I (named normally algebraically general). However, as Linet pointed out in [97], a particular choice of the constants K_0, K_1 and K_2, makes the solution to be an algebraically special spacetime of type D. We find that, with the inclusion of the scalar field, i.e. by means of the constant α, the Petrov type D spacetimes are no longer determined only by those particular values of K_i, but by a range of values driven by α. Namely, Petrov type D spacetimes are found for values of K_i taken as any ordering of $\pm\frac{2}{3}\sqrt{4-3\alpha}$, $\mp\frac{1}{3}\sqrt{4-3\alpha}$, and $\mp\frac{1}{3}\sqrt{4-3\alpha}$, provided $0 \leq \alpha < 4/3$. These type D spacetimes have a planar section (two K_i are equal), which allows an additional symmetry. This fourth Killing vector corresponds to a rotation or a boost in this plane depending on its signature.

A novel feature introduced by the scalar field, is a nontrivial Petrov type O subfamily. In fact, for $\alpha = \frac{4}{3}$, $b \neq 0$ and vanishing K_i, a conformally flat spacetime arises, and it is given by

$$ds^2 = dr^2 + (e^{3r/l} - b^2 e^{-3r/l})^{2/3}(-dt^2 + dz^2 + l^2 d\phi^2). \tag{8.86}$$

In other words, the scalar field gives rise to a wider family of spacetimes. This Petrov type O is a new subfamily parametrized by b, which strictly emerges due to the scalar field. In this case the number of isometries is enlarged to six since we are dealing with a conformally flat spacetime. It is remarkable to have such a number of symmetries in a space endowed with a matter source, in particular since for the vacuum (nontrivial) case there are at most four Killing vectors [99]. For $b = 0$ the scalar field is trivial —it is a constant— and (8.86) reduces to the locally AdS spacetime (8.85).

Studying the Weyl and Ricci scalars of the Newman-Penrose formalism it is shown that they are singular at the axis for the whole family of solutions, except in two cases. The first one, corresponds to the CSI spacetimes, which will be discussed in Section 8.4.2. The second case appears for a constant scalar field ($\alpha = 0$) provided the constants K_i take the values $\{\pm\frac{4}{3}, \mp\frac{2}{3}, \mp\frac{2}{3}\}$, or any permutation of them [97]. Since this special solution is regular at the axis, a change of the radial coordinate r can be performed to prove that this type D solution is a black string. In fact, for $K_0 = 4/3$, and $K_1 = K_2 = -2/3$ the transformation reads

$$r = \frac{2l}{3} \log \left[\frac{\rho^{3/2} + \sqrt{\rho^3 - 4bl^3}}{2l^{3/2}} \right], \tag{8.87}$$

yielding the black string

$$ds^2 = -\left(\frac{\rho^2}{l^2} - \frac{4lb}{\rho}\right) dt^2 + \frac{d\rho^2}{\frac{\rho^2}{l^2} - \frac{4lb}{\rho}} + \frac{\rho^2}{l^2} dz^2 + \rho^2 d\phi^2. \tag{8.88}$$

Note that the original axis of symmetry at $r_0 = l/3 \log |b|$ is mapped to the horizon $\rho_+ = 2^{2/3} l b^{1/3}$, and the new axis of symmetry is located at $\rho = 0$. This black string was previously found by solving the Einstein field equations in [28], and by using an adequate coordinate transformation in [100].

8.3.2. Topological construction of the rotating solution from a static one

As explained in [108], a diagonal static metric with dependence on the spacelike coordinates r and z, and with the "angular" coordinate stretched to infinity, can be locally equivalent but globally different to a stationary axisymmetric metric obtained from a topological identification in the static spacetime. This identification is defined by two essential parameters. This kind of essential parameters can not be removed by a permissible change of coordinates since they encode topological information. In this section we are going to build the stationary solution (8.76) with the metric coefficients (8.77), using the procedure presented in [108] in the particular case of cylindrical symmetry.

Let us consider the static solution with scalar field

$$ds^2 = -g_0(r) d\hat{t}^2 + g_1(r) l^2 d\hat{\phi}^2 + g_2(r) dz^2 + dr^2, \tag{8.89}$$

where g_i is given by (8.78) in a coordinate system $(\hat{t}, r, z, \hat{\phi})$ with $\hat{t} \in (-\infty, \infty)$, $r \in [0, \infty)$, $z \in (-\infty, \infty)$ and $\hat{\phi} \in (-\infty, \infty)$. Note that $\hat{\phi}$ is not a compact coordinate. We perform a coordinate transformation on the $(\hat{t}, \hat{\phi})$ plane given by

$$\hat{t} = \beta_0 \phi + \beta_1 t, \qquad \hat{\phi} = \alpha_0 \phi + \alpha_1 t, \tag{8.90}$$

where α_0, α_1, β_0 and β_1 are parameters. This transforms (8.89) into (8.77) by defining these parameters as follows

$$\begin{aligned} \alpha_0 &= \frac{\sqrt{b_1}}{l}, \quad \alpha_1 = -\frac{\sqrt{a_1}}{l}, \\ \beta_0 &= -\sqrt{b_0}, \quad \beta_1 = \sqrt{a_0}. \end{aligned} \tag{8.91}$$

As shown in [108], α_1 and β_1 are not essential parameters, and they can be set as $\alpha_1 = 0$ and $\beta_1 = 1$. On the contrary, α_0 and β_0 are essential. However, after a topological identification, which transforms the $(\hat{t}, \hat{\phi})$ plane into a cylinder, one can fix the period of the angular coordinate ϕ to 2π by choosing $\alpha_0 = 1$. Since that in (8.89) all the coordinates are not compact, b can be absorbed by rescaling the coordinates. After identification, $\hat{\phi}$ becomes periodic and b has a topological meaning. The parameter α_0 plays the same topological role, and in fact it redefines b. Therefore, without loss of generality α_0 can be fixed, but not simultaneously with b. In other words, since from the beginning the static solution contains an arbitrary conicity parameter b, the constant α_0 can be fixed. Going back to relations (8.91) we find that $a_0 = 1, a_1 = 0$ and $b_1 = l^2$ reproduce the set of values chosen for α_0, α_1 and β_1. Then, after fixing the period as 2π there is just one essential parameter β_0 in the transformation, which will be named $-a$ hereafter. Then, the transformation (8.90) reduces to

$$\hat{t} = t - a\phi, \qquad \hat{\phi} = \phi. \tag{8.92}$$

In summary, a topological construction can bring the solution (8.89) into a locally equivalent, but globally different, solution by doing the transformation (8.92) to get

$$ds^2 = -g_0(r)(dt - a d\phi)^2 + g_1(r) l^2 d\phi^2 + g_2(r) dz^2 + dr^2. \tag{8.93}$$

Transformation (8.92) is not a proper coordinate transformation, since it converts an exact 1-form into a closed but not exact 1-form, as was discussed in detail in [110]. Hence, (8.92) only preserves the local geometry, but not the global one. Therefore, the resulting manifold is globally stationary but locally static. Hereafter, we will consider (8.93) instead of (8.77) as the general solution, because it already contains all the local and global essential information.

8.3.3. Asymptotic behavior

In order to display the asymptotic behavior of the fields, it is convenient to use the coordinate $\rho = le^{r/l}$. In this way, the behavior at large ρ is given by

$$\begin{aligned}
g_{tt}(\rho) &= -\frac{\rho^2}{l^2} + \frac{2blK_0}{\rho} + O(\rho^{-4}), \\
g_{\phi\phi}(\rho) &= \rho^2(1 - \frac{a^2}{l^2}) + \frac{2lb(-l^2K_1 + a^2K_0)}{\rho} + O(\rho^{-4}), \\
g_{t\phi}(\rho) &= \frac{\rho^2 a}{l^2} - \frac{2blaK_0}{\rho} + O(\rho^{-4}), \\
g_{zz}(\rho) &= \frac{\rho^2}{l^2} - \frac{2blK_2}{\rho} + O(\rho^{-4}), \qquad g_{\rho\rho}(\rho) = \frac{l^2}{\rho^2}, \\
\Phi(\rho) &= \Phi_0 + \sqrt{\frac{2\alpha}{\kappa}}\frac{bl^3}{\rho^3} + O(\rho^{-9}).
\end{aligned} \tag{8.94}$$

One can note that the metric asymptotically approaches a locally AdS spacetime, as the scalar field becomes constant. The background is fixed by setting $a = b = \alpha = \Phi_0 = 0$, which corresponds to a locally AdS spacetime.

8.3.4. Mass and angular momentum

The mass and angular momentum of the solutions are determined using the Regge-Teitelboim method [117]. In the canonical formalism, the generator of an asymptotic symmetry associated to the vector $\xi = (\xi^\perp, \xi^i)$ is built as a linear combination of the constraints $\mathcal{H}_\perp, \mathcal{H}_i$, with an additional surface term $Q[\xi]$

$$H[\xi] = \int d^3x \left(\xi^\perp \mathcal{H}_\perp + \xi^i \mathcal{H}_i\right) + Q[\xi]. \tag{8.95}$$

A suitable choice of this surface term attains the generator has well-defined functional derivatives with respect to the canonical variables [117]. The surface term $Q[\xi]$ is the conserved charge under deformations ξ provided the constraints vanish. For the action (8.1), the variation of $Q[\xi]$ is given by

$$\begin{aligned}
\delta Q[\xi] = \oint d^2 S_l \bigg[&\frac{G^{ijkl}}{2\kappa}(\xi^\perp \delta g_{ij;k} - \xi^\perp{}_{,k}\delta g_{ij}) + 2\xi_k \delta\pi^{kl} \\
&+ (2\xi^k \pi^{jl} - \xi^l \pi^{jk})\delta g_{jk} - (\sqrt{g}\xi^\perp g^{lj}\Phi_{,j} + \xi^l \pi_\Phi)\delta\Phi \bigg],
\end{aligned} \tag{8.96}$$

where $G^{ijkl} \equiv \sqrt{g}(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl})/2$. The canonical variables are the spatial metric g_{ij} and the scalar field Φ together with their respective conjugate momenta π^{ij} and π_Φ.

To evaluate $\delta Q[\xi]$ we consider as asymptotic conditions just the asymptotic behavior of the solutions with a negative cosmological constant (8.94), where the integration constants K_i, a, b, α are allowed to be varied. The additive constant of the scalar field Φ_0 is considered as a fixed constant without variation, in order to save the asymptotic scale invariance[4]. Since the solution is in the comoving frame along z, the corresponding momentum $Q[\partial_z]$ vanishes. Then, the only nonvanishing charges are those associated to symmetry under time translations and the rotational invariance, the mass and angular momentum, respectively. Defining $q[\xi]$ as the charge by unit length $Q[\xi] = \int q[\xi]dz$, we can obtain from (8.94) and (8.96), the explicit form of $\delta q[\xi]$

$$\delta q[\xi] = \frac{6\pi}{\kappa}\left[-\xi^t \delta(b(K_1 + K_2)) + \xi^\phi \delta(ab(K_1 - K_0))\right]. \tag{8.97}$$

Thus, using $\kappa = 8\pi G$, the mass $M = q[\partial_t]$ and angular momentum $J = q[\partial_\phi]$ per unit length are

$$M = \frac{3b}{4G}K_0, \qquad\qquad J = \frac{3ab}{4G}(K_1 - K_0). \tag{8.98}$$

These global charges are defined up to an additive constant without variation. In order to set the locally AdS spacetime (8.85) as a background, these additive constants must be chosen to be null.

As we can see from the expression for the angular momentum, there are two manners of turning off the angular momentum. The first one is by doing $a = 0$, which cancels the off-diagonal term $g_{t\phi}$ in the metric. The second way is less obvious, since it is achieved by considering $K_0 = K_1$. Indeed, this particular choice of the parameters yields a static solution of type D. This can be shown from the coordinate transformation

$$d\phi \to d\phi + \frac{a}{(a^2 - l^2)}dt, \qquad dt \to dt. \tag{8.99}$$

As analyzed in [108], this transformation contains an inessential parameter $\alpha_1 = a/(a^2 - l^2)$, which does not change the topology. Therefore, the solution with $K_0 = K_1$ is no just locally equivalent to the static solution, but also globally.

8.4. Analysis of the solutions with $\Lambda = 0$

The limit $\Lambda \to 0$, or equivalently $l \to \infty$, in the configurations given by Eqs. (8.78) and (8.79) in Section 8.2, does not provide all the solutions coming from a

[4]For $\delta\Phi_0 \neq 0$, $\delta Q[\xi]$ contains a term proportional to $\oint d^2S\xi^t\sqrt{\alpha}b\delta\Phi_0$. The integration of this term requires a boundary condition relating Φ_0 with α and b.

direct integration of the field equations. In fact, as shown in Section 8.1, two classes of solutions are obtained. The first type corresponds to solutions that match the limit $\Lambda \to 0$ in the configurations introduced in Section 8.3, and they are dubbed as Levi-Civita type spacetimes. The second type is formed by spacetimes having all their invariants constant. These two types will be analyzed in detail below. The discussion in this section is focused on static solutions. The topological construction explained in Section 8.3 does not depend on the value of the cosmological constant, and in consequence, the stationary solutions for $\Lambda = 0$ can be obtained from the improper transformation (8.92). Since (8.92) is a local transformation, the static configuration and its stationary counterpart share the same local properties.

8.4.1. Levi-Civita type spacetimes

In this subsection, we show analyze a Levi-Civita type spacetime in presence of a massless scalar field. The algebraic relations (8.82) and (8.83) determine two essential constants related to the gravitational and scalar field strengths. Since ϕ is an angular coordinate with a given period, the constant b_1 in (8.77) cannot be absorbed by a rescaling of this coordinate keeping the same period. Then, b_1 is a third essential parameter and plays a topological role in the same way as b in Section 8.3. The transformation (8.92) provides the fourth essential parameter for the stationary solution.

As in Section 8.3, we study the local properties through the Petrov classification. Normally the solution is algebraically general as occurs in vacuum [118], but algebraically special spacetimes are also possible to be found. The scalar field parametrizes three families of type D spacetimes, which will be described in Table 1. Two of these families (S_1 and S_2) are allowed only for a nonvanishing scalar field, while the third one (S_3) reduces to the three known vacuum type D Levi-Civita spacetimes by switching off the scalar field and by circular permutations of K_i. A nontrivial type O spacetime emerges strictly from the scalar field. In this case $K_0 = K_1 = K_2 = 0$ and $\alpha = 4/3$ yielding the conformally flat metric

$$ds^2 = dr^2 + r^{2/3}(-dt^2 + dz^2 + g_1^0 d\phi^2). \tag{8.100}$$

This is the counterpart with $\Lambda = 0$ of the conformally flat spacetime described in (8.86).

It is found that the nonvanishing components of the Riemann tensor $R^{\mu\nu}{}_{\lambda\rho}$ and Kretschmann scalar are proportional to r^{-2} and r^{-4}, respectively. Then, the spacetime is asymptotically locally flat.

Until now, we have assumed a nonvanishing constant u_1, defined by (8.41) in Section 8.1 . However, when we consider $u_1 = 0$, the functional form of $g_i(r)$ is

	K_0	K_1	K_2	α
S_1	$\frac{2}{3}\sqrt{4-3\alpha}$	$-\frac{1}{3}\sqrt{4-3\alpha}$	$-\frac{1}{3}\sqrt{4-3\alpha}$	$(0,\frac{4}{3})$
S_2	$-\frac{2}{3}$	$\frac{1}{3}\pm\sqrt{1-\alpha}$	$\frac{1}{3}\mp\sqrt{1-\alpha}$	$(0,1]$
S_3	$-\frac{2}{3}\sqrt{4-3\alpha}$	$\frac{1}{3}\sqrt{4-3\alpha}$	$\frac{1}{3}\sqrt{4-3\alpha}$	$[0,\frac{4}{3})$

Table 8.1: Petrov D spacetimes for $\Lambda = 0$. The constants K_i are classified in three sets, and depend on the amplitude of the scalar field α. Within each set K_0, K_1 and K_2 can be taken in any order. The last column shows the range of α allowed for each set. The first two sets are exclusive for a non-constant scalar field ($\alpha \neq 0$), and the third one also includes a trivial scalar field.

drastically modified. This new branch of solutions, which is not directly provided by the limit $\Lambda \to 0$ in Section 8.3, are analyzed in next subsection.

8.4.2. CSI spacetimes

In general, the Levi-Civita type spacetimes discussed above possess curvature invariants which are singular at $r = 0$. However, it is possible to find regular spacetimes, i.e spacetimes free of any curvature singularity, where in addition, all polynomial scalar invariants constructed from the Riemann tensor and its covariant derivatives are constant. These spacetimes are known as constant scalar invariant (CSI) spacetimes. In this subsection, a nontrivial CSI spacetime due to the presence of the scalar field is presented. It is found that it is required to switch off the cosmological constant in order to get this class of spacetimes. This case is of particular interest since it provides a non-vacuum solution with constant curvature scalars. For simplicity, only the static cases will be considered, since the stationary CSI spacetimes containing a static limit can be obtained by performing the coordinate transformation (8.92).

From the field equations 8.3 one can obtain the Ricci scalar, which reads

$$R = 4\Lambda + \kappa\Phi'^2 = 4\Lambda + \frac{P_0^2}{2U^2}\ , \tag{8.101}$$

where the last equality comes from (8.31). Assuming $P_0 \neq 0$, i.e., a nontrivial scalar field, $U = u_0 = \textit{constant}$ is a necessary condition for a CSI spacetime. Since $U' = 0$ is not a solution for a nonvanishing Λ, there are no CSI spacetimes in this case. However, for $\Lambda = 0$ the function U becomes a constant by setting $u_1 = 0$ in Eq. (8.41), and consequently, a must be negative. Thus, the candidates to CSI spacetimes are the ones lacking a static limit. Nevertheless, if a phantom scalar field is considered, i.e., if we replace P_0^2 by $-P_0^2$, there are no restrictions on the sign of a for $u_1 = 0$. In this way, it is possible to find the static CSI solution with $a \geqslant 0$.

The solution in this case is given by exponentials,

$$g_i(r) = g_i^0 e^{K_i r}, \tag{8.102}$$

and the scalar field is a linear function, $\Phi(r) = \Phi_0 + \sqrt{\alpha/(2\kappa)}r$, where $\alpha = -P_0^2/u_0^2$. The integration constants satisfy the algebraic relations,

$$\begin{aligned} K_0 + K_1 + K_2 = 0, \\ K_0K_1 + K_1K_2 + K_2K_0 = \alpha. \end{aligned} \tag{8.103}$$

Thus, from (8.103) we obtain,

$$K_0 = \frac{1}{2}(-K_2 \pm \sqrt{-3(K_2)^2 - 4\alpha}), \tag{8.104}$$

$$K_1 = \frac{1}{2}(-K_2 \mp \sqrt{-3(K_2)^2 - 4\alpha}). \tag{8.105}$$

Note that the reality condition of the line element demands $\alpha < -\frac{3}{4}(K_2)^2$ and as a consequence the scalar field becomes imaginary. This means that the presence of a phantom scalar field makes possible to remove curvature singularities present in the vacuum solutions. The Petrov classification indicates that these spacetimes are type D.

In order to verify that these spacetimes are indeed CSI spacetimes, we make use of a theorem proved in [119, 120]. The theorem states that any four dimensional locally homogeneous spacetime is a CSI spacetime. The static line element with the metric coefficients (8.102) has three trivial Killing vectors ∂_t, ∂_z, and ∂_ϕ. However, it is possible to find a fourth Killing vector given by

$$\xi^{(4)} = (-\frac{1}{2}K_0 t, -\frac{1}{2}K_1\phi, -\frac{1}{2}K_2 z, 1), \tag{8.106}$$

in the coordinate system (t, ϕ, z, r), which in addition to the trivial ones, form a transitive group of isometries. Therefore, this spacetime is locally homogeneous.

8.5. Concluding remarks

In the presented chapter, the general stationary cylindrically symmetric solution of Einstein-massless scalar field system with a non-positive cosmological constant has been found, and its local and global properties has been studied. As shown in Section 8.1, there is an additional class of solutions, which fail in having a static limit.

Four integration constants are essential parameters for the general solution. This means that these parameters encode all the relevant physical information. One is the amplitude of the scalar field, which beside a second one present in the metric, characterize the gravitational field strength. The other two parameters have a topological origin, since they appearing in the improper gauge transformation that allow us to obtain the stationary solution from the static one. The meaning of these parameters can be also analyzed from the expressions for the mass and angular momentum of the solutions with a negative cosmological constant.

The Petrov classification was performed to explore the effects of the scalar field on the vacuum solutions for a negative and a vanishing cosmological constant. The inclusion of the scalar field enlarges the family of solutions in comparison with the vacuum case. Thus, type D solutions are now parametrized by the amplitude of the scalar field and nontrivial type O solutions have been found in presence of nonvanishing scalar field. These conformally flat solutions endowed with a matter field have six Killing vectors. Note that in the vacuum case, there are not type O solutions apart from the trivial ones, the locally Minkowski (for $\Lambda = 0$) and the locally AdS spacetime (for $\Lambda < 0$).

Other interesting case occurs for $\Lambda = 0$. There are special type D solutions which are possible only if the scalar field is present. We have shown that these spacetimes have a fourth Killing vector, which completes a transitive group of isometries, and consequently they are locally homogeneous. Thus, these solutions become CSI spacetimes dressed by a phantom scalar field.

Chapter 9

Conclusions

In this thesis gravitational solutions have been explored in order to study the influence of the scalar field and its coupling in two different aspect of relevant interest: as a degree of freedom in Hordenski theory and as a minimally coupled matter field.

First we constructed and studied black hole and boson stars solutions in the Horndeski scenario when the non-minimal kinetic coupling is considered. The inclusion of a Maxwell field confers remarkable features to the solution. Black hole solutions were found for arbitrary dimensions $n \geqslant 4$ with a real scalar field in the domain of outer communications satisfying the weak energy condition. The asymptotically locally AdS configurations exhibit an effective cosmological constant determined by the coupling constants and a Coulombic electric field. Although the solutions are not smoothly connected to maximally symmetric spaces, they contain a spherically symmetric asymptotically AdS soliton spacetime which could be used to obtain the thermodynamical properties of the black holes in the Hawking-Page approach. When the minimal coupling is switched off, i.e. when only the non-minimal kinetic coupling is present we obtain an asymptotically AdS solution for $\Lambda < 0$ provided some constraints on the electric charge and non-minimal coupling constant. The electric field is constant and its supported by the cosmological constant at infinity. For vanishing cosmological constant the black hole exhibits an asymptotically flat behavior and the electric field is zero at infinity. At the same time, switching off the electric charge the scalar field is constants and the solution recovers Schwarzschild black hole.

In the same context of Horndeski theory, we constructed a four dimensional boson star configuration for the non-minimal kinetic sector in its bi-scalar extension. In this way, complex scalar field supports a boson star solution and represents two degrees of freedom of the theory. We investigated the mini-boson star as well as the boson star with sixth order self-interacting potential when it possesses two degenerate local vacua. The determinant of the system of equations is the parameter that determines the existence of this kind of solutions. The analysis is carried out by its value at the origin Δ, where this quantity reach its minimum and the scalar field reaches its central amplitude. A vanishing value for Δ implies a singular system of equations and a non regular solution. Unlike minimally coupled mini-boson star solution, we found that mini-boson stars in this setup exist for a range of central amplitudes of the scalar field which is bounded from above when the non-minimal coupling parameter is positive. This is because in this situation Δ decreases drastically. In other words, Δ goes to zero after that value for the central amplitude. When the non-minimal coupling factor is negative the situation is different, since Δ decreases monotonically but not reaches zero while the central amplitude of the scalar field increases. Therefore, in principle the central amplitude is unbounded as long as Δ do not take values within the range of tolerance imposed by the numerical integrator. However, for large values of this quantity, Δ becomes very sensitive to the non-minimal coupling factor decreasing the range where solutions exist. We analyzed the stability of the solutions by computing their binding energy, finding that the presence of the non-minimal kinetic coupling tends to enhance the stability of the solutions compared with minimally coupled boson stars. Qualitatively, the same analysis is valid for the self-interacting case but the space of parameters in which solutions exist is reduced.

In the second part of this work we considered a minimally coupled scalar field to gravitation in four dimensions with cosmological constant. The equations of motion were exhaustively solved, finding two classes of solutions in which one of them fails in having a static limit, and then focusing our analysis in the class with static limit. We found the most general stationary cylindrically symmetric solution and gave an interpretation to the integration constants. Two integration constants give account of the gravitational field strength encoding information of the local properties, while the other two contain topological information as they parametrize an improper transformation that allows to get the stationary solution from the static one. These constants are related with the mass and the angular momentum of the solution when a negative cosmological constant is considered. By performing the Petrov classification we clearly observe that one of the effects of the scalar field is to enlarge the family of solutions since type D solutions are parametrized by the scalar amplitude and, unlike the vacuum case, nontrivial type O conformally flat solutions are found. In the case of vanishing cosmological constant the presence of a phantom scalar field allows the existence of CSI spacetimes which are locally homogeneous.

Chapter 10

Conclusiones

El trabajo presentado en esta tesis ha explorado soluciones gravitacionales para estudiar la influencia del campo escalar y su acoplamiento en dos aspectos de interés relevantes: como un grado de libertad en la teoría de Horndeski y como un campo de materia minimalmente acoplado.

Primero, construimos y estudiamos soluciones de agujeros negros y estrellas bosónicas en el escenario de Horndeski cuando el acoplamiento cinético no-minimal es considerado. La inclusión de un campo de Maxwell confiere características destacables a la solución. Las soluciones de agujero negro que fueron encontradas para dimensión arbitraria $n \geqslant 4$, contienen un campo escalar real fuera del horizonte de eventos y satisfacen la condición de energía débil. Las configuraciones asintóticamente localmente AdS exhiben una constante cosmológica efectiva determinada por las constantes de acoplamiento y un campo eléctrico de Coulomb. Aunque las soluciones no están suavemente conectadas a los espaciotiempo maximalmente simétricos, ellas contienen un espaciotiempo solitónico asintóticamente localmente AdS el cual puede ser usado para obtener las propiedades termodinámicas de los agujeros negros en el enfoque de Hawking-Page. Cuando el acoplamiento minimal es anulado, i.e. cuando solamente el acoplamiento cinético no-minimal está presente, obtenemos una solución asintóticamente AdS para $\Lambda < 0$ siempre cuando existan algunas restricciones sobre la carga eléctrica y la constante de acoplamiento no-minimal. El campo eléctrico es constante y es sustentado por la constante cosmológica en infinito. Para constante cosmológica nula el agujero negro presenta un comportamiento asintótico plano y el campo eléctrico es nulo en infinito. Al mismo tiempo, anulando la carga eléctrica, el campo escalar es constante y la solución recupera el agujero negro de Schwarzschild.

En el mismo contexto de la teoría de Horndeski, construimos una configuración de estrella bosónica para el sector cinético no-minimal en su extensión bi-escalar. De esta manera, el campo escalar complejo provee una solución de estrella bosónica y representa dos grados de libertad de la teoría. Nosotros investigamos mini-estrellas bosónicas así como también estrellas bosónicas con potencial auto-interactuante de sexto orden cuando este posee dos vacíos locales degenerados. El determinante del sistema de ecuaciones es el parámetro que determina la existencia de este tipo de soluciones. El análisis es llevado a cabo mediante su valor en el origen Δ, donde esta cantidad alcanza un mínimo y el campo escalar su amplitud central. Un valor nulo para Δ implica un sistema de ecuaciones singular y una solución no regular. A diferencia de la mini-estrella bosónica minimalmente acoplada, en nuestro caso encontramos que las mini-estrellas bosónicas existen para un rango de valores de la amplitud central del campo escalar, esto es, el módulo del campo escalar evaluado en el origen, el cual es acotado por arriba cuando el parámetro de acoplamiento no-minimal es positivo. Esto es porque en esta situación Δ disminuye drásticamente. En otra palabras, Δ va a cero después de aquel valor para la amplitud central. Cuando el factor de acoplamiento no-minimal es negativo la situación es diferente, puesto que Δ decrece monótonamente pero no alcanza el valor cero mientras la amplitud central del campo escalar aumenta. Por tanto, en principio la amplitud central no es acotada siempre cuando Δ no alcance valores dentro del rango de tolerancia impuesto por el integrador numérico. Sin embargo, para valores grandes de esta cantidad, Δ se torna muy sensible al factor de acoplamiento no-minimal, disminuyendo el rango donde las soluciones existen. Analizamos la estabilidad de las soluciones calculando la energía de enlace encontrando que la presencia del acoplamiento cinético no-minimal tiende a mejorar la estabilidad de las soluciones comparada con las estrellas bosónica minimalmente acopladas. Cualitativamente, el mismo análisis es válido para el caso auto-interactuante pero el espacio de parámetros en los cuales las soluciones existen es reducido.

En la segunda parte de este trabajo consideramos un campo escalar minimalmente acoplado a gravitación en cuatro dimensiones con constante cosmológica. Las ecuaciones de movimiento fueron resueltas exhaustivamente, encontrando dos clases de soluciones, una de las cuales no tiene un limite estático, concentrando nuestro análisis en la clase con límite estático. Encontramos la solución más general estacionaria con simetría cilíndrica y dimos una interpretación a las constantes de integración. Dos constantes de integración dan cuenta de la intensidad del campo gravitacional conteniendo información de la propiedades locales, mientras las otras dos tienen información topológica ya que parametrizan una transformación impropia que permite obtener la solución estacionaria a partir de la estática. Estas constantes fueron relacionadas con la masa y el momento angular de la solución cuando unja constante cosmológica es negativa es considerada. Realizando la clasificación de Petrov claramente observamos que uno de los efectos del campo escalar es aumentar la familia de soluciones puesto que ahora las soluciones tipo D son parametrizadas por la amplitud del campo escalar y, a diferencia del caso en vacío, soluciones conformalmente planas notriviales tipo O son encontradas. En el caso de constante cosmológica nula la presencia de un campo escalar fantasma permite la existencia de espaciotiempo CSI, los cuales son localmente homogéneos.

Appendix A

Field equations for boson stars in Horndeski gravity

Our field equations can be cast in the following matrix form

$$AB = C \tag{A.1}$$

where, before the rescaling made in Section 6.1.4, we have defined

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{A.2}$$

$$B = \begin{bmatrix} F'' \\ G'' \\ \phi'' \end{bmatrix} \tag{A.3}$$

$$C = \begin{bmatrix} K_1(F, F', G, G', \phi, \phi') \\ K_2(F, F', G, G', \phi, \phi') \\ K_3(F, F', G, G', \phi, \phi') \end{bmatrix} \tag{A.4}$$

with

$$A_{11} = \frac{r}{4}\left(\frac{-6\omega^2\phi^2\eta G^3 F r + 4\kappa F^2 G^3 r + 2rG^2F^3\phi'^2\eta}{G^{5/2}F^4}\right)$$

$$A_{12} = \left(\frac{r^2\omega^2\phi^2\eta}{\sqrt{G}F^2}\right)$$

$$A_{13} = \frac{r}{4}\left(\frac{4rG^2F^3\phi'\eta F' + 8rG^3F^2\omega^2\phi\eta}{G^{5/2}F^4}\right)$$

$$A_{21} = \left(\frac{r^2\omega^2\phi^2\eta}{\sqrt{G}F^2}\right) = A_{12}$$

$$A_{22} = -\frac{1}{8}\left(\frac{8r^2G^2\kappa F^3 + 4r^2G^2\omega^2\phi^2\eta F^2 + 4r^2G\phi'^2\eta F^4}{G^{7/2}F^3}\right)$$

$$A_{23} = -\frac{1}{8}\left(\frac{16rG^2\phi'\eta F^4 + 16r^2G^3\omega^2\phi\eta F^2 + 8r^2G\phi'\eta F^4G'}{G^{7/2}F^3}\right)$$

$$A_{31} = \frac{r}{4}\left(\frac{4rG^2F^3\phi'\eta F' + 8rG^3F^2\omega^2\phi\eta}{G^{5/2}F^4}\right) = A_{13}$$

$$A_{32} = -\frac{1}{8}\left(\frac{16rG^2\phi'\eta F^4 + 16r^2G^3\omega^2\phi\eta F^2 + 8r^2G\phi'\eta F^4G'}{G^{7/2}F^3}\right) = A_{23}$$

$$A_{33} = -\frac{1}{4}\left(\frac{-8r^2F^3G^4\alpha + 2r^2F^4G\eta G'^2 - 2r^2F^2G^3\eta F'^2 + 8rF^4G^2\eta G'}{G^{7/2}F^3}\right)$$

$$\begin{aligned}
K_1 = &-\frac{r}{4G^{5/2}F^4}(-4\lambda_2\phi^4F^2G^4r + 4\lambda_3\phi^6F^2G^4r \\
&+ 2\kappa F^2G^2G'rF' - 8\omega^2\phi^2\alpha FG^4r + 9\omega^2\phi^2\eta G^3F'^2r \\
&- 12\omega^2\phi^2\eta G^3F'F + 8\omega^2\phi^2\eta G^2G'F^2 - \phi'^2\eta F^2G^2F'^2r \\
&+ \phi'^2\eta F^4G'^2r - 4\kappa FG^3F'^2r + 16G^3F^2\omega^2\phi\eta\phi' \\
&+ 4\lambda_1\phi^2F^2G^4r - 3\omega^2\phi^2\eta G^2G'FrF' + 4G^2F^3\phi'^2\eta F' \\
&+ 4\phi'^2\eta F^4G'G + 8\kappa F^2G^3F' + 8rG^3F^2\omega^2\phi'^2\eta \\
&+ 4rG^2F^2\omega^2\phi\eta G'\phi' - rGF^3\phi'^2\eta F'G' \\
&- 12rG^3F\omega^2\phi\eta F'\phi' - 3\omega^2\phi^2\eta GG'^2F^2r)
\end{aligned}$$

$$\begin{aligned}
K_2 = &\frac{1}{8G^{7/2}F^3}(8rG\phi'^2\eta F^4G' - 5r^2\phi'^2\eta F^4G'^2 - 12r^2G^4\omega^2\phi^2\alpha F \\
&+ 13r^2G^3\omega^2\phi^2\eta F'^2 - r^2G^2\phi'^2\eta F^2F'^2 + 4r^2G\phi'^2\eta F^3G'F' \\
&+ 8rG^2\phi'^2\eta F^3F' + 16r^2G^3\omega^2\phi'^2\eta F^2 - 6r^2F^3GG'^2\kappa \\
&+ 16rG^2\kappa F^3G' + 12r^2G^4\lambda_1\phi^2F^2 - 12r^2G^4\lambda_2\phi^4F^2 \\
&+ 4r^2G^3\phi'^2\alpha F^3 + 8F^4G^2\phi'^2\eta - 3r^2F^2GG'^2\omega^2\phi^2\eta \\
&+ 32rG^3\omega^2\phi\eta F^2\phi' - 16rG^3\omega^2\phi^2\eta F'F + 8rG^2\omega^2\phi^2\eta G'F^2 \\
&+ 8r^2G^2\omega^2\phi\eta G'F^2\phi' - 4r^2G^2\omega^2\phi^2\eta G'FF' \\
&- 24r^2G^3\omega^2\phi\eta FF'\phi' + 2r^2G^3\kappa FF'^2 + 12r^2G^4\lambda_3\phi^6F^2)
\end{aligned}$$

$$
\begin{aligned}
K_3 =& \frac{1}{4G^{7/2}F^3}(r^2\phi' F^2 G'\eta G^2 F'^2 + 2r^2F^3G\phi'\eta G'^2 F' \\
& - 4r^2F^3G^3\phi'\alpha G' - 8\phi' F^4 G\eta G'^2 r - 8r^2\phi G^5\omega^2\alpha F \\
& + 10r^2\phi G^4\omega^2\eta F'^2 - 4\phi' F^2 G^3\eta F'^2 r + 16r\phi G^3\omega^2\eta G' F^2 \\
& + 2r^2FG^3\phi'\eta F'^3 + 8r^2\phi G^5\lambda_1 F^2 - 6r^2\phi G^2\omega^2\eta G'^2 F^2 \\
& + 24r^2\phi^5G^5\lambda_3F^2 - 5r^2\phi' F^4 G'^3\eta + 8\phi' F^4G^2\eta G' \\
& - 4r^2\phi G^3\omega^2\eta G' F F' - 16r\phi G^4\omega^2\eta F' F - 16\phi' F^3G^4\alpha r \\
& + 8rF^3G^2\phi'\eta G' F' - 16r^2\phi^3G^5\lambda_2F^2)
\end{aligned}
$$

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