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On deformations of T-varieties

(Sobre deformaciones de T-variedades)

Tesis para optar al grado de Doctor en Matemática

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Resumen

El principal concepto a estudiarse en la presente tesis es el de T-variedad: una variedad algebraica normal X de dimensión n definida sobre un campo $\mathbb{K}$ algebraicamente cerrado y de característica cero, junto con una acción efectiva del toro algebraico $(\mathbb{K}^*)^m$ sobre X (ver Definición 2.0.2). El número $n - m$ es la complejidad de la T-variedad X y las T-variedades de complejidad cero se llaman variedades tóricas. Las variedades tóricas fueron estudiadas por primera vez por M. Demazure en [9] en 1970, en donde se muestra que éstas pueden describirse de manera puramente combinatorial usando el lenguaje de abanico de conos: una cierta colección de conos poliedrales racionales en un espacio vectorial racional (ver Definición 1.1.8). En 2006, K. Altmann, J. Hausen y H. Süss generalizan esta idea a T-variedades en [4] con la introducción del lenguaje de abanicos divisoriales (ver Definición 2.2.1). Por otro lado, las T-variedades de complejidad pueden describirse por medio de su anillo de Cox como muestra D. Cox en [7] y J. Hausen, H. Süss en [12]. Los anillos de Cox son anillos de coordenadas homogéneas globales para variedades algebraicas que permiten describir la variedad original como un cociente categórico de un conjunto cuasiafín por la acción de un cierto grupo abeliano (ver Sección 1.2.4 o [2] para una explicación más completa).

El principal fin de esta tesis es hacer uso de ambos lenguajes, abanicos divisoriales y anillos de Cox, para estudiar deformaciones de variedades tóricas y variedades de complejidad uno. En este aspecto, se ha logrado generalizar la sucesión de Euler usual para variedades tóricas. Como consecuencia de esto, se puede describir cuáles T-superficies suaves de complejidad uno son infinitesimalmente rígidas. Por otro lado, con el Teorema 4.2.1 describimos familias de deformación para variedades tóricas usando el lenguaje de anillos de Cox, de manera similar a como lo hace A. Mavlyutov en [24], demostrando así la equivalencia entre la construcción de A. Mavlyutov y la de N. Ilten con R. Vollmert. Como último resultado, describimos puntos triples racionales en superficies con acción de $\mathbb{K}^*$ calculando sus anillos de Cox y la matriz P de la construcción de Cox. Esto se ha hecho con el propósito de construir deformaciones de dichas singularidades en un proyecto futuro, adaptando nuestro resultado anterior al caso de variedades singulares.

Introduction

The main concept to be studied in the present thesis is that of a T-variety: a normal algebraic variety X of dimension n defined over an algebraically closed field $\mathbb{K}$ of characteristic zero, together with an effective action of the algebraic torus $(\mathbb{K}^*)^m$ on X (see Definition 2.0.2). The number $n-m$ is the complexity of the T-variety X and T-varieties of complexity zero are called toric varieties. Toric varieties were first studied by M. Demazure in [9] in 1970 where he shows that they can be described in a purely combinatorial way by using the language of fan of cones: certain collections of rational polyhedral cones in a rational vector space (see Definition 1.1.8). In 2006, K. Altmann, J. Hausen and H. Süss generalize this idea to T-varieties in [4] with the introduction of the language of divisorial fans (see Definition 2.2.1). On the other hand T-varieties of complexity at most one can be described by means of their Cox ring as shown by D. Cox in [7] and J. Hausen, H. Süss in [12]. Cox rings are global homogeneous coordinate rings for algebraic varieties which allow to describe the original variety as a categorical quotient of a quasi-affine set by the action of a certain abelian group (see Section 1.2.4 or [2] for a more complete explanation). The main aim of this thesis is to make use of both languages, divisorial fans and Cox rings, to the study of deformations of toric varieties and varieties of complexity one. In these regard our main results are the following. In Theorem 1 we construct a Euler-type sequence for the cotangent sheaf of a smooth complexity one T-surface which does not admit torus-invariant elliptic points, i.e. isolated torus-invariant points which lies in the closure of infinitely many orbits. Given such a surface X there is a morphism $\pi\colon X \to Y$ onto a smooth curve Y which is a categorical quotient for the torus action. The reducible fibers of π consist of certain torus invariant curves $E_1, \ldots, E_s$ whose intersection points are called hyperbolic fixed points. Moreover the fibration π admits two sections whose images are called the source F^+ and the sink F^- of X respectively.

Theorem 1. *Let X be a complete smooth $\mathbb{K}^*$-surface without elliptic points, with quotient morphism $\pi\colon X \to Y$, source divisor F^+, sink divisor F^-, r hyperbolic fixed points which are intersection of pairs of the prime invariant divisors $E_1, \ldots, E_s$. There is an exact sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \pi^*\Omega_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(E_{\mathcal{S}}) \longrightarrow \Omega_X \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X^{r+1} \longrightarrow 0$$

where $\mathcal{G}$ is the quotient of $\oplus_{i\in I}\mathcal{O}_X(E_i)\oplus\mathcal{O}_X(F^+)\oplus\mathcal{O}_X(F^-)$ by the subsheaf $\oplus_{i<j}\mathcal{O}_X(-E_i-E_j)$, where the second sum is taken over all the pairs (i,j) such that $E_i\cap E_j\neq\emptyset$.x

This generalizes the usual Euler sequence for toric surfaces. As a consequence we are able to describe which smooth complexity one T-surfaces are infinitesimally rigid.

Theorem 2. *Let X be a complete smooth $\mathbb{K}^*$-surface without elliptic points. Then the following are equivalent:*

1. *The equality $h^1(X,T_X)=0$ holds;*
2. *X is toric Fano.*

Recall that the space of infinitesimal deformations, i.e. over $\operatorname{Spec}\mathbb{K}[\varepsilon]/\langle\varepsilon^2\rangle$, of a smooth variety X is parametrized by the vector space $\mathrm{H}^1(X,T_X)$, where T_X is the tangent sheaf. If X is a smooth complete toric variety the latter vector space admits a generating subset which can be described in a combinatorial way by means of the fan of X. For each such generator $\xi\in\mathrm{H}^1(X,T_X)$ N. Ilten and R. Vollmert produce a one parameter flat family of complexity one T-varieties $\mathcal{X}_\xi\to\mathbb{A}^1$ whose central fiber is isomorphic to X and whose Kodaira-Spencer map value is ξ. In Theorem 4.2.1 we describe such a family in the language of Cox rings in a similar way as A. Mavlyutov did in [24], proving in this way that the equivalence of A. Mavlyutov construction with that of N. Ilten and R. Vollmert. The Cox ring for the total deformation space is given by a trinomial in Cox coordinates:

$$\lambda\prod_{(1,j)\in U_1}T_{1j}^{a_j}-\prod_{(2,j)\in U_2}T_{2j}^{-a_j}+\prod_{(3,j)\in U_3}T_{3j}^{-a_j},$$

where each T_{ij} is in correspondence with the j-th ray of the fan of X and $\lambda\in\mathbb{A}^1$ is the coordinate of the base space of the one-dimensional deformation (see Section 4.2 for more details). As a last result, we describe rational triple points in surfaces with $\mathbb{K}^*$-action by computing their Cox ring and the P-matrix of the Cox construction. This is done with the purpose of constructing deformations of these singularities in a future project by adapting our previous result to the case of singular varieties.

Theorem 3. *For each triple point we exhibit an affine $\mathbb{K}^*$-surface X with divisor class group $\mathrm{Cl}(X)$ of rank zero, having the triple point as its unique singular point. The table below shows, for each case, the defining matrix P of X, the grading of the Cox ring and the divisor class group of X, with the Cox ring being given by $\mathcal{R}(X)$ is given by*

$$\mathbb{K}[T_1,T_2,T_3]/\langle T_1^{-p_{11}}-T_2^{p_{12}}+T_3^{p_{23}}\rangle$$

where p_{ij} is the (i,j)-th entry of P.

Type	*P-matrix*	*Grading*	*Cl(X)*
$A_{n,m,p}$	$\begin{bmatrix} -(n+1) & m+1 & 0 \\ -(n+1) & 0 & p+1 \\ -1 & -1 & -1 \end{bmatrix}$	*See* 5.2.9	
$B_{m,n}$	$\begin{bmatrix} -(2m+3) & 2 & 0 \\ -(2m+3) & 0 & n+1 \\ -(m+2) & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 4m+n+5 & 2m+n+4 & 1 \end{bmatrix}$	$\frac{\mathbb{Z}}{(4m+n+7)\mathbb{Z}}$
$C_{m,n}$	$\begin{bmatrix} -\alpha & 2 & 0 \\ -\alpha & 0 & 2 \\ -(n+2) & 1 & -1 \end{bmatrix}$ $\alpha = (n+2)m + (2n+3)$	*See* 5.2.2	
$D_{n,5}$	$\begin{bmatrix} -(3n+5) & 2 & 0 \\ -(3n+5) & 0 & 3 \\ -(n+2) & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 3n+6 & 2n+4 \end{bmatrix}$	$\frac{\mathbb{Z}}{(3n+7)\mathbb{Z}}$
$E_{6,0}$	$\begin{bmatrix} -4 & 2 & 0 \\ -4 & 0 & 5 \\ -1 & 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 5 & 2 \end{bmatrix}$	$\frac{\mathbb{Z}}{6\mathbb{Z}}$
$E_{7,0}$	$\begin{bmatrix} -5 & 2 & 0 \\ -5 & 0 & 5 \\ -1 & 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$	$\frac{\mathbb{Z}}{5\mathbb{Z}}$
$E_{0,7}$	$\begin{bmatrix} -3 & 2 & 0 \\ -3 & 0 & 9 \\ -3 & 1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$	$\frac{\mathbb{Z}}{3\mathbb{Z}}$
$F_{n,6}$	$\begin{bmatrix} -(4n+7) & 2 & 0 \\ -(4n+7) & 0 & 3 \\ -(n+2) & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 2n+4 & n+4 & 1 \end{bmatrix}$	$\frac{\mathbb{Z}}{(2n+5)\mathbb{Z}}$
$G_{n,0}$	$\begin{bmatrix} -(n+1) & 3 & 0 \\ -(n+1) & 0 & 3 \\ -1 & 1 & -1 \end{bmatrix}$	*See* 5.2.8	

Table 1: $\mathbb{K}^*$-surfaces associated to triple points (Source: own elaboration)

The thesis is organized as follows: Chapter 1 is a discussion on the topic of toric varieties and Cox rings. We start the chapter by going over the basic definitions in toric geometry and explain the correspondence between toric varieties and fans of cones. After that, we discuss Cox rings, explaining how they are constructed in general and then looking at the particular case of Cox rings of toric varieties. In Chapter 2 we briefly recall the language of divisorial fans for T-varieties as was presented in [4] and a few other related works. We also treat the specific situation of a two dimensional variety with an action of $\mathbb{K}^*$. In Chapter 3 we introduce some basic definition of deformation theory, after which we prove the existence of the Euler-like sequence mentioned above. The chapter ends with a few applications of this sequence to the study of the cohomology group $\mathrm{H}^1(X, T_X)$. Chapter 4 is dedicated to the construction of one-parameter deformations of toric varieties from the point of view of Cox rings, as well as showing that this construction is equivalent to the one done in [20] using divisorial fans. We then apply our results in the study of deformation of scrolls and deformation of subvarieties. In Chapter 5 we recall how to express the data of a surface with $\mathbb{K}^*$-action using P-matrices and how to obtain resolutions of singularities. Then we construct, case by case, surfaces with singularities of multiplicity 3.

Chapter 1

Cox Rings and toric varieties

In this chapter we recall the basic facts about toric varieties and their combinatorial description with the language of fan of cones, following [8, 10]. Then we recall the definition of the Cox sheaf and Cox ring of a normal complete variety with finitely generated divisor class group following [2]. We further describe the Cox construction of a toric variety, via Cox rings, and make use of it to describe hypersurfaces of toric varieties by means of global homogeneous coordinates. In this chapter, and through the whole thesis, we always consider $\mathbb{K}$ to be an algebraically closed field of characteristic zero.

1.1 Toric varieties

We begin by defining the concept of a toric variety [8, 10], as well as explaining the one-to-one correspondence to sets of polyhedral cones.

Consider the algebraic group $(\mathbb{K}^*)^n$, with component-wise multiplication. This is called the *n-dimensional torus*.

Definition 1.1.1. An n-dimensional normal variety X is called a *toric variety* if it contains an n-dimensional torus as a Zariski open subset in such a way that the action of the torus on itself extends to an action of the torus on X.

Example 1.1.2. A common example is the projective space $\mathbb{P}^n$. It admits a $(\mathbb{K}^*)^n$-action given by

$$(t_1, \ldots, t_n) \cdot [x_1 : \ldots : x_{n+1}] = [t_1 x_1 : t_2 x_2 : \ldots : t_n x_n : x_{n+1}].$$

When restricted to the open set

$$\{[x_1 : \ldots : x_{n+1}] : x_i \neq 0 \ \forall i\} \cong (\mathbb{K}^*)^n,$$

the action reduces to component-wise multiplication on the torus.

We fix some more notation for this section. Let $N \cong \mathbb{Z}^n$ be a lattice of rank n and let $M = \mathrm{Hom}(N, \mathbb{Z})$ be its dual. We denote by $\langle -, - \rangle \colon M \times N \to \mathbb{Z}$ the perfect pairing defined by $(u, v) \mapsto \langle u, v \rangle := u(v)$ and by $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ the corresponding rational vector spaces.

Definition 1.1.3. Let $S \subseteq M$ be a commutative finitely generated semigroup. The *semigroup algebra*, denoted by $\mathbb{K}[S]$, is the following $\mathbb{K}$-vector space

$$\mathbb{K}[S] = \left\{ \sum_{m \in S} \lambda_m \chi^m : \lambda_m \in \mathbb{K},\ c_m = 0 \text{ for all but finitely many } m \right\},$$

with multiplication induced by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$

Definition 1.1.4. Let S be a finite subset of $N_{\mathbb{Q}}$. The *cone* $\sigma \subset N_{\mathbb{Q}}$ generated by S is defined as

$$\sigma = \mathrm{cone}\,(S) := \left\{ \sum_{v \in S} \lambda_v v : \lambda_v \in \mathbb{Q}_{\geq 0} \right\}.$$

A *cone* in $N_{\mathbb{Q}}$ is any cone generated by some finite subset. The *dimension* of σ is the dimension of the smallest linear subspace of $N_{\mathbb{Q}}$ that contains σ. A cone is said to be *pointed* if it does not contain any linear space. The *dual cone* of σ, denoted by $\sigma^\vee$, is defined as

$$\sigma^\vee = \{u \in M : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

Given a cone σ, we can construct an affine toric variety as follows. First, we introduce the following notation:

$$S_\sigma := \sigma^\vee \cap M.$$

Note that S_σ is a semigroup with vector addition. The map

$$\mathbb{K}[S_\sigma] \to \mathbb{K}[S_\sigma] \otimes_{\mathbb{K}} \mathbb{K}[M],\ \chi^m \mapsto \chi^m \otimes \chi^m$$

induces an action of $\mathrm{Spec}\,\mathbb{K}[M] \cong (\mathbb{K}^*)^n$ on $\mathrm{Spec}\,\mathbb{K}[S_\sigma]$. Thus, $U_\sigma := \mathrm{Spec}\,\mathbb{K}[S_\sigma]$ is an affine toric variety.

To construct non-affine toric varieties, we will need to introduce a few more definitions. Let $m \in M_{\mathbb{Q}}$ such that $m \neq 0$, we define the hyperplane

$$m^\perp := \{v \in N_{\mathbb{Q}} : \langle m, v \rangle = 0\}.$$

Definition 1.1.5. Let $\sigma \subset N_{\mathbb{Q}}$ be a cone. A *face* of σ is any subset of the form $m^\perp \cap \sigma$ for some $m \in \sigma^\vee \cap M$. The notation $\tau \preceq \sigma$ is often used to mean 'τ is a face of σ'.

Lemma 1.1.6. [8, Prop 1.3.16] *Let $\sigma \subset N_{\mathbb{Q}}$ be a cone and let $\tau = \sigma \cap m^\perp$ be a face of σ with $m \in \sigma^\vee \cap M$. Then*

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m).$$

Lemma 1.1.7. [8, Prop 1.2.13] *Given two cones $\sigma_1, \sigma_2 \subset N_{\mathbb{Q}}$, if $\tau = \sigma_1 \cap \sigma_2$ is a face of each, then*

$$\tau = \sigma_1 \cap m^{\perp} = \sigma_2 \cap m^{\perp}$$

for all m in the relative interior of $\sigma_1^{\vee} \cap (-\sigma_2)^{\vee}$.

Definition 1.1.8. A fan Σ in $N_{\mathbb{Q}}$ is a finite collection of pointed cones in $N_{\mathbb{Q}}$ such that

1. For each cone $\sigma \in \Sigma$, all faces of σ are also in Σ.
2. If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of both cones.

Given a fan Σ, for each pair of cones $\sigma_1, \sigma_2 \in \Sigma$ that share a face τ, we have by Lemma 1.1.6 that for some $m \in M$

$$\mathbb{K}[S_\tau] = \mathbb{K}[S_{\sigma_1} + \mathbb{Z}(-m)] \cong \mathbb{K}[S_{\sigma_1}]_{\chi^m}.$$

By Lemma 1.1.7, we can assume that for this m the following holds

$$U_{\sigma_1} \supset (U_{\sigma_1})_{\chi^m} = U_\tau = (U_{\sigma_2})_{\chi^{-m}} \subset U_{\sigma_2}.$$

Therefore, the affine varieties $\{U_\sigma\}_{\sigma \in \Sigma}$ can be glued into a toric variety which will be denoted by X_Σ.

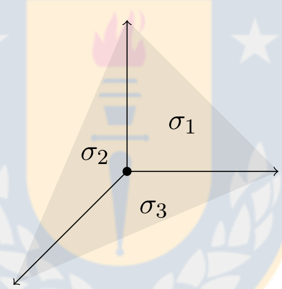


Figure 1.1: Fan of $\mathbb{P}^2$
(Source: own elaboration)

Example 1.1.9. A classic example is the fan Σ (see Figure 1.1) with rays generated by $(1,0)$, $(0,1)$ and $(-1,-1)$. The duals of the cones are $\sigma_1^{\vee} = \text{cone}\langle (1,0),(0,1)\rangle$, $\sigma_2^{\vee} = \text{cone}\langle(-1,1),(-1,0)\rangle$ and $\sigma_3^{\vee} = \text{cone}\langle(0,-1),(1,-1)\rangle$. Then, by definition we have

$$\begin{aligned} U_{\sigma_1} &= \operatorname{Spec} \mathbb{K}[x,y] \\ U_{\sigma_2} &= \operatorname{Spec} \mathbb{K}[x^{-1}y, x^{-1}] \\ U_{\sigma_3} &= \operatorname{Spec} \mathbb{K}[y^{-1}, xy^{-1}] \end{aligned}$$

which, by setting $x \mapsto z_1/z_0$ and $y \mapsto z_2/z_0$, correspond to the standard affine charts of $\mathbb{P}^2$. Indeed, it can be shown that $X_\Sigma \cong \mathbb{P}^2$.

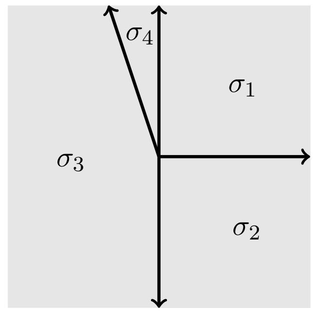


Figure 1.2: Fan of Hirzebruch surface (Source: own elaboration)

Example 1.1.10. Let r be a non-negative integer and consider the bidimensional fan depicted in Figure 1.2 with four bidimensional cones and four rays passing through $(0,-1)$, $(1,0)$, $(0,1)$ and $(-1,r)$.

This induces a toric variety by gluing four toric affine varieties:

$$\begin{aligned} U_{\sigma_1} &= \operatorname{Spec} \mathbb{K}[x,y] \\ U_{\sigma_2} &= \operatorname{Spec} \mathbb{K}[x,y^{-1}] \\ U_{\sigma_3} &= \operatorname{Spec} \mathbb{K}[x^{-1},x^{-r}y^{-1}] \\ U_{\sigma_4} &= \operatorname{Spec} \mathbb{K}[x^{-1},x^{r}y]. \end{aligned}$$

This toric variety is called the *Hirzebruch surface* $\mathbb{F}_r$.

Having explained how to obtain a toric variety from a fan, we now explain the inverse process, i.e. constructing a fan from a given toric variety.

Definition 1.1.11. A *one-parameter subgroup* of a torus is a group homomorphism $\lambda\colon \mathbb{K}^* \to (\mathbb{K}^*)^n$. A *character* is a group homomorphism $\chi\colon (\mathbb{K}^*)^n \to \mathbb{K}^*$.

Proposition 1.1.12. [14, §16] *All one-parameter subgroups of a torus T are of the form*

$$\rho\colon \mathbb{K}^* \to T, \quad \rho(t) = (t^{a_1}, \dots, t^{a_n}),$$

where $a_1, \dots, a_n \in \mathbb{Z}$. Thus, there is a 1-1 correspondence

$$\left\{ \begin{array}{c} \textit{One-parameter subgroups} \\ \textit{of } T \end{array} \right\} \longleftrightarrow \mathbb{Z}^n$$

Let X be a toric variety of dimension n and let U be the open subset that is isomorphic to the torus T. Choose a point $x_0 \in U$ and for each one-parameter subgroup ρ of T we check if the orbit map $\mathbb{K}^* \to X,\ t \mapsto \rho(t)\cdot x_0$ can be extended to a map $\mathbb{K} \to X$. In this case we define the limit

$$\lim_{t\to 0} \rho(t)\cdot x_0 := \rho(0)\cdot x_0,$$

where the righthand side is the extended map. By identifying a one-parameter subgroup ρ with an element of $\mathbb{Z}^n$, we see that the set of all $\rho' \in \mathbb{Z}^n$ such that

$$\lim_{t\to 0} \rho'\cdot x_0 = \lim_{t\to 0} \rho\cdot x_0$$

corresponds to the set of integer points of some cone σ in $\mathbb{Q}^n$. After constructing all possible cones with this method, we obtain a fan on $\mathbb{Q}^n$ that we denote by Σ_X.

Example 1.1.13. We take the toric variety $\mathbb{P}^2$, whose torus action is

$$(s,t)\cdot[x_1 : x_2 : x_3] = [sx_1 : tx_2 : x_3].$$

If we take

$$x_0 = [1:1:1]$$

and a one-parameter subgroup ρ with its corresponding element $(a,b)\in\mathbb{Z}^2$, we have

$$\lim_{t\to 0} \rho(t)\cdot x_0 = \lim_{t\to 0}[t^a : t^b : 1] = \begin{cases} [0:0:1] & \text{if } a>0,\ b>0 \\ [0:1:0] & \text{if } a>b,\ b<0 \\ [1:0:0] & \text{if } a<0, b>a \\ [1:0:1] & \text{if } a=0,\ b>0 \\ [0:1:1] & \text{if } a>0,\ b=0 \\ [1:1:0] & \text{if } a=b,\ b<0 \\ [1:1:1] & \text{if } a=b=0 \end{cases}$$

In this example, we see that the limit $[0:0:1]$ is obtained by taking all integer points (a,b) in the interior of $\operatorname{cone}((1,0),(0,1)) \subset \mathbb{Q}^2$; the limit $[0:1:0]$ is obtained by integer points in the interior of $\operatorname{cone}((-1,-1),(1,0))$, and so on, with all of them forming a fan $\Sigma_{\mathbb{P}^2}$. Note that this fan is exactly the same we saw in example 1.1.9.

Remark 1.1.14. Each fixed point of the toric variety is given by a limit of the form $\lim_{t\to 0}\rho(t)\cdot x_0$ as above. From the construction of Σ_X, we see that there is a one-to-one correspondence

$$\{\text{Maximal cones of } \Sigma_X\} \longleftrightarrow \{\text{Fixed points of } X\}$$

More generally, we have the following.

Proposition 1.1.15. [2, Prop 2.1.2.2] *Let X be a toric variety with torus T and Σ its corresponding fan. For each maximal cone $\sigma \in \Sigma$, denote by x_σ the corresponding fixed point of X.*

There is a bijection

$$\Sigma \to \{T\text{−orbits of } X\}, \quad \sigma \mapsto T \cdot x_\sigma$$

Moreover, for any two $\sigma_1, \sigma_2 \in \Sigma$, we have $\sigma_1 \preceq \sigma_2$ if and only if $\overline{T \cdot \sigma_1} \supseteq \overline{T \cdot \sigma_2}$. For the affine chart X_σ defined by $\sigma \in \Sigma$ we have

$$X_\sigma = \bigcup_{\tau \preceq \sigma} T \cdot x_\tau.$$

1.2 Cox rings

In this section we introduce Cox rings together with the Cox construction of a toric variety as a good quotient of an affine space by a quasitorus action. Our main reference for this section is [2].

1.2.1 Divisors

Let X be an irreducible normal variety defined over $\mathbb{K}$. Denote by $\mathrm{WDiv}(X)$ the group of *Weil divisors* of X, that is the free abelian group generated by the irreducible hypersurfaces of X. An element $D \in \mathrm{WDiv}(X)$ is a finite sum $\sum_i a_i D_i$, where $a_i \in \mathbb{Z}$ and D_i is an irreducible hypersurface of X for each i. In this case the union $\cup_i D_i \subseteq X$ is the *support* of the divisor D. Given a non-zero rational function $f \in \mathbb{K}(X)^*$ and a prime hypersurface D of X we define the order of f at D, denoted by $\mathrm{ord}_D(f)$, as follows. If f belongs to the local ring $\mathcal{O}_{D,X}$, (cf. [11, Ch. I, Ex 3.13]) then $\mathrm{ord}_D(f)$ is the length of the $\mathcal{O}_{D,X}$-module $\mathcal{O}_{D,X}/\langle f \rangle$, and otherwise one writes $f = g/h$ with $g, h \in \mathcal{O}_{D,X}$ and defines the order of f to be the difference of the orders of g and h. Denote by

$$\mathrm{div}(f) := \sum_{D \text{ prime}} \mathrm{ord}_D(f) \cdot D$$

the principal divisor defined by f. The map $\mathbb{K}(X)^* \to \mathrm{WDiv}(X)$, $f \mapsto \mathrm{div}(f)$ is a homomorphism of groups and its image, denoted by $\mathrm{PDiv}(X)$, is called the subgroup of *principal divisors of X*. Two divisors of X are *linearly equivalent* if their difference is a principal divisor. Divisors which are locally principal are called *Cartier divisors*, denoted by $\mathrm{CDiv}(X)$. Cartier divisors form a subgroup of the group of Weil divisors which clearly contains the subgroup $\mathrm{PDiv}(X)$. The variety X is *factorial* if $\mathrm{CDiv}(X)$ equals $\mathrm{WDiv}(X)$ and it is $\mathbb{Q}$-factorial if $\mathrm{CDiv}(X)$ has finite index in $\mathrm{WDiv}(X)$. Denote by

$$\mathrm{Cl}(X) := \frac{\mathrm{WDiv}(X)}{\mathrm{PDiv}(X)} \qquad \mathrm{Pic}(X) := \frac{\mathrm{CDiv}(X)}{\mathrm{PDiv}(X)}$$

the *divisor class group* and the *Picard group* of X respectively.

1.2.2 Sheaves of divisorial algebras

A Weil divisor $D = \sum_i a_i D_i$ is *effective* if $a_i \geq 0$ for any i. To any Weil divisor D one associates a sheaf $\mathcal{O}_X(D)$ of $\mathcal{O}_X$ modules as follows: given an open subset $U \subseteq X$ define

$$\Gamma(U, \mathcal{O}_X(D)) := \{f \in \mathbb{K}(X)^* : (\operatorname{div}(f) + D)|_U \geq 0\} \cup \{0\},$$

where the meaning of the above restriction to U is to remove all the prime divisors of the support of $\operatorname{div}(f) + D$ which are contained in $X \setminus U$. To a subgroup $K \subseteq \operatorname{WDiv}(X)$ we associate the *sheaf of divisorial algebras*:

$$\mathcal{S} := \bigoplus_{D \in K} \mathcal{S}_D \qquad \mathcal{S}_D := \mathcal{O}_K(D).$$

The multiplication map is the multiplication of rational functions of X. Observe that the sheaf $\mathcal{S}$ is graded by the abelian group K and its degree D part is the sheaf $\mathcal{S}_D$. We recall [2, Remark 1.3.1.5] that if K has finite rank, for example s, then the algebra of global sections $\Gamma(X, \mathcal{S})$ is isomorphic to a K-graded subalgebra of the ring $\mathbb{K}(X)[T_1^{\pm 1}, \dots, t_s^{\pm 1}]$ and in particular it is a domain. Whenever K is finitely generated and $\mathcal{S}$ is locally of finite type we define the relative spectrum of the sheaf $\mathcal{S}$:

$$\widetilde{X} := \operatorname{Spec}_X \mathcal{S}$$

as the gluing of the spectra of algebras of sections of $\mathcal{S}$ over an affine open cover of X (see [2, § 1.3.2]). The K-grading of $\mathcal{S}$ is equivalent to an action of $H := \operatorname{Spec} \mathbb{K}[K]$ on the variety $\widetilde{X}$. There is a canonical surjective affine morphism $p \colon \widetilde{X} \to X$ that is invariant under the H-action and the pullback $p^* \colon \mathcal{O}_X \to (p_* \mathcal{O}_{\widetilde{X}})^H$ is an isomorphism. This matches the following definition.

Definition 1.2.1. Let G be an algebraic group acting on a variety U. A morphism of varieties $\pi \colon U \to V$ is called a *good quotient* for the action if it is affine and G-invariant and the pullback $\pi^* \colon \mathcal{O}_V \to (\pi_* \mathcal{O}_U)^G$ is an isomorphism.

The variety X is a good quotient of $\widetilde{X}$ by the action of H, this means that there exists an affine H-invariant surjective morphism $p \colon \widetilde{X} \to X$ such that the pullback $p^* \colon \mathcal{O}_X \to (p_* \mathcal{O}_{\widetilde{X}})^H$ is an isomorphism.

Example 1.2.2. Let $X = \mathbb{P}^1$ be the projective line and let $K := \langle D \rangle$, where $D := \{\infty\}$. Let $X_0 = \mathbb{K}$ and $X_1 = \mathbb{K}^* \cup \{\infty\}$ be the two affine charts. There are two isomorphisms

$$\begin{array}{ll} \mathbb{K}[T_0^{\pm 1}, T_1] \to \Gamma(X_0, \mathcal{S}) & \mathbb{K}[T_0^{\pm 1}, T_1]_n \ni f \mapsto f(1, z) \in \Gamma(X_0, \mathcal{S}_{nD}) \\ \mathbb{K}[T_0, T_1^{\pm 1}] \to \Gamma(X_1, \mathcal{S}) & \mathbb{K}[T_0, T_1^{\pm 1}]_n \ni f \mapsto f(z, 1) \in \Gamma(X_0, \mathcal{S}_{nD}) \end{array}$$

Thus, both the corresponding spectra are isomorphic to $\mathbb{K}^2$, the gluing takes place along $(\mathbb{K}^*)^2$ and gives $\widetilde{X} = \mathbb{K}^2 \setminus \{(0, 0)\}$.

1.2.3 Cox sheaves

Our next aim is to define the *Cox sheaf* of a normal algebraic variety with finitely generated divisor class group $\mathrm{Cl}(X)$ and whose global invertible regular functions are just the constants. As a module the Cox sheaf is the following

$$\mathcal{R} = \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{O}_X(D)$$

where a representative divisor is chosen for each divisor class. If the divisor class group is free abelian then any choice of a subgroup $K \subseteq \mathrm{WDiv}(X)$, such that the class map $K \to \mathrm{Cl}(X)$ is an isomorphism, gives a sheaf of divisorial algebras $\mathcal{S}$ which is isomorphic to $\mathcal{R}$. This provides a structure of algebra to $\mathcal{R}$ and it possible to show that the isomorphism class of these sheaf of algebras does not vary with the choice of such a subgroup K. If on the other hand the divisor class group is not free abelian, then the construction of the Cox sheaf is more involved. We briefly recall it here. Let $K \subseteq \mathrm{WDiv}(X)$ be a finitely generated subgroup such that the class map $K \to \mathrm{Cl}(X)$ is surjective and let K^0 be its kernel. This means that each element of K^0 is a principal divisor. Fix a homomorphism of groups $\chi\colon K^0 \to \mathbb{K}(X)^*$ such that

$$\mathrm{div}(\chi(D)) = D \quad \text{for all } D \in K^0.$$

Such isomorphism can be easily given by fixing a basis of K^0 (which, being the subgroup of the free abelian group $\mathrm{WDiv}(X)$, is free abelian itself) and assigning to each element D of this basis a rational function whose divisor is D itself. Let $\mathcal{S}$ be the divisorial algebra defined by K and define the ideal sheaf $\mathcal{I}$ of $\mathcal{S}$ which is locally generated by elements of the form $1 - \chi(D)$, for $D \in K^0$. The Cox sheaf is the quotient $\mathcal{R} = \mathcal{S}/\mathcal{I}$. It is graded by the divisor class group $\mathrm{Cl}(X)$ since $\mathcal{S}$, which is graded by K, admits a coarsening of the grading given by the surjection $K \to \mathrm{Cl}(X)$ and the ideal sheaf $\mathcal{I}$ is generated by degree zero homogeneous elements, with respect to the $\mathrm{Cl}(X)$-grading. The effect of taking quotient for $\mathcal{I}$ is that of identifying two graded parts $\mathcal{S}_D$ and $\mathcal{S}_{D'}$ whenever D is linearly equivalent to D'. In this way in the Cox sheaf survives exactly one addendum $\mathcal{O}_X(D)$ for each class $[D] \in \mathrm{Cl}(X)$. It is possible to show [2, Proposition 1.4.2.2] that the isomorphism class of the Cox sheaf does not depend on the choice of the subgroup $K \subseteq \mathrm{WDiv}(X)$ or on the choice of the character χ. Moreover for any open subset $U \subseteq X$ one has an isomorphism $\Gamma(U, \mathcal{S})/\Gamma(U, \mathcal{I}) \simeq \Gamma(U, \mathcal{S}/\mathcal{I})$.

1.2.4 Cox rings

Let X be a normal algebraic variety with finitely generated divisor class group $\mathrm{Cl}(X)$ and whose global invertible regular functions are just the constants. The *Cox ring* of X is the ring of global sections of the Cox sheaf:

$$\mathcal{R}(X) := \Gamma(X, \mathcal{R}).$$

It is possible to show that the Cox sheaf is locally of finite type if either the Cox ring $\mathcal{R}(X)$ is finitely generated or X is $\mathbb{Q}$-factorial. If this is the case then it is possible to show that taking the relative spectrum gives an irreducible normal variety

$$\widehat{X} := \operatorname{Spec}_X(\mathcal{R})$$

called the *characteristic space* of X. The $\mathrm{Cl}(X)$-grading of the Cox sheaf is equivalent to an action of the quasitorus $H_X := \operatorname{Spec}\mathbb{K}[\mathrm{Cl}(X)]$ on the variety $\widehat{X}$. The corresponding good quotient morphism is denoted by

$$p_X\colon \widehat{X} \to X.$$

It is possible to show [2, Proposition 1.6.1.6] that the morphism p_X is an étale H_X-principal bundle over the subset X_{reg} of smooth points of X. Moreover the preimage of a codimension ≥ 2 subset of X has codimension ≥ 2 in $\widehat{X}$.

In case the Cox ring is finitely generated its spectrum $\overline{X}$ is acted as well by the quasitorus H_X and there is an equivariant embedding $\widehat{X} \subseteq \overline{X}$ induced by the sheaf defined by $\mathcal{R}(X)$ to the sheaf $\mathcal{R}$. It is possible to show that the Zariski closed subset $\overline{X}\setminus\widehat{X}$, called the *irrelevant locus*, has codimension at least two. The defining ideal of the irrelevant locus is the *irrelevant ideal* $\mathcal{J}_{\mathrm{irr}}(X) \subseteq \mathcal{R}(X)$.

1.2.5 Toric varieties

In this section we discuss Cox rings of toric varieties. Recall that for us by definition a toric variety $X = X(\Sigma)$ is a normal algebraic variety equipped with an effective action of an algebraic torus $T = (\mathbb{K}^*)^n$ of the same dimension of X. In order to define the Cox ring of X we further require that the only global invertible functions of X are constants, this is equivalent to ask the support of the defining fan $\Sigma \subseteq N_{\mathbb{Q}}$ of X to be full-dimensional. The divisor class group of X is always finitely generated as can be deduced by the following diagram with exact rows and columns [2, Proposition 2.1.2.7]:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "P^*"] \arrow[d, "u\mapsto \mathrm{div}(\chi^u)"] & E \arrow[r, "Q"] \arrow[d, "e\mapsto D(e)"] & K \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \mathrm{PDiv}^T(X) \arrow[r] & \mathrm{WDiv}^T(X) \arrow[r] & \mathrm{Cl}(X) \arrow[r] & 0,
\end{tikzcd} \tag{1.2.1}$$

where $\mathrm{PDiv}^T(X)$, $\mathrm{WDiv}^T(X)$ are the T-invariant principal and Weil divisors of X respectively, $M \simeq \mathbb{Z}^n$ is the group of characters of the torus T and $E \simeq \mathbb{Z}^{|\Sigma(1)|}$, where $\Sigma(1)$ are the one-dimensional cones of the fan Σ. P is the matrix whose columns are the generators of the one dimensional cones of Σ. The Cox sheaf of X can be constructed by forming the E-graded sheaf of divisorial algebras $\mathcal{S}$ and the character $\chi\colon M \to \mathbb{K}(X)^*$ defined by

$u \mapsto \chi^u$. Let $F = \mathrm{Hom}(E, \mathbb{Z})$ be the dual of E and let δ be the positive orthant of the rational vector space $F_{\mathbb{Q}}$. Define the fan $\widehat{\Sigma} \subseteq F_{\mathbb{Q}}$ as

$$\widehat{\Sigma} := \{\widehat{\sigma} \subseteq \delta \,:\, P(\widehat{\sigma}) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

By [2, Theorem 2.1.3.2] the Cox ring of X is isomorphic to the K-graded polynomial ring

$$\mathcal{R}(X) \simeq \mathbb{K}[E \cap \delta^{\vee}].$$

Moreover, the characteristic space is the toric morphism $p_X \colon X(\widehat{\Sigma}) \to X(\Sigma)$ induced by the natural fan map $\widehat{\Sigma} \to \Sigma$. A basic example of Cox ring of a toric variety is provided by the construction of the projective space $\mathbb{P}^n$. In this case $M \simeq \mathbb{Z}^n$, $E \simeq \mathbb{Z}^{n+1}$ and $K \simeq \mathbb{Z}$. The characteristic space is the toric morphism $\mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ defined by $x \mapsto [x]$.

1.3 Subvarieties of toric varieties

Using the language of Cox rings we have the following lemma.

Proposition 1.3.1. *Let Z be a toric variety with Cox ring $\mathcal{R}(Z) = \mathbb{K}[\delta^{\vee} \cap E]$ and fan Σ. Let X be a subvariety of Z defined by a homogeneous binomial $g \in \mathcal{R}(Z)$, let $v \in E$ be the difference of the exponents of the two monomials of g and let $u \in M$ be such that $P^*(u) = v$. Then X is a toric variety with fan $u^{\perp} \cap \Sigma$.*

Proof. Let Z be a toric variety with characteristic space $p \colon \widehat{Z} \to Z$ and total coordinate space $\overline{Z} = \mathbb{A}^r$. Recall that the characteristic space is a good quotient with respect to the action of the torus $H_X := \operatorname{Spec} \mathbb{K}[\mathrm{Cl}(Z)]$. A subvariety of $\overline{Z}$ defined by an H_X-invariant binomial $g \in \mathbb{K}[T_1, \ldots, T_r]$ defines a toric subvariety X of Z which is the image of $\widehat{X} = V(g) \cap \widehat{Z}$ via p, as shown in the following commutative diagram

$$\begin{tikzcd} \widehat{X} \arrow[r] \arrow[d, "p"] & \widehat{Z} \arrow[d, "p"] \\ X \arrow[r] & Z \end{tikzcd}$$

We explain how to construct the fan of X. The difference v of the exponents of the binomial g is an element of the character lattice E of the torus $\mathbb{T}^r \subseteq \overline{Z}$. The quotient lattice $E/\langle v \rangle$ is the character lattice of the toric variety $V(g) \subseteq \overline{Z}$. Since g is H_X-invariant, the vector $v \in E$ is in the kernel of the grading map $Q \colon E \to \mathrm{Cl}(Z)$ and thus we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Cl}(Z) \arrow[r, equal] & \mathrm{Cl}(Z) \\
E \arrow[u, "Q"] \arrow[r] & E/\langle v\rangle \arrow[u, "Q_v"'] \\
M \arrow[u, "P*"] \arrow[r] & M/\langle u\rangle \arrow[u, "P_v^*"']
\end{tikzcd}$$

where the left column is the dual of the Cox construction for Z and $u \in M$ is such that $P^*(u) = v$. The image of the positive orthant $\delta^\vee$ of $E_{\mathbb{Q}}$ is the cone $\sigma^\vee \subseteq (E/\langle v\rangle)_{\mathbb{Q}}$ which defines the affine toric variety $V(g)$, that is $\Gamma(V(g), \mathcal{O}) \cong \mathbb{K}[\sigma^\vee \cap E/\langle v\rangle]$. Dualizing we get the following commutative diagram

$$\begin{tikzcd}
v^\perp \arrow[r] \arrow[d, "P_v"'] & F \arrow[d, "P"] \\
u^\perp \arrow[r] & N
\end{tikzcd}$$

The dual cone σ of $\sigma^\vee$ equals the intersection $v^\perp \cap (F_{\mathbb{Q}})_{\geq 0}$. □

Chapter 2

T-varieties and polyhedral data

Recall from the previos chapter that the algebraic group $T := (\mathbb{K}^*)^m$, with component-wise multiplication, is called the m-dimensional torus. A group action of T on a set X is called *effective* if the identity is the only element $t \in T$ for which $t \cdot x = x$ holds for all $x \in X$. In this chapter we will be looking at the following type of variety

Definition 2.0.2. A *T-variety* is a normal algebraic variety X of dimension n coming with an effective group action of an m-dimensional torus T. The *complexity* of the T-variety is defined as $n - m$.

These varieties serve as a generalization of toric varieties, since a toric variety is simply a T-variety of complexity 0. T-varieties also admit a polyhedral description, which was given by K. Altmann and J. Hausen for the affine case in [3] and later, together with H. Suß, in [4] for the non-affine case. The purpose of this chapter is to serve as a summary of their main definitions and theorems, which will be needed for the later sections of this work. We will also recall a consturcion by N. Ilten and H. Suß in [19] that helps to simplify notation in the case of complexity one. As a final addition, we will treat the particular case of two-dimensional T-varieties.

2.1 Polyhedral data for affine T-varieties

In this section we explain how a set of polyhedra defines an affine T-variety. The main reference is [3].

2.1.1 Polyhedral divisors

Let N be a lattice of rank n, and $M = \operatorname{Hom}(N, \mathbb{Z})$ its dual. We denote by $\langle -, - \rangle \colon M \times N \to \mathbb{Z}$ the perfect pairing defined by $(u, v) \mapsto \langle u, v \rangle := u(v)$ and by $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ the corresponding rational vector spaces. A *polyhedron* in $N_{\mathbb{Q}}$ is an intersection of finitely many affine half spaces in $N_{\mathbb{Q}}$. If we require the supporting hyperplane of any half space to be a linear subspace, the polyhedron is called a *cone*. This is consistent with our previous definition of cone in 1.1.4.

Let $\Delta \subseteq N_{\mathbb{Q}}$ be a polyhedron. The set

$$\mathrm{tail}(\Delta) := \{v \in N_{\mathbb{Q}} : tv + \Delta \subseteq \Delta,\ \forall t \in \mathbb{Q}\}$$

is a cone called the *tailcone* of Δ. If $\mathrm{tail}(\Delta) = \sigma$, Δ is called a *σ-polyhedron.*

Given two polyhedrons Δ, Δ', we define their *Minkowski sum* as the polyhedron

$$\Delta + \Delta' = \{v + v' : v \in \Delta, v' \in \Delta'\}.$$

For any cone σ, we consider the empty set as a σ-polyhedron, with the rule

$$\Delta + \emptyset = \emptyset + \Delta = \emptyset$$

for all polyhedrons Δ.

Definition 2.1.1. Let Y be a normal variety and σ a cone. A *polyhedral divisor* on Y is a formal sum

$$\mathcal{D} := \sum_P \Delta_P \otimes P,$$

where P runs over all prime divisors of Y and the Δ_P are all σ-polyhedrons such that $\Delta_P = \sigma$ for all but finitely many P.

The sum of two polyhedral divisors $\mathcal{D} := \sum \Delta_P \otimes P$, $\mathcal{D}' := \sum \Delta'_P \otimes P$ is defined naturally as

$$\mathcal{D} + \mathcal{D}' := \sum (\Delta_P + \Delta'_P) \otimes P.$$

Let $\mathcal{D} := \sum \Delta_P \otimes P$ be a polyhedral divisor on Y, with tailcone σ. For every $u \in \sigma^\vee$ we define the evaluation

$$\mathcal{D}(u) := \sum_{\substack{P \subset Y \\ \Delta_P \neq \emptyset}} \min_{v \in \Delta_P} \langle u, v \rangle\, P \in \mathrm{WDiv}_{\mathbb{Q}}(\mathrm{Loc}\,\mathcal{D})$$

where $\mathrm{Loc}\,\mathcal{D} := Y \setminus (\cup_{\Delta_P = \emptyset} P)$ is the *locus* of $\mathcal{D}$.

We will now give the definition of a special type of polyhedral divisor.

Definition 2.1.2. Let Y be a normal variety. A *proper polyhedral divisor*, also called a *pp-divisor* is a polyhedral divisor $\mathcal{D}$ on Y, such that

(i) $\mathcal{D}(u)$ is Cartier and semiample for every $u \in \sigma^\vee \cap M$.

(ii) $\mathcal{D}(u)$ is big for every $u \in (\operatorname{relint} \sigma^\vee) \cap M$.

Now, let $\mathcal{D}$ be a polyhedral divisor on a semiprojective (i.e. projective over some affine variety) variety Y, and $\mathcal{D}$ having tailcone $\sigma \subseteq N_{\mathbb{Q}}$. This defines an M-graded algebra

$$A(\mathcal{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(\operatorname{Loc} \mathcal{D}, \mathcal{O}(\mathcal{D}(u))).$$

The affine scheme $X(\mathcal{D}) := \operatorname{Spec} A(\mathcal{D})$ comes with a natural action of $\operatorname{Spec} \mathbb{K}[M]$. If $\mathcal{D}$ is a proper polyhedral divisor, then the following result holds.

Theorem 2.1.3. [3, Theorem 3.1 and Theorem 3.4]. *Let $\mathcal{D}$ be a pp-divisor on a normal variety Y. Then $X(\mathcal{D})$ is an affine T-variety of complexity equal to* $\dim Y$. *Moreover, every affine T-variety arises like this.*

Example 2.1.4. To make it a bit more clear that the fact that $\mathcal{D}$ is proper cannot be ovelooked, consider the following situation. Let N be a two dimensional lattice and $\sigma \subset N_{\mathbb{Q}}$ be the cone generated by the rays passing through $(1, 0)$ and $(1, 1)$. Let $L \subset N_{\mathbb{Q}}$ be the line segment from $(1, 0)$ to $(1, 1)$ and take Δ as the σ-polyhedron $L + \sigma$.

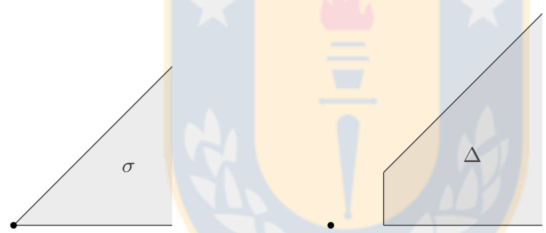


Figure 2.1: The cone σ and the σ-polyhedron Δ
(Source: own elaboration)

Consider the polyhedral divisor over $\mathbb{P}^1$

$$\mathcal{D} = \Delta \otimes V(x),$$

where x is the standard coordinate of $\mathbb{P}^1$. The dual cone $\sigma^\vee$ is generated by the vectors $(0, 1)$ and $(1, -1)$. The subcones where the evaluation $\min_{v \in \Delta} \langle -, v \rangle$ is linear are $\operatorname{cone}\langle (1, 0), (0, 1) \rangle$ and $\operatorname{cone}\langle (1, -1), (1, 0) \rangle$. Thus

$$A(\mathcal{D}) = \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(\mathbb{P}^1, \mathcal{O}(\mathcal{D}(u))) = \mathbb{K}\left[\tfrac{1}{x}\chi^{(1,0)}, \chi^{(0,1)}, \chi^{(1,-1)}\right].$$

Therefore, the corresponding T-variety $X(\mathcal{D})$ is $\mathbb{A}^3$ with an action of the 2-dimensional torus given by

$$(t,s)\cdot(x,y,z) = (tx, sy, ts^{-1}z).$$

Consider now $\Delta' := L' + \sigma$, where L' is the segment joining $(0,0)$ and $(0,1)$. The polyhedral divisor $\mathcal{D}' := \Delta' \otimes V(x)$ is not proper, because by taking $u_0 := (1,-1) \in \sigma^\vee \cap M$, we have $\mathcal{D}'(u_0) = -V(x)$, which is not a big divisor on $\mathbb{P}^1$. In this case we get

$$A(\mathcal{D}') = \mathbb{K}[\chi^{(1,0)}, \chi^{(0,1)}],$$

so $X(\mathcal{D}')$ is 2-dimensional with an action of the 2-dimensional torus and thus the statement of Theorem 2.1.3 does not hold for such a $\mathcal{D}$.

2.2 The non-affine case

We now study non-affine T-varieties by taking sets of polyhedral divisros as shown in [4], much like we did with fans of cones for toric varieties.

2.2.1 Divisorial fans

Non-affine T-varieties are obtained by gluing affine T-varieties coming from pp-divisors in a combinatorial way that we will briefly recall in this subsection.

Consider two polyhedral divisors $\mathcal{D} = \sum \Delta_P \otimes P$ and $\mathcal{D}' = \sum \Delta'_P \otimes P$ on Y, with tailcones σ and σ' respectively and such that $\Delta_P \subseteq \Delta'_P$ for every P. We then have an inclusion

$$\bigoplus_{u \in \sigma'^\vee \cap M} \Gamma(\operatorname{Loc}\mathcal{D}, \mathcal{O}(\mathcal{D}'(u))) \subseteq \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(\operatorname{Loc}\mathcal{D}, \mathcal{O}(\mathcal{D}(u))),$$

which induces a morphism $X(\mathcal{D}) \to X(\mathcal{D}')$. We say that $\mathcal{D}$ is a *face* of $\mathcal{D}'$, denoted by $\mathcal{D} \prec \mathcal{D}'$, if this morphism is an open embedding.

Definition 2.2.1. A *divisorial fan* on Y is a finite set $\mathcal{S}$ of pp-divisors on Y such that for every pair of divisors $\mathcal{D} = \sum \Delta_P \otimes P$ and $\mathcal{D}' = \sum \Delta'_P \otimes P$ in $\mathcal{S}$, we have $\mathcal{D} \cap \mathcal{D}' \in \mathcal{S}$ and $\mathcal{D} \succ \mathcal{D} \cap \mathcal{D}' \prec \mathcal{D}'$, where $\mathcal{D} \cap \mathcal{D}' := \sum (\Delta_P \cap \Delta'_P) \otimes P$.

This definition allows us to glue affine T-varieties via

$$X(\mathcal{D}) \longleftarrow X(\mathcal{D} \cap \mathcal{D}') \longrightarrow X(\mathcal{D}'),$$

thus resulting in a scheme $X(\mathcal{S})$. For the following theorem see [4, Theorem 5.3 and Theorem 5.6].

Theorem 2.2.2. *The scheme $X(\mathcal{S})$ constructed above is a T-variety of complexity equal to $\dim Y$. Every T-variety can be constructed like this.*

2.2.2 Constructing divisorial fans

Given a toric variety X with fan Σ, denote by T the torus acting on it. Taking a subtorus action, i.e an injective map $T' \hookrightarrow T$, we can see X as a T'-variety of positive complexity. There is a simple method for constructing a divisorial fan for X from Σ, first shown in [3, §11].

Let P be the cokernel of the map $N' \to N$ associated to the subtorus injection. We have the exact sequence of lattices

$$0 \longrightarrow N' \underset{s}{\rightleftarrows} N \xrightarrow{P} N'' \longrightarrow 0,$$

where s is some projection. Also note that N'' is torsion-free because T' acts effectively. Let $\Sigma' \subset N''_{\mathbb{Q}}$ be the coarsest fan refining the images by P of all the cones in Σ. Define the toric variety $Y := Y_{\Sigma'}$. For each cone $\sigma \in \Sigma$, consider the polyhedral divisor on Y

$$\mathcal{D}^\sigma := \sum_{\rho \in \Sigma'(1)} s(P^{-1}(\rho) \cap \sigma) \otimes D_\rho,$$

where D_ρ is the invariant prime divisor on Y corresponding to ρ. The set $\{\mathcal{D}^\sigma\}_{\sigma \in \Sigma}$ constitutes a divisorial fan for X with an effective action of T'.

Example 2.2.3. Note that for any toric fan Σ corresponding to a complete toric surface, the method described above gives $Y = \mathbb{P}^1$ and the slices of the resulting divisorial fan $\mathcal{S} = \{\mathcal{D}^\sigma\}_{\sigma \in \Sigma}$ can easily be obtained by tracing two lines parallel to the direction of the subtorus action.

Consider, for example, the fan of $\mathbb{P}^2$ with a subtorus action given by $(1, 0)$, as shown in the image below.

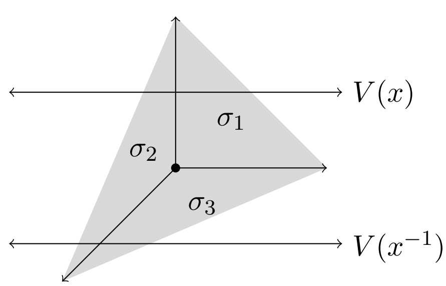


Figure 2.2: The fan of $\mathbb{P}^2$ with a subtorus action
(Source: own elaboration)

We can easily see that the corresponding divisorial fan has only one non-trivial slice. More specifically, we can list the pp-divisors that make it up. Namely,

$$\begin{aligned}
\mathcal{D}_1 &= \emptyset \otimes V(x^{-1}) \\
\mathcal{D}_2 &= (-\infty, -1] \otimes V(x^{-1}) \\
\mathcal{D}_3 &= \emptyset \otimes V(x) + [-1, \infty) \otimes V(x^-1)
\end{aligned}$$

and their intersections.

2.2.3 Marked fansy divisors on curves

Let Y be a projective variety. Let $\mathcal{D} = \sum_P \Delta_P \otimes P$ be a pp-divisor on Y, and let $y \in Y$, set

$$\mathcal{D}_y := \sum_{P \ni y} \Delta_P.$$

Then for a divisorial fan $\mathcal{S}$ on Y, we define the *slice* of $\mathcal{S}$ at y as $\{\mathcal{D}_y : \mathcal{D} \in \mathcal{S}\}$.

Consider the following example.

Example 2.2.4. Let y be the variable on $\mathbb{P}^1$. Take the divisorial fan $\mathcal{S}_1$ over $\mathbb{P}^1$ generated by

$$\begin{aligned}
\mathcal{D}_1 &= [1, \infty) \otimes V(y) + \emptyset \otimes V(y^{-1}) \\
\mathcal{D}_1' &= \emptyset \otimes V(y) + [1, \infty) \otimes V(y^{-1})
\end{aligned}$$

And let $\mathcal{S}_2$ be the divisorial fan on $\mathbb{P}^1$ generated by

$$\mathcal{D}_2 = [1, \infty) \otimes V(y) + [1, \infty) \otimes V(y^{-1}).$$

We can easily see that for every $y \in \mathbb{P}^1$, the slices of $\mathcal{S}_1$ and $\mathcal{S}_2$ on y agree. However, $X(\mathcal{S}_2)$ is an affine T-variety, whereas $X(\mathcal{S}_1)$ is not affine.

This example illustrates the fact that the slices of a divisorial fan do not give enough information about the corresponding T-variety. Two divisorial fans $\mathcal{S}$ and $\mathcal{S}'$ can have the same slices, yet $X(\mathcal{S}) \neq X(\mathcal{S}')$. In [19], this issue is fixed for the case of complete complexity-one T-varieties with the following definition.

Definition 2.2.5. A *marked fansy divisor* on a curve Y is a formal sum $\Xi = \sum_{P \in Y} \Xi_P \otimes P$ together with a fan Σ and a subset $C \subseteq \Sigma$, such that

(i) Ξ_P is a complete polyhedral subdivision of $N_{\mathbb{Q}}$, and $\operatorname{tail}(\Xi_P) = \Sigma$ for all $P \in Y$.

(ii) For $\sigma \in C$ of full dimension, $\sum \Delta_P^\sigma \otimes P$ is a pp-divisor, where Δ_P^σ is the only σ-polyhedron of Ξ_P.

(iii) For $\sigma \in C$ of full dimension and $\tau \prec \sigma$, we have $\tau \in C$ if and only if $(\sum_P \Delta_P^\sigma) \cap \tau \neq \emptyset$.

(iv) If $\tau \prec \sigma$ and $\tau \in C$, then $\sigma \in C$.

We say that the cones in C are *marked*.

Given a complete divisorial fan $\mathcal{S}$ on a curve Y, we can define the marked fansy divisor $\Xi = \sum_P \mathcal{S}_P \otimes P$ with marks on all the tailcones of divisors $\mathcal{D} \in \mathcal{S}$ having complete locus. We denote it by $\Xi(\mathcal{S})$. The following is proved in [19, Proposition 1.6].

Proposition 2.2.6. *For any marked fansy divisor Ξ, there exists a complete divisorial fan $\mathcal{S}$ with $\Xi(\mathcal{S}) = \Xi$. If two divisorial fans $\mathcal{S}$, $\mathcal{S}'$ satisfy $\Xi(\mathcal{S}) = \Xi(\mathcal{S}')$, then $X(\mathcal{S}) = X(\mathcal{S}')$.*

2.2.4 Cox ring from a divisorial fan

Definition 2.2.7. Lef Ξ be a fansy divisor over Y and $Z \subset Y$ be a prime divisor. Denote by Ξ_Z the fan of all tailcones of Ξ.

1. The *index* of a vertex $v \in \Xi_Z$ is the minimal positive integer $\mu(v)$ such that $\mu(v)v \in N$.

2. A vertex $v \in \Xi_Z$ is called extremal if there is a $\mathcal{D} \in \Xi$ with $v \in \mathcal{D}_Z$ such that $\mathcal{O}(\mathcal{D}(u))$ is big on Z for any $u \in ((\mathcal{D}_Z - v)^\vee)^\circ$. The set of all extremal vertices $v \in \Xi_Z$ is denoted by $\Xi_Z^\times$.

3. We call a ray $\varrho \in \Xi_Y$ extremal if there is a $\mathcal{D} \in \Xi$ with $\varrho \in \mathcal{D}_Y$ such that $\mathcal{O}(\mathcal{D}(u))$ is big on Y for any $u \in ((\varrho^\perp \cap \omega)^\vee)^\circ$. The set of all extremal rays $\varrho \in \Xi_Y$ is denoted by $\Xi_Y^\times$.

4. We say that the prime divisor Z is irrelevant if $\Xi_Z^\times$ is empty, and we denote by $Y^\circ \subset \cup_{\mathcal{D}\in\Xi} \mathrm{Loc}\,(\mathcal{D})$ the open subset obtained by removing all irrelevant Z.

Theorem 2.2.8. [12, Theorem 4.8] *Let Ξ be a fansy divisor on $Y = \mathbb{P}^1$ having non trivial slices $\Xi_{a_0}, \ldots, \Xi_{a_r}$. Then the cox ring of $X = X(\Xi)$ is given by*

$$\mathbb{K}[S_\varrho, T_v; \varrho \in \Xi_Y^\times,\ v \in \Xi_{a_0}^\times \dot\cup \ldots \dot\cup \Xi_{a_r}] \Big/ \left\langle \sum_{i=0}^{r} \beta_i T^{\mu_i};\ \beta \in \mathrm{Rel}(\widetilde{a_0}, \ldots \widetilde{a_r}) \right\rangle$$

where $\widetilde{a_i} \in \mathbb{K}^2$ represents $a_i \in \mathbb{P}^1$, we set $T^{\mu_i} := \prod_{v \in \Xi_{a_i}^\times} T_v^{\mu(v)}$ and $\mathrm{Rel}(\widetilde{a_0}, \ldots, \widetilde{a_r})$ is a basis for the space of linear relations among $\widetilde{a_0}, \ldots, \widetilde{a_r}$.

2.3 $\mathbb{K}^*$-surfaces

We now look at the case of a T-variety of dimension two and complexity one. We call this type of variety a $\mathbb{K}^*$-*surface.*

Definition 2.3.1. Let X be a $\mathbb{K}^*$-surface and let $x \in X$ be a fixed point for the torus action. We say that the fixed point x is:

- *elliptic* if there is an invariant open neighborhood U of x such that this point lies in the closure of every orbit of U,
- *parabolic* if x lies on a curve made entirely of fixed points,
- *hyperbolic* otherwise.

Before proceeding recall that a *categorical quotient* is a morphism $\pi\colon X \to Y$, where X is a T-variety, which is invariant with respect to the T-action and satisfies the following universal property: any other T-invariant morphism $X_0 \to Y$ where X_0 is a T-variety factors uniquely through π.

Let $\pi : X \to Y$ be a morphism of T-varieties and $\{U_i\}_i$ an invariant affine cover for X. If $\pi|_{U_i}$ is a good quotient for all i, then π is a categorical quotient.

Proposition 2.3.2. *Let X be a complete $\mathbb{K}^*$-surface. The following statements are equivalent.*

(i) There exists a morphism $X \to Y$ onto a smooth projective curve Y that is a categorical quotient for the $\mathbb{K}^*$-action.

(ii) X has no elliptic fixed points.

(iii) X is given by a marked fansy divisor without marks.

Proof. We prove three implications.

(i) $\Rightarrow$ *(ii)*. Let $x \in X$ be an elliptic fixed point. There is an open neighborhood $x \in U$ such that x lies in the closure of every orbit in U. Therefore, there cannot be a categorical quotient $X \to Y$ because the open set U would be mapped to a single point; a contradiction.

(ii) $\Rightarrow$ *(iii)*. Assume that the marked fansy divisor defining X has a mark. There is then an open affine chart of X given by a pp-divisor $\mathcal{D}$ with complete locus Y. The zero-degree component of $A(\mathcal{D})$ is $\Gamma(Y, \mathcal{O}) \cong \mathbb{K}$, meaning that the degrees of the generators of $A(\mathcal{D})$ as an algebra are either all positive (if $\operatorname{tail}\mathcal{D} = \mathbb{Q}_{\geq 0}$) or all negative (if $\operatorname{tail}\mathcal{D} = \mathbb{Q}_{\leq 0}$). In either case, we can take local coordinates such that the origin x_0 lies in the closure of every orbit, i.e. x_0 is an elliptic fixed point.

(iii) $\Rightarrow$ *(i)*. Let $\mathcal{S}$ be a divisorial fan on a smooth projective curve Y such that the marked fansy divisor for X is $\Xi(\mathcal{S})$. Since there are no marks, each $\mathcal{D} \in \mathcal{S}$ has

an affine locus, so there is a morphism $\pi_{\mathcal{D}} : X(\mathcal{D}) \to \operatorname{Loc}\mathcal{D}$ coming from the inclusion $A(\mathcal{D})_0 := \Gamma(\operatorname{Loc}\mathcal{D}, \mathcal{O}) \subseteq A(\mathcal{D})$. These maps glue together to create a morphism $\pi\colon X \to Y$ because the completeness of X implies that $\{\operatorname{Loc}\mathcal{D} : \mathcal{D} \in \mathcal{S}\}$ is an affine open covering of Y. Each $\pi_{\mathcal{D}}$ is a good quotient because $A(\mathcal{D})_0$ is precisely the subalgebra of invariants of $A(\mathcal{D})$. Thus π is a categorical quotient. $\square$

Let X be a complete $\mathbb{K}^*$-surface. There exist two invariant subsets $F^- \subseteq X$ and $F^+ \subseteq X$, called *sink* and *source* respectively, such that there is an open set $U \subseteq X$ where the closure in X of every orbit in U intersects both F^- and F^+. There are finitely many orbits outside of $U \cup F^- \cup F^+$, that are called the *special* orbits. The source can be either an elliptic point or an irreducible curve of parabolic points; the same holds true for the sink. Every fixed point outside of $F^+ \cup F^-$ is hyperbolic.

Now, consider a complete smooth $\mathbb{K}^*$-surface having no elliptic points. Denote by $E_1, \dots, E_r$ the closures of the special orbits. F. Orlik and P. Wagreich (cf. [25]) construct a graph having vertex set $\{E_1, \dots, E_r, F^+, F^-\}$ and two vertices are joined by an edge if and only if the two corresponding curves intersect. Each vertex carries a weight equal to the self-intersection number of the curve that it represents. This graph takes the following form.

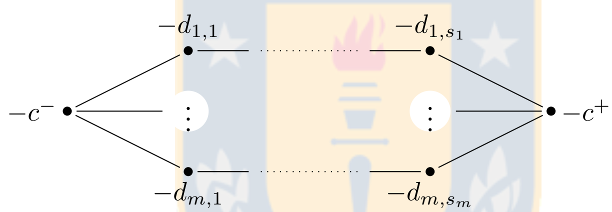


Figure 2.3: The graph of the $\mathbb{K}^*$-surface (Source: own elaboration)

The $d_{i,j}$'s are all positive and satisfy that the Hirzebruch-Jung continued fraction $[d_{i,1}, d_{i,2}, \dots, d_{i,s_i}]$ equals 0 for every $1 \le i \le m$. On the other hand, our surface is given by a marked fansy divisor, without marks, on a smooth projective curve Y.

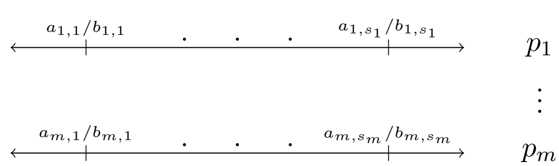


Figure 2.4: Marked fansy divisor without marks (Source: own elaboration)

Where the smoothness of the surface implies that $b_{i,j}a_{i,j+1} - a_{i,j}b_{i,j+1} = 1$ for every

$1 \leq i \leq m$, $1 \leq j < s_i$, as well as $b_{i,1} = b_{i,s_i} = 1$, as shown in [30, Theorem 3.3]. It turns out, as well, that each $b_{i,j}$ with $j > 1$ equals the Hirzebruch-Jung continued fraction $[d_{i_1}, d_{i,2}, \ldots, d_{i,j-1}]$.

Chapter 3

Euler sequence for T-varieties

In [11, V.8.13], we are presented with an exact sequence, namely

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0,$$

which allows the author to make several computations regarding differentials. This sequence is named *Euler sequence* and in [8] it is generalized to any smooth toric variety X coming from a fan Σ whose rays span the whole ambient lattice. The exact sequence in question is

$$0 \longrightarrow \Omega_X \longrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_{\mathbb{P}^n}(-D_\rho) \longrightarrow \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \longrightarrow 0, \tag{3.0.1}$$

where D_ρ is the invariant divisor associated to the ray ρ.

In this section we attempt to obtain a similar result in the case of $\mathbb{K}^*$-surfaces. We briefly recall that a $\mathbb{K}^*$-surface X without elliptic points (see Definition 2.3.1) is equipped with an equivariant morphism $\pi\colon X \to Y$, onto a smooth projective curve Y, which admits two distinguished sections called the source F^+ and the sink F^- of X. This allows us to form a divisor $E_{\mathcal{S}}$, which is numerically equivalent to $mF - \sum_{i\in I} E_i$, where m is the number of reducible fibers of π and $\{E_i : i \in I\}$ is the set of the prime components of such fibers (see Definition 3.2.1). This chapter's main results are Theorem 1 where we construct a Euler-type exact sequence for the cotangent sheaf of a $\mathbb{K}^*$-surface and Theorem 2, where we show that the only rigid rational $\mathbb{K}^*$-surfaces without elliptic points are Fano. This is a partial generalization of [17, Corollary 2.8] where the case of toric surfaces is considered.

The results of this chapter are contained in the paper [22].

3.1 Generalities about deformations

We will go over some basic definition about deformations. We are especially interested in the importance of the first cohomology group of the tangent sheaf.

3.1.1 Basic concepts in deformation theory

Definition 3.1.1. Let $f\colon \mathcal{X} \to S$ be a morphism of schemes. Let $s \in S$ be a point. In the following fibre product diagram

$$\begin{tikzcd}
\mathcal{X}(s) = \operatorname{Spec}(\mathbb{K}(s)) \times_S \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\operatorname{Spec}(\mathbb{K}(s)) \arrow[r] & S
\end{tikzcd}$$

the scheme $\mathcal{X}(s)$ is called the *scheme-theoretic fiber* of f over s.

Example 3.1.2. In the following morphism of schemes

$$f\colon \mathcal{X} := \operatorname{Spec}\left(\mathbb{K}[t,x,y]/\langle y^2 - x^3 - x - t\rangle\right) \;\longrightarrow\; S := \operatorname{Spec}\left(\mathbb{K}[t]\right),$$

if $a \in \mathbb{K}$, let $s_a \in S$ be the closed point corresponding to the maximal ideal $\langle t-a\rangle \subset \mathbb{K}[t]$. Then the fiber of f over s_a is the curve $\mathcal{X}(s_a) = \operatorname{Spec}\left(\mathbb{K}[x,y]/\langle y^2 - x^3 - x - a\rangle\right)$.

Definition 3.1.3. Let X be a scheme over $\mathbb{K}$. A *deformation* of X over a scheme S is a flat surjective morphism of schemes $\pi : \mathcal{X} \to S$ that fits in a cartesian diagram

$$\begin{tikzcd}
X = \operatorname{Spec}(\mathbb{K}) \times_S \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\
\operatorname{Spec}(\mathbb{K}) \arrow[r, "s"] & S
\end{tikzcd}$$

If S is algebraic, then for each rational point $t \in S$, the scheme-theoretic fiber $\mathcal{X}(t)$ is also called a *deformation* of X.

The deformation is called *local* if $S = \operatorname{Spec}(A)$ where A is a local $\mathbb{K}$-algebra with residue field $\mathbb{K}$ and $s \in S$ is the closed point. A local deformation is *infinitesimal* if A is artinian. It is a *first-order* deformation if $A = \mathbb{K}[t]/\langle t^2\rangle$

Definition 3.1.4. If $\pi\colon \mathcal{X} \to S$ and $\pi'\colon \mathcal{X}' \to S$ are two deformations of X, we say that π and π' are *isomorphic* if there is a morphism $\phi : \mathcal{X} \to \mathcal{X}'$ inducing the identity over X and such that the diagram

$$\begin{tikzcd}
& \mathcal{X} \arrow[dr, "\phi"] \arrow[dd, "\pi"] & \\
X \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{X}' \arrow[dd, "\pi'"] \\
& S \arrow[dr, equal] & \\
\operatorname{Spec}(\mathbb{K}) \arrow[ur] \arrow[rr] & & S
\end{tikzcd}$$

is commutative.

Example 3.1.5. Let $C = \{(x,y) \in \mathbb{A}^2 : \ xy - 1 = 0\}$ and $W = \{(x,y,t) \in \mathbb{A}^3 : \ xy - t = 0, \ t \neq 0\}$. Let π and π' be the projections of W and $C \times (\mathbb{A} - \{0\})$ respectively over the first coordinate. Then the deformations

$$\begin{tikzcd} C \arrow[r] \arrow[d] & W \arrow[d, "\pi"] \\ \operatorname{Spec}(\mathbb{K}) \arrow[r, "1"] & \mathbb{A}^1 - \{0\} \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r] \arrow[d] & C \times (\mathbb{A} - \{0\}) \arrow[d, "\pi'"] \\ \operatorname{Spec}(\mathbb{K}) \arrow[r, "1"] & \mathbb{A}^1 - \{0\} \end{tikzcd}$$

of C over $\mathbb{A}^1 - \{0\}$, are clearly isomorphic by taking

$$\phi \colon W \to C \times (\mathbb{A} - \{0\}), \quad (x,y,t) \mapsto ((x,y),t)\,.$$

By $\operatorname{Def}_X(S)$, we denote the set of isomorphism classes of deformations of X over S. When X is a T-variety, a deformation $\pi : \mathcal{X} \to S$ is called *T-invariant* if the torus action of X extends over the whole π-diagram, such that it acts trivially on S and such that all maps are equivariant.

Theorem 3.1.6. [28, Theorem 2.4.1] *If X is a smooth scheme, there is an isomorphism of vector spaces*

$$\kappa : \operatorname{Def}_X(\mathbb{K}[t]/(t^2)) \xrightarrow{\sim} \mathrm{H}^1(X, T_X),$$

where T_X is the tangent sheaf of X.

Let X be a smooth algebraic variety and consider a deformation $\pi\colon \mathcal{X} \to S$ of X. Giving $\varphi \in T_{S,s}$ is equivalent to giving a morphism $\varphi\colon \operatorname{Spec}\big(\mathbb{K}[t]/\langle t^2\rangle\big) \to S$ with image s. Pulling back the deformation by φ, we obtain a deformation

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \times_S \operatorname{Spec}\big(\mathbb{K}[t]/\langle t^2\rangle\big) \arrow[d, "\pi'"] \\ \operatorname{Spec}(\mathbb{K}) \arrow[r, "s"] & \operatorname{Spec}\big(\mathbb{K}[t]/\langle t^2\rangle\big) \end{tikzcd}$$

which, by Theorem 3.1.6, corresponds to an element of $\mathrm{H}^1(X, T_X)$.

Definition 3.1.7. The above construction gives a map

$$T_{S,s} \to \mathrm{H}^1(X, T_X)$$

called the *Kodaira-Spencer map* of the deformation π.

Note that in the case $S = \operatorname{Spec}\mathbb{K}[x]$, the morphism φ above is uniquely defined up to scalar multiplication, so the image of the Kodaira-Spencer map is determined by a single element in $\mathrm{H}^1(X, T_X)$.

3.2 Euler sequence for $\mathbb{K}^*$-surfaces

3.2.1 Constructing the sequence

Let X be a complete smooth $\mathbb{K}^*$-surface having no elliptic points given by a marked fansy divisor Ξ with no marks, like the one depicted in Figure 2.4. Each fraction $a_{i,j}/b_{i,j}$ in the figure corresponds to a divisor $E_{i,j}$ which is the closure of a special orbit of X.

Definition 3.2.1. The multiplicity of the divisor $E_{i,j}$ is the non-negative integer

$$\mu(E_{i,j}) := b_{i,j} - 1.$$

According to the definition of the divisor $E_{\mathcal{S}}$ given in the introduction of the chapter, the equality $E_{\mathcal{S}} := \sum_{i,j} \mu(E_{i,j}) \cdot E_{i,j}$ holds, where $1 \leq i \leq m$ and $1 \leq j \leq s_i$.

Let Ω_X and Ω_Y be the cotangent sheaves of X and Y respectively, where Y is defined as in Proposition 2.3.2. As in Section 2.3, let F^- and F^+ denote the source and sink of X. Let $Z \subseteq X$ be the set of hyperbolic fixed points of X. In what follows we will call $\mathcal{F}_X$ the sheaf $\mathcal{O}_{F^-} \oplus \mathcal{O}_{F^+} \oplus \mathcal{O}_Z$.

Lemma 3.2.2. *Let X be a complete smooth $\mathbb{K}^*$-surface without elliptic points. There is an exact sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \pi^*\Omega_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(E_{\mathcal{S}}) \xrightarrow{\imath} \Omega_X \xrightarrow{\alpha} \mathcal{O}_X \longrightarrow \mathcal{F}_X \longrightarrow 0$$

where α is defined by $fdz \mapsto \deg(z)fz$, for any homogeneous local coordinate z with respect to the $\mathbb{Z}$-grading of $\mathcal{O}_X$ induced by the $\mathbb{K}^*$-action and $\imath$ is defined by $dt \otimes f \mapsto fdt$, where t is the pull-back of a local coordinate on Y.

Proof. Let Ξ be a fansy divisor describing X. Each affine chart of X, or an intersection of them, is given by a polyhedron Δ on some slice of Ξ. In other words, it is given by the pp-divisor

$$\mathcal{D} = \Delta \otimes p + \sum_{\mathcal{P}-\{p\}} \emptyset \otimes p,$$

where $\mathcal{P}$ is the set of points of Y with non-trivial slices. We analyze each possible Δ separately.

Case 1. $\Delta = [a_1/b_1, a_2/b_2]$ with $b_1 b_2 \neq 0$. In this case $\mathcal{D}$ defines an open affine subset $X_{\mathcal{D}}$ of X which is the spectrum of the algebra

$$\bigoplus_{u \in \mathbb{Z}} \Gamma(\mathrm{loc}(\mathcal{D}), \mathcal{O}(\mathcal{D}(u))) = S^{-1}\mathbb{K}[x^{a_2}\chi^{-b_2}, x^{-a_1}\chi^{b_1}] \cong S^{-1}\mathbb{K}[z, w] =: R,$$

where x is a regular function of $\mathrm{loc}(\mathcal{D})$ which has a simple zero at p and is non-zero at $\mathrm{loc}(\mathcal{D}) - \{p\}$ and $S \subseteq \mathbb{K}[x^{a_2}\chi^{-b_2}, x^{-a_1}\chi^{b_1}]$ is the multiplicative system defined by degree zero homogeneous polynomials which do not vanish on $\mathrm{loc}(\mathcal{D})$. We have an exact sequence

$$Rdz \oplus Rdw \cong \Omega_R \xrightarrow{\alpha} R \longrightarrow R/I \longrightarrow 0$$

where $I \subseteq R$ is the ideal $\langle \deg(z)z, \deg(w)w\rangle$. The restriction of the quotient map $\pi\colon X \to Y$ to the open subset $X_{\mathcal{D}}$ is defined by the inclusion $\mathbb{K}[x] \subseteq R$. Since $x = z^{\deg(w)}w^{\deg(z)} = z^{b_1}w^{b_2}$, the curve $\pi^{-1}(p) \cap X_{\mathcal{D}}$ has two irreducible components which are vertical curves intersecting at the fixed point $q \in Z$ of local coordinates $z = w = 0$. Thus R/I defines the skyscraper sheaf $\mathcal{O}_q$ and we get the first exact sequence from $\mathcal{F}|_{X_{\mathcal{D}}} \cong \mathcal{O}_q$. The sheaf $\pi^*\Omega_Y$ is locally generated by $dx = z^{b_1-1}w^{b_2-1}(b_1wdz + b_2zdw)$, thus we have the desired isomorphism

$$\pi^*\Omega_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X((b_1-1)E_1 + (b_2-1)E_2)|_{X_{\mathcal{D}}} \to \ker(\alpha)|_{X_{\mathcal{D}}},$$

where for $i = 1, 2$, the divisor E_i is the one associated to the fraction a_i/b_i, as explained in the beginning of this section.

Case 2. $\Delta = [a_1, \infty)$. In this case $\mathcal{D}$ defines an open affine subset $X_{\mathcal{D}}$ of X which is the spectrum of the algebra

$$\bigoplus_{u\in\mathbb{Z}_{\geq 0}} \Gamma(\mathrm{loc}(\mathcal{D}), \mathcal{O}(\mathcal{D}(u))) = S^{-1}\mathbb{K}[x, x^{-a_1}\chi] \cong S^{-1}\mathbb{K}[z, w] =: R,$$

where x and S are defined in a similar way as in the first case. Since $x = z$, the curve $\pi^{-1}(p) \cap X_{\mathcal{D}}$ has one irreducible component which is a vertical curve intersecting F^+ at one point. Again we got an exact sequence as above and observe that now $I = \langle w\rangle$. In this case R/I defines the sheaf $\mathcal{O}_{F^+}|_{X_{\mathcal{D}}}$ and we get the first exact sequence from $\mathcal{F}|_{X_{\mathcal{D}}} \cong \mathcal{O}_{F^+}|_{X_{\mathcal{D}}}$. The sheaf $\pi^*\Omega_Y$ is locally generated by $dx = dz$, thus we have an isomorphism

$$\pi^*\Omega_Y|_{X_{\mathcal{D}}} \to \ker(\alpha)|_{X_{\mathcal{D}}}.$$

Case 3. $\Delta = (-\infty, a_2]$. This is similar to the previous case and we omit the details.

Case 4. $\Delta = \{a/b\}$. In this case $\mathcal{D}$ defines an open affine subset $X_{\mathcal{D}}$ of X which is the spectrum of the algebra

$$\bigoplus_{u\in\mathbb{Z}} \Gamma(\mathrm{loc}(\mathcal{D}), \mathcal{O}(\mathcal{D}(u))) = S^{-1}\mathbb{K}[x^k\chi^{-l}, (x^{-a}\chi^b)^{\pm 1}] \cong S^{-1}\mathbb{K}[z, w^{\pm 1}] =: R,$$

where x and S are defined in a similar way as in the first case and c, d are integers such that $bk - la = 1$. Since $x = z^{\deg(w)}w^{\deg(z)} = z^b w^l$, the curve $\pi^{-1}(p) \cap X_{\mathcal{D}}$ has an irreducible component which is a vertical curve which has empty intersection with $F^+ \cup F^- \cup Z$. Again we get an exact sequence as in the first case with $I = \langle z, w^{\pm 1}\rangle = R$. In this case $R/I = 0$

and we get the first exact sequence from $\mathcal{F}|_{X_{\mathcal{D}}} = 0$. The sheaf $\pi^*\Omega_Y$ is locally generated by $dx = z^{b-1}w^{l-1}(bwdz + lzdw)$, thus we have an isomorphism (since w is a unit in this chart)

$$\pi^*\Omega_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X((b-1)E)|_{X_{\mathcal{D}}} \to \ker(\alpha)|_{X_{\mathcal{D}}},$$

where E is the divisor associated to the fraction a/b.

□

We define the sheaf $\mathcal{Q}_\alpha := \Omega_X / \ker(\alpha)$. Now, maintaining the same hypothesis and notation as above, we prove the following.

Lemma 3.2.3. *There is a short exact sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \mathcal{Q}_\alpha \longrightarrow \mathcal{G} \longrightarrow \mathbb{Z}^{r+1} \otimes \mathcal{O}_X \longrightarrow 0,$$

where

$$\mathcal{G} = \mathcal{O}(-F^-) \oplus \mathcal{O}(-F^+) \oplus \Big(\, (\oplus_{i,j}\mathcal{O}(-E_{i,j})) \,/\, (\oplus_{i=1}^{m} \oplus_{j=1}^{s_i} \mathcal{O}(-E_{i,j} - E_{i,j+1}))\, \Big).$$

Proof. Let us consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d, dotted] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{Q}_\alpha \arrow[r] \arrow[d, dotted] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{F}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, dotted] & \mathbb{Z}^{r+2} \otimes \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{F}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}^{r+1} \otimes \mathcal{O}_X \arrow[r, equal] \arrow[d, dotted] & \mathbb{Z}^{r+1} \otimes \mathcal{O}_X \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

where the top row comes from Lemma 3.2.2. The middle row is the direct sum of the fundamental short exact sequences

$$0 \longrightarrow \mathcal{O}(-F^{\pm}) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{F^{\pm}} \longrightarrow 0$$

together with the following exact sequences (cf. [6]) for each $E_i \cap E_j = p \in Z$

$$0 \longrightarrow \mathcal{O}(-E_i - E_j) \longrightarrow \mathcal{O}(-E_i) \oplus \mathcal{O}(-E_j) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_p \longrightarrow 0$$

where we replace the first two sheaves with their quotient to obtain short sequences. The middle column is simply the exact sequence of modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(1,\dots,1)} \mathbb{Z}^{r+2} \longrightarrow \mathbb{Z}^{r+1} \longrightarrow 0$$

after tensoring by $\mathcal{O}_X$. The exactness of the sequences, together with the commutativity of both squares (easy to check), ensures the existence of an exact sequence on the left column. □

We can now prove the first theorem stated in the introduction.

Proof of Theorem 1. It is direct from Lemmas 3.2.2 and 3.2.3 since $\operatorname{im}(\alpha)$ in $\mathcal{O}_X$ is isomorphic to $\mathcal{Q}_\alpha$. □

Proposition 3.2.4. *The following holds:* $\mathcal{E}xt^1(\mathcal{Q}_\alpha, \mathcal{O}_X) \cong \mathcal{O}_Z$.

Proof. By the definition of $\mathcal{Q}_\alpha$, Lemma 3.2.2 and the long exact sequence for ext sheaves we have $\mathcal{E}xt^1(\mathcal{Q}_\alpha, \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{F}, \mathcal{O}_X)$. Since the functor $\mathcal{E}xt^i$ commutes with finite direct sums, it is enough to show that $\mathcal{E}xt^2(\mathcal{O}_{F^\pm}, \mathcal{O}_X) = 0$ and $\mathcal{E}xt^2(\mathcal{O}_{p_i}, \mathcal{O}_X) \cong \mathcal{O}_p$ for any $p \in Z$. Taking the long exact $\mathcal{E}xt$-sequence of the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-F^\pm) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{F^\pm} \longrightarrow 0$$

and using the fact that $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{O}_X) = 0$ for any $i > 0$, by [11, Pro. III.6.3(b)], we get $\mathcal{E}xt^1(\mathcal{O}_X(-F^\pm), \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{O}_{F^\pm}, \mathcal{O}_X)$. By [11, Pro. III.6.7] we conclude that the these sheaves are the zero sheaf, proving the first vanishing. To prove the second isomorphism observe that for each $p \in Z$ lying in the intersection $E_i \cap E_j$ we have the following exact sequence of sheaves (cf. [6])

$$0 \longrightarrow \mathcal{O}_X(-E_i - E_j) \longrightarrow \mathcal{O}_X(-E_i) \oplus \mathcal{O}_X(-E_j) \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

Denoting by $\mathcal{N}$ the quotient sheaf $\mathcal{O}_X(-E_i) \oplus \mathcal{O}_X(-E_j)/\mathcal{O}_X(-E_i - E_j)$ we deduce $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{O}_p, \mathcal{O}_X)$ and the fact that $\mathcal{E}xt^1(\mathcal{N}, \mathcal{O}_X)$ is the cokernel of the map $\mathcal{O}_X(E_i) \oplus \mathcal{O}_X(E_j) \to \mathcal{O}_X(E_i + E_j)$ induced by φ taking tensor product with $\mathcal{O}_X(E_i + E_j)$. This proves the statement. □

3.3 Applications

We present some applications of Theorem 1 to the study of $\mathrm{H}^1(X, T_X)$.

3.3.1 Results on $\mathrm{H}^1(X, T_X)$

Lemma 3.3.1. *Let $\varphi\colon \tilde{X} \to X$ be the blow-up of a smooth projective variety at a point $p \in X$. Then $h^1(\tilde{X}, T_{\tilde{X}}) \geq h^1(X, T_X)$.*

Proof. Since φ is a blow-up it follows that $R^i\varphi_* T_{\tilde{X}}$ vanishes for any $i > 0$. Thus the equality $h^i(\tilde{X}, T_{\tilde{X}}) = h^i(X, \varphi_* T_{\tilde{X}})$ holds for any i by [11, Exercise III.8.1] and we conclude by the following exact sequence of sheaves

$$0 \longrightarrow \varphi_* T_{\tilde{X}} \longrightarrow T_X \longrightarrow T_p \longrightarrow 0.$$

$\square$

Proof of Theorem 2. We begin by showing (1) $\Rightarrow$ (2). Consider the good quotient map $\pi\colon X \to Y$. Assume first that the curve Y has positive genus. If π has only irreducible fibers, X is a ruled surface so by [27, Theorem 4] we have $h^1(X, T_X) > 0$. This still holds if there are reducible fibers, by Lemma 3.3.1, because X would be a blow-up of one of such ruled surfaces. Thus, Y must necessarily be rational.

We show now that X contains no invariant rational curves C with $C^2 = -n \leq -2$. Suppose such a curve exists. From Lemma 3.2.2, after tensoring by $\mathcal{O}(K_X + C)$, we have an exact sequence

$$0 \to \pi^*(\Omega_{\mathbb{P}^1}) \otimes \mathcal{O}(E_{\mathcal{S}} + K_X + C) \to \Omega_X(K_X + C) \to \mathrm{im}(\alpha) \otimes \mathcal{O}(K_X + C) \to 0.$$

Let us compute some cohomology groups for these sheaves. Assume that $K_X + C$ is linearly equivalent to an effective divisor. From the genus formula we have

$$(K_X + C) \cdot C = 2\mathrm{g}(C) - 2 = -2 < 0,$$

so by applying [2, Proposition V.1.1.2] we see that C must be in the base locus of $|K_X + C|$, meaning $K_X + C \sim C + E'$ for some effective divisor E'. This would imply that K_X is linearly equivalent to an effective divisor, a contradiction because X is rational and smooth. Thus $h^0(X, K_X + C) = 0$. Since $\mathrm{im}(\alpha) \otimes \mathcal{O}(C + K)$ injects into $\mathcal{O}(C + K)$, then also

$$h^0(X, \mathrm{im}(\alpha) \otimes \mathcal{O}(C + K)) = 0.$$

If F is a general fiber of π, the genus formula yields $F \cdot K_X = -2$. The product $F \cdot C$ equals at most 1 (where the equality holds if C is the source or sink curve), so

$$F \cdot (-2F + E_{\mathcal{S}} + K_X + C) = F \cdot K_X + F \cdot C < 0.$$

Then $h^0(X, \pi^*(\Omega_{\mathbb{P}^1}) \otimes \mathcal{O}(E_{\mathcal{S}} + K_X + C)) = h^0(X, -2F + E_{\mathcal{S}} + K_X + C) = 0$. Going back to the exact sequence, we can now deduce that $h^0(X, \Omega_X(K_X + C)) = 0$, and due to Serre's duality we conclude $h^2(X, T_X(-C)) = 0$.

Consider now the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Tensoring by T_X gives a new exact sequence

$$0 \longrightarrow T_X(-C) \longrightarrow T_X \longrightarrow T_X|_C \longrightarrow 0.$$

From the vanishing at H^2 shown above, there is a surjection $H^1(X, T_X) \to H^1(X, T_X|_C)$, so it suffices to show that $h^1(X, T_X|_C) \neq 0$ to prove the non-existence of this curve C, but this comes directly from the exact sequence

$$0 \longrightarrow T_C \longrightarrow T_X|_C \longrightarrow N_{C|X} \longrightarrow 0$$

and the fact that $h^1(X, T_C) = h^2(X, T_C) = 0$ and $h^1(X, N_{C|X}) = n - 1$.

We showed that any invariant rational curve of X has self-intersection ≥ -1. Since the classes of these curves generate the Mori cone of X (see [2]) we conclude that $-K_X$ is ample and thus X is del Pezzo. Moreover by [13, Proposition 5.9] del Pezzo $\mathbb{K}^*$-surfaces without elliptic fixed points are toric.

The proof of (2) $\Rightarrow$ (1) is a consequence of [17, Corollary 2.8]. $\square$

Chapter 4

Deformations of smooth toric varieties

The topic of deformations of toric varieties has been studied by K. Altmann in [5] and A. Mavlyutov in [24]. In the affine case they describe toric deformations in a combinatorial way via polyhedral decompositions of linear sections of the defining cone of the toric variety. The theory of polyhedral divisors is later developed in [3] and [4] as a generalization of toric varieties to T-varieties, i.e. varieties coming with a torus action. N. Ilten and R. Vollmert make use of the language of polyhedral divisors in [20] to describe deformations of T-varieties of complexity one. Their method involves decomposing the polyhedral data of the varieties, similar to what Altmann did for the affine case. Moreover, in the case of smooth toric varieties they also prove that such deformations are in correspondence with a generating set of the space of infinitesimal deformations of the starting variety. In [12], we are presented with an explicit way to compute the Cox ring of a T-variety, starting from its polyhedral representation. This serves as the main connection between the work of N. Ilten and R. Vollmert and the work shown in this chapter.

This chapter is devoted to the study of deformations of smooth toric varieties from the point of view of Cox rings, in a similar spirit of [24]. Starting from a toric variety X and some extra combinatorial data, we describe the Cox ring of a complexity one variety $\mathcal{X}$ which fits into to a one-parameter deformation $\mathcal{X} \to \mathbb{A}^1$ of X, as shown in Theorem 4.2.1. After that, we proceed to study the corresponding Kodaira-Spencer map in Theorem 4.2.3.

We show that the variety $\mathcal{X}$ matches the variety introduced in [20] which was described with the language of polyhedral divisors. Moreover, and much like what was done for polyhedral divisors, we show that the deformations we describe generate the space of infinitesimal deformations of X. As applications of this theory we study deformations of scrolls and deformations of hypersurfaces of smooth toric varieties.

The results of this chapter are contained in the paper [23].

4.1 Preliminaries

We fix some of the notations that we will be using for the rest of the chapter.

4.1.1 The tangent sheaf of a toric variety

Let X be a smooth complete toric variety with defining fan $\Sigma \subseteq N_{\mathbb{Q}}$ and character group M. By the Euler exact sequence that we saw in (3.0.1), the cohomology group of T_X are graded by M. In particular

$$H^1(X, T_X) = \bigoplus_{m \in M} H^1(X, T_X)_m.$$

Definition 4.1.1. (cf. [18, §2.1]) Let $m \in M$ be such that there exists $\varrho \in \Sigma(1)$ with $m(\varrho) = -1$, where with abuse of notation we identify the one dimensional cone ϱ with its primitive generator. Define the graph $\Gamma_\varrho(m)$ whose set of vertices is

$$\mathrm{Vertices}(\Gamma_\varrho(m)) := \{\varrho' \in \Sigma(1) \setminus \{\varrho\} \, : \, m(\varrho') < 0\},$$

and two vertices are joined by an edge if and only if they are rays of a common cone of Σ. If C is a proper component of $\Gamma_\varrho(m)$ we say that the triple (m, ϱ, C) is *admissible.*

To any admissible triple (m, ϱ, C) one can associate a cocycle of $H^1(X, T_X)_m$ in the following way. Define a derivation $\partial_{m,\varrho} \in \mathrm{Der}(\mathbb{K}[M], \mathbb{K}[M])$ by

$$\partial_{m,\varrho} \colon \mathbb{K}[M] \to \mathbb{K}[M] \qquad \chi^u \mapsto u(\varrho)\chi^{u+m}.$$

The announced cocycle is

$$\xi(m, \varrho, C) = \{\alpha(\sigma, \tau) \cdot \partial_{m,\varrho} \, : \, \sigma, \tau \in \Sigma(n)\} \in H^1(X, T_X)_m \tag{4.1.1}$$

where $\alpha(\sigma, \tau)$ equals 1 if $\sigma(1) \cap C$ is non-empty and $\tau(1) \cap C$ is empty, it equals -1 if the roles of σ and τ are exchanged and it equals 0 otherwise.

Proposition 4.1.2. [20, Thm 6.5] *The cocycles $\xi(m, \varrho, C)$ span the vector space $H^1(X, T_X)$.*

4.2 Deformations of smooth toric varieties

In what follows, X is a smooth toric variety. Our aim now is to show how to associate to any admissible triple a one parameter deformation $\pi \colon \mathcal{X} \to \mathbb{A}^1$ such that the image of the Kodaira-Spencer map $T_{\mathbb{A}^1} \to H^1(X, T_X)$ associated to π is the original admissible triple.

4.2.1 The deformation space

Let X be a smooth toric variety and let (m, ϱ, C) be an admissible triple. The element $m \in \mathrm{Hom}(N, \mathbb{Z})$ is a homomorphism $N \to \mathbb{Z}$ with kernel K. We let $\gamma\colon \mathbb{Z} \to N$ be the section of m defined by $\gamma(-1) = v_\varrho$ and let $\pi\colon N \to K$ be the corresponding projection. The above maps are encoded in the following exact sequence:

$$0 \longrightarrow K \underset{\pi}{\leftrightarrows} N \underset{\gamma}{\overset{m}{\leftrightarrows}} \mathbb{Z} \longrightarrow 0. \tag{4.2.1}$$

Let $\tilde{N} = \mathbb{Z}^2 \oplus K \oplus \mathbb{Z}$ and $\tilde{M}$ its dual. We define the map

$$\imath\colon N \to \tilde{N} \qquad v \mapsto [m(v), m(v), \pi(v), 0]. \tag{4.2.2}$$

Let $\varrho_1, \dots, \varrho_r$ be the primitive generators of the one-dimensional cones of the fan Σ, let $a_i = m(\varrho_i)$ for any i and let

$$U_1 = \{(1,i) \,:\, a_i > 0\} \qquad U_2 = \{(2,i) \,:\, a_i < 0 \text{ and } \varrho_i \in C \cup \{\varrho\}\}$$
$$U_4 = \{(4,i) \,:\, a_i = 0\} \qquad U_3 = \{(3,i) \,:\, a_i < 0 \text{ and } \varrho_i \notin C\}$$

and let U be the union $U_1 \cup U_2 \cup U_3 \cup U_4$. For any $(j,i) \in U$ we define the row vector $v_j = [a_i \,:\, (j,i) \in U_j]$. Define the matrix $A_j = [\pi(\varrho_i) \,:\, (j,i) \in U_j]$ whose columns are the vectors $\pi(\varrho_j)$. Finally we define the following block matrix

$$P(m, \varrho, C) := \begin{bmatrix} 1 & v_1 & v_2 & 0 & 0 \\ 1 & v_1 & 0 & v_3 & 0 \\ 0 & A_1 & A_2 & A_3 & A_4 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.2.3}$$

where a 0 represents a zero matrix of adequate dimensions, whereas a 1 is simply the number one (1×1 matrix).

From here on, given a one dimensional ray ϱ_s we denote by $i(\varrho_s)$ its index s and by $k(\varrho_s) \in \{1, 2, 3, 4\}$ the index of the set U_k corresponding to the sign of a_s. Given a maximal cone $\sigma \in \Sigma_X$, we will now define the cone indices of a new cone $\tilde{\sigma}$ in such a way that the cones $\{\tilde{\sigma}\}_{\sigma \in \Sigma}$ form the fan of the ambient toric variety in which $\mathcal{X}$ is embedded. The indices are defined as follows: For every $\varrho_i \in \sigma(1)$ we add every possible $(k, i) \in U$ as a cone index for $\tilde{\sigma}$. We also add, if it is not added already, the index $(2, i(\varrho))$ if $\sigma(1) \cap C = \emptyset$ or the index $(3, i(\varrho))$ if $\sigma(1) \cap C \neq \emptyset$. Lastly, we always add 1 as an index for $\tilde{\sigma}$. All this can be summarized as follows:

- if $\sigma(1) \cap C$ is empty then the cone indices for $\tilde{\sigma}$ are:

$$\begin{aligned} &\{1, (2, i(\varrho))\} \cup \{(1, s) \,:\, \varrho_s \in \sigma(1) \text{ y } a_s > 0\} \\ \cup &\{(4, s) \,:\, \varrho_s \in \sigma(1) \text{ y } a_s = 0\} \cup \{(3, s) \,:\, \varrho_s \in \sigma(1) \text{ y } a_s < 0\} \end{aligned}$$

- if $\sigma(1) \cap C$ is non-empty then the cone indices for $\tilde{\sigma}$ are:

$$\{1, (3, i(\varrho))\} \cup \{(1, s) \,:\, \varrho_s \in \sigma(1) \text{ y } a_s > 0\}$$
$$\cup \{(4, s) \,:\, \varrho_s \in \sigma(1) \text{ y } a_s = 0\} \cup \{(2, s) \,:\, \varrho_s \in \sigma(1) \text{ y } a_s < 0\}.$$

Denote by $\tilde{X}$ the toric variety whose fan $\Sigma_{\tilde{X}}$, defined on $\tilde{N}$, is given by the cones $\tilde{\sigma}$ for every $\sigma \in \Sigma_X$. We then define $\mathcal{X} = \mathcal{X}(m, \varrho, C)$ as the T-variety of complexity one embedded in $\tilde{X}$ whose equation in Cox coordinates T_1, T_{ij} with $(i, j) \in U$, is the following trinomial

$$T_1 \prod_{(1,j) \in U_1} T_{1j}^{a_j} - \prod_{(2,j) \in U_2} T_{2j}^{-a_j} + \prod_{(3,j) \in U_3} T_{3j}^{-a_j}. \tag{4.2.4}$$

Recall from subsection 1.2.4 that we denote by $\bar{\mathcal{X}}$ the affine subvariety defined by this trinomial equation in Cox coordinates.

Theorem 4.2.1. *Let (m, ϱ, C) be an admissible triple and let $\mathcal{X} = \mathcal{X}(m, \varrho, C)$ be the T-variety defined above. The inclusion $\mathbb{K}[T_1] \to \mathbb{K}[\bar{\mathcal{X}}]$ defines a T-equivariant morphism $\pi \colon \mathcal{X} \to \mathbb{A}^1$ which is a one-parameter deformation of X, i.e. X is isomorphic to the fiber of π over $0 \in \mathbb{A}^1$.*

Proof. To see that π is indeed a morphism, recall that $\mathcal{X}$ is embedded in a toric variety $\tilde{X}$ whose toric fan $\Sigma_{\tilde{X}}$ has the columns of P as ray generators and maximal cones given by $\{\tilde{\sigma}\}_{\sigma \in \Sigma(n)}$. By taking the projection onto the last coordinate, we map every ray of $\Sigma_{\tilde{X}}$ to 0, except for the one corresponding to T_1, which is mapped onto $\mathbb{Q}_{\geq 0}$. Thus, we have a morphism of toric varieties

$$\mathcal{X} \to \mathbb{A}^1.$$

Let $\mathcal{X}_0$ be the fiber of π resulting by setting $T_1 = 0$. The trinomial in (4.2.4) becomes a binomial $\chi^{v_1} - \chi^{v_2}$, with $v_1, v_2 \in \tilde{N}$. Let $v = v_1 - v_2$ and let $u \in \tilde{M}$ be such that $P^*(u) = v$. Now, $\mathcal{X}_0$ admits an action of the subtorus defined by $u^\perp$. Recall that $T_1 = 0$, so for the action to be effective we must take the subtorus $N_0 := u^\perp \cap (e_{n+2}^*)^\perp$. Since N_0 has the same dimension as $\mathcal{X}_0$, this fiber can be seen as a toric variety having

$$\Sigma_{\mathcal{X}_0} := \Sigma_{\tilde{X}} \cap N_0$$

as fan. It can be shown that $\imath(N) = N_0$: Indeed, a vector $[a, b] \oplus w \oplus [d] \in \tilde{N}$ belongs to N_0 if and only if $a = b$ and $d = 0$, so it is clear that $\imath(N) \subseteq N_0$. Conversely, if the vetor is of the form $[a, a] \oplus w \oplus [0] \in \tilde{N}$, then it is equal to $\imath\left(w + \gamma(a)\right)$. We now wish to prove that the following equality holds

$$\Sigma_X = \Sigma_{\mathcal{X}_0}.$$

Take a cone $\sigma \in \Sigma_X$ and a ray $\tau \in \sigma(1)$ and let v_τ be its primitive generator. If $\tau \in U_1$ or $\tau \in U_4$, then $\imath(v) \in \tilde{\sigma}$ because it is a column of $P(m, \varrho, C)$. Otherwise, $\imath(v)$ is a linear combination of columns of $P(m, \varrho, C)$, one of index $(j_1, \imath(\tau))$ and one of index $(j_2, \imath(\varrho))$ with $\{j_1, j_2\} = \{2, 3\}$, thus we still have $\imath(v) \in \tilde{\sigma}$ in this case. We conclude that $\imath(v) \in \tilde{\sigma} \cap \imath(N) = \tilde{\sigma} \cap N_0$. Due to the completeness of the fans, the fact that $\imath(\sigma) \subset \tilde{\sigma} \cap N_0$ implies that $\Sigma_X = \Sigma_{\mathcal{X}_0}$ as claimed. □

4.2.2 The central fiber

We now describe the embedding $X \to \mathcal{X}$ at the level of Cox rings. We define the following homomorphism of polynomial rings

$$\eta\colon \mathbb{K}[T_{ij} \,:\, (i,j) \in U] \to \mathbb{K}[S_1, \ldots, S_r] \qquad T_{ij} \mapsto \begin{cases} \prod_{(3,j)\in U_2} S_j^{-a_j} & \text{if } i = 2 \text{ and } \varrho_j = \varrho \\ \prod_{(2,j)\in U_3} S_j^{-a_j} & \text{if } i = 3 \text{ and } \varrho_j = \varrho \\ S_j & \text{otherwise} \end{cases}$$

and $T_1 \mapsto 0$. Observe that the variable T_1 is the variable which gives the coordinate on the base $\mathbb{A}^1$ of the deformation.

Proposition 4.2.2. *The homomorphism of polynomial rings η induces an isomorphism $\eta'\colon \mathbb{K}[\bar{\mathcal{X}}]/\langle T_1 \rangle \to \mathcal{R}(X)$ which induces the inclusion $X \to \mathcal{X}$ in Cox coordinates.*

Proof. First of all we observe that the binomal $\prod_{(2,j)\in S_2} T_{2j}^{-a_j} - \prod_{(3,j)\in S_3} T_{3j}^{-a_j}$ is contained in the kernel of η. Moreover since the kernel is a prime principal ideal we conclude that it is generated by the above binomial. Thus, after identifying $\mathbb{K}[T_{ij} \,:\, (i,j) \in U]$ with $\mathbb{K}[\bar{\mathcal{X}}]/\langle T_1 \rangle$, the homomorphism η induces an isomorphism $\eta'\colon \mathbb{K}[\bar{\mathcal{X}}]/\langle T_1 \rangle \to \mathcal{R}(X)$ as claimed. Observe that η' is a graded map with respect to the $\mathrm{Cl}(\mathcal{X})$-grading on the domain and the $\mathrm{Cl}(X)$-grading on the codomain. Denote by

$$P_0 := \begin{bmatrix} v_1 & v_2 & 0 & 0 \\ v_1 & 0 & v_3 & 0 \\ A_1 & A_2 & A_3 & A_4 \end{bmatrix}.$$

the matrix obtained by removing the first column and the first row from $P(m, \varrho, C)$. Define the homomorphism

$$\psi\colon \mathbb{Z}^r \to \mathbb{Z}^{r+1} \qquad e_j \mapsto \begin{cases} e_{(1,j)} & \text{if } a_j > 0 \\ e_{(2,j)} - a_j e_{(3,\varrho)} & \text{if } \varrho_j \in C \cup \{\varrho\} \\ e_{(3,j)} - a_j e_{(2,\varrho)} & \text{if } \varrho_j \in (\Gamma_\varrho(m) \setminus C) \cup \{\varrho\} \\ e_{(4,j)} & \text{if } a_j = 0 \end{cases}. \tag{4.2.5}$$

Observe that ψ fits in the following commutative diagram

$$\begin{tikzcd} \mathbb{Z}^r \arrow[r, "-\psi"] \arrow[d, "P_X"] & \mathbb{Z}^{r+1} \arrow[d, "P_0"] \\ \mathbb{Z}^n \arrow[r, "{(a_1,\ldots,a_n)\mapsto(-a_n,-a_n,a_1,\ldots,a_{n-1})}"] & \mathbb{Z}^{n+1} \end{tikzcd} \tag{4.2.6}$$

where P_X is the P-matrix of the Cox construction of X (see (1.2.1)). Moreover ψ maps the positive orthant of $\mathbb{Z}^r$ into the positive orthant of $\mathbb{Z}^{r+1}$, it maps cones of X into cones of $\mathcal{X}$ and it induces η'. The statement follows. □

4.2.3 The Kodaira Spencer map

Theorem 4.2.3. *Let (m, ϱ, C) be an admissible triple and let $\pi\colon \mathcal{X} \to \mathbb{A}^1$ be the corresponding one-parameter family. The image of π via the Kodaira-Spencer map is the cocycle $\xi(m, \varrho, C) \in H^1(X, T_X)_m$ defined in (4.1.1).*

Proof. The complexity one variety $\mathcal{X}$ is canonically embedded into the toric variety $\tilde{X}$ and the morphism $\pi\colon \mathcal{X} \to \mathbb{A}^1$ is induced by a toric morphism $\tilde{X} \to \mathbb{A}^1$. Let Σ be the fan of the toric variety X. We denote by $\tilde{\Sigma} \subseteq \tilde{N}_{\mathbb{Q}}$ the fan of $\tilde{Z}$. Given a cone $\sigma \in \Sigma$ we denote by $\tilde{\sigma}$ the corresponding cone of $\tilde{\Sigma}$, that is $\tilde{\sigma} \cap \imath(N) = \sigma$, where the map $\imath$ is the one defined in (4.2.2). Let $\tilde{M}$ be the dual of $\tilde{N}$. The trinomial (4.2.4) is locally described in $\mathbb{K}[\tilde{\sigma}^\vee \cap \tilde{M}]$ by a polynomial of the form

$$\chi^{u_4+u_1} - \chi^{u_2} + \chi^{u_3},$$

where $u_4 = [0, \dots, 0, 1]$, so that $\varepsilon = \chi^{u_4}$. We denote by ϱ_σ the primitive generator of the extremal ray of the cone $\tilde{\sigma}$ which is one of the column of the matrix $P(m, \varrho, C)$ whose index is $(2, i(\varrho))$ if $\sigma \cap C$ is non-empty and it is $(3, i(\varrho))$ otherwise. Assume we are in the first case then the following equation holds

$$P(m, \varrho, C)^*(u_2) = v_2,$$

where T^{v_2} is the monomial in Cox coordinates which corresponds to the character χ^{u_2}. The monomial T^{v_2} does not contain any variable $T_{(k,i)}$ such that $\varrho_i \in \sigma(1)$ with the only exception of the variable $T_{(2,i(\varrho))}$ which appears with exponent 1. This implies that u_2 has scalar product 0 with each column of the $P(m, \varrho, C)$ of index (k, i) when $\varrho_i \in \sigma(1) \setminus \{\varrho\}$ and it has scalar product 1 with the column of index $(2, i(\varrho))$. In particular u_2 generates an extremal ray of the smooth cone $\tilde{\sigma}$ and then χ^{u_2} is a variable of the polynomial ring $\mathbb{K}[\tilde{\sigma}^\vee \cap \tilde{M}]$. Analogously, if $\sigma \cap C$ is empty, the character χ^{u_3} is a variable of the ring. Both cases establish an isomorphism

$$\frac{\mathbb{K}[\tilde{\sigma}^\vee \cap \tilde{M}]}{\langle \chi^{u_4+u_1} - \chi^{u_2} + \chi^{u_3} \rangle} \to \mathbb{K}[\tilde{\sigma}^\vee \cap \tilde{M} \cap \varrho_\sigma^\perp].$$

For the rest of this proof, we fix two cones $\sigma, \tau \in \Sigma$. The isomorphism above leads to the following diagram

$$\begin{tikzcd}
\frac{\mathbb{K}[(\tilde\sigma\cap\tilde\tau)^\vee\cap\tilde M]}{\langle\chi^{2u_4},\chi^{u_4+u_1}-\chi^{u_2}+\chi^{u_3}\rangle} \arrow[r, equal] \arrow[d, "\simeq"] & \frac{\mathbb{K}[(\tilde\sigma\cap\tilde\tau)^\vee\cap\tilde M]}{\langle\chi^{2u_4},\chi^{u_4+u_1'}-\chi^{u_2'}+\chi^{u_3'}\rangle} \arrow[d, "\simeq"] \\
\frac{\mathbb{K}[(\tilde\sigma\cap\tilde\tau)^\vee\cap\tilde M\cap\varrho_\sigma^\perp]}{\langle\chi^{2u_4}\rangle} \arrow[d, "\beta"', "\simeq"] & \frac{\mathbb{K}[(\tilde\sigma\cap\tilde\tau)^\vee\cap\tilde M\cap\varrho_\tau^\perp]}{\langle\chi^{2u_4}\rangle} \arrow[d, "\simeq"] \\
\mathbb{K}[(\sigma\cap\tau)^\vee\cap M]\otimes_{\mathbb{K}}\mathbb{K}[\varepsilon] & \mathbb{K}[(\sigma\cap\tau)^\vee\cap M]\otimes_{\mathbb{K}}\mathbb{K}[\varepsilon]
\end{tikzcd}$$

where the map β is defined by the composition $\beta^*\colon M\cap\varrho_\sigma^\perp\to M\to M/\langle u\rangle$ of the inclusion with the projection and observing that $\beta^*(\tilde\sigma)=\sigma$ and $\beta^*(\tilde\tau)=\tau$. Moreover β^* is an isomorphism being $u(\varrho_\sigma)=\pm1$. We have thus constructed an isomorphism

$$\varphi\colon \mathbb{K}[(\sigma\cap\tau)^\vee\cap M]\otimes_{\mathbb{K}}\mathbb{K}[\varepsilon]\to\mathbb{K}[(\sigma\cap\tau)^\vee\cap M]\otimes_{\mathbb{K}}\mathbb{K}[\varepsilon].$$

Let $a\in(\tilde\sigma\cap\tilde\tau)^\vee\cap M$. We will assume that χ^{u_2} is a variable in $\mathbb{K}[\tilde\sigma^\vee\cap\tilde M]$ and $\chi^{u_3'}$ is a variable in $\mathbb{K}[\tilde\tau^\vee\cap\tilde M]$. The other cases work similarly so analyzing only this case is enough. Since $\tau^\vee$ is smooth, we can write $a=v+a(\varrho_\tau)u_3'$ where v is a linear combination of the rays of $\tau^\vee$ different from u_3'. Hence the following hold

$$\begin{aligned}
\beta(\chi^a) &= \beta(\chi^v\chi^{a(\varrho_\tau)u_3'})\\
&= \beta(\chi^v(\varepsilon\chi^{u_1'}+\chi^{u_3'})^n)\\
&= \chi^{\imath^*(a)}+a(\varrho_\tau)\varepsilon\chi^{\imath^*(a)+\imath^*(u_1'-u_3')}
\end{aligned}$$

where the last equality is due to the fact that the argument of β does not contain $\chi^{u_2'}$. Now, observe that $\chi^{u_1-u_3}$ is the monomial T^w where w is the difference between the second and last row of $P(m,\varrho,C)$. This means $u_1-u_3=[0,1,0,\dots,0,-1]$ and therefore $\imath^*(u_1-u_3)=m$. By setting $u=\imath^*(a)$, this shows that φ is defined as

$$\varphi(\chi^u)=a(\varrho_\tau)\chi^{u+m}.$$

The only thing left to prove is that $a(\varrho_\tau)=u(\varrho)$. Simply notice that $a(\varrho_\sigma)=0$, so

$$a(\varrho_\tau)=a(\varrho_\tau+\varrho_\sigma)=a\left(\imath(\varrho)\right)=\imath^*(a)\left(\varrho\right)=u(\varrho).$$

The coefficient $\alpha(\sigma,\tau)$ from (4.1.1) equals 1 in this case and is easily seen to appear when checking the other cases. $\square$

4.3 Polyhedral description

In this section we describe deformations of toric varieties as shown in [20]. Their results are very closely related to the ones found in Section 4.2, but they use a completely different language. Namely, the language of polyhedral divisors, which we summarize here.

4.3.1 Deformations via polyhedral decompositions

In [20, §6], Ilten and Vollmert define one-parameter deformations of smooth toric varieties denoted by $\pi = \pi(m, \varrho, C)$, where m is a lattice vector, ϱ is a ray in a fan and C is a connected component of some graph. The construction is as follows.

Let Σ be a smooth complete fan giving rise to a toric variety $X = X_\Sigma$. By choosing a $m \in M$ and intersecting the hyperplanes $\{v \in N : m(v) = -1\}$ and $\{v \in N : m(v) = 1\}$ with Σ we get two polyheral subdivisions corresponding to the slices $\mathcal{S}_0$ and $\mathcal{S}_\infty$ of a divisorial fan $\mathcal{S}$ on $\mathbb{P}^1$, describing X as a variety of complexity one. Now, choose $\varrho \in \Sigma(1)$ such that $m(\varrho) = -1$ and recall Definition 4.1.1 of the graph $\Gamma_\varrho(m)$, whose set of vertices is

$$\{\tau \in \Sigma(1) : \tau \neq \varrho, m(\tau) < 0\}$$

and whose edges join two vertices whose corresponding one-dimensional rays lie in a common cone. Assume $\Gamma_\varrho(m)$ has at least two connected components, and let C be one of them. This choice induces a one parameter deformation on X as follows.

Each polyhedron $\Delta \in \mathcal{S}_0$ will be decomposed as $\Delta = \Delta^0 + \Delta^1$. If Δ contains a vertex coming from a ray in C, take $\Delta^0 = \operatorname{tail}\Delta$ and $\Delta^1 = \Delta$. If Δ contains no such vertex, take $\Delta^0 = \Delta$ and $\Delta^1 = \operatorname{tail}\Delta$. The sets $\{\Delta^0\}_{\Delta\in\mathcal{S}_0}$ and $\{\Delta^1\}_{\Delta\in\mathcal{S}_0}$ define new polyhedral subdivisions $\mathcal{S}_0^0$ and $\mathcal{S}_0^1$ such that $\mathcal{S} = \mathcal{S}_0^0 + \mathcal{S}_0^1$. Let $\tilde{\mathcal{S}}$ be the divisorial fan on $\mathbb{A}^1 \times \mathbb{P}^1$ whose only non-trivial slices are $\mathcal{S}_0^0$ at $V(y)$, $\mathcal{S}_0^1$ at $V(y-x)$ and $\mathcal{S}_\infty$ at $V(y^{-1})$, where we are using coordinates $(x, y) \in \mathbb{A}^1 \times \mathbb{P}^1$. Then $\mathcal{X} := X(\tilde{\mathcal{S}})$ comes with a morphism $\pi : \mathcal{X} \to \mathbb{A}^1$ which is a one-parameter deformation of X. This deformation is called $\pi(m, \varrho, C)$.

Proposition 4.3.1. *The deformation $\pi(m, \varrho, C)$ is the same as the one described in Theorem 4.2.1.*

Proof. We consider the $\mathbb{K}^*$-action on $\mathbb{A}^1 \times \mathbb{P}^1$ given by $t \cdot (x, y) = (tx, ty)$. This allows us to describe this surface with a divisorial fan $\mathcal{Z}$ on $\mathbb{P}^1$ whose tailfan is given by a single ray on the positive axis and whose only non-trivial slice has vertices in 0 and 1. By applying [21, Prop 2.1], we describe $\mathcal{X}$ with a new divisorial fan $\mathcal{S}'$ on $\mathbb{P}^1$, having three non-trivial slices. One non-trivial slice of $\mathcal{S}'$ contains $\mathcal{S}_0^0$ at height 0 and a single vertex at height 1, whereas the other two non-trivial slice of $\mathcal{S}'$ are simply $\mathcal{S}_0^1$ and $\mathcal{S}_\infty$ embedded in the corresponding higher dimensional space. Thus, by [12, Corollary 4.9] we see that the cox ring of $\mathcal{X}$ is given precisely by (4.2.4). The matrix $P(m, \varrho, C)$ can be obtained from [12, Pop 4.7]. $\square$

4.4 Applications

We use the language developed in subsection 4.1.1 and in section 4.2 to study deformations of scrolls (see also [29]) and deformations of hypersurfaces of smooth toric varieties (see also [16]).

4.4.1 Deformations of scrolls

Let $n > 1$ be an integer and let $a_1, \ldots, a_n$ be integers. We denote by $\mathbb{F}(a_1, \ldots, a_n)$ the $\mathbb{P}^{n-1}$-bundle (i.e. a scroll) over $\mathbb{P}^1$ associated to the sheaf $\mathcal{O}_{\mathbb{P}^1}(a_1) + \ldots + \mathcal{O}_{\mathbb{P}^1}(a_n)$. It can be defined as the quotient of the space $(\mathbb{A}^2 \backslash 0) \times (\mathbb{A}^n \backslash 0)$ by the following $(\mathbb{K}^*)^2$-action.

$$\begin{aligned} (\lambda, 1) \cdot (t_1, t_2, x_1, \ldots, x_n) &= (\lambda t_1, \lambda t_2, \lambda^{-a_1} x_1, \ldots, \lambda^{-a_n} x_n) \\ (1, \mu) \cdot (t_1, t_2, x_1, \ldots x_n) &= (t_1, t_2, \mu x_1, \ldots, \mu x_n). \end{aligned}$$

The action on the fist two coordinates gives $\mathbb{F}(a_1, \ldots, a_n)$ a morphism over $\mathbb{P}^1$ by projecting on the first factor

$$\begin{tikzcd} (\mathbb{A}^2 \backslash 0) \times (\mathbb{A}^n \backslash 0) \arrow[r] \arrow[d] & \mathbb{F}(a_1, \ldots, a_n) \arrow[d] \\ (\mathbb{A}^2 \backslash 0) \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

Remark 4.4.1. It can be shown (cf. [26, Ch. 2]) that $\mathbb{F}(a_1, \ldots, a_n) \cong \mathbb{F}(b_1, \ldots, b_n)$ if and only if there exists $c \in \mathbb{Z}$ and a permutation $\sigma \in S_n$ such that for every i we have $a_i = b_{\sigma(i)} + c$.

Proposition 4.4.2.

(a) A scroll over $\mathbb{P}^1$ is rigid if and only if it is isomorphic to $\mathbb{F}(a_1, \ldots, a_n)$ where $\{a_i\}_{i=1}^n \subseteq \{0, 1\}$.

(b) Let $X = \mathbb{F}(a_1, \ldots, a_n)$, such that $d := a_1 - a_2 > 2$. For any $d' < d$, the scroll X admits a deformation to $\mathbb{F}(a_1 - d', a_2 + d', a_3, \ldots, a_n)$.

Proof. If $n = 2$, we have a Hirzebruch surface and these results are well known. They can be found for example in [18, §3]. Therefore, we will assume $n \geq 3$. The degree matrix of $\mathbb{F}(a_1, \ldots, a_n)$ is

$$Q = \begin{bmatrix} 1 & 1 & -a_1 & \cdots & -a_n \\ 0 & 0 & 1 & \cdots & 1 \end{bmatrix}.$$

Since the components of the irrelevant ideal are $(1, 2)$ and $(3, \ldots, n+2)$ and $n \geq 3$, it is clear that (ϱ_1, ϱ_2) is the only pair of rays in Σ that are not in a common cone. Thus, if we choose an admissible triple (m, ϱ, C) we must have that

- $m(\varrho_1) < 0$ and $m(\varrho_2) < 0$.
- $m(\varrho_k) = -1$ for some $3 \leq k \leq n+2$. This ϱ_k will be ϱ.
- $m(\varrho_i) \geq 0$ for every $i = 3, \dots, n+2$, $i \neq k$.

since this conditions are the only way to ensure that $\Gamma_\varrho(m)$ has at least two connected components. Now, define $u_i := m(\varrho_i)$ and form the column vector

$$u := (u_1, \dots, u_{n+2})^t.$$

The conditions above become

$$u_1, u_2 < 0; \qquad u_k = -1; \qquad u_i \geq 0,\ i \notin \{1, 2, k\}. \tag{4.4.1}$$

From (1.2.1) we see that $Q(u) = 0$, which when written as a system of equations is equivalent to

$$\begin{cases} u_1 + u_2 - \displaystyle\sum_{i=3}^{n+2} u_i a_{i-2} = 0 & (4.4.2) \\ \displaystyle\sum_{i=3}^{n+2} u_i = 0 & (4.4.3) \end{cases}$$

From (4.4.1) and (4.4.3) we deduce there exists $3 \leq \ell \leq n+2$, with $\ell \neq k$, such that

$$u_\ell = 1 \text{ and } u_i = 0 \text{ for } i \notin \{1, 2, k, \ell\}. \tag{4.4.4}$$

Aditionally, by (4.4.2),

$$a_{k-2} - a_{\ell-2} = -u_1 - u_2 \geq 2. \tag{4.4.5}$$

Thus, $\mathbb{F}(a_1, \dots, a_n)$ has an admissible triple if and only if two of the a_i have distance at least 2, proving part (a).

Assume now that we are in the case where the admissible triple (m, ϱ, C) exists. We will set $C := \{\varrho_1\}$ From now on, to simplify notation without loss of generality, let $k = 3$ and $\ell = 4$. Recall that $u_i = m(\varrho_i)$. Therefore, (4.4.1) and (4.4.4) imply that the trinomial of the Cox ring of the total deformation space $\mathcal{X}$ is

$$T_1 T_{(1,4)} - T_{(2,3)} T_{(2,1)}^{-u_1} + T_{(3,3)} T_{(3,2)}^{-u_2} \tag{4.4.6}$$

The irrelevant ideal $\mathcal{I}$ of $\mathcal{X}$ is given by the following components

$$\begin{aligned} \mathcal{I}_1 &= \langle T_{(2,1)}, T_{(3,2)} \rangle \\ \mathcal{I}_2 &= \langle T_{(2,1)}, T_{(2,3)}, T_{(1,4)} \rangle + \langle T_{(4,5)}, T_{(4,6)}, \dots, T_{(4,n+2)} \rangle \\ \mathcal{I}_3 &= \langle T_{(3,2)}, T_{(3,3)}, T_{(1,4)} \rangle + \langle T_{(4,5)}, T_{(4,6)}, \dots, T_{(4,n+2)} \rangle \\ \mathcal{I}_4 &= \langle T_{(2,3)}, T_{(3,3)}, T_{(1,4)} \rangle + \langle T_{(4,5)}, T_{(4,6)}, \dots, T_{(4,n+2)} \rangle \end{aligned}$$

In the general fiber of the deformation we have $T_1 = t \in \mathbb{K}^*$, so by (4.4.6), the variable $T_{(1,4)}$ can be replaced by the other variables. This reduces $\mathcal{I}$ to

$$\langle T_{(2,1)}, T_{(3,2)} \rangle \ , \ \left\langle T_{(2,3)}, T_{(3,3)} \right\rangle + \left\langle T_{(4,5)}, T_{(4,6)}, \dots, T_{(4,n+2)} \right\rangle .$$

By Proposition 4.2.2, and using the same notation, we have

$$\deg\left(T_{(2,3)}\right) = \deg\left(S_2^{-u_2} S_3\right); \quad \deg\left(T_{(3,3)}\right) = \deg\left(S_1^{-u_1} S_3\right).$$

This means that both the irrelevant ideal and the degree matrix of the general fiber of the deformation match that of $\mathbb{F}(a_1 + u_1, a_1 + u_2, a_3, \dots, a_n)$. Then (b) follows after noticing that (4.4.5) implies $a_1 + u_2 = a_2 - u_1$. $\square$

Proposition 4.4.3. *The scroll $\mathbb{F}(a_1, \dots, a_n)$ can be deformed to*

$$\mathbb{F}(\underbrace{1, 1, \dots, 1}_{r}, \underbrace{0, 0 \dots, 0}_{n-r})$$

where

$$r \equiv \sum_{i=1}^{n} a_i \pmod{n}.$$

Proof. By Remark 4.4.1, we can assume that the sequence $a_1, \dots, a_n$ is decreasing and non-negative, with $a_n = 0$. We proceed by induction over a_1. The cases $a_1 = 0$ and $a_1 = 1$ are trivial. Assume now that $a_1 \geq 2$. Let

$$M = \#\{i : a_i = a_1\}, \quad m = \#\{i : a_i = 0\}.$$

If $M < m$, then by Proposition 4.4.2 the scroll can be deformed by subtracting 1 from each $a_1, \dots, a_M$ and adding 1 to M of the a_i that equal 0.

If $M \geq m$, the scroll can be deformed by subtracting 1 from each $a_1, \dots, a_m$ and adding 1 to every a_i that equals 0. Then we subtract 1 from every a_i (recall that this does not change the variety).

Note that in both cases, and after just a permutation of indices, we have deformed the original scroll to $\mathbb{F}(b_1, \dots, b_n)$ where $b_i \geq b_{i+1}$ for every i, the b_i are all non-negative and $b_n = 0$. Furthermore $\sum a_i \equiv \sum b_i \pmod{n}$ and $b_1 < a_1$ so the induction is complete. $\square$

4.4.2 Deformation of hypersurfaces

Let X be a toric variety and let $\mathcal{X} \to \mathbb{A}^1$ be a one-parameter deformation of X. In this subsection we study how a hypersurface of X deforms in $\mathcal{X}$ (see also [16] for the deformation of the Picard group of X).

By subsection 1.2.5, the Cox ring of X is a polynomial ring $\mathbb{K}[S_1, \dots, S_r]$. Choose an admissible triple (m, ϱ, C) and construct the corresponding deformation as explained in section 4.2.1. The map η given in section 4.2.2 is defined by a semigroup homomorphism $\nu_+ : \mathbb{Z}^{r+1}_{\geq 0} \to \mathbb{Z}^r_{\geq 0}$ which can be naturally extended to a group homomorphism $\nu : \mathbb{Z}^{r+1} \to \mathbb{Z}^r$. Notice that ν is the transpose of ψ defined in (4.2.5). A homogeneous polynomial $f \in \mathbb{K}[S_1, \ldots, S_r]$ can be written as a sum

$$f = c_1\mathfrak{m}_1 + \ldots + c_k\mathfrak{m}_k$$

where $c_i \in \mathbb{K}$ and $\mathfrak{m}_i$ is a monomial for all i. A homogeneous polynomial $\tilde{f} \in \mathbb{K}[T_1, T_{i,j}]$ such that $f = \eta(\tilde{f})$ will exist if and only if the exponent vector of each $\mathfrak{m}_i$ is in the image of ν_+. In this case, if we let $g \in \mathbb{K}[T_1, T_{ij}]$ be the trinomial (4.2.4) corresponding to (m, ϱ, C), the subvariety

$$V(\tilde{f}, g) \subset \tilde{X}$$

defines a one-parameter deformation of X. Observe that if $\tilde{f}' \in \mathbb{K}[T_1, T_{ij}]$ is another lifting of f, i.e. $\eta(\tilde{f}') = f$, then $\tilde{f}' - \tilde{f} \in \langle g, T_1 \rangle$ so that the equality $V(\tilde{f}, g, T_1) = V(\tilde{f}', g, T_1)$ holds.

Let $Q_X : \mathbb{Z}^r \to \mathrm{Cl}(X)$ be the grading map of the toric variety X, i.e Q_X maps $e \in \mathbb{Z}^r$ to the class of the divisor $\sum_{i=1}^r e_i D_i$, where D_i is the i-th invariant prime divisor of X. Given a class $w \in \mathrm{Cl}(X)$ and an equivariant divisor D of X such that $[D] = w$, a monomial basis of the Riemann-Roch space of D is in bijection with the set

$$Q_X^{-1}(w) \cap \mathbb{Z}^r_{\geq 0}.$$

The subset of monomials that can be lifted to monomials of $\mathbb{K}[T_1, T_{ij}]$ via η is in bijection with

$$\mathrm{im}(\nu_+) \cap Q_X^{-1}(w) \cap \mathbb{Z}^r_{\geq 0}.$$

Proposition 4.4.4. *The set* $\mathrm{im}(\nu_+)$ *is the Hilbert basis of the rational polyhedral cone that it generates.*

Proof. Let A_ν be the matrix associated to the map ν and let j_ϱ be the index such that S_{j_ϱ} corresponds to the ray ϱ in Σ_X. Due to the way η is defined, it is clear that by removing the $(2, j_\varrho)$-th column from A_ν, and after an adequate rearrangement of its columns, we obtain a matrix with the following properties:

- All the entries in the diagonal are 1.
- Only one column has non-zero entries outside of the diagonal.

It is easy to see that such a matrix has determinant equal to 1.

Similarly, we can remove the $(3, j_\varrho)$-th column from A_ν to get a matrix with determinant 1. This shows that the cone generated by the columns of A_ν is the union of two smooth cones (in the sense of toric geometry), which proves the statement.

□

Let P_X be the matrix whose columns are the generators of the rays of Σ_X. Let P_0 be the minor of $P(m, \varrho, C)$ resulting from removing the leftmost column and bottom row. Let $\tilde{Q}$ be the cokernel of P_0^*, i.e. the grading matrix of $\tilde{X}$ after removing the null vector column corresponding to T_1. From the Cox construction seen in (1.2.1), we get the following commutative diagram of group homomorphisms with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \tilde{M} \arrow[r, "P_0^*"] \arrow[d, "\xi"] & \mathbb{Z}^{r+1} \arrow[r, "\tilde{Q}"] \arrow[d, "\nu"] & \mathrm{Cl}(\tilde{X}) \arrow[r] \arrow[d, "\bar{\nu}"] & 0 \\
0 \arrow[r] & M \arrow[r, "P_X^*"] & \mathbb{Z}^r \arrow[r, "Q_X"] & \mathrm{Cl}(X) \arrow[r] & 0
\end{tikzcd}$$

where the square on the left is the dual of (4.2.6) and $\bar{\nu}$ is uniquely defined by ν. Denote the exponent vector of a monomial $\mathfrak{m}$ by $\mathbf{v}(\mathfrak{m})$. Then we have

$$\ker \nu = \mathbf{v}\left(\prod_{(2,j)\in U_2} T_{2j}^{-a_j}\right) - \mathbf{v}\left(\prod_{(3,j)\in U_3} T_{3j}^{-a_j}\right) \subseteq \ker \tilde{Q},$$

which together with the surjectivity of ξ and ν, imply that $\bar{\nu}$ is an isomorphism.

Example 4.4.5. We now turn our attention to the case of Hirzebruch surfaces, i.e. $X = \mathbb{F}_n$. The fan Σ_X has four rays $\rho_1, \rho_2, \rho_3, \rho_4$, generated respectively by

$$v_1 = (1, 0),\ v_2 = (0, 1),\ v_3 = (-1, n),\ v_4 = (0, -1).$$

Let D_1, D_2, D_3, D_4 be the corresponding invariant divisors. In this case we have $\mathrm{Cl}(X) \cong \mathbb{Z}^2$ generated by $[D_1]$ and $[D_2]$, along with

$$P_X = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & n & -1 \end{bmatrix}, \quad Q_X = \begin{bmatrix} 1 & 0 & 1 & n \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

plus a section s for Q and a projection π for P^* given by

$$s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \pi = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

We choose $\omega = (a, b) \in \mathrm{Cl}(X)$, corresponding to the class $a[D_1] + b[D_2]$, with $a > bn > 0$ to guarantee ampleness. Then, by [8, §9.1], the polyhedron $\pi(Q_X^{-1}(\omega) \cap \mathbb{Z}^r_{\geq 0})$ is a trapezoid with vertices (in counterclockwise order)

$$(0, 0),\ (-a, 0),\ (-a, -b),\ (-nb, -b).$$

Applying $P^* + (a, b, 0, 0)$ to it, we obtain the trapezoid $Q_X^{-1}(\omega) \cap \mathbb{Z}^r_{\geq 0}$, whose vertices are

$$(a, b, 0, 0),\ (0, b, a, 0),\ (0, 0, a - bn, b),\ (a - bn, 0, 0, b).$$

If we now consider the deformation given by the admissible triple (m, ϱ, C) where $m = [-\alpha, -1]$, $\varrho = \rho_2$, $C = \{\rho_1\}$ and $0 < \alpha < n$, we get

$$P_0 = \begin{bmatrix} 1 & -\alpha & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & {\scriptstyle \alpha-n} \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}, \quad \nu = \begin{bmatrix} 0 & 1 & 0 & \alpha & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & {\scriptstyle n-\alpha} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that the vertices of $Q_X^{-1}(\omega) \cap \mathbb{Z}^r_{\geq 0}$ can now be written as

$$\nu(0, a - b\alpha, 0, b, 0),\ \nu(0, 0, b, 0, a - bn + b\alpha),\ \nu(b, 0, 0, 0, a - bn),\ \nu(b, a - nb, 0, 0, 0),$$

which shows that the trapezoid is contained in $\mathrm{im}(\nu_+)$. This means that when we deform Hirzebruch surfaces, every function in the Riemann-Roch space of the class ω can be lifted via η.

Chapter 5

Resolutions of triple points

A *triple point* is a rational singularity of multiplicity 3 in a surface. In [1], M. Artin classifies the dual resolution graphs of triple points into 9 cases (see Section 5.2). He also proves every triple point can be embedded in $\mathbb{C}^4$. Our motivation for this chapter is to study these singularities using Cox coordinates and finding deformations of triple points. For this end, we start the chapter by explaining a description found in [2] which associates every $\mathbb{K}^*$-surface to a pair of matrices A and P, with P being the matrix that fits in (1.2.1). We also explain how to use this data to find resolutions of singularities in $\mathbb{K}^*$-surfaces (cf. [2, §5.4.3]). We make use of these methods to construct a surface for each triple point that contains it. Our main result in this section is Theorem 3.

5.1 Combinatorial data for $\mathbb{K}^*$-surfaces

We begin by recalling the combinatorial description of complete $\mathbb{K}^*$-surfaces given in [2, §5.4]. We begin with the construction of a graded algebra which will be the Cox ring of the $\mathbb{K}^*$-surface.

Construction 5.1.1. Let r and $n_0, \ldots, n_r$ be positive integers. Consider vectors $l_i := (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}^{n_i}_{\geq 1}$ and $d_i := (d_{i1}, \ldots, d_{in_i}) \in \mathbb{Z}^{n_i}$ satisfying

$$\frac{d_{i1}}{l_{i1}} < \ldots < \frac{d_{in_i}}{l_{in_i}} \text{ for all } i, \qquad \gcd(l_{ij}, d_{ij}) \text{ for all } i, j.$$

With this data, define the following block matrices

$$L := \begin{bmatrix} -l_0 & l_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ -l_0 & 0 & \cdots & l_r \end{bmatrix}, \qquad d := \begin{bmatrix} d_0 & d_1 & \cdots & d_r \end{bmatrix}.$$

Set $n := n_1 + \ldots + n_r$. Then we define four types of $(r+1) \times (n+m)$ matrices P, where $m = 0, 1, 2$ according to the following cases:

Type (e-e)	Type (p-e)	Type (e-p)	Type (p-p)
$P = \begin{bmatrix} L \\ d \end{bmatrix}$	$P = \begin{bmatrix} L & 0 \\ d & 1 \end{bmatrix}$	$P = \begin{bmatrix} L & 0 \\ d & -1 \end{bmatrix}$	$P = \begin{bmatrix} L & 0 & 0 \\ d & 1 & -1 \end{bmatrix}$

and require that the columns of P generate $\mathbb{Q}^{r+1}$ as a cone. Choose also a matrix $A = [a_0, \ldots, a_r]$ with pairwise linearly independent columns $a_0, \ldots, a_r \in \mathbb{K}^2$.
We define the $\mathbb{K}$-algebra

$$R(A,P) := \mathbb{K}[T_{ij}, S_k : \ 0 \le i \le r, \ 1 \le j \le n_i, \ 1 \le k \le m] / \langle g_I : \ I \in \mathfrak{J} \rangle$$

where $\mathfrak{J}$ is the set of all triples $I = (i_1, i_2, i_3)$, with $0 \le i_1 \le i_2 \le i_3 \le r$, and

$$g_I := g_{i_1,i_2,i_3} := \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}, \qquad T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}}.$$

Theorem 5.1.2. *The $\mathbb{K}$-algebra $R(A,P)$ is the Cox ring of a $\mathbb{Q}$-factorial projective $\mathbb{K}^*$-surface $X(A,P)$, which is uniquely determined, up to isomorphism, by A and P. Moreover, every rational normal complete $\mathbb{K}^*$-surface is isomorphic to some $X(A,P)$.*

Proposition 5.1.3. *Let $X = X(A,P)$ be a $\mathbb{K}^*$-surface as constructed above. For each of the four types of P we have the following*

(e-e) *Both the source and the sink are elliptic fixed points.*

(p-e) *The source is a smooth rational curve and the sink is an elliptic fixed point.*

(e-p) *The source is an elliptic fixed point and the sink is a smooth rational curve.*

(p-p) *Both the source and the sink are smooth rational curves.*

This description of $\mathbb{K}^*$-surfaces by matrices allows for a simple method of resolving singularities. The procedure consist of two steps.

Construction 5.1.4. Let $X = X(A,P)$ be a $\mathbb{K}^*$-surface. We recall the procedure given in [2, Construction 5.4.3.2] to desingularize X in an equivariant way.

Tropical Step. Enlarge P to a matrix P' by adding the columns $[0, \ldots, 0, 1]$ and $[0, \ldots, 0, -1]$ if not already present, so that P' is a matrix of type (p-p). This step is represented by the following picture (taken from [2]).

Observe that in this way the new surface is of type (e-e).

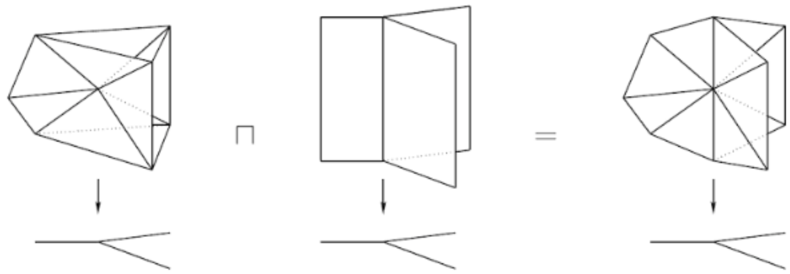


Figure 5.1: Tropical step (Source: [2])

Toric Step. Enlarge P' to a matrix P'', which will have columns equal to the primitive generators of the cones generated by the following pairs of vectors (where, if subindices appear, every possible pair must be considered):

(i) The column $[0, \dots, 0, 1]$ and the column containing l_{i1}

(ii) The column $[0, \dots, 0, -1]$ and the column containing l_{in_i}

(iii) The column containing l_{ij} and the one containing $l_{i(j+1)}$

The columns in P'' are ordered according to the requirements set in Construction 5.1.1. This step is represented by the following picture (taken from [2]).

Figure 5.2: Toric step (Source: [2])

Observe that in this way the new surface is of type (e-e) and it is smooth.

The resulting surface $X'' = X(A, P'')$ is then a (canonical) resolution of singularities of X.

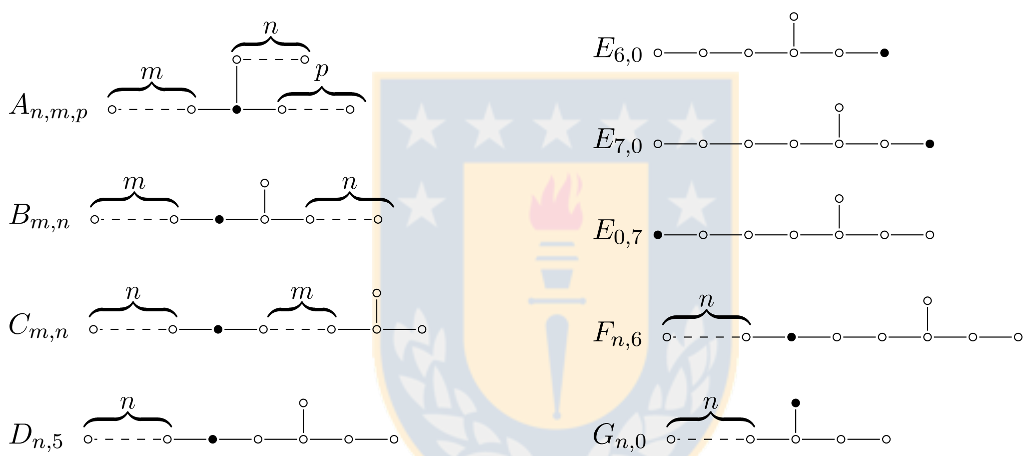
Remark 5.1.5. Observe that, as shown in the picture, the toric step is a toric resolution of singularities on each "leaf" above the tropical $\mathbb{P}^1$. As a consequence, given three primitive generators v_{i-1}, v_i, v_{i+1} of three consecutive one dimensional rays on the same leaf the following integer relation holds:

$$-b_i v_i = v_{i-1} + v_{i+1},$$

where b_i is the self-intersection number of the prime invariant curve which corresponds to v_i.

5.2 Resolution of triple points

We recall Artin's classification of triple points, according to their resolution dual graphs. We use a white circle to represent a curve of self-intersection -2 and a black circle to represents a curve of self-intersection -3. There are 9 cases.



We will treat each triple point separately, leaving $A_{m,n,p}$ for the end as the calculations involved with this case proved to be slightly more complicated than the rest.

5.2.1 The case $B_{m,n}$

Take the triple point $B_{m,n}$. We will construct a complete $\mathbb{K}^*$-surface Y such that the intersection graph of its prime invariant curves contains the intersection graph of the triple point. The situation is displayed in the following picture.

Let P_Y be the matrix corresponding to Y. We know, according to Construction 5.1.1, that P_Y is constructed from six vectors $l_0, l_1, l_2, d_0, d_1, d_2$ in the following way

$$P_Y = \begin{bmatrix} -l_0 & l_1 & 0 & 0 & 0 \\ -l_0 & 0 & l_2 & 0 & 0 \\ d_0 & d_1 & d_2 & 1 & -1 \end{bmatrix}. \tag{5.2.1}$$

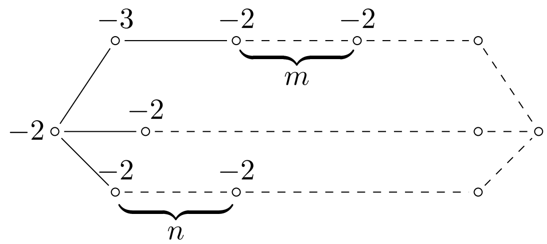


Figure 5.3: The case $B_{m,n}$ (Source: own elaboration)

We begin by showing that for all i the following hold:

$$l_{i1} = l_{in_i} = 1. \tag{5.2.2}$$

Indeed, let $b_{i1}, \ldots, b_{is_i}$ be the self-intersection numbers of one of the three horizontal branches of the above intersection graph (excluding the nodes of valency three). The smoothness of Y implies that the Hirzebruch-Jung continued fraction $[b_{i1}, \ldots, b_{is_i}]$ equals 0 or equivalently the following equality

$$b_{i1} - \frac{1}{[b_{i2}, \ldots, b_{in_i}]} = 0.$$

Since b_{i1} is integer we deduce that the numerator of $[b_{i2}, \ldots, b_{in_i}]$ equals 1. Recalling that the vector $l_i = (l_{i1}, \ldots, l_{in_i})$ consists of the numerators of the subsequent convergents of the continued fraction $[b_{i1}, \ldots, b_{in_i}]$ we deduce $l_{i1} = 1$. The equality $l_{in_i} = 1$ is obtained in a similar way as a consequence of the equality $[b_{in_i}, \ldots, b_{i1}] = 0$. On the other hand, let D_Y^- be the (-2)-curve which is represented by the left point of valency three in the above graph. According to [2, Corollary 5.4.2.2] the first of the following equalities holds

$$\frac{d_{01}}{l_{01}} + \frac{d_{11}}{l_{11}} + \frac{d_{21}}{l_{21}} = (D_Y^-)^2 = -2,$$

which reduces to

$$d_{01} + d_{11} + d_{21} = -2$$

by the above observation. By appying suitable elementary row operations to the matrix P_Y we can assume without loss of generality that

$$d_{01} = -1, \quad d_{11} = 0, \quad d_{21} = -1. \tag{5.2.3}$$

Now, according to Remark 5.1.5, we see that the self-intersection numbers $-b_{ij}$ must fit in the equations

$$b_{ij}\begin{bmatrix} l_{ij} \\ d_{ij} \end{bmatrix} = \begin{bmatrix} l_{i(j-1)} \\ d_{i(j-1)} \end{bmatrix} + \begin{bmatrix} l_{i(j+1)} \\ d_{i(j+1)} \end{bmatrix} \tag{5.2.4}$$

where we are considering $l_{i0} = 0$ and $d_{i0} = -1$ for all i, because these correspond to the column $(0, 0, -1)$ of P_Y. This allows us to inductively compute the entries of l_i and d_i up to the $k_i - 1$-th coordinate, where $k_i - 2$ is the number of nodes of valency two on the i-th branch of the graph of $B_{m,n}$. We deduce the following

$$\begin{aligned} l_0 &= (1, 3, 5, \dots, 2m+3, l_{0k_0}, \dots, l_{0n_0}) \\ l_1 &= (1, 2, l_{1k_1}, \dots, l_{1n_1}) \\ l_2 &= (1, 2, 3, \dots, n+1, l_{2k_2}, \dots, l_{2n_2}) \\ d_0 &= (-1, -2, -3, \dots, -(m+2), d_{0k_0}, \dots, d_{0n_0}) \\ d_1 &= (0, 1, d_{1k_1}, \dots, d_{1n_1}) \\ d_2 &= (\underbrace{-1, -1, \dots, -1}_{n+1 \text{ times}}, d_{2k_2}, \dots, d_{2n_2}). \end{aligned}$$

Then, since for all i we see that $(b_{i0}, d_{i0}) = (0, -1)$ and (b_{i1}, d_{i1}) equals either $(1, 0)$ or $(1, -1)$ we have that these two vectos span $\mathbb{Z}^2$. Using (5.2.4) and induction it is easy to see that the vectors (b_{ij}, d_{ij}) and $(b_{i(j+1)}, d_{i(j+1)})$ also span $\mathbb{Z}^2$, meaning that they generate a smooth cone. This shows that, for all i, the vectors

$$(b_{i0}, d_{i0}), (d_{i1}, d_{i1}), \dots, (b_{ik_i-1}, d_{ik_i-1})$$

form a Hilbert basis of

$$\operatorname{Cone}\left((0, -1)\ , (b_{ik_i-1}, d_{ik_i-1})\right)$$

because none of the curves in the dual graph of a triple point have a self-intersection less than -2 or less. Observe that the unknown entries in the matrix P_Y can be reconstructed by producing the Hilbert basis of $\operatorname{Cone}\left((b_{ik_i-1}, d_{ik_i-1})\,(0, 1)\right)$ for each i. Let us now construct the following matrices

$$P := \begin{bmatrix} -l_{0k_0} & l_{1k_1} & 0 & 0 \\ -l_{0k_0} & 0 & l_{2k_2} & 0 \\ d_{0k_0} & d_{1k_1} & d_{2k_2} & 1 \end{bmatrix}, \quad A := \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \tag{5.2.5}$$

and let $X = X(P, A)$ be the complete $\mathbb{K}^*$-surface associated to them as in Theorem 5.1.2. We can construct a resolution of X using the Construction 5.1.4, which will restore the vectors l_i and d_i described above. This shows that $Y \to X$ is an equivariant resolution of singularities and thus X contains a singular point whose dual graph is the same as the one we started with, and therefore the triple point in question is in X. In the present case of the point $B_{m,n}$, we have

$$P := \begin{bmatrix} -(2m+3) & 2 & 0 & 0 \\ -(2m+3) & 0 & n+1 & 0 \\ -(m+2) & 1 & -1 & 1 \end{bmatrix}.$$

The affine open invariant subset $U \subseteq X$ obtained by removing the invariant divisor which corresponds to the last column of the above matrix is an affine $\mathbb{K}^*$-surface which has a unique singular point and the singularity type of this point is $B_{m,n}$. By [15] the surface U is described as well by means of a matrix which is obtained by removing the last column of the above matrix P. The corresponding Cox ring together with its grading matrix and divisor class group are:

$$\mathcal{R}(U) = \frac{\mathbb{K}[T_1, T_2, T_3]}{\langle T_1^{2m+3} - T_2^2 + T_3^{n+1} \rangle} \qquad \begin{bmatrix} 4m+n+5 & 2m+n+4 & 1 \end{bmatrix} \qquad \mathbb{Z}/(4m+n+7)\mathbb{Z}$$

5.2.2 The case $C_{m,n}$

Take the triple point $C_{m,n}$. We will construct a complete $\mathbb{K}^*$-surface Y such that the intersection graph of its prime invariant curves contains the intersection graph of the triple point. The situation is displayed in the following picture.

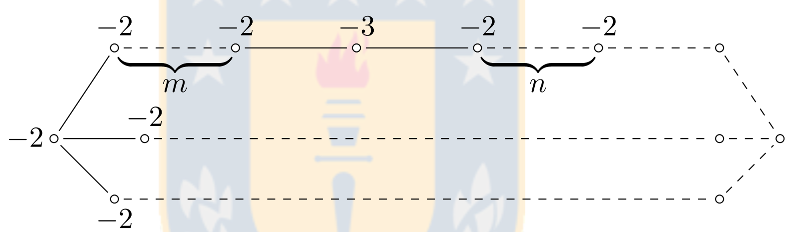


Figure 5.4: The case $C_{m,n}$ (Source: own elaboration)

We denote by P_Y the matrix constructed as in (5.2.1). Reasoning as in the case $B_{m,n}$ we deduce the equalities (5.2.2) and (5.2.3). This allows us to inductively compute the entries of l_i and d_i up to the $k_i - 1$-th coordinate, where $k_i - 2$ is the number of nodes of valency two on the i-th branch of the graph of $C_{m,n}$. We deduce the following

$$\begin{aligned}
l_0 &= (1, 2, 3, \ldots, m, m+1, 2m+3, 3m+5, \ldots, (n+2)m+2n+3, l_{0k_0}, \ldots, l_{0n_0}) \\
l_1 &= (1, 2, l_{1k_1}, \ldots, l_{1n_1}) \\
l_2 &= (1, 2, l_{2k_2}, \ldots, l_{2n_2}) \\
d_0 &= (-1, \ldots, -1, -2, -3, \ldots, -(n+2), d_{0k_0}, \ldots, d_{0n_0}) \\
d_1 &= (0, 1, d_{1k_1}, \ldots, d_{1n_1}) \\
d_2 &= (-1, -1, d_{2k_2}, \ldots, d_{2n_2}).
\end{aligned}$$

By the above we deduce that the matrix P given in (5.2.5) is the following

$$P = \begin{bmatrix} -\alpha & 2 & 0 & 0 \\ -\alpha & 0 & 2 & 0 \\ -(n+2) & 1 & -1 & 1 \end{bmatrix}$$

where $\alpha = (n+2)m+2n+3$. Let $X = X(P,A)$ be the complete $\mathbb{K}^*$-surface defined by the above matrix and the matrix A given in (5.2.5). The affine open invariant subset $U \subseteq X$ obtained by removing the invariant divisor which corresponds to the last column of the above matrix is an affine $\mathbb{K}^*$-surface which has a unique singular point and the singularity type of this point is $C_{m,n}$. By [15] the surface U is described as well by means of a matrix which is obtained by removing the last column of the above matrix P. The corresponding Cox ring is:

$$\mathcal{R}(U) = \frac{\mathbb{K}[T_1, T_2, T_3]}{\langle T_1^{\alpha} - T_2^2 + T_3^2 \rangle}$$

The grading matrix and divisor class group are

$$\begin{array}{rcr} \text{if } 2|m: & \begin{bmatrix} 4n+6 & 2n+5 & 1 \end{bmatrix} & \mathbb{Z}/(4n+8)\mathbb{Z} \\ \text{if } 2\nmid m,\ 2|n: & \begin{bmatrix} 2n+2 & 2n+5 & 1 \end{bmatrix} & \mathbb{Z}/(4n+8)\mathbb{Z} \\ \text{if } 2\nmid m,\ 2\nmid n: & \begin{bmatrix} 1 & 1 & 0 \\ n & n+3 & 1 \end{bmatrix} & (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/(2n+4)\mathbb{Z}) \end{array}$$

5.2.3 The case $D_{n,5}$

Take the triple point $D_{n,5}$. We will construct a complete $\mathbb{K}^*$-surface Y such that the intersection graph of its prime invariant curves contains the intersection graph of the triple point. The situation is displayed in the following picture.

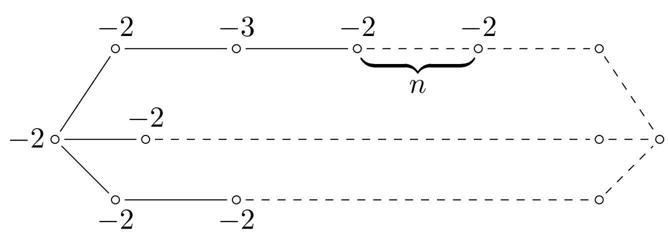


Figure 5.5: The case $D_{n,5}$ (Source: own elaboration)

We denote by P_Y the matrix constructed as in (5.2.1). Reasoning as in the case $B_{m,n}$ we deduce the equalities (5.2.2) and (5.2.3). This allows us to inductively compute the entries of l_i and d_i up to the $k_i - 1$-th coordinate, where $k_i - 2$ is the number of nodes of valency two on the i-th branch of the graph of $D_{n,5}$. We deduce the following

$$\begin{aligned}
l_0 &= (1, 2, 5, 8, 11, \dots, 3n+5, l_{0k_0}, \dots, l_{l_{0n_0}}) \\
l_1 &= (1, 2, l_{1k_1}, \dots, l_{1n_1}) \\
l_2 &= (1, 2, 3, l_{2k_2}, \dots, l_{2n_2}) \\
d_0 &= (-1, -1, -2, -3, -4, \dots, -(n+2), d_{0k_0}, \dots, d_{0n_0}) \\
d_1 &= (0, 1, d_{1k_1} \dots, d_{1n_1}) \\
d_2 &= (-1, -1, -1, d_{2k_2}, \dots, d_{2n_2})
\end{aligned}$$

By the above we deduce that the matrix P given in (5.2.5) is the following

$$P = \begin{bmatrix} -(3n+5) & 2 & 0 & 0 \\ -(3n+5) & 0 & 3 & 0 \\ -(n+2) & 1 & -1 & 1 \end{bmatrix}$$

Let $X = X(P, A)$ be the complete $\mathbb{K}^*$-surface defined by the above matrix and the matrix A given in (5.2.5). The affine open invariant subset $U \subseteq X$ obtained by removing the invariant divisor which corresponds to the last column of the above matrix is an affine $\mathbb{K}^*$-surface which has a unique singular point and the singularity type of this point is $D_{n,5}$. By [15] the surface U is described as well by means of a matrix which is obtained by removing the last column of the above matrix P. The corresponding Cox ring together with its grading matrix and divisor class group are:

$$\mathcal{R}(U) = \frac{\mathbb{K}[T_1, T_2, T_3]}{\langle T_1^{3n+5} - T_2^2 + T_3^3 \rangle} \qquad \begin{bmatrix} 1 & 3n+6 & 2n+4 \end{bmatrix} \qquad \mathbb{Z}/(3n+7)\mathbb{Z}$$

5.2.4 The case $E_{6,0}$

Take the triple point $E_{6,0}$. We will construct a complete $\mathbb{K}^*$-surface Y such that the intersection graph of its prime invariant curves contains the intersection graph of the triple point. The situation is displayed in the following picture.

We denote by P_Y the matrix constructed as in (5.2.1). Reasoning as in the case $B_{m,n}$ we deduce the equalities (5.2.2) and (5.2.3). This allows us to inductively compute the entries of l_i and d_i up to the $k_i - 1$-th coordinate, where $k_i - 2$ is the number of nodes of

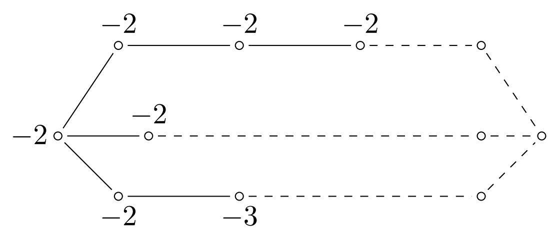


Figure 5.6: The case $E_{6,0}$ (Source: own elaboration)

valency two on the i-th branch of the graph of $E_{6,0}$. We deduce the following

$$\begin{aligned} l_0 &= (1,2,3,4,l_{0k_0},\ldots,l_{l_{0n_0}}) \\ l_1 &= (1,2,l_{1k_1},\ldots,l_{1n_1}) \\ l_2 &= (1,2,5,l_{2k_2},\ldots,l_{2n_2}) \\ d_0 &= (-1,-1,-1,-1,d_{0k_0},\ldots,d_{0n_0}) \\ d_1 &= (0,1,d_{1k_1}\ldots,d_{1n_1}) \\ d_2 &= (-1,-1,-2,d_{2k_2},\ldots,d_{2n_2}) \end{aligned}$$

By the above we deduce that the matrix P given in (5.2.5) is the following

$$P = \begin{bmatrix} -4 & 2 & 0 & 0 \\ -4 & 0 & 5 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$$

Let $X = X(P,A)$ be the complete $\mathbb{K}^*$-surface defined by the above matrix and the matrix A given in (5.2.5). The affine open invariant subset $U \subseteq X$ obtained by removing the invariant divisor which corresponds to the last column of the above matrix is an affine $\mathbb{K}^*$-surface which has a unique singular point and the singularity type of this point is $E_{6,0}$. By [15] the surface U is described as well by means of a matrix which is obtained by removing the last column of the above matrix P. The corresponding Cox ring together with its grading matrix and divisor class group are:

$$\mathcal{R}(U) = \frac{\mathbb{K}[T_1,T_2,T_3]}{\langle T_1^4 - T_2^2 + T_3^5\rangle} \qquad \begin{bmatrix} 1 & 5 & 2 \end{bmatrix} \qquad \mathbb{Z}/6\mathbb{Z}$$

5.2.5 The case $E_{7,0}$

Take the triple point $E_{7,0}$. We will construct a complete $\mathbb{K}^*$-surface Y such that the intersection graph of its prime invariant curves contains the intersection graph of the triple point. The situation is displayed in the following picture.

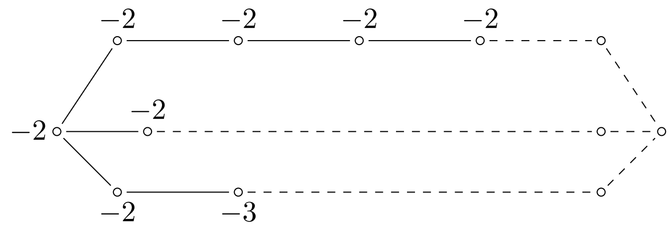


Figure 5.7: The case $E_{7,0}$ (Source: own elaboration)

We denote by P_Y the matrix constructed as in (5.2.1). Reasoning as in the case $B_{m,n}$ we deduce the equalities (5.2.2) and (5.2.3). This allows us to inductively compute the entries of l_i and d_i up to the $k_i - 1$-th coordinate, where $k_i - 2$ is the number of nodes of valency two on the i-th branch of the graph of $E_{7,0}$. We deduce the following

$$\begin{array}{rcl} l_0 & = & (1,2,3,4,5,l_{0k_0},\dots,l_{l_{0n_0}}) \\ l_1 & = & (1,2,l_{1k_1},\dots,l_{1n_1}) \\ l_2 & = & (1,2,5,l_{2k_2},\dots,l_{2n_2}) \\ d_0 & = & (-1,-1,-1,-1,-1,d_{0k_0},\dots,d_{0n_0}) \\ d_1 & = & (0,1,d_{1k_1}\dots,d_{1n_1}) \\ d_2 & = & (-1,-1,-2,d_{2k_2},\dots,d_{2n_2}) \end{array}$$

By the above we deduce that the matrix P given in (5.2.5) is the following

$$P = \begin{bmatrix} -5 & 2 & 0 & 0 \\ -5 & 0 & 5 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$$

Let $X = X(P, A)$ be the complete $\mathbb{K}^*$-surface defined by the above matrix and the matrix A given in (5.2.5). The affine open invariant subset $U \subseteq X$ obtained by removing the invariant divisor which corresponds to the last column of the above matrix is an affine $\mathbb{K}^*$-surface which has a unique singular point and the singularity type of this point is $E_{7,0}$. By [15] the surface U is described as well by means of a matrix which is obtained by removing the last column of the above matrix P. The corresponding Cox ring together with its grading matrix and divisor class group are:

$$\mathcal{R}(U) = \frac{\mathbb{K}[T_1,T_2,T_3]}{\langle T_1^5 - T_2^2 + T_3^5\rangle} \qquad \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \qquad \mathbb{Z}/5\mathbb{Z}$$

5.2.6 The case $E_{0,7}$

Take the triple point $E_{0,7}$. We will construct a complete $\mathbb{K}^*$-surface Y such that the intersection graph of its prime invariant curves contains the intersection graph of the triple point. The situation is displayed in the following picture.

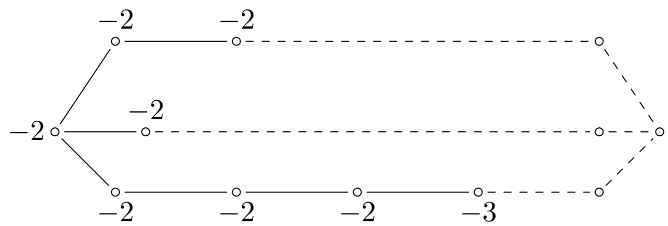


Figure 5.8: The case $E_{0,7}$ (Source: own elaboration)

We denote by P_Y the matrix constructed as in (5.2.1). Reasoning as in the case $B_{m,n}$ we deduce the equalities (5.2.2) and (5.2.3). This allows us to inductively compute the entries of l_i and d_i up to the $k_i - 1$-th coordinate, where $k_i - 2$ is the number of nodes of valency two on the i-th branch of the graph of $E_{0,7}$. We deduce the following

$$\begin{aligned}
l_0 &= (1, 2, 3, l_{0k_0}, \ldots, l_{l_{0n_0}}) \\
l_1 &= (1, 2, l_{1k_1}, \ldots, l_{1n_1}) \\
l_2 &= (1, 2, 3, 4, 9, l_{2k_2}, \ldots, l_{2n_2}) \\
d_0 &= (-1, -1, -1, d_{0k_0}, \ldots, d_{0n_0}) \\
d_1 &= (0, 1, d_{1k_1} \ldots, d_{1n_1}) \\
d_2 &= (-1, -1, -1, -1, -2, d_{2k_2}, \ldots, d_{2n_2})
\end{aligned}$$

By the above we deduce that the matrix P given in (5.2.5) is the following

$$P = \begin{bmatrix} -3 & 2 & 0 & 0 \\ -3 & 0 & 9 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$$

Let $X = X(P, A)$ be the complete $\mathbb{K}^*$-surface defined by the above matrix and the matrix A given in (5.2.5). The affine open invariant subset $U \subseteq X$ obtained by removing the invariant divisor which corresponds to the last column of the above matrix is an affine $\mathbb{K}^*$-surface which has a unique singular point and the singularity type of this point is $E_{0,7}$. By [15] the surface U is described as well by means of a matrix which is obtained by removing the last column of the above matrix P. The corresponding Cox ring together with its grading matrix and divisor class group are:

$$\mathcal{R}(U) = \frac{\mathbb{K}[T_1, T_2, T_3]}{\langle T_1^3 - T_2^2 + T_3^9 \rangle} \qquad \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \qquad \mathbb{Z}/3\mathbb{Z}$$

5.2.7 The case $F_{n,6}$

Take the triple point $F_{n,6}$. We will construct a complete $\mathbb{K}^*$-surface Y such that the intersection graph of its prime invariant curves contains the intersection graph of the triple point. The situation is displayed in the following picture.

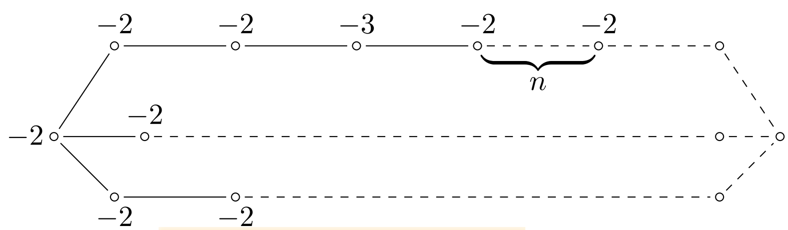


Figure 5.9: The case $F_{n,6}$ (Source: own elaboration)

We denote by P_Y the matrix constructed as in (5.2.1). Reasoning as in the case $B_{m,n}$ we deduce the equalities (5.2.2) and (5.2.3). This allows us to inductively compute the entries of l_i and d_i up to the $k_i - 1$-th coordinate, where $k_i - 2$ is the number of nodes of valency two on the i-th branch of the graph of $F_{n,6}$. We deduce the following

$$\begin{aligned} l_0 &= (1, 2, 3, 7, 11, 15, \ldots, 4n+7, l_{0k_0}, \ldots, l_{l_{0n_0}}) \\ l_1 &= (1, 2, l_{1k_1}, \ldots, l_{1n_1}) \\ l_2 &= (1, 2, 3, l_{2k_2}, \ldots, l_{2n_2}) \\ d_0 &= (-1, -1, -1, -2, -3, -4, \ldots, -(n+2), d_{0k_0}, \ldots, d_{0n_0}) \\ d_1 &= (0, 1, d_{1k_1} \ldots, d_{1n_1}) \\ d_2 &= (-1, -1, -1, d_{2k_2}, \ldots, d_{2n_2}) \end{aligned}$$

By the above we deduce that the matrix P given in (5.2.5) is the following

$$P = \begin{bmatrix} -(4n+7) & 2 & 0 & 0 \\ -(4n+7) & 0 & 3 & 0 \\ -(n+2) & 1 & -1 & 1 \end{bmatrix}$$

Let $X = X(P, A)$ be the complete $\mathbb{K}^*$-surface defined by the above matrix and the matrix A given in (5.2.5). The affine open invariant subset $U \subseteq X$ obtained by removing the invariant divisor which corresponds to the last column of the above matrix is an affine

$\mathbb{K}^*$-surface which has a unique singular point and the singularity type of this point is $F_{n,6}$. By [15] the surface U is described as well by means of a matrix which is obtained by removing the last column of the above matrix P. The corresponding Cox ring together with its grading matrix and divisor class group are:

$$\mathcal{R}(U) = \frac{\mathbb{K}[T_1, T_2, T_3]}{\langle T_1^{4n+7} - T_2^2 + T_3^3 \rangle} \qquad \begin{bmatrix} 2n+4 & n+4 & 1 \end{bmatrix} \qquad \mathbb{Z}/(2n+5)\mathbb{Z}$$

5.2.8 The case $G_{n,0}$

Take the triple point $G_{n,0}$. We will construct a complete $\mathbb{K}^*$-surface Y such that the intersection graph of its prime invariant curves contains the intersection graph of the triple point. The situation is displayed in the following picture.

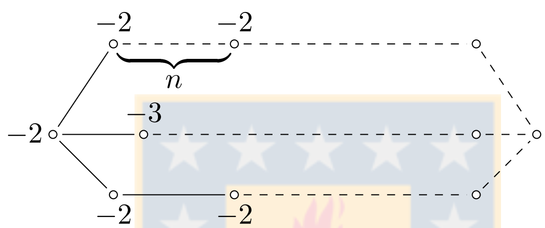


Figure 5.10: The case $G_{n,0}$ (Source: own elaboration)

We denote by P_Y the matrix constructed as in (5.2.1). Reasoning as in the case $B_{m,n}$ we deduce the equalities (5.2.2) and (5.2.3). This allows us to inductively compute the entries of l_i and d_i up to the $k_i - 1$-th coordinate, where $k_i - 2$ is the number of nodes of valency two on the i-th branch of the graph of $G_{n,0}$. We deduce the following

$$\begin{aligned}
l_0 &= (1, 2, 3, \ldots, n+1, l_{0k_0}, \ldots, l_{l_{0n_0}}) \\
l_1 &= (1, 3, l_{1k_1}, \ldots, l_{1n_1}) \\
l_2 &= (1, 2, 3, l_{2k_2}, \ldots, l_{2n_2}) \\
d_0 &= (\underbrace{-1, \ldots, -1}_{n+1 \text{ times}}, d_{0k_0}, \ldots, d_{0n_0}) \\
d_1 &= (0, 1, d_{1k_1} \ldots, d_{1n_1}) \\
d_2 &= (-1, -1, -1, d_{2k_2}, \ldots, d_{2n_2})
\end{aligned}$$

By the above we deduce that the matrix P given in (5.2.5) is the following

$$P = \begin{bmatrix} -(n+1) & 3 & 0 & 0 \\ -(n+1) & 0 & 3 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Let $X = X(P, A)$ be the complete $\mathbb{K}^*$-surface defined by the above matrix and the matrix A given in (5.2.5). The affine open invariant subset $U \subseteq X$ obtained by removing the invariant divisor which corresponds to the last column of the above matrix is an affine $\mathbb{K}^*$-surface which has a unique singular point and the singularity type of this point is $G_{n,0}$. By [15] the surface U is described as well by means of a matrix which is obtained by removing the last column of the above matrix P. The corresponding Cox ring is:

$$\mathcal{R}(U) = \frac{\mathbb{K}[T_1, T_2, T_3]}{\langle T_1^{n+1} - T_2^3 + T_3^3 \rangle}$$

its grading matrix and divisor class group are

$$\begin{array}{lll}
\text{for } n \equiv 0 \,(\text{mod } 3): & \begin{bmatrix} 3 & 4 & 1 \end{bmatrix}, & \mathbb{Z}/9\mathbb{Z} \\[2ex]
\text{for } n \equiv 1 \,(\text{mod } 3): & \begin{bmatrix} 6 & 7 & 1 \end{bmatrix}, & \mathbb{Z}/9\mathbb{Z} \\[2ex]
\text{for } n \equiv 2 \,(\text{mod } 3): & \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, & (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})
\end{array}$$

5.2.9 The case $A_{m,n,p}$

Take the triple point $A_{m,n,p}$. We will construct a complete $\mathbb{K}^*$-surface Y such that the intersection graph of its prime invariant curves contains the intersection graph of the triple point. The situation is displayed in the following picture.

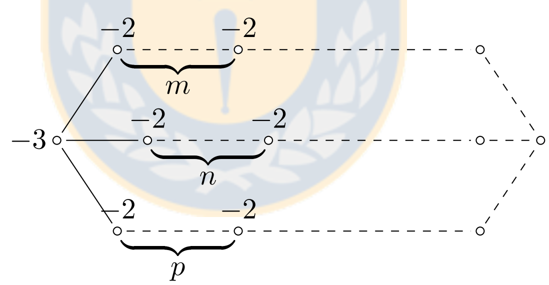


Figure 5.11: The case $A_{m,n,p}$ (Source: own elaboration)

We denote by P_Y the matrix constructed as in (5.2.1). Reasoning as in the case $B_{m,n}$ we deduce the equality (5.2.2). According to [2, Corollary 5.4.2.2] the first of the following equalities holds

$$\frac{d_{01}}{l_{01}} + \frac{d_{11}}{l_{11}} + \frac{d_{21}}{l_{21}} = (D_Y^-)^2 = -3,$$

which reduces to

$$d_{01} + d_{11} + d_{21} = -3.$$

By appying suitable elementary row operations to the matrix P_Y we can assume without loss of generality that

$$d_{01} = d_{11} = d_{21} = -1.$$

This allows us to inductively compute the entries of l_i and d_i up to the k_i-1-th coordinate, where k_i-2 is the number of nodes of valency two on the i-th branch of the graph of $A_{m,n,p}$. We deduce the following

$$\begin{aligned}
l_0 &= (1,2,3,\dots,m+1, l_{0k_0},\dots,l_{l_{0n_0}}) \\
l_1 &= (1,2,3,\dots,n+1, l_{1k_1},\dots,l_{1n_1}) \\
l_2 &= (1,2,3,\dots,p+1, l_{2k_2},\dots,l_{2n_2}) \\
d_0 &= (\underbrace{-1,\dots,-1}_{n+1 \text{ times}}, d_{0k_0},\dots,d_{0n_0}) \\
d_1 &= (\underbrace{-1,\dots,-1}_{m+1 \text{ times}}, d_{1k_1}\dots,d_{1n_1}) \\
d_2 &= (\underbrace{-1,\dots,-1}_{p+1 \text{ times}}, d_{2k_2},\dots,d_{2n_2})
\end{aligned}$$

By the above we deduce that the matrix P given in (5.2.5) is the following

$$P = \begin{bmatrix} -(n+1) & m+1 & 0 & 0 \\ -(n+1) & 0 & p+1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Let $X = X(P,A)$ be the complete $\mathbb{K}^*$-surface defined by the above matrix and the matrix A given in (5.2.5). The affine open invariant subset $U \subseteq X$ obtained by removing the invariant divisor which corresponds to the last column of the above matrix is an affine $\mathbb{K}^*$-surface which has a unique singular point and the singularity type of this point is $A_{m,n,p}$. By [15] the surface U is described as well by means of a matrix which is obtained by removing the last column of the above matrix P. Computing the grading of the Cox ring for this case is still a work in progress. However, we can give the class group of X by the following rule: let

$$A := mn + mp + np + 2m + 2n + 2p + 3$$

and let B be the greatest common divisor of $m+1$, $n+1$ and $p+1$. Using now the fact that $m \equiv n \equiv p \equiv -1$ modulo B, it is easy to check that A is divisible by B. Let $C = A/B$, then

$$\mathrm{Cl}(X) \cong \frac{\mathbb{Z}}{B\mathbb{Z}} \times \frac{\mathbb{Z}}{C\mathbb{Z}}.$$

Conclusion

The main objective of this thesis was to study deformations of T-varieties of complexity at most 1. We have done this mainly in two different ways.

Firstly, for the case of a bidimensional, smooth, complexity-one T-variety we have managed to construct an Euler-type sequence (Theorem 1) which helps in studying the cohomology group $\mathrm{H}^1(X, T_X)$. Recall that this group parametrizes infinitesimal deformations of X and as such it is directly related to our initial goal. Using this sequence, we were also able to answer the question of whether a smooth complexity-one T-surface is infinitesimally rigid (Theorem 2).

Secondly, we used the language of Cox rings to give a combinatorial method to construct one-parameter deformations of smooth toric varieties (Theorem 4.2.1) and to compute their image under the Kodaira-Spencer map (Theorem 4.2.3). We then used this construction to understand when a scroll over $\mathbb{P}^1$ is rigid and whether it can deform to another scroll. We have also used our construction to study deformations of hypersurfaces.

For future projects, we plan to get a deeper understanding of how closely related are M. Mavlyutov's method [24] and N. Ilten, R. Vollmert's method [20] for constructing deformations of toric varieties. More precisely, we would like to prove that these two methods are equivalent by using our results from Chapter 4. Afterwards, we will attempt to generalize Theorem 4.2.1 to the case of singular $\mathbb{K}^*$-surfaces, with the intent of applying it to the surfaces found in Chapter 5 and thus obtaining deformations of $\mathbb{K}^*$-surfaces containing rational triple points as singularities.

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