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Sobre superficies de Büchi generalizadas.

On generalized Büchi surfaces.

Tesis para optar al grado de Magíster en Matemática

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Introducción

En esta tesis estudiamos una clase especial de superficies algebraicas proyectivas complejas, llamadas *Superficies Generalizadas de Büchi*, abreviadas GBS. Dado un entero $n \geq 3$ y un conjunto de puntos distintos ordenados $\alpha := \{[1 : \alpha_0], \ldots, [1 : \alpha_n]\} \subseteq \mathbb{P}^1$, la *Superficie Generalizada de Büchi* $S_n(\alpha)$ es la intersección completa de las $n-2$ cuadricas diagonales del espacio proyectivo complejo $\mathbb{P}^n$:

$$x_i^2 - \beta_2^i x_2^2 - \beta_1^i x_1^2 - \beta_0^i x_0^2 = 0, \tag{0.1}$$

donde $3 \leq i \leq n$ y

$$\beta_0^i = \frac{(\alpha_1 - \alpha_i)(\alpha_2 - \alpha_i)}{(\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)},\ \beta_1^i = -\frac{(\alpha_0 - \alpha_i)(\alpha_2 - \alpha_i)}{(\alpha_0 - \alpha_1)(\alpha_1 - \alpha_2)},\ \beta_2^i = \frac{(\alpha_0 - \alpha_i)(\alpha_1 - \alpha_i)}{(\alpha_0 - \alpha_2)(\alpha_1 - \alpha_2)}.$$

La motivación primitiva de este trabajo es el *Problema de Büchi*, un problema de naturaleza aritmetica el cual su formulación geométrica es el punto de partida para esta tesis. A continuación describimos el problema de Büchi.

Dado $n \geq 3$ entero, una sucesión de números enteros $\{x_i\}_{i=1}^n$ es una *sucesión de Büchi* si cumple la relación:

$$(x_i^2 - x_{i-1}^2) - (x_{i-1}^2 - x_{i-2}^2) = 2 \qquad \text{para } 3 \leq i \leq n. \tag{0.2}$$

Para cualquier entero x, la sucesión de enteros consecutivos $\{x_i = x+i\}_{i=1}^n$ es una sucesión de Büchi. Este tipo de sucesiones son llamadas las soluciones triviales. También se pueden obtener nuevas sucesiones de Büchi a partir de una sucesión de Büchi dada por cambios de signo de cualquiera de sus elementos u ordenando estos de forma inversa.

El problema de Büchi en su formulación aritmetica, pregunta si existe un entero $n > 0$ tal que todas las soluciones enteras al sistema (0.2) sean triviales. Para $n \geq 3$ el sistema (0.2) define las superficies proyectivas X_n[1] en $\mathbb{P}^n$, definida por las $n-2$ cuadricas:

$$x_i^2 - 2x_{i-1}^2 + x_{i-2}^2 - 2x_0^2 = 0. \qquad \text{con } 3 \leq i \leq n.$$

El problema de Büchi en su formulación geométrica, pregunta si existe un entero $n > 0$ tal que todos los puntos racionales de X_n de la forma $[1 : \alpha_1 : \cdots : \alpha_n]$, con

[1]Note que X_n se corresponde con $S_n(\alpha)$, donde $\alpha = [\infty, \alpha_1, \ldots, \alpha_n]$ con α_i enteros consecutivos. Ver Remark 3.1.3.

$\alpha_i \in \mathbb{Z}$, pertenecen a las rectas $x_i = \pm(x+i)$, con $i \in \{1, \dots, n\}$ y $x \in \mathbb{Z}$. Estas rectas se llaman las *rectas triviales* de X_n. Debido a la fomulación geometrica del problema, se estudia el conjunto $X_n(\mathbb{Q})$ de puntos racionales de X_n. La importancia de estudiar los puntos racionales es el hecho que si el problema de Büchi tiene una respuesta positiva en un anillo, esto también es cierto en cualquier subanillo (ver [18, 5]). Luego la pregunta natural (considerando la conjetura de Bombieri [18, 9]) es si $X_n(\mathbb{Q})$ está contenido en las rectas triviales para algún $n > 0$.

Se sabe que para $n \leq 4$ existen soluciones enteras no triviales para (0.2). Para una descripcion explicita de sucesiones de Büchi de largo 3 y 4 se puede consultar [20] y [6] respectivamente. Por otro lado, puesto que la superficie X_3 es una cuadrica con un punto racional y X_4 es una superficie de del Pezzo, ambas racionales, los conjuntos $X_3(\mathbb{Q})$ y $X_4(\mathbb{Q})$ son Zariski-densos en X_3 y X_4 respectivamente, esto tambien lo cumple la superficie $K3$ X_5 pero hace uso del hecho que X_5 admite una fibración eliptica definida sobre $\mathbb{Q}$ con grupo de secciones infinito [2, Prop. 5.3]. Para $n \geq 8$, un teorema de Vojta (ver [23] y [18, §9]) implica, asumiendo cierta la conjetura de Bombieri, que el problema de Büchi tiene una respuesta positiva. Para un survey del problema de Büchi el lector puede recurrir a [18].

En el paper *On Büchi's K3 surface* [2] los autores estudian la geometría de la superficie $K3$ de Büchi X_5, la cual se corresponde con $S_5(\alpha)$, donde $\alpha = [\infty, -2, -1, 0, 1, 2]$ de las GBS definidas en (0.1), esto se hace por medio de la clasica correspondencia entre curvas hiperelípticas de genero 2 y superficies de Kummer [11, §10.3.3]. Especificamente, reconstruyen la superficie $S_5(\alpha)$ a partir de la curva hiperelíptica de genero 2 con ecuación $y^2 = (x+2)(x+1)x(x-1)(x-2)$. Por otro lado, recuperar la curva a partir de la superficie $S_5(\alpha)$ es sencillo y solo se necesita conocer las rectas triviales de la superficie, a saber, cada recta trivial L de $S_5(\alpha)$ corta 6 otras rectas triviales, el cubrimiento doble de L ramificado a lo largo de dichos puntos de intersección es la curva inicial.

Es esta última idea la que se logra generalizar en este trabajo, la cual se desarrolla en el Capítulo 3. Dada una GBS $S_n(\alpha)$, mediante el conocimiento de sus rectas triviales se pueden recuperar, a menos de cambios de coordenadas, los $n+1$ numeros complejos no ordenados que forman α. Viceversa, dados los $n+1$ numeros complejos que forman α podemos obtener una $S_n(\alpha)$. Cada GBS $S_n(\alpha)$ contiene las 2^n *rectas triviales* parametrizadas por $t \mapsto [\pm(t-\alpha_0) : \cdots : \pm(t-\alpha_n)]$. Como mostrado en la Proposición 3.2.3 estas son las únicas rectas de la superficie y la determinan completamente (ver Proposiciónes 3.1.5 y 3.2.2). Cada recta corta exactamente $n+1$ otras rectas a lo largo de un subconjunto de puntos proyectivamente equivalentes a α (ver Proposición 3.2.5). En el caso de ser n impar el grafo de intersección de las rectas es bipartito de tipo $(2^{n-1}, 2^{n-1})$ (ver Proposición 3.2.6). Si $n = 2g+1$, con $g \geq 1$, cada GBS lleva asociada una curva hipereliptica de genero g (ver Observación 3.3.1). El resultado es sutilmente mas profundo, pues en realidad se prueba que salvo proyectividades, existe una biyección entre $n+1$ puntos no ordenados de $\mathbb{P}^1$ y el conjunto de las GBS (ver Teorema 3.2.1).

Por otro lado, en el Capítulo 4 estudiamos el caso $n = 2g + 1$ impar. Acá relacionamos la superficie $S := S_{2g+1}(\alpha)$ con una curva hipereliptica C de genero g de la siguiente manera. Construimos la superficie Y que se obtiene al cuocientar S por el subgrupo G_0 del grupo de automorfismos de S de los cambios de signo de una cantidad par de coordenadas. El morfismo cuociente $S \to Y$ está definido sobre $\mathbb{Q}$, así que $S(\mathbb{Q})$ se mapea en $Y(\mathbb{Q})$. Por otro lado, consideramos la segunda potencia simetrica $C^{(2)}$ de C y su involución hiperéliptica $\imath$ y mostramos que las superficies $C^{(2)}/\langle\imath\rangle$ e Y son isomorfas (ver Teorema 4.2.1 y Corolario 4.2.2) sobre el campo $\mathbb{Q}(\alpha_0, \alpha_1, \alpha_2)$ que es $\mathbb{Q}$ para las superficies de Büchi. En este sentido seria ideal sustituir el estudio de los puntos racionales de S con propiedades aritmeticas de la curva C (ver Sección 4.3). Entre las propiedades estudiadas de Y, destacamos que Y es una superficie de tipo general si $g \geq 3$ (ver Proposición 4.1.1), no birracional a S (ver Proposición 4.1.2) y es la cubierta doble de $\mathbb{P}^2$ ramificada a lo largo de la unión de $2g + 2$ rectas tangentes a una cónica Γ (4.2), donde los puntos de tangencia son las coordenadas del vector α (ver Proposición 4.1.3). En la subsección 4.2.1 vemos algunos hechos básicos del grupo de Neron-Severi de $C^{(2)}$, especificamente calculamos la matriz de intersección de las curvas C_p, Δ y E (ver Proposición 4.2.4), con el proposito de describir a futuro algunas propiedades geométricas de la superficie Y.

Esta tesis esta organizada de la siguiente manera. En el capítulo 1 se definen las variedades algebraicas complejas afines y projectivas, y sus hechos básicos, tales como, dimensión, suavidad, variedades normales, divisores y los morfismos entre estas variedades. Se particularizan estos hechos en curvas y superficies, se destaca en curvas, el Teorema de Riemann-Roch, formula de Riemann-Hurwitz, producto simetrico. En el caso de superficies, destacamos el grupo de Neron-Severi, la formula del genero y clasificación de Kodaira, entre otros hechos relaciónados que no mencionamos aquí. El capítulo 2 estudiamos teoría basica de Galois y curvas hiperelípticas. En la sección de Galois principalmente se definen las extensiones de campo (finita, normal, separable), grupo de Galois y Galois absoluto. En la sección de curvas hiperelípticas, definimos el espacio proyectivo pesado complejo, las curvas hiperelíticas de genero g, involución hiperelíptica y algunos hechos sobre puntos racionales sobre curvas. En general los capítulos 1 y 2 son los preliminares necesarios para el desarrollo de este trabajo. Los capítulos 3 y 4 son de resultados, principalmente lo descrito en los últimos dos parrafos. Finalmente hay un apéndice de algunos programas Magma utilizados en calculos necesarios.

Introduction

In this thesis we study a class of special complex projective algebraic surfaces, called *Generalized Büchi Surfaces*, GBS for short. Given an integer $n \geq 3$ and a set of distinct ordered points $\alpha := \{[1 : \alpha_0], \dots, [1 : \alpha_n]\} \subseteq \mathbb{P}^1$, the *Generalized Büchi Surface* $S_n(\alpha)$ is the complete intersection of the $n-2$ diagonal quadrics of the complex projective space $\mathbb{P}^n$:

$$x_i^2 - \beta_2^i x_2^2 - \beta_1^i x_1^2 - \beta_0^i x_0^2 = 0, \tag{0.1}$$

where $3 \leq i \leq n$ and

$$\beta_0^i = \frac{(\alpha_1 - \alpha_i)(\alpha_2 - \alpha_i)}{(\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)},\ \beta_1^i = -\frac{(\alpha_0 - \alpha_i)(\alpha_2 - \alpha_i)}{(\alpha_0 - \alpha_1)(\alpha_1 - \alpha_2)},\ \beta_2^i = \frac{(\alpha_0 - \alpha_i)(\alpha_1 - \alpha_i)}{(\alpha_0 - \alpha_2)(\alpha_1 - \alpha_2)}.$$

The primitive motivation for this work is the *Büchi's problem*, a problem of arithmetic nature whose geometric formulation is the starting point for this thesis. In the next paragraph we describe the Büchi's problem.

Let $n \geq 3$ be an integer, a sequence of integer numbers $\{x_i\}_{i=1}^n$ is a *Büchi's sequence* if it is satisfies the relation:

$$(x_i^2 - x_{i-1}^2) - (x_{i-1}^2 - x_{i-2}^2) = 2 \qquad \text{with } 3 \leq i \leq n. \tag{0.2}$$

For any integer x, the sequence of consecutive integers $\{x_i = x+i\}_{i=1}^n$ is a Büchi's sequence. These types of sequences are called the trivial solutions. New Büchi's sequences can also be obtained from a given Büchi sequence by changes of signs of any of its elements or reversing the order.

The Büchi's problem in its arithmetic formulation, asks if there is an integer $n > 0$ such that all integer solutions to the system (0.2) are trivial. For $n \geq 3$ the system (0.2) defines the projective surface X_n[1] on $\mathbb{P}^n$, which is complete intersection of the $n-2$ quadrics:

$$x_i^2 - 2x_{i-1}^2 + x_{i-2}^2 - 2x_0^2 = 0. \qquad \text{with } 3 \leq i \leq n.$$

The Büchi's problem in its geometric formulation, asks if there is an integer $n > 0$ such that all rational points of X_n of the form $[1 : \alpha_1 : \cdots : \alpha_n]$, with $\alpha_i \in \mathbb{Z}$, belong to the lines $x_i = \pm(x+i)$, with $i \in \{1, \dots, n\}$ and $x \in \mathbb{Z}$. These lines are

[1]Note that X_n corresponds to $S_n(\alpha)$, where $\alpha = [\infty, \alpha_1, \dots, \alpha_n]$ with α_i consecutive integers. See Remark 3.1.3.

called the *trivial lines* of X_n. Due to the geometric formulation of the problem, the set $X_n(\mathbb{Q})$ of rational points of X_n must be studied. The importance of studying rational points it is the fact that if the Büchi problem has a positive answer in a ring, this too is true in any subring (see [18, 5]). Then the natural question (considering Bombieri's conjecture [18, 9]) is if $X_n(\mathbb{Q})$ is contained in the trivial lines for some $n > 0$.

It is known that for $n \leq 4$ there are non-trivial integer solutions of (0.2). For an explicit description of Büchi sequences of length 3 and 4 see [20] and [6] respectively. On the other hand, since the surface X_3 is a quadric with a rational point and X_4 is a del Pezzo surface, both rational surfaces, the sets $X_3(\mathbb{Q})$ and $X_4(\mathbb{Q})$ are Zariski-dense in X_3 and X_4 respectively. It is possible to show that also $X_5(\mathbb{Q})$ is Zarisky-dense in X_5, but this makes use an elliptic fibration defined over $\mathbb{Q}$ with infinite group of sections [2, Prop. 5.3]. For $n \geq 8$, a theorem of Vojta (see [23] and [18, §9]) implies, assuming the Bombieri's conjecture, that the Büchi's problem has a positive answer. For a survey of Büchi problem the reader can consult [18].

In the paper *On Büchi's K3 surface* [2] the authors study the geometry of the Büchi's K3 surface X_5, which corresponds with the GBS $S_5(\alpha)$, defined in (0.1), where $\alpha = [\infty, -2, -1, 0, 1, 2]$. This is done by means of the classic correspondence between hyperelliptic curves of genus 2 and Kummer surfaces [11, §10.3.3]. Specifically, they reconstruct the surface $S_5(\alpha)$ from the hyperelliptic curve of genus 2 of equation $y^2 = (x+2)(x+1)x(x-1)(x-2)$. On the other hand, recovering the curve from the surface $S_5(\alpha)$ is simple and we just need to known the trivial lines of the surface, namely, each trivial line L meets other 6 trivial lines, the double cover of L branched along the intersection points is the initial curve.

It is this last idea that is generalized in this work, which is developed in Chapter 3. Given a GBS $S_n(\alpha)$, by knowing its trivial lines, the unordered $n+1$ complex entries of α can be recovered up to changes of coordinates. Viceversa, given the $n+1$ complex numbers that form α we can obtain $S_n(\alpha)$. Each GBS $S_n(\alpha)$ contains the 2^n *trivial lines* parametrized by $t \mapsto [\pm(t-\alpha_0) : \cdots : \pm(t-\alpha_n)]$. These are the only lines of this surface (see Proposition 3.2.3) and determine it completely (see Propositions 3.1.5 and 3.2.2). Each line meet exactly $n+1$ other lines along a subset of points projectively equivalent to α (see Proposition 3.2.5). In the case n odd the intersection graph of the lines is bipartite of type $(2^{n-1}, 2^{n-1})$ (see Proposition 3.2.6). In $n = 2g+1$, with $g \geq 1$, each GBS has associated a hyperelliptic curve of genus g (see Remark 3.3.1). The result is subtly deeper, because in fact it is proven that there is a bijection between $n+1$ unordered points of $\mathbb{P}^1$ and the set of the GBS unless projectivities (see Theorem 3.2.1).

On the other hand, in Chapter 4 we study the case $n = 2g+1$ odd, here we relate the surface $S := S_{2g+1}(\alpha)$ with a hyperelliptic curve C of genus g as follows. We build the surface Y which is quotient of S by the subgroup G_0 of the automorphism group of S of the sign changes of an even number of coordinates. The quotient morphism $S \to Y$ is defined over $\mathbb{Q}$ so that $S(\mathbb{Q})$ is mapped on $Y(\mathbb{Q})$. On the other hand, we consider the second symmetrical power $C^{(2)}$ of C and its hyperelliptical involution $\imath$ and we show that the surfaces $C^{(2)}/\langle\imath\rangle$ and Y are isomorphic (see Theorem 4.2.1 and Corollary 4.2.2) over the field $\mathbb{Q}(\alpha_0, \alpha_1, \alpha_2)$ which is $\mathbb{Q}$ for the Büchi surfaces. In this sense it would be ideal to replace the study of the rational points of S with arithmetic properties of the curve C (see Section 4.3). Within the studied properties of Y, we highlight that Y is a general type surface if $g \geq 3$ (see Proposition 4.1.1), not birracional to S (see Proposition 4.1.2) and it is the double cover of $\mathbb{P}^2$ branched along the union of $2g+2$ lines tangent to a conic Γ (4.2), where the tangency points are the coordinates of the vector α (see Proposition 4.1.3). In the subsection 4.2.1 we see some basic facts of Neron-Severi group of $C^{(2)}$, specifically we calculate the intersection matrix of the curves C_p, Δ and E (see Proposition 4.2.4), whit the purpose of describing some geometric properties of the surface Y.

This thesis is organized as follows. Chapter 1 defines affine and projective complex algebraic varieties, and some basic facts, such as, dimension, smoothness, normal varieties, divisors and morphisms between these varieties. For curves we discuss the Riemann-Roch theorem, the Riemann-Hurwitz formula and symmetric powers. For surfaces, we discuss the Neron-Severi group, the genus formula and the Enriques-Kodaira classification, among other basic facts that we do not mention here. In Chapter 2 we study basic Galois theory and hyperelliptic curves. In the Galois section we define field extensions (finite, normal, separable), the Galois group and the absolute Galois group. In the section of hyperelliptic curves we define the complex weighted projective space, the hyperelliptic involution and some facts about rational points on curves. In general the chapters 1 and 2 are the preliminary ones necessary for the development of this work. The chapters 3 and 4 contains new results, as described above. Finally there is an appendix of some Magma programs used for the necessary calculations.

1 Algebraic varieties

1.1 Affine and projective algebraic varieties

An *affine algebraic variety* X is the zero locus in $\mathbb{C}^n$, for some $n > 0$, of a radical ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$. Denote by $I(X)$ the ideal I and by

$$\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_n]/I(X)$$

the *coordinate ring* of X. The variety X is *irreducible* if the ideal $I(X)$ is prime. In this case $\mathbb{C}[X]$ is an integral domain and its field of fractions is denoted by $\mathbb{C}(X)$. A *projective algebraic variety* X is the zero locus in $\mathbb{P}^n$ of a homogeneous radical ideal $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$. The *homogeneous coordinate ring* of X is the quotient $\mathbb{C}[x_0, \ldots, x_n]/I$, denoted by $R(X)$. The topology which we endow affine (projective) algebraic varieties is the *Zariski topology*, which is the coarsest topology on $\mathbb{C}^n$ ($\mathbb{P}^n$) such that all zero sets of polynomials (homogeneous polynomials) are closed, all open sets will be considered in this topology unless otherwise indicated. Observe that, given the projective variety $X \subseteq \mathbb{P}^n$ and the affine open subset $U_i \subseteq \mathbb{P}^n$, of points whose i-th coordinate does not vanish, the intersection $X \cap U_i$ is an affine variety whose defining ideal is obtained by evaluating the polynomials of the homogeneous ideal at $x_i = 1$. This is called the i-th *affine patch* of X. This process of passing from projective to affine can also be inverted and a projective variety can be obtained by "gluing" $n+1$ affine varieties $X_i \subseteq U_i$ such that $X_i \cap U_j = X_j \cap U_i$ for any i, j. More generally one can define an *algebraic variety* to be obtained by gluing affine varieties. Here however we will stick to affine or projective varieties, so that the word algebraic variety is to be intended to refer to one of these two. Given an algebraic variety X and two open affine subsets $U, V \subseteq X$ it is possible to show that $\mathbb{C}(U) \simeq \mathbb{C}(V)$. It is then natural to define the field $\mathbb{C}(X)$ as $\mathbb{C}(U)$ for an affine $U \subseteq X$.

Definition 1.1.1. The *dimension* of an algebraic variety X, denoted by $\dim X$, is the transcendence degree of the field $\mathbb{C}(X)$ over the base field $\mathbb{C}$.

It is possible to show that the dimension of X is the number of inclusions in a maximal chain of irreducible subvarieties $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = X$. Varieties of dimension one are called *curves*, varieties of dimension two are called *surfaces* and more generally varieties of dimension n are called n-folds.

Given an irreducible affine algebraic variety X and a point $p \in X$ one denotes by

$m_p \subseteq \mathbb{C}[X]$ the maximal ideal of functions which vanish at p. Denote by

$$\mathcal{O}_{X,p} := \{f/g \in \mathbb{C}(X) \,:\, g \notin m_p\},$$

the localization of $\mathbb{C}[X]$ at m_p. The above ring is a local ring with maximal ideal $\mathfrak{m}_p := m_p \mathcal{O}_{X,p}$. Both $\mathfrak{m}_p$ and its square $\mathfrak{m}_p^2$ are complex vector spaces. The *tangent space* of X at p is the dual vector space

$$T_{X,p} := (\mathfrak{m}_p/\mathfrak{m}_p^2)^*.$$

Definition 1.1.2. A point $p \in X$ is *smooth* if $\dim_{\mathbb{C}} T_{X,p} = \dim X$ and *singular* otherwise. An affine algebraic variety X is *smooth* if each point of X is smooth. More generally an algebraic variety is smooth if any affine patch is smooth.

We recall the following useful jacobian criterion for determining when a point of an affine variety is smooth.

Proposition 1.1.3. *Let $X \subseteq \mathbb{C}^n$ be an irreducible affine variety with defining ideal $I(X) = \langle f_1, \ldots, f_k\rangle$ and $p \in X$. Then*

$$\dim T_{X,p} = n - \mathrm{rk} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1}(p) & \cdots & \frac{\partial f_k}{\partial x_n}(p) \end{pmatrix}.$$

In particular X is smooth at p if and only if the the above jacobian matrix has rank $n - \dim X$.

Proof. Denote by $\mathfrak{m}$ the maximal ideal of the local ring $\mathcal{O}_{\mathbb{C}^n,p}$, so that $\mathfrak{m}_p = \mathfrak{m} + I$. The homomorphism $\mathfrak{m} \to (\mathfrak{m}+I)/(\mathfrak{m}^2+I)$, obtained by composing the inclusion with the projection onto the quotient, is surjective with kernel $\mathfrak{m}^2 + \mathfrak{m} \cap I$. It follows that $\mathfrak{m}_p/\mathfrak{m}_p^2$ is isomorphic to $\mathfrak{m}/\mathfrak{m}^2 + \mathfrak{m} \cap I$. Observe that $\mathfrak{m} = \langle x_1 - p_1, \ldots, x_n - p_n\rangle$, where $p_1, \ldots, p_n$ are the coordinates of $p \in X \subseteq \mathbb{C}^n$. Given a function $f \in \mathfrak{m}$ we have

$$f \equiv \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - p_i) \pmod{\mathfrak{m}^2}.$$

Thus $\mathfrak{m} \cap I$, mod $\mathfrak{m}^2$, is generated by $\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(p)(x_i - p_i)$, for $1 \leq j \leq k$. □

The jacobian criterion in the projective case is the following.

Corollary 1.1.4. *Let $X \subseteq \mathbb{P}^n$ be a projective variety with homogeneous ideal $I(X) = \langle f_1, \ldots, f_k\rangle$, and $p \in X$. Then X is smooth at p if only if the rank of the $k \times (n+1)$ Jacobian matrix is $n - \dim X$.*

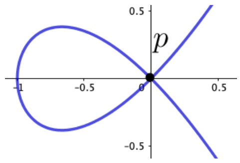

Figure 1.1: Plane cubic of equation $x_2^2 - x_1^3 - x_1^2 = 0$, singular at the origin.

The Proposition 1.1.3 provides a computational tool to describe the locus of singular points of an affine variety. For example if X is a hypersurface, or equivalently if $I(X) = \langle f \rangle$ is a principal ideal, then the singular locus of X is cut out by the equations $f = \partial f/\partial x_1 = \cdots = \partial f/\partial x_n = 0$.

Recall that a domain R is *integrally closed* into its field of fractions $\mathrm{Frac}(R)$ if given $f/g \in \mathrm{Frac}(R)$ such that $(f/g)^r + a_1(f/g)^{r-1} + \cdots + a_{r-1}(f/g) + a_r = 0$ for some $a_1, \ldots, a_r \in R$ then $f/g \in R$.

Definition 1.1.5. An affine variety is *normal* if its coordinate ring is integrally closed into its fraction field. A variety is normal if any of its affine patches is normal.

The condition of being normal is crucial for developing a theory of Weil divisors on the algebraic variety, as we will see later. At the moment we recall two basic facts about normal varieties: any smooth variety is normal and the singular locus of a normal variety has codimension at least two. The second statement in particular implies that a curve is normal if and only if it is smooth. For example in Fig. 1.1 the function $t := x_2/x_1 \in \mathbb{C}(X)$ is root of the polynomial $t^2 - x_1 - 1$, whose coefficients are in $\mathbb{C}[X]$, but $t \notin \mathbb{C}[X]$.

Given a normal algebraic variety X one defines the group of *Weil divisors* of X, denoted by $\mathrm{WDiv}(X)$, to be the free abelian group generated by irreducible subvarieties of X of codimension one. In particular a Weil divisor of X is a finite sum $n_1D_1 + \cdots + n_kD_k$, where all the n_i are integers and the D_i are irreducible hypersurfaces, also called *prime divisors*. The set $\{D_1, \ldots, D_k\}$ is the *support* of the divisor D. To any rational function $f \in \mathbb{C}(X)$ one can associate a Weil divisor in the following way. First of all assume that X is affine and let D be a prime divisor. Define the local ring

$$\mathcal{O}_{X,D} := \{a/b \in \mathbb{C}(X) \,:\, b \notin I(D)\}.$$

Define $\mathrm{ord}_D(a)$ as the length of the $\mathcal{O}_{X,D}$-module $\mathcal{O}_{X,D}/\langle a \rangle$ and $\mathrm{ord}_D(a/b) := \mathrm{ord}_D(a) - \mathrm{ord}_D(b)$. If the singular locus of X is of codimension at least 2, then $\mathcal{O}_{X,D}$ is a discrete valuation ring with valuation $f \mapsto \mathrm{ord}_D(f)$. For more general X one can provide a similar valuation on any open affine chart which has non-empty intersection with D. Then one defines

$$\mathrm{div}(f) := \sum_D \mathrm{ord}_D(f) \cdot D.$$

Divisors of the form $\mathrm{div}(f)$ are called *principal divisors*, they form a subgroup of $\mathrm{WDiv}(X)$ denoted by $\mathrm{PDiv}(X)$. The quotient

$$\mathrm{Cl}(X) := \frac{\mathrm{WDiv}(X)}{\mathrm{PDiv}(X)}$$

is the *divisor class group* of X. Two divisors D_1 and D_2 with the same class are called *linearly equivalent*, denoted by $D_1 \sim D_2$. There is another important subgroup of the group of Weil divisors: the group of *Cartier divisors*, denoted by $\mathrm{CDiv}(X)$. It consists of divisors which are locally principal, that is of those $D \in \mathrm{WDiv}(X)$ such that for any $p \in X$ there is a neighbourhood U of p where $D|_U = \mathrm{div}(f)$ for some $f \in \mathbb{C}(X)$. The quotient

$$\mathrm{Pic}(X) := \frac{\mathrm{CDiv}(X)}{\mathrm{PDiv}(X)}$$

is the *Picard group* of X. It is possible to show that any Weil divisor is a Cartier divisor if the variety X is *locally factorial*, that is for any $p \in X$ the ring $\mathcal{O}_{X,p}$ is a unique factorization domain. The last condition holds for example if X is smooth.

Let X be a normal variety of dimension n, let $X^0 \subseteq X$ be the subset of smooth points of X and let $w \in \Omega^n(X^0)$ be a holomorphic n-form. Cover X^0 with open holomorphic charts and let (U, φ) be one of these charts with coordinates $z_1, \ldots, z_n$. Then $w = f dz_1 \wedge \cdots \wedge dz_n$ on U, with f holomorphic. Define $\mathrm{div}(w)$ to be the closure in X of the Cartier divisor of X^0 which on each such chart U equals $\mathrm{div}(f)$.

Definition 1.1.6. A canonical divisor of a normal variety X of dimension n is $K_X := \mathrm{div}(w)$, where $w \in \Omega^n(X^0)$ is a meromorphic n-form.

Given two canonical divisors $\mathrm{div}(w)$, and $\mathrm{div}(w')$ their difference is a principal divisor because the rational function which represents $\mathrm{div}(w)$ in a chart is multiplied by the jacobian determinant of the coordinate change when passing from one chart to the other, and the same holds for $\mathrm{div}(w')$. It follows that there is a unique *canonical class* $[K_X] \in \mathrm{Cl}(X)$.

Let X be a normal variety. A Weil divisor $D := \sum_i n_i D_i$ is *effective*, denoted by $D \geq 0$, if $n_i \geq 0$ for any i. Given an open subset $U \subseteq X$ the restricted divisor $D|_U$ is the sum $\sum_i n_i(D_i \cap U)$, where $D_i \cap U$ is to be intended 0 if the intersection is empty. To any Weil divisor D on a normal variety X and structure sheaf $\mathcal{O}_X$ we can associate the following sheaf of $\mathcal{O}_X$-modules

$$\mathcal{O}_X(D)(U) := \{f \in \mathbb{C}(X)^* \,:\, \mathrm{div}(f) + D|_U \geq 0\} \cup \{0\}.$$

The Riemann-Roch space of D is the space of global sections of the sheaf $\mathcal{O}_X(D)$. The dimension of Riemann-Roch space of D is denoted by $l(D)$.

Theorem 1.1.7 (Adjunction formula)**.** *Let $Y \subseteq X$ be a smooth subvariety of a smooth variety X the following* adjunction formula *relates the two canonical divisors:*

$$K_Y = (K_X + Y)|_Y.$$

Proof. See [14, Prop. II.8.20] □

To correctly compute the above restriction the divisor $K_X + Y$ can be substituted with any linear equivalent divisor which do not have Y into its support.

1.1.1 Morphisms between algebraic varieties

Let X be an affine variety, and let U be an open subset of X. A *regular function* on U is a map $\varphi : U \to \mathbb{C}$, with the following property: for every $a \in U$ there are polynomial functions $f, g \in \mathbb{C}[X]$ with $f(x) \neq 0$ and $\varphi(x) = \frac{g(x)}{f(x)}$ for all x in an open subset U_a with $a \in U_a \subset U$. The set of all such regular functions on U will be denoted by $\mathcal{O}_X(U)$. The set $\mathcal{O}_X(U)$ is a ring with pointwise addition and multiplication, it is also a $\mathbb{C}$-vector space since we can multiply a regular function pointwise with a fixed scalar in $\mathbb{C}$, i.e., $\mathcal{O}_X(U)$ is a $\mathbb{C}$-algebra.

The rings $\mathcal{O}_X(U)$ of regular functions on open subsets $U \subset X$, together with the usual restriction maps of functions, form a sheaf (see [13, Def. 3.16]) $\mathcal{O}_X$ on X. We call $\mathcal{O}_X$ the *sheaf of regular functions on* X.

A *ringed space* is a topological space X together with a sheaf of rings on X. An affine variety will always be considered as a ringed space together with its sheaf of regular functions. An open subset U of a ringed space X will always be considered as a ringed space with the structure sheaf being the restriction of $\mathcal{O}_X|_U$ of sheaf (see [13, Def. 3.18]) $\mathcal{O}_X$.

Let $f : X \to Y$ be a map of ringed spaces. For any map $\varphi : U \to \mathbb{C}$, with $U \subset Y$ we denote the composition $\varphi \circ f : f^{-1}(U) \to \mathbb{C}$ by $f^*\varphi$. It is called the *pull-back* of φ by f. The map f is called a *morphism* (of ringed spaces) if it is continuous, and if for all open subsets $U \subset Y$ and $\varphi \in \mathcal{O}_Y(U)$ we have $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$. So in this case pulling back by f yields $\mathbb{C}$-algebra homomorphisms

$$f^* \colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U)), \quad \varphi \mapsto f^*\varphi.$$

We say that f is an isomorphisms (of ringed spaces) if it has a two-sided inverse, i.e., if it is bijective, and both $f : X \to Y$ and $f^{-1} : Y \to X$ are morphisms. Morphisms and isomorphisms of (open subsets of) affine varieties are morphisms (resp. isomorphisms) as ringed spaces. Observe that the composition and the restriction of morphisms also are morphisms, further the morphisms satisfy the "gluing property", i.e., if $f : X \to Y$ is a map of ringed spaces and exists an open cover $\{U_j : j \in J\}$ of X such that all restrictions $f_{U_j} : U_j \to Y$ are morphisms, then f is a morphism (see [13, Lemma 4.6]).

Proposition 1.1.8. *Let U be an open subset of an affine variety X, and let $Y \subset \mathbb{A}^n$ be another affine variety. Then the morphisms $f : U \to Y$ are exactly the maps of the form*

$$f = (\varphi_1, \dots, \varphi_n) : U \to Y, \quad x \mapsto (\varphi_1(x), \dots, \varphi_n(x)),$$

whit $\varphi_i \in \mathcal{O}_X(U)$ for all $i = 1, \dots, n$.. In particular the morphism from U to $\mathbb{A}^1$ are exactly the regular functions in $\mathcal{O}_X(U)$.

Proof. See [13, Prop. 4.7] □

Corollary 1.1.9. *For any two affine varieties X and Y there is a one-to-one correspondence*

$$\{\textit{morphisms } X \to Y\} \leftrightarrow \{\mathbb{C}\textit{-algebra homomorphisms } \mathbb{C}[Y] \to \mathbb{C}[X]\}$$

$$f \mapsto f^*$$

In particular, isomorphisms of affine varieties correspond exactly to $\mathbb{C}$-algebra isomorphisms in this way.

Proof. See [13, Cor. 4.8]. □

An example of bijective morphism which is not an isomorphism is the following. Let $X = V(x_1^2 - x_2^3) \subset \mathbb{A}^2$ be the curve as in the Fig. 1.2. It has a singular point at the origin. Now consider the map $f : \mathbb{A}^1 \to X$, $t \mapsto (t^3, t^2)$ which is a morphism. Its corresponding $\mathbb{C}$-algebra homomorphism $f^* : \mathbb{C}[X] \to \mathbb{C}[\mathbb{A}^1]$ is given by

$$\mathbb{C}[x_1, x_2]/(x_1^2 - x_2^3) \to \mathbb{C}[t], \qquad \overline{x_1} \mapsto t^3, \quad \overline{x_2} \mapsto t^2$$

which can be seen by composing f with the two coordinate functions of $\mathbb{A}^2$. Note that f is bijective with inverse map

$$f^{-1} : X \to \mathbb{A}^1, \quad (x_1, x_2) \mapsto \begin{cases} \frac{x_1}{x_2} & \text{if} \quad x_2 \neq 0 \\ 0 & \text{if} \quad x_2 = 0, \end{cases}$$

but f^{-1} is not a morphism, since otherwise the map f^* above would have to be an isomorphism as well, which is false since the lineal polynomial t is not in its image. Therefore f is not an isomorphism.

The concept of morphism for projective varieties is analogous with some details to considered (see [13, Ch. 7]). So as not to make this tedious exhibition, we will only give the results that interest us.

Let $X \subset \mathbb{P}^n$ be a projective variety, and let $f_0, \dots, f_m \in R(X) := \mathbb{C}[x_0, \dots, x_n]/I(X)$ be homogeneous elements of the same degree. Then on the open subset $U := X \backslash V(f_0, \dots, f_m)$ these elements define a morphism $f : U \to \mathbb{P}^m$, $x \mapsto [f_0(x) : \dots : f_m(x)]$ (see [13, Lemma 7.5]).

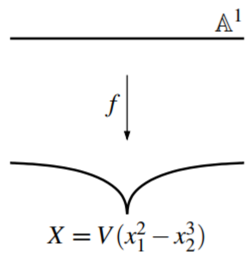

Figure 1.2: Isomorphisms $\neq$ bijective morphisms.

Some examples of morphisms between projective varieties are: $A : \mathbb{P}^n \to \mathbb{P}^n$, $x \mapsto Ax$ where A is an invertible matrix, in fact this is an automorphism of $\mathbb{P}^n$. Moreover these are the only isomorphisms of $\mathbb{P}^n$ (see [13, Prop. 13.4]). For another example, consider $a = [0 : \cdots : 1] \in \mathbb{P}^n$ and $V(x_n) \cong \mathbb{P}^{n-1}$, then the map $f : \mathbb{P}^n \backslash \{a\} \to \mathbb{P}^{n-1}$, $[x_0 : \cdots : x_n] \mapsto [x_1 : \cdots : x_{n-1}]$ is a morphism. The obvious geometric interpretation is that f is the projection from a to the lineal subspace $V(x_n) \cong \mathbb{P}^{n-1}$.

Let X and Y be irreducible varieties. A *rational map*, denoted by $f : X \dashrightarrow Y$ is a morphism $f : U \to Y$ from a non-empty open subset $U \subset X$ to Y. We say that two such rational maps $f_1 : U_1 \to Y$ and $f_2 : U_2 \to Y$, with $U_1, U_2 \subset X$ are the same if $f_1 = f_2$ on a non empty open subset of $U_1 \cap U_2$. The rational map f is called *dominant* if its image contains a non-empty open $U \subset Y$. In this case, if $g : Y \dashrightarrow Z$ is another rational map, defined on a non-empty open $V \subset Y$, we can construct the composition we can construct the composition $g \circ f : X \dashrightarrow Z$ as a rational map since we have such a composition of ordinary morphisms on the non-empty open subset $f^{-1}(U \cap V)$. The rational map f is called *birational* if it is dominant, and if there is another dominant rational map $g : Y \dashrightarrow X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. We say that X and Y are birational if there is a birational map $f : X \dashrightarrow Y$ between them. Observe that by definition two irreducible varieties are birational if and only if they contains isomorphic non-empty open subsets. In particular this implies that birational irreducible varieties have the same dimension.

Let X be an irreducible variety. A rational map $\varphi : X \dashrightarrow \mathbb{A}^1 = \mathbb{C}$ is called a *rational function* on X. In other words, a rational function on X is given by a regular function $\varphi \in \mathcal{O}_X(U)$ on some non-empty open subset $U \subset X$, with two such regular functions defining the same rational function if and only if they agree on a non-empty open subset. The set of all rational functions on X is a field and will be denoted by $\mathbb{C}(X)$ and called the *function field* of X. If $U \subset X$ is a non-empty open subset of an irreducible variety X then $\mathbb{C}(U) \cong \mathbb{C}(X)$. An isomorphism is given by:

$$\mathbb{C}(U) \to \mathbb{C}(X), \quad \varphi \in \mathcal{O}_U(V) \mapsto \varphi \in \mathcal{O}_X(V),$$

whit inverse

$$\mathbb{C}(X) \to \mathbb{C}(U), \quad \varphi \in \mathcal{O}_X(V) \mapsto \varphi|_{V\cap U} \in \mathcal{O}_U(V \cap U).$$

In particular, birational irreducible varieties have isomorphic function fields.

The following theorem summarize the above results.

Theorem 1.1.10. *For any two varieties X, Y the following conditions are equivalent*

1. *X and Y are birationally equivalent.*
2. *There are open subsets $U \subseteq X$ and $V \subseteq Y$ with U isomorphic to V.*
3. *$\mathbb{C}(X) \cong \mathbb{C}(Y)$ as k-algebras.*

Proof. See [14, pp. 26, Cor. 4.5]. □

In this work we need some basic definitions of the Graph theory, which we give below. For more details it can be consult the book of R. Diestel, Graph theory [10, Ch. 1, §1.1 and §1.6].

Definition 1.1.11. A *graph* is a pair $Gr := (V, E)$ of sets such that $E \subseteq [V]^2$, where $[V]^2$ denote the set of all 2-element subsets of V. Note that $V \cap E = \emptyset$. The elements of V are the *vertices* of the graph Gr, the elements of E are its *edges*.

Definition 1.1.12. Let $r \geq 2$ be an integer. A graph $Gr = (V, E)$ is called *r-partite* if V admits a partition into r classes such that every edge has its ends in different classes: vertices in the same partition class must not be adjacent. Instead of 2-*partite* one usually says *bipartite*.

Definition 1.1.13. A bipartite graph $Gr = (V, E)$ is said to be of type (a, b) if a and b are the cardinalities of the two classes of V.

1.2 Algebraic curves

In this work a curve C is a smooth projective algebraic variety over $\mathbb{C}$ of dimension one. This section is based on the books [1], [12] and [14].

The basic facts about the divisors of a curve C are described in Section 1.1.

Given a divisor D, the *complete linear series*, or *system* $|D|$ is the set of effective divisors linearly equivalent to D. Given two meromorphic functions f and g, we have that $\operatorname{div}(f) = \operatorname{div}(g)$ if only if there is a non-zero constant λ such that $f = \lambda g$. Let $\mathcal{L}(D)$ be the Riemann-Roch space of D and $l(D)$ its dimension (see

Section 1.1). We then have an identification $|D| = \mathbb{P}\mathcal{L}(D)$ obtained by associating to each non-zero $f \in \mathcal{L}(D)$ the divisor $\operatorname{div}(f) + D$. A complete linear series is therefore a projective space.

Theorem 1.2.1 (Riemann-Roch)**.** *Let D be a divisor on a curve C of genus g. Then*

$$l(D) - l(K_C - D) = \deg D + 1 - g,$$

where K_C is the canonical divisor of C.

Proposition 1.2.2. *Let X and Y be connected Riemann surfaces and $F : X \to Y$ a non-constant holomorphic map. For each point x, there is a unique integer $k = k_x \geq 1$ such that we can find charts around $x \in X$ and $F(x) \in Y$ which F is represented by the map $z \mapsto z^k$.*

Proof. See [12, Ch. 4, §4.1, Prop. 5]. □

Proposition 1.2.3. *Let $F : X \to Y$ be a non-constant holomorphic map between connected Riemann surfaces. Let $R \subset X$ be the set of points x where $k_x > 1$*

1. *R is a discrete subset of X.*
2. *If F is proper, then the image $\Delta = F(R)$ is discrete in Y.*
3. *If F is proper, then for any $y \in Y$ the pre-image $F^{-1}(y)$ is a finite subset of X.*

Proof. See [12, Ch. 4, §4.1, Prop. 6]. □

We call the points of the set R the *ramifications points* of F and the points of Δ *branch points*. For $x \in X$ we call the integer k_x the *multiplicity* of F at x.

Proposition 1.2.4. *Let $F : X \to Y$ be a proper non-constant holomorphic map between connected Riemann surfaces. Then the integer*

$$d(y) := \sum_{x \in F^{-1}(y)} k_x$$

does not depend on $y \in F(X)$.

Proof. See [12, Ch. 4, §4.1, Prop. 7]. □

We call the number $d(y)$ the *degree* of F and the sum $R_F := \sum_{x \in X}(k_x - 1)$ the *total ramification index* of F.

Theorem 1.2.5 (Riemann-Hurwitz formula)**.** *Let $F : X \to Y$ be a non-constant holomorphic map of degree d between connected compact Riemann surfaces and X. Then the genus $g(X)$ of X and the genus $g(Y)$ of Y are related by*

$$2g(X) - 2 = d(2g(Y) - 2) + R_F.$$

Proof. See [12, Ch. 7, §7.2, Prop. 19]. □

Definition 1.2.6 (Symmetric product)**.** Let C be a Riemann surface, the d-th *symmetric product* $C^{(d)}$ of C is the quotient of the d-fold product $C^d = C \times \cdots \times C$ of C with itself d times by the action of the symmetric group $\mathfrak{S}_d$ on d letters

Proposition 1.2.7. *If C is a Riemann surface, then $C^{(d)}$ is a complex manifold of complex dimension d.*

Proof. See [12, Ch. 12, §12.2.3, Prop. 40]. □

Proposition 1.2.8. $(\mathbb{P}^1)^{(2)} \simeq \mathbb{P}^2$.

Proof. The morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ defined by $([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0y_0 : x_0y_1 + x_1y_0 : x_1y_1]$ has degree two and factorizes through $(\mathbb{P}^1)^{(2)}$. Since the latter surface is smooth by Proposition 1.2.7, the above map induces the claimed isomorphism. □

1.3 Algebraic surfaces

This section is devoted to recall some basic facts about algebraic surfaces.

Let X be a smooth projective algebraic surface. Given two curves $C_1, C_2 \subseteq X$ and a point $p \in C_1 \cap C_2$ the *intersection multiplicity* m_p of C_1 and C_2 at p is the dimension of the complex vector space $\mathcal{O}_{X,p}/\langle f_1, f_2\rangle$, where $f_i \in \mathcal{O}_{X,p}$ locally defines C_i near p. Then one defined the *intersection number*

$$C_1 \cdot C_2 := \sum_{p \in C_1 \cap C_2} m_p.$$

It is possible to show that the above pairing does not vary if one substitute the curve C_i with a linearly equivalent one. The proof makes use of the fact that the intersection number is the degree of the restriction of one curve (seen as divisor) to the other (seen as a subvariety), in formulas

$$C_1 \cdot C_2 = \deg(C_1|_{C_2}).$$

Moreover the intersection number is clearly $\mathbb{Z}$-linear on each factor, so that it descends to a bilinear pairing

$$\mathrm{Cl}(X) \times \mathrm{Cl}(X) \to \mathbb{Z}, \qquad ([D_1], [D_2]) \mapsto D_1 \cdot D_2.$$

Denote by $\mathrm{Cl}^0(X) \subseteq \mathrm{Cl}(X)$ the subgroup consisting of classes $[D]$ such that $D \cdot D' = 0$ for any Weil divisor D'. The quotient

$$N^1(X) := \frac{\mathrm{Cl}(X)}{\mathrm{Cl}^0(X)}$$

is the *Néron-Severi* group of X which is known to be finitely generated, free abelian [17, Prop. 1.1.14]. Its dimension $\varrho(X)$ is the *Picard number* of X. A divisor D is *nef* if $D \cdot C \geq 0$ for any curve C. It is possible to show that any ample divisor is nef and that the class of a nef divisor is limit of ample classes. Moreover the subset of nef classes of $N^1(X)$, denoted by $\mathrm{Nef}(X)$, form a convex cone of maximal dimension. The Generalized inequality of Hodge type [17, Thm. 1.6.1] states that given two nef divisors D_1 and D_2 the following inequality holds:

$$(D_1 \cdot D_2)^2 \geq D_1^2 \, D_2^2.$$

As a consequence of the above inequality and the fact that the nef cone is full dimensional, the signature of the intersection form on $N^1(X)$ is $(1, \varrho - 1)$, where ϱ is the Picard number of X. This is called the *Hodge index theorem*. The below picture displays, when $\varrho(X) = 3$, the cone of classes $[D]$ with $D^2 = 0$. The self-intersection is positive in the interior of the cone and negative outside.

Figure 1.3: The light cone of $N^1(X)$

Recall that if C is a smooth projective curve of genus $g := g(C)$, then the degree of a canonical divisor of C is $\deg(K_C) = 2g - 2$. If $C \subseteq X$ is a smooth curve on a smooth projective surface X then, by the adjunction formula, we have that

$$2g - 2 = \deg(K_C) = \deg((K_X + C)|_C) = (K_X + C) \cdot C. \tag{1.1}$$

The above is also know as the *genus formula* [4, Ch. I, Thm. 6.4].

Let X be a smooth projective surface with structure sheaf $\mathcal{O}_X$. Recall that for any divisor D of X the sheaf cohomology groups $H^i(X, \mathcal{O}_X(D))$ are finite dimensional and vanish for all but $i \in \{0, 1, 2\}$. Thus the following *Euler characteristic* of the sheaf $\mathcal{O}_X(D)$ is well defined

$$\chi(\mathcal{O}_X(D)) := \sum_{i=0}^{2} (-1)^i h^i(X, \mathcal{O}_X(D)).$$

The Riemann-Roch theorem provides a formula for the above Euler characteristic in terms of a canonical divisor of X and the topological Euler characteristic $e(X)$ of X:

$$\chi(\mathcal{O}_X(D)) = \frac{D^2 - D \cdot K_X}{2} + \frac{K_X^2 + e(X)}{12}. \tag{1.2}$$

1.3.1 Smooth algebraic surfaces according to their Kodaira dimension

Definition 1.3.1. Given a normal algebraic variety X, a Weil divisor $D \in \mathrm{WDiv}(X)$ and a basis $f_0, \ldots, f_n$ for the space of global sections of the sheaf $\mathcal{O}_X(D)$ one can define the rational map $\phi_D \colon X \to \mathbb{P}^n$ by $p \mapsto [f_0(p) : \cdots : f_n(p)]$, whose indeterminacy is the common zero locus of the f_i. The *Iitaka dimension* of D is [17, Definition 2.1.3]:

$$\kappa(X, D) := \max_{n \in \mathbb{N}} \{\dim \phi_{nD}(X)\}.$$

Assuming X to be smooth its *Kodaira dimension* $\kappa(X) := \kappa(X, K_X)$ is the Iitaka dimension of a canonical divisor of X.

Observe that $\kappa(X) \leq \dim X$ and when the equality hods X is said to be of *general type*. On the other hand, if the sheaf $\mathcal{O}_X(nK_X)$ has no global sections for any integer $n > 0$, then one conventionally puts $k(X) = -\infty$. Smooth algebraic curves are classified according to their Kodaira dimension [5, Ex. VII.2] summarized in the following table

Genus of the curve	0	1	≥ 2
Kodaira dimension	$-\infty$	0	1

Table 1.1: Kodaira dimension for smooth algebraic curves.

In particular the only curve of negative Kodaira dimension is $\mathbb{P}^1$. The situation for surfaces is more subtle. First of all one observes that, due to a criterion of Castelnuovo, if a smooth algebraic surface X contains a smooth rational curve E with $E^2 = -1$ then there exists a birational morphism $\pi \colon X \to Y$, with Y smooth, such that $p = \pi(E)$ is a point and π induces an isomorphism $X \setminus E \to Y \setminus \{p\}$. Such a curve E is called a (-1)-curve, the morphism π is the *contraction* of E or the *blow-up* of $p \in Y$. One can show that the Picard rank of Y is one less than that of X, so that only a finite number of (-1)-curve can be contracted. What is fundamental is that, due to the formula

$$K_X = \pi^* K_Y + E$$

one deduces that the Kodaira dimension of X equals that of Y. A surface without (-1)-curves is *minimal* and the Enriques-Kodaira classification of smooth projective surfaces describes the minimal surfaces according to their Kodaira dimension. See [4, 5] for a complete discussion of the subject.

- Kodaira dimension $-\infty$. Here the minimal models are *ruled surfaces*, that is surfaces which admit a surjective morphism $X \to C$ to a smooth curve C

and whose fibers are smooth rational curves. In case C is a rational curve the surface X is rational as well. This class of surfaces contains the del Pezzo surfaces, whose anticanonical class is ample. Del Pezzo surfaces are blow-ups of $\mathbb{P}^2$ at $0 \leq r \leq 8$ points in general position or $\mathbb{P}^1 \times \mathbb{P}^1$.

- Kodaira dimension 0. The minimal models are $K3$ surfaces: simply connected surfaces with trivial canonical class, Enriques surfaces, Abelian and hyperelliptic surfaces. Here we have specified something more only for K3 surfaces because we will meet them in this work. An example of $K3$ surface is a smooth hypersurface of degree $a_0 + a_1 + a_2 + a_3$ of the weighted projective space $\mathbb{P}(a_0, a_1, a_2, a_3)$.

- Kodaira dimension 1. The minimal models are surfaces which admit a morphism with connected fibers $X \to C$, where C is a smooth curve and the general fiber is a smooth genus one curve.

- Kodaira dimension 2. These are called surfaces of *general type*. Their minimal models have a nef canonical divisor K_X such that $K_X^2 > 0$. For each such minimal surface X there exists a positive integer $n \leq 6$ such that the map $\phi_{nK_X} \colon X \to X'$ is a birational morphism which contracts only a finite number of (-2)-curves of X, i.e. smooth rational curves with self-intersection -2. The surface X' is the *canonical model* of X and its singularities are du Val singularities [15, §7.5].

2 Rational points

2.1 Basic Galois theory

This section is based on the book of D. Cox, Galois theory [7].

Let $(k, +, \cdot)$ be a field, for short k, denote by k^* the multiplicative group $(k\backslash\{0\}, \cdot)$. Let $1 \in k$ be the multiplicative unity. The *characteristic* of k is the number $\mathrm{char}(k) = p$ if p is the smallest positive integer such that $1 \cdot p = 0$ and $\mathrm{char}(k) = 0$ otherwise. If k is a field with $\mathrm{char}(k) = p > 0$, then p is a prime number.

Definition 2.1.1. A *field extension* is an inclusion of a field k in other field K. We denote this extension by K/k.

We observe that K is a k-vector space.

Definition 2.1.2. The *degree* of the extension K/k, is defined as the dimension $\dim_k K$of K as a k-vector space and it is denoted by $[K : k]$. If $\dim_k K$ is finite we say that K/k is a *finite extension*.

Lemma 2.1.3. *A field extension L/K has degree $[L : K] = 1$ if only if $K = L$.*

Proof. If $[L : K] = 1$, then any nonzero element of L, say $1 \in L$, is a basis. Thus $L = \{a \cdot 1 : a \in K\} = K$. The reciprocal is obvious. □

An example of field extension of degree infinite is $\mathbb{R}/\mathbb{Q}$, since $\mathbb{R}$ it is uncountable.

Theorem 2.1.4 (Tower law)**.** *If K/F and F/k are fields extensions of finite degree then $[K : F][F : k] = [K : k]$.*

Proof. See [7, Thm. 4.3.8]. □

Let K/k be a field extension, $u_1, \ldots, u_n \in K$ and x a independent variable. Denote by $k(u_1, \ldots, u_n)$ the smallest subfield of K that contains k and the elements $u_1, \ldots, u_n$. Given $u \in K$, the map $v_u : k[x] \to K$ defined by $P(x) \to P(u)$ is a homomorphism of rings. As K is an integral domain then the kernel of v_u is a prime ideal. Moreover, since $k[x]$ is a principal ideal domain, $\ker v_u$ is either trivial or generated by an irreducible monic polynomial. Denote by $k[u]$ the image of v_u. We observe that $k(u)$ is the field of fractions of $k[u]$, that is it consists of the fractions $a/b \in K$ such that $a, b \in k[u]$, with $b \neq 0$.

Definition 2.1.5. Let K/k be a field extension and $u \in K$. The element u is:

- *transcendental* if the kernel of v_u is trivial;
- *algebraic* if the kernel of v_u is non trivial.

If u is algebraic the kernel of v_u is generated by an irreducible monic polynomial $P_u(x)$. We call this polynomial the *minimal polynomial* of u over k.

An alternative definition, equivalent of course, for the minimal polynomial is given in [7, Lemma 4.1.3], that we can also take as a definition.

Definition 2.1.6. If $u \in K$ is algebraic over k, then there is a unique nonconstant monic polynomial $P_u \in k[x]$ with the following two properties.

1. u is a root of P_u, i.e., $P_u(u) = 0$.
2. If $f \in k[x]$ is any polynomial with u as a root, then f is a multiple of P_u.

Definition 2.1.7. A field extension K/k is *simple* if there exists $u \in K$ such that $K = k(u)$.

Theorem 2.1.8. *Let $k(u)/k$ be a simple field extension, where u is algebraic over k. Then the following are true.*

1. *$k[u]$ is isomorphic to $k[x]/\langle P_u(x)\rangle$.*
2. *$k[u] = k(u)$.*
3. *$[k(u) : k] = \deg P_u(x)$.*

Proof. Consider the homomorphism of rings $v_u : k[x] \to k(u)$, defined by $P(x) \mapsto P(u)$ and let $k[u]$ be its image, so by the first isomorphism theorem and the fact that the kernel of v_u, equal to the ideal$\langle P_u(x)\rangle$, is a prime ideal, $k[u] \cong k[x]/\langle P_u(x)\rangle$ is a field, then $k[u] = k(u)$. This proves (1) and (2).

To show (3) the procedure is the following. Suppose $\deg P_u(x) = n$. Now we will show that $1, u, \dots, u^{n-1}$ form a basis of $k(u)$ over k. Since $k[u] = k(u)$, every element of $k(u)$ is of the form $g(u)$ for some $g \in k[x]$. Dividing g by P_u gives

$$g = qP_u + a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$$

where $q \in k[x]$ and $a_0, \dots, a_{n-1} \in k$, and evaluating this at $x = u$ yields

$$g(u) = a_0 + a_1u + \cdots + a_{n-1}u^{n-1},$$

since u is a root of P_u. Thus the set $U := \{1, u, \dots, u^{n-1}\}$ span $k(u)$ over k. To show linear independence of U, suppose that

$$a_0 + a_1u + \cdots + a_{n-1}u^{n-1} = 0$$

where $a_0, \dots, a_{n-1} \in k$. Then u is a root of $a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \in k[x]$, which is the zero polynomial, since its degree is less than the $\deg P_u(x)$. Hence $a_i = 0$ for each i. Then by Definition 2.1.2 $[k(u) : k] = n$. □

Definition 2.1.9. A field extension K/k is *algebraic* if every element of K is algebraic over k.

We observe that a finite extension K/k is algebraic, since if $u \in K$, so the set $\{1, u, \dots, u^m\}$ is linearly dependent for some $m > 0$, then there are $b_i \in k$ such that $b_m u^m + \cdots + b_1 u + b_0 = 0$, hence u is algebraic over k. The reciprocal is not true, a counterexample is the extension $\overline{\mathbb{Q}}/\mathbb{Q}$, this is by definition algebraic, but its degree is infinite.

Proposition 2.1.10. *Let K/k be a field extension and F the set of the elements of K that are algebraic over k. Then F is a subfield of K.*

Proof. Let a, b be elements of F, with $b \neq 0$. We have to prove that $a \pm b, ab, ab^{-1} \in F$. Consider the inclusions $k \subseteq k(a) \subseteq k(a, b)$. The first and second extensions are finite because a is algebraic over k and b is algebraic over $k(a)$. Then by the tower law the extension $k(a, b)/k$ is finite, so each $u \in k(a, b)$ is algebraic over k. In particular $k(a, b) \subseteq F$ and then $a \pm b, ab, ab^{-1} \in F$. □

Proposition 2.1.11. *Let K/k be a field extension. Then the following are equivalent.*

1. *K/k is finite.*

2. *K/k is algebraic and finitely generated.*

Proof. (1) ⇒ (2). If K/k is finite, then $K = k(u_1, \dots, u_n)$, where $\{u_1, \dots, u_n\}$ is a basis of K over k. Then the extension K/k is finitely generated and algebraic.

(2) ⇒ (1). If K/k is algebraic and finitely generated, then $K = k(u_1, \dots, u_n)$ for some u_i algebraic over k. Define $K_i := k(u_1, \dots, u_i)$, then $k = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$. Where each extension $K_{i-1} \subseteq K_i = K_{i-1}(u_i)$ is algebraic, so K_i/K_{i-1} is finite and then K/k is finite by Tower law. □

Definition 2.1.12. Let K/k be a field extension.

1. The extension K/k is *normal* if every irreducible polynomial in $k[x]$ that has a root in K has all its roots in K.

2. One says that K is a *splitting field* of a polynomial $f \in k[x]$ if K is the smallest field that contains every roots of f.

An example of extension no normal is $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$, since the polynomial $x^3 - 2 \in \mathbb{Q}[x]$ is irreducible but not all its roots are in $\mathbb{Q}(\sqrt[3]{2})$.

The following theorem can be seen in [7, Thm. 5.2.4].

Theorem 2.1.13. *Let K/k be a field extension. Then the following are equivalent.*

1. *The extension K/k is finite and normal.*
2. *K is the splitting field of a polynomial $f \in k[x]$.*

Proof. (1) $\Rightarrow$ (2). Let K/k be a normal and finite extension. Then $K = k(u_1, \dots, u_n)$ for some $u_i \in K$. Let $g_i(x) \in k[x]$ be the minimal polynomial of u_i over k. We observe that each $g_i(x)$ is irreducible and has the root $u_i \in K$. Then by the normality of the extension, $g_i(x)$ has all its roots in K and hence K is the splitting field of the polynomial $g_1(x) \cdots g_n(x)$.

(2) $\Rightarrow$ (1), see [7, Thm. 5.1.5 and Prop. 5.2.1]. □

Definition 2.1.14. Let K/k be a field extension, the *Galois group* of the extension K/k is the following subgroup of $\mathrm{Aut}(K)$.

$$\mathrm{Gal}(K/k) := \{\sigma \in \mathrm{Aut(K)} : \sigma(a) = a, \text{ for each } a \in k\}.$$

Lemma 2.1.15. *Let $u \in K$ and $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial with coefficients in k. If u is a root of $p \in k[x]$. Then for each $\sigma \in \mathrm{Gal}(K/k)$ the element $\sigma(u) \in K$ is a root of p.*

Proof. Since σ is a homomorphism that fixed the elements of k and u is a root of p, we have the following identities

$$0 = \sigma(0) = \sigma\Big(\sum_{i=0}^{n} a_i u^i\Big) = \sum_{i=0}^{n} \sigma(a_i)\sigma(u)^i = \sum_{i=0}^{n} a_i \sigma(u)^i,$$

Hence the element $\sigma(u)$ is a root of p. □

A similar argument shows that if $K = k(\alpha_1, \dots, \alpha_n)$, then $\sigma \in \mathrm{Gal}(K/k)$ is uniquely determined by its values on $\alpha_1, \dots, \alpha_n$. Further the previous lemma is true for the minimal polynomial $P_u(x)$ of u over k, defined before.

Definition 2.1.16. Let k be a field, a polynomial $f \in k[x]$ is *separable* if it is nonconstants and its roots in a splitting field are all distinct.

Definition 2.1.17. Let K/k be an algebraic extension.

1. $\alpha \in K$ is separable over k if its minimal polynomial over k is separable.
2. K/k is a *separable extension* if every $\alpha \in K$ is separable over F.

Theorem 2.1.18. *If K is the splitting field of a separable polynomial in $k[x]$, then $|\mathrm{Gal}(K/k)| = [K : k]$.*

Proof. See [7, Thm. 6.2.1] □

Definition 2.1.19. Let K/k be a field extension and H a subgroup of the Galois group $\mathrm{Gal}(K/k)$, the *fixed subfield* of K for H is defined by

$$K^H := \{u \in K \ : \ \sigma(u) = u \text{ for each } \sigma \in H\}.$$

Theorem 2.1.20. *Let K/k be a finite extension. Then the following are equivalent:*

1. *K is the splitting field of a separable polynomial in $k[x]$.*
2. *k is the fixed field of $\mathrm{Gal}(K/k)$ acting on K.*
3. *K/k is a normal separable extension.*

Proof. See [7, Thm. 7.1.1] □

Definition 2.1.21. A field extension K/k is Galois if it is a finite extension satisfying any of the equivalent conditions of Theorem 2.1.20

Theorem 2.1.22. *Let K/k be a finite extension. Then:*

1. *$|\mathrm{Gal}(K/k)|$ divides $[K : k]$.*
2. *$|\mathrm{Gal}(K/k)| \leq [K : k]$.*
3. *K/k is Galois if and only if $|\mathrm{Gal}(K/k)| = [K : k]$.*

Proof. See [7, Thm. 7.1.5] □

Definition 2.1.23. Let K be a field and $\overline{K}$ be the separable algebraic closure of K. The *absolute Galois group* of K is the group $\mathrm{Gal}(\overline{K}/K)$. In other words, it is the group of all automorphism of the separable algebraic closure of K that fix K.

We observe that if K is algebraically closed then $\mathrm{Gal}(\overline{K}/K)$ is trivial.

Proposition 2.1.24. *If $\sigma \in \mathrm{Gal}(\overline{k}/k)$ and $\sigma(K) = K$, then K/k is Galois.*

Proof. If $\alpha \in K \setminus k$ and $P(x)$ be its minimal polynomial. Then the surjection $k[x] \to k[\alpha]$, defined by $x \mapsto \alpha$, induces an isomorphism $\phi_\alpha \colon k[x]/\langle P(x)\rangle \to k(\alpha)$. Given another root β of $P(x)$ one shows that the isomorphism $\phi_\beta \circ \phi_\alpha^{-1} \colon k(\alpha) \to k(\beta)$ lifts to an element $\sigma \in Gal(\overline{k}/k)$. Since by hypothesis $\sigma(K) = K$, the restriction $\sigma|_K$ is an element of $Gal(K/k)$. It follows that the fixed field of $Gal(K/k)$ is k and thus the extension K/k is Galois. □

2.2 Hyperelliptic curves

This section is based in the books [11, §5.2.1], [21] and [8].

2.2.1 Weighted projective spaces

Let $a := (a_0, \dots, a_n)$ be an $(n+1)$-tuple of positive integers. We say that a is *well-formed* if the greatest common divisor of any n entries of a of them is 1.

Definition 2.2.1. Let a be a well-formed $(n+1)$-tuple of positive integers. Define the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}\backslash\{0\}$ by $\lambda \cdot (x_0, \dots, x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n)$ and denote by $\sim_a$ the corresponding equivalence relation on $\mathbb{C}^{n+1}\backslash\{0\}$. The quotient space

$$\mathbb{P}(a_0, \dots, a_n) := (\mathbb{C}^{n+1}\backslash\{0\}) / \sim_a$$

is the *weighted projective space* with weights $(a_0, \dots, a_n)$.

If $a = (a_0, \dots, a_n) = (1, \dots, 1)$ we obtain that $\mathbb{P}(a)$ is the usual projective space $\mathbb{P}^n$. The reason for considering only well formed $(n+1)$-tuples of positive integers is to avoid repetitions, like e.g. $\mathbb{P}(1,2,2) \simeq \mathbb{P}(1,1,1)$ via the map $[x_0 : x_1 : x_2] \mapsto [x_0^2 : x_1 : x_2]$.

Denote by $[x_0 : \cdots : x_n]$ the class of the element $(x_0, \dots, x_n) \in \mathbb{C}^{n+1}\backslash\{0\}$. We sometimes will use the notation $\mathbb{P}(a)$ for $\mathbb{P}(a_0, \dots, a_n)$. The coordinate ring of $\mathbb{P}(a)$ is the graded ring $\mathbb{C}[x_0, \dots, x_n]$ with the grading $\deg(x_i) = a_i$, for any $i \in \{0, \dots, n\}$. A *homogeneous polynomial* f is a linear combination, with complex coefficients, of monomials of the same degree. This degree is called the *degree of the polynomial* f. In particular a homogeneous polynomial of degree d has the form

$$f = \sum_{a_0 i_0 + \cdots + a_n i_n = d} b_{i_0 \cdots i_n} x_0^{i_0} \cdots x_n^{i_n}$$

with coefficients $b_{i_0 \cdots i_n} \in \mathbb{C}$ and non-negative exponents. For any $0 \le i \le n$ denote by

$$U_i(a) := \{[x_0 : \cdots : x_i : \cdots : x_n] : x_i \neq 0\}$$

the i-th *standard affine patch* of $\mathbb{P}(a)$. Observe that $\mathbb{P}^n(a) = \bigcup_{i=0}^n U_i(a)$. The intersection $\bigcap_{i=0}^n U_i(a)$ can be described as follows. It is the affine algebraic variety whose coordinate ring is $\mathbb{C}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]_0$, the subring of homogeneous Laurent polynomials of degree 0. Since a subgroup of a free abelian group if free abelian we have an exact sequence

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{P} \mathbb{Z}^{n+1} \xrightarrow{\cdot a} \mathbb{Z} \longrightarrow 0,$$

where P is given by an $(n+1)\times n$ matrix with integer coefficients whose image is the kernel of the scalar product with a. The above exact sequence provides us with an isomorphism

$$\mathbb{C}[u_1^{\pm 1},\dots,u_n^{\pm n}] \to \mathbb{C}[x_0^{\pm 1},\dots,x_n^{\pm 1}]_0 \qquad u_i \mapsto x^{P(e_i)}.$$

In particular the above map induces an isomorphism $\bigcap_{i=0}^n U_i(a) \to (\mathbb{C}^*)^n$, showing that $\mathbb{P}(a)$ has dimension n and it is birational to $\mathbb{P}^n$. In general one has to compute the matrix P to move to torus coordinates. In the following lemma we consider the special case when one of the coordinates of a is equal to 1. In this case we get local coordinates in a simple form.

Lemma 2.2.2. *If $a_0 = 1$ then the isomorphism is*

$$\mathbb{C}[u_1^{\pm 1},\dots,u_n^{\pm n}] \to \mathbb{C}[x_0^{\pm 1},\dots,x_n^{\pm 1}]_0 \qquad u_i \mapsto x_i/x_0^{a_i}.$$

Moreover the corresponding isomorphism of algebraic varieties extends to an isomorphism

$$\varphi_0\colon U_0(a) \to \mathbb{C}^n \qquad [x_0 : \cdots : x_n] \mapsto \left(\frac{x_1}{x_0^{a_1}},\dots,\frac{x_n}{x_0^{a_n}}\right).$$

Proof. It suffice to show that the inverse of φ_0 is $\nu_0\colon (x_1,\dots,x_n) \mapsto [1 : x_1 : \cdots : x_n]$. Indeed $\varphi_0 \circ \nu_0$ is clearly the identity and $(\nu_0 \circ \varphi_0)([x_0 : \cdots : x_n]) = [1 : x_1/x_0^{a_1} : \cdots : x_n/x_0^{a_n}] = [x_0 : \cdots : x_n]$, where the last equality is obtained by acting with $\lambda = x_0$ and recalling that $a_0 = 1$ by hypothesis. □

Remark 2.2.3. In this work appear weighted projective spaces of the form $\mathbb{P}(a)$, where a is the $(n+1)$-tuple $(1,\dots,1,a_n)$. In this case the affine patches $U_i(a)$ are isomorphic to $\mathbb{C}^n$ for $i \in \{0,\dots,n-1\}$ and covers all $\mathbb{P}(a)$ except for the point $[0 : \cdots : 0 : 1]$ which is the only singular one. To see this observe that the coordinate ring of the affine chart $U_n(a)$ is isomorphic to the invariant ring $\mathbb{C}[x_0,\dots,x_{n-1}]_0$, where each variable has degree $[1] \in \mathbb{Z}/a_n\mathbb{Z}$. This ring is generated by the monomials of degree a_n and in particular it has a singularity at $x_0 = \cdots = x_{n-1} = 0$. For example, when $n = 2$ and $a_3 = k$ we have

$$\mathbb{C}[x_0,x_1]_0 \simeq \mathbb{C}[x_0^k, x_0^{k-1}x_1,\dots,x_1^k] \simeq \mathbb{C}[z_0,\dots,z_k]/\langle z_iz_j - z_{i+1}z_{j-1} \,:\, 0 \le i < j \le k\rangle,$$

where the last is the coordinate ring of the affine cone over the rational normal curve of degree k. The minimal resolution of this singularity has a unique irreducible exceptional curve E with $E^2 = -k$.

Next we define what is an hyperelliptic curve and the properties that we use in this work.

2.2.2 Hyperelliptic curve

Definition 2.2.4. A *hyperelliptic curve* of genus $g \geq 2$ over a field K not of characteristic 2 is the subvariety of $\mathbb{P}(1, 1, g+1)$ defined by an equation of the form $w^2 = F(u, v)$, where $F \in K[u, v]$ is a homogeneous polynomial of degree $2g+2$ and is not divisible by the square of a homogeneous polynomial of positive degree.

The standard affine patches of C are the intersections of C with the affine patches of $\mathbb{P}(1, 1, g+1)$, i.e., $C \cap \{[1 : v : w] \subset \mathbb{P}(1, 1, g+1)\}$ and $C \cap \{[u : 1 : w] \subset \mathbb{P}(1, 1, g+1)\}$, they are affine plane curves with equations

$$w^2 = F(1, v) \quad \text{and} \quad w^2 = F(u, 1)$$

respectively.

Is usual think the curve C how the hyperelliptic curve defined by $w^2 = f(v)$, where $f(v) = F(1, v)$, but will always consider C as a curve in its corresponding weighted projective space $\mathbb{P}(1, 1, g+1)$.

Let

$$F(u, v) = \alpha_{2g+2}v^{2g+2} + \alpha_{2g+1}v^{2g+1}u + \cdots + \alpha_1 vu^{2g+1} + \alpha_0 u^{2g+2} \tag{2.1}$$

be the homogeneous polynomial of degree $2g+2$, where the $\alpha_i \in \mathbb{C}$ are its coefficient. We have that $f(v) = F(1, v)$ can have degree $2g+1$ or degree $2g+2$, so we can reconstruct $F(u, v)$ from $f(v)$.

The points $[u : v : w] \in C$ such that $u \neq 0$ are of the form $[1 : v : w]$, where $w^2 = f(v)$ which corresponds with the affine point (v, w). The remains points of C are called *points at infinity*. Which are obtained by setting $u = 0$ and $v = 1$ in the equation $w^2 = F(u, v)$ (see Equation (2.1)), i.e., $w^2 = \alpha_{2g+2}$. If $\alpha_{2g+2} = 0$, which means that $\deg f = 2g+1$, then there is one such point, namely $[0 : 1 : 0]$. If $\alpha_{2g+2} \neq 0$, i.e., $\deg f$ is $2g+2$, then there are two points at infinity, namely $[0 : 1 : s]$ and $[0 : 1 : -s]$. Note that the point $[0 : 0 : 1]$ is never a point on an hyperelliptic curve C. Recall that the point $[0 : 0 : 1]$ is a singular point in $\mathbb{P}(1, 1, g+1)$.

2.2.3 Hyperelliptic involution

Consider the map defined on the curve C

$$\pi : C \to \mathbb{P}^1, \qquad [u : v : w] \mapsto [u : v]$$

and note that the map π is well defined since the point $[0 : 0 : 1] \notin C$ and is a surjective morphism. The morphism π is clearly of degree two.

In fact, an alternative way of defining a hyperelliptic curve of genus g is: A nonsingular projective curve C of genus $g > 1$ that admits a degree two map $\pi : C \to \mathbb{P}^1$.

Every hyperelliptic curve C has a no-trivial automorphism, that we denote by $\imath$ and is defined as

$$\imath : C \to C, \qquad [u : v : w] \mapsto [u : v : -w].$$

The map $\imath$ is an involution of the curve C, where its fixed points are the $2g + 2$ points $[u : v : 0]$, with $[u : v] \in \mathbb{P}^1$ a root of the homogeneous polynomial $F(u, v)$.

Definition 2.2.5. The automorphism $\imath$ and the fixed points of C under $\imath$ defined above are called, the *hyperelliptic involution* of C and the *Weirstrass points* of C respectively.

2.2.4 Rational points of the hyperelliptic curve C

The rational points of the hyperelliptic curve C are the points of the set:

$$C(\mathbb{Q}) = \{[u : v : w] \in \mathbb{P}(1, 1, g + 1)(\mathbb{Q}) : w^2 = F(u, v)\}.$$

Example 2.2.6.

1. The curve C defined by the equation $y^2 = x^5 + 1$, has genus $g = 2$ and its projective form is $y^2 = x^5z + z^6$, its point at infinity is $\infty = [1 : 0 : 0]$ and some rational points of C are $[0 : 1 : 1]$, $[0 : -1 : 1]$, $[-1 : 0 : 1]$, also ∞. One can asked if these are all the rational points of C. But in general this question is not easy.

2. In Chapter 4 of this work we will see that the curve C defined by the equation
$$C : \quad y^2 = \prod_{i=-4}^{4} (x - i)$$
has exactly 10 rational points (see Theorem 4.3.2).

From the paper [21] we extract some important results for the topic of rational points of curves.

Theorem 2.2.7 (Faltings)**.** *If C is a smooth, projective and absolutely irreducible curve over $\mathbb{Q}$ of genus $g \geq 2$, then $C(\mathbb{Q})$ is finite.*

Theorem 2.2.8. *Let C be a smooth, projective and geometrically irreducible curve over $\mathbb{Q}$, of genus g and with Jacobian variety J. Assume that the rank r of $J(\mathbb{Q})$ is strictly less than g. Then $C(\mathbb{Q})$ is finite.*

More precisely, let p be an odd prime of good reduction for C. Then

$$\#C(\mathbb{Q}) \leq \#\bar{C}(\mathbb{F}_p) + 2g - 2 + \left\lfloor \frac{2g-2}{p-2} \right\rfloor.$$

Using the fact that $r < g$ another bound for the rational points of C is

$$\#C(\mathbb{Q}) \leq \#\bar{C}(\mathbb{F}_p) + 2r + \left\lfloor \frac{2r}{p-2} \right\rfloor,$$

where $\bar{C}$ is the reduction of C modulo the prime p of good reduction.

3 Generalized Büchi surfaces

In this chapter we study a special class of complex projective algebraic surfaces in $\mathbb{P}^n$, with $n \geq 3$ integer, which are complete intersection of diagonals quadrics. These surfaces are called *Generalized Büchi Surfaces*, GBS for short. In Section 3.1 we define the GBS and its lines, also we state Theorem 3.1.4 which describes basic aspects of a GBS (smoothness, irreducibility, type of surface) and its configuration of lines, paying special attention to the case of an odd n. In the same section we show the smoothness and irreducibility of the GBS. Section 3.2 is dedicated to study of the lines of the GBS, also we state Theorem 3.2.1, which gives a bijection between $n+1$ unordered points of $\mathbb{P}^1$ and the GBS modulo projectivities. If n is odd the bijection is between the moduli space of genus $\frac{1}{2}(n-1)$ curves and the GBS modulo projectivities. Finally in Section 3.3 we prove Theorems 3.1.4 and 3.2.1.

3.1 Definition, smoothness and irreducibility

Definition 3.1.1. Let $n \geq 3$ be an integer, and let $\alpha_0, \ldots, \alpha_n$ be $n+1$ distinct complex numbers. Let $\alpha := \{[1 : \alpha_0], \ldots, [1 : \alpha_n]\} \subseteq \mathbb{P}^1$ be an ordered set of cardinality $n+1$. The *Generalized Büchi surface*, GBS for short, $S_n(\alpha)$ is the zero locus in $\mathbb{P}^n$ of the $n-2$ diagonal quadrics of equations

$$x_i^2 - \beta_2^i x_2^2 - \beta_1^i x_1^2 - \beta_0^i x_0^2 = 0,$$

where $i \in \{3, \ldots, n\}$ and

$$\beta_0^i = \frac{(\alpha_1 - \alpha_i)(\alpha_2 - \alpha_i)}{(\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)}, \qquad \beta_1^i = -\frac{(\alpha_0 - \alpha_i)(\alpha_2 - \alpha_i)}{(\alpha_0 - \alpha_1)(\alpha_1 - \alpha_2)}, \qquad \beta_2^i = \frac{(\alpha_0 - \alpha_i)(\alpha_1 - \alpha_i)}{(\alpha_0 - \alpha_2)(\alpha_1 - \alpha_2)}.$$

Definition 3.1.2. For any choice of signs the image of the morphism $\mathbb{A}^1 \to \mathbb{P}^n$ defined by

$$t \mapsto [\pm(t - \alpha_0) : \cdots : \pm(t - \alpha_n)] \tag{3.1}$$

is a line of $S_n(\alpha)$. There are 2^n such lines which are named the *trivial lines* of $S_n(\alpha)$.

Remark 3.1.3. Observe that if we send one of the $n+1$ points, say α_0 to ∞ the above lines seem to collapse to a point. A way to compute the correct limit positions for the lines is to use the action of the general projective linear group. More specifically, by applying a projectivity one maps the above lines to $t \mapsto [\pm\frac{1}{\alpha_0}(t-\alpha_0) : \cdots : \pm(t-\alpha_n)]$. Then, as α_0 goes to ∞ the set of lines goes to $t \mapsto [1 : \pm(t-\alpha_1) : \cdots : \pm(t-\alpha_n)]$. The three coefficient $\beta_0^i, \beta_1^i, \beta_2^i$ go, respectively, to

$$\beta_0^i = (\alpha_1 - \alpha_i)(\alpha_2 - \alpha_i), \qquad \beta_1^i = -\frac{\alpha_2 - \alpha_i}{\alpha_1 - \alpha_2}, \qquad \beta_2^i = \frac{\alpha_1 - \alpha_i}{\alpha_1 - \alpha_2},$$

where the first one takes this form because β_0^i is replaced with $\alpha_0^2\beta_0^i$ after applying the projecivity.

Theorem 3.1.4. *$S_n(\alpha)$ is a smooth irreducible surface which is rational for $n = 4$, $K3$ for $n = 5$ and of general type for $n \geq 6$. The only lines in $S_n(\alpha)$ are the trivial lines. The group of sign changes acts transitively on the set of lines of $S_n(\alpha)$ and each such line meet exactly other $n+1$ lines along a subset of points which is projectively equivalent to α. Moreover if n is odd then the intersection graph of the lines is bipartite of type $(2^{n-1}, 2^{n-1})$.*

Proposition 3.1.5. *$S_n(\alpha)$ is the zero locus of the ideal generated by the quadrics which vanish along the trivial lines.*

Proof. We call $\mathcal{L}$ the set of the trivial lines of $S_n(\alpha)$. First we show that the quadratic part $\mathcal{I}(\mathcal{L})_2$ of the ideal $\mathcal{I}(\mathcal{L})$ is generated by the following homogeneous polynomials

$$\mathcal{Q}_{i-2} := x_i^2 - \beta_2^i x_2^2 - \beta_1^i x_1^2 - \beta_0^i x_0^2,$$

with $\beta_0^i, \beta_1^i, \beta_2^i$ as in Definition 3.1.1 and $i \in \{3, \ldots, n\}$. Let $\langle \mathcal{Q}_1, ..., \mathcal{Q}_{n-2} \rangle$ be the linear span of the above quadratic polynomials. The inclusion $\langle \mathcal{Q}_1, ..., \mathcal{Q}_{n-2} \rangle \subseteq \mathcal{I}(\mathcal{L})_2$ is clear because each polynomial $\mathcal{Q}_i$ vanishes at $\mathcal{L}$. We now prove the opposite inclusion. First of all let $\sigma_k \colon \mathbb{P}^n \to \mathbb{P}^n$ be the involution which exchanges the sign of the coordinate x_k. Observe that $p \in \mathcal{L}$ if and only if $\sigma_k(p) \in \mathcal{L}$ for any $0 \leq k \leq n$ by Definition 3.1.2. Now assume that all the coordinates of $p \in \mathcal{L}$ are non-zero. Given $Q \in \mathcal{I}(\mathcal{L})_2$ we have

$$Q = x_k^2 + x_k l_k + g_k, \qquad l_k, g_k \in \mathbb{C}[x_0, ..., \hat{x_k}, ..., x_n].$$

Since $Q(p) = Q(\sigma_k(p)) = 0$, we deduce that $l_k(p) = 0$. Since this holds for any such general $p \in \mathcal{L}$ and $\mathcal{L}$ is not degenerate, i.e. it is not contained in a linear subspace, we conclude that l_k vanishes identically. Repeating the argument for each k, we have that Q is a degree two diagonal homogeneous polynomial, let us say $Q = \sum_{i=0}^n C_i x_i^2$. Then

$$Q - \sum_{i=1}^{n-2} C_i \mathcal{Q}_i = \gamma_0 x_0^2 + \gamma_1 x_1^2 + \gamma_2 x_2^2 \in \mathcal{I}(\mathcal{L})_2.$$

By evaluating the above polynomial at any line of $\mathcal{L}$ one gets a linear combination of the following three polynomials $(t-\alpha_0)^2, (t-\alpha_1)^2, (t-\alpha_2)^2$ of $\mathbb{C}[t]$. Since these polynomials are linearly independent we conclude that $\gamma_0 = \gamma_1 = \gamma_2 = 0$. Therefore $Q \in \langle \mathcal{Q}_1, ..., \mathcal{Q}_{n-2} \rangle$. In conclusion $\mathcal{I}(\mathcal{L})_2 = \langle \mathcal{Q}_1, ..., \mathcal{Q}_{n-2} \rangle$ and so $S_n(\alpha) = \mathcal{V}(\langle \mathcal{I}(\mathcal{L})_2 \rangle)$. □

Corollary 3.1.6. *Let α be an ordered $(n+1)$-tuple of distinct points of $\mathbb{P}^1$ and let α' be a permutation of α. Then $S_n(\alpha)$ is projectively equivalent to $S_n(\alpha')$.*

Proof. Let $\sigma \in \mathfrak{S}_{n+1}$ be the permutation such that $\alpha'_i = \alpha_{\sigma(i)}$. Let $\phi \in \mathrm{PGL}(n+1, \mathbb{C})$ be the projectivity defined by $\phi([x_0 : \cdots : x_n]) = [x_{\sigma(0)} : \cdots : x_{\sigma(n)}]$. Then $\phi(S_n(\alpha')) = S_n(\alpha)$. □

Proposition 3.1.7. *$S_n(\alpha)$ is an irreducible smooth surface in $\mathbb{P}^n$.*

Proof. We have that $S_n(\alpha)$ is the complete intersection of the $n-2$ diagonal quadrics $\mathcal{Q}_i$ in $\mathbb{P}^n$, that is

$$\mathcal{Q}_1 = \cdots = \mathcal{Q}_{n-2} = 0.$$

Thus $S_n(\alpha)$ is a surface in $\mathbb{P}^n$.

To prove the smoothness of $S_n(\alpha)$ we will use the jacobian criterion (see Corollary 1.1.4). The $(n-2) \times (n+1)$ Jacobian matrix $\left(\dfrac{\partial \mathcal{Q}_{i-2}}{\partial x_l}\right)_{i,l}$, for $i \in \{3, ..., n\}$ and $l \in \{0, ..., n\}$, is

$$2\begin{pmatrix} -\beta_0^3 x_0 & -\beta_1^3 x_1 & -\beta_2^3 x_2 & x_3 & & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ -\beta_0^i x_0 & -\beta_1^i x_1 & -\beta_2^i x_2 & & & x_i & & \\ \vdots & \vdots & \vdots & & & & \ddots & \\ -\beta_0^n x_0 & -\beta_1^n x_1 & -\beta_2^n x_2 & & & & & x_n \end{pmatrix}. \tag{3.2}$$

We thus have to show that the above Jacobian matrix has maximal rank. First of all we claim that a point of a General Büchi surface has at most two coordinates equal to zero. Indeed assume that three coordinates are zero. From the equations of a General Büchi surface, given in Definition 3.1.1, it immediately follows that these coordinates cannot be x_0, x_1, x_2, otherwise all the coordinates would vanish. Similarly it cannot be that two of the three vanishing coordinates are among the first three. Suppose now that only one of the vanishing coordinates is among the first three, let us say x_0, and the remaining two are x_i, x_j. Then

$$\beta_1^i x_1^2 + \beta_2^i x_2^2 = 0 \qquad \text{and} \qquad \beta_1^j x_1^2 + \beta_2^j x_2^2 = 0.$$

Since the determinant of the 2×2 matrix is (see Magma Program 4.3)

$$\begin{vmatrix} \beta_1^i & \beta_2^i \\ \beta_1^j & \beta_2^j \end{vmatrix} = \frac{(\alpha_i - \alpha_j)(\alpha_0 - \alpha_j)(\alpha_0 - \alpha_i)}{(\alpha_1 - \alpha_2)(\alpha_0 - \alpha_2)(\alpha_0 - \alpha_1)} \neq 0,$$

the only solution of the above equations is $x_1 = x_2 = 0$, so that again the first three variables would vanish giving a contradiction. Finally if none of the three vanishing variables x_i, x_j, x_k is among the first three then

$$\beta_0^i x_0^2 + \beta_1^i x_1^2 + \beta_2^i x_2^2 = 0, \quad \beta_0^j x_0^2 + \beta_1^j x_1^2 + \beta_2^j x_2^2 = 0 \quad \text{and} \quad \beta_0^k x_0^2 + \beta_1^k x_1^2 + \beta_2^k x_2^2 = 0.$$

Since the determinant of the 3×3 matrix is (see Magma Program 4.3)

$$\begin{vmatrix} \beta_0^i & \beta_1^i & \beta_2^i \\ \beta_0^j & \beta_1^j & \beta_2^j \\ \beta_0^k & \beta_1^k & \beta_2^k \end{vmatrix} = \frac{(\alpha_j - \alpha_k)(\alpha_i - \alpha_k)(\alpha_i - \alpha_j)}{(\alpha_1 - \alpha_2)(\alpha_0 - \alpha_2)(\alpha_0 - \alpha_1)} \neq 0,$$

the only solution of the above equations is again $x_0 = x_1 = x_2 = 0$, a contradiction. The claim is proved. As a consequence of the claim one can always find $n-2$ columns of the above Jacobian matrix whose determinant is non-zero, so that the matrix has maximal rank.

Now we will show the irreducibility of $S_n(\alpha)$. We have that for each integer $n \geq 4$ the surface $S_n(\alpha) \subset \mathbb{P}^n$ is the double cover of $S_{n-1}(\alpha) \subset \mathbb{P}^{n-1}$ with covering map

$$S_n(\alpha) \to S_{n-1}(\alpha), \qquad [x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_{n-1}],$$

where the branch divisor B is the curve of $S_{n-1}(\alpha) \subset \mathbb{P}^{n-1}$ given by the equations

$$\beta_2^n x_2^2 + \beta_1^n x_1^2 + \beta_0^n x_0^2 = \mathcal{Q}_{n-3} = \cdots = \mathcal{Q}_1 = 0.$$

In order to prove that $S_n(\alpha)$ is irreducible it suffice to show that a Zariski open subset of $S_n(\alpha)$ is irreducible or equivalently that the covering is non-trivial in the neighbourhood of some point $P \in B$. To check this it is enough to show that B has a smooth point. Let $P \in B$ be a point such that $x_3 \cdots x_{n-1} \neq 0$ (just take a point $[x_0 : x_1 : x_2]$ on the conic $\beta_2^n x_2^2 + \beta_1^n x_1^2 + \beta_0^n x_0^2 = 0$ with $x_0 x_1 x_2 \neq 0$ which lies outside the union of the conics $\beta_2^i x_2^2 + \beta_1^i x_1^2 + \beta_0^i x_0^2 = 0$ for any $i < n$). Now apply the jacobian criterion to B in the point P, we have that the $(n-2) \times n$ Jacobian matrix is

$$2 \begin{pmatrix} -\beta_0^3 x_0 & -\beta_1^3 x_1 & -\beta_2^3 x_2 & x_3 & & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ -\beta_0^i x_0 & -\beta_1^i x_1 & -\beta_2^i x_2 & & & x_i & \\ \vdots & \vdots & \vdots & & & & \ddots \\ -\beta_0^{n-1} x_0 & -\beta_1^{n-1} x_1 & -\beta_2^{n-1} x_2 & & & & x_{n-1} \\ -\beta_0^n x_0 & -\beta_1^n x_1 & -\beta_2^n x_2 & 0 & & \cdots & 0 \end{pmatrix}. \tag{3.3}$$

Observe that the above matrix has maximal rank. Indeed the first $n-3$ rows are linearly independent because the same holds for the first $n-3$ rows of matrix (3.2). Moreover the last row is not in the row space of the first $n-3$ due to the condition $x_3 \cdots x_{n-1} \neq 0$. One concludes that B is smooth at P. $\square$

3.2 Lines

The previous section shows that there is a morphism

$$(\mathbb{P}^1)^{n+1} \setminus \text{diagonals} \to \text{GBS} \qquad \alpha \mapsto S_n(\alpha),$$

which maps an $(n+1)$-tuple of distinct points to the corresponding General Büchi surface. In this section we show that two projectively equivalent unordered $(n+1)$-tuples α, α' gives projectively equivalent General Büchi surfaces $S_n(\alpha)$ and $S_n(\alpha')$. As a consequence the above morphism descends to a morphism

$$\Phi_n \colon \mathcal{M}_{0,n+1}/\mathfrak{S}_{n+1} \to \mathcal{GBS}_n := \mathrm{GBS}/\mathrm{PGL}(n+1,\mathbb{C}),$$

where $\mathcal{M}_{0,n+1}$ is the quotient of $(\mathbb{P}^1)^{n+1} \setminus$ diagonals by the natural action of $\mathrm{PGL}(2,\mathbb{C})$.

Theorem 3.2.1. *For any positive integer $n > 3$ the morphism Φ_n is a bijection. In particular if n is odd the map Φ_n gives a bijection from the moduli space of genus $\frac{1}{2}(n-1)$ curves to $\mathcal{GBS}_n$.*

We begin by describing the lines of a General Büchi surface and will show how these determine the surface up to projectivities. This will be the main step in establishing Theorem 3.2.1.

Proposition 3.2.2. *Let α and α' be two projectively equivalent ordered $(n+1)$-tuples of distinct points of $\mathbb{P}^1$. Then the surfaces $S_n(\alpha)$ and $S_n(\alpha')$ are projectively equivalent.*

Proof. We claim that the corresponding unions of lines $\mathcal{L}$, $\mathcal{L}'$ of $\mathbb{P}^n$, given as in Definition 3.1.2, are projectively equivalent. Recall that projectivities of $\mathbb{P}^1$ are Möbius transformations and the group of Möbius transformations is generated by the maps

$$z \mapsto z+u, \qquad z \mapsto uz \qquad \text{and} \qquad z \mapsto \frac{1}{z},$$

where z is the complex coordinate and $u \in \mathbb{C}$. The first transformation maps $\mathcal{L}$ to the union of lines parametrized by $t \mapsto [\pm(t-u-\alpha_0) : \cdots : \pm(t-u-\alpha_n)]$. The change of coordinates $t \mapsto t+u$ in the parametrization shows that $\mathcal{L}$ stays unchanged. The second transformation maps $\mathcal{L}$ to the union of lines parametrized by $t \mapsto [\pm(t-u\alpha_0) : \cdots : \pm(t-u\alpha_n)]$. The change of coordinates $t \mapsto ut$ in the parametrization shows that $\mathcal{L}$ stays unchanged. Finally the third transformation maps $\mathcal{L}$ to the union of lines parametrized by

$$t \mapsto \Big[\pm\Big(t-\frac{1}{\alpha_0}\Big) : \cdots : \pm\Big(t-\frac{1}{\alpha_n}\Big)\Big].$$

Then reparameterizing the lines via $t \mapsto t^{-1}$ we get

$$t \mapsto \Big[\pm\Big(\frac{1}{t}-\frac{1}{\alpha_0}\Big) : \cdots : \pm\Big(\frac{1}{t}-\frac{1}{\alpha_n}\Big)\Big] = \Big[\pm\frac{(t-\alpha_0)}{\alpha_0} : \cdots : \pm\frac{(t-\alpha_n)}{\alpha_n}\Big].$$

Then the projectivity $\mathbb{P}^n \to \mathbb{P}^n$, $[x_0 : \cdots : x_n] \mapsto [\alpha_0 x_0 : \cdots : \alpha_n x_n]$ maps this set of lines back to $\mathcal{L}$. So the claim is proved. By Proposition 3.1.5

$$S_n(\alpha) = \mathcal{V}(\mathcal{I}(\mathcal{L})_2) \qquad \text{and} \qquad S_n(\alpha') = \mathcal{V}(\mathcal{I}(\mathcal{L}')_2).$$

The statement follows. □

Proposition 3.2.3. *Let $n \geq 4$ be an integer. The only lines of $S_n(\alpha)$ are the 2^n trivial lines.*

Proof. The proof is by induction on n. The case $n = 4$ is a consequence of the fact that $S_4(\alpha) \subset \mathbb{P}^4$ is a del Pezzo surface of degree four which thus contains exactly $2^4 = 16$ lines [16], the trivial ones. Now we assume that statement holds for $n \leq k$ and we prove it for $n = k + 1$. Since $S_{k+1}(\alpha)$ is a double cover of $S_k(\alpha)$ and the covering map $[x_0 : \cdots : x_{k+1}] \mapsto [x_0 : \cdots : x_k]$ is lineal, each line of $S_{k+1}(\alpha)$ is mapped to a line of $S_k(\alpha)$. By induction hypothesis $S_k(\alpha)$ contains exactly 2^k lines, so that $S_{k+1}(\alpha)$ contains at most 2^{k+1} lines. Since we know $S_{k+1}(\alpha)$ to contain 2^{k+1} trivial lines the statement follows. □

Proposition 3.2.4. *Each line of $S_n(\alpha)$ meets exactly $n + 1$ other lines along a subset of points which is projectively equivalent to α.*

Proof. Since the lines of $S_n(\alpha)$ form one orbit with respect to the group of sign changes, it suffice to prove the statement for the line L parametrized by $t \mapsto [t - \alpha_0 : \cdots : t - \alpha_n]$. Let L' be another trivial line parametrized by $u \mapsto [\varepsilon_0(u - \alpha_0) : \cdots : \varepsilon_n(u - \alpha_n)]$, where $\varepsilon_i \in \{-1, 1\}$ for any i. Then $L \cap L'$ is non empty if and only if the following matrix has rank one for some values of $t, u \in \mathbb{P}^1$

$$\begin{pmatrix} t - \alpha_0 & \ldots & t - \alpha_n \\ \varepsilon_0(u - \alpha_0) & \ldots & \varepsilon_n(u - \alpha_n) \end{pmatrix}.$$

Assume now that the intersection is non-empty and let i, j be two indices such that $\varepsilon_i = \varepsilon_j$. The corresponding 2×2 minor is $\varepsilon_i(t-\alpha_i)(u-\alpha_j) - \varepsilon_i(t-\alpha_j)(u-\alpha_i) = \varepsilon_i(t - u)(\alpha_i - \alpha_j)$, so we deduce $u = t$. Applying this substitution, all the 2×2 minors with $\varepsilon_i = \varepsilon_j$ vanish, while each minor with $\varepsilon_i = -\varepsilon_j$ is equal to $\varepsilon_j(t-\alpha_i)(t-\alpha_j) - \varepsilon_i(t-\alpha_j)(t-\alpha_i) = 2\varepsilon_j(t-\alpha_i)(t-\alpha_j)$. Thus in this last case we conclude $t \in \{\alpha_i, \alpha_j\}$. Assume $t = \alpha_i$, then the i-th column of the above matrix is the zero vector and, from the above discussion, we conclude that $\varepsilon_j = \varepsilon_k$ for any j, k different from i. The intersection point is $[\alpha_i - \alpha_0 : \cdots : \alpha_i - \alpha_n]$. This proves that L intersects exactly $n + 1$ trivial lines. By applying the morphism

$$L \to \mathbb{P}^1 \qquad [z_0 : \cdots : z_n] \mapsto [z_0 - z_1 : \alpha_1 z_0 - \alpha_0 z_1],$$

to the above intersection point we get $[1 : \alpha_i]$, which proves the second statement. □

Corollary 3.2.5. *Two lines of $S_n(\alpha)$ intersect if and only if they have only one coordinate with a different sign.*

Proposition 3.2.6. *Let $n \geq 5$ be an odd integer, then the intersection graph of the lines of $S_n(\alpha)$ is bipartite of type $(2^{n-1}, 2^{n-1})$.*

Proof. Denote by $[n + 1]$ the set $\{0, \ldots, n\}$. For each subset $\mathcal{P} \subset [n + 1]$, denote by $\sigma_{\mathcal{P}}$ the automorphism of $S_n(\alpha)$ that changes the sign of all the variables with indices in $\mathcal{P}$, we observe that $\sigma_{\mathcal{P}} = \sigma_{[n+1]\setminus\mathcal{P}}$. Then each line of $S_n(\alpha)$ can de identified with a subset $\mathcal{P}$ of $[n+1]$. Also we note that the complement of $\mathcal{P}$ is identified with the same line. Let us denote by $\mathfrak{P}$ the power set of $[n+1]$ and let

$$\mathfrak{P}_i := \{\mathcal{P} \in \mathfrak{P} : |\mathcal{P}| \equiv i \pmod 2\}.$$

The partition $\mathfrak{P} = \mathfrak{P}_0 \cup \mathfrak{P}_1$ induces a partition $\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1$ of the set of lines of $S_n(\alpha)$ because $\mathcal{P} \in \mathfrak{P}_i$ if and only if $[n+1] \setminus \mathcal{P} \in \mathfrak{P}_i$, being n odd. A consequence of Corollary 3.2.5, is that given $\mathcal{P}, \mathcal{P}' \in \mathfrak{P}$ distinct subsets, the corresponding lines, say $L_{\mathcal{P}}$ and $L_{\mathcal{P}'}$ intersect if and only if the automorphism $\sigma_{\mathcal{P}} \circ \sigma_{\mathcal{P}'}$ is the sign change of a single variable, equivalently if, up to relabelling and taking complements, $\mathcal{P}' \subset \mathcal{P}$ and $|\mathcal{P} \setminus \mathcal{P}'| = 1$. From this it follows that $L_{\mathcal{P}}$ and $L_{\mathcal{P}'}$ can not belong to $\mathcal{L}_0$ or $\mathcal{L}_1$ at the same time, since in both cases the lines $L_{\mathcal{P}}$ and $L_{\mathcal{P}'}$ have opposite signs in at least two coordinates. This shows that the graph is bipartite.

To show that the graph is of type $(2^{n-1}, 2^{n-1})$, we observe that the identity

$$0 = (1-1)^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^i$$

implies that $|\mathfrak{P}_0| = |\mathfrak{P}_1|$. Since the total number of lines is 2^n we are done. □

Proposition 3.2.7. *Let $n > 5$ be an integer and let Z be a surface isomorphic to a General Büchi surface $S_n(\alpha)$. Then Z is projectively equivalent to $S_n(\alpha)$ and in particular the two surfaces have the same configuration of lines.*

Proof. By the adjunction formula a canonical divisor of $S_n(\alpha)$ is $K = (n-5)H$, where H is a hyperplane section. Since $S_n(\alpha)$ is a complete intersection, its Picard group is torsion-free by [9, Thm. 1.8, pag. 49] or [3, Thm. B] and the same holds for Z. In particular there is a unique divisor class $[D]$ on Z such that $K_Z := (n-5)D$ is a canonical divisor. An isomorphism $f \colon S_n(\alpha) \to Z$ maps $[K]$ to $[K_Z]$ and thus, by the above unicity, must map $[D]$ to $[H]$. It follows that $f^*(D)$ is linearly equivalent to H, so that there exists a rational function h on $S_n(\alpha)$ with $\mathrm{div}(h) = f^*(D) - H$. The following map

$$H^0(Z, D) \to H^0(S_n(\alpha), H) \qquad \gamma \mapsto h \cdot (\gamma \circ f)$$

is linear so it induces a projectivity between Z and $S_n(\alpha)$. □

3.3 Proof of Theorem 3.1.4 and Theorem 3.2.1

Proof of Theorem 3.1.4. The surface $S_n(\alpha)$ is smooth irreducible by Propositions 3.1.5 and 3.1.7. Let $n := |\alpha| - 1$. Since $S_n(\alpha)$ is a complete intersection of $n-2$ quadrics of $\mathbb{P}^n$ a canonical divisor can be computed by applying several times the adjunction formula:

$$K_{S_n(\alpha)} = (n-5)H.$$

It follows that $S_n(\alpha)$ is a del Pezzo surface when $n = 4$ and in particular it is rational. When $n = 5$ it is a K3 surface and it is of general type for $n > 5$ (see Section 1.3). The only lines in $S_n(\alpha)$ are the trivial lines by Proposition 3.2.3. The group of sign changes acts transitively on the set of lines of $S_n(\alpha)$ and each trivial line meets exactly other $n+1$ trivial lines along a subset of points which is projectively equivalent to α, by Proposition 3.2.4. Finally the statement about the configuration of the lines in the case n odd is by Proposition 3.2.6. □

Proof of Theorem 3.2.1. Observe that Φ_n is well-defined by Proposition 3.2.2 and Corollary 3.1.6. Moreover it is surjective by definition. We now show that Φ_n is injective. Let α, α' be two ordered n-tuples such that $\Phi_n(\alpha) = \Phi_n(\alpha')$, that is $S_n(\alpha)$ is projectively equivalent to $S_n(\alpha')$. In particular the union of lines of the two surfaces are projectively equivalent and thus α and α' are projectively equivalent by Proposition 3.2.4.

When n is odd the moduli space $\mathcal{M}_{0,n+1}/\mathfrak{S}_{n+1}$ of unordered $(n+1)$-tuples of points of $\mathbb{P}^1$ is isomorphic to the moduli space of hyperelliptic curves of genus $\frac{1}{2}(n-1)$, the isomorphism being given by taking the double cover of $\mathbb{P}^1$ at the $n+1$ points. This competes the proof of the theorem.

□

Remark 3.3.1. Summarizing we have that from a Generalized Büchi surface $S \subseteq \mathbb{P}^n$ we can recover an unordered $(n+1)$-tuple of points $\{\alpha_0, \dots, \alpha_n\}$ of $\mathbb{P}^1$, which are the intersection points of a given line L of S with the other lines of S. When $n = 2g+1$ is odd one associates to S the genus g hyperelliptic curve C, of affine equation

$$y^2 = (x-\alpha_0)\cdots(x-\alpha_{2g+1}). \tag{3.1}$$

This is the double cover of L branched along the $2g+2$ points.

On the other hand, let $g \geq 1$ be an integer and let C be a genus g hyperelliptic curve with affine equation as define before in (3.1). Consider the set $2g+2$ Weierstraß points, $\alpha_0, \dots, \alpha_{2g+1}$, and built the trivial lines with these points. Proceeding as in the Proposition 3.1.5 we get a Generalized Büchi surface $S_{2g+1}(\alpha)$ in $\mathbb{P}^{2g+1}$. By reordering the Weierstraß points we get another, possibly distinct, Generalized Büchi surface which anyway is projectively equivalent to $S_{2g+1}(\alpha)$ by Corollary 3.1.6.

4 Hyperelliptic curves and GBS

In this chapter we study General Büchi Surfaces $S_{2g+1}(\alpha) \subset \mathbb{P}^{2g+1}$, with $g \geq 2$ integer and its relation with a hyperelliptic curve C of genus g. In Section 4.1 we will study the surface Y that is obtained by taking quotient of $S_{2g+1}(\alpha)$ by the subgroup G_0 of the automorphism group of $S_{2g+1}(\alpha)$ of the sign changes of an even number of coordinates. In general we will see basic aspect of the geometry of Y and highlight Proposition 4.1.3, which shows that Y is the double cover of $\mathbb{P}^2$ branched along the union of $2g+2$ lines tangent to a conic Γ, where the tangency points are the coordinates of the vector α (seen in the previous chapter). In Section 4.2, given a hyperelliptic curve C of genus $g \geq 2$, we study the second symmetric power $C^{(2)}$ of C and the natural quotient of $C^{(2)}$, induced by the hyperelliptic involution $\imath$ of C. The main result of this section is Theorem 4.2.1, which describes the geometry of the surface $C^{(2)}/\langle \imath \rangle$ and has as a consequence Corollary 4.2.2, which shows that the surfaces Y and $C^{(2)}/\langle \imath \rangle$ are isomorphic. In Subsection 4.2.1 we observe some basic facts over the Neron-Severi group of the surface $C^{(2)}$. Finally in Section 4.3 we make an observation on the rational points over Y.

4.1 The quotient by the group of even sign changes

Let $g \geq 2$ be an integer and let $S := S_{2g+1}(\alpha) \subseteq \mathbb{P}^{2g+1}$ be a General Büchi surface. The automorphisms group of S contains the subgroup G of coordinates sign changes. Let

$$G_0 := G \cap \mathrm{SL}(2g+2, \mathbb{C})$$

be the subgroup whose elements contains an even number of sign changes. Observe that, being $2g+1$ an odd number, the subgroup G_0 is proper and in particular it has index two in G. Denote by

$$Y_{2g+1}(\alpha) := S/G_0$$

the corresponding quotient. Whenever it will be clear from the context we will denote by Y the surface $Y_{2g+1}(\alpha)$.

Proposition 4.1.1. *The surface Y is isomorphic to the following hypersurface of degree $2g+2$ of $\mathbb{P}(1,1,1,g+1)$:*

$$w^2 = z_0 z_1 z_2 \prod_{i=3}^{2g+1} (\beta_2^i z_2 + \beta_1^i z_1 + \beta_0^i z_0), \tag{4.1}$$

where the coefficients are as in Definition 3.1.1. In particular Y has $\binom{2g+2}{2}$ ordinary double points of type A_1 and it is a surface of general type for $g \geq 3$.

Proof. First of all observe that the invariant ring $\mathbb{C}[x_0, \ldots, x_{2g+1}]^{G_0}$ is generated by the monomials $x_0^2, \ldots, x_{2g+1}^2, x_0 \cdots x_{2g+1}$. Indeed, these monomials are invariant, the first $2g+2$ monomials generate $\mathbb{C}[x_0, \ldots, x_{2g+1}]^G$ and the morphism $S/G_0 \to S/G$ is of degree two between normal varieties. The above invariant monomials define Y as a subvariety of the weighted projective space $\mathbb{P}(1, \ldots, 1, g+1)$ of dimension $2g+2$, whose equation is $w^2 = z_0 \cdots z_{2g+1}$. The quadratic polynomial $x_i^2 - \beta_2^i x_2^2 - \beta_1^i x_1^2 - \beta_0^i x_0^2$, appearing among the defining equations of S, becomes $z_i - \beta_2^i z_2 - \beta_1^i z_1 - \beta_0^i z_0$ in the new variables. This gives the claimed equation for Y in $\mathbb{P}(1, 1, 1, g+1)$.

From the equation one deduces that Y is a double covering of $\mathbb{P}^2$ branched along the union of $2g+2$ lines. In particular Y is singular at each intersection point of two such lines and the singularity is an ordinary double point of type A_1. To compute a canonical divisor K_Y for Y we apply the adjunction formula together with the observation that, being Y a normal surface, its canonical divisor is the closure of a canonical divisor of the smooth locus. First of all we recall that a canonical divisor of the weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ is $K_{\mathbb{P}} = -(a_0 + \cdots + a_n)H$, where H is degree 1 divisor [8, Thm. 8.2.3]. In particular a canonical divisor of $\mathbb{P}(1, 1, 1, g+1)$ is $(-g-4)H$. Thus we get

$$K_Y = K_{\mathbb{P}} + Y|_Y \sim (-g-4)H + (2g+2)H|_Y = (g-2)H|_Y.$$

The statement follows. □

Proposition 4.1.2. *The surfaces Y and S are birational if and only if $g = 2$.*

Proof. If $g = 2$ then both Y and S are K3 surfaces and the statement is classically known to be true, see e.g. [11, Thm. 10.3.16].

Assume now $g > 2$. Denote by K_S and $e(S)$ a canonical divisor and the Euler characteristic of the surface S, respectively. Since $S \subseteq \mathbb{P}^{2g+1}$ is a complete intersection of $2g-1$ quadrics then by [19, Exa. 2.3] we have

$$K_S^2 = 4(g-2)^2 2^{2g-1} \qquad e(S) = (2g^2 - 5g + 5)2^{2g-1}.$$

Replacing these values in the formula (1.2), with $D = 0$, and using $h^1(S, \mathcal{O}_S) = 0$ [9, Thm. 1.5 $(iii)_a$] one deduces

$$h^0(S, K_S) = h^2(S, \mathcal{O}_S) = \chi(\mathcal{O}_S) - 1 = (2g^2 - 7g + 7)2^{2g-3} - 1,$$

where the first equality is by Serre's duality.

On the other side, consider the surface $Y \subseteq \mathbb{P} := \mathbb{P}(1, 1, 1, g+1)$. The fundamental sequence of Y is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-Y) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Taking tensor product with $\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}} + Y)$, passing to the long exact sequence in cohomology and recalling that $K_Y \sim K_{\mathbb{P}} + Y|_Y$, we get

$$H^0(\mathbb{P}, K_{\mathbb{P}}) \longrightarrow H^0(\mathbb{P}, K_{\mathbb{P}} + Y) \longrightarrow H^0(Y, K_Y) \longrightarrow H^1(\mathbb{P}, K_{\mathbb{P}}).$$

Since $K_{\mathbb{P}}$ has degree $-g-4<0$ we have $H^0(\mathbb{P}, K_{\mathbb{P}}) = 0$. Moreover by Batyrev-Borisov vanishing [8, Thm. 9.2.7] we have $H^1(\mathbb{P}, K_{\mathbb{P}}) = 0$. It follows that $h^0(Y, K_Y) = h^0(\mathbb{P}, K_{\mathbb{P}} + Y)$, where the last number equals the number of monomials of degree $2g+2-(g+4) = g-2$ in $\mathbb{P}(1,1,1,g+1)$. Thus we conclude

$$h^0(Y, K_Y) = \binom{g}{2} = \frac{g(g-1)}{2}.$$

Then, it is a brief calculation to see that $h^0(Y, K_Y) < h^0(S, K_S)$ if $g > 2$. Recalling, by Proposition 4.1.1, that the only singularities of Y are du Val singularities, then, if $\pi\colon \tilde{Y} \to Y$ is the minimal resolution of Y one has $K_{\tilde{Y}} = \pi^* K_Y$ by [15, Thm. 7.5.1]. As a consequence $p_g(\tilde{Y}) = h^0(\tilde{Y}, K_{\tilde{Y}}) = h^0(Y, K_Y)$ because any birational morphisms between smooth surfaces is a composition of smooth blow-ups [5] and p_g is invariant for smooth blow-ups. Therefore, since $p_g(S) > p_g(\tilde{Y})$ the two surfaces cannot be birational. □

Proposition 4.1.3. *The quotient surface Y is isomorphic to the double cover of $\mathbb{P}^2$ branched along the union of $2g+2$ lines tangent to a conic Γ, where the tangency points are projectively equivalent to the set of points $\{\alpha_0, \dots, \alpha_{2g+1}\}$ in $\mathbb{P}^1$.*

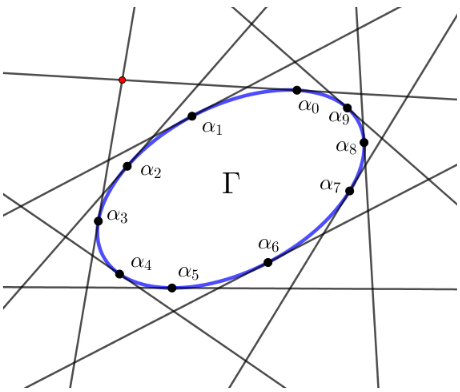

Figure 4.1: Sketch conic Γ and tangency points for $g = 4$.

Proof. To show that the $2g+2$ lines are tangent to a conic Γ we prove that the points corresponding to these lines, in the dual projective plane, lies on a conic Γ^*. The first three points, corresponding to z_0 z_1 and z_2, are the fundamental points so that Γ^* must have equation $c_1x_0x_1 + c_2x_0x_1 + c_3x_1x_2 = 0$. By evaluating at the point $[\beta_0^i : \beta_1^i : \beta_2^i]$, where the β_k^i are given in Definition 3.1.1, an easy computation gives the following equation for Γ^*:

$$(\alpha_0 - \alpha_1)^2 x_0 x_1 + (\alpha_0 - \alpha_2)^2 x_0 x_2 + (\alpha_1 - \alpha_2)^2 x_1 x_2 = 0.$$

The equation of the dual conic Γ is obtained by taking the inverse of the symmetric defining matrix of Γ^*. To better display the equation we introduce the notation $\alpha_{ij} := \alpha_i - \alpha_j$.

$$\alpha_{12}^4 z_0^2 - 2\alpha_{02}^2\alpha_{12}^2 z_0 z_1 - 2\alpha_{01}^2\alpha_{12}^2 z_0 z_2 + \alpha_{02}^4 z_1^2 - 2\alpha_{01}^2\alpha_{02}^2 z_1 z_2 + \alpha_{01}^4 z_2^2 = 0. \tag{4.2}$$

A tedious but elementary calculation shows that the conic Γ is tangent to the line $\beta_0^i z_0 + \beta_1^i z_1 + \beta_2^i z_2 = 0$ at the point $p_i := [(\alpha_0 - \alpha_i)^2 : (\alpha_1 - \alpha_i)^2 : (\alpha_2 - \alpha_i)^2]$. By applying the isomorphism $\phi\colon \mathbb{P}^2 \to \mathbb{P}^2$ defined by $[z_0 : z_1 : z_2] \mapsto [f_0 : f_1 : f_2]$, where

$$\begin{aligned}
f_0 &:= (\alpha_1 - \alpha_2) z_0 - (\alpha_0 - \alpha_2) z_1 + (\alpha_0 - \alpha_1) z_2 \\
f_1 &:= (\alpha_1^2 - \alpha_2^2) z_0 - (\alpha_0^2 - \alpha_2^2) z_1 + (\alpha_0^2 - \alpha_1^2) z_2 \\
f_2 &:= \alpha_1\alpha_2(\alpha_1 - \alpha_2) z_0 - \alpha_0\alpha_2(\alpha_0 - \alpha_2) z_1 + \alpha_0\alpha_1(\alpha_0 - \alpha_1) z_2,
\end{aligned} \tag{4.3}$$

we see that p_i is mapped to $[1 : 2\alpha_i : \alpha_i^2]$. Projecting on the first two coordinates and eventually dividing by 2 one proves the statement.

□

Remark 4.1.4. Observe that each trivial line of S is mapped to Γ by the quotient map $S \to S/G$. Indeed the map is defined by $[x_0 : \cdots : x_{2g+1}] \mapsto [x_0^2 : \cdots : x_{2g+1}^2]$ and using the equations of S one expresses all the coordinates as functions of the first three. Thus the parametrized line $t \mapsto [\pm(t - \alpha_0) : \cdots : \pm(t - \alpha_{2g+1})]$ is sent to the parametrized conic $t \mapsto [(t - \alpha_0)^2 : (t - \alpha_1)^2 : (t - \alpha_2)^2]$, whose equation is (4.2). By replacing this parametrization into (4.1) we deduce that the double cover $Y = S/G_0 \to S/G$ is trivial over Γ since the curve has the following two preimages of parametric equation

$$t \mapsto \left[(t - \alpha_0)^2 : (t - \alpha_1)^2 : (t - \alpha_2)^2 : \pm \prod_{i=0}^{2g+1} (t - \alpha_i) \right].$$

In particular if we denote by $\Gamma_0, \Gamma_1 \subseteq Y$ the above two curves, corresponding respectively to the sign $+$ and $-$, then the trivial lines of S mapped to Γ_0 are exactly those with an even number of negative signs.

4.2 Symmetric product of an hyperelliptic curve

Let C be an hyperelliptic curve of genus $g \geq 2$ and let $\imath$ be the hyperelliptic involution. Let N be the subgroup of $\mathrm{Aut}(C \times C)$ generated by the action of $\imath$

on each factor. Observe that N is isomorphic to the Klein group $(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$. Denote by H the subgroup of $\mathrm{Aut}(C\times C)$ generated by the involution $(p,q)\mapsto(q,p)$. A direct calculation shows that H is in the normalizer of N so that NH is a group which contains N as a normal subgroup. Let

$$C^{(2)} := (C\times C)/H$$

be the second symmetric power of C with itself, let $C\times C\to C^{(2)}$ be the quotient map and let $p+q$ be the image of (p,q). Observe that H conmutes with the subgroup of N generated by the involution which acts on both factors. As a consequence this involution descends to $C^{(2)}$ acting as $p+q\mapsto\imath(p)+\imath(q)$. We denote by $C^{(2)}/\langle\imath\rangle$ the corresponding quotient surface. Recalling that the symmetric product of $\mathbb{P}^1$ with itself is $\mathbb{P}^2$ we summarize the above construction in the following commutative diagram

$$\begin{tikzcd} C\times C \arrow[r, "/H"] \arrow[d, "/N"', "\pi"] & C^{(2)} \arrow[r, "/\imath"] \arrow[d, "\pi^{(2)}"] & C^{(2)}/\langle\imath\rangle \arrow[d, "\pi^{(2)}_\imath"] \\ \mathbb{P}^1\times\mathbb{P}^1 \arrow[r, "/H"] & \mathbb{P}^2 \arrow[r, equal] & \mathbb{P}^2 \end{tikzcd} \tag{4.1}$$

where $\pi^{(2)}$ is the degree 4 morphism induced by π. Observe that $\pi^{(2)}$ is *not* a quotient morphism by a group action. Moreover it factorizes as the composition of the two displayed degree two morphisms.

The aim of this section is to prove the following.

Theorem 4.2.1. *Let C be a hyperelliptic curve of genus $g\geq 2$ with hyperelliptic involution $\imath$ and let $\alpha_0,\ldots,\alpha_{2g+1}\in\mathbb{P}^1$ be the images of the Weierstraß points of C. Then*

$$\pi^{(2)}_\imath\colon C^{(2)}/\langle\imath\rangle\to\mathbb{P}^2$$

is a double cover branched along the union of the $2g+2$ lines of equations $z_2-\alpha_i z_1+\alpha_i^2 z_0=0$, which are images of the curves $\{p\}\times C$, where p varies along the Weierstraß points of C.

Proof. We describe the morphisms of (4.1) in an invariant affine chart of $C\times C$. An affine equation of the curve C is $y^2=f(x)$, where

$$f(x)=\prod_{i=1}^{2g+1}(x-\alpha_i).$$

The cartesian product $C\times C$ is $\{(x,u,y,v)\in\mathbb{C}^4 : y^2=f(x)\text{ and }v^2=f(u)\}$. The generators of the group N act by exchanging the signs of y and v respectively, while the generator of H maps (x,u,y,v) to (u,x,v,y). We can use invariant theory, see e.g. [22], to explicitly compute the quotient morphisms in (4.2.1).

This is done in Program 4.1. Thus the morphisms are

$$\begin{tikzcd}
(x,u,y,v) \arrow[r] \arrow[d, "\pi"] & (x+u,xu,y+v,yv) \arrow[r] \arrow[d, "\pi^{(2)}"] & (x+u,xu,yv) \arrow[d, "\pi_i^{(2)}"] \\
(x,u) \arrow[r] & (x+u,xu) \arrow[r, equal] & (x+u,xu).
\end{tikzcd} \tag{4.2}$$

From the above description of $\pi_i^{(2)}$ we see that it ramifies along the curves with $yv = 0$. By the equation of $C \times C$ these are the images of the curves $\{p\} \times C$, where p is a Weierstraß point. These are the lines of parametric equation $t \mapsto (\alpha_i + t, \alpha_i t)$, whose homogeneous cartesian equation is $z_2 - \alpha_i z_1 + \alpha_i^2 z_0 = 0$. $\square$

As an immediate consequence we are able to show that the surface $C^{(2)}/\langle \imath \rangle$ is isomorphic to the surface Y defined in Section 4.1.

Corollary 4.2.2. *With the same notation of Theorem 4.2.1, let* $\alpha := (\alpha_0, \ldots, \alpha_{2g+1})$ *and let* f_0, f_1, f_2 *be the three linear polynomials defined in* (4.3). *The automorphism of* $\mathbb{P}(1,1,1,g+1)$*, defined by* $[z_0, z_1, z_2, z_3] \mapsto [f_0, f_1, f_2, z_3]$*, maps* $Y_{2g+1}(\alpha)$ *to* $C^{(2)}/\langle \imath \rangle$*.*

Proof. By the proof of Theorem 4.2.1 a defining equation for $C^{(2)}/\langle \imath \rangle$ in the weighted projective space $\mathbb{P}(1,1,1,g+1)$ is the following

$$w^2 = \prod_{i=1}^{2g+1} (z_2 - \alpha_i z_1 + \alpha_i^2 z_0). \tag{4.3}$$

A direct calculation proves the statement. $\square$

Remark 4.2.3. The $2g+2$ lines which form the branch divisor of the morphism $\pi_i^{(2)}$ are tangent to the conic Γ_C of equation

$$4z_0 z_2 - z_1^2 = 0.$$

The line of equation $z_2 - \alpha_i z_1 + \alpha_i^2 z_0 = 0$ is tangent to the above conic at the point $[1 : 2\alpha_i : \alpha_i^2]$. The restriction of $\pi_i^{(2)}$ to the plane defined by the first three coordinates, that is the morphism $[z_0 : z_1 : z_2] \mapsto [f_0 : f_1 : f_2]$, maps the conic Γ of equation (4.2) to Γ_C. Thus the given isomorphism

$$Y_{2g+1}(\alpha) \simeq C^{(2)}/\langle \imath \rangle$$

is defined over the field $\mathbb{Q}(\alpha_0, \alpha_1, \alpha_2)$ and in particular if the vector α has at least three rational entries then one can produce such an isomorphism defined over the rationals, allowing one to identify the rational points of both surfaces.

4.2.1 The Néron-Severi group of $C^{(2)}$

Here we recall some basic facts about the Nerón-Severi group of the surface $C^{(2)}$. First of all we define three curves on the surface via the surjection $C \times C \to C^{(2)}$. The first curve, named C_p is the image of $C \times \{p\}$, or equivalently of $\{p\} \times C$. The second one, Δ is the image of the diagonal and the third E is the image of the graph of the hyperelliptic involution.

Proposition 4.2.4. *The intersection matrix of the three curves C_p, Δ and E is*

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4-4g & 2g+2 \\ 1 & 2g+2 & 1-g \end{pmatrix}.$$

In particular $4C_p - \Delta - 2E \equiv 0$.

Lemma 4.2.5. *The curves E and C_p intersect transversely.*

Proof. Consider the morphism $C \times C \to C^{(2)}$. In an affine chart we have that $C \times C = \{(x,u,y,v) : y^2 = f(x) \text{ and } v^2 = f(u)\}$ and the above morphism is

$$(x,u,y,v) \mapsto (x+u, xu, y+v, yv).$$

The graph of the hyperelliptic involution is the following curve of $C \times C$

$$\tilde{E} = \{(x,u,y,v) : y^2 = f(x), \quad x = u, \quad v = -y\}.$$

Since $\Gamma_\imath$ is a local complete intersection, its tangent space at $(x,u,y,v) \in C \times C$ is the left kernel of the following matrix, whose rows are the gradients of the hypersurfaces which define the curve

$$\begin{pmatrix} -f'(x) & 0 & 2y & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The kernel is generated by the vector $(2y \ 2y \ f'(x) \ -f'(x))$. In the same affine chart we have the curve

$$C \times \{p\} = \{(x,u,y,v) : y^2 = f(x), \quad u = u_0, \quad v = v_0\}$$

whose tangent space at $(x,u,y,v) \in C \times C$ is the left kernel of the matrix

$$\begin{pmatrix} -f'(x) & 0 & 2y & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The kernel is generated by the vector $(2y \ 0 \ f'(x) \ 0)$. The two generators are linearly independent unless $y = 0$ and $f'(x) = 0$. But in this case one would deduce that both $f(x)$ and $f'(x)$ vanish at the same point, a contradiction since $f(x)$ by hypothesis has distinct roots. This show that $\Gamma_\imath$ intersects transversely $C \times \{p\}$ in $C \times C$. Since the intersection points lie outside the ramification divisor of the covering $C \times C \to C^{(2)}$, we conclude that E meets C_p transversely. $\square$

Lemma 4.2.6. *The curve E and Δ intersect transversely as well as C_p and C_q for $p \neq q$.*

Proof. It suffice to show that the images of the curves E and Δ via the map $C^{(2)} \to C^{(2)}/\langle\imath\rangle$ intersect transversely. Both curves are mapped to the unique conic $\Gamma_C \subseteq \mathbb{P}^2$ which is tangent to all the branch lines of the double covering $C^{(2)}/\langle\imath\rangle \to \mathbb{P}^2$. Thus locally, over one of these tangency points, we can assume Γ_C to be defined by $y = x^2$ and the tangent line by $y = 0$. The double covering has local equation $z^2 = y$ and so the images of E and Δ have local equations $z - x = 0$ and $z + x = 0$, so that they meet transversely. A similar argument proves the second transversality. □

Lemma 4.2.7. $E \cdot C_p = 1$ *and* $E \cdot \Delta = 2g + 2$.

Proof. By Lemma 4.2.5 and Lemma 4.2.6 it is enough to show that $E \cap C_p$ has cardinality 1 and $E \cap \Delta$ has cardinality $2g + 2$. The first equality follows by observing that the equation $q + i(q) = p + q$ has $p + i(p)$ as unique solution. The second equality follows by observing that the equation $q + i(q) = 2p$ has solution only when $q = i(q)$, that is when q is one of the $2g+2$ Weierstraß points of C. □

Proof of Prop. 4.2.4. Let $K := K_{C^{(2)}}$ be a canonical divisor of $C^{(2)}$. Since $K_{C\times C} = \pi_1^* K_C + \pi_2^* K_C \equiv (2g-2)(\{p\}\times C) + (2g-2)(C \times \{p\})$, by the ramification formula $K_{C\times C} \sim \pi^* K + \pi^*(\frac{1}{2}\Delta)$ we deduce

$$K \equiv (2g-2)C_p - \frac{1}{2}\Delta. \tag{4.4}$$

The curves C_p and C_q are numerically equivalent and they intersect transversally at $p + q$, so that $C_p^2 = C_p \cdot C_q = 1$. Then by the genus formula (1.1) we deduce $C_p \cdot K = 2g-3$, so that $\Delta \cdot C_p = 2$ by (4.4). Now, by applying the genus formula to the curve Δ and using the last intersection product, we deduce that $\Delta^2 = 4 - 4g$. By Lemmas 4.2.7, 4.2.5, 4.2.6 and the genus formula applied to the curve E we deduce that $E^2 = 1$. So the statement follows. □

Remark 4.2.8. Some possible expectations that would complement this work would be to describe the geometry of the double covering

$$C^{(2)} \to Y.$$

It is not difficult to see that the $2g + 2$ lines tangent to the conic Γ, which form the branch divisor of the double covering $Y \to \mathbb{P}^2$, have as preimages in $C^{(2)}$ the $2g+2$ curves of the form C_p, where $p \in C$ is a Weierstraß point. The preimages of these lines in the covering $S \to Y$ are the $2g + 2$ curves, each of which is defined by the vanishing of a coordinate variable of the ambient vector space.

In particular we want to say something about the following:

- The preimages in $C^{(2)}$ of the two curves $\Gamma_0, \Gamma_1 \subset Y$ which are preimages of the conic Γ (see Remark 4.1.4).
- The intersection matrix of the curves we know in Y.
- The Néron-Severi group of Y.

These problems will remain as pending problems in this work.

4.3 Rational points

Let us denote by $F : C \times C \to C^{(2)}/\langle \imath \rangle$ the degree four quotient map. In affine coordinates, according to Diagram 4.2, the map F is

$$(x, u, y, v) \mapsto (xu, x + u, yv).$$

Given a point $(p, q) \in C \times C$ we denote by $\mathbb{Q}(p, q)$ the extension of the rationals obtained by adding the affine coordinates of the points p and q. Thus if $p = (x, y)$ and $q = (u, v)$ then $\mathbb{Q}(p, q) := \mathbb{Q}(x, u, y, v)$. We say that a point of $C^{(2)}/\langle \imath \rangle$ is *rational* if all of its coordinates are rational numbers.

Observe that if $F(p, q)$ is a rational point of $C^{(2)}/\langle \imath \rangle$ then $xu, x + u, yv$ are all rational numbers so that $\mathbb{Q}(x, u, y, v) = \mathbb{Q}(x, y)$.

Proposition 4.3.1. *Let $(p, q) \in C \times C$ be such that its image via F is a rational point of $C^{(2)}/\langle \imath \rangle$ then the field extension $\mathbb{Q}(p, q)/\mathbb{Q}$ is Galois and its Galois group is one of the following: $\langle \mathrm{Id} \rangle$, $\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$.*

Proof. First of all recall the equalities $\mathbb{Q}(p, q) := \mathbb{Q}(x, u, y, v) = \mathbb{Q}(x, y)$, where the second is by the hypothesis. The degree of the extension is a divisor of 4 by applying the tower law to $\mathbb{Q} \subseteq \mathbb{Q}(x) \subseteq \mathbb{Q}(x, y)$ and recalling that $y^2 = f(x)$, where f has rational coefficients.

To prove that $\mathbb{Q}(p, q)/\mathbb{Q}$ is Galois, by Proposition 2.1.24 it suffice to show that any automorphism σ of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ maps the field $\mathbb{Q}(p, q)$ to itself. Indeed by Lemma 2.1.15 we have that σ preserves the set $\{x, u\}$. On other side, applying σ to the equation $y^2 = f(x)$, and recalling that f has rational coefficients we get

$$\sigma(y)^2 = \sigma(f(x)) = f(\sigma(x)) \in \{f(x), f(u)\}.$$

Thus $\sigma(y) \in \{-y, y, -v, v\}$ and analogously we have $\sigma(v) \in \{-y, y, -v, v\}$. This shows that the extension $\mathbb{Q}(p, q)/\mathbb{Q}$ is Galois. Since its degree is a divisor of 4, the only possibilities for the Galois group are $\langle \mathrm{Id} \rangle$, $\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^2$ and $\mathbb{Z}/4\mathbb{Z}$. We now show that the last one cannot realize concluding the proof. Indeed if this were the case, then the Galois group would contain an order four element which should act as

$$x \mapsto u, \quad u \mapsto x, \quad y \mapsto v, \quad v \mapsto -y$$

so that yv would not be invariant, a contradiction. $\square$

We observe that in the case the Galois group of $\mathbb{Q}(p,q)/\mathbb{Q}$ is trivial, then both p and q are rational points of C. We show in the following proposition that when $g = 4$ we know all such points.

Theorem 4.3.2. *If C is the hyperelliptic curve $y^2 = \prod_{i=-4}^{4}(x-i)$ then the only rational points of C are the Weierstraß points.*

Proof. By [21, pp. 15] if p is a prime of good reduction of C then

$$\#C(\mathbb{Q}) \leq \#\overline{C}(\mathbb{F}_p) + 2r + \left\lfloor \frac{2r}{p-2} \right\rfloor,$$

where $\overline{C}$ is the reduction of C modulo p and r is the rank of the group $JC(\mathbb{Q})$ of rational points on the Jacobian variety. The Magma program 4.2 compute the discriminant of C which is $2^{62}3^{18}5^{8}7^{4}$, thus the prime $p = 13$ is of good reduction for C, also compute the numbers $\#\overline{C}(\mathbb{F}_{13}) = 10$ and $r = 0$. Since $C(\mathbb{Q})$ contains the nine Weierstraß points together with the point at infinity, the statement follows. $\square$

Appendix: Magma programs

In this Magma program we compute the subrings of invariants used in Section 4.2, specifically when building the diagram (4.2).

Program 4.1

```
> Q:= Rationals();
> U := Matrix(Q,2,2,[0,1,1,0]);
> M := DiagonalJoin(U,U);
> G1 := MatrixGroup<4, Q | M>;
> R1 := InvariantRing(G1);
> FundamentalInvariants(R1);
[
    x1 + x2,
    x3 + x4,
    x1^2 + x2^2,
    x1*x3 + x2*x4,
    x3^2 + x4^2
]
> I := IdentityMatrix(Q,2);
> N := DiagonalJoin(I,-I);
> G2 := MatrixGroup<4,Q | M,N>;
> R2 := InvariantRing(G2);
> FundamentalInvariants(R2);
[
    x1 + x2,
    x1^2 + x2^2,
    x3^2 + x4^2,
    x3*x4,
    x1*x3^2 + x2*x4^2
]
```

In this Magma program we define the hyperelliptic curve C of affine equation

$$y^2 = \prod_{i=-4}^{4}(x-i).$$

Compute its discriminant for deduce that the prime $p = 13$ is the good reduction, later compute the number of points of the set $\overline{C}(\mathbb{F}_{13})$ and also the rank r of the group $JC(\mathbb{Q})$. This data is used in Theorem 4.3.2.

Program 4.2

```
> R<x> := PolynomialRing(Rationals());
> C := HyperellipticCurve(&*[x-i : i in [-4..4]]);
> a := Discriminant(C);
> a;
1675693213808209968850639257600000000
> Factorization(Numerator(a));
[ <2, 62>, <3, 18>, <5, 8>, <7, 4> ]
> #Points(ChangeRing(C,GF(13)));
10
> JC := Jacobian(C);
> RankBounds(JC);
0 0
```

In this Magma program we compute the determinants of the associated matrices to the following systems of equations

$$\begin{aligned} \beta_1^i x_1^2 + \beta_2^i x_2^2 &= 0 \\ \beta_1^j x_1^2 + \beta_2^j x_2^2 &= 0. \end{aligned} \qquad \begin{aligned} \beta_0^i x_0^2 + \beta_1^i x_1^2 + \beta_2^i x_2^2 &= 0 \\ \beta_0^j x_0^2 + \beta_1^j x_1^2 + \beta_2^j x_2^2 &= 0 \\ \beta_0^k x_0^2 + \beta_1^k x_1^2 + \beta_2^k x_2^2 &= 0. \end{aligned}$$

These results are used in Proposition 3.1.7.

Program 4.3

```
> R<a0,a1,a2,ai,aj,ak> := FunctionField(Rationals(),6);
> b0i := (a1-ai)*(a2-ai)/((a0-a1)*(a0-a2));
> b1i := -(a0-ai)*(a2-ai)/((a0-a1)*(a1-a2));
> b2i := (a0-ai)*(a1-ai)/((a0-a2)*(a1-a2));
> b0j := (a1-aj)*(a2-aj)/((a0-a1)*(a0-a2));
> b1j := -(a0-aj)*(a2-aj)/((a0-a1)*(a1-a2));
> b2j := (a0-aj)*(a1-aj)/((a0-a2)*(a1-a2));
> b0k := (a1-ak)*(a2-ak)/((a0-a1)*(a0-a2));
> b1k := -(a0-ak)*(a2-ak)/((a0-a1)*(a1-a2));
> b2k := (a0-ak)*(a1-ak)/((a0-a2)*(a1-a2));
>
> M2 := Matrix(2,2,[b1i,b2i,b1j,b2j]);
> Factorization(Numerator(Determinant(M2)));
[
    <ai - aj, 1>,
    <a0 - aj, 1>,
    <a0 - ai, 1>
]
> Factorization(Denominator(Determinant(M2)));
[
    <a1 - a2, 1>,
    <a0 - a2, 1>,
    <a0 - a1, 1>
]
> M3 := Matrix(3,3,[b0i,b1i,b2i,b0j,b1j,b2j,b0k,b1k,b2k]);
> Factorization(Numerator(Determinant(M3)));
[
    <aj - ak, 1>,
    <ai - ak, 1>,
    <ai - aj, 1>
]
> Factorization(Denominator(Determinant(M3)));
[
    <a1 - a2, 1>,
    <a0 - a2, 1>,
    <a0 - a1, 1>
]
```

References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 267, Springer-Verlag, New York, 1985. ↑14

[2] Michela Artebani, Antonio Laface, and Damiano Testa, *On Büchi's K3 surface*, Math. Z. **278** (2014), no. 3-4, 1113–1131. ↑2, 5

[3] Lucian Bădescu, *A remark on the Grothendieck-Lefschetz theorem about the Picard group*, Nagoya Math. J. **71** (1978), 169–179. ↑36

[4] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven, *Compact complex surfaces*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 4, Springer-Verlag, Berlin, 2004. ↑17, 18

[5] Arnaud Beauville, *Complex algebraic surfaces*, 2nd ed., London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. ↑18, 40

[6] Duncan A. Buell, *Integer squares with constant second difference*, Math. Comp. **49** (1987), no. 180, 635–644. ↑2, 5

[7] David A. Cox, *Galois theory*, 2nd ed., Pure and Applied Mathematics (Hoboken), John Wiley & Sons, Inc., Hoboken, NJ, 2012. ↑20, 21, 22, 23, 24

[8] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. ↑25, 39, 40

[9] *Groupes de monodromie en géométrie algébrique. II*, Lecture Notes in Mathematics, Vol. 340, Springer-Verlag, Berlin-New York, 1973 (French). Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II); Dirigé par P. Deligne et N. Katz. ↑36, 39

[10] Reinhard Diestel, *Graph theory*, 3rd ed., Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Berlin, 2005. ↑14

[11] Igor V. Dolgachev, *Classical algebraic geometry*, Cambridge University Press, Cambridge, 2012. A modern view. ↑2, 5, 25, 39

[12] Simon Donaldson, *Riemann surfaces*, Oxford Graduate Texts in Mathematics, vol. 22, Oxford University Press, Oxford, 2011. ↑14, 15, 16

[13] Andreas Gathmann, *Algebraic Geometry, Class Notes TU Kaiserslautern 2014*, Online notes. ↑11, 12, 13

[14] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. ↑11, 14

[15] Shihoko Ishii, *Introduction to singularities*, Springer, Tokyo, 2018. Second edition. ↑19, 40

[16] B. Kunyavskij, A. Skorobogatov, and M. Tsfasman, *Del pezzo surfaces of degree four*, Mémoires de la S. M. F. **37** (1989), Link. ↑35

[17] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series. ↑17, 18

[18] H. Pasten, T. Pheidas, and X. Vidaux, *A survey on Büchi's problem: new presentations and open problems*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **377** (2010), no. Issledovaniya po Teorii Chisel. 10, 111–140, 243 (English, with English and Russian summaries); English transl., J. Math. Sci. (N.Y.) **171** (2010), no. 6, 765–781. ↑2, 5

[19] Chris Peters and Hans Sterk, *Complete intersections of quadrics and complete intersections on Segre varieties with common specializations* (2017). ↑39

[20] Pablo Sáez and Xavier Vidaux, *A characterization of Büchi's integer sequences of length 3*, Acta Arith. **149** (2011), no. 1, 37–56, DOI 10.4064/aa149-1-3. MR2805541 ↑2, 5

[21] Michael Stoll, *Rational points on curves*, J. Théor. Nombres Bordeaux **23** (2011), no. 1, 257–277 (English, with English and French summaries). ↑25, 28, 47

[22] Bernd Sturmfels, *Algorithms in invariant theory*, 2nd ed., Texts and Monographs in Symbolic Computation, SpringerWienNewYork, Vienna, 2008. ↑42

[23] P. Vojta, *Diagonal quadratic forms and Hilbert's Tenth Problem*, Contemp. Math. **270** (2000), 261-274. ↑2, 5