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Superficies de Riemann Pseudoreales de género pequeño.

Pseudoreal Riemann Surfaces of small genus.

Tesis para optar al grado de Magíster en Matemática

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Introduction

Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a smooth complex projective curve defined as the zero locus of homogeneous polynomials $P_1, \ldots, P_r \in \mathbb{C}[x_0, \ldots, x_n]$ and let $\overline{X}$ be its conjugate, i.e. the zero locus of the polynomials obtained conjugating the coefficients of every polynomial P_i. The curve X is called *pseudoreal* if it is isomorphic to $\overline{X}$ but is not isomorphic to a curve defined by polynomials with coefficients in the field $\mathbb{R}$ of real numbers. Because of the equivalence between isomorphism classes of smooth projective complex curves with conformal classes of compact Riemann surfaces [Har77, Theorem 3.1, p. 441], together with the fact that the definability of a curve over a field only depends on the isomorphism class of X, we can define the concept of pseudoreal also for compact Riemann surfaces.

A different but equivalent definition of pseudoreal Riemann surface can be given as follows. The association $X \to \overline{X}$ defines an involution on the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g (see [Sch89, Chapter 7]). The fixed locus of such involution contains the conformal classes of *real Riemann surfaces* (Definition 1.2.7), i.e. Riemann surfaces admitting a projective model defined over $\mathbb{R}$, and the conformal classes of *pseudoreal Riemann surfaces*, which are Riemann surfaces (seen as Klein surfaces, Definition 1.2.8) carrying anticonformal automorphisms but no anticonformal involutions (Definition 1.3.1). In particular, it is known that the locus of pseudoreal Riemann surfaces is contained in the singular locus of $\mathcal{M}_g$ (Lemma 3.2.3).

It can be easily proved that Riemann surfaces of genus 0 and 1 are not pseudoreal (Proposition 1.1). The first examples of pseudoreal Riemann surfaces of genus $g \geq 2$ are due to C. Earle [Ear71, p. 126] and G. Shimura [Shi72, p. 177] and they are hyperelliptic curves of even genus.

In literature, one can find two main approaches to the study of pseudoreal Riemann surfaces: a number-theoretical approach and an approach through NEC groups. The first approach deals, more generally, with the problem of deciding whether the field of moduli of a curve (Definition 1.1.10) is a field of definition (Definition 1.1.1). In this setting, pseudoreal curves are complex curves whose field of moduli is contained in $\mathbb{R}$, but it has not $\mathbb{R}$ as a field of definition (Definition 1.1.14). A fundamental tool in this approach is a classical theorem by A. Weil (Theorem 1.1.7), which provides a necessary and sufficient conditions for a projective variety defined over a field L, to be definable over a subfield $K \subseteq L$ when the extension is Galois. More recently, P. Dèbes and M. Emsalem proved that $X/\mathrm{Aut}(X)$ can always be defined over the field of moduli of X and that X has the same property when a suitable model B of $X/\mathrm{Aut}(X)$ over the subfield $K \subseteq L$ has a K-rational point (see [DE99, Corollary 4.3 (c)]). In particular, this result turns out to be useful when $X/\mathrm{Aut}(X)$ has genus zero: this has been applied by B. Huggins to complete the classification of pseudoreal hyperelliptic curves (see [Hug05, Proposition 5.0.5]) and it was later generalized by A. Kontogeorgis in [Kon09] by studying p-gonal curves. Unfortunately, the result of Dèbes-Emsalem is not easy to apply as soon as $X/\mathrm{Aut}(X)$ has genus not equal to zero.

A second approach, specific of compact Riemann surfaces, is through the theory of Fuchsian groups, and more generally of non-euclidean crystallographic (NEC) groups (Definition 1.2.19), which are discrete subgroups Δ of the full automorphism group of the hyperbolic plane $\mathbb{H}$ such that $\mathbb{H}/\Delta$ is a compact Klein surface (Definition 1.2.8). In fact, by the uniformization theorem (see [Sch89, Chapter 7]) any Riemann surface X of genus $g \geq 2$ is the quotient of $\mathbb{H}$ by a torsion free Fuchsian group Δ. Moreover, the full automorphism group $\mathrm{Aut}^{\pm}(\mathbb{H}/\Delta)$ of $\mathbb{H}/\Delta$ is the quotient $N_{\mathrm{Aut}^{\pm}(\mathbb{H})}(\Delta)/\Delta$, where $N_{\mathrm{Aut}^{\pm}(\mathbb{H})}(\Delta)$ is the normalizer of Δ in $\mathrm{Aut}^{\pm}(\mathbb{H})$ (Theorem 1.2.22), and its conformal automorphism group $\mathrm{Aut}^{+}(\mathbb{H}/\Delta)$ is $N_{\mathrm{Aut}^{\pm}(\mathbb{H})}(\Delta)^{+}/\Delta$, where $N_{\mathrm{Aut}^{\pm}(\mathbb{H})}(\Delta)^{+}$ is the canonical Fuchsian subgroup of $N_{\mathrm{Aut}^{\pm}(\mathbb{H})}(\Delta)$ (Definition 1.2.19). If G is the full automorphism group of a pseudoreal Riemann surface, then the conformal automorphism group G^{+} of

G is an index two subgroup such that $G \backslash G^{+}$ contains no involutions. Moreover, there exists an epimorphism $\varphi : \Gamma \to G$ from a NEC group Γ such that $\ker(\varphi)$ is a torsion free group and $\varphi(\Gamma^{+}) = G^{+}$, where Γ^{+} is the canonical free Fuchsian subgroup of Γ. This idea allowed D. Singerman to prove the existence of pseudoreal Riemann surfaces of any genus (Theorem 3.1.1). Moreover, it has been used by E. Bujalance, M. Conder and A.F. Costa in [BCC10] and [BC14] to classify the full automorphism groups of pseudoreal Riemann surfaces up to genus 4.

The aim of this thesis is to provide an introduction to both approaches and to show some known and new results in this topic. The thesis is organized as follows. In Chapter 1, we give the background material for both approaches, defining and dealing with the concepts of field of definition and fields of moduli (Section 1.1), Riemann and Klein surfaces, the automorphism groups of such surfaces, Fuchsian and NEC groups, their signatures (Section 1.2), and the concept of pseudoreal Riemann surface (Section 1.3). In Chapter 2, we provide the principal tools and results in the problem of the definability of a curve X over its field of moduli, when $X/\mathrm{Aut}(X)$ has genus 0, considering X defined over a field not necessarily equal to $\mathbb{C}$. We review the main known theorems and we show the results obtained for hyperelliptic and p-gonal curves. Along this way, we would like to point out that Section 2.3 contains our first new results, which are the following:

Theorem 1. (Theorem 2.3.5) *Let F be an infinite perfect field of characteristic $q \neq 2$ and let $\overline{F}$ be an algebraic closure of F. Let X be a curve of genus $g \geq 2$ defined over $\overline{F}$ and let $Z(G)$ the center of the automorphism group G of X. Suppose $X/Z(G)$ has genus 0, and $G/Z(G)$ is neither trivial, nor cyclic (if $q = 0$), nor cyclic of order relatively prime to q (if $q \neq 0$). In that case X can be defined over $M_{\overline{F}/F}(X)$.*

Corollary 1. (Corollary 2.3.6) *If X is a pseudoreal Riemann surface such that the quotient $X/Z\left(\mathrm{Aut}(X)\right)$ has genus 0, then $\mathrm{Aut}(X)$ must be an Abelian group.*

In Chapter 3, we provide the principal tools and results in the NEC group approach, such as the existence of pseudoreal Riemann surfaces in every genus, the characterization of the full automorphism groups of pseudoreal Riemann surfaces, and the known classifications of conformal and full automorphism groups of Riemann surfaces with respect to a fixed genus. We provide new results (Corollary 3.3.5) which give conditions on the existence of a group which extends another group with degree two, allowing us to obtain an easier proof of [DE99, Corollary 4.3 (b)] and sufficient conditions for a group to be the conformal automorphism group of a pseudoreal Riemann surface.

Theorem 2. (Theorem 3.3.7) *If G is a group such that $Z(G) = \{1\}$ and $\mathrm{Inn}(G)$ has group complement in $\mathrm{Aut}(G)$, then any degree two extension of G will be a semidirect product of C_2 and G.*

Corollary 2. (Corollary 3.3.8) *Let G be the conformal automorphism group of a Riemann surface X. Suppose that $Z(G) = \{1\}$ and that $\mathrm{Inn}(G)$ has group complement in $\mathrm{Aut}(G)$. Then X cannot be a pseudoreal Riemann surface.*

Theorem 3. (Corollary 3.3.6) *If G is a group such that $Z(G) = \{1\}$ and $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ has no involutions — where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G —, then any extension of G by C_2 is a direct product of G and C_2.*

Corollary 3. (Corollary 3.3.9) *If G is the conformal automorphism group of a Riemann surface X such that $Z(G) = \{1\}$ and $\mathrm{Out}(G)$ has no involutions, then X cannot be pseudoreal.*

In Section 3.4.3 we complete the work in [BG10] identifying necessary and sufficient conditions to find full automorphism groups of pseudoreal Riemann surfaces with NEC signature $(3; -; [-]; \{-\})$ (Lemma 3.4.5). Finally, we consider the maximal full automorphism groups of pseudoreal Riemann surfaces (see Theorem 3.4.6) and we prove our following results.

Theorem 4. (Theorem 3.4.8) *If a pseudoreal Riemann surface X has maximal full automorphism group, then its conformal automorphism group is not Abelian.*

Corollary 4. (Corollary 3.4.12) *If X is a pseudoreal Riemann surface with maximal full automorphism group, then it cannot be generalized superelliptic* (see Definiton 3.4.11).

Chapter 4 contains a summary of the known full automorphism groups of pseudoreal Riemann surfaces of genus $2 \leq g \leq 4$, together with algebraic models that we found in the literature (Section 4.1). Moreover, we extend the classification of full automorphism groups until genus 10 in our next result.

Theorem 5. (Theorem 4.2.1) *Two finite groups G and $\overline{G}$ are the conformal and full automorphism groups of a pseudoreal Riemann surface X of genus $5 \leq g \leq 10$ if and only if $G = \mathrm{Aut}^+(X)$ and $\overline{G} = \mathrm{Aut}^{\pm}(X)$ in the corresponding table by genus among Table 5.2, 5.3, 5.4, 5.5, 5.6, 5.3.4, and 5.7.*

In Section 4.3 we describe the known algebraic models for pseudoreal Riemann surfaces in genus $5 \leq g \leq 10$. We finish the chapter proving our following theorems.

Theorem 6. (Theorem 4.4.1) *If X is a pseudoreal plane quintic X, then $\mathrm{Aut}^+(X)$ and $\mathrm{Aut}^{\pm}(X)$ must be in a row in Table 4.3.*

Theorem 7. (Theorem 4.5.1) *Two finite groups G and $\overline{G}$ are the conformal and full automorphism groups of a pseudoreal generalized superelliptic curve X of genus $3 \leq g \leq 10$ and cental element τ* (remember Definition 3.4.11) *if and only if $G = \mathrm{Aut}^+(X)$ and $\overline{G} = \mathrm{Aut}^{\pm}(X)$ in the corresponding table by genus among Table 5.8, 5.9, 5.10, 5.11, 5.12, 5.13, 5.14* and *5.15.*

In Chapter 5 we provide the Magma [BCP97] programs we wrote to carry out the above classification and to make conjectures about the conformal and full automorphism groups of pseudoreal Riemann surfaces.

Finally, in Appendix A we give the list of all the groups we used in this thesis, together with a presentation, their orders and ID numbers when possible. In Appendix B we give the classification tables of conformal and full automorphism groups of pseudoreal Riemann surfaces of genus $5 \leq g \leq 10$. Appendix C contains the classification tables of conformal and full automorphism groups of pseudoreal generalized superelliptic curves of genus $3 \leq g \leq 10$.

Introducción

Sea $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ una curva proyectiva compleja suave definida como el lugar de ceros de los polinomios homogéneos $P_1, \ldots, P_r \in \mathbb{C}[x_0, \ldots, x_n]$ y sea $\overline{X}$ su conjugada, es decir, el lugar de ceros de los polinomios obtenidos al conjugar los coeficientes de cada polinomio P_i. La curva X se dice *pseudoreal* si es isomorfa a $\overline{X}$ pero no es isomorfa a una curva definida por polinomios con coeficientes en el cuerpo $\mathbb{R}$ de los números reales. Por la equivalencia entre clases de isomorfismos de curvas complejas proyectivas suaves y clases conformales de superficies de Riemann compactas [Har77, Teorema 3.1, p. 441], junto con el hecho de que la definibilidad de una curva sobre un cuerpo solo depende de la clase de isomorfismo de X, también podemos definir el concepto de pseudoreal para superficies de Riemann compactas.

Una definición distinta pero equivalente de superficie de Riemann pseudoreal puede ser dada como sigue. La aplicación $X \to \overline{X}$ define una involución en el espacio de moduli $\mathcal{M}_g$ de superficies de Riemann compactas de género g (ver [Sch89, Capítulo 7]). El lugar fijo de dicha involución contiene las clases conformales de *superficies de Riemann reles* (Definición 1.2.7), es decir, superficies de Riemann que admiten un modelo proyectivo definido sobre $\mathbb{R}$, y las clases conformales de *superficies de Riemann pseudoreales*, las cuales son superficies de Riemann (vistas como superficies de Klein, Definición 1.2.8) que poseen automorfismos anticonformales pero ninguna involución anticonformal (Definición 1.3.1). En particular, se sabe que el lugar de superficies de Riemann pseudoreales está contenido en el lugar singular de $\mathcal{M}_g$ (Lema 3.2.3).

Se puede probar fácilmente que las superficies de Riemann de género 0 y 1 no son pseudoreales (Proposición 1.1). Los primeros ejemplos de superficies de Riemann pseudoreales de género $g \geq 2$ son de C. Earle [Ear71, p. 126] y G. Shimura [Shi72, p. 177] y son curvas hiperelípticas de género par.

En la literatura uno puede encontrar dos grandes enfoques al estudio de superficies de Riemann pseudoreales: un enfoque de teoría de números y un enfoque a través de los grupos NEC. El primer enfoque trata, de manera general, el problema de decidir cuando el cuerpo de moduli de una curva (Definición 1.1.10) es un cuerpo de definición (Definición 1.1.1). En este contexto, las curvas pseudoreales son curvas complejas cuyo cuerpo de moduli está contenido en $\mathbb{R}$, pero que no tienen a $\mathbb{R}$ como un cuerpo de definición (Definición 1.1.14). Una herramienta fundamental en este enfoque es un teorema clásico de A. Weil (Teorema 1.1.7), que proporciona condiciones necesarias y suficientes para que una variedad proyectiva definida sobre un cuerpo L sea definible sobre un subcuerpo $K \subseteq L$ cuando la extensión es Galois. Más recientemente, P. Dèbes y M. Emsalem probaron que $X/\mathrm{Aut}(X)$ siempre se puede definir sobre el cuerpo de moduli de X y que X tiene la misma propiedad cuando un determinado modelo B de $X/\mathrm{Aut}(X)$ sobre el subcuerpo $K \subseteq L$ tiene un punto K-racional (ver [DE99, Corolario 4.3 (c)]). En particular, este resultado resulta muy útil cuando $X/\mathrm{Aut}(X)$ tiene género 0: esto ha sido usado por B. Huggins para completar la clasificación de las curvas hiperelípticas pseudoreales (ver [Hug05, Proposición 5.0.5]) y fue posteriormente generalizado por A. Kontogeorgis en [Kon09] estudiando curvas p-gonales. Desafortunadamente, el resultado de Dèbes-Emsalem nos es fácil de aplicar si $X/\mathrm{Aut}(X)$ tiene género distinto de 0.

Un segundo enfoque, especifico de las superficies de Riemann compactas, es a través de la teoría de los grupos Fuchsianos, y más generalmente de los grupos cristalográficos no-euclidianos (grupos NEC, Definición 1.2.19), los cuales son subgrupos discretos Δ del grupo full de automorfismos del plano hiperbólico $\mathbb{H}$ tal que $\mathbb{H}/\Delta$ es una superficie de Klein compacta (Definición 1.2.8). De hecho, por el teorema de uniformización (ver [Sch89, Capítulo 7]) cualquier superficie de Riemann X de género $g \geq 2$ es el cociente de $\mathbb{H}$ por un grupo Fuchsiano libre de torsión Δ. Más aún, el grupo full de automorfismos $\mathrm{Aut}^{\pm}(\mathbb{H}/\Delta)$ de $\mathbb{H}/\Delta$ es el cociente $N_{\mathrm{Aut}^{\pm}(\mathbb{H})}(\Delta)/\Delta$, donde $N_{\mathrm{Aut}^{\pm}(\mathbb{H})}(\Delta)$ es el normalizador de Δ en $\mathrm{Aut}^{\pm}(\mathbb{H})$ (Teorema 1.2.22), y su grupo conformal de automorfismos $\mathrm{Aut}^{+}(\mathbb{H}/\Delta)$

es $N_{\mathrm{Aut}^{\pm}(\mathbb{H})}(\Delta)^{+}/\Delta$, donde $N_{\mathrm{Aut}^{\pm}(\mathbb{H})}(\Delta)^{+}$ es el subgrupo Fuchsiano canónico de $N_{\mathrm{Aut}^{\pm}(\mathbb{H})}(\Delta)$ (Definición 1.2.19). Si G es el grupo full de automorfismos de una superficie de Riemann pseudoreal, entonces el grupo conformal de automorfismos G^{+} de G es un subgrupo de índice dos tal que $G\backslash G^{+}$ no contiene involuciones. Más aún, existe un epimorfismo $\varphi : \Gamma \to G$ de un grupo NEC Γ tal que $\ker(\varphi)$ es un grupo libre de torsión y $\varphi(\Gamma^{+}) = G^{+}$, donde Γ^{+} es el subgrupo Fuchsiano canónico libre de Γ. Esta idea le permitió a D. Singerman probar la existencia de superficies de Riemann pseudoreales para cada género (Teorema 3.1.1). Más aún, ha sido usada por E. Bujalance, M. Conder y A.F. Costa en [BCC10] y [BC14] para clasificar los grupos full de automorfismos de superficies de Riemann pseudoreales hasta género 4.

El objetivo de esta tesis es proporcionar una introducción a ambos enfoques y mostrar nuevos resultados en estos tópicos. La tesis está organizada de la siguiente manera. En el Capítulo 1 entregamos los contenidos básicos de ambos enfoques, definiendo y explicando los conceptos de cuerpo de definición y cuerpo de moduli (Sección 1.1), superficies de Riemann y de Klein, los grupos de automorfismos de dichas superficies, grupos Fuchsianos y NEC, sus signaturas (Sección 1.2), y el concepto de superficie de Riemann pseudoreal (Sección 1.3). En el Capítulo 2 entregamos las principales herramientas y resultados del problema de la definibilidad de una curva X sobre su cuerpo de moduli, cuando $X/\mathrm{Aut}(X)$ tiene género 0, considerando X definida sobre un cuerpo no necesariamente igual a $\mathbb{C}$. Revisamos los principales teoremas y mostramos los resultados obtenidos para curvas hiperelípticas y p-gonales. Nos gustaría puntualizar que la Sección 2.3 contiene nuestros primeros resultados nuevos, que son los siguientes:

Teorema 1. (Teorema 2.3.5) *Sea F un cuerpo perfecto infinito de característica $q \neq 2$ y sea $\overline{F}$ una clausura algebraica de F. Sea X una curva de género $g \geq 2$ definida sobre $\overline{F}$ y sea $Z(G)$ el centro del grupo de automorfismos G de X. Supongamos que $X/Z(G)$ tiene género 0, y que $G/Z(G)$ no es ni trivial ni cíclico (si $q = 0$), ni tampoco cíclico de orden coprimo con q (si $q \neq 0$). En este caso X se puede definir sobre $M_{\overline{F}/F}(X)$.*

Corolario 1. (Corolario 2.3.6) *Si X es una superficie de Riemann pseudoreal tal que el cociente $X/Z(\mathrm{Aut}(X))$ tiene género 0, entonces $\mathrm{Aut}(X)$ debe ser un grupo Abeliano.*

En el Capítulo 3 entregamos las principales herramientas y resultados en el enfoque de grupos NEC, como la existencia de superficies de Riemann pseudoreales en cada género, la caracterización de los grupos full de automorfismos de superficies de Riemann pseudoreales, y las clasificaciones conocidas de grupos conformales y grupos full de automorfismos de superficies de Riemann con respecto a un género fijo. Entregamos nuevos resultados (Corolario 3.3.5) que nos dan condiciones para la existencia de una extensión de grado dos de un grupo dado, lo que nos permite obtener una demostración más simple de [DE99, Corolario 4.3 (a)], y condiciones suficientes para que un grupo sea el grupo conformal de automorfismos de una superficie de Riemann pseudoreal.

Teorema 2. (Teorema 3.3.7) *Si G es un grupo tal que $Z(G) = \{1\}$ y $\mathrm{Inn}(G)$ tiene complemento de grupo en $\mathrm{Aut}(G)$, entonces cualquier extensión de grado dos de G será un producto semidirecto de C_2 y G.*

Corolario 2. (Corolario 3.3.8) *Sea G el grupo conformal de automorfismos de una superficie de Riemann X. Supongamos que $Z(G) = \{1\}$ y que $\mathrm{Inn}(G)$ tiene complemento de grupo en $\mathrm{Aut}(G)$. Entonces X no puede ser una superficie de Riemann pseudoreal.*

Teorema 3. (Corolario 3.3.6) *Si G es un grupo tal que $Z(G) = \{1\}$ y $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ no tiene involuciones — donde $\mathrm{Inn}(G)$ es el grupo de automorfismos internos de G —, entonces cualquier extensión de G por C_2 es un producto directo de G y C_2.*

Corolario 3. (Corolario 3.3.9) *Si G es el grupo conformal de automorfismos de una superficie de Riemann X tal que $Z(G) = \{1\}$ y $\mathrm{Out}(G)$ no tiene involuciones, entonces X no puede ser pseudoreal.*

En la Sección 3.4.3 completamos el trabajo realizado en [BG10] identificando condiciones necesarias y suficientes para encontrar grupos full de automorfismos de superficies de Riemann pseudoreales con signatura NEC $(3; -; [-]; \{-\})$ (Lema 3.4.5). Finalmente, consideramos los grupos full maximales de automorfismos de superficies de Riemann pseudoreales (ver Teorema 3.4.6) y demostramos los siguientes resultados.

Teorema 4. (Teorema 3.4.8) *Si una superficie de Riemann pseudoreal X tiene grupo full maximal de automorfismos, entonces su grupo conformal de automorfismos no es Abeliano.*

Corolario 4. (Corolario 3.4.12) *Si X es una superficie de Riemann pseudoreal con grupo full maximal de automorfismos, entonces no puede ser una curva superelíptica generalizada* (ver Definición 3.4.11).

El Capítulo 4 contiene un resumen de todos los grupos full de automorfismos de superficies de Riemann pseudoreales de género $2 \leq g \leq 4$ que se conocen, junto con modelos algebraicos que encontramos en la literatura (Sección 4.1). Más aún, extendimos la clasificación de los grupos full de automorfismos hasta género 10 en nuestro siguiente resultado.

Teorema 5. (Teorema 4.2.1) *Dos grupos finitos G y $\overline{G}$ son el grupos conformal y el grupo full de automorfismos de una superficie de Riemann pseudoreal X de género $5 \leq g \leq 10$ sí y solo si $G = \mathrm{Aut}^{+}(X)$ y $\overline{G} = \mathrm{Aut}^{\pm}(X)$ en la correspondiente tabla por género de entre las tablas 5.2, 5.3, 5.4, 5.5, 5.6, 5.3.4, y 5.7.*

En la Sección 4.3 describimos los modelos algebraicos conocidos para superficies de Riemann pseudoreales de género $5 \leq g \leq 10$. Finalizamos el capítulo proporcionando nuestros siguientes teoremas.

Teorema 6. (Teorema 4.4.1) *Si X es una quíntica plana pseudoreal X, entonces $G = \mathrm{Aut}^+(X)$ y $\overline{G} = \mathrm{Aut}^{\pm}(X)$ deben estar en una fila de la Tabla 4.3.*

Teorema 7. (Teorema 4.5.1) *Dos grupos finitos G y $\overline{G}$ son el grupo conformal y el grupo full de automorfismos de una curva superelíptica generalizada pseudoreal X de género $3 \leq g \leq 10$ y con elemento central τ* (recuerde Definición 3.4.11) *sí y solo si $G = \mathrm{Aut}^+(X)$ y $\overline{G} = \mathrm{Aut}^{\pm}(X)$ en la correspondiente tabla por género de entre las tablas 5.8, 5.9, 5.10, 5.11, 5.12, 5.13, 5.14 and 5.15.*

En el Capítulo 5 entregamos los programas de Magma [BCP97] que escribimos para llevar a cabo la clasificación en el capítulo anterior, y para hacer conjeturas sobre los grupos conformales y grupos full de automorfismos de superficies de Riemann pseudoreales.

Finalmente, en el Apéndice A damos una lista de todos los grupos que utilizamos en esta tesis, junto con una presentación, sus órdenes y sus ID number cuando es posible. En el Apéndice B damos las tablas de clasificación de grupos conformales y grupos full de automorfismos de superficies de Riemann pseudoreales de género $5 \leq g \leq 10$. El Apéndice C contiene las tablas de clasificación de grupos conformales y grupos full de curvas superelípticas generalizadas pseudoreales de género $3 \leq g \leq 10$.

Chapter 1

Preliminaries

1.1 Fields of Moduli of projective curves

In this thesis *curve* is a *smooth projective irreducible algebraic curve*, a *cyclic group* is a monogenous group, and $A \backslash B$ means $\{x \in A : \ x \notin B\}$.

Definition 1.1.1. Let $\overline{F}$ be an algebraically closed field and let $X \subseteq \mathbb{P}^n_{\overline{F}}$ be a curve defined as the zero locus of some homogeneous polynomials $\{p_i\}_{i=0}^r \subseteq \overline{F}[x_0, \dots, x_n]$, i.e.

$$X = \{x \in \mathbb{P}^n_{\overline{F}} : \quad p_i(x) = 0, \quad i = 0, \dots, r\}.$$

If $\overline{F}/K$ is a field extension, we say that K is a *field of definition* of X if there exists a curve $Y \subseteq \mathbb{P}^m_{\overline{F}}$ defined by

$$Y = \{x \in \mathbb{P}^m_{\overline{F}} : \quad q_i(x) = 0, \quad i = 0, \dots, s\},$$

such that $\{q_i\}_{i=0}^s \subseteq K[x_0, \dots, x_n]$ and X is isomorphic to Y over $\overline{F}$. In this case, we say that Y is a *K-model* for X.

Definition 1.1.2. If $f : X \longrightarrow Y$ is a morphism between the curves X and Y, then we say that f *is defined over the field* F if the polynomials defining f have all their coefficients in F.

Example 1.1.3. The curve $X \subseteq \mathbb{P}^2_{\mathbb{C}}$ given by

$$X : \quad x^3 + y^3 + iz^3 = 0$$

is clearly defined over $\mathbb{C}$, but it has also $\mathbb{R}$ (or $\mathbb{Q}$) as a field of definition because we have the isomorphism

$$f : X \longrightarrow Y \quad , \quad [x : y : z] \mapsto [x : y : -zi]$$

defined over $\mathbb{C}$, where Y is the curve in $\mathbb{P}^2_{\mathbb{C}}$ defined by

$$Y : \quad x^3 + y^3 + z^3 = 0.$$

Definition 1.1.4. If F/K is a field extension, then we define

$$\mathrm{Aut}(F/K) := \{\sigma \in \mathrm{Aut}(F) \ : \ \sigma|_K = \mathrm{Id}_K\},$$

where $\mathrm{Aut}(F)$ is the group of all automorphisms of F.

Definition 1.1.5. Let $X \subseteq \mathbb{P}^n_F$ be a curve defined as the zero locus of some homogeneous polynomials $\{p_i\}_{i=0}^r \subseteq F[x_0, \dots, x_n]$, and let $K \subseteq F$ be a subfield. If $\sigma \in \mathrm{Aut}(F/K)$, we denote by X^σ the curve defined by the zero locus of the homogeneous polynomials $\{p_i^\sigma\}_{i=0}^r \subseteq F[x_0, \dots, x_n]$, where p_i^σ is the polynomial given by applying σ to all the coefficients of p_i, for $i \in \{0, \dots, r\}$.

Let us know now that we can also define an action of $\mathrm{Aut}(F/K)$ on a morphism f as follows.

Definition 1.1.6. If $\sigma \in \mathrm{Aut}(F/K)$ and $f : X \to Y$ is a morphism, we define

$$f^\sigma : X^\sigma \longrightarrow Y^\sigma, \qquad z \mapsto \sigma(f(\sigma^{-1}(z))),$$

where $\sigma[x_0 : x_1 : \ldots : x_n] = [\sigma(x_0) : \sigma(x_1) : \ldots : \sigma(x_n)]$.

By the definition of f^σ, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \sigma \big\downarrow & & \big\downarrow \sigma \\ X^\sigma & \xrightarrow[\ f^\sigma\ ]{} & Y^\sigma \end{array}$$

A very important result in this theme is the following

Theorem 1.1.7. ***(Weil's Theorem)*** [Wei56, Theorem 1] *Let X be a (not necessarily smooth) curve defined over F and let F/K be a Galois extension. If for every $\rho \in \mathrm{Aut}(F/K)$ there exists a birational map $f_\rho : X \longrightarrow X^\rho$ defined over F such that*

$$f_{\sigma\tau} = f_\tau^\sigma \circ f_\sigma, \qquad \forall \sigma, \tau \in \mathrm{Aut}(F/K),$$

then there exist a curve Y defined over K and a birational map $g : X \longrightarrow Y$ defined over F such that $Y = Y^\mu$ and $g^\mu \circ f_\mu = g, \forall \mu \in \mathrm{Aut}(F/K)$.

Let us give here a brief sketch of the proof for the existence of the curve Y of the above theorem. One uses the fact that for every $\sigma \in \mathrm{Aut}(F/K)$ there exists a birational map $f_\sigma : X \longrightarrow X^\sigma$, and one can build a map $\sigma^* : F(X) \longrightarrow F(X)$ defined by $\phi \mapsto \phi^\sigma \circ f_\sigma$, where $F(X)$ is the rational function field of the curve X (see [Har77, p. 16]). Then one can build a monomorphism $\Phi : \mathrm{Aut}(F/K) \longrightarrow \mathrm{Aut}(F(X)/K)$ defined by $\sigma \mapsto \sigma^*$ (see [Hid10, Lemma 3.3.1]). The group $\Phi(\mathrm{Aut}(F/K))$ has a fixed field $\mathbb{F}$ in $F(X)$, which is finitely generated over K (see [Wei55, Proposition 3]), say $\mathbb{F} = \langle a_1, \dots, a_m \rangle$. Then the homomorphism

$$\theta : K[x_1, \dots, x_m] \longrightarrow \mathbb{F}, \qquad x_i \mapsto a_i$$

has a kernel $\mathrm{Ker}(\theta)$ which determines a curve Y defined over K with the same rational function field as X, so they are birational. For more details see [Hid10, Chapter 3].

Remark 1.1.8. In Weil's theorem, if we assume the $f_\rho : X \longrightarrow X^\rho$ are biregular maps, then we can find a biregular isomorphism $g : X \longrightarrow Y$ defined over F such that $Y = Y^{\mu}$ and $g^{\mu} \circ f_{\mu} = g, \forall \mu \in \mathrm{Aut}(F/K)$ (see [Hid10, Chapter 3]).

It is natural to ask for the smallest field of definition of a curve, and an obvious candidate could be the intersection of all its fields of definition.

Definition 1.1.9. [Koi72, Definition 1.1] Let X be a curve defined over a field F, and consider it inside $\mathbb{P}^n_{\overline{\mathbb{F}}}$. The *field of moduli* F_X of X is the intersection of all the fields of definition of X.

Furthermore, one can define the concept of field of moduli of a curve relative to a field extension as follows:

Definition 1.1.10. The *field of moduli of a curve X relative to a Galois extension* F/K is

$$M_{F/K}(X) := \mathrm{Fix}(F_K(X)) := \{a \in F : \sigma(a) = a \ , \ \forall \sigma \in F_K(X)\},$$

where $F_K(X) = \{\sigma \in \mathrm{Aut}(F/K) : X \cong_{\overline{F}} X^{\sigma}\}$ and $X \cong_{\overline{F}} X^{\sigma}$ means there exists an isomorphism between X and X^{σ} defined over $\overline{F}$.

Koizumi in [Koi72] proved that $M_{\overline{F}/P}(X)$ is a purely inseparable extension of F_X [Koi72, Proposition 2.3 (ii)], where P is the prime field of $\overline{F}$. The two previous definitions coincide if F is a perfect field [DF04, Definition, p. 549], so in this case

$$M_{\overline{F}/P}(X) \cong F_X.$$

The main relation between the field of moduli in Definition 1.1.9 and Definition 1.1.10 is the following theorem:

Theorem 1.1.11. [Hug05, Theorem 1.6.9] *Let X be a curve defined over a field K and let K_X be the field of moduli of X [*in the sense of Definition 1.1.9*]. The curve X is definable over K_X if and only if given any algebraically closed field F such that $K \subseteq F$,* *and any subfield $L \subseteq F$ with F/L Galois, X_F [X* seen as a curve defined over *F] can be defined over its field of moduli relative to the extension F/L.*

Proposition 1.1.12. *If F/K is a Galois extension and X is a curve defined over the field F, then we have the following properties:*

1. *$K \subseteq M_{F/K}(X) \subseteq F$.*
2. *$M_{F/R}(X) = R$, where $R = M_{F/K}(X)$.*
3. *If $X \cong_{\overline{F}} Y$ then $M_{F/K}(X) = M_{F/K}(Y)$.*
4. *If F' is any field of definition of X such that $K \subseteq F' \subseteq F$, then $M_{F/K}(X) \subseteq F'$.*

Proof. To prove *1,2* and *3* we only have to understand the definition of field of moduli relative to a field extension. We will prove here only *4.* to show the relevance of the Galois extension hypothesis.

If F' is a field of definition of X, then there exists a curve $Y \cong_{\overline{F}} X$ such that Y has all its coefficients in F'. Then if we take $\sigma \in \text{Aut}(F/F')$, because of $\sigma|_{F'} = \text{Id}_{F'}$, we know that $\sigma \in \text{Aut}(F/K)$ and also $Y = Y^\sigma$, so we have $\sigma \in F_K(Y)$. Hence $\text{Aut}(F/F') \leq F_K(Y)$ and by definition of field of moduli relative to the extension F/K we have

$$M_{F/K}(Y) = \text{Fix}(F_K(Y)) \subseteq \text{Fix}(\text{Aut}(F/F')) = F',$$

where the last equality comes from the fact that F/K is Galois implies F/F' is also Galois, and we conclude by the Galois correspondence. Finally, since $X \cong_{\overline{F}} Y$, by *3.* we have $M_{F/K}(X) \subseteq F'$. □

Proposition 1.1.13. *If X is a curve defined over a field F, F/K is a Galois extension and* $\text{Aut}(X)$ *is trivial, then X can be defined over its field of moduli $M_{F/K}(X)$.*

Proof. Let $R = M_{F/K}(X)$. Since F/K is a Galois extensions, then F/R is also a Galois extension. We have

$$\text{Fix}(\text{Aut}(F/R)) = \text{Fix}(F_R(X)),$$

because both fields are R. Then by the Galois correspondence we have

$$\mathrm{Aut}(F/R) = F_R(X) = \{\sigma \in \mathrm{Aut}(F/K) : \ X \cong_{\overline{F}} X^\sigma\},$$

so for every $\sigma, \tau \in \mathrm{Aut}(F/R)$ there exist isomorphisms

$$f_{\tau\sigma} : X \longrightarrow X^{\tau\sigma}, \quad f_\sigma^\tau : X^\tau \longrightarrow (X^\sigma)^\tau = X^{\tau\sigma}, \quad f_\tau : X \longrightarrow X^\tau.$$

Since $(f_{\tau\sigma})^{-1} \circ f_\sigma^\tau \circ f_\tau : X \longrightarrow X$ is an automorphism of X and $\mathrm{Aut}(X) = \{\mathrm{Id}_X\}$, then $f_{\tau\sigma} = f_\sigma^\tau \circ f_\tau$, and by Weil's theorem we conclude that X can be defined over $R = M_{F/K}(X)$. □

The field of moduli F_X of a curve X defined over a field F may be properly contained in every field of definition of X, but they coincide when we consider curves of genus 0 and 1.

1. **Genus 0 case.** If X is a curve of genus 0 defined over a field F, and if we have a Galois extension F/K, then X is isomorphic to $\mathbb{P}^1_F$ and can be seen as the zero set of the polynomial

$$P(x_0, x_1, x_2) = x_2 \in K[x_0, x_1, x_2]$$

in $\mathbb{P}^2_F$. Hence it can clearly be defined over K, i.e. K is a field of definition of X. By Proposition 1.1.12 *1* and *4*, we obtain that $M_{F/K}(X) = K$.

2. **Genus 1 case.** Let X be a curve of genus 1 defined over $\overline{F}$, where F is a perfect field. It is known that $\overline{F}/F$ is a Galois extension if and only if F is a perfect field. We will prove that $M_{\overline{F}/F}(X)$ is a field of definition of X.
We know that $X \cong_{\overline{F}} C_\lambda$, where $C_\lambda \subseteq \mathbb{P}^2_{\overline{F}}$ is a smooth model of X defined by the zero locus of the polynomial

$$P_\lambda(x_0, x_1, x_2) = x_1^2 x_2 - x_0(x_0 - x_2)(x_0 - \lambda x_2) \in \overline{F}[x_0, x_1, x_2],$$

and $\lambda \in \overline{F} - \{0,1\}$. In fact, two models C_λ and C_μ are isomorphic if and only if $j(\lambda) = j(\mu)$, where j is the j-invariant

$$j(x) = 256\frac{(1-x+x^2)^3}{x^2(1-x)^2}$$

(for more details see [Har77, Chapter 4, Section 4]). Then we have

$$M_{\overline{F}/F}(X) = M_{\overline{F}/F}(C_\lambda) := \mathrm{Fix}(\{\sigma \in \mathrm{Aut}(\overline{F}/F) : C_\lambda \cong_{\overline{F}} C_\lambda^\sigma\}),$$

where the first equality is due to Proposition 1.1.12 *3*. Note that $C_\lambda^\sigma = C_{\sigma(\lambda)}$, so

$$M_{\overline{F}/F}(X) := \mathrm{Fix}(\{\sigma \in \mathrm{Aut}(\overline{F}/F) : j(\lambda) = j(\sigma(\lambda)) = \sigma(j(\lambda))\}),$$

where the last equality is valid because $j(\lambda)$ could be thought as a function of λ with coefficients in F; so

$$M_{\overline{F}/F}(X) = F(j(\lambda)),$$

where $F(j(\lambda))$ is the smallest field which contains F and $j(\lambda)$. For every $\lambda \in \overline{F}$, by [Hid10, p. 44] we can find a smooth plane model for C_λ which is defined over $F(j(\lambda))$, so $M_{\overline{F}/F}(X)$ is also a field of definition of X.

During the last decades there has been a lot of work in determining conditions for the curves of genus $g \geq 2$ defined over a perfect field F to be defined over their field of moduli relative to the Galois extension $\overline{F}/F$, or how far is this field from being a field of definition. Remembering that the generic curve of genus $g > 2$ has trivial automorphism group (see [Gre63, Theorem 2]), by Proposition 1.1.13 and Theorem 1.1.11 we deduce that X is always defined over its field of moduli F_X. Therefore, the problem of finding special curves which are not definable over their field of moduli reduces to study curves with non trivial automorphisms and to check if they satisfy or not the Weil's theorem, but in fact this is very hard to verify in general.

In this thesis we will focus on complex curves X and the conditions under which its field of moduli $M_{\mathbb{C}/\mathbb{R}}(X)$ with respect to the Galois extension $\mathbb{C}/\mathbb{R}$ is not a field of definition. We resume this concept in the next

Definition 1.1.14. A *pseudoreal curve* is a complex curve X such that $M_{\mathbb{C}/\mathbb{R}}(X) = \mathbb{R}$ but X does not have $\mathbb{R}$ as a field of definition.

1.2 Riemann surfaces, Klein surfaces and their automorphism groups

1.2.1 Riemann and Klein surfaces

In this section we give the definitions and some basic facts about Riemann surfaces, Klein surfaces, and their automorphism groups. For more details see [BEGG90, Chapter 0, Chapter 1].

Definition 1.2.1. A holomorphic map $f : A \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ is said to be *conformal* if $f'(z) \neq 0, \forall z \in A$, which is equivalent to say that f preserve oriented angles locally.

Definition 1.2.2. A *complex atlas* on a topological space X is a collection of pairs $\{(U_i, \phi_i)\}_{i\in I}$ such that $\{U_i\}_{i\in I}$ is an open covering of X and $\phi_i : U_i \longrightarrow V_i$ are homeomorphisms, where V_i are open sets of $\mathbb{C}$, such that every function

$$\phi_i^{-1} \circ \phi_j : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

is a conformal function.

Two complex atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent* if their union is still a complex atlas. This is an equivalence relation, and every equivalence class of complex atlases is called a *complex structure* on X.

Definition 1.2.3. A *Riemann surface* is a connected Hausdorff second countable topological space with a complex structure.

Definition 1.2.4. If X and Y are Riemann surfaces with complex structures $\{(U_i, \phi_i)\}$ and $\{(V_j, \varphi_j)\}$ respectively, we say that a continuous map $F : X \longrightarrow Y$ is a *conformal map* if for every $p \in X$ and for every U_i and V_j containing p and $F(p)$ respectively, the *transition function*

$$\varphi_j \circ F \circ \phi_i^{-1} : \phi_i(F^{-1}(V_j) \cap U_i) \longrightarrow \mathbb{C}$$

is a conformal function.

A *conformal isomorphism* between two Riemann surfaces X and Y is a conformal map $F : X \longrightarrow Y$ such that there exists a conformal map $G : Y \longrightarrow X$ which satisfies $F \circ G = \mathrm{Id}_Y$ and $G \circ F = \mathrm{Id}_X$.

Definition 1.2.5. An *conformal automorphism* of a Riemann surface X is a conformal isomorphism from X to itself. The set of all conformal automorphisms of X is a group with respect to composition, and it will be called the *conformal automorphism group of* X, denoted by $\mathrm{Aut}^+(X)$.

We will study the conformal and anticonformal groups of automorphisms of Riemann surfaces, so we need to enlarge the structure of Riemann surface to admit the anticonformal maps.

Definition 1.2.6. Let f be an analytic function, considered as a function of real variables $f(x, y) = u(x, y) + iv(x, y)$. We define the *complex conjugate of* f as the map $\overline{f}(x, y) = u(x, y) - iv(x, y)$.

Definition 1.2.7. A *dianalytic atlas* on a topological space X is a collection of pairs $\{(U_i, \phi_i)\}_{i \in I}$ such that $\{U_i\}_{i \in I}$ is an open covering of X and $\phi_i : U_i \longrightarrow V_i$ are homeomorphisms, where V_i are open sets of $\mathbb{C}$ or $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$, such that every function

$$\phi_i^{-1} \circ \phi_j : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

is a conformal function, or the complex conjugate of a conformal function.

Two dianalytic atlases $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent* if their union is still a dianalytic atlas. This is an equivalence relation and every equivalence class of dianalytic atlases is called the *dianalytic structure* on X.

Definition 1.2.8. A *Klein surface* is a connected Hausdorff second countable topological space X with a dianalytic structure $\{(U_i, \phi_i)\}_{i\in I}$.

Definition 1.2.9. The *boundary* of a Klein surface X with dianalytic structure $\{(U_i, \phi_i)\}_{i\in I}$ is the set

$$\partial X := \{x \in X : \; x \in U_i, \; \phi_i(x) \in \mathbb{R}, \; \phi(U_i) \subseteq \mathbb{C}^+, \text{for some } i \in I\}$$

Klein surfaces are a generalization of Riemann surfaces in the following sense.

Theorem 1.2.10. [BEGG90, Proposition 1.2.2] *Riemann surfaces of genus $g \geq 2$ are precisely the orientable and unbordered Klein surfaces of genus $g \geq 2$.*

In general, Klein surfaces admit border and could be non-orientable. Here are some examples of Klein surfaces:

1. The Riemann sphere $\hat{\mathbb{C}}$ with the dianalytic atlas

$$\{(\mathbb{C}, \mathrm{Id}_{\mathbb{C}}), (\mathbb{C} - \{0\} \cup \{\infty\}, [z \mapsto \frac{1}{z}, \infty \mapsto 0])$$

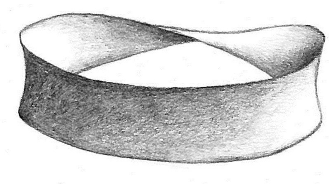
2. The closed disc $D := \{z \in \mathbb{C} : \; |z| \leq 1\}$. This is a Klein surface which is not a Riemann surface, because it has a border $\partial D = \{z \in \mathbb{C} : \; |z| = 1\}$.

3. The Mobius strip (see Figure 1.1). This is a Klein surface which is not a Riemann surface, because it is bordered and it is non-orientable.

4. The upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \; \mathrm{Im}(z) > 0\}$. This is a Riemann surface, hence a Klein surface, which will be very important in this thesis.

Definition 1.2.11. If X and Y are Klein surfaces with dianalytic structures $\{(U_i, \phi_i)\}$ and $\{(V_j, \varphi_j)\}$ respectively, we say that a continuous map $F : X \longrightarrow Y$ is a *Klein*

Figure 1.1: The Mobius strip



morphism if $F(\partial X) \subseteq \partial Y$ and for every $p \in X$ and for every U_i and V_j containing p and $F(p)$ respectively, the *transition function*

$$\varphi_j \circ F \circ \phi_i^{-1} : \phi_i(F^{-1}(V_j) \cap U_i) \longrightarrow \mathbb{C}$$

is a conformal function or the conjugate of a conformal function.
A *conformal Klein morphism* is a Klein morphism such that the transition maps are conformal functions. An *anticonformal Klein morphism* is a Klein morphism such that the transition functions are either conformal functions or the complex conjugate of a conformal function.
A *Klein isomorphism* between two Klein surfaces X and Y is a Klein morphism $F : X \longrightarrow Y$ such that there exists a Klein morphism $G : Y \longrightarrow X$ which satisfies $F \circ G = \mathrm{Id}_Y$ and $G \circ F = \mathrm{Id}_X$.

Definition 1.2.12. A *Klein automorphism* of a Klein surface X is a Klein isomorphism from X to itself. The set of all Klein automorphisms of X will be called the *full automorphism group of* X and will be denoted by $\mathrm{Aut}^{\pm}(X)$.

The topological genus of a Klein surface can be defined in terms of the Euler characteristic of the surface as follows.

Definition 1.2.13. The topological genus of a Klein surface X is

$$g(X) := \begin{cases} \dfrac{2-\chi(X)-k(X)}{2} & \text{if X is orientable,} \\ 2-\chi(X)-k(X) & \text{if X is not orientable,} \end{cases}$$

where $\chi(X)$ is the Euler characteristic of X (see [Mas77, Chapter 8]) and $k(X)$ is the number of connected components of the boundary of X.

Remark 1.2.14. In the case of Riemann surfaces X considered as Klein surfaces, let us note that it is possible to have $\mathrm{Aut}^{\pm}(X)$ different from its conformal automorphism group $\mathrm{Aut}^{+}(X)$. In such case, $\mathrm{Aut}^{\pm}(X)$ contains $\mathrm{Aut}^{+}(X)$ as a subgroup of index 2, because the composition of two anticonformal automorphisms is a conformal automorphism. The full automorphism group $\mathrm{Aut}^{\pm}(X)$ will contain then two cosets: the conformal automorphisms and the anticonformal automorphisms.

Remark 1.2.15. From here onwards, for every curve X we will use the notations $\mathrm{Aut}^{+}(X)$ or $\mathrm{Aut}(X)$ indistinctly for its conformal automorphism group (biregular automorphisms).

Theorem 1.2.16. [BEGG90, Corollary 1.3.5] *If X is a Klein surface of genus $g \geq 2$, then its full automorphism group $\mathrm{Aut}^{\pm}(X)$ is finite.*

In particular, by Theorem 1.2.16 every Riemann surface X (seen as a Klein surface) of genus $g \geq 2$ will have a full automorphism group $\mathrm{Aut}^{\pm}(X)$ of finite order, and by Remark 1.2.14 $\mathrm{Aut}^{+}(X)$ is either equal or is an index two subgroup of $\mathrm{Aut}^{\pm}(X)$.

Theorem 1.2.17. [BEGG90, Theorem 0.1.15] *The full automorphism group $\mathrm{Aut}^{\pm}(\mathbb{H})$ of the upper half plane $\mathbb{H}$ is isomorphic to $\mathrm{PGL}(2,\mathbb{R}) := GL(2,\mathbb{R})/\sim$, where $A \sim B$ if and only if there exists some real number $\lambda \neq 0$ such that $A = \lambda B$, via the group isomorphism*

$$\mathrm{PGL}(2,\mathbb{R}) \longrightarrow \mathrm{Aut}^{\pm}(\mathbb{H}) \quad , \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left[f_A \ : \ z \mapsto \begin{cases} \dfrac{az+b}{cz+d} & \text{if } det(A) = 1 \\ \dfrac{a\overline{z}+b}{c\overline{z}+d} & \text{if } det(A) = -1 \end{cases} \right].$$

In particular, the conformal automorphism group $\mathrm{Aut}^{+}(\mathbb{H})$ of the upper half plane $\mathbb{H}$ is $\mathrm{PSL}(2,\mathbb{R})$, sending the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R})$ to the map $z \mapsto \dfrac{az+b}{cz+d}$ (see [GGD12, Proposition 1.27]).

The following theorem will be very useful in the next chapter.

Theorem 1.2.18. [MV80, Theorem 1] *Let $\overline{F}$ be an algebraically closed field of characteristic p. If G a finite subgroup of* $\mathrm{PGL}(2,\overline{F})$, *then G is isomorphic to one of the following groups*

$$C_n,\ D_n,\ A_4,\ S_4,\ \textit{or}\ A_5 \quad \text{when } p=0 \text{ or when } gcd(|G|,p)=1,$$

$$C_p^t,\ C_p^t \rtimes C_m,\ \mathrm{PGL}(2,\mathbb{F}_{p^r}),\ \textit{or}\ \mathrm{PSL}(2,\mathbb{F}_{p^r}) \quad \text{when } p \text{ divides } |G|,$$

where $\mathbb{F}_{p^r}$ is the field of p^r elements, $gcd(n,p)=1$, $r>0$, $t\le r$, and m divides p^t-1. Moreover, the signature of the quotient $\mathbb{P}^1_{\overline{F}}/G$ is given in Table 1.1, where $\alpha=\dfrac{p^r(p^r-1)}{2}$ and $\beta=\dfrac{p^r+1}{2}$.

Table 1.1: Finite subgroups $G\le \mathrm{PGL}(2,\overline{F})$

Group G	Signature of $\mathbb{P}^1_{\overline{F}}/G$
C_n	(n,n)
D_n	$(2,2,n)$
$A_4,\ p\neq 2,3$	$(2,3,3)$
$S_4,\ p\neq 2,3$	$(2,3,4)$
$A_5,\ p\neq 2,3,5$	$(2,3,5)$
$A_5,\ p=3$	$(6,5)$
C_p^t	(p^t)
$C_p^t\rtimes C_m$	(mp^t,m)
$\mathrm{PSL}(2,\mathbb{F}_{p^r}),\ p\neq 2$	(α,β)
$\mathrm{PGL}(2,\mathbb{F}_{p^r})$	$(2\alpha,2\beta)$

1.2.2 NEC and Fuchsian groups

Definition 1.2.19. A *NEC group* is a discrete subgroup Δ of $\mathrm{PGL}(2,\mathbb{R})$ such that $\mathbb{H}/\Delta$ is a compact Klein surface. A *Fuchsian group* is a NEC group contained in $\mathrm{PSL}(2,\mathbb{R})$, the group of conformal automorphisms of $\mathbb{H}$. If Γ is a NEC group which is not a Fuchsian group, it is called a *proper NEC group* and the index 2 subgroup $\Gamma \cap \mathrm{PSL}(2,\mathbb{R})$ of Γ is called the *canonical Fuchsian subgroup of* Γ.

Because of Theorem 1.2.17 we can consider the full automorphism group $\mathrm{Aut}^{\pm}(\mathbb{H})$ as a topological group and the previous definition leads to the following theorem:

Theorem 1.2.20. [BEGG90, Proposition 1.2.3] *Every compact Klein surface X of genus $g \geq 2$ is Klein isomorphic to a quotient $\mathbb{H}/\Delta$, where Δ is a NEC group. We say that X is* uniformized by the NEC group Δ*.*

We can classify the elements in a NEC group depending on the fixed points they have in $\mathbb{H}$. If we take $f \in \mathrm{Aut}^{\pm}(\mathbb{H}) = \mathrm{PGL}(2,\mathbb{R})$, then there is only one matrix $A \in \mathrm{GL}(2,\mathbb{R})$ whose determinant is equal to 1 such that $f = f_A = f_{-A}$. This allows us to define $\det(f) := \det(A)$ and $\mathrm{tr}(f) := |\mathrm{tr}(A)|$.
If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $det(f) = 1$, then the fixed points $z \in \mathbb{H}$ of f satisfy

$$\frac{az+b}{cz+d} = z,$$

so we have

$$z = \frac{a-d \pm \sqrt{(a-d)^2 + 4bc}}{2c} = \frac{a-d \pm \sqrt{(d+a)^2 - 4}}{2c}$$

where the last equality comes from $ad - bc = 1$. The set of fixed points of f depends on the value of the discriminant $(d+a)^2 - 4 = tr(A)^2 - 4$ of the previous equation.
If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $det(f) = -1$, then the fixed points $z \in \mathbb{H}$ of f satisfy

$$\frac{a\overline{z}+b}{c\overline{z}+d} = z,$$

so we have $a\overline{z} + b = cz\overline{z} + dz$, which is equivalent to $a\overline{z} + b - dz = cz\overline{z}$. Since $\overline{cz\overline{z}} = cz\overline{z}$, we must have $a\overline{z} + b - dz = \overline{a\overline{z} + b - dz}$, which is equivalent to $(a+d)(z - \overline{z}) = 0$. We say that f is:

1. *hyperbolic* if $\det(f) = 1$ and $\mathrm{tr}(f) > 2$.

2. *elliptic* if $\det(f) = 1$ and $\operatorname{tr}(f) < 2$.

3. *a boundary element* if $\det(f) = 1$ and $\operatorname{tr}(f) = 2$.

4. *a glide reflection* if $\det(f) = -1$ and $\operatorname{tr}(f) \neq 0$.

5. *a reflection* if $\det(f) = -1$ and $\operatorname{tr}(f) = 0$.

Suppose we have a NEC group Δ such that $\mathbb{H}/\Delta$ has genus g, the projection map $\pi : \mathbb{H} \longrightarrow \mathbb{H}/\Delta$ has r branch points in the interior with ramification index $m_i \geq 2$, and the ramification index of π in the i-th boundary component of $\mathbb{H}/\Delta$, denoted by C_i, are s_i-uples of integers

$$C_i := (n_{i1}, \dots n_{is_i}),$$

such that $n_{ij} \geq 2$. The NEC group Δ has a presentation $(*)$ given by the following generators (see [BEGG90, p. 14]).

1. Hyperbolic elements $a_1, b_1, \dots, a_g, b_g$ if $\mathbb{H}/\Delta$ is orientable.

2. Elliptic elements $x_1, \dots, x_r$.

3. Boundary elements $e_1, \dots, e_k$.

4. $d_1, d_2, \dots, d_g$ if $\mathbb{H}/\Delta$ is not orientable.

5. Reflections $c_{10}, \dots, c_{1s_1}, \dots, c_{k0}, \dots, c_{ks_k}$.

These elements satisfy the relations

$$x_i^{m_i} = 1 \quad \forall i \in \{1 \dots r\},$$

$$c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1, \quad c_{is_i} = e_i^{-1} c_{i0} e_i, \quad \forall i \in \{1, \dots k\}, \quad \forall j \in \{0, \dots s_i\},$$

$$x_1 \dots x_r e_1 \dots e_k [a_1, b_1] \dots [a_g, b_g] = 1 \quad \text{(if } \mathbb{H}/\Delta \text{ is orientable)},$$

$$x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_g^2 = 1 \quad \text{(if } \mathbb{H}/\Delta \text{ is not orientable)},$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

In particular, if Δ is a NEC group without reflections, then it does not have boundary elements and it must satisfy just the next relations

$$x_i^{m_i} = 1, \quad \forall i \in \{1 \dots r\},$$

$$x_1 \dots x_r [a_1, b_1] \dots [a_g, b_g] = 1 \quad \text{(if } \mathbb{H}/\Delta \text{ is orientable)},$$

$$x_1 \dots x_r d_1^2 \dots d_g^2 = 1 \quad \text{(if } \mathbb{H}/\Delta \text{ is not orientable)}.$$

Keeping in mind Theorem 1.2.20, we can distinguish between two Klein surfaces in terms of their uniformizing NEC groups as follows

Theorem 1.2.21. [BEGG90, Theorem 1.3.2 (2)] *Consider two Klein surfaces X and Y uniformized by the NEC groups Δ_1 and Δ_2 respectively. Then X and Y are Klein isomorphic if and only if Δ_1 and Δ_2 are conjugate subgroups of* $\mathrm{PGL}(2, \mathbb{R})$.

We can also study the full automorphism group of a Klein surface in terms of the NEC group which uniformizes it.

Theorem 1.2.22. [BEGG90, Theorem 1.3.2 (3)] *If $X = \mathbb{H}/\Delta$ is a Klein surface uniformized by the NEC group Δ, then its full automorphism group $\mathrm{Aut}^{\pm}(X)$ is isomorphic to $N_{\mathrm{PGL}(2,\mathbb{R})}(\Delta)/\Delta$, where $N_{\mathrm{PGL}(2,\mathbb{R})}(\Delta)$ is the normalizer of Δ in* $\mathrm{PGL}(2, \mathbb{R})$.

Using the same notations, every subgroup $G \leq \mathrm{Aut}^{\pm}(X)$ will be a subgroup of $N_{\mathrm{PGL}(2,\mathbb{R})}(\Delta)/\Delta$, so $G \cong \Gamma/\Delta$ for some subgroup Γ such that $\Delta \leq \Gamma \leq N_{\mathrm{PGL}(2,\mathbb{R})}(\Delta)$. Note that this Γ is also a NEC group (see [BEGG90, Remark 1.3.6]) and we have the following corollary:

Corollary 1.2.23. *If $X = \mathbb{H}/\Delta$ is a Klein surface uniformized by the NEC group Δ, then $G \leq \mathrm{Aut}^{\pm}(X)$ if and only if $G \cong \Gamma/\Delta$ for some NEC group Γ which contains Δ as a normal subgroup.*

For that reason, the study of groups G acting on Klein surfaces can be done through the finite extendability of NEC groups Δ. With the previous notations, for every finite group G acting on a Klein surface $X = \mathbb{H}/\Delta$, we have the following diagram:

$$\begin{array}{ccc} \mathbb{H} & \longrightarrow & X := \mathbb{H}/\Delta \\ & \searrow \quad \swarrow & \\ & X/G \cong \mathbb{H}/\Gamma & \end{array}$$

The study of NEC groups Γ which represent the actions on X by finite groups G can be translated to the study of the possible epimorphisms from a NEC group Γ to the group G with Δ as its kernel, whose canonical Fuchsian subgroup Δ^+ must be torsion free, and we get the exact sequence of groups

$$1 \to \Delta \to \Gamma \xrightarrow{\theta} G \to 1,$$

where θ is an epimorphism that must preserve the orders of the ellipic elements, and must send conformal (anticonformal) to conformal (anticonformal) elements.

The case of Riemann surfaces with their conformal structure is analogous, as one can see in the following theorem:

Theorem 1.2.24. *Consider two Riemann surfaces $X = \mathbb{H}/\Delta_1$ and $Y = \mathbb{H}/\Delta_2$, where Δ_1, Δ_2 are the Fuchsian groups which uniformize them. We have:*

1. *X and Y are conformal isomorphic if and only if Δ_1 and Δ_2 are conjugate subgroups of* $\mathrm{PSL}(2,\mathbb{R})$ *(see* [GGD12, Proposition 2.25]*).*

2. *The conformal automorphism group $\mathrm{Aut}^+(X)$ of X is isomorphic to $N_{\mathrm{PSL}(2,\mathbb{R})}(\Delta)/\Delta$, where $N_{\mathrm{PSL}(2,\mathbb{R})}(\Delta)$ is the normalizer of Δ in* $\mathrm{PSL}(2,\mathbb{R})$ *(see* [GGD12, Proposition 2.35]*)*

3. *$G \leq \mathrm{Aut}^+(X)$ if and only if $G \cong \Gamma/\Delta$ for some Fuchsian group Γ which contains Δ as a normal subgroup of finite index (see* [GGD12, Corollary 2.38].

1.2.3 NEC and Fuchsian signatures

Because of the importance of NEC groups in studying the Klein surfaces, it is convenient to define a new concept which encapsulates the presentation of a NEC group Δ given after Theorem 1.2.20.

Definition 1.2.25. Let Δ be a NEC group with the above presentation 1.2.2 $(*)$. The *signature of the NEC group* Δ (also called *NEC signature*) is the vector given by

$$(g; \pm; [m_1, \ldots, m_r], \{C_1, \ldots C_k\}),$$

where the sign $+$ or $-$ depends on the quotient $\mathbb{H}/\Delta$ being orientable or not. If $r = 0$, we write $[-]$ instead of $[m_1, \ldots, m_r]$. If $k = 0$, we write $\{-\}$ instead of $\{C_1, \ldots, C_k\}$. If some C_i is empty, we write $(-)$ instead of C_i. If some m_i appears n times, then we write $[\ldots, m_i^n, \ldots]$ instead of $[\ldots, \underbrace{m_i, \ldots, m_i}_{n}, \ldots]$.

Definition 1.2.26. A NEC group Δ is called *surface NEC group* if it has signature

$$(g; \pm; [-]; \{\underbrace{(-), \ldots, (-)}_{k \text{ times}}\}),$$

with $k \geq 0$. In the Fuchsian case, we call *surface Fuchsian group* a NEC group with signature $(g; +; [-]; \{-\})$, and we will denote this by $(g; +; [-])$.

In the case of Fuchsian groups, we have no anticonformal elements and no boundary components, so the signature of a Fuchsian group will always have the form $(g; +; [m_1, \ldots, m_r])$. In this case we will denote it by $(g; [m_1, \ldots, m_r])$.

Remark 1.2.27. The surface NEC groups are precisely the NEC groups which uniformize Klein surfaces, and surface Fuchsian groups are the ones which uniformize Riemann surfaces.

Remark 1.2.28. It can be proved (see [BCG10, p. 12]) that a NEC group is a surface NEC group if and only if it has no non-trivial conformal elements of finite order.

Definition 1.2.29. Let us consider the following vectors

$$v := (g; \pm; [m_1, \dots, m_r], \{C_1, \dots, C_k\}),$$

$$v' := (g; +; [m_1, \dots, m_r]).$$

We define

$$\mu(v) := 2\pi \left(\epsilon g - 2 + k + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right)\right),$$

$$\mu(v') = 2\pi \left(2g - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)\right),$$

where $\epsilon = 2$ if the sign of v is $+$ and $\epsilon = 1$ otherwise.

We can build NEC signatures for which there exists a NEC group with that signature, as we see in the following theorem:

Theorem 1.2.30. [BEGG90, Theorem 0.2.8] *Let us consider the vector*

$$v := (g; \pm; [m_1, \dots, m_r], \{C_1, \dots, C_k\})$$

such that $g \geq 0, m_i \geq 2, n_{ij} \geq 2, k \geq 0,$ and $s_i \geq 0$. The vector v is the NEC signature of some NEC group Δ if and only if $\mu(s(\Delta))$ is positive and $\epsilon + g \geq 2$.
In the case of an arbitrary vector $v' := (g; [m_1, \dots, m_r])$ such that $g \geq 0$, $m_i \geq 2$, we have v' is the Fuchsian signature of some Fuchsian group Δ if and only if $\mu(s(\Delta))$ is positive.

We have a generalization of Riemann-Hurwitz formula in the following theorem.

Theorem 1.2.31. [BEGG90, Remark 0.2.9] *If $\Delta \leq \Gamma$ are NEC groups such that $[\Gamma : \Delta]$ is finite, then*

$$[\Gamma : \Delta] = \frac{\mu(s(\Delta))}{\mu(s(\Gamma))},$$

where μ is defined above in Definition 1.2.29.

We can know the signature of the canonical Fuchsian group (remember Definition 1.2.19) of a given NEC group thanks to [Sin74a, Theorem 2]. We will use a particular case of his result, as we see in the following theorem:

Theorem 1.2.32. *Let X/Δ be a Riemann surface (considered as a Klein surface) and denote by Γ the NEC group $N_{\mathrm{PGL}(2,\mathbb{R})}(\Delta)$ which corresponds to its full automorphism group. If Γ is a proper NEC group which has no reflections, then its signature has the form*

$$(g; -; [m_1, \ldots, m_r]),$$

and the signature of his canonical Fuchsian subgroup Γ^+ will be

$$(g-1; +; [m_1, m_1, \ldots, m_r, m_r]),$$

where every m_i appears two times.

Definition 1.2.33. Consider an epimorphism $\theta : \Delta \longrightarrow G$ from a NEC group Δ onto a finite group G which defines an action on a Riemann surface uniformized by $\mathrm{Ker}(\theta)$. If $s(\Delta) = (g; \pm; [m_1, \ldots, m_r])$, then we say that v is a *generating vector for the action* θ if

$$v = (\theta(d_1), \ldots, \theta(d_g), \theta(x_1), \ldots, \theta(x_r)), \quad \text{in } - \text{ case},$$

$$v = (\theta(a_1), \theta(b_1) \ldots, \theta(a_g), \theta(b_g), \theta(x_1), \ldots, \theta(x_r)), \quad \text{in } + \text{ case}.$$

Because of Theorem 1.2.24*3.* and Corollary 1.2.23, it is important to study the finite extendability of Fuchsian and NEC groups. The concept of signature of such groups is very useful, because just knowing the signature of a Fuchsian or NEC group Δ it is possible to determine if a group G for which there exists a torsion free kernel epimorphism $\theta : \Delta \longrightarrow G$ can be the conformal or full automorphism group of some Riemann surface.

For this reason, we need to introduce also the following concepts.

Definition 1.2.34. A Fuchsian group Δ is *finitely maximal* if it is not contained properly in another Fuchsian group with finite index. Denoting by $s(G)$ the signature of a NEC group G, the signature $(g; [m_1, \ldots, m_r])$ of a Fuchsian group Δ (a Fuchsian signature) is *finitely maximal* if for every Fuchsian group Γ containing Δ as a proper subgroup, we have $d(s(\Gamma)) \neq d(s(\Delta))$, where $d(g; [m_1, \ldots, m_r]) := 6g - 6 + 2r$ is the real dimension of the Teichmuller space of the signature $(g; [m_1, \ldots, m_r])$ (see [Sin74b, p. 19]).

Almost all Fuchsian signatures are finitely maximal, and those which are not finitely maximal were found by L. Greenberg and D. Singerman in the articles [Gre63] and [Sin72]. The authors determine there the so called **Singerman List** (see Table 1.2), which contains the only 19 non finitely maximal Fuchsian signatures.

As a consequence of this result, we deduce that if the signature of a Fuchsian group Γ such that $\Gamma/\Delta \cong G$ *is not in the Singerman list*, then G is the conformal automorphism group of X, where $X := \mathbb{H}/\Delta'$, where Δ' is a Fuchsian group isomorphic to Δ.

In the case that some group acts on a Riemann surface with a signature that appears in the Singerman list, there exist some theorems which allow one to determine whether the group is the conformal automorphism group of some Riemann surface or not. Some works in line with this are [BC99], where the authors work on cyclic groups, and the paper [BCC03], where the authors work in the general case giving sufficient conditions for the group to be the conformal automorphism group of some Riemann surface.

The analogous of Singerman List for NEC groups was developed and completed in [Buj82] and [EI06]. The lists are very long, so we recomend the reader to see them in their original two articles. We will use these lists to find NEC signatures which correspond to full automorphism groups of pseudoreal Riemann surfaces.

The following result is very important because it will allow us to conclude that a NEC signature is a maximal signature in terms of the maximality of its canonical

Table 1.2: Pairs of non-finitely maximal Fuchsian signatures

Singerman List		
σ_1	σ_2	$[\sigma_2 : \sigma_1]$
$(2; [-])$	$(0; [2, 2, 2, 2, 2, 2])$	2
$(1; [t, t])$	$(0; [2, 2, 2, 2, t])$	2
$(1; [t])$	$(0; [2, 2, 2, 2t])$	2
$(0; [t, t, t, t])$ $t \geq 3$	$(0; [2, 2, 2, t])$	4
$(0; [t_1, t_1, t_2, t_2])$ $t_1 + t_2 \geq 5$	$(0; [2, 2, t_1, t_2])$	2
$(0; [t, t, t])$ $t \geq 4$	$(0; [3, 3, t])$	3
$(0; [t, t, t])$ $t \geq 4$	$(0; [2, 3, 2t])$	6
$(0; [t_1, t_1, t_2])$ $t_1 \geq 3, t_1 + t_2 \geq 7$	$(0; [2, t_1, 2t_2])$	2
$(0; [7, 7, 7])$	$(0; [2, 3, 7])$	24
$(0; [2, 7, 7])$	$(0; [2, 3, 7])$	9
$(0; [3, 3, 7])$	$(0; [2, 3, 7])$	8
$(0; [4, 8, 8])$	$(0; [2, 3, 8])$	12
$(0; [3, 8, 8])$	$(0; [2, 3, 8])$	10
$(0; [9, 9, 9])$	$(0; [2, 3, 9])$	12
$(0; [4, 4, 5])$	$(0; [2, 4, 5])$	6
$(0; [n, 4n, 4n])$ $n \geq 2$	$(0; [2, 3, 4n])$	6
$(0; [n, 2n, 2n])$ $n \geq 3$	$(0; [2, 4, 2n])$	4
$(0; [3, n, 3n])$ $n \geq 3$	$(0; [2, 3, 3n])$	4
$(0; [2, n, 2n])$ $n \geq 4$	$(0; [2, 3, 2n])$	3

Fuchsian signature.

Theorem 1.2.35. [BCG10, Remark 1.4.7] *Let s be the signature of a proper NEC group. If the signature s^+ of its canonical Fuchsian subgroup if maximal, then so is s.*

1.3 Pseudoreal Riemann surfaces

The space $\mathcal{M}_g$ of conformal isomorphism classes of Riemann surfaces of genus g can be embedded into a complex projective space $\mathbb{P}^N_{\mathbb{C}}$ such that $\mathcal{M}_g \subseteq \mathbb{P}^N_{\mathbb{C}}$ becomes a quasi projective variety defined by polynomials with rational coefficients. For that reason, the complex conjugation in $\mathbb{P}^N_{\mathbb{C}}$ induces an anticonformal involution $\sigma^* : \mathcal{M}_g \longrightarrow \mathcal{M}_g$, where σ^* takes the conformal isomorphism class of a Riemann surface to the class of its complex conjugate.

Because of the dianalytic structure of the Klein surfaces, we can consider the set $\mathcal{M}_g^K$ of isomorphism classes of Riemann surfaces seen as Klein surfaces as the quotient space $\mathcal{M}_g/\langle\sigma^*\rangle$. The fixed point set $\mathrm{Fix}(\sigma^*)$ of σ^* is the preimage of the ramification locus of the projection $\mathcal{M}_g \longrightarrow \mathcal{M}_g^K$, and it consists of all Riemann surfaces of genus g which are isomorphic to their conjugate.

A Riemann surface in this branch locus which admits an anticonformal involution is called a *real Riemann surface* (see [AG69]). These surfaces have been widely studied in the last decades (see [Nat04], [BCG10], [Sin74b]). On the other hand, the Riemann surfaces in this branch locus which are not real Riemann surfaces are the main topic of this thesis, and for this reason we isolate the concept in the following definition:

Definition 1.3.1. A *pseudoreal Riemann surface* is a Riemann surface, seen as a Klein surface, which admits anticonformal automorphisms, but does not admit anticonformal involutions.

Definition 1.3.2. A NEC signature $(g; \pm; [m_1, \dots, m_r])$ is called an *even signature* if every m_i appear exactly an even number of times. Otherwise it is called *odd signature.*

Corollary 1.3.3. *If X is a pseudoreal Riemann surface uniformized by a Fuchsian group Δ, then $s(N_{\mathrm{PGL}(2,\mathbb{R})}(\Delta)^+)$ is an even signature.*

Proof. If $N_{\mathrm{PGL}(2,\mathbb{R})}(\Delta)$ has a reflection, then the epimorphism

$$\theta : N_{\mathrm{PGL}(2,\mathbb{R})}(\Delta) \longrightarrow N_{\mathrm{PGL}(2,\mathbb{R})}(\Delta)/\Delta \cong \mathrm{Aut}^{\pm}(X)$$

will send such reflection to an anticonformal automorphism of order 2, and this contradicts the fact that X is pseudoreal. In that case $N_{\mathrm{PGL}(2,\mathbb{R})}(\Delta)$ has neither reflections nor boundary elements. By Theorem 1.2.32 we get the result. □

Remark 1.3.4. This fact was used extensively in [BCC10] and [BC14] to classify the full automorphism groups of pseudoreal Riemann surfaces of low genus.

We can define $J : \mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$ as

$$J[x_0 : \ldots : x_n] := [\overline{x_0} : \ldots : \overline{x_n}].$$

If we have a complex curve $X \subseteq \mathbb{P}^n_{\mathbb{C}}$, we can restrict J to X and we get the anticonformal map $J|_X : X \longrightarrow J|_X(X) = \overline{X}$, where $\overline{X}$ is the complex conjugate of the curve X. Considering the transition functions of an automorphism of a complex curve (see [Mir97, Chapter I, Proposition 3.11]) it is possible to prove that $f : X \longrightarrow X$ is an anticonformal automorphism of X if and only if $J_X \circ f : X \longrightarrow \overline{X}$ is an isomorphism between X and $\overline{X}$.

If we have a Riemann surface, we can embed it in many ways in projectives spaces. All these embeddings give isomorphic curves (in the biregular sense), so the following definition makes sense:

Definition 1.3.5. If X is a Riemann surface, we define the *field of moduli* F_X *of* X as the field of moduli relative to the extension $\mathbb{C}/\mathbb{R}$ of any complex curve $Y \subseteq \mathbb{P}^n_{\mathbb{C}}$ which is the image of an embedding of X in $\mathbb{P}^n_{\mathbb{C}}$.

Theorem 1.3.6. [Hid10, Theorem 6.3.1] *A Riemann surface X has field of moduli $F_X = \mathbb{R}$ if and only if it has an anticonformal automorphism, and it has field of definition $\mathbb{R}$ if and only if it has some anticonformal involution.*

Proof. For the first part, we note that $M_{\mathbb{C}/\mathbb{R}}(X) = \mathbb{R}$ if and only if $X \cong_{\mathbb{C}} \overline{X}$. If X is isomorphic to $\overline{X}$, then there exists an isomorphism $f : \overline{X} \longrightarrow X$, and we can consider $f \circ J|_X : X \longrightarrow X$, where $J|_X : X \longrightarrow \overline{X}$ is the complex conjugation. This is an anticonformal automorphism because $J|_X \circ (f \circ J|_X)$ is an isomorphism between X and $\overline{X}$. Conversely, if there exists an anticonformal automorphism of X, say f, we see that $J|_X \circ f : X \longrightarrow \overline{X}$ is an isomorphism over $\mathbb{C}$ between X and $\overline{X}$.

For the second part of the statement, if X has field of definition $\mathbb{R}$, then $X \cong_{\mathbb{C}} \overline{X}$, and we can assume without loss of generality that $X = \overline{X}$. If we take the map $J|_X : X \longrightarrow \overline{X} = X$, it will be an anticonformal involution of X because $J \circ J : X \longrightarrow X$ is the identity isomorphism. Conversely, if X has an anticonformal involution $\tau : X \longrightarrow X$, then $J|_X \circ \tau : X \longrightarrow \overline{X}$ is an isomorphism. If $\mathrm{Aut}(\mathbb{C}/\mathbb{R}) = \{e, \sigma\}$, let $f_e : X \longrightarrow X^e = X$ be the identity map Id_X of X. For the complex conjugation σ, taking $f_\sigma := J|_X \circ \tau : X \longrightarrow X^\sigma = \overline{X}$, we have

$$f_\sigma = (f_\sigma)^e \circ f_e, \qquad f_\sigma = (f_e)^\sigma \circ f_\sigma, \qquad f_e = (f_e)^e \circ f_e,$$

and we see that

$$(f_\sigma)^\sigma \circ f_\sigma = (J|_X \circ \tau)^\sigma \circ (J|_X \circ \tau) = ((J|_X)^\sigma \circ \tau^\sigma) \circ (J|_X \circ \tau)$$

$$= ((J|_X)^\sigma \circ \tau^\sigma \circ J|_X) \circ \tau = \tau \circ \tau = \tau^2 = \mathrm{Id}_X = f_e,$$

so the associations $e \mapsto f_e$ and $\sigma \mapsto f_\sigma$ satisfy the Weil's theorem. Hence X can be defined over $\mathbb{R}$. □

Summing up Definition 1.3.1 and Theorem 1.3.6, we obtain the following corollary:

Corollary 1.3.7. *Pseudoreal Riemann surfaces are precisely those which have field of moduli $\mathbb{R}$, but cannot be defined over $\mathbb{R}$.*

Chapter 2

When $X/\mathrm{Aut}(X)$ has genus zero

2.1 Dèbes-Emsalem theorem

A very useful result that gives sufficient conditions for the problem of definability of a curve over its field of moduli is [DE99, Corollary 4.3]. The authors observe that for a curve X of genus $g \geq 2$ and a Galois extension F/K such that $K = M_{F/K}(X)$, for every $\sigma \in \mathrm{Aut}(F/K)$ there exists an isomorphism $f_\sigma : X \longrightarrow X^\sigma$ that induces an isomorphism

$$\varphi_\sigma : X/\mathrm{Aut}(X) \longrightarrow X^\sigma/\mathrm{Aut}(X^\sigma),$$

which makes the following diagram $(*)$ commute

$$\begin{array}{ccc} X & \xrightarrow{\quad f_\sigma \quad} & X^\sigma \\ \pi \Big\downarrow & & \Big\downarrow \pi^\sigma \\ X/\mathrm{Aut}(X) & \xrightarrow[\varphi_\sigma]{} & X^\sigma/\mathrm{Aut}(X^\sigma). \end{array}$$

Composing φ_σ with the canonical isomorphism $i_\sigma : X^\sigma/\mathrm{Aut}(X^\sigma) \longrightarrow (X/\mathrm{Aut}(X))^\sigma$, which maps $x^\sigma\mathrm{Aut}(X^\sigma)$ to $x^\sigma\mathrm{Aut}(X)^\sigma$, we get a family of isomorphisms

$$\{i_\sigma \circ \varphi_\sigma =: \overline{\varphi_\sigma} \ : \ X/\mathrm{Aut}(X) \longrightarrow (X/\mathrm{Aut}(X))^\sigma\}_{\sigma \in \mathrm{Aut}(F/K)} \, .$$

For every $\sigma, \tau \in \mathrm{Aut}(F/K)$ we have $f_\tau^\sigma \circ f_\sigma \circ f_{\sigma\tau}^{-1} \in \mathrm{Aut}(X^{\tau\sigma})$. Considering the map $\pi^{\sigma\tau} : X^{\sigma\tau} \longrightarrow X^{\sigma\tau}/\mathrm{Aut}(X^{\sigma\tau})$ we have

$$\begin{aligned}
\pi^{\sigma\tau} = \pi^{\sigma\tau} \circ f_\tau^\sigma \circ f_\sigma \circ f_{\sigma\tau}^{-1} \Leftrightarrow \quad & \pi^{\sigma\tau} \circ f_{\sigma\tau} = \pi^{\sigma\tau} \circ f_\tau^\sigma \circ f_\sigma \\
\Leftrightarrow \quad & \overline{\varphi}_{\sigma\tau} \circ \pi = (\pi^\tau \circ f_\tau)^\sigma \circ f_\sigma \\
\overset{(*)}{\Leftrightarrow} \quad & \overline{\varphi}_{\sigma\tau} \circ \pi = (\overline{\varphi}_\tau \circ \pi)^\sigma \circ f_\sigma \\
\Leftrightarrow \quad & \overline{\varphi}_{\sigma\tau} \circ \pi = \overline{\varphi}_\tau^\sigma \circ (\pi^\sigma \circ f_\sigma) \\
\overset{(*)}{\Leftrightarrow} \quad & \overline{\varphi}_{\sigma\tau} \circ \pi = \overline{\varphi}_\tau^\sigma \circ (\overline{\varphi}_\sigma \circ \pi) \\
\Leftrightarrow \quad & \overline{\varphi}_{\sigma\tau} = \overline{\varphi}_\tau^\sigma \circ \overline{\varphi}_\sigma,
\end{aligned}$$

so the family $\{\overline{\varphi_\sigma} : \quad X/\mathrm{Aut}(X) \longrightarrow (X/\mathrm{Aut}(X))^\sigma\}_{\sigma\in\mathrm{Aut}(F/K)}$ satisfies the conditions of Theorem 1.1.7, thus there exists a curve Y isomorphic to $X/\mathrm{Aut}(X)$ over F which is defined over K, and an isomorphism $R : X/\mathrm{Aut}(X) \longrightarrow Y$ such that $R = R^\sigma \circ \overline{\varphi_\sigma}$. This curve Y is the called the *canonical K-model* of $X/\mathrm{Aut}(X)$.

In this case $X/\mathrm{Aut}(X)$ can be defined over its field of moduli $M_{F/K}(X)$. The definability of the original curve X over its field of moduli depends on the conditions given in the following theorem:

Theorem 2.1.1. ***(Dèbes-Emsalem theorem)*** [DE99, Corollary 4.3] *Let F/K be a Galois extension, and let X be a smooth projective curve of genus $g \geq 2$ defined over F. Suppose that the order of $\mathrm{Aut}(X)$ is not divisible by the characteristic of K. Moreover, assume that $M_{F/K}(X) = K$. The curve $X/\mathrm{Aut}(X)$ can be defined over K. Furthermore, K will be a field of definition of X in each of the following situations:*

a) the automorphism group $\mathrm{Aut}(X)$ of X has no center and it has group complement (see [DF04, Definition, p. 180]) in $\mathrm{Aut}(\mathrm{Aut}(X))$;

b) the canonical K-model of $X/\mathrm{Aut}(X)$ given by the Weil's theorem has at least one K-rational point outside the branch locus of $R \circ \pi$, where $\pi : X \longrightarrow X/\mathrm{Aut}(X)$ is the canonical projection and R is as g in Theorem 1.1.7.

For the part *a)* of the above theorem, the authors base their proof on another previous paper [DD97] which is quite technical. We provide in Theorem 3.3.8 a more friendly proof for the case of the Galois extension $\mathbb{C}/\mathbb{R}$, which is a consequence of a study of group extensions of degree 2.

2.2 Hyperelliptic pseudoreal curves

The most studied pseudoreal curves have been the hyperelliptic and p-gonal curves. In fact, the first examples of pseudoreal Riemann surfaces were published in [Ear71, Theorem 2], where the author shows the existence of a genus five pseudoreal Riemann surface, in addition to suggesting how to find pseudoreal Riemann surfaces of every genus of the form $4k+1$, and to giving the following family of hyperelliptic pseudoreal curves X of genus two

$$X \ : \ y^2 = x(x^2-a^2)(x^2+ta^2x-a),$$

where $a = e^{\frac{2\pi i}{3}}$ and $t \in \mathbb{R}^+ - \{1\}$. Earle showed that X has an anticonformal automorphism $(x,y) \mapsto \left(\frac{-1}{\overline{x}}, \frac{i\overline{y}}{\overline{x}^3}\right)$ of order 4, but it has no anticonformal involution except for $t=1$.

The next examples of pseudoreal curves can be found in [Shi72, p. 77], where the author proves that any complex algebraic curve X defined by

$$X \ : \ y^2 = a_0x^m + \sum_{r=1}^{m}(a_rx^{m+r} + (-1)^r a_r^{\sigma} x^{m-r}), \qquad a_m = 1,$$

has field of moduli $\mathbb{R}$, but it cannot be defined over $\mathbb{R}$ (here m is odd, σ is the complex conjugation, a_0 is a real number and all the a_i's and a_j^{σ}'s are complex numbers which are algebraically independent over $\mathbb{Q}$). Note that an isomorphism μ between the curves X and X^{σ} is given by

$$\mu(x,y) = (-x^{-1}, i\cdot x^{-m}y).$$

The conditions over the coefficients guarantee that $\mathrm{Aut}(X)$ contains only the identity Id_X of X and the hyperelliptic involution i. Moreover, we have $\mathrm{Aut}(\mathbb{C}/\mathbb{R}) = \{e, \sigma\}$, where e is the identity of $\mathbb{C}$ and σ is the complex conjugation. To show that X cannot be defined over $\mathbb{R}$, it is sufficient to prove that no anticonformal automorphism of X has order two.

For this, note that every isomorphism $f_\sigma : X \longrightarrow X^\sigma$ induces a conformal automorphism of X given by the composition $X \xrightarrow{f_\sigma} X^\sigma \xrightarrow{\mu^{-1}} X$, so $\mu^{-1} \circ f_\sigma \in \{\mathrm{Id}_X, i\}$, i.e. $f_\sigma \in \{\mu, \mu \circ i\}$. Thus the only anticonformal automorphisms of X are $J \circ \mu$ and $J \circ \mu \circ i$ (Section 1.3). But we have

$$(J \circ \mu)^2(x, y) = (x, -y) = (J \circ \mu \circ i)^2(x, y),$$

so we conclude that there are not anticonformal automorphisms of order two.

The previous example given by Shimura is a family of hyperelliptic pseudoreal curves of even genus and conformal automorphism group C_2, for every even genus. The problem of finding hyperelliptic pseudoreal curves of odd genus and conformal automorphism group C_2 was solved in [LR12, Proposition 4.14], where the authors show that this is impossible by proving the following theorem:

Theorem 2.2.1. *Assume that X is a hyperelliptic curve of odd genus defined over a perfect field F such that $\mathrm{Aut}(X) = \{\mathrm{Id}_X, i\} \cong C_2$, where i is the hyperelliptic involution. Then X can be defined over its field of moduli $M_{\overline{F}/F}(X)$.*

By Dèbes-Emsalem's theorem, in the case that $X/\mathrm{Aut}(X)$ has genus zero, a sufficient condition for X to be definable over its field of moduli K is the existence of a K-rational point on the canonical model Y of $X/\mathrm{Aut}(X)$. So, if we assume that $X/\mathrm{Aut}(X)$ has genus 0 (as it happens in the case of hyperelliptic curves), we can study the K-rational points of Y as points of $\mathbb{P}^1_F$. More precisely, we have the following theorem:

Theorem 2.2.2. [Hug05, Theorem 4.1.1] *Let F be a perfect field of characteristic $p \neq 2$*

and let $\overline{F}$ be its algebraic closure. Let X be a hyperelliptic curve defined over $\overline{F}$ and let $G = \mathrm{Aut}(X)/\langle i\rangle$, where i is the hyperelliptic involution of X. If the group G is not cyclic, or if it is cyclic of order not divisible by p, then X can be defined over $M_{\overline{F}/F}(X)$.

When X is a hyperelliptic curve defined over a field F, $\mathrm{Aut}(X)/\langle i\rangle$ is a finite subgroup of $\mathrm{PGL}(2, F)$, so a priori it could be one of the following groups:

$$C_n, D_n, A_4, A_5, S_4,$$

$$C_p^t, C_p^t \rtimes_\varphi C_m, \mathrm{PGL}(2, \mathbb{F}_{p^r}), \mathrm{PSL}(2, \mathbb{F}_{p^r}),$$

depending on whether the characteristic of F is 0 or not (see 1.2.18). B. Huggins showed that in all previous cases, except for the cyclic case, one can find a rational point on the canonical model of $X/\mathrm{Aut}(X)$, so the hyperellitpic curve X is not a pseudoreal curve, because it can be defined over its field of moduli F_X (see [Hug05, Theorem 4.1.2]).

For example, when $\mathrm{Aut}(X)/\langle i\rangle \cong D_n$, where D_n is the dihedral group of $2n$ elements (with $n > 2$), Huggins proved that the function field of $X/\mathrm{Aut}(X)$ is $F(t)$, where $t = x^{2n} + \dfrac{1}{x^{2n}}$, and that for every $\sigma \in \mathrm{Aut}(\overline{F}/F)$ we have $\sigma^*(t) = \pm t$ (see the sketch of the proof of Theorem 1.1.7). So $t = 0$ is a K-rational point on the canonical model B of $X/\mathrm{Aut}(X)$, because B is the fixed field of $F(t)$ under $\{\sigma^*\}_{\sigma\in \mathrm{Aut}(F/K)}$ (see [Hug05, p. 68]).

Theorem 2.2.2 generalizes the result given in [CQ05, Theorem 2], which says that a genus two curve X can be defined over $M_{\overline{F}/F}(X)$ if $\mathrm{Aut}(X)/\langle i\rangle$ is not trivial.

In [GSS05, Table 1], the authors determine the groups $\mathrm{Aut}(X)/\langle i\rangle$ for hyperelliptic curves of genus three. By Huggins' theorem we know that if such a curve X has automorphism group among

$$C_2^3, C_2 \times D_8, C_2 \times C_4, D_{12}, U_6, V_8, C_2 \times S_4,$$

then it cannot be the conformal automorphism group of a pseudoreal curve because the quotient group $\mathrm{Aut}(X)/\langle i \rangle$ would be

$$C_2 \times C_2, D_8, C_2 \times C_2, D_6, D_{12}, D_{16}, S_4$$

respectively, and none of them is cyclic. In the same Table 1 of [GSS05], one can discard also the case $\mathrm{Aut}(X) \cong C_{14}$, because in this case X has a model given by $y^2 = x^7 - 1$, which is clearly defined over $\mathbb{R}$ (in fact, over $\mathbb{Q}$).

Huggins also applied her theorem for completing the work in [BT02], and in [Hug05, Theorem 5.0.5] she was able to classify all the pseudoreal hyperelliptic Riemann surfaces, by showing the following theorem:

Theorem 2.2.3. [Hug05, Theorem 5.0.5] *Let X be a hyperelliptic curve defined over $\mathbb{C}$ such that $M_{\mathbb{C}/\mathbb{R}}(X) = \mathbb{R}$. Then X is not definable over $\mathbb{R}$ if and only if it is isomorphic to either $y^2 = f(x)$, or $y^2 = g(x)$, where*

$$f(x) = \prod_{i=1}^{r}(x^n - a_i)\left(x^n + \frac{1}{\overline{a_i}}\right), \qquad g(x) = x\prod_{i=1}^{s}(x^m - b_i)\left(x^m + \frac{1}{\overline{b_i}}\right),$$

with m, n, r, s non negative integers such that $2nr > 5$, sm is even, if n is odd then r is odd, and some conditions on the a_i's and b_i's that can be found in [Hug05, Page 82]. *Moreover, these two curves have automorphism groups isomorphic to $C_2 \times C_n$ and C_{2n}, respectively.*

2.3 Non-hyperelliptic pseudoreal curves

In [Kon09, Theorem 1.1], A. Kontogeorgis generalized Huggins' theorem (Theorem 2.2.2) considering normal cyclic p-gonal curves such that X/H has genus 0 for some cyclic group $H \cong C_p$ contained in $\mathrm{Aut}(X)$, where p is a prime number. Moreover, the author shows an example of a cyclic p-gonal curve which is pseudoreal, explicitely

$$X \ : \ y^p = \prod_{i=1}^{m}(x^n - a_i)\left(x^n + \frac{1}{\overline{a_i}}\right),$$

such that $\mathrm{Aut}(X) \cong C_p \times C_n$, where m is odd, $a_i = (i+1)\zeta_m^i$ for $i = 1, \ldots, m$, ζ_m is a primitive m-th root of unity, $p < mn$ and $p \mid 2m$.

Kontogeorgis' result was generalized even further in [HQ16], where the authors consider some particular subgroups of the automorphism group of a curve, defined as follows.

Definition 2.3.1. A subgroup $H \leq \mathrm{Aut}(X)$ is said *unique up to conjugation* if for any subgroup $K \leq \mathrm{Aut}(X)$ isomorphic to H such that the signatures of the covers $\pi_H : X \longrightarrow X/H$ and $\pi_K : X \longrightarrow X/K$ are the same, then there is an element $\alpha \in \mathrm{Aut}(X)$ such that $H = \alpha^{-1}K\alpha$.

Lemma 2.3.2. [Hug05, Lemma 4.0.4] *Let B be a curve of genus 0 defined over an infinite field L, and suppose that B has an L-rational divisor D of odd degree. Then B has infinitely many L-rational points.*

With Definition 2.3.1 and Lemma 2.3.2, the most general version of Huggins and Kontogeorgis' theorem is the following

Theorem 2.3.3. [HQ16, Theorem 1.2] *Let F be an infinite perfect field of characteristic $q \neq 2$ and let $\overline{F}$ be an algebraic closure of F. Let X be a curve of genus $g \geq 2$ defined over $\overline{F}$ and let H be a subgroup of $\mathrm{Aut}(X)$ which is unique up to conjugation, such that X/H has genus 0. If $N_{\mathrm{Aut}(X)}(H)/H$ is neither trivial, nor cyclic (if $q = 0$), nor cyclic of order relatively prime to q (if $q \neq 0$), then X can be defined over $M_{\overline{F}/F}(X)$.*

Remark 2.3.4. The center $Z(\mathrm{Aut}(X))$ of $\mathrm{Aut}(X)$ is not, in general, unique up to conjugation in $\mathrm{Aut}(X)$. To see this, we note that

$$h^{-1}Z(\mathrm{Aut}(X))h = Z(\mathrm{Aut}(X))$$

for every $h \in \mathrm{Aut}(X)$, because $Z(\mathrm{Aut}(X))$ is normal in $\mathrm{Aut}(X)$. If $Z(\mathrm{Aut}(X))$ were unique up to conjugation in $\mathrm{Aut}(X)$, then every subgroup $H \leq \mathrm{Aut}(X)$ isomorphic to $Z(\mathrm{Aut}(X))$ such that $s(X/H) = s(X/Z(\mathrm{Aut}(X))$ should be *equal* to $Z(\mathrm{Aut}(X))$. This is not always true, as the following example shows. Consider a curve

$$X: \quad x^4 + y^4 + z^4 + ax^2y^2 + bxyz^2 = 0$$

with $ab \neq 0$ as in [Bar05, Theorem 29], which has conformal automorphism group

$$\mathrm{Aut}(X) = \langle f : [x : y : z] \mapsto [y : x : z],\ g : [x : y : z] \mapsto [ix : -iy : z] \rangle \cong D_4.$$

The map g has order 4, and its square $g^2 : [x : y : z] \mapsto [x : y : -z]$ generates the center of $\mathrm{Aut}(X)$, which has order 2. The group $\langle f \rangle$ also has order 2, and the quotients $X/\langle f \rangle$ and $X/\langle g^2 \rangle$ both have signature $(1; [2, 2, 2, 2])$, thus $Z(\mathrm{Aut}(X))$ is not unique up to conjugation in this case.

Following the ideas in the proof of Theorem 2.3.3, we proved the analogous of the previous theorem under the same hypotheses but replacing H with $Z(\mathrm{Aut}(X))$.

Theorem 2.3.5. *Let F be an infinite perfect field of characteristic $q \neq 2$ and let $\overline{F}$ be an algebraic closure of F. Let X be a curve of genus $g \geq 2$ defined over $\overline{F}$ and let $Z(G)$ the center of the automorphism group G of X. Suppose $X/Z(G)$ has genus 0, and $G/Z(G)$ is neither trivial, nor cyclic (if $q = 0$), nor cyclic of order relatively prime to q (if $q \neq 0$). In that case X can be defined over $M_{\overline{F}/F}(X)$.*

Proof. We will prove the theorem for the case $q = 0$. The case $q \neq 0$ is the same as [HQ16].

Without loss of generality we can assume $F = M_{\overline{F}/F}(X)$ (see [DE99, Proposition 2.1]). Let $\sigma \in \mathrm{Aut}(\overline{F}/F)$. Since $F = M_{\overline{F}/F}(X)$, there exists an isomorphism $f_\sigma : X \longrightarrow X^\sigma$ (see the first part of the proof of Theorem 1.3.6).

Claim 1. $f_\sigma Z(G) f_\sigma^{-1} = Z(G)^\sigma$.

For every $a \in Z(G)$ and $b \in G^\sigma$ we have

$$(f_\sigma a f_\sigma^{-1})b = f_\sigma a(\underbrace{f_\sigma^{-1} b f_\sigma}_{b' \in G}) f_\sigma^{-1} = f_\sigma a b' f_\sigma^{-1} \underbrace{=}_{a \in Z(G)} f_\sigma b' a f_\sigma^{-1} = b(f_\sigma a f_\sigma^{-1}).$$

This says that $f_\sigma a f_\sigma^{-1}$ commutes with every element of G^σ for every $a \in Z(G)$, i.e. $f_\sigma Z(G) f_\sigma^{-1} \subseteq Z(G^\sigma)$, which is equal to $Z(G)^\sigma$. The other inclusion is obtained analogously. Q.E.D.

By *Claim 1.* and the fact that $f_\sigma G f_\sigma^{-1} = G^\sigma$, there exist two isomorphisms g_σ and h_σ, respectively, such that the following diagram commutes:

$$\begin{array}{ccc}
X & \xrightarrow{\quad f_\sigma \quad} & X^\sigma \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1^\sigma} \\
X/Z(G) & \xrightarrow[g_\sigma]{} & (X/Z(G))^\sigma \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2^\sigma} \\
X/G & \xrightarrow[h_\sigma]{} & (X/G)^\sigma .
\end{array}$$

Claim 2. Without loss of generality, we can assume that the branch locus of π_2 is $\mathcal{B} = \{[1:0],[0:1],[1:1]\}$.

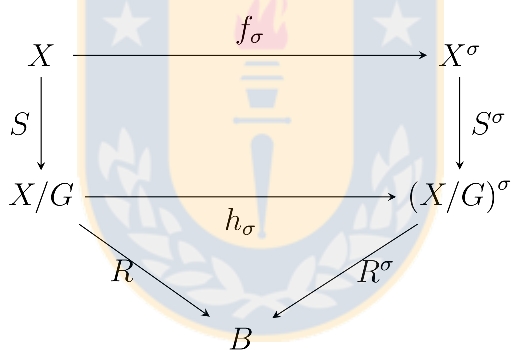
Since $X/Z(G)$ has genus 0, then $(X/Z(G))^\sigma$, X/G and $(X/G)^\sigma$ also have genus 0. We have that the covering group of π_2 is $\mathrm{Deck}(\pi_2) \cong G/Z(G)$, where $\mathrm{Deck}(\pi_2) = \{f \in \mathrm{Aut}(X) : \ \pi_2 \circ f = \pi_2\}$. This last group is a finite group acting on the projective line, so it must be isomorphic to either C_n, D_{2n}, A_4, A_5 or S_4 (see Theorem 1.2.18). By our hypothesis, $G/Z(G)$ is not a cyclic group, so it must be isomorphic to either D_{2n}, A_4, A_5 or S_4. Note that in any of these four cases π_2 has a branch locus $\mathcal{B} = \{b_1, b_2, b_3\}$ which contains 3 elements, and since any element of $\mathrm{PGL}(2, \overline{F})$ acts as a Mobius transformation on the complex projective line, so we can assume $b_1 = [1:0], b_2 = [0:1]$ and $b_3 = [1:1]$. Q.E.D.

Claim 3. There exists an isomorphism $R : X/G \longrightarrow B$ over the canonical F-model B of X/G such that $R = R^\sigma \circ h_\sigma$.

Write $S := \pi_2 \circ \pi_1$. If there exist other isomorphisms f'_σ and h'_σ such that $f'_\sigma Z(G) f'^{-1}_\sigma = Z(G)^\sigma$ and $h'_\sigma \circ S = S^\sigma \circ f'_\sigma$, then $f^{-1}_\sigma \circ f'_\sigma \in G$ so there exists $F \in G$ such that $f'_\sigma = f_\sigma \circ F$, and we have

$$h'_\sigma \circ S = S^\sigma \circ f'_\sigma = S^\sigma \circ f_\sigma \circ F = S^\sigma \circ F' \circ f_\sigma = S^\sigma \circ f_\sigma = h_\sigma \circ S,$$

where the third equality follows from the fact that $f_\sigma \circ F \circ f_\sigma^{-1} \in G^\sigma$, so we define it as F', and the fourth equality follows from the fact that $S^\sigma = \pi_2^\sigma \circ \pi_1^\sigma : X^\sigma \longrightarrow X^\sigma/G^\sigma$. So $h'_\sigma = h_\sigma$. This means that h_σ is uniquely determined by σ. Thus $\{h_\sigma\}_{\sigma\in \mathrm{Aut}(\overline{F}/F)}$ satisfies Weil's Theorem 1.1.7. Then there exists an isomorphism $R : X/G \longrightarrow B$ such that B is a curve of genus 0 defined over F and we have the following commutative diagram $(*)$:



that is, $R = R^\sigma \circ h_\sigma$. Q.E.D.

Since g_σ and h_σ are isomorphisms, we have $h_\sigma(\mathcal{B}) = \sigma(\mathcal{B})$. Note that

$$\sigma(\mathcal{B}) = \{\sigma([1:0]), \sigma([0:1]), \sigma([1:1])\} = \mathcal{B},$$

so $h_\sigma(\mathcal{B}) = \mathcal{B}$ $(**)$.

Claim 4. B has an F-rational point r outside the branch locus of $R \circ S$.

We can consider the branch divisor $D = R(b_1) + R(b_2) + R(b_3)$, which satisfies

$$D^\sigma = \sigma(R(b_1)) + \sigma(R(b_2)) + \sigma(R(b_3)) = R^\sigma(\sigma(b_1)) + R^\sigma(\sigma(b_2)) + R^\sigma(\sigma(b_3))$$

$$= R^\sigma(b_1) + R^\sigma(b_2) + R^\sigma(b_3) \overset{(*)}{=} R \circ h_\sigma^{-1}(b_1) + R \circ h_\sigma^{-1}(b_2) + R \circ h_\sigma^{-1}(b_3) \overset{(**)}{=} D,$$

so B has an F-rational divisor of degree 3. By Lemma 2.3.2 we see that B has infinitely F-rational points. In particular, B must have an F-rational point r outside the branch locus of $R \circ S$. Q.E.D.

Claim 5. X can be defined over F.

By Dèbes-Emsalem theorem (Theorem 2.1.1(b)) and *Claim 4.* we conclude that F is a field of definition of X. □

Corollary 2.3.6. *If X is a pseudoreal Riemann surface such that $X/Z(\mathrm{Aut}(X))$ has genus 0, then $\mathrm{Aut}(X)$ must be an Abelian group.*

Proof. Since X is a pseudoreal Riemann surface, then $M_{\mathbb{C}/\mathbb{R}}(X) = \mathbb{R}$ and X cannot be defined over $\mathbb{R}$. Having in mind the hypothesis, by Theorem 2.3.5 we deduce that $\mathrm{Aut}(X)/Z(\mathrm{Aut}(X))$ is a cyclic group. But it is known that if a group G is such that $G/Z(G)$ is a cyclic group, then G must be Abelian. So $\mathrm{Aut}(X)$ must be Abelian. □

Using the same polynomials which define her examples of pseudoreal hyperelliptic curves, Huggins proved the existence of the first non-hyperelliptic pseudoreal curves, which are in fact complex plane curves (see [Hug05, Section 7.1]). One of the examples she gave among the possible pseudoreal plane curves is the curve X defined by the homogeneous polynomial equation

$$X \ : \ X_2^{2nr} = \prod_{i=1}^{r} (X_0^n - a_i X_1^n)(X_0^n + a_i^c X_1^n),$$

with certain conditions on the coefficients a_i and a_i^c given in [Hug05, p. 131]. She proved that X has conformal automorphism group generated by

$$E = \begin{bmatrix} \zeta_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_n & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_{2nr} \end{bmatrix},$$

where ζ_i is a primitive i-th root of unity. Finally, she proved that $M_{\mathbb{C}/\mathbb{R}}(X) = \mathbb{R}$ but X cannot be defined over $\mathbb{R}$ (see [Hug05, Proposition 7.1.2]).

In [Hid09], Hidalgo found another family C_{λ_1,λ_2} of non-hyperelliptic pseudoreal curves in $\mathbb{P}^5_{\mathbb{C}}$ defined by

$$C_{\lambda_1,\lambda_2} := \left\{ \begin{array}{ll} x_1^2 + x_2^2 + x_3^2 & = 0 \\ \lambda_1 x_1^2 + x_2^2 + x_4^2 & = 0 \\ \lambda_2 x_1^2 + x_2^2 + x_5^2 & = 0 \\ -\lambda_2 x_1^2 + x_2^2 + x_6^2 & = 0 \end{array} \right\},$$

where $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \in \mathbb{C}$ are such that $\lambda_1 < -3 + 2\sqrt{2}$, $\mathrm{Re}(\lambda_2) < 0, \mathrm{Im}(\lambda_2) > 0$ and $|\lambda_2|^2 = -\lambda_1$. Hidalgo proved that these curves are of genus 17 with conformal automorphism group C_2^5, and they admit an anticonformal automorphism $\eta : \mathbb{P}^5_{\mathbb{C}} \longrightarrow \mathbb{P}^5_{\mathbb{C}}$ of order 4 defined by

$$\eta[x_1 : x_2 : x_3 : x_4 : x_5 : x_6] := [\overline{x_2} : \sqrt{\lambda_1}\overline{x_1} : \overline{x_4} : \sqrt{\lambda_1}\overline{x_3} : \sqrt{\lambda_2}\overline{x_6} : i\sqrt{\lambda_2}\overline{x_5}],$$

but they have no anticonformal involutions. He also showed that C_{λ_1,λ_2} is a covering of the pseudoreal hyperelliptic curve given by Earle in [Ear71, p. 126].

2.4 Odd signature curves

Let X be a curve of genus $g \geq 2$ defined over an algebraically closed field F. We can determine if X is definable over its field of moduli $M_{F/K}(X)$, where F/K is a Galois extension, by studying the ramification indices of the branch points of the projection map $\pi : X \longrightarrow X/\mathrm{Aut}(X)$, provided that this quotient has genus 0. In fact, we need this last condition if we want to use Lemma 2.3.2.

Definition 2.4.1. Consider a curve X of genus $g \geq 2$ and let $\pi : X \longrightarrow X/G$ be the projection map, where $G \leq \mathrm{Aut}(X)$. Denoting by $q_1, q_2, \ldots, q_r$ the branch points of π, we define the *signature of* π as the vector

$$s(\pi) := (g_0; m_1, m_2, \ldots, m_r),$$

where g_0 is the genus of the curve X/G and m_i is the ramification index of the branch point q_i for $i = 1, \ldots, r$. Moreover, if $s(\pi) = (0; m_1, m_2, \ldots m_r)$ is such that every m_i appears exactly an odd number of times, then X is called an *odd signature curve* and $s(\pi)$ is called an *odd signature*.

Using Lemma 2.3.2 and the fact that the projection map $\pi : X \longrightarrow X/\mathrm{Aut}(X)$ has finitely many branch points, we have the following theorem (a weaker version was published in [AQ12, Theorem 0.1], before the publication of the PhD thesis of the second author):

Theorem 2.4.2. [Qui13, Theorem 2.10] *Let X be a curve of genus $g \geq 2$ defined over an algebraically closed field F, and let F/L be a Galois extension. If $H \leq \mathrm{Aut}(X)$ is unique up to conjugation and $\pi_N : X \longrightarrow X/N$ is an odd signature cover, where N is the normalizer of H in* $\mathrm{Aut}(X)$*, then $M_{F/L}(X)$ is a field of definition for X.*

Remark 2.4.3. In the proof of Theorem 2.10 of [Qui13], S. Quispe used Lemma 2.3.2 to prove his result in the case when F is infinite, while for the case of a finite field F, the result was proved in [Hug05, Corollary 1.6.6].

Remark 2.4.4. Theorem 2.4.2 says that if we consider a subgroup $H \leq \mathrm{Aut}(X)$ of a Riemann surface X such that $s(\pi)$ is an odd signature, then X cannot be a pseudoreal Riemann surface. Let us note here the similarity between this theorem and Corollary 1.3.3, where the conclusions are *almost* the same, but in the latter one we do not need that the first entry of the signature is 0, while in the former one it is necessary.

In a particular case, in [AQ12, Corollary 3.5] the authors prove that every non-normal cyclic p-gonal curve defined over a zero characteristic field F can be defined over its field of moduli F_X, using the fact that the signature of every covering $\pi_X : X \longrightarrow X/\mathrm{Aut}(X)$ is an odd signature. For the convenience of the reader, we give in Table 2.1 the signatures and the automorphism groups of all non normal p-gonal curves with p a prime number (see [Woo07, Theorem 8.1]).

Table 2.1: Signatures and automorphism groups of non-normal p-gonal curves

p	signature of π_G	g	G
3	$(0; [2, 3, 8])$	2	$\mathrm{GL}(2, 3)$
3	$(0; [2, 3, 12])$	3	$\mathrm{SL}(2, 3)/CD$
5	$(0; [2, 4, 5])$	4	S_5
7	$(0; [2, 3, 7])$	3	$\mathrm{PSL}(2, 7)$
$p \geq 5$	$(0; [2, 3, 2p])$	$\frac{(p-1)(p-2)}{2}$	$(C_p \times C_p) \rtimes S_3$
$p \geq 3$	$(0; [2, 2, 2, p])$	$(p-1)^2$	$(C_p \times C_p) \rtimes V_4$
$p \geq 3$	$(0; [2, 4, 2p])$	$(p-1)^2$	$(C_p \times C_p) \rtimes D_4$

Theorem 2.4.2 is important to us because it allows one to prove also the following theorem:

Theorem 2.4.5. [Qui13, Proposition 2.14] *Let X be a smooth curve of genus $g \geq 2$ defined over a field F. If $X/\mathrm{Aut}(X)$ has genus 0, then X can be defined over its field of moduli F_X, or over an extension of degree 2 of F_X.*

Remark 2.4.6. If we discard the genus 0 condition on $X/\mathrm{Aut}(X)$ in the Theorem 2.4.5, the problem is still open.

Chapter 3

Pseudoreal Riemann surfaces and NEC groups

3.1 Existence for any genus

As we saw in Section 2.2, in [Shi72, p. 177] the author finds pseudoreal curves for every even genus, and in [Ear71, Theorem 2] the author finds a genus 5 pseudoreal Riemann surface and he also suggested the existence of pseudoreal Riemann surfaces of every genus of the form $4k+1$. The problem of proving the existence of pseudoreal Riemann surfaces in every genus $g \geq 2$ was solved in [Sin80, Theorem 1 and p. 48], where the author proves the following result.

Theorem 3.1.1. *There exist pseudoreal Riemann surfaces for every genus $g \geq 2$.*

Proof. Consider a NEC group Δ with signature $(1; -; [2^{g+1}]; \{-\})$, where g is even. If $\{x_i\}_{i=1}^{g+1}$ is the set of elliptic generators and d_1 is the glide reflection, which together generate Δ, we can define an epimorphism $\theta : \Delta \longrightarrow C_4 = \langle a : a^4 = 1 \rangle$ given by

$$\theta(x_i) = a^2, \quad \forall i \in \{1, \ldots, g+1\}, \qquad \theta(d_1) = a.$$

Since θ preserves the orders of the elliptic generators, $\mathrm{Ker}(\theta)$ is torsion free, so the quotient $X = \mathbb{H}/\operatorname{Ker}(\theta)$ is a Riemann surface such that $\operatorname{Aut}^{\pm}(X)$ contains the group $\Delta/\operatorname{Ker}(\theta) \cong C_4$. Since this group has finitely maximal signature (see 1.2.34), we can conclude by Proposition 1.2.35 that $\operatorname{Aut}^{\pm}(X) \cong C_4$. This Riemann surface X has genus g and has anticonformal automorphisms, but no anticonformal involutions, because a^2, the only element of order 2 in C_4, is in the conformal part $\operatorname{Aut}^{+}(X) \cong \{1, a^2\} \leq C_4$. Then X is pseudoreal.

For odd g, we can do the same work beginning with the NEC signature $(2; -; [2^{g-1}]; \{-\})$, and considering the epimorphism $\theta : \Delta \longrightarrow C_4 = \langle a : a^4 = 1 \rangle$ given by

$$\theta(x_i) = a^2, \quad \forall i \in \{1, \ldots, g-1\}, \qquad \theta(d_1) = \theta(d_2) = a. \qquad \square$$

Remark 3.1.2. The above proof, in contrast with the first results on pseudoreal curves, is an *existence proof*: no algebraic model for the pseudoreal Riemann surfaces are given.

3.2 Characterization of full groups of pseudoreal Riemann surfaces

We know from L. Greenberg [Gre74, Theorem 4] that every finite group is the conformal automorphism group of some Riemann surface. In the case of pseudoreal Riemann surfaces, in [BG10] the authors study the *essential groups* which act on pseudoreal Riemann surfaces, i.e. those which contain anticonformal elements. For every action of the full automorphism group of a Riemann surface which has anticonformal automorphisms, we have the following non-split exact sequence of groups

$$1 \longrightarrow \operatorname{Aut}^{+}(X) \longrightarrow \operatorname{Aut}^{\pm}(X) \longrightarrow C_2 \longrightarrow 0.$$

Lemma 3.2.1. *An exact sequence of groups*

$$1 \longrightarrow G \longrightarrow \overline{G} \xrightarrow{\pi} C_2 \longrightarrow 0$$

is split if and only if there exists an order 2 element in $\overline{G}\backslash G$.

Proof. We know that the exact sequence in the statement is a split sequence if and only if there exists a group homomorphism $f : C_2 \longrightarrow \overline{G}$ such that $\pi \circ f = \mathrm{Id}_{C_2}$. If we have an order 2 element $p \in \overline{G}\backslash G$, then we can consider the group homomorphism $f : C_2 \longrightarrow \overline{G}$ which sends $0 \in C_2$ to $e \in \overline{G}$, and $1 \in C_2$ to p. Conversely, if we have the group homomorphism $f : C_2 \longrightarrow \overline{G}$ such that $\pi \circ f = \mathrm{Id}_{C_2}$, then $f(1)$ will be an element in $\overline{G}$ such that $\pi(f(1)) = 1$, so $f(1)$ is an order 2 element in $\overline{G}\backslash G$. □

Lemma 3.2.2. *In every non-split exact sequence*

$$1 \longrightarrow G \longrightarrow \overline{G} \longrightarrow C_2 \longrightarrow 0,$$

the group G *has even order.*

Proof. By Lemma 3.2.1 we know that $\overline{G}\backslash G$ has no involutions. Because G is an index two subgroup of $\overline{G}$, we have $\overline{G}$ has even order, so by Cauchy's theorem it has an order 2 element, which must be in G. By Lagrange's theorem we conclude that G has even order. □

Corollary 3.2.3. *If* X *is a pseudoreal Riemann surface, then* $\mathrm{Aut}^+(X)$ *has even order.*

With the previous lemmas, we are prepared to prove the following theorem:

Theorem 3.2.4. [BG10, Theorem 3.3] *A finite group* $\overline{G}$ *acts as an essential group on a pseudoreal Riemann surface* X *if and only if it is a non-split extension of a group of even order by the cyclic group of order* 2*. Furthermore there exists a Riemann surface* X *having* $\overline{G}$ *as its full automorphism group.*

Proof. Suppose we have a pseudoreal Riemann surface X such that $\overline{G}$ is its full automorphism group and G is its conformal automorphism group. By Lemma 3.2.1 we know that $G \leq \overline{G}$ is a non-split extension and by Corollary 3.2.3 we know that G has even order.

Conversely, suppose we have an exact non-split extension

$$1 \longrightarrow G \longrightarrow \overline{G} \longrightarrow C_2 \longrightarrow 0.$$

By Corollary 3.2.3 we know that G has even order. We will find a pseudoreal Riemann surface X such that $\mathrm{Aut}^{\pm}(X) \cong \overline{G}$. We can assume $G = \langle g_1, \ldots, g_r \rangle$ where g_i has order $m_i \geq 2$ for a fixed $r \geq 3$, and take $d \in \overline{G} \backslash G$. Let m_{r+1} be the order of $d^2 g_1 \ldots g_r$. We have 2 cases:

1. If $m_{r+1} = 1$, then $d^2 g_1 \ldots g_r = 1$. We can consider a NEC group Δ with signature $s(\Delta) = (1; -; [m_1, \ldots m_r]; \{-\})$ which is a finitely maximal NEC signature (see [Buj82]). We have an epimorphism $\theta : \Delta \longrightarrow \overline{G}$ given by $\theta(d_1) = d$, $\theta(x_i) = g_i, \forall i \in \{1, \ldots, r\}$. We have that θ preserves the order of the elliptic generators, so $\mathrm{Ker}(\theta)$ is a surface Fuchsian group which uniformizes a Riemann surface $\mathbb{H}/\mathrm{Ker}(\theta)$. Because the signature $s(\Delta)$ is finitely maximal, then $\mathrm{Aut}^{\pm}(\mathbb{H}/\mathrm{Ker}(\theta)) \cong \overline{G}$. Since $\alpha : \overline{G} \longrightarrow C_2$ is non split, then $\overline{G} \backslash G$ has no elements of order 2, so the Riemann surface is pseudoreal.

2. If $m_{r+1} > 1$, then we can consider a NEC group Δ with signature $s(\Delta) = (1; -; [m_1, \ldots m_r, m_{r+1}]; \{-\})$, which is a finitely maximal NEC signature (see [Buj82]). We have an epimorphism $\theta : \Delta \longrightarrow \overline{G}$ given by $\theta(d_1) = d$, $\theta(x_i) = g_i, \forall i \in \{1, \ldots, r\}$ and $\theta(x_{r+1}) = (d^2 g_1 \ldots g_r)^{-1}$. We have that θ preserves the order of the elliptic generators, so $\mathrm{Ker}(\theta)$ is a surface Fuchsian group which uniformizes a Riemann surface $\mathbb{H}/\mathrm{Ker}(\theta)$. Because the signature $s(\Delta)$ is finitely maximal, then $\mathrm{Aut}^{\pm}(\mathbb{H}/\mathrm{Ker}(\theta)) \cong \overline{G}$. Since $\alpha : \overline{G} \longrightarrow C_2$ is non split, then $\overline{G} \backslash G$ has no elements of order 2, so the Riemann surface is pseudoreal. □

Corollary 3.2.5. *No symmetric or dihedral group can be the full automorphism group of a pseudoreal Riemann surface.*

Proof. The only subgroup G of the symmetric group S_n such that $[S_n : G] = 2$ is $G = A_n$, the alternating group, and $S_n = A_n \rtimes C_2$.
For every subgroup G of the dihedral group D_{2n} such that $[D_{2n} : G] = 2$ we have that $D_{2n} \backslash G$ has involutions, because D_{2n} has $2n + 1$ involutions. □

Corollary 3.2.6. *The dicyclic group* Dic_{4n} *acts as a full automorphism group of some pseudoreal Riemann surface, for every* $n \geq 2$*.*

Proof. For every $n \geq 2$ we have the following exact non-split extension

$$1 \longrightarrow C_{2n} \longrightarrow \mathrm{Dic}_{4n} \longrightarrow C_2 \longrightarrow 0,$$

because Dic_{4n} has only 1 involution, a^n, which is inside C_{2n}. □

3.3 Group extensions of degree 2

We have seen the importance of knowing the possible extensions of degree 2 of a given group G to another group $\overline{G}$ because of the relation between the conformal automorphism group and the full automorphism group of pseudoreal Riemann surfaces. The most general approach is through cohomology of finite groups (see [AM04, Chapter 1]), but we will use easier tools to deal with the problem.
We will study the extensions of a group G by the cyclic group of order 2, and this will allow us to prove Theorem 2.1.1 (a) for the field extension $\mathbb{C}/\mathbb{R}$ in an easier way.

A group G admits a degree 2 extension to a group $\overline{G}$ if there exists an exact sequence of groups

$$1 \longrightarrow G \longrightarrow \overline{G} \longrightarrow C_2 \longrightarrow 0.$$

For any such extension, we can consider an element $x \in \overline{G} \backslash G$, which induces an automorphism ϕ_x of G defined by conjugation by x (from now on, we will denote ϕ_p the conjugation by the element p), because G is normal in $\overline{G}$. We have $[\overline{G} : G] = 2$ so $g = x^2 \in G$, the map ϕ_x^2 is the conjugation by $g = x^2$, and g is fixed by ϕ_x.

Any other element in $\overline{G}\backslash G$ can be written as xh, where $h \in G$. In this case

$$\phi_{xh}(g) = (xh)g(xh)^{-1} = x(hgh^{-1})x^{-1} = \phi_x(\phi_h(g)),$$

so $\phi_{xh} = \phi_x \circ \phi_h$, and we have

$$(xh)^2 = xhxh = xhx^{-1}x^2h = \phi_x(h)x^2h = \phi_x(h)gh.$$

Let $P(G)$ be the subset of $\mathrm{Aut}(G) \times G$ defined by

$$P(G) := \{(\phi, g) \in \mathrm{Aut}(G) \times G : \ \phi^2 = \phi_g, \ \ \phi(g) = g\}.$$

We can define an equivalence relation on $P(G)$ by

$$(\phi, g) \sim (\phi \circ \phi_h, \phi(h)gh), \qquad \forall h \in G.$$

We leave to the reader verifying that it is indeed an equivalence relation. Let $E(G)$ be the quotient set $P(G)/\sim$.

Lemma 3.3.1. *Given a group G, there exists a well defined function from the set of group extensions*

$$1 \longrightarrow G \longrightarrow \overline{G} \longrightarrow C_2 \longrightarrow 0,$$

to $E(G)$.

Proof. For any such extension we can take an element $x \in \overline{G}\backslash G$ and construct the pair (ϕ_x, x^2). To be sure that any other pair of that form will be equivalent to it, we note that $[\overline{G} : G] = 2$, so $\overline{G} = G \cup xG$ (disjoint union) and any element in $\overline{G}\backslash G$ will be of the form xh with $h \in G$. But we have which shows us that for every $h \in G$ we have

$$(\phi_x, x^2) \sim (\phi_{xh}, (xh)^2).$$ □

Lemma 3.3.2. *Given an element* $(\phi, g) \in P(G)$*, the group* $\overline{G} := (G \rtimes_F \mathbb{Z})/\langle (g^{-1}, z^2) \rangle$ *fits in the group extension sequence*

$$1 \longrightarrow G \longrightarrow \overline{G} \longrightarrow C_2 \longrightarrow 0,$$

where z *is any generator of* $(\mathbb{Z};+)$*, and there exists an element* $x \in \overline{G} \backslash G$ *such that* $\phi = \phi_x$ *and* $x^2 = g$*.*

Proof. Consider the homomorphism induced by

$$F : \mathbb{Z} \longrightarrow \mathrm{Aut}(G), \qquad z \mapsto \phi.$$

The subgroup $\langle (g^{-1}, z^2) \rangle$ is normal in $G \rtimes_F \mathbb{Z}$ because

$$[(1, z) \cdot (g^{-1}, z^2)] \cdot (1, z)^{-1} = (F(z)(g^{-1}), z^3) \cdot (1, z^{-1}) = (\phi(g^{-1}), z^3) \cdot (1, z^{-1})$$

$$= (g^{-1} F(z^3)(1), z^2) = (g^{-1}, z^2),$$

and

$$[(h, 1) \cdot (g^{-1}, z^2)] \cdot (h, 1)^{-1} = (hF(1)(g^{-1}), z^2) \cdot (h^{-1}, 1) = (hg^{-1}, z^2) \cdot (h^{-1}, 1)$$

$$= (hg^{-1}F(z^2)(h^{-1}), z^2) = (hg^{-1}\phi^2(h^{-1}), z^2) = (hg^{-1}(gh^{-1}g^{-1}), z^2) = (g^{-1}, z^2).$$

Clearly G injects into $\overline{G}$ through $a \mapsto (a, 1)$, and we have that

$$\overline{G} = \{(g, 1)\ ,\ g \in G\} \cup \{(g, z)\ ,\ g \in G\},$$

because for $(p, z^m) \in G \rtimes_F \mathbb{Z}$ we have two cases

$$[(p, z^m)] = [(p, z^m) \cdot (g, z^{-2})^{\frac{m}{2}}] = [(pg^{\frac{m}{2}}, 1)] \qquad \text{for even } m,$$

$$[(p, z^m)] = [(p, z^m) \cdot (g, z^{-2})^{\frac{m-1}{2}}] = [(pg^{\frac{m-1}{2}}, z)] \qquad \text{for odd } m,$$

so $|\overline{G}| = 2|G|$ and we have $\overline{G}/G \cong C_2$. Moreover

$$\phi_{(1,z)}(h,1) = [(1,z)\cdot(h,1)]\cdot(1,z)^{-1} = (F(z)(h),z)\cdot(1,z^{-1}) = (\phi(h),z)\cdot(1,z^{-1})$$

$$= (\phi(h)F(z)(1),1) = (\phi(h),1),$$

and

$$(1,z)^2 = (1,z^2) = (1,z^2)\cdot(g,z^{-2}) = (F(z^2)(g), z^2z^{-2}) = (g,1),$$

so we can choose x as $(1,z)$. □

Definition 3.3.3. We say that two exact sequences

$$1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1,$$

$$1 \longrightarrow H \longrightarrow G' \longrightarrow K \longrightarrow 1,$$

are *isomorphic* if there exists a group isomorphism $\alpha : G \longrightarrow G'$ such that the following diagram commmutes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & K & \longrightarrow & 1 \\ & & {\scriptstyle \mathrm{Id}_H}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \mathrm{Id}_K}\downarrow & & \\ 1 & \longrightarrow & H & \longrightarrow & G' & \longrightarrow & K & \longrightarrow & 1. \end{array}$$

Theorem 3.3.4. *There is a bijection between the set of isomorphism classes of extensions of G by C_2, and $E(G)$.*

Proof. Given an extension

$$1 \longrightarrow G \longrightarrow \overline{G} \longrightarrow C_2 \longrightarrow 0$$

we can associate to it the class $[(\phi_x, x^2)]$ by Lemma 3.3.1. If we have another extension

$$1 \longrightarrow G \longrightarrow \overline{G}' \longrightarrow C_2 \longrightarrow 0$$

isomorphic to the previous one, there exists an isomorphisms $\alpha : \overline{G} \longrightarrow \overline{G}'$ which is the identity in G. We can idenfity $\overline{G}$ with $\overline{G}'$, so to this last extension we can associate the same pair $[(\phi_x, x^2)]$.

Conversely, by Lemma 3.3.2, we can associate to every pair $(\phi, g) \in P(G)$ an extension of G defined by A as is the group extension $\overline{G}$ in Theorem 3.3.2. Every pair $(\phi \circ \phi_h, \phi(h)gh)$ equivalent to (ϕ, g) will give us another group

$$B = (G \rtimes_{F'} \mathbb{Z})/\langle((\phi(h)gh)^{-1}, y^2)\rangle,$$

where $\mathbb{Z} = \langle y \rangle$, $h \in G$ and $F' : \mathbb{Z} \longrightarrow \mathrm{Aut}(G)$ is induced by $y \mapsto \phi \circ \phi_h$. An isomorphism $\alpha : A \longrightarrow B$ is induced by $\alpha(g, 1) = (g, 1)$, $\alpha(1, x) = (\phi(h)^{-1}, y)$. It is well defined because

$$\alpha(g^{-1}, x^2) = (g^{-1}, 1)(\phi(h)^{-1}, y)(\phi(h)^{-1}, y) = (g^{-1}, 1)(\phi(h)^{-1}F'(y)(\phi(h)^{-1}), y^2)$$

$$= (g^{-1}, 1)(\phi \circ \phi_h(\phi(h)^{-1}), y^2) = (g^{-1}, 1)(gh^{-1}g^{-1}\phi(h)^{-1}, y^2) = ((\phi(h)gh)^{-1}, y^2)$$

and clearly $\alpha|_G = \mathrm{Id}_G$. □

Corollary 3.3.5. *The extension associated to $(\phi, g) \in P(G)$ is split if and only if it is equivalent to some pair (ϕ', e), and it is a direct product if and only if it is equivalent to the pair (Id_G, e).*

Proof. The exact sequence $1 \to G \to \overline{G} \to C_2 \to 0$ is split if and only if $\overline{G} \backslash G$ has an order 2 element p, which gives us the desired pair (ϕ_p, e). If $\overline{G} = G \times C_2$ one can choose $p = (e, 1)$, which satisfies $\phi_p = \mathrm{Id}_G$. □

Corollary 3.3.6. *If G is a group such that $Z(G) = \{1\}$ and $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ has no involutions — where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G —, then any extension of G by C_2 is a direct product of G and C_2.*

Proof. Suppose that we have a group extension $G \leq \overline{G}$ or order 2. By Lemma 3.3.1 we can associate to it a pair (ϕ, g), such that $\phi^2 = \phi_g$. The class $[\phi]$ of ϕ in $\mathrm{Out}(G) = \mathrm{Aut}(G)/$

$\mathrm{Inn}(G)$ satisfies $[\phi]^2 = [\phi^2] = [\phi_g] = [1]$, but $\mathrm{Out}(G)$ has no order 2 elements, so $[\phi] = [1]$, and then $\phi \in \mathrm{Inn}(G)$. In that case $(\phi, g) \sim (\phi \circ \phi^{-1}, g') = (\mathrm{Id}_G, g')$ for some $g' \in G$. But we must have $\mathrm{Id}_G^2 = \phi_{g'}$, so $g' = e$ because $Z(G) = \{1\}$. In that case $(\phi, g) \sim (\mathrm{Id}_G, e)$, so every extension of G by C_2 will be the direct product $G \times C_2$. □

Using the previous theorem we can prove the most important theorem of this section

Theorem 3.3.7. *If G is a group such that $Z(G) = \{1\}$ and $\mathrm{Inn}(G)$ has group complement in $\mathrm{Aut}(G)$, then any degree 2 extension of G is split, i.e. $\overline{G}$ is a semidirect product of C_2 and G.*

Proof. Suppose the group complement of $\mathrm{Inn}(G)$ inside $\mathrm{Aut}(G)$ is H, that is

$$\mathrm{Aut}(G) = H \cdot \mathrm{Inn}(G), \qquad H \cap \mathrm{Inn}(G) = \{1\}.$$

We will prove that with these conditions on G, every degree 2 extension of it must be a semidirect product $G \rtimes_F C_2$ for some homomorphism $F : C_2 \longrightarrow \mathrm{Aut}(G)$.

Thanks to the previous bijection between isomorphism classes of degree 2 extensions of a group G and the elements of the quotient set $E(G) = P(G)/\sim$, we will be able to prove that any pair $(\phi, g) \in P(G)$ is equivalent to a pair (ϕ', e), and to conclude by Corollary 3.3.5 that the degree 2 extensions of G are all semidirect products.

If $(\phi, g) \in P(G)$, then $\phi^2 = \phi_g$ and $\phi(g) = g$. We have $\mathrm{Aut}(G) = H \cdot \mathrm{Inn}(G)$, so $\phi \in \mathrm{Aut}(G)$ can be written as $\phi = \varphi \circ \phi_h$ with $\varphi \in H$ and $\phi_h \in \mathrm{Inn}(G)$, so $(\phi, g) = (\varphi \circ \phi_h, g) \sim (\varphi, g')$ for some $g' \in G$. We also have $\varphi^2 \in H \cap \mathrm{Inn}(G) = \{1\}$ so $\varphi^2 = 1$, but $\varphi^2 = \phi_{g'}$ so $\phi_{g'} = 1$, which is equivalent to $g' = e$ because $Z(G) = \{1\}$. In that case $(\varphi, g') = (\varphi, e)$ so we get the desired equality $[(\phi, g)] = [(\varphi, e)]$. □

If we translate the previous results to pseudoreal Riemann surfaces, we get the following corollaries:

Corollary 3.3.8. *Let G be the conformal automorphism group of a Riemann surface X. Suppose that $Z(G) = \{1\}$ and that $\mathrm{Inn}(G)$ has group complement in $\mathrm{Aut}(G)$. Then X cannot be a pseudoreal Riemann surface.*

Corollary 3.3.9. *If G is the conformal automorphism group of a Riemann surface X such that $Z(G) = \{1\}$ and $\mathrm{Out}(G)$ has no involutions, then X cannot be pseudoreal.*

3.4 Automorphism groups of pseudoreal Riemann surfaces

3.4.1 Conformal groups of Riemann surfaces

In the previous section we saw the importance of the structure of the conformal automorphism group of a Riemann surface to see if a Riemann surface is pseudoreal. For this reason it is of our interest to know the results about such actions and the tools that they require.

In the literature there exist classifications of groups of automorphisms for many types of Riemann surfaces. For example, the problem is solved in the case of hyperelliptic surfaces in [BGG93], where the authors consider all the possible conformal actions on them and they classified their possible complete automorphism groups. There exist also some explicit classifications of groups (not necessarily complete automorphism groups) in small genera, concretely the cases of genus $2, 3, 4$ and 5 (see [Bar05], [Bog97], [Bro91], [CGLR99], [Kim03], [KK90], [KK90], [MSSV02]). In these works, the authors use tools from Fuchsian groups and their signatures, and also the representation of those groups as subgroups of $\mathrm{PGL}(n, F)$.

As the genus increases, the classification of conformal actions on Riemann surfaces becomes intractable by hand, so it is unavoidable the use of computers to make the classification in higher genera. In the last turn of the century T. Breuer devised an algorithm to generate a list of all groups acting on a Riemann surface of a given genus, and he implemented it in GAP [TGG16]. His program depends on databases of groups of a given order which are available in GAP. He ran the codes in GAP finding actions on Riemann surfaces of genus $g \leq 48$ (see [Bre00] and [Bre11]). In 2015, J. Paulhus used the codes of T. Breuer and wrote a program for Magma [BCP97], which gives a list of all the possible conformal actions on Riemann surfaces of a given genus $g \leq 20$, together with the signature of the action and generating vectors (see [Pau15]).

3.4.2 Actions with conformal and anticonformal elements

The classification of actions on Riemann surfaces which admit anticonformal elements has taken many directions. For example, we have seen the problem of studying such actions on particular types of Riemann surfaces (hyperelliptic, p-gonal) in Section 2.2 and Section 2.3. There is a theorem of D. Singerman which tells us about cyclic actions on Riemann surfaces.

Theorem 3.4.1. [Sin74b, Corollary 1] *Let S be a Riemann surface of genus g which admits a conformal automorphism of order $N > 2g+2$. Then S admits an anticonformal involution, so it cannot be pseudoreal.*

In particular, if X is a pseudoreal Riemann surface and the conformal automorphism group of X is isomorphic to C_n, then $n \leq 2g+2$.

In this thesis we will need some results in the following directions: the study of the minimal genus of Riemann surfaces on which a given group can act, and the classification of all the actions on a given genus.

In the first direction, we have the work [Gor85], where the author studies the actions of cyclic groups with anticonformal elements through NEC groups. He computed the minimal genus in which a cyclic group acts as the (not necessarily full) conformal and anticonformal group of some Riemann surface, depending if the group has anticonformal involutions (see [Gor85, Theorem 3]) or not (see [Gor85, Theorem 5]).
In [BG10] the authors define the *essential actions* of groups on Riemann surfaces, which are the actions having anticonformal elements. They solved the minimal genus problem for the cyclic *full automorphism groups* of pseudoreal Riemann surfaces, proving the following theorem:

Theorem 3.4.2. [BG10, Theorem 6.1] *The minimal genus of a pseudoreal Riemann surface, admitting a cyclic group of order* $4n$ *as an essential group of automorphisms, equals*

$$\begin{aligned} &2n \quad \textit{if } n=1,2,4, \\ &\frac{3n}{2}+1 \quad \textit{if } 8 \mid n \textit{ and } 3 \nmid n, \\ &\frac{2(p-1)n}{p}+1 \quad \textit{otherwise}, \end{aligned}$$

where p *is the smallest odd prime divisor of* n*. Furthermore, it is possible to choose a pseudoreal Riemann surface of that genus with* C_{4n} *as its full automorphism group.*

In [KWT15] the authors generalize the previous theorem as we see in the following theorem:

Theorem 3.4.3. [KWT15, Theorem 3.7] *The cyclic group* C_{4n} *is the full automorphism group of a pseudoreal Riemann surface of genus* $g \geq 2$ *if and only if there exists a sequence of integers* $(\gamma, q, a, k) \neq (1,1,1,-1), (0,0,1,-1)$ *and* $(0,0,2,0)$ *such that* $\gamma, q \geq 0$, $a, k \geq -1$, $g = qn + a(n-1)$*,* a *and* k *have the same parity,* q *and* γ *have the same parity and* $k \leq \min(a, q-2\gamma)$*. In particular,* $g \neq 2n-1, n-1$ *and* $2n-2$*.*

In the second direction, we have the classification of full groups of pseudoreal Riemann surfaces of genus $2 \leq g \leq 4$ contained in the papers [BCC10] and [BC14]. In the former, the authors coin the term *pseudoreal Riemann surfaces* and classified all the possible full groups of those surfaces for genus 2 and 3, and in the latter they corrected part of their previous work, added the generating vectors for the actions, and added the case of genus 4. We summarize their work in the next tables.

Table 3.1: Automorphism groups of pseudoreal Riemann surfaces of genus 2

Genus 2				
$\mathrm{Aut}^+(X)$	Fuchsian signature	$\mathrm{Aut}^{\pm}(X)$	NEC signature	Generating Vector
C_2	(2^6)	C_4	$(1; -; [2^3])$	$(a; a^2, a^2, a^2)$

Table 3.2: Automorphism groups of pseudoreal Riemann surfaces of genus 3

Genus 3				
$\mathrm{Aut}^+(X)$	Fuchsian signature	$\mathrm{Aut}^\pm(X)$	NEC signature	Generating Vector
C_2	$(1;[2^4])$	C_4	$(2;-;[2^2])$	$(a,a;a^2,a^2)$
$C_2\times C_2$	$(0;[2^6])$	$C_4\times C_2$	$(1;-;[2^3])$	$(a;b,b,a^2)$

Table 3.3: Automorphism groups of pseudoreal Riemann surfaces of genus 4

Genus 4				
$\mathrm{Aut}^+(X)$	Fuchsian signature	$\mathrm{Aut}^\pm(X)$	NEC signature	Generating Vector
C_2	$(0;[2^{10}])$	C_4	$(1;-;[2^5])$	$(a;a^2,a^2,a^2,a^2,a^2)$
C_2	$(2;[2^2])$	C_4	$(3;-;[2])$	$(a,a,a;a^2)$
C_4	$(0;[2^4,4^2])$	C_8	$(1;-;[2^2,4])$	$(a^3;a^4,a^4,a^2)$
D_5	$(0;[2^2,5^2])$	$F20$	$(1;-;[2,5])$	$(b;b^2a,a^4)$

There are other works we want to mention, as the article [BC15] where the authors determine that the full automorphism groups for cyclic p-gonal pseudoreal surfaces of genus g such that $g>(p-1)^2$ are always cyclic or semidirect products of the form $C_n\rtimes_\phi C_p$, where $p>2$ is prime.

3.4.3 Full groups with non finitely maximal NEC signatures

In [BG10, Section 4], the authors study under which conditions a finite group G with a given non finitely maximal NEC signature can act as the full automorphism group of a pseudoreal Riemann surface. The 3 NEC signatures they studied are

$$(1;-;[k,l];\{-\}),\qquad (1;-;[k,k];\{-\}),\qquad (2;-;[k];\{-\}),$$

which are associated to the non finitely maximal Fuchsian *even* signatures

$$(0;[k,k,l,l]),\qquad k+l\geq 5,$$

$$(0; [k, k, k, k]), \qquad k \geq 3,$$

$$(1; [k, k])$$

of the Singerman list (see Table 1.2 and Theorem 1.2.32). They proved one theorem for each one of the three signatures, giving sufficient and necessary conditions for a group G to be a full automorphism group acting with that signature. We will show just the first one because they have very similar statements and proofs, and this theorem in particular will be important in the next section.

Theorem 3.4.4. [BG10, Lemma 4.1] *Let Δ be a NEC group with signature*

$$(1; -; [k, l]; \{-\}),$$

where $k \neq l$. There exists an epimorphism $\theta : \Delta \longrightarrow G$ onto a finite group G defining an essential action of G on a pseudoreal Riemann surface, if and only if G is a non-split extension of some of its subgroups H of index 2, G is generated by two elements d, x such that x and d^2x have orders k and l, respectively, $d \notin H$ and the map $x \mapsto x^{-1}$, $d \mapsto d^{-1}$ does not induce an automorphism of G. Furthermore, such a group G is necessarily the full automorphism group of a pseudoreal Riemann surface on which it acts.

We observed that they did not study the non finitely maximal NEC signature

$$(3; -; [-]; \{-\}),$$

and we needed it to complete the classification of possible automorphism groups for pseudoreal Riemann surfaces, so we proved the following lemma:

Lemma 3.4.5. *Let Δ be a NEC group with signature*

$$(3; -; [-]; \{-\}).$$

There exists an epimorphism $\theta : \Delta \longrightarrow G$ onto a finite group G defining an essential *action of G on a pseudoreal Riemann surface, if and only if G is a non-split extension of some of its subgroups H of index 2, G is generated by three elements d', d'', d''' such that $d', d'', d''' \notin H$ such that $(d')^2(d'')^2(d''')^2 = 1$ and the map*

$$d' \mapsto (d')^{-1}, \quad d'' \mapsto (d')^2(d'')^{-1}(d')^{-2}, \quad d''' \mapsto (d''')^{-1}$$

does not induce an automorphism of G. Furthermore, such a group G is necessarily the full automorphism group of a pseudoreal Riemann surface on which it acts.

Proof. Suppose we have an epimorphism $\theta : \Delta \longrightarrow G$ onto a finite group G defining an essential action on the pseudoreal Riemann surface $X = \mathbb{H}/\mathrm{Ker}(\theta)$. The group $H := \theta(\Delta^+)$ is an index 2 subgroup of G, because G has anticonformal elements. The extension $H \leq G$ is non-split because if it were a split extension, then $G \backslash H$ would contain anticonformal involutions, which cannot occur because X is pseudoreal. We have $\Delta = \langle d_1, d_2, d_3 : \; d_1^2 d_2^2 d_3^2 = 1 \rangle$ where the d_i's are glide reflections, so the anticonformal elements $d' := \theta(d_1), d'' := \theta(d_2)$ and $d''' := \theta(d_3)$ cannot belong to H. To prove the statement we need to show that the map

$$d' \mapsto (d')^{-1}, \quad d'' \mapsto (d')^2(d'')^{-1}(d')^{-2}, \quad d''' \mapsto (d''')^{-1}$$

does not induce an automorphism of G. To see this, observe that by [Buj82, p. 529-30] there is a NEC group Δ' with the unique signature $(0; +; [2, 2, 2], \{(-)\})$ containing Δ as a subgroup of index 2. By [Buj82, Proposition 4.8] we know that if

$$\Delta' = \langle x_1, x_2, x_3, c_1, e_1 : x_1 x_2 x_3 e_1 = 1, \; e_1^{-1} c_1 e_1 c_1 = 1, \; x_1^2 = x_2^2 = x_3^2 = 1, \; c_1^2 = 1 \rangle$$

then Δ can be written as

$$\Delta = \langle d_1 := c_1 x_1, \; d_2 := x_1 c_1 x_1 x_2, \; d_3 := x_2 x_1 c_1 x_1 x_2 x_3 \rangle \leq \Delta'.$$

If we conjugate every generator of Δ by c_1 we get

$$c_1^{-1}d_1c_1 = d_1^{-1}, \quad c_1^{-1}d_2c_1 = d_1^2d_2^{-1}d_1^{-2}, \quad c_1^{-1}d_3c_1 = d_3^{-1},$$

so $\mathrm{Ker}(\theta)$ would be a normal subgroup of Δ' if and only if the images of d_1, d_2 and d_3 through θ satisfy that the map

$$d' \mapsto (d')^{-1}, \quad d'' \mapsto (d')^2(d'')^{-1}(d')^{-2}, \quad d''' \mapsto (d''')^{-1}$$

induces an automorphism of $\Delta/\mathrm{Ker}(\theta) = G$. So the assertion follows, since if $\mathrm{Ker}(\theta)$ is a normal subgroup of Δ', then $\Delta'/\mathrm{Ker}(\theta) \cong \mathrm{Aut}^{\pm}(X)$ and it will contain $c_1\mathrm{Ker}(\theta)$, which is an anticonformal involution, contradicting the hypothesis that X is pseudoreal.

Conversely, for a NEC group Δ with signature $(3; -; [-], \{-\})$ and a non-split extension $H \leq G$ of degree 2, we can consider the map $\theta(d_1) = d'$, $\theta(d_2) = d''$, $\theta(d_3) = d'''$ which induces an epimorphism $\theta : \Delta \longrightarrow G$, defining an essential action on $X := \mathbb{H}/\mathrm{Ker}(\theta)$. The group G is the full automorphism group of X, because if not, then $\mathrm{Ker}(\theta)$ would be a normal subgroup of a NEC group Δ' with signature $(0; +; [2, 2, 2], \{(-)\})$, and so by the previous part of the proof, the mapping

$$d' \mapsto (d')^{-1}, \quad d'' \mapsto (d')^2(d'')^{-1}(d')^{-2}, \quad d''' \mapsto (d''')^{-1}$$

would define an automorphism of G, contradicting our assumptions. Finally, since G is a non-split extension of H, then $G \backslash H$ contains no involutions, then X is a pseudoreal Riemann surface. □

We wrote a program in Magma for each one of the 3 lemmas [BG10, Lemma 4.1, Lemma 4.2, Lemma 4.3], and for Lemma 3.4.5 in Section 5.3.3.

3.4.4 Maximal full groups

The signature $(1; -; [2, 3])$ will be very important in our thesis, because it is linked with the best upper bound of the order of full automorphism groups of pseudoreal Riemann surfaces. The study of bounds for the order of automorphism groups or Riemann surfaces began with A. Hurwitz, who proved that the orders of the conformal automorphism groups of the Riemann surfaces of genus $g \geq 2$ are bounded above by $84(g-1)$ (see [Hur93, p. 424]), and the bound is *sharp* in the sense that there are infinitely many Riemann surfaces for which their conformal automorphism group attain that bound (see [Mac61, Corollary in p. 96]). These groups are called the *Hurwitz groups*, and Hurwitz proved that a group G will be a Hurwitz group if and only if G is a finite quotient group of

$$\langle a, b \ : \ a^2 = b^3 = (ab)^7 = 1 \rangle.$$

The first example of a Hurwitz group is the order 168 group $\mathrm{PSL}(2, 7)$, which is the conformal automorphism group of the Klein's quartic

$$\{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} \ : \ x^3y + y^3z + z^3x = 0\},$$

which has genus 3. In the case of pseudoreal Riemann surfaces, the Hurwitz bound is never attained because all such surfaces have conformal automorphism groups of signature $(0; [2, 3, 7])$, which is an odd signature. For pseudoreal Riemann surfaces there is a better upper bound, as we see in the following theorem:

Theorem 3.4.6. [BCC10, Theorem 5.1] *If $X = \mathbb{H}/\Gamma$ is a pseudoreal Riemann surface of genus g with full automorphism group G, then $|G| \leq 12(g-1)$. Moreover, if $|G| = 12(g-1)$ and $G = \Delta/\Gamma$ then the signature of Δ is $(1; -; [2, 3])$.*

If a pseudoreal Riemann surface X has genus g and full group of order $12(g-1)$, we will say that X has *maximal full group*. In particular, if we have a Riemann surface X with maximal full automorphism group $\mathrm{Aut}^{\pm}(X)$, then $\mathrm{Aut}^{+}(X)$ will be a group of order $6(g-1)$ and the signature of the Fuchsian group associated with the conformal automorphism group will be $(0; +; [2, 2, 3, 3])$. Using the program **BG1** in Section 5.3.3 and the data of conformal actions of J. Paulhus, we found the minimum genus for which there exists a pseudoreal Riemann surface with maximal full group, which is $g = 14$, with conformal automorphism group ID$(78, 1)$ and full automorphism group is ID$(156, 7)$, which is non Abelian.

The authors of [BCC10] also prove the following theorem:

Theorem 3.4.7. [BCC10, Theorem 5.5] *There exist pseudoreal Riemann surfaces with automorphism group of maximal order, for infinitely many genera. In particular, there are infinitely many pseudoreal Riemann surface with maximal automorphism group.*

As the groups that they obtained are non Abelian, this inspired us to prove the following result:

Theorem 3.4.8. *If a pseudoreal Riemann surface X has maximal full automorphism group, then its conformal automorphism group is not Abelian.*

Proof. Let us suppose that the conformal automorphism group G is Abelian. First observe that the Fuchsian signature associated to G is $(0; [2, 2, 3, 3])$ (Theorem 3.4.6 and Theorem 1.2.32). If G is a non cyclic Abelian group, then by [BCC03, Theorem 7.1] we must have that G is a quotient of $C_2 \times C_{gcd(2,3)} \times C_3 = C_6$, so the order of G must divide 6. But we know that $|G| = 6(g-1)$ so $|G|$ is divisible by 6, then $|G| = 6$. The only 2 groups of order 6 are C_6 and S_3, but none of these groups is a conformal automorphism group of a pseudoreal Riemann surface of genus 2 (see Table 3.1).

Then if G is Abelian it must be a cyclic group of order $6(g-1)$, so $G \cong C_{6(g-1)}$. By Table 3.4.2 we know that there is no conformal automorphism group of a pseudoreal Riemann surface of order 6 in genus 2, so we can assume $g > 2$. However, in this case we have the following inequality

$$6(g-1) > 2g + 2,$$

and then any generator of G will be an element of order $> 2g+2$. By Theorem 3.4.1 X is not pseudoreal, contradicting the hypothesis, thus G must be a non Abelian group. □

Corollary 3.4.9. *If a pseudoreal Riemann surface X admits a maximal full group, then $\mathrm{Aut}^{\pm}(X)$ is non Abelian.*

An alternative proof of this result was given by R. Hidalgo, as we see in the following

Theorem 3.4.10. *If X is a compact Riemann surface of genus $g \geq 2$ admitting an Abelian group of conformal automorphisms G so that X/G has signature $(0; [2,2,3,3])$, then $g = 2$ and X admits a group G' of conformal automorphisms containing G as a normal subgroup such that $G'/G \cong C_2 \times C_2$. In particular, X cannot be pseudoreal.*

Proof. Assume X is a compact Riemann surface of genus $g \geq 2$ admitting an Abelian group G of conformal automorphisms so that X/G has signature $(0; [2,2,3,3])$. Let Δ be the Fuchsian group uniformizing X/G, so it has a presentation

$$\Delta = \langle x_1, x_2, x_3, x_4 : \ x_1^2 = x_2^2 = x_3^3 = x_4^3 = x_1x_2x_3x_4 = 1\rangle$$

If Δ' is its derived (commutator) subgroup, then $\Delta/\Delta' \cong C_6$. We know that there is a (torsion free) normal subgroup $\Gamma \trianglelefteq \Delta$ so that $X = \mathbb{H}/\Gamma$ and $\Delta/\Gamma \cong G$. As G is Abelian, $\Delta' \leq \Gamma$. It follows that Δ' is a torsion free Fuchsian group; set $X' := \mathbb{H}/\Delta'$; which is a compact Riemann surface of genus 2. As the regular cover $X' \longrightarrow X/G$ has deck group C_6 and it factors through $X \longrightarrow X/G$, we have that there is a subgroup $N \leq C_6$ (acting freely on the fixed points of X') so that $X = X'/N$. In this way, as we are assuming X of genus $g \geq 2$, we must have $N = \{1\}$, $X = X'$ and $G = C_6$. But it can be seen that (as Δ' is a characteristic subgroup of Δ) that the Klein group $C_2 \times C_2$ (keeping invariant the collection of four cone points of X/G) must lift to X as a group of conformal automorphisms with G as a subgroup of index 4. □

Now we apply Theorem 3.4.8 to prove that the *generalized superelliptic curves*, which we will define immediately, cannot have maximal full automorphism group.

Definition 3.4.11. A curve X is said to be *generalized superelliptic* if there exists some $\tau \in Z\left(\mathrm{Aut}(X)\right)$ such that $X/\langle\tau\rangle$ has genus 0.

For those curves we have the following result.

Corollary 3.4.12. *If X is a pseudoreal Riemann surface with maximal full automorphism group, then it cannot be a generalized superelliptic curve.*

Proof. If X is a superelliptic generalized pseudoreal Riemann surface, then there exists an element $\tau \in Z(G)$ such that $X/\langle\tau\rangle$ has genus 0. We can consider the projection map $X/\langle\tau\rangle \longrightarrow X/Z(G)$, so $X/Z(G)$ has genus 0 also. Because of Theorem 2.3.6 G is an Abelian group, which contradicts the previous theorem. Then X is a non generalized superelliptic Riemann surface. □

Chapter 4

Classification

4.1 Summary of known pseudoreal Riemann surfaces in low genus

In this section we will make a summary of all the possible conformal and full automorphism groups of pseudoreal Riemann surfaces of small genus, and we will show explicit algebraic models if they exist in the literature.

1. **Genus 2.** For a curve X of genus 2, from [CQ05, Theorem 2] we know that if X is defined over a field of characteristic not equal to 2, and $\mathrm{Aut}(X) \ncong C_2$, then X can be defined over its field of moduli, but when $\mathrm{Aut}(X) \cong C_2$, it is possible for the curve not to be definable over its field of moduli. In [CNP05, Theorem 5] the authors prove that in characteristic 2, a genus 2 curve is always definable over its field of moduli. In that case, if X is a pseudoreal Riemann surface of genus 2, then $\mathrm{Aut}^+(X) \cong C_2$. The latter result was obtained in [BCC10, Theorem 4.1] via NEC groups and epimorphisms, obtaining C_4 as the only possible full automorphism group in genus 2 (see Table 3.1).
   In fact, an algebraic model for a pseudoreal curve of genus 2 is Earle's example

$$X \;:\; y^2 = x(x^2 - a^2)(x^2 + ta^2x - a),$$

where $a = e^{\frac{2\pi i}{3}}$ and $t \in \mathbb{R}^+ - \{1\}$. The full automorphism group of X is $\mathrm{Aut}^{\pm}(X) \cong C_4$ (see [Ear71, p. 126]).

2. **Genus 3.** In [GSS05, Corollary 2] the authors prove that if we have a hyperelliptic complex curve X of genus 3 such that $|\mathrm{Aut}(X)| > 2$, then it can be defined over its field of moduli. This result had a small error, because in [Hug07, Proposition 5.6] the author shows examples of hyperelliptic curves X with $\mathrm{Aut}(X) \cong C_2 \times C_2$ with field of moduli $\mathbb{R}$ but which cannot be defined over $\mathbb{R}$. In fact, it is true for the hyperelliptic case of genus 3 that if $|\mathrm{Aut}(X)| > 4$, then X can be defined over its field of moduli as we saw above after 2.2.2.

   For the non-hyperelliptic case, in [AQ12, Theorem 0.2] the authors prove that if X is a smooth plane quartic such that $|\mathrm{Aut}(X)| > 4$, then X can be defined over its field of moduli, because all the other groups have odd signatures, as we see in Table 4.1 (see [Bar05, Theorem 16] for details).

   Table 4.1: Automorphism groups of smooth plane quartics

$\mathrm{Aut}(X)$	Signature of $X/\mathrm{Aut}(X)$
$\mathrm{PSL}(2,7)$	$(0; [2,3,7])$
S_3	$(0; [2^4, 3])$
$C_2 \times C_2$	$(0; [2^6])$
D_4	$(0; [2^5])$
S_4	$(0; [2^3, 3])$
$(C_4 \times C_4) \rtimes S_3$	$(0; [2,3,8])$
$C_4 \circ (C_2 \times C_2)$	$(0; [2^3, 4])$
$C_4 \circ A_4$	$(0; [2,3,12])$
C_6	$(0; [2,3^2,6])$
C_9	$(0; [3,9^2])$
C_3	$(0; [3^5])$
C_2	$(1; [2^4])$

   If X is a smooth plane complex quartic such that $\mathrm{Aut}(X) \cong C_2 \times C_2$, then X

must be isomorphic to some curve in the 3 complex parameters family

$$X_{a,b,c} \ : \ x^4 + y^4 + z^4 + ax^2y^2 + by^2z^2 + cz^2x^2 = 0,$$

with $a, b, c \in \mathbb{C}$, $a^2 + b^2 + c^2 - abc \neq 4$ and such that no a^2, b^2, c^2 is 4, to get a smooth curve (see [Bar05, Theorem 16]). In [AQ12, Corollary 4.5], the authors prove that if this curve has $\mathbb{R}$ as field of moduli, it will also have $\mathbb{R}$ as a field of definition. Then if a genus 3 complex curve is pseudoreal, it must have conformal automorphism group isomorphic to either C_2 or $C_2 \times C_2$, and in the latter case it will be hyperelliptic. The same result was obtained in [BC14, Proposition 3.5] (after fixing some mistake in [BCC10, Theorem 4.2]) via NEC groups and epimorphisms, obtaining C_4 and $C_4 \times C_2$ as the only possible full automorphism groups in genus 3 (see Table 3.2).

We have algebraic models of pseudoreal curves for every case: in the case of $\mathrm{Aut}^+(X) \cong C_2$, we have an explicit non-hyperelliptic pseudoreal curve in [AQ12, Proposition 4.3] given by

$$X \ : \ y^4 + y^2(x - a_1 z)\left(x + \frac{1}{\overline{a_1}} z\right) + (x - a_2 z)\left(x + \frac{1}{\overline{a_2}} z\right)(x - a_3 z)\left(x + \frac{1}{\overline{a_3}} z\right) = 0,$$

where $a_1 \in \mathbb{R}$, $a_2 a_3 \in \mathbb{R}$,in which case $\mathrm{Aut}^{\pm}(X) \cong C_4$. In the case $\mathrm{Aut}^+(X) \cong C_2 \times C_2$ we have Huggins' example in [Hug05, p. 82] given by

$$X \ : \ y^2 = (x^2 - a_1)\left(x^2 + \frac{1}{\overline{a_1}}\right)(x^2 - a_2)\left(x^2 + \frac{1}{\overline{a_2}}\right),$$

where $\mathrm{Aut}^{\pm}(X) \cong C_4 \times C_2$.

3. **Genus 4.** The classification of automorphism groups is done in [BC14, Theorem 4.3], where the authors find that the only possible full automorphism groups for pseudoreal Riemann surfaces are C_4, C_8 and the Frobenius group $F20$ (see Table 3.3). The proof gives no model for pseudoreal curves with these full automorphism groups, but only the existence of models.

We have algebraic models of pseudoreal curves when $\mathrm{Aut}^+(X)$ is C_2 or C_4. When $\mathrm{Aut}^+(X)$ is C_2, we have Shimura's example

$$y^2 = x^5 + (a_1x^6 - \overline{a_1}x^4) + (a_2x^7 + \overline{a_2}x^3) + (a_3x^8 - \overline{a_3}x^2) + (a_4x^9 + \overline{a_4}x) + (x^{10} - 1),$$

which has full group C_4, where the coefficients a_i and $\overline{a_j}$ are algebraically independent over $\mathbb{Q}$. When $\mathrm{Aut}^+(X)$ is C_4, we have a hyperelliptic example in [Hug05, p. 82] given by

$$y^2 = x(x^4 - b_i)\left(x^4 + \frac{1}{\overline{b_i}}\right),$$

which has full group C_8.

Remark 4.1.1. There is no explicit model yet of pseudoreal Riemann surface of genus 4 with full automorphism group $F20$ or with conformal automorphism group C_2 with signature $(2; [2, 2])$.

In the following section we will extend the classification until genus 10.

4.2 Full groups for pseudoreal Riemann surfaces of genus $5 \leq g \leq 10$

Theorem 4.2.1. *Two finite groups G and $\overline{G}$ are the conformal and full automorphism groups of a pseudoreal Riemann surface X of genus $5 \leq g \leq 10$ if and only if $G = \mathrm{Aut}^+(X)$ and $\overline{G} = \mathrm{Aut}^{\pm}(X)$ in the corresponding table by genus among Table 5.2, 5.3, 5.4, 5.5, 5.6, 5.3.4, and 5.7.*

To carry out the classification, we follow the next steps.

1. We fix a genus $5 \leq g \leq 10$. Using the programs in Section 5.1 we consider the complete list of conformal actions $\mathrm{Aut}^+(X)$ for Riemann surfaces X of genus g,

which is given to us by Magma with the program of J. Paulhus (see [Pau15]). From that list, programs in Section 5.3 select only the groups of even order and even signature (see Theorem 3.2.2 and Theorem 1.3.3).

2. From the previous list, programs in Section 5.3.4 separate the finitely maximal and the non finitely maximal signatures.

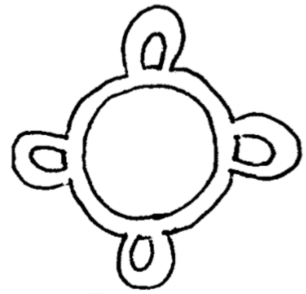
   2.1 In the first case, the program **IsPseudoreal** given at the end of Section 5.3 gives us the possible full automorphism groups, conformal automorphism groups and Fuchsian signatures for pseudoreal Riemann surfaces which have maximal NEC signature. To do this, we input a conformal action G with finitely maximal Fuchsian signature s, and the program considers all the degree 2 extensions $\overline{G}$ such that $G \leq \overline{G}$ is non-split (Theorem 3.2.4), and it also checks all the possible generating vectors of an epimorphism θ from a NEC group Δ with finitely maximal NEC signature $\overline{s}$ (Theorem 1.2.35) such that the canonical Fuchsian group Δ^+ of Δ has signature s (Theorem 1.2.32), onto the groups $\overline{G}$. The program gives the results only when there is at least one generating vector.

   2.2 In the second case, the programs **BG1, BG2, BG3** and **BG4** in Section 5.3.3 give us the same information for non finitely maximal NEC groups, based on Lemmas [BG10, Lemma 4.1, Lemma 4.2, Lemma 4.3] and Lemma 3.4.5. □

4.3 Further examples

1. **Genus 5.** In [Ear71, Theorem 2.] the author gives an example of a pseudoreal Riemann surface X of genus 5 with an order 4 anticonformal element called f, which generates $\mathrm{Aut}^{\pm}(X) \cong C_4$ (see Figure 4.1). There are exactly 2 possible conformal actions of C_2 on pseudoreal Riemann surfaces, having signatures $(3; [-])$ and $(1; [2^8])$. Since X has the conformal automorphism f^2 which has no fixed

Figure 4.1: Earle's picture of his genus 5 example



points, the conformal action is C_2 with signature $(3;[-])$ (see Table 5.2).

We can consider Huggin's example in [Hug05, p. 82] given by

$$y^2 = (x^2 - a_1)\left(x^2 + \frac{1}{\overline{a_1}}\right)(x^2 - a_2)\left(x^2 + \frac{1}{\overline{a_2}}\right)(x^2 - a_3)\left(x^2 + \frac{1}{\overline{a_3}}\right),$$

which is the only pseudoreal hyperelliptic curve with conformal automorphism group $C_2 \times C_2$ in this genus.

There are non-hyperelliptic examples of pseudoreal curves with conformal automorphism group $C_2 \times C_2$ in [ACHQ16]. For example, we can consider the curve defined by the equations

$$w_2^2 = 1-2re^{i\theta}w_1, \quad w_3^2 = w_1(re^{i\theta}w_1-1), \quad w_4^2 = (1-w_1(1+re^{i\theta}))(1-w_1(re^{i\theta}-r^2)),$$

such that $\theta \in]0,\pi[$, $\theta \neq \frac{\pi}{2}$, $r \in]1,+\infty[$ and $r \neq \sqrt{1+\cos^2(\theta)} \pm \cos(\theta)$ (see [ACHQ16, p. 9-10] for more examples).

2. **Genus 6.** An example of a pseudoreal Riemann surface X such that $\mathrm{Aut}^+(X) = C_2$, we can consider Shimura's example

$$y^2 = x^7 + (a_1x^8 - \overline{a_1}x^6) + \ldots + (a_6x^{13} + \overline{a_6}x) + (x^{14} - 1),$$

which is a hyperelliptic curve, taking a_i and a_j^σ algebraically independent over $\mathbb{Q}$. An example of a pseudoreal Riemann surface X such that $\mathrm{Aut}^+(X) = C_4$ (with signature $(0;[2^6,4^2]))$ is Huggins' example in [Hug05, p. 82], given by

$$y^2 = x(x^2-b_1)\left(x^2+\frac{1}{\overline{b_1}}\right)\dots(x^2-b_3)\left(x^2+\frac{1}{\overline{b_3}}\right),$$

which is a pseudoreal hyperelliptic curve. An example of a pseudoreal Riemann surface X such that $\mathrm{Aut}^+(X) = C_6$ is Huggins' example in [Hug05, p. 82], given by

$$y^2 = (x^3-b_1)\left(x^3+\frac{1}{\overline{b_1}}\right)(x^3-b_2)\left(x^3+\frac{1}{\overline{b_2}}\right),$$

which is a pseudoreal hyperelliptic curve, and by [BT02, Theorem 1.2] its full automorphism group $\mathrm{Aut}^{\pm}(X)$ is C_{12}.

3. **Genus 7.** An example of a pseudoreal Riemann surface X such that $\mathrm{Aut}^+(X) = C_4 \times C_2$ is Huggins' example in [Hug05, p. 82] given by

$$y^2 = (x^4-a_1)\left(x^4+\frac{1}{\overline{a_1}}\right)(x^4-a_2)\left(x^4+\frac{1}{\overline{a_2}}\right),$$

which is the only pseudoreal hyperelliptic curve in this case and its full automorphism group is $\mathrm{Aut}^{\pm}(X) = C_8 \times C_2$.

4. **Genus 8.** An example of a pseudoreal Riemann surface X such that $\mathrm{Aut}^+(X) = C_2$ is Shimura's example

$$y^2 = x^9 + (a_1x^{10} - \overline{a_1}x^8) + \ldots + (a_8x^{17} + \overline{a_8}x) + (x^{18}-1),$$

which is a pseudoreal hyperelliptic curve, taking a_i and a_j^{σ} algebraically independent over $\mathbb{Q}$. An example of a pseudoreal Riemann surface X such that $\mathrm{Aut}^+(X) = C_4$ is Huggins' example in [Hug05, p. 82] given by

$$y^2 = x(x^2-b_1)\left(x^2+\frac{1}{\overline{b_1}}\right)\dots(x^2-b_4)\left(x^2+\frac{1}{\overline{b_4}}\right),$$

which is a pseudoreal hyperelliptic curve. An example of a pseudoreal Riemann surface X such that $\mathrm{Aut}^+(X) = C_8$ is Huggins' example in [Hug05, p. 82], given by

$$y^2 = x(x^4 - b_1)\left(x^4 + \frac{1}{\overline{b_1}}\right)(x^4 - b_2)\left(x^4 + \frac{1}{\overline{b_2}}\right),$$

which is a pseudoreal hyperelliptic curve.

5. **Genus 9.** An example of a pseudoreal Riemann surface X such that $\mathrm{Aut}^+(X) = C_2 \times C_2$ is Huggins' example in [Hug05, p. 82] given by

$$y^2 = (x^2 - a_1)\left(x^2 + \frac{1}{\overline{a_1}}\right) \dots (x^2 - a_5)\left(x^2 + \frac{1}{\overline{a_5}}\right),$$

which is the only pseudoreal hyperelliptic case in this genus.

A non-hyperelliptic example in this genus appears in [ACHQ16], where the authors find the curve defined by the equations

$$w_2^2 = 1 - w_1(1 + re^{i\theta}), \quad w_3^2 = 1 - w_1(re^{i\theta} - r^2),$$

$$w_4^2 = 1 - 2re^{i\theta}w_1, \quad w_5^2 = w_1(re^{i\theta}w_1 - 1),$$

which has conformal automorphism group C_2^4 and full automorphism group $\mathrm{ID}(32, 22)$ according to Table 5.3.4 (see [ACHQ16, p. 11] for more examples).

6. **Genus 10.** An example of a pseudoreal Riemann surface X such that $\mathrm{Aut}^+(X) = C_2$ is Shimura's example

$$y^2 = x^{11} + (a_1x^{12} - \overline{a_1}x^{10}) + \ldots + (a_{10}x^{21} + \overline{a_{10}}x) + (x^{22} - 1),$$

which is a pseudoreal hyperelliptic curve, taking a_i and a_j^σ algebraically independent over $\mathbb{Q}$.

An example of a pseudoreal Riemann surface X such that $\mathrm{Aut}^+(X) = C_4$ is

Huggins' example in [Hug05, p. 82] given by

$$y^2 = x(x^2 - a_1)\left(x^2 + \frac{1}{\overline{a_1}}\right)\dots(x^2 - a_5)\left(x^2 + \frac{1}{\overline{a_5}}\right),$$

which is a pseudoreal hyperelliptic curve.

An example of a pseudoreal Riemann surface X such that $\mathrm{Aut}^+(X) = C_6$ is Kontogeorgis' example in [Kon09, Section 5] given by

$$y^3 = (x^2 - a_1)\left(x^2 + \frac{1}{\overline{a_1}}\right)(x^2 - a_2)\left(x^2 + \frac{1}{\overline{a_2}}\right)(x^2 - a_3)\left(x^2 + \frac{1}{\overline{a_3}}\right),$$

which is a 3-gonal pseudoreal curve, taking $a_i = (i+1)\epsilon_3^i,\ i \in \{0,1,2\}$, where ϵ_3 is a primitive 3rd root of unity. The conformal automorphism of X is generated by the map f defined by

$$(x, y) \mapsto (-x, \epsilon_3 y),$$

and we have the anticonformal automorphism g defined by

$$(x, y) \mapsto \left(\frac{i}{\overline{x}}, \frac{\epsilon_3^{-1}\overline{y}}{\overline{x}^4}\right).$$

Since $f \circ g \neq g \circ f$, the full automorphism group of X is non Abelian, so it must be Dic_{12}.

Finally, for C_{10} we have Huggins' example in [Hug05, p. 82] given by

$$y^2 = x(x^5 - a_1)\left(x^5 + \frac{1}{\overline{a_1}}\right)(x^5 - a_2)\left(x^5 + \frac{1}{\overline{a_2}}\right),$$

which is a pseudoreal hyperelliptic curve.

4.4 Pseudoreal plane quintics

A few months ago, E. Badr and F. Bars [BB16] classified the automorphism groups of plane quintics defined over an algebraically closed field K of zero characteristic, giving a smooth plane model for every group (see Table 4.2). We used this classification, taking $K=\mathbb{C}$, together with our classification of conformal and full automorphism groups of pseudoreal Riemann surfaces of genus 6 (see Table 5.3), to get the possible conformal and full groups of pseudoreal plane quintics.

Table 4.2: Automorphism groups of smooth plane quintics

Group	Generators	Polynomial of the smooth plane model
ID(150, 5)	$[\epsilon_5 x : y : z], [x : \epsilon_5 y : z],$ $[x : z : y], [y : z : x]$	$x^5 + y^5 + z^5$
ID(39, 1)	$[x : \epsilon_{13} y : \epsilon_{13}^{10} z], [y : z : x]$	$x^4 y + y^4 z + z^4 x$
ID(30, 1)	$[x : \epsilon_{15} y : \epsilon_{15}^{11} z], [x : z : y]$	$x^5 + y^4 z + z^4 y$
C_{20}	$[x : \epsilon_{20}^4 y : \epsilon_{20}^5 z]$	$x^5 + y^5 + xz^4$
C_{16}	$[x : \epsilon_{16} y : \epsilon_{16}^{12} z]$	$x^5 + y^4 z + xz^4$
C_{10}	$[x : \epsilon_{10}^2 y : \epsilon_{10}^5 z]$	$x^5 + y^5 + xz^4 + \beta_{2,0} x^3 z^2$ $\beta_{2,0} \neq 0, \beta_{2,0}^2 \neq 20$
D_5	$[x : \epsilon_5 y : \epsilon_5^2 z], [z : y : x]$	$x^5 + y^5 + z^5 + \beta_{3,1} x^2 y z^2 + \beta_{4,3} x y^3 z$ $(\beta_{3,1}, \beta_{4,3}) \neq (0, 0)$
C_8	$[x : \epsilon_8 y : \epsilon_8^4 z]$	$x^5 + y^4 z + xz^4 + \beta_{2,0} x^3 z^2$ $\beta_{2,0} \neq 0, \pm 2$
S_3	$[x : \epsilon_3 y : \epsilon_3^2 z], [x : z : y]$	$x^5 + y^4 z + yz^4 + \beta_{2,1} x^3 yz +$ $\beta_{3,3} x^2 (y^3 + z^3) + \beta_{4,2} x y^2 z^2$ (not above)
C_5	$[x : y : \epsilon_5 z]$	$z^5 + L_{5,z}$ (not above)
C_4	$[x : \epsilon_4 y : \epsilon_4^2 z]$	$x^5 + x(z^4 + \alpha y^4) + \beta_{2,0} x^3 z^2 + \beta_{3,2} x^2 y^2 z +$ $+ \beta_{5,2} y^2 z^3, \beta_{5,2} \neq 0$ (not above)
C_4	$[x : y : \epsilon_4 z]$	$z^4 L_{1,z} + L_{5,z}$ (not above)
C_3	$[x : \epsilon_3 y : \epsilon_3^2 z]$	$x^5 + y^4 z + \alpha y z^4 + \beta_{2,1} x^3 yz +$ $+ x^2 (\beta_{3,0} z^3 + \beta_{3,3} y^3) + \beta_{4,2} x y^2 z^2$ (not above)
C_2	$[x : y : \epsilon_2 z]$	$z^4 L_{1,z} + z^2 L_{3,z} + L_{5,z}$ (not above)
$L_{n,z}$ denotes a homogeneous polynomial of degree n in $K[x, y]$ and ϵ_n is a primitive nth rooth of unity		

Theorem 4.4.1. *Two finite groups G and $\overline{G}$ are the conformal and full automorphism groups of a pseudoreal plane quintic X if and only if $G = \mathrm{Aut}^{+}(X)$ and $\overline{G} = \mathrm{Aut}^{\pm}(X)$ in a row of Table 4.3.*

Table 4.3: Possible automorphism groups for pseudoreal plane quintics

Aut$^+$(X)	Fuchsian signature	Aut$^\pm$(X)	NEC signature	Generating Vector
C_4	$(0; 4^6)$	C_8	$(1; -; [4^3])$	$(a; a^2, a^2, a^2)$
C_2	$(2; [2^6])$	C_4	$(3; -; [2^3])$	$(a, a, a; a^2, a^2, a^2)$

Proof. We know that the conformal and full automorphism group of a genus 6 pseudoreal Riemann surface, together with their signatures, must be in Table 5.3. For the remaining groups we can use Riemann-Hurwitz formula to calculate the signature of the covering $X \longrightarrow X/\mathrm{Aut}(X)$, obtaining the following cases:

1. $\mathrm{Aut}(X) = D_5$, with group generators $[x : \epsilon_5 y : \epsilon_5^2 z]$ and $[z : y : x]$, with a smooth plane model
$$x^5 + y^5 + z^5 + ax^2yz^2 + bxy^3z = 0,$$
with $(a, b) \neq (0, 0)$. In this case the covering $X \longrightarrow X/\mathrm{Aut}(X)$ has signature $(0; [2^6])$.

2. $\mathrm{Aut}(X) = C_4$, with generator $[x : y : \epsilon_4 z]$, with smooth plane model
$$z^4 L_{1,z} + L_{5,z} = 0.$$
In this case the covering $X \longrightarrow X/\mathrm{Aut}(X)$ has signature $(0; [4^6])$.

3. $\mathrm{Aut}(X) = C_2$, with generator $[x : y : -z]$, with smooth plane model
$$z^4 L_{1,z} + z^2 L_{3,z} + L_{5,z} = 0.$$
In this case the covering $X \longrightarrow X/\mathrm{Aut}(X)$ has signature $(2; [2^6])$.

We will prove that Case 1 cannot occur. Suppose we have an isomorphism f between

$$X: \; x^5 + y^5 + z^5 + ax^2yz^2 + bxy^3z = 0$$

and

$$\overline{X}: \; x^5 + y^5 + z^5 + \overline{a}x^2yz^2 + \overline{b}xy^3z = 0.$$

Then f must have a representation as a 3×3 matrix (see [BB16, p. 4328]) and it must preserve the fixed points of the subgroup $C_5 \leq D_5$. We see that the fixed points of $[x : \epsilon_5 y : \epsilon_5^2 z]$ are $[1 : 0 : 0], [0 : 1 : 0]$ and $[0 : 0 : 1]$, so the matrix representing f must be the composition of a permutation matrix with a diagonal matrix. Since $[z : y : x]$ and the identity permutation $[x : y : z]$ are the only possible permutations in X, the only possible matrix representations for f are the following

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & r & 0 \\ s & 0 & 0 \end{bmatrix},$$

for complex entries r, s. Since the coefficients of X remain fixed through f, the complex numbers r and s must be of the form ϵ_5^m and ϵ_5^n, respectively. So $f[x : y : z]$ must be $[x : \epsilon_5^m y : \epsilon_5^n z]$ or $[z : \epsilon_5^m y : \epsilon_5^n x]$, but in both cases we have

$$(J \circ f)^2[x : y : z] = [x : y : z],$$

so the curve admits anticonformal involutions, and it cannot be pseudoreal. □

4.5 Pseudoreal generalized superelliptic curves of low genus

In [BSZ15], the authors motivate the study of generalized superelliptic curves (Definition 3.4.11), which have very nice properties: they are a natural generalization of hyperelliptic curves, they have nice affine models, and their automorphism groups are not too hard to compute. They propose the problem of finding the minimal field of definition of those curves when they are considered as complex curves. In [HS16] the authors study the definability of generalized superelliptic curves over their field of moduli, and they give a partial classification up to genus 10. In this section we will discuss the case of the Galois extension $\mathbb{C}/\mathbb{R}$ for generalized superelliptic curves of genus $3 \leq g \leq 10$, because we have all the possible conformal and full automorphism groups of pseudoreal Riemann surfaces in such genera, and in [MPRZ14] we have the classification of groups of generalized superelliptic curves in genus $5 \leq g \leq 10$. This is interesting because most of the genus 4 curves which have non trivial automorphism groups are generalized superelliptic curves ($70\% - 80\%$, see [BSZ15, Page 2]), and probably in bigger genus they are also an important subset of the curves which have non trivial automorphism groups.

Theorem 4.5.1. *Two finite groups G and $\overline{G}$ are the conformal and full automorphism groups of a pseudoreal generalized superelliptic curve X of genus $3 \leq g \leq 10$ and cental element τ (remember Definition 3.4.11) if and only if $G = \mathrm{Aut}^+(X)$ and $\overline{G} = \mathrm{Aut}^{\pm}(X)$ in the corresponding table by genus among Table 5.8, 5.9, 5.10, 5.11, 5.12, 5.13, 5.14 and 5.15.*

By Corollary 2.3.6, every conformal automorphism group of a pseudoreal generalized superelliptic curve is Abelian. Since in [MPRZ14] we have all the possible automorphism groups of generalized superelliptic curves in genus $5 \leq g \leq 10$, from those tables we can select genus by genus the groups which appear also in Section 4.2. When $|\tau| = 2$, i.e. when the generalized superelliptic curve is hyperelliptic, we know by [BT02, Theorem

1.2] that its full automorphism group is Abelian, so we can discard the case in genus 6 where we have a full automorphism group $C_8 \rtimes_\phi C_2$. For all the hyperelliptic cases and for the 3-gonal curve in genus 10 we have explicit examples by [Hug05] and [Kon09] (see Section 4.3), but in the other non-hyperelliptic cases we have no certainty of the existence of a pseudoreal generalized superelliptic curve with those automorphism groups.

Chapter 5

Magma Programs

In this chapter we will present the implementation of all the programs that we used to carry out the classification of full automorphism groups of pseudoreal Riemann surfaces done in Section 4.2.

5.1 J. Paulhus' program

Our program relies on Jennifer Pauhlus' program **GenVectMagmaToGenus20**, which is available at

http://www.math.grinnell.edu/~paulhusj/monodromy.html

and is based on the paper [Pau15]. To run our program one first needs to download the packages **genvectors.m**, **searchroutines.m**, **GenVectMagmaToGenus20** and save all of them in the same folder. To access the data in Pauhlus' program, for example for genus 4, one has to write in Magma [BCP97]

```
load "genvectors.m";
load "searchroutines.m";
L:=ReadData("Fullg20/grpmono04", test);
```

where "test" is a function taking as input a permutation group, a signature (as a vector) and a generating vector (as a vector whose entries are permutations). For example when using the following function, the program gives the list of all triples (G, s, v), where G is a group of order bigger than 7 acting on a Riemann surface of genus 4 with signature s and generating vector v.

```
test:=function(G,s,Lmonod)
 return  Order(G) gt 7;
end function;
```

Thus this program allows to analyse the automorphisms groups of all Riemann surfaces up to genus 20, looking for certain properties specified by the function "test" [1] Observe that G is not necessarily the complete automorphism group of some Riemann surface of the chosen genus (this will be one of the main issues in our program).

5.2 Our program

Given a genus $2 \leq g \leq 20$ our program describes the automorphism group of all the pseudoreal Riemann surfaces of genus g. More precisely it gives the full automorphism group, the conformal automorphism group and its Fuchsian signature. For each entry of the output there exists a pseudoreal Riemann surface of genus g with such properties.

To run the program one needs to download the file **pseudoreal.m**, which is available here

https://www.dropbox.com/s/k786b7a2vrmt22i/pseudoreal.m?dl=0

and save it in the same folder as Pauhlus' programs. The program (again in the case of genus 4) consists of the following lines

```
load "genvectors.m";
load "searchroutines.m";
load "pseudoreal.m";
```

[1] The data was computed using Magma V2.19-9. In newer versions of Magma an error may be returned for some genera (at least 5 and 9). See the warning in J. Paulhus's web page.

```
L:=ReadData("Fullg20/grpmono04", testpr);
PR(L);
```

The output is a list whose entries are of the form $\langle\langle\,,\,\rangle,\langle\,,\,\rangle,[\ldots]\rangle$, where the first bracket contains the ID number of the full automorphism group, the second bracket contains the ID number of the conformal automorphism group and the final sequence is the corresponding Fuchsian signature (the first entry is the genus of the quotient by the conformal automorphism group). In what follows we will describe each of the functions contained in pseudoreal.m.

5.3 The package pseudoreal.m

5.3.1 Basic functions

This function embeds a given group G in a symmetric group S_n.

```
converttoperm:=function(G)
 SL:= Subgroups(G);
 T := {X`subgroup: X in SL};
 TrivCore := {H:H in T| #Core(G,H) eq 1};
 mdeg := Min({Index(G,H):H in TrivCore});
 Good := {H: H in TrivCore| Index(G,H) eq mdeg};
 H := Rep(Good);
 f,P,K := CosetAction(G,H);
 return P;
end function;
```

This function takes a group G and returns the list of all subgroups of G.

```
subg:=function(G)
 S:=Subgroups(G);
```

```
 S1:=[S[i]`subgroup: i in [1..#S]];
 return S1;
end function;
```

This function takes a group G and returns the list of all its non-split group extensions of degree 2.

```
nonsplitext:=function(K)
 A:=SmallGroups(2*Order(K));
 T:=[g: g in A| IdentifyGroup(K) in [IdentifyGroup(p): p in subg(g)]
 and #[p: p in g|Order(p) eq 2] eq #[p: p in K| Order(p) eq 2]];
 return T;
end function;
```

This function takes a signature vector v and returns True or False depending if it is an even signature or not (including the cases without elliptic elements).

```
evensign:=function(v)
a:=[#[i: i in [1..#v]|v[j] eq v[i]]: j in [1..#v]];
 if {IsEven(a[i]): i in [1..#v]} eq {true} or #v eq 0
 then return true;
 else return false;
 end if;
end function;
```

This is the function we enter in J. Paulhus program to select only the even order groups with even signature.

```
testpr:=function(G,s,Lmonod)
return evensign([s[i]: i in [2..#s]])
and IsEven(Order(G));
end function;
```

This function is an intermediate step for the program IsPseudoreal. It separates the cases where we have or do not have elliptic elements in our Fuchsian signature.

```
gencond:=function(v,t,k,s,h)
 if k gt t then return
  &*[v[i]^2: i in [1..t]]*&*[v[j]: j in [t+1..k]] eq Identity(h)
  and [Order(v[i]) : i in [t+1..k]] eq [s[2*i]: i in [1..(#s-1)/2]]
  and {v[i] in h: i in [t+1..k]} eq {true};
 else return &*[v[i]^2: i in [1..t]] eq Identity(h);
 end if;
end function;
```

5.3.2 The function IsPseudoreal

This function takes a group G and a signature s and it returns true if G has even order, s is even and there exists an epimorphism from a NEC group with canonical Fuchsian signature s to a (possible full automorphism) group $\overline{G}$, where $\overline{G}$ is a degree two non-split extension of G. Otherwise it returns false. In case the answer is true, it returns the full automorphism group $\overline{G}$, the conformal automorphism group G and the Fuchsian signature s.

```
IsPseudoreal:=function(G,s)
 s := [s[1]] cat Sort([s[i]: i in [2..#s]]);
  if {IsEven(Order(G))} eq {false} and {evensign([s[i]: i in [2..#s]])}
  eq {false} then return false;
  end if;
 G := converttoperm(G);
 Ext := nonsplitext(G);
 Ext2 := [converttoperm(g): g in Ext];
 L := <>;
  for g in Ext2 do
```

```
 H := [K: K in subg(g)| IdentifyGroup(K) eq IdentifyGroup(G)];
   for h in H do
```

```
   t := s[1]+1;
   k := (#s-1)/2+s[1]+1;
   gen := [v: v in Subsequences(Set(g),Numerator(k))|
   #sub<g|[v[i]: i in [1..#v]]> eq #g
   and {v[i] in h: i in [1..t]} eq {false}
   and gencond(v,t,k,s,h) eq true];
    if #gen gt 0
    then Append(~L,<IdentifyGroup(g), IdentifyGroup(h), s>);
    end if;
   end for;
 end for;
  if #L eq 0 then return <false>;
  else return <true, L>;
  end if;
end function;
```

5.3.3 Lemmas in Baginski-Gromadzki's paper

This section contains functions based on [BG10, Lemma 4.1, Lemma 4.2, Lemma 4.3] and Lemma 3.4.5. The first functions are intermediate steps for the next 4 ones, and they check if a certain map is a group isomorphism, which is a condition of the 3 lemmas in [BG10] and Lemma 3.4.5.

```
auto1:=function(G,v)
S:=sub<G|v>;
f:=IsHomomorphism(S,S,[S.1^-1, S.2^-1]);
return f;
end function;
```

```
auto2:=function(G,v)
S:=sub<G|v>;
f1:=IsHomomorphism(S,S,[S.1^-1, S.2^-1]);
f2:=IsHomomorphism(S,S,[S.1^-1*S.2^-2, S.2]);
return [f1,f2];
end function;
```

```
auto3:=function(G,v)
S:=sub<G|v>;
t:=hom<S->S|[S.2^-2*S.1^-1, S.2^-1*S.1^-2]>;
f:=IsHomomorphism(S,S,[S.2^-2*S.1^-1, S.2^-1*S.1^-2]);
if f eq true and #t(S) eq #(S)
then return true;
else return false;
end if;
end function;
```

```
auto4:=function(G,v)
S:=sub<G|v>;
f:=IsHomomorphism(S,S,[S.1^-1,S.1^2*S.2^-1*S.1^-2,S.3^-1]);
return f;
end function;
```

The following 4 functions take a group G, a subgroup $H \leq G$ and some natural numbers (except in IsFull4) and check the conditions of Baginski-Gromadzki's Lemmas and Lemma 3.4.5.

```
IsFull1:=function(G,H,k,l)
G:=converttoperm(G);
gen:=[[g,h]: g,h in G|sub<G|g,h> eq G and Order(g) eq k
and Order(h^2*g) eq l and h notin H];
```

```
if false in {auto1(G,v): v in gen}
then return true;
else return false;
end if;
end function;
```

```
IsFull2:=function(G,H,k)
 G:=converttoperm(G);
 gen:=[[g,h]: g,h in G|sub<G|g,h> eq G and Order(g) eq k
 and Order(h^2*g) eq k and h notin H];
if [false, false] in {auto2(G,v): v in gen}
then return true;
else return false;
end if;
end function;
```

```
IsFull3:=function(G,H,k)
gen:=[[g,h]: g,h in G|sub<G|g,h> eq G and Order(g^2*h^2) eq k
and g notin H and h notin H];
if false in {auto3(G,v): v in gen}
then return true;
else return false;
end if;
end function;
```

```
IsFull4:=function(G,H)
G:=converttoperm(G);
gen:=[[a,b,c]: a,b,c in G|sub<G|a,b,c> eq G and a notin H and b notin H
and c notin H and a^2*b^2*c^2 eq Identity(G)];
if false in {auto4(G,v): v in gen}
then return true;
```

```
else return false;
end if;
end function;
```

These 4 functions take a group G and some natural numbers (except in BG4), and they apply the previous 4 functions, checking all the possible non-split extensions $G \leq \overline{G}$ of degree two.

```
BG1:=function(G,k,l)
 Ext := nonsplitext(G);
 Ext2 := [converttoperm(g): g in Ext];
  L := [];
  for g in Ext2 do
  H := [K: K in subg(g)| IdentifyGroup(K) eq IdentifyGroup(G)];
   for h in H do
    if IsFull1(g,h,k,l)
    then Append(~L,IdentifyGroup(g));
    end if;
   end for;
  end for;
    return L;
    end function;
```

```
BG2:=function(G,k)
 Ext := nonsplitext(G);
 Ext2 := [converttoperm(g): g in Ext];
  L := [];
  for g in Ext2 do
  H := [K: K in subg(g)| IdentifyGroup(K) eq IdentifyGroup(G)];
   for h in H do
    if IsFull2(g,h,k)
```

```
    then Append(~L,IdentifyGroup(g));
    end if;
   end for;
  end for;
    return L;
    end function;
```

```
BG3:=function(G,k)
 Ext := nonsplitext(G);
 Ext2 := [converttoperm(g): g in Ext];
  L := [];
  for g in Ext2 do
  H := [K: K in subg(g)| IdentifyGroup(K) eq IdentifyGroup(G)];
   for h in H do
    if IsFull3(g,h,k)
    then Append(~L,IdentifyGroup(g));
    end if;
   end for;
  end for;
    return L;
    end function;
```

```
BG4:=function(G)
 Ext := nonsplitext(G);
 Ext2 := [converttoperm(g): g in Ext];
 L := [];
  for g in Ext2 do
  H := [K: K in subg(g)| IdentifyGroup(K) eq IdentifyGroup(G)];
   for h in H do
    if IsFull4(g,h)
```

```
    then Append(~L,IdentifyGroup(g));
    end if;
   end for;
  end for;
 return L;
 end function;
```

5.3.4 The function PR

Finally we analyse the function PR step by step. The function takes as an input a list L (which will be the output of Pauhlus' program) and gives as an output the final result of our program, i.e. the list of all the triples describing the automorphism group of the pseudoreal Riemann surfaces of a given genus.

These sentences transform the list L in a list which contains just the pairs of groups and signatures given in L defined previously.

```
gps:=[L[i][1]: i in [1..#L]];
sign:=[L[i][2]: i in [1..#L]];
gpsn:=[[IdentifyGroup(gps[i])[j]: j in [1,2]]: i in [1..#gps]];
list:={[gpsn[i],sign[i]]: i in [1..#L]};
ll:=SetToSequence(list);
```

These sentences separate the maximal and non-maximal Fuchsian signatures.

```
X1:=[i: i in [1..#ll]| #(ll[i][2]) eq 5 and ll[i][2][1] eq 0
and &+[ll[i][2][k]: k in [2,4]] ge 5 and ll[i][2][2] ne ll[i][2][4]];
X2:=[i: i in [1..#ll]| #(ll[i][2]) eq 5 and ll[i][2][1] eq 0
and &+[ll[i][2][k]: k in [2,4]] ge 5 and ll[i][2][2] eq ll[i][2][4]];
X3:=[i: i in [1..#ll]| #(ll[i][2]) eq 3 and ll[i][2][1] eq 1
and ll[i][2][2] eq ll[i][2][3]];
Y:=[i: i in [1..#ll]| #(ll[i][2]) eq 1 and ll[i][2][1] eq 2];
```

```
Z:=[i: i in [1..#ll]| i notin X1 and i notin X2 and i notin X3
and i notin Y];
```

The following programs check the conditions for non maximal signatures given in the three lemmas of [BG10] and Lemma 3.4.5. The last program uses the function IsPseudoreal for maximal signatures.

```
Pr1:=[];
 for i in X1 do
 k:=ll[i][2][2];
 l:=ll[i][2][4];
 G:=SmallGroup(ll[i][1][1],ll[i][1][2]);
 F:=BG1(G,k,l);
 for f in F do
 Append(~Pr1,<f,IdentifyGroup(G),[0,k,k,l,l]>);
 end for;
end for;
```

```
Pr2:=[];
 for i in X2 do
 k:=ll[i][2][2];
 G:=SmallGroup(ll[i][1][1],ll[i][1][2]);
 F:=BG2(G,k);
 for f in F do
 Append(~Pr2,<f,IdentifyGroup(G),[0,k,k,k,k]>);
 end for;
end for;
```

```
Pr3:=[];
 for i in X3 do
 k:=ll[i][2][2];
```

```
 G:=SmallGroup(ll[i][1][1],ll[i][1][2]);
 F:=BG3(G,k);
 for f in F do
 Append(~Pr3,<f,IdentifyGroup(G),[1,k,k]>);
 end for;
end for;
```

```
Pr4:=[];
 for i in Y do
 G:=SmallGroup(ll[i][1][1],ll[i][1][2]);
 F:=BG4(G);
 for f in F do
 Append(~Pr4,<f,IdentifyGroup(G),[2]>);
 end for;
end for;
```

```
Prmax:=[];
 for i in Z do
  q:= IsPseudoreal(SmallGroup(ll[i][1][1],ll[i][1][2]),ll[i][2]);
  if q[1] eq true
  then for j in [1..(#q[2])] do
Append(~Prmax, q[2][j]);
  end for;
  end if;
 end for;
```

These last sentences give us a list of 3-uples, whose first entry is a full automorphism group for a pseudoreal Riemann surface in that genus, the second entry is the corresponding conformal automorphism group, and the third entry is the Fuchsian signature of the conformal action. The list is exhaustive.

```
PrCasi1:= Pr1 cat Pr2 cat Pr3 cat Pr4 cat Prmax;
Pr:={a: a in PrCasi1};
return Pr;
```

Appendix A: List of groups

We present the groups used in this thesis and their ID number when pertinent, which is a pair of numbers whose first entry is the order of the group, and the second entry is the position the group has in the Magma database.

Table 5.1: Groups used in this thesis

Group	Order	Presentation/Name	ID number
C_n	n	$\langle a : a^n = 1\rangle$	–
D_n	$2n$	$\langle r, s : r^n = s^2 = 1, srs^{-1} = r^{-1}\rangle$	–
S_n	$n!$	Permutations group of n symbols	–
$C_m \times C_n$	mn	$\langle a, b : a^m = b^n = 1, ab = ba\rangle$	–
$F20$	20	$\langle a, b : a^5 = b^4 = 1, bab^{-1} = a^2\rangle$	$\mathrm{ID}(20, 3)$
Q_8	8	$\langle i, j, k : i^2 = j^2 = k^2 = ijk = -1\rangle$	$\mathrm{ID}(8, 4)$
QD_8	16	$\langle a, x : a^8 = x^2 = 1, xax^{-1} = a^3\rangle$	$\mathrm{ID}(16, 8)$
Dic_{4n}	$4n$	$\langle a, x : a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1}\rangle$	–
$C_i \times C_j \times C_k$	ijk	$\langle a, b, c : a^i = b^j = c^k = 1, ab = ba, bc = cb, ac = ca\rangle$	–
$C_8 \rtimes_\phi C_2$	16	$\langle a, x : a^8 = x^2 = 1, xax^{-1} = a^5\rangle$	$\mathrm{ID}(16, 6)$
U_6	24	$\langle x, y : \ x^2, y^6, xyxy^4\rangle$	$\mathrm{ID}(24, 5)$
V_8	32	$\langle x, y : \ x^4, y^4, (xy)^2, (x^{-1}y)^2\rangle$	$\mathrm{ID}(32, 9)$

Appendix B: Classification tables

Table 5.2: Automorphism groups of pseudoreal Riemann surfaces of genus 5

Genus 5				
$\mathrm{Aut}^+(\mathrm{X})$	Fuchsian signature	$\mathrm{Aut}^{\pm}(\mathrm{X})$	NEC signature	Generating Vector
C_2	$(3; [-])$	C_4	$(4; -; [-])$	$(a, a, a, a; [-])$
C_2	$(1; [2^8])$	C_4	$(2; -; [2^4])$	$(a, a; a^2, a^2, a^2, a^2)$
C_4	$(1; [2^4])$	C_8	$(2; -; [2^2])$	$(a, a^3; a^4, a^4)$
C_4	$(1; [2^4])$	Q_8	$(2; -; [2^2])$	$(j, k; -1, -1)$
C_4	$(0; [2^2, 4^4])$	Q_8	$(1; -; [2, 4^2])$	$(j; -1, i, -i)$
$C_2 \times C_2$	$(0; [2^8])$	$C_4 \times C_2$	$(1; -; [2^4])$	$(a; b, b, b, a^2b)$
$C_2 \times C_2$	$(1; [2^4])$	$C_4 \times C_2$	$(2; -; [2^2])$	$(a, a; b, b)$
C_6	$(1; [3^2])$	C_{12}	$(2; -; [3])$	$(a, a; a^8)$
D_4	$(0; [2^6])$	QD_8	$(1; -; [2^3])$	$(xa; a^4, a^4, a^4)$
$C_2 \times C_2 \times C_2$	$(0; [2^6])$	$C_4 \times C_2 \times C_2$	$(1; -; [2^3])$	$(a; b, c, a^2bc)$
	$-$	$C_4 \times C_2 \rtimes_{\phi} C_2$	$(1; -; [2^3])$	$(a; b, c, a^2bc)$

Table 5.3: Automorphism groups of pseudoreal Riemann surfaces of genus 6

Genus 6				
$\mathrm{Aut}^+(\mathrm{X})$	Fuchsian signature	$\mathrm{Aut}^\pm(\mathrm{X})$	NEC signature	Generating Vector
C_2	$(0;[2^{14}])$	C_4	$(1;-;[2^7])$	$(a;a^2,a^2,a^2,a^2,a^2,a^2,a^2)$
C_2	$(2;[2^6])$	C_4	$(3;-;[2^3])$	$(a,a,a;a^2,a^2,a^2)$
C_4	$(0;[2^6,4^2])$	C_8	$(1;-;[2^3,4])$	$(a;a^4,a^4,a^4,a^2)$
C_4	$(0;[4^6])$	C_8	$(1;-;[4^3])$	$(a;a^2,a^2,a^2)$
C_6	$(0;[2^4,6^2])$	C_{12}	$(1;-;[2^2,6])$	$(a;a^6,a^6,a^{10})$
C_6	$(0;[2^2,3^4])$	C_{12}	$(1;-;[2,3^2])$	$(a^5;a^6,a^4,a^4)$
	$-$	Dic_{12}	$(1;-;[2,3^2])$	$(x;a^3,a^2,a^4)$
D_5	$(0;[2^6])$	$F20$	$(1;-;[2^3])$	$(b;ab^2,ab^2,b^2)$

Table 5.4: Automorphism groups of pseudoreal Riemann surfaces of genus 7

Genus 7				
$\mathrm{Aut}^+(\mathrm{X})$	Fuchsian signature	$\mathrm{Aut}^\pm(\mathrm{X})$	NEC signature	Generating Vector
C_2	$(1;[2^{12}])$	C_4	$(2;-;[2^6])$	$(a,a;a^2,a^2,a^2,a^2,a^2,a^2)$
C_2	$(3;[2^4])$	C_4	$(4;-;[2^2])$	$(a,a,a,a;a^2,a^2)$
C_4	$(0;[2^4,4^4])$	Q_8	$(1;-;[2^2,4^2])$	$(j;-1,-1,i,i)$
C_4	$(1;[2^6])$	C_8	$(2;-;[2^3])$	$(a,a;a^4,a^4,a^4)$
C_4	$(1;[4^4])$	C_8	$(2;-;[4^2])$	$(a,a;a^2,a^2)$
	$-$	Q_8	$(2;-;[4^2])$	$(j,j;i,-i)$
C_4	$(2;[2^2])$	Q_8	$(3;-;[2])$	$(j,j,j;-1)$
$C_2\times C_2$	$(0;[2^{10}])$	$C_4\times C_2$	$(1;-;[2^5])$	$(a;b,b,a^2,a^2,a^2)$
$C_2\times C_2$	$(1;[2^6])$	$C_4\times C_2$	$(2;-;[2^3])$	$(a,a;a^2,a^2b,b)$
$C_2\times C_2$	$(2;[2^2])$	$C_4\times C_2$	$(3;-;[2])$	$(a,a,ab;a^2)$
C_6	$(1;[2^4])$	C_{12}	$(2;-;[2^2])$	$(a^3,a^3;a^6,a^6)$
	$(1;[2^4])$	Dic_{12}	$(2;-;[2^2])$	$(ax,x;a^3,a^3)$
$C_4\times C_2$	$(0;[2^4,4^2])$	$C_8\times C_2$	$(1;-;[2^2,4])$	$(a;b,b,a^6)$
	$-$	$C_8\rtimes_\phi C_2$	$(1;-;[2^2,4])$	$(ax;a^4x,a^4x,a^2)$
D_4	$(0;[2^4,4^2])$	QD_8	$(1;-;[2^2,4])$	$(a;x,x,a^6)$
D_6	$(0;[2^6])$	$C_3\times S_3$	$(1;-;[2^3])$	$-$

Table 5.5: Automorphism groups of pseudoreal Riemann surfaces of genus 8

Genus 8				
$\mathrm{Aut}^+(\mathrm{X})$	Fuchsian signature	$\mathrm{Aut}^{\pm}(\mathrm{X})$	NEC signature	Generating Vector
C_2	$(0; [2^{18}])$	C_4	$(1; -; [2^9])$	$(a; a^2, a^2, a^2, a^2, a^2, a^2, a^2, a^2, a^2)$
C_2	$(2; [2^{10}])$	C_4	$(3; -; [2^5])$	$(a, a, a; a^2, a^2, a^2, a^2, a^2)$
C_2	$(4; [2^2])$	C_4	$(5; -; [2])$	$(a, a, a, a, a; a^2)$
C_4	$(0; [2^8, 4^2])$	C_8	$(1; -; [2^4, 4])$	$(a^3; a^4, a^4, a^4, a^4, a^2)$
C_4	$(0; [2^2, 4^6])$	C_8	$(1; -; [2, 4^3])$	$(a^3; a^4, a^2, a^2, a^2)$
C_4	$(2; [4^2])$	C_8	$(3; -; [4])$	$(a, a, a; a^2)$
C_6	$(0; [3^4, 6^2])$	C_{12}	$(1; -; [3^2, 6])$	$(a; a^4, a^4, a^2)$
	$-$	Dic_{12}	$(1; -; [3^2, 6])$	$(x; a^2, a^2, a^5)$
C_6	$(0; [2^6, 3^2])$	C_{12}	$(1; -; [2^3, 3])$	$(a; a^6, a^6, a^6, a^4)$
C_6	$(0; [2^2, 6^4])$	C_{12}	$(1; -; [2, 6^2])$	$(a; a^6, a^2, a^2)$
	$-$	Dic_{12}	$(1; -; [2, 6^2])$	$(x; a^3, a, a^5)$
C_8	$(0; [2^4, 8^2])$	C_{16}	$(1; -; [2^2, 8])$	$(a; a^8, a^8, a^{14})$

Table 5.6: Automorphism groups of pseudoreal Riemann surfaces of genus 9

Genus 9			
$\mathrm{Aut}^+(\mathrm{X})$	Fuchsian signature	$\mathrm{Aut}^\pm(\mathrm{X})$	NEC signature
C_2	$(1;[2^{16}])$	C_4	$(2;-;[2^8])$
C_2	$(3;[2^8])$	C_4	$(4;-;[2^4])$
C_2	$(5;[-])$	C_4	$(6;-;[-])$
C_4	$(0;[4^8])$	Q_8	$(1;-;[4^4])$
C_4	$(0;[2^6,4^4])$	Q_8	$(1;-;[2^3,4^2])$
C_4	$(1;[2^8])$	C_8	$(2;-;[2^4])$
	$-$	Q_8	$(2;-;[2^4])$
C_4	$(1;[2^2,4^4])$	C_8	$(2;-;[2,4^2])$
	$-$	Q_8	$(2;-;[2,4^2])$
C_4	$(3;[-])$	C_8	$(4;-;[-])$
	$-$	Q_8	$(4;-;[-])$
$C_2\times C_2$	$(0;[2^{12}])$	$C_4\times C_2$	$(1;-;[2^6])$
$C_2\times C_2$	$(1;[2^8])$	$C_4\times C_2$	$(2;-;[2^4])$
$C_2\times C_2$	$(2;[2^4])$	$C_4\times C_2$	$(3;-;[2^2])$
$C_2\times C_2$	$(3;[-])$	$C_4\times C_2$	$(4;-;[-])$
C_6	$(1;[2^2,6^2])$	C_{12}	$(1;-;[2^6])$
C_6	$(1;[3^4])$	Dic_{12}	$(2;-;[3^2])$
	$-$	C_{12}	$(2;-;[3^2])$
D_4	$(0;[2^8])$	QD_8	$(1;-;[2^4])$
C_2^3	$(0;[2^8])$	$\mathrm{ID}(16,3)$	$(1;-;[2^4])$
	$-$	$C_4\times C_2^2$	$(1;-;[2^4])$
$C_4\times C_2$	$(0;[2^2,4^4])$	$C_4\times C_4$	$(1;-;[2,4^2])$
	$-$	$\mathrm{ID}(16,4)$	$(1;-;[2,4^2])$
	$-$	$C_2\times Q_8$	$(1;-;[2,4^2])$
Q_8	$(0;[2^2,4^4])$	Q_{16}	$(1;-;[2,4^2])$
C_8	$(1;[2^4])$	C_{16}	$(2;-;[2^2])$
	$-$	Q_{16}	$(2;-;[2^2])$

Genus 9 (continuation)			
Aut$^+$(X)	Fuchsian signature	Aut$^\pm$(X)	NEC signature
$C_4 \times C_2$	$(1; [2^4])$	$C_4 \times C_4$	$(2; -; [2^2])$
	$-$	ID$(16, 4)$	$(2; -; [2^2])$
	$-$	$C_8 \times C_2$	$(2; -; [2^2])$
	$-$	ID$(16, 6)$	$(2; -; [2^2])$
	$-$	$C_2 \times Q_8$	$(2; -; [2^2])$
D_4	$(1; [2^4])$	QD_8	$(2; -; [2^2])$
C_2^3	$(1; [2^4])$	ID$(16, 3)$	$(2; -; [2^2])$
	$-$	$C_4 \times C_2^2$	$(2; -; [2^2])$
$C_4 \times C_2$	$(2; [-])$	ID$(16, 6)$	$(3; -; [-])$
D_4	$(2; [-])$	QD_8	$(3; -; [-])$
D_5	$(1; [5^2])$	$F20$	$(2; -; [5])$
C_{10}	$(1; [5^2])$	C_{20}	$(2; -; [5])$
D_6	$(0; [2^4, 3^2])$	$C_4 \times S_3$	$(2; -; [2^2, 3])$
$C_6 \times C_2$	$(0; [2^4, 3^2])$	$C_{12} \times C_2$	$(2; -; [2^2, 3])$
C_{12}	$(1; [3^2])$	C_{24}	$(2; -; [3])$
	$-$	$C_3 \times Q_8$	$(2; -; [3])$
$C_6 \times C_2$	$(1; [3^2])$	$C_{12} \times C_2$	$(2; -; [3])$
D_8	$(0; [2^6])$	ID$(32, 19)$	$(1; -; [2^3])$
$C_2 \times D_4$	$(0; [2^6])$	ID$(32, 6)$	$(1; -; [2^3])$
	$-$	ID$(32, 7)$	$(1; -; [2^3])$
	$-$	ID$(32, 9)$	$(1; -; [2^3])$
	$-$	$C_2 \times QD_8$	$(1; -; [2^3])$
ID$(16, 13)$	$(0; [2^6])$	ID$(32, 11)$	$(1; -; [2^3])$
	$-$	ID$(32, 38)$	$(1; -; [2^3])$
C_2^4	$(0; [2^6])$	ID$(32, 22)$	$(1; -; [2^3])$
ID$(16, 6)$	$(1; [2^2])$	ID$(32, 15)$	$(2; -; [2])$
D_{10}	$(0; [2^2, 10^2])$	ID$(40, 12)$	$(1; -; [2, 10])$

Table 5.7: Automorphism groups of pseudoreal Riemann surfaces of genus 10

Genus 10			
$\mathrm{Aut}^+(X)$	Fuchsian signature	$\mathrm{Aut}^\pm(X)$	NEC signature
C_2	$(0; [2^{22}])$	C_4	$(1; -; [2^{11}])$
C_2	$(2; [2^{14}])$	C_4	$(3; -; [2^7])$
C_2	$(4; [2^6])$	C_4	$(5; -; [2^3])$
C_4	$(0; [2^{10}, 4^2])$	C_8	$(1; -; [2^5, 4])$
C_4	$(0; [2^4, 4^6])$	C_8	$(1; -; [2^2, 4^3])$
C_4	$(2; [2^2, 4^2])$	C_8	$(3; -; [2, 4])$
C_6	$(0; [2^4, 3^2, 6^2])$	Dic_{12}	$(1; -; [2^2, 3, 6])$
	$-$	C_{12}	$(1; -; [2^2, 3, 6])$
C_6	$(0; [2^2, 3^6])$	Dic_{12}	$(1; -; [2, 3^3])$
	$-$	C_{12}	$(1; -; [2, 3^3])$
C_6	$(0; [6^6])$	Dic_{12}	$(1; -; [6^3])$
	$-$	C_{12}	$(1; -; [6^3])$
C_6	$(2; [2^2])$	Dic_{12}	$(3; -; [2])$
	$-$	C_{12}	$(3; -; [2])$
C_8	$(0; [2^2, 4^2, 8^2])$	C_{16}	$(1; -; [2, 4, 8])$
C_{10}	$(0; [2^4, 10^2])$	C_{20}	$(1; -; [2^2, 10])$
$\mathrm{ID}(18, 4)$	$(0; [2^6])$	$\mathrm{ID}(36, 9)$	$(1; -; [2^3])$
$\mathrm{ID}(36, 9)$	$(0; [2^2, 4^2])$	$\mathrm{ID}(72, 39)$	$(1; -; [2, 4])$

Appendix C: Pseudoreal generalized superelliptic curves

Table 5.8: Pseudoreal generalized superelliptic curves of genus 3

$\mathrm{Aut}^+(X)$	Fuchsian signature	$\mathrm{Aut}^\pm(X)$	NEC signature	Generating Vector	$\|\tau\|$
$C_2 \times C_2$	$(0; [2^6])$	$C_4 \times C_2$	$(1; -; [2^3])$	$(a; b, b, a^2)$	2

Table 5.9: Pseudoreal generalized superelliptic curves of genus 4

$\mathrm{Aut}^+(X)$	Fuchsian signature	$\mathrm{Aut}^\pm(X)$	NEC signature	Generating Vector	$\|\tau\|$
C_2	$(0; [2^{10}])$	C_4	$(1; -; [2^5])$	$(a; a^2, a^2, a^2, a^2, a^2)$	2
C_4	$(0; [2^4, 4^2])$	C_8	$(1; -; [2^2, 4])$	$(a^3; a^4, a^4, a^2)$	2

Table 5.10: Pseudoreal generalized superelliptic curves of genus 5

$\mathrm{Aut}^+(X)$	Fuchsian signature	$\mathrm{Aut}^\pm(X)$	NEC signature	Generating Vector	$\|\tau\|$
$C_2 \times C_2$	$(0; [2^8])$	$C_4 \times C_2$	$(1; -; [2^4])$	$(a; b, b, b, a^2 b)$	2

Table 5.11: Possible pseudoreal generalized superelliptic curves of genus 6

$\mathrm{Aut}^+(\mathrm{X})$	Fuchsian signature	$\mathrm{Aut}^\pm(\mathrm{X})$	NEC signature	Generating Vector	$\lvert\tau\rvert$
C_2	(2^{14})	C_4	$(1; -; [2^7])$	$(a; a^2, a^2, a^2, a^2, a^2, a^2, a^2)$	2
C_4	(4^6)	C_8	$(1; -; [4^3])$	$(a; a^2, a^2, a^2)$	4
C_4	$(2^6, 4^2)$	C_8	$(1; -; [2^3, 4])$	$(a; a^4, a^4, a^4, a^2)$	2
C_6	$(2^4, 6^2)$	C_{12}	$(1; -; [2^2, 6])$	$(a; a^6, a^6, a^{10})$	2

Table 5.12: Pseudoreal generalized superelliptic curves of genus 7

$\mathrm{Aut}^+(\mathrm{X})$	Fuchsian signature	$\mathrm{Aut}^\pm(\mathrm{X})$	NEC signature	Generating Vector	$\lvert\tau\rvert$
$C_4 \times C_2$	$(2^4, 4^2)$	$C_8 \times C_2$	$(1; -; [2^2, 4])$	$(a; b, b, a^6)$	2

Table 5.13: Pseudoreal generalized superelliptic curves of genus 8

$\mathrm{Aut}^+(\mathrm{X})$	Fuchsian signature	$\mathrm{Aut}^\pm(\mathrm{X})$	NEC signature	Generating Vector	$\lvert\tau\rvert$
C_2	(2^{18})	C_4	$(1; -; [2^9])$	$(a; a^2, ..^{(10)}.., a^2)$	2
C_4	$(2^8, 4^2)$	C_8	$(1; -; [2^4, 4])$	$(a^3; a^4, a^4, a^4, a^4, a^2)$	2
C_8	$(2^4, 8^2)$	C_{16}	$(1; -; [2^2, 8])$	$(a; a^8, a^8, a^{14})$	2

Table 5.14: Possible pseudoreal generalized superelliptic curves of genus 9

$\mathrm{Aut}^+(\mathrm{X})$	Fuchsian signature	$\mathrm{Aut}^\pm(\mathrm{X})$	NEC signature	$\lvert\tau\rvert$
C_4	(4^8)	Q_8	$(1; -; [4^4])$	4
$C_2 \times C_2$	(2^{12})	C_8	$(1; -; [2^6])$	2
$C_4 \times C_2$	$(2^2, 4^4)$	$C_4 \times C_4$	$(1; -; [2^2, 8])$	4
	–	ID(16, 4)	$(1; -; [2^2, 8])$	4
	–	$C_2 \times Q_8$	$(1; -; [2^2, 8])$	4

Table 5.15: Possible pseudoreal generalized superelliptic curves of genus 10

$\mathrm{Aut}^+(\mathrm{X})$	Fuchsian signature	$\mathrm{Aut}^{\pm}(\mathrm{X})$	NEC signature	$\|\tau\|$
C_2	(2^{22})	C_4	$(1; -; [2^{11}])$	2
C_4	$(2^{10}, 4^2)$	C_8	$(1; -; [2^5])$	2
C_6	$(2^2, 3^6)$	Dic_{12}	$(1; -; [2, 3^3])$	3
	–	C_{12}	$(1; -; [2, 3^3])$	3
C_6	(6^6)	Dic_{12}	$(1; -; [6^3])$	6
	–	C_{12}	$(1; -; [6^3])$	6
C_{10}	$(2^4, 10^2)$	C_{20}	$(1; -; [2^2, 10])$	2

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