UNIVERSIDAD DE CONCEPCIÓN
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DOCTORADO EN CIENCIAS FÍSICAS

INTEGRABLE SYSTEMS AND THE BOUNDARY DYNAMICS OF (HIGHER SPIN) GRAVITY ON AdS_3

Thesis presented to the Facultad de Ciencias Físicas y Matemáticas of the Universidad de Concepción to qualify for the academic degree of Doctor en Ciencias Físicas

EMILIO GERMÁN OJEDA LEYTON

Thesis Advisor : Dr. Alfredo Pérez Donoso
Centro de Estudios Científicos (CECs)

January, 2021
Concepción, Chile

Thesis Advisor : Dr. Alfredo Pérez Donoso
Laboratorio de Física Teórica,
Centro de Estudios Científicos (CECs),
Valdivia, Chile.

Assessment Committee : Dr. Ricardo Troncoso Pérez
Laboratorio de Física Teórica,
Centro de Estudios Científicos (CECs),
Valdivia, Chile.

Dr. Fernando Izaurieta Aranda
Departamento de Física,
Universidad de Concepción (UDEC),
Concepción, Chile.

Dedicated to my parents.

Acknowledgments

Throughout the writing of this thesis I have received a great deal of support and assistance.

First of all, my parents, who supported me with love and understanding. All my achievements would not have been possible without them.

I would also like to thank to my thesis advisor Alfredo Pérez, for providing guidance and feedback throughout this project and the whole doctoral process.

I am also very grateful with the members of the Centro de Estudios Científicos, specially with professors Ricardo Troncoso, Jorge Zanelli and David Tempo, my colleagues in the attic, where I spent much of my time in Valdivia, and the administrative staff, mainly to Patricia Fernandoy and Angélica Treimún.

Finally, to Camila, who has been at my side from the very beginning, and whose friendship and company has been undoubtedly the greatest gift.

Abstract

This thesis extends a previously found relation between the integrable KdV hierarchy and the boundary dynamics of pure gravity on AdS_3 described in the highest weight gauge, to a more general class of integrable systems associated to three-dimensional gravity on AdS_3 and higher spin gravity with gauge group $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ in the diagonal gauge. We present new sets of boundary conditions for the (higher spin) gravitational theories on AdS_3, where the dynamics of the boundary degrees of freedom is described by two independent left and right members of a hierarchy of integrable equations. For the pure gravity case, the associated hierarchy corresponds to the Gardner hierarchy, also known as the "mixed KdV-mKdV" one, while for the case of higher spin gravity, they are identified with the "modified Gelfand-Dickey" hierarchies. The complete integrable structure of the hierarchies, i.e., the phase space, the Poisson brackets and the infinite number of commuting conserved charges, are directly obtained from the asymptotic structure and the conserved surface integrals in the gravitational theories. Consequently, the corresponding Miura transformation is recovered from a purely geometric construction in the bulk. Black hole solutions that fit within our boundary conditions, the Hamiltonian reduction at the boundary and more general thermodynamic ensembles called "Generalized Gibbs Ensemble" (GGE) are also discussed.

Contents

Introduction

The asymptotic structure of spacetime plays a fundamental role in the description of General Relativity in three dimensions. Since this is a topological theory, it does not possess local propagating degrees of freedom, and consequently, its dynamics is completely dictated by the choice of boundary conditions. In the case of a negative cosmological constant, in 1986, J. Brown and M. Henneaux presented a set of boundary conditions [1], whose asymptotic symmetries are spanned by two copies of the Virasoro algebra with central charge $c = 3l/2G$. Interestingly, this choice of boundary conditions is not unique. There are other possible sensible choices that can be consistently used, with different physical consequences [2–9]. In particular, as was first shown in ref. [6] and later extended in refs. [8, 30, 35], the dynamics of the gravitational excitations at the boundary are related with two-dimensional integrable systems, and therefore, the asymptotic symmetries are spanned by an infinite set of commuting charges.

The key of the relationship between three-dimensional gravity and two-dimensional integrable systems is the precise way in which the length and time scales are fixed in the asymptotic region of spacetime in the gravitational theory. Indeed, the asymptotic values of the lapse and shift functions in an ADM decomposition of the metric, can be chosen to depend explicitly on the dynamical fields in a very tight form, compatible with the action principle.

In the case of the KdV hierarchy analyzed in ref. [6], the fall-off of the metric extends the one of Brown and Henneaux when one includes the most general form of the Lagrange multipliers [12,13]. However, the lapse and shift functions, and consequently the boundary metric of AdS_3, are no longer kept fixed at infinity, because they are now allowed to depend explicitly on the Virasoro currents. As a consequence of this particular choice of chemical potentials, Einstein equations in the asymptotic region precisely reduce to two independent left and right copies of the k-th element of the KdV hierarchy. Furthermore, the complete integrable structure of the KdV system, i.e., the phase space, the Poisson brackets and the infinite number of commuting conserved charges, are directly obtained from the asymptotic analysis and the conserved surface integrals in the gravitational theory. The standard Brown–Henneaux boundary conditions [1] are contained as a particular case ($k = 0$) of the KdV hierarchy.

It is worth pointing out that there exists a deep relation between the KdV hierarchy and two-dimensional conformal field theories (CFT_2), indeed, the infinite (quantum) commuting KdV charges can be expressed as composite operators in terms of the stress tensor of a CFT_2 [14–16]. This fact has been recently used to describe Generalized Gibbs ensembles in these theories [6,17–28], as well as constructing black hole solutions carrying non-trivial KdV charges [29].

Different extensions of this new duality between a three-dimensional gravitational theory in the bulk with certain specific boundary conditions, and a two-dimensional integrable system at the boundary, have been reported in the literature. In ref. [30], the case of General Relativity without cosmological constant was studied. The associated integrable system corresponds to a generalization of the Hirota-Satsuma coupled KdV system [31], which possesses a BMS_3 Poisson structure. Another interesting generalization was proposed in ref. [32], where it was shown that the integrable system associated to gravity on AdS_3 coupled to two $U(1)$ Chern-Simons fields is the Benjamin-Ono_2 integrable system.

The main objective of this thesis is to exhibit two new generalizations of this duality that were reported in refs. [8] and [35]. In particular, in ref. [8] a new set of boundary conditions for pure gravity on AdS_3 was proposed, such that the boundary dynamics is described by the so called Gardner (or mixed KdV-mKdV) hierarchy. These boundary conditions are closely related with the soft hairy ones in ref. [5], but where now the chemical potentials are chosen to depend on the dynamical fields in a very precise way. This nontrivial way of fixing the chemical potentials is the key to make contact with the integrable system. In ref. [35] these results were further extended to the case of higher spin gravity on AdS_3 with gauge group $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$, where the corresponding integrable system was shown to be the N-th modified Gelfand-Dickey hierarchy.

Thesis plan

In chapter 1, a brief review of some general properties of integrable systems is presented. In particular, we focus on the integrable systems which are relevant for the main goal of this thesis, i.e., the (modified) KdV and the family of (modified) Gelfand-Dickey hierarchies. In chapter 2 we discuss the new boundary conditions for pure gravity on AdS_3 reported in [8] which were shown to be related with the Gardner (or mixed KdV-mKdV) hierarchy. The higher spin generalization proposed in ref. [35] is described in chapter 3. Finally we include a section devoted to the conclusions. Seven appendices with some technical details are also included.

Chapter 1

An introduction to integrable systems

The integrability of a differential equation is determined by the existence of integrals of motion or conserved quantities. Formally, Liouville's theorem in classical mechanics states the conditions under which the equations of motion of a dynamical system can always be solved by a mathematical procedure. For a system with a finite number of degrees of freedom, the Liouville theorem says:

"*A Hamiltonian system with a $2N$-dimensional phase space is integrable by the method of quadratures if and only if there exist N, functionally* ***independent*** *conserved quantities which are in* ***involution***."

For a system with an infinite number of degrees of freedom, it is necessary to consider an infinite number of independent conserved quantities in involution.

In this first chapter, the most relevant aspects of non-linear integrable systems are reviewed. It will be addressed using two equivalent formulations. The first corresponds to the bi-Hamiltonian formulation needed to make contact with the asymptotic dynamics of gravitational theories in $2+1$ dimensions, and the second one is the Lax pair formulation, which clarifies the role played by the Miura transformation and allows us to generalize the results in ref. [6,8] to the three-dimensional gravitational theory endowed with higher spin fields [35].

1.1 Korteweg-de Vries hierarchy

The Korteweg-de Vries equation is one of the most important non-linear partial differential equations, which historically helped to understand in detail the problem of integrability. This is a nonlinear dispersive wave equation which allows to describe “solitary waves,” observed by first time in 1834 by the Scottish engineer John Scott Russell (1808–1882). Remarkably, the KdV equation appears in many different branches of mathematics and physics [34, 41].

1.1.1 KdV equation as a Hamiltonian system

The Korteweg-de Vries equation is given by

$$\dot{\mathcal{L}} = \frac{3}{2}\mathcal{L}\mathcal{L}' + \frac{1}{4}\mathcal{L}''', \tag{1.1}$$

where dots and primes denote derivatives with respect to the temporal and spatial coordinates, respectively.

The dynamics of eq. (1.1) can be described using the Hamiltonian formalism, where the Poisson bracket of two arbitrary functionals F and G is given by

$$\{F, G\}_1 = \int d\phi \left(\frac{\delta F}{\delta \mathcal{L}} \mathcal{D}_{(1)} \frac{\delta G}{\delta \mathcal{L}} \right), \tag{1.2}$$

where $\mathcal{D}_{(1)} := 2\partial_\phi$.

The Poisson bracket (1.2), together with the Hamiltonian

$$H_{(1)}^{\mathrm{KdV}} = \oint d\phi \left(\frac{1}{8}\mathcal{L}^3 - \frac{1}{16}\mathcal{L}'^2 \right), \tag{1.3}$$

precisely reproduce the KdV equation (1.1), which can be rewritten as

$$\dot{\mathcal{L}} = \left\{\mathcal{L}, H^{\text{KdV}}_{(1)}\right\}_1 = \mathcal{D}_{(1)} \left(\frac{\delta H^{\text{KdV}}_{(1)}}{\delta \mathcal{L}}\right). \tag{1.4}$$

The integrability of the KdV equation and the existence of a hierarchy of integrable equations, rely on the fact that this system is actually bi-Hamiltonian. Indeed, there exists an alternative symplectic structure characterized by the following operator

$$\mathcal{D}_{(2)} := \partial_\phi \mathcal{L} + 2\mathcal{L}\partial_\phi + \frac{1}{2}\partial_\phi^3. \tag{1.5}$$

Consequently, the Poisson bracket of two arbitrary functionals F and G associated with the operator $\mathcal{D}_{(2)}$ is

$$\{F, G\}_2 = \int d\phi \left(\frac{\delta F}{\delta \mathcal{L}} \mathcal{D}_{(2)} \frac{\delta G}{\delta \mathcal{L}}\right). \tag{1.6}$$

Using the second Poisson bracket (1.6) and the Hamiltonian

$$H^{\text{KdV}}_{(0)} = \oint d\phi \left(\frac{1}{4}\mathcal{L}^2\right), \tag{1.7}$$

we can alternatively rewrite the KdV equation as

$$\dot{\mathcal{L}} = \left\{\mathcal{L}, H^{\text{KdV}}_{(0)}\right\}_2 = \mathcal{D}_{(2)} \left(\frac{\delta H^{\text{KdV}}_{(0)}}{\delta \mathcal{L}}\right). \tag{1.8}$$

1.1.2 Recursion relation and conserved charges of the KdV hierarchy

From eqs. (1.4) and (1.8), the following relation is obtained

$$\dot{\mathcal{L}} = \mathcal{D}_{(1)} \left(\frac{\delta H^{\text{KdV}}_{(1)}}{\delta \mathcal{L}}\right) = \mathcal{D}_{(2)} \left(\frac{\delta H^{\text{KdV}}_{(0)}}{\delta \mathcal{L}}\right),$$

which relates the "gradient" of the Hamiltonian $H^{\text{KdV}}_{(1)}$ with the "gradient" of the Hamiltonian $H^{\text{KdV}}_{(0)}$ as

$$\left(\frac{\delta H^{\text{KdV}}_{(1)}}{\delta \mathcal{L}}\right) = \mathcal{D}^{-1}_{(1)} \mathcal{D}_{(2)} \left(\frac{\delta H^{\text{KdV}}_{(0)}}{\delta \mathcal{L}}\right). \tag{1.9}$$

This can be used to define a recursion relation that associates the gradient of the k-th Hamiltonian $H^{\rm KdV}_{(k)}$ with the gradient of the $(k-1)$-th Hamiltonian $H^{\rm KdV}_{(k-1)}$ through

$$R_{(k)} = \mathcal{D}^{-1}_{(1)}\mathcal{D}_{(2)}R_{(k-1)}, \tag{1.10}$$

with

$$R_{(k)} := \frac{\delta H^{\rm KdV}_{(k)}}{\delta \mathcal{L}}, \tag{1.11}$$

which are the so called Gelfand-Dickey polynomials associated to the k-th Hamiltonian of the hierarchy.

Using the recursion relation (1.10), one can construct an infinite number of Hamiltonians $H^{\rm KdV}_{(k)}$. Each of these Hamiltonians defines an independent conserved charge of the KdV equation (1.1).

Finally, one can prove that the conserved charges $H^{\rm KdV}_{(k)}$ are in involution with respect to both Poisson brackets. In order to see that, let us consider $k > m$, so that the Poisson bracket associated to the operator $\mathcal{D}_{(2)}$ of two conserved charges can be written as

$$\begin{aligned}
\left\{H^{\rm KdV}_{(k)}, H^{\rm KdV}_{(m)}\right\}_2 &= \int d\phi \left(R_{(k)}\mathcal{D}_{(2)}R_{(m)}\right), \\
&= \int d\phi \left(R_{(k)}\mathcal{D}_{(1)}R_{(m+1)}\right), \\
&= -\int d\phi \left(\mathcal{D}_{(1)}R_{(k)}R_{(m+1)}\right), \\
&= -\int d\phi \left(R_{(m+1)}\mathcal{D}_{(2)}R_{(k-1)}\right),
\end{aligned} \tag{1.12}$$

where we have used the recursion relation (1.10). Thus, eq. (1.12) can be rewritten as

$$\begin{aligned}
\left\{H^{\rm KdV}_{(k)}, H^{\rm KdV}_{(m)}\right\}_2 &= -\left\{H^{\rm KdV}_{(m+1)}, H^{\rm KdV}_{(k-1)}\right\}_2, \\
&= \left\{H^{\rm KdV}_{(k-1)}, H^{\rm KdV}_{(m+1)}\right\}_2.
\end{aligned}$$

Applying this procedure $k-m$ times, one finally gets

$$\left\{H_{(k)}^{\text{KdV}}, H_{(m)}^{\text{KdV}}\right\}_2 = \left\{H_{(m)}^{\text{KdV}}, H_{(k)}^{\text{KdV}}\right\}_2 = 0,$$

which proves that all the conserved quantities are in involution with respect to the second Poisson bracket. The involution with respect to the first Poisson bracket can be proved in a similar way, so that

$$\left\{H_{(k)}^{\text{KdV}}, H_{(m)}^{\text{KdV}}\right\}_1 = 0.$$

Furthermore, we can use these conserved quantities $H_{(k)}^{\text{KdV}}$ as Hamiltonians of new differential equations. Hence, it is then possible to define a complete hierarchy of integrable equations. The k-th equation has the form

$$\dot{\mathcal{L}}_{(k)} = \left\{\mathcal{L}, H_{(k)}^{\text{KdV}}\right\}_1 = \left\{\mathcal{L}, H_{(k-1)}^{\text{KdV}}\right\}_2.$$

For example, the second member of the hierarchy can be obtained with the operator $\mathcal{D}_{(2)}$ and the Hamiltonian $H_{(1)}^{\text{KdV}}$. It is given by

$$\dot{\mathcal{L}} = \frac{15}{8}\mathcal{L}^2\mathcal{L}' + \frac{5}{4}\mathcal{L}'\mathcal{L}'' + \frac{5}{8}\mathcal{L}\mathcal{L}''' + \frac{1}{16}\mathcal{L}'''''. \tag{1.13}$$

1.2 Lax pairs

In 1967, Gardner, Greene, Kruskal, and Miura introduced in ref. [36] the inverse scattering transformation for the study of exact solutions of the initial-value problem for the KdV equation. This led to a new method for solving non-linear partial differential equations called the inverse scattering method, which can be regarded as an analogue of the Fourier transformation method for linear partial differential equations. One year after that, based on this method, Peter Lax showed in ref. [37] that the following pair of differential operators

$$L = \partial_\phi^2 + \mathcal{L}, \tag{1.14}$$

$$P = \partial_\phi^3 + \frac{3}{2}\mathcal{L}\partial_\phi + \frac{3}{4}\mathcal{L}', \tag{1.15}$$

together with the equation

$$\partial_t L = [P, L]\,, \tag{1.16}$$

reproduce the KdV equation.

Equation (1.16) is called the isospectral Lax equation, and relates an integrable non-linear partial differential equation like (1.1) with the following pair of equations

$$L\psi = \lambda\psi, \tag{1.17}$$

$$\dot{\psi} = P\psi. \tag{1.18}$$

The first equation is called the spectral equation of L, and the second one describes the time evolution for the eigenfunction ψ. Eq. (1.16) guarantees that the eigenvalue λ is time independent, as it can be seen by taking the time derivative of eq. (1.17)

$$\partial_t L\psi + L\partial_t\psi = \partial_t\lambda\psi + \lambda\partial_t\psi. \tag{1.19}$$

If one replaces eqs. (1.17) and (1.18) in eq. (1.19), the following condition is obtained

$$\left(\partial_t L - [P, L]\right)\psi = \partial_t\lambda\psi.$$

Therefore, if eq. (1.16) is satisfied, one concludes that $\partial_t\lambda = 0$, i.e., the KdV equation preserves the spectrum of the operator L. Furthermore, any member of the KdV hierarchy fulfills this property.

Besides, note that the operator P can be written as

$$P = \partial_\phi^3 + \frac{3}{2}\mathcal{L}\partial_\phi + \frac{3}{4}\mathcal{L}' = \left(L^{\frac{3}{2}}\right)_{\geq 0},$$

where $(\ldots)_{\geq 0}$ corresponds to the purely differential part of the pseudo-differential operator (See Appendix A).

As it was discussed above, the entire KdV hierarchy preserves the spectrum of the operator L. Thus, the k-th equation of the hierarchy is obtained by using the following commutator

$$\dot{\mathcal{L}} = \left[\left(L^{k+\frac{1}{2}}\right)_{\geq 0}, L\right].$$

1.2.1 Gelfand-Dickey hierarchies

In order to generalize the previous construction to higher order operators, one can consider the following n-th order differential operator[1]

$$L = \partial_\phi^n + \mathcal{U}_{n-2}\partial_\phi^{n-2} + \ldots + \mathcal{U}_1\partial_\phi + \mathcal{U}_0, \tag{1.20}$$

where the coefficients $\mathcal{U}_{n-2}, \ldots, \mathcal{U}_0$ are the $n-1$ real or complex fields of the theory under discussion.

The operator (1.20), together with

$$P_m = \left(L^{\frac{m}{n}}\right)_{\geq 0},$$

are called a Lax pair (L, P_m). The isospectral Lax equation then takes the form

$$\dot{L} = \left[\left(L^{\frac{m}{n}}\right)_{\geq 0}, L\right], \tag{1.21}$$

defining the equations associated to the Gelfand-Dickey hierarchy. In particular, for the

[1] More general cases can be found in ref. [38]

choice of integers ($n = 2,\ m = 3$), one recovers the KdV equation.

Boussinesq equation ($n = 3,\ m = 2$)

As an example, let us consider the following pair of operators:

$$L = \partial_\phi^3 + \mathcal{L}\partial_\phi + \left(\mathcal{W} + \frac{1}{2}\mathcal{L}'\right), \tag{1.22}$$

$$P_2 = \partial_\phi^2 + \frac{2}{3}\mathcal{L}. \tag{1.23}$$

Equation (1.21) then implies

$$\begin{aligned} \dot{\mathcal{L}} &= 2\mathcal{W}', \\ \dot{\mathcal{W}} &= -\frac{2}{3}\mathcal{L}\mathcal{L}' - \frac{1}{6}\mathcal{L}'''. \end{aligned} \tag{1.24}$$

Combining both equations and eliminating the field $\mathcal{W}$, one finds the following equation that must be satisfied by $\mathcal{L}$

$$\ddot{\mathcal{L}} = -\frac{4}{3}\left(\mathcal{L}\mathcal{L}'\right)' - \frac{1}{3}\mathcal{L}''''. \tag{1.25}$$

This is known as the "Good" Boussinesq equation (see e.g. [33]).

1.2.2 Poisson structures

The Poisson structures of the Hamiltonian formulation presented in sec. 1.1, can be recovered by using the Lax operators. Following [38], the first Poisson structure associated to the n-th order differential operator L in eq. (1.20) is given by the commutator

$$\dot{L} = \Theta_1(L)\nabla H = [L, \nabla H]_{\geq 0}, \tag{1.26}$$

where ∇H is a pseudo-differential operator defined by[2]

$$\nabla H = \sum_{i=0}^{n-2} \partial_\phi^{-i-1} \frac{\delta H}{\delta \mathcal{U}_i}.$$

The second Poisson structure is given by

$$\dot{L} = \Theta_2(L)\nabla H = (L\nabla H)_{\geq 0} L - L(\nabla H\, L)_{\geq 0} + \frac{1}{n}\left[\partial^{-1}\text{res}\left([\nabla H, L]\right), L\right], \tag{1.27}$$

where $\text{res}(\ldots)$ means the residue of the operator, i.e., the coefficient of the order ∂^{-1}. Equations (1.26) and (1.27) can be rewritten in the following form

$$\dot{\mathcal{U}}_{(k)} = \mathscr{O}_1 \frac{\delta H_{(k+1)}}{\delta \mathcal{U}} = \mathscr{O}_2 \frac{\delta H_{(k)}}{\delta \mathcal{U}}, \tag{1.28}$$

where $\mathcal{U}$ denotes the set of fields $(\mathcal{U}_{n-2}, \ldots, \mathcal{U}_0)$ and the operators $\mathscr{O}_1$ and $\mathscr{O}_2$ are certain differential $(n-1) \times (n-1)$ matrix operators.

1.2.3 Modified Gelfand-Dickey hierarchies

The n-th order differential operator in eq. (1.20) can be factorized as

$$L = (\partial_\phi - \mathcal{V}_{n-1})(\partial_\phi - \mathcal{V}_{n-2})\cdots(\partial_\phi - \mathcal{V}_0), \tag{1.29}$$

If we compare this expression with (1.20), we find an equation which allow us to write the field $\mathcal{V}_{n-1}$ in terms of the remaining $(n-1)$-$\mathcal{V}$ fields. This can be traced back to the absence of a term in the order ∂^{n-1} in eq. (1.20). Now we can write a set of $(n-1)$-differential equations which relate the fields $\mathcal{U}$ with the fields $\mathcal{V}$ and its spatial derivatives.

$$\mathcal{U}_i = \mathcal{U}_i(\mathcal{V}_0, \ldots, \mathcal{V}_{n-2}), \qquad i = 0, \ldots, n-2. \tag{1.30}$$

[2]See Appendix A for some properties of the pseudo-differential operators.

These equations define the Miura transformation. It was discovered in 1968 by Robert Miura in the study of the KdV equation [39].

The fields $\mathcal{V}$ also satisfy a system of non-linear partial differential equations, known as modified Gelfand-Dickey equations. These equations are also bi-Hamiltonian, i.e., they possess two compatible Poisson structures.

In order to obtain the Poisson structures, let us take the time derivative of the Miura transformation in eq. (1.30)

$$\dot{\mathcal{U}}_{(k)} = \left(\frac{\partial \mathcal{U}}{\partial \mathcal{V}}\right) \dot{\mathcal{V}}_{(k)} = M \dot{\mathcal{V}}_{(k)}, \tag{1.31}$$

where $M = \left(\frac{\partial \mathcal{U}}{\partial \mathcal{V}}\right)$ is a $(n-1) \times (n-1)$-matrix differential operator called the Fréchet derivative, whose formal adjoint $M^\dagger$ relates the Gelfand-Dickey polynomials of both systems

$$\frac{\delta H_{(k)}}{\delta \mathcal{V}} = M^\dagger \frac{\delta H_{(k)}}{\delta \mathcal{U}}. \tag{1.32}$$

By means of eqs. (1.28), (1.31) and (1.32) we can obtain the equations of the modified Gelfand-Dickey hierarchy

$$\dot{\mathcal{V}}_{(k)} = \mathscr{D}_1 \frac{\delta H_{(k)}}{\delta \mathcal{V}} = \mathscr{D}_2 \frac{\delta H_{(k-1)}}{\delta \mathcal{V}}, \tag{1.33}$$

where $\mathscr{D}_1$ and $\mathscr{D}_2$ are $(n-1) \times (n-1)$ matrix differential operators given by

$$\mathscr{D}_1 = M^{-1} \mathscr{O}_2 \left(M^\dagger\right)^{-1}, \tag{1.34}$$

$$\mathscr{D}_2 = \mathscr{D}_1 M^\dagger \mathscr{O}_1^{-1} M \mathscr{D}_1. \tag{1.35}$$

Modified KdV hierarchy

The KdV Lax operator (1.14), rewritten as in eq. (1.29), takes the form

$$L=\partial_\phi^2+\mathcal{L}=\left(\partial_\phi-\mathcal{J}\right)\left(\partial_\phi+\mathcal{J}\right).$$

From here one can read the Miura transformation

$$\mathcal{L}=\mathcal{J}'-\mathcal{J}^2. \tag{1.36}$$

Besides, the Fréchet derivative and its formal ajdoint are given by

$$\begin{aligned} M &= \partial_\phi-2\mathcal{J}, \\ M^\dagger &= -\partial_\phi-2\mathcal{J}. \end{aligned}$$

The modified KdV equation can then be obtained by replacing the Miura transformation (1.36) in the KdV equation (1.1)

$$\dot{\mathcal{J}}=-\frac{3}{2}\mathcal{J}^2\mathcal{J}'+\frac{1}{4}\mathcal{J}'''. \tag{1.37}$$

Finally, the entire modified KdV hierarchy can be obtained using the recursion relation and the following two Poisson structures

$$\mathscr{D}_1 = -\frac{1}{2}\partial_\phi, \tag{1.38}$$

$$\mathscr{D}_2 = -2\partial_\phi\left(\mathcal{J}\partial_\phi^{-1}\left(\mathcal{J}\partial_\phi\right)\right)+\frac{1}{2}\partial_\phi^3. \tag{1.39}$$

Remarkably, there is another set of integrable systems called the Gardner hierarchy, which combines both, the KdV and mKdV hierarchies. The first member of the Gardner hierarchy is given by

$$\dot{\mathcal{J}}=\frac{3}{4}\mathcal{J}\mathcal{J}'-\frac{3}{4}\mathcal{J}^2\mathcal{J}'+\frac{1}{4}\mathcal{J}''', \tag{1.40}$$

which is known as the Gardner equation.

It is worth mentioning that there is an alternative formulation of the integrable systems discussed in this chapter, known as the "zero curvature formulation." In this framework the Lax pair is described by a two-dimensional connection with vanishing field strength which is valued on a certain Lie algebra [73]. In the case of the (modified) KdV hierarchy the corresponding Lie algebra is $SL(2, \mathbb{R})$, while for the case of the N-th (modified) Gelfand-Dickey hierarchy is $SL(N, \mathbb{R})$.

The zero curvature formulation turns out to be the natural one that allows us to make contact with gravitational theories. This is widely discussed in the next chapters.

Chapter 2

Boundary conditions for gravity on AdS_3 and the Gardner hierarchy

In this chapter we elaborate on the deep relationship that exists between integrable systems and gravitational theories with certain boundary conditions. In particular, we explore the consequences of extending the “soft hairy” boundary conditions on AdS_3 introduced in refs. [5, 40] by choosing the chemical potentials as appropriate *local* functions of the fields, and its relation with integrable systems. For this class of boundary conditions, the fall-off of the metric near infinity differs from the one of Brown and Henneaux. The boundary conditions have the particular property that, by virtue of the topological nature of three-dimensional General Relativity, they can also be interpreted as being defined in the near horizon region of spacetimes with event horizons.

The corresponding integrable system is shown to be the Gardner hierarchy, also known as the “mixed KdV-mKdV” hierarchy (see e.g. [41]), whose first member is given by [1]

$$\dot{\mathcal{J}} = 3a\mathcal{J}\mathcal{J}' + 3b\mathcal{J}^2\mathcal{J}' - 2\mathcal{J}'''. \tag{2.1}$$

[1] Eq. (1.40) is recovered from eq. (2.1) by applying the following rescaling $\mathcal{J} \to -\frac{a}{b}\mathcal{J}$, $t \to -\sqrt{2b}\frac{b}{4a^3}t$, $\phi \to \frac{\sqrt{2b}}{a}\phi$.

Here, a, b are arbitrary constants. Equation (2.1) has the very special property that simultaneously combines both, KdV and modified KdV (mKdV) equations. Remarkably, in spite of the deformations generated by the arbitrary parameters a and b, the integrability of this equation and of the complete hierarchy associated to it, is maintained. Indeed, when $b = 0$ and $a \neq 0$, eq. (2.1) precisely reduces to the KdV equation, while when $a = 0$ and $b \neq 0$ it coincides with the mKdV equation.

From the point of view of the gravitational theory, there exists a precise choice of chemical potentials, that extends the boundary conditions in [5, 40], such that Einstein equations reduce to two (left and right) copies of eq. (2.1). Moreover, the infinite number of abelian conserved quantities of the Gardner hierarchy are obtained from the asymptotic symmetries and the canonical generators in General Relativity using the Regge-Teitelboim approach [42]. These results are generalized to the whole hierarchy, and can also be applied to the case of vanishing cosmological constant. We also show that if the hierarchy is "extended backwards" one recovers the soft hairy boundary conditions of refs. [5, 40] as a particular case.

For a negative cosmological constant, black hole solutions are naturally accommodated within our boundary conditions. They are described by static configurations associated to the corresponding member of the Gardner hierarchy. In particular, BTZ black holes [43, 44] correspond to constant values of the left and right fields $\mathcal{J}_{\pm}$. Static solutions with nonconstants $\mathcal{J}_{\pm}$ can also be found, describing black holes carrying nontrivial conserved charges. The thermodynamic properties of the black holes in the ensembles defined by our boundary conditions are also briefly discussed.

2.1 Asymptotic behavior of the gravitational field

2.1.1 Chern-Simons formulation of gravity on AdS$_3$

In this section we describe the asymptotic behavior (fall-off) of the gravitational field without specifying yet what is fixed at the boundary of spacetime, i.e., without imposing at this step of the analysis a precise boundary condition. For the purpose of simplicity and clarity in the presentation, we mostly work in the Chern–Simons formulation of three-dimensional Einstein gravity with a negative cosmological constant [45, 46].

The action for General Relativity on AdS$_3$ can be written as a Chern–Simons action for the gauge group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

$$I = I_{CS}\left[A^{+}\right] - I_{CS}\left[A^{-}\right],$$

where

$$I_{CS}\left[A\right] = \frac{\kappa}{4\pi}\int\left\langle AdA + \frac{2}{3}A^{3}\right\rangle . \tag{2.2}$$

Here, the gauge connections $A^{\pm}$ are 1-forms valued on the $sl(2,\mathbb{R})$ algebra, and are related to the vielbein e and spin connection ω through $A^{\pm} = \omega \pm e\ell^{-1}$. The level in (2.2) is given by $\kappa = \ell/4G$, where ℓ is the AdS radius and G is the Newton constant, while the bilinear form $\langle , \rangle$ is defined by the trace in the fundamental representation of $sl(2,\mathbb{R})$. The $sl(2,\mathbb{R})$ generators L_n with $n = 0, \pm 1$, obey the commutation relations $[L_n, L_m] = (n-m)\,L_{n+m}$, with non-vanishing components of the bilinear form given by $\langle L_1 L_{-1}\rangle = -1$, and $\left\langle L_0^2\right\rangle = 1/2$.

2.1.2 Asymptotic form of the gauge field

In order to describe the fall-off of the gauge connection, we closely follow the analysis in refs. [5, 40]. We will assume that the gauge fields in the asymptotic region take the form

$$A^{\pm} = b_{\pm}^{-1}\left(d + \mathfrak{a}^{\pm}\right) b_{\pm}, \tag{2.3}$$

where $b_{\pm}$ are gauge group elements depending only on the radial coordinate, and $\mathfrak{a}^{\pm} = \mathfrak{a}_t^{\pm} dt + \mathfrak{a}_{\phi}^{\pm} d\phi$ correspond to auxiliary connections that only depend on the temporal and angular coordinates t and ϕ, respectively [47]. It is worth pointing out that, by virtue of the lack of local propagating degrees of freedom of three-dimensional Einstein gravity, all the relevant physical information necessary for the asymptotic analysis is completely encoded in the auxiliary connections $\mathfrak{a}^{\pm}$, independently of the precise choice of $b_{\pm}$. We will not specify any particular $b_{\pm}$ until section 2.3, where the metric formulation will be discussed.

The auxiliary connections $\mathfrak{a}^{\pm}$ are chosen to be given by

$$\mathfrak{a}^{\pm} = L_0\left(\zeta^{\pm} dt \pm \mathcal{J}^{\pm} d\phi\right). \tag{2.4}$$

Here, $\mathfrak{a}^{\pm}$ are diagonal matrices in the fundamental representation of $sl(2,\mathbb{R})$, and by that reason we say that the connections $\mathfrak{a}^{\pm}$ in eq. (2.4) are written in the "diagonal gauge." The fields $\zeta^{\pm}$ are defined along the temporal components of $\mathfrak{a}^{\pm}$, and consequently they correspond to Lagrange multipliers (chemical potentials). On the other hand, $\mathcal{J}^{\pm}$ belong to the spatial components of the gauge connections and therefore they are identified as the dynamical fields.

The field equations are determined by the vanishing of the field strength $F^{\pm} = dA^{\pm} + A^{\pm 2} = 0$, and take the form

$$\dot{\mathcal{J}}_{\pm} = \pm\zeta'_{\pm}. \tag{2.5}$$

Note that there are no time derivatives associated to the fields $\zeta_\pm$, which is consistent with the fact that they are chemical potentials.

It is important to emphasize that, as was pointed out in refs. [5, 40], one can also interpret the connections in (2.3) and (2.4) as describing the behavior of the fields in a region near a horizon. Indeed, the reconstructed metric can always be expressed (in a co-rotating frame) as the direct product of the two-dimensional Rindler metric times S^1, together with appropriate deviations from it. Furthermore, the charges are independent of the gauge group element $b_\pm(r)$, and consequently they do not depend on the precise value of the radial coordinate where they are evaluated. In this sense, one can also interpret the present analysis as describing "near horizon boundary conditions." This near horizon interpretation can also be extended to higher spacetime dimensions [48].

2.1.3 Consistency with the action principle and canonical generators

A fundamental requirement that must be guaranteed is the consistency of the asymptotic form of the gauge connections in eqs. (2.3) and (2.4) with the action principle. In order to analyze the physical consequences of this requirement, we will use the Regge-Teitelboim approach [42] in the canonical formulation of Chern–Simons theory.

The canonical action can be written as

$$I_{can}\left[A^\pm\right] = -\frac{\kappa}{4\pi}\int dt d^2x \varepsilon^{ij}\left\langle A_i^\pm \dot{A}_j^\pm - A_t^\pm F_{ij}^\pm\right\rangle + B_\infty^\pm,$$

where $B_\infty^\pm$ are boundary terms needed to ensure that the action principle attains an extremum. Their variations are then given by

$$\delta B_\infty^\pm = -\frac{\kappa}{2\pi}\int dt d\phi \left\langle A_t^\pm \delta A_\phi^\pm\right\rangle,$$

so that, when one evaluates them for the asymptotic form of the connections (2.3) and (2.4) yields

$$\delta B_\infty^\pm = \mp\frac{\kappa}{4\pi}\int dt d\phi \zeta_\pm \delta\mathcal{J}_\pm. \tag{2.6}$$

Consistency of the analysis requires that one must be able to "take the delta outside" in the variation of the boundary terms (2.6), i.e., they must be integrable in the functional space[2]. This can only be achieved provided one specifies a precise boundary condition, which turns out to be equivalent to specify what fields are fixed, i.e., without functional variation, at the boundary of spacetime. Following ref. [6], a generic possible choice of boundary conditions is to assume that the chemical potentials $\zeta_{\pm}$ depend on the dynamical fields $\mathcal{J}_{\pm}$ through

$$\zeta_{\pm} = \frac{4\pi}{\kappa} \frac{\delta H^{\pm}}{\delta \mathcal{J}_{\pm}}, \tag{2.7}$$

where $H^{\pm} = \int d\phi \mathcal{H}^{\pm}[\mathcal{J}_{\pm}, \mathcal{J}_{\pm}', \mathcal{J}_{\pm}'', \dots]$ are functionals depending *locally* on the fields $\mathcal{J}_{\pm}$ and their spatial derivatives. Here we also assume that the left and right sectors are decoupled. With the particular choice of boundary conditions given by (2.7), the delta in (2.6) can be immediately taken outside, and consequently the boundary terms necessary to improve the canonical action take the form

$$B^{\pm} = \mp \int dt H^{\pm}.$$

An immediate consequence of this choice is that the total energy of the system, defined as the on-shell value of the generator of translations in time, can be directly written in terms of the "Hamiltonians" $H^{\pm}$ as

$$E = H^{+} + H^{-}. \tag{2.8}$$

The conserved charges associated to the asymptotic symmetries are also sensitive to the choice of boundary conditions, but in a more subtle way. The form of the connections $\mathfrak{a}^{\pm}$ in eq. (2.4) is preserved under gauge transformations $\delta\mathfrak{a}^{\pm} = d\lambda^{\pm} + [\mathfrak{a}^{\pm}, \lambda^{\pm}]$, with gauge

[2]The requirement of integrability of the boundary terms can relaxed in the presence of ingoing or outgoing radiation, where their lack of integrability precisely gives the rate of change of the charges in time [49–51]. In the present case this possibility is not at hand, because General Relativity in 2+1-dimensions does not have local propagating degrees of freedom.

parameters $\lambda^{\pm}$ given by[3]

$$\lambda^{\pm} = \eta_{\pm} L_0,$$

provided the fields $\mathcal{J}^{\pm}$ and the chemical potentials $\zeta^{\pm}$ transform as

$$\delta \mathcal{J}_{\pm} = \pm \eta'_{\pm}, \tag{2.9}$$

$$\delta \zeta_{\pm} = \dot{\eta}_{\pm}. \tag{2.10}$$

Following the Regge-Teitelboim approach [42], the variation of the conserved charges reads

$$\delta Q^{\pm}\left[\eta_{\pm}\right] = \pm \frac{\kappa}{4\pi} \oint d\phi \eta_{\pm} \delta \mathcal{J}_{\pm}. \tag{2.11}$$

Here, the parameters $\eta^{\pm}$ are not arbitrary, because now the chemical potentials $\zeta^{\pm}$ depend on the fields $\mathcal{J}^{\pm}$ and their spatial derivatives. Hence, one must use the chain rule in the variation at the left hand-side of eq. (2.10), giving the following first order differential equations in time for $\eta^{\pm}$

$$\dot{\eta}_{\pm} = \pm \frac{4\pi}{\kappa} \frac{\delta}{\delta \mathcal{J}_{\pm}} \int d\phi \frac{\delta H^{\pm}}{\delta \mathcal{J}_{\pm}} \partial_{\phi} \eta_{\pm}. \tag{2.12}$$

Generically, these differential equations will depend explicitly on $\mathcal{J}_{\pm}$, and consequently finding explicit solutions to them becomes a very hard task. However, this can be achieved for certain special choices of boundary conditions, which are related to the integrable hierarchy associated to the Gardner equation. This will be discussed in detail in the next section.

[3]Extra terms along L_1 and L_{-1} might also be added, however they are pure gauge.

2.2 Different choices of boundary conditions in the diagonal gauge

2.2.1 Soft hairy boundary conditions

A simple choice of boundary conditions corresponds to fix the chemical potentials $\zeta_\pm$ at boundary, such that they are arbitrary functions without variation, i.e., $\delta\zeta_\pm = 0$. This possibility was analyzed in detail in refs. [5, 40], and it was termed "soft hairy boundary conditions." The asymptotic symmetry algebra is then spanned by the generators

$$Q^{\pm}_{\text{soft hairy}}\left[\eta_\pm\right] = \pm\frac{\kappa}{4\pi}\oint d\phi \eta_\pm \mathcal{J}_\pm,$$

where, by the consistency with time evolution, the parameters $\eta_\pm$ are arbitrary functions without variation $(\delta\eta_\pm = 0)$. The global charges are then characterized by $\mathcal{J}_\pm$, which obey the following Poisson brackets

$$\left\{\mathcal{J}_\pm\left(\phi\right), \mathcal{J}_\pm\left(\bar{\phi}\right)\right\} = \pm\frac{4\pi}{\kappa}\partial_\phi\delta\left(\phi-\bar{\phi}\right). \tag{2.13}$$

The asymptotic symmetry algebra is then given by two copies of $\hat{u}\left(1\right)$ current algebras.

In a co-rotating frame, $\zeta_+ = \zeta_- = const.$, the generator of time evolution is identified with the sum of the left and right zero modes of $\mathcal{J}_\pm$, that commute with all the members of the algebra. In this sense, one can say that the higher modes describe "soft hair excitations" in the sense of Hawking, Perry and Strominger [52, 53], because they do not change the energy of the gravitational configuration. Besides, in this frame, it is possible to find solutions with non-extremal horizons which are diffeomorphic to BTZ black holes, and that are endowed with not trivial soft hair charges. These solutions were called "black flowers"[4] [5, 40].

[4] A different type of black flower solution was found in ref. [54] in the context of new massive gravity [55, 56].

2.2.2 Gardner equation

A different choice of boundary conditions that makes contact with the Gardner (mixed KdV-mKdV) equation is $\zeta^{\pm} = \frac{3}{2} a {\mathcal{J}_{\pm}}^2 + b {\mathcal{J}_{\pm}}^3 - 2\mathcal{J}_{\pm}''$, which according to eq. (2.7) corresponds to use the following Hamiltonians

$$H_{(1)}^{\pm} = \frac{\kappa}{4\pi} \oint d\phi \left(\frac{1}{2} a \mathcal{J}_{\pm}^3 + \frac{1}{4} b \mathcal{J}_{\pm}^4 + \mathcal{J}_{\pm}'^2 \right) . \tag{2.14}$$

With this particular choice of chemical potentials, Einstein equations (2.5) precisely reduce to two independent left and right copies of the Gardner equation (2.1), that read

$$\dot{\mathcal{J}}_{\pm} = \pm \left(3a \mathcal{J}_{\pm} \mathcal{J}_{\pm}' + 3b {\mathcal{J}_{\pm}}^2 \mathcal{J}_{\pm}' - 2\mathcal{J}_{\pm}''' \right) . \tag{2.15}$$

Equations (2.12), describing the time evolution of the gauge parameters $\eta^{\pm}$, now take the form

$$\dot{\eta}_{\pm} = \pm \left(3a \mathcal{J}_{\pm} \eta_{\pm}' + 3b {\mathcal{J}_{\pm}}^2 \eta_{\pm}' - 2\eta_{\pm}''' \right) , \tag{2.16}$$

where the explicit dependence on $\mathcal{J}_{\pm}$ is manifest. These equations are linear in $\eta_{\pm}$, and by virtue of the integrability of the system, it is possible to find their general solutions under the assumption that they depend locally on $\mathcal{J}_{\pm}$ and their spatial derivatives. Indeed, the gauge parameters $\eta_{\pm}$ obeying eq. (2.16), are expressed as a linear combination of "generalized Gelfand-Dickey polynomials" $R_{(n)}^{\pm}$, i.e.,

$$\eta_{\pm} = \frac{4\pi}{\kappa} \sum_{n \geq 0} \alpha_{(n)}^{\pm} R_{(n)}^{\pm}, \tag{2.17}$$

where $\alpha_{(n)}^{\pm}$ are arbitrary constants, and the polynomials $R_{(n)}^{\pm}$ are defined through the following recursion relation

$$\partial_{\phi} R_{(n+1)}^{\pm} = \mathcal{D}_{\phi} R_{(n)}^{\pm}, \tag{2.18}$$

where $\mathcal{D}_{\phi}$ is a non–local operator given by

$$\mathcal{D}_\phi := a\left(\partial_\phi \mathcal{J}_\pm + 2\mathcal{J}_\pm \partial_\phi\right) + 2b\partial_\phi \left(\mathcal{J}_\pm \partial_\phi^{-1}\left(\mathcal{J}_\pm \partial_\phi\right)\right) - 2\partial_\phi^3. \tag{2.19}$$

The polynomials $R^\pm_{(n)}$ can be expressed as the "gradient" of the Hamiltonians $H^\pm_{(n)}$ of the integrable system, i.e.,

$$R^\pm_{(n)} = \frac{\delta H^\pm_{(n)}}{\delta \mathcal{J}_\pm}. \tag{2.20}$$

The first generalized Gelfand-Dickey polynomials together with their corresponding Hamiltonians are explicitly displayed in appendix B.

If one replaces the general solution of eqs. (2.16), given by (2.17), into the expression for the variation of the canonical generators (2.11), one can take immediately the delta outside, and write the conserved charges as

$$Q^\pm = \pm \sum_{n\geq 0} \alpha^\pm_{(n)} H^\pm_{(n)}, \tag{2.21}$$

where the Hamiltonians $H^\pm_{(n)}$ are in involution with respect to the Poisson brackets (2.13),

$$\left\{H^\pm_{(n)}, H^\pm_{(m)}\right\} = 0.$$

The presence of the operator $\mathcal{D}_\phi$ in the recurrence relation for the generalized Gelfand-Dickey polynomials (2.18), is related to the fact that this integrable system is bi–Hamiltonian. Consequently, it is possible to define a second Poisson bracket between the dynamical fields

$$\left\{\mathcal{J}_\pm\left(\phi\right), \mathcal{J}_\pm\left(\bar{\phi}\right)\right\}_2 := \pm\frac{4\pi}{\kappa}\mathcal{D}_\phi \delta\left(\phi - \bar{\phi}\right), \tag{2.22}$$

so that the Gardner equations can be written as

$$\dot{\mathcal{J}}_\pm = \left\{\mathcal{J}_\pm, H^\pm_{(1)}\right\} = \left\{\mathcal{J}_\pm, H^\pm_{(0)}\right\}_2,$$

where the Hamiltonians $H^\pm_{(1)}$ are given by eq. (2.14), while the Hamiltonians $H^\pm_{(0)}$ take the form

$$H^{\pm}_{(0)} = \frac{\kappa}{4\pi} \oint d\phi \left(\frac{1}{2} \mathcal{J}^2_{\pm} \right). \tag{2.23}$$

It is worth pointing out that only the first Poisson bracket structure, given by eq. (2.13), is obtained from the gravitational theory using the Dirac method for constrained systems. It is not clear how to obtain the second Poisson structure (2.22) directly from the canonical structure of General Relativity.

As it was discussed in the introduction, the (left/right) Gardner equation in (2.15) has the special property that when $a = 0$ and $b \neq 0$, the equation and its whole integrable structure, precisely reduce to those of the mKdV integrable system. Conversely, when $b = 0$ and $a \neq 0$, the integrable system reduces to KdV. In this case, the operator $\mathcal{D}_\phi$ in (2.19) becomes a local differential operator, and the second Poisson structure associated to it through eq. (2.22) corresponds to two copies of the Virasoro algebra with left and right central charges given by $c^{\pm} = 3\ell/\left(Ga^2\right)$. Note that these central charges do not coincide with the ones of Brown and Henneaux. This is not surprising because, as we say before, this Poisson structure does not come directly from the canonical structure of the gravitational theory.

The Gardner equation does not have additional symmetries besides the ones generated by the infinite number of commuting Hamiltonians described above. In particular, it is not invariant under Galilean boosts unless $b = 0$ (KdV case). Nevertheless, it is possible to show that there exists a particular Galilean boost with parameter $\omega = 3a^2/\left(4b\right)$, together with a shift in $\mathcal{J}_{\pm}$ given by $\mathcal{J}_{\pm} \to \mathcal{J}_{\pm} - a/\left(2b\right)$, such that the Gardner equation reduces to the mKdV equation. However, the conserved charges are not mapped into each other, which is a direct consequence of the fact that this transformation does not map the higher members of the Gardner and mKdV hierarchies.

In the next section we will show how to extend the previous results in order to incorporate an arbitrary member of the Gardner hierarchy.

2.2.3 Extension to the Gardner hierarchy

It is possible to find precise boundary conditions for General Relativity on AdS_3, such that the dynamics of the boundary degrees of freedom are described by the k-th member of the Gardner hierarchy. This is achieved by choosing the chemical potentials $\zeta_\pm$ as follows

$$\zeta_\pm = \frac{4\pi}{\kappa} R^\pm_{(k)}, \tag{2.24}$$

where $R^\pm_{(k)}$ are the k-th generalized Gelfand-Dickey polynomials. The functionals $H^\pm$ in eq. (2.7), are identified with the k-th Hamiltonian of the hierarchy defined by eq. (2.20). The particular case $k = 1$ corresponds to the one developed in the previous section.

With the boundary condition specified by eq. (2.24), Einstein equations in (2.5) take the form

$$\dot{\mathcal{J}}_\pm = \pm\frac{4\pi}{\kappa}\partial_\phi R^\pm_{(k)}, \tag{2.25}$$

which coincide with two (left/right) copies of the k-th element of the Gardner hierarchy. By virtue of the bi-Hamiltonian character of the system, eq. (2.25) can also be written as

$$\dot{\mathcal{J}}_\pm = \left\{\mathcal{J}_\pm, H^\pm_{(k)}\right\} = \left\{\mathcal{J}_\pm, H^\pm_{(k-1)}\right\}_2. \tag{2.26}$$

The equations that describe the time evolution of the gauge parameters $\eta_\pm$ take the form of eq. (2.12), but with $H^\pm \to H^\pm_{(k)}$, whose general solution that depends locally on $\mathcal{J}_\pm$ and their spatial derivatives coincides with eq. (2.17). Consequently, the global charges integrate as (2.21), which as it was discussed in the previous section, commute among them with both Poisson structures.

Lifshitz scaling.

The members of the Gardner hierarchy are not invariant under Lifshitz scaling. However, in the particular cases when they belong to KdV or mKdV hierarchies, the anisotropic scaling, with dynamical exponent $z = 2k + 1$, is restored.

Case 1: $a = 0$ *(mKdV)*

When $a = 0$, the k-th member of the hierarchy is invariant under

$$t \to \lambda^z t \quad , \quad \phi \to \lambda\phi \quad , \quad \mathcal{J}_\pm \to \lambda^{-1}\mathcal{J}_\pm . \tag{2.27}$$

Case 2: $b = 0$ *(KdV)*

When $b = 0$, the k-th member of the hierarchy is invariant under

$$t \to \lambda^z t \quad , \quad \phi \to \lambda\phi \quad , \quad \mathcal{J}_\pm \to \lambda^{-2}\mathcal{J}_\pm . \tag{2.28}$$

Note that the dynamical exponent is the same in both cases, but the fields $\mathcal{J}_\pm$ scale in different ways.

Extending the hierarchy backwards.

The first nonlinear members of the Gardner hierarchy correspond to the case $k = 1$, and are given by eq. (2.15). However, if one uses the Hamiltonians $H^\pm_{(0)}$ defined in eq. (2.23) together with the first Poisson bracket structure (2.13), one obtains a linear equation describing left and right chiral movers

$$\dot{\mathcal{J}}_\pm = \pm\mathcal{J}'_\pm ,$$

which can be considered as the member with $k = 0$ of the hierarchy.

From the point of view of the gravitational theory, it is useful to extend the hierarchy an additional step backwards by using the recursion relation (2.18) in the opposite direction. Hence, one obtains a new Hamiltonian for each copy of the form

$$H^\pm_{(-1)} = \frac{\kappa}{4\pi}\oint d\phi \left(a^{-1}\mathcal{J}_\pm\right) . \tag{2.29}$$

These Hamiltonians can only be defined when $a \neq 0$, and their corresponding generalized Gelfand-Dickey polynomials $R^\pm_{(-1)} = \kappa/\left(4\pi a\right)$ can be used as a seed that generates the whole hierarchy through the recursion relation.

Using the Hamiltonians $H^{\pm}_{(-1)}$ in (2.29), together with the first Poisson bracket structure (2.13), the soft hairy boundary conditions of refs. [5, 40], reviewed in section 2.2.1, are recovered in a co-rotating frame. In this case, the chemical potentials $\zeta_{\pm}$ are constants and given by

$$\zeta_{\pm} = \frac{4\pi}{\kappa} R^{\pm}_{(-1)} = a^{-1},$$

while the equations of motion become $\dot{\mathcal{J}}_{\pm} = 0$. In this sense, one can consider the soft hairy boundary conditions as being part of the hierarchy, and besides as the first member of it.

2.3 Metric formulation

In this section, we provide a metric description of the results previously obtained in the context of the Chern–Simons formulation of General Relativity on AdS_3. As was discussed in sec. 2.1.2, the boundary conditions that describe the Gardner hierarchy can be interpreted as being defined either at infinity or in the near horizon region. We will analyze these two possible interpretations in the metric formalism following the lines of ref. [40].

2.3.1 Asymptotic behavior

The spacetime metric can be directly reconstructed from the Chern–Simons fields (2.3), (2.4), provided a particular gauge group element $b_{\pm}(r)$ is specified. In order to describe the metric in the asymptotic region, it is useful to choose

$$b_{\pm}(r) = \exp\left[\pm\frac{1}{2}\log\left(\frac{2r}{\ell}\right)(L_1 - L_{-1})\right].$$

The expansion of the metric for $r \to \infty$ then reads

$$
\begin{aligned}
g_{tt} &= -\zeta_+\zeta_- r^2 + \frac{\ell^2}{4}\left(\zeta_+^2+\zeta_-^2\right) + \mathcal{O}\left(r^{-1}\right), \\
g_{tr} &= \mathcal{O}\left(r^{-2}\right), \\
g_{t\phi} &= \left(\zeta_+\mathcal{J}_- - \zeta_-\mathcal{J}_+\right)\frac{r^2}{2} + \frac{\ell^2}{4}\left(\zeta_+\mathcal{J}_+ - \zeta_-\mathcal{J}_-\right) + \mathcal{O}\left(r^{-1}\right), \\
g_{rr} &= \frac{\ell^2}{r^2} + \mathcal{O}\left(r^{-5}\right), \\
g_{r\phi} &= \mathcal{O}\left(r^{-2}\right), \\
g_{\phi\phi} &= \mathcal{J}_+\mathcal{J}_- r^2 + \frac{\ell^2}{4}\left(\mathcal{J}_+^2+\mathcal{J}_-^2\right) + \mathcal{O}\left(r^{-1}\right).
\end{aligned}
\tag{2.30}
$$

As it was explained in sec. 2.2.3, in order to implement the boundary conditions associated to the Gardner hierarchy we must choose the chemical potentials $\zeta_\pm$ according to eq. (2.24), i.e.,

$$
\zeta_\pm = \frac{4\pi}{\kappa} R^\pm_{(k)}.
$$

The differential equations associated to the (left/right) k-th element of the hierarchy are precisely recovered if one imposes that the metric in eq. (2.30) obeys Einstein equations with a negative cosmological constant in the asymptotic region of spacetime.

The fall-off in (2.30) is preserved under the asymptotic symmetries generated by the following Killing vectors

$$
\begin{aligned}
\xi^t &= \frac{\eta^+\mathcal{J}_- + \eta^-\mathcal{J}_+}{\zeta_+\mathcal{J}_- + \zeta_-\mathcal{J}_+} + \mathcal{O}\left(\frac{1}{r^3}\right), \\
\xi^r &= \mathcal{O}\left(\frac{1}{r^2}\right), \\
\xi^\phi &= \frac{\eta^+\zeta_- - \eta^-\zeta_+}{\zeta_+\mathcal{J}_- + \zeta_-\mathcal{J}_+} + \mathcal{O}\left(\frac{1}{r^3}\right).
\end{aligned}
\tag{2.31}
$$

The conserved charges can be directly computed using the Regge-Teitelboim approach [42], and as expected coincide with the expression in eq. (2.11) obtained using the Chern–Simons formulation.

Note that the boundary metric

$$d\bar{s}^2 = r^2 \left(-\zeta_+\zeta_- dt^2 + \left(\zeta_+\mathcal{J}_- - \zeta_-\mathcal{J}_+\right) dt d\phi + \mathcal{J}_+\mathcal{J}_- d\phi^2\right),$$

explicitly depends on the dynamical fields $\mathcal{J}_\pm$ and consequently *it is not fixed* at the boundary of spacetime, i.e., it has a nontrivial functional variation.

2.3.2 Near horizon behavior

Following ref. [40], the metric in the near horizon region can be reconstructed using

$$b_\pm(r) = \exp\left(\pm\frac{r}{2\ell}\left(L_1 - L_{-1}\right)\right),$$

and considering an expansion around $r = 0$. The metric then reads

$$\begin{aligned}
g_{tt} &= \frac{\ell^2}{4}\left(\zeta_+ - \zeta_-\right)^2 - \zeta_+\zeta_- r^2 + \mathcal{O}\left(r^3\right), \\
g_{tr} &= \mathcal{O}\left(r^2\right), \\
g_{t\phi} &= \frac{\ell^2}{4}\left(\mathcal{J}_+ + \mathcal{J}_-\right)\left(\zeta_+ - \zeta_-\right) + \left(\zeta_+\mathcal{J}_- - \zeta_-\mathcal{J}_+\right)\frac{r^2}{2} + \mathcal{O}\left(r^3\right), \\
g_{rr} &= 1 + \mathcal{O}\left(r^2\right), \\
g_{r\phi} &= \mathcal{O}\left(r^2\right), \\
g_{\phi\phi} &= \frac{\ell^2}{4}\left(\mathcal{J}_+ + \mathcal{J}_-\right)^2 + \mathcal{J}_+\mathcal{J}_- r^2 + \mathcal{O}\left(r^3\right).
\end{aligned} \tag{2.32}$$

Again, the chemical potentials $\zeta_\pm$ are expressed in terms of the generalized Gelfand-Dickey polynomials according to eq. (2.24)[5].

The behavior of the metric near the horizon is preserved under the action of the following Killing vectors

[5] Note that with our choice of boundary conditions it is not possible to write the metric (2.32) in a co-rotating frame ($\zeta_+ = \zeta_- = const.$), because $\zeta_\pm$ have a very precise dependence on the fields $\mathcal{J}_\pm$, and generically cannot be set to be equal to constants.

$$\begin{aligned}
\xi^t &= \frac{\eta^+\mathcal{J}_- + \eta^-\mathcal{J}_+}{\zeta_+\mathcal{J}_- + \zeta_-\mathcal{J}_+} + \mathcal{O}\left(r^3\right), \\
\xi^r &= \mathcal{O}\left(r^3\right), \\
\xi^\phi &= \frac{\eta^+\zeta_- - \eta^-\zeta_+}{\zeta_+\mathcal{J}_- + \zeta_-\mathcal{J}_+} + \mathcal{O}\left(r^3\right).
\end{aligned} \tag{2.33}$$

The conserved charges can be obtained using the Regge-Teitelboim approach, and evaluating them at $r = 0$. The results coincide with eq. (2.11), as expected.

2.3.3 General solution

In ref. [5, 40], it was shown that it is possible to construct the general solution of Einstein equations obeying the fall-off described in (2.30). It is given by

$$\begin{aligned}
ds^2 = \; & dr^2 + \tfrac{\ell^2}{4}\cosh^2(r/\ell)\left[(\zeta_+ - \zeta_-)\,dt + (\mathcal{J}_+ + \mathcal{J}_-)\,d\phi\right]^2 \\
& - \tfrac{\ell^2}{4}\sinh^2(r/\ell)\left[(\zeta_+ + \zeta_-)\,dt + (\mathcal{J}_+ - \mathcal{J}_-)\,d\phi\right]^2,
\end{aligned} \tag{2.34}$$

and satisfies Einstein equations provided that $\mathcal{J}_\pm$ obey the differential equations associated to the k-th member of the hierarchy when $\zeta_\pm$ is fixed according to eq. (2.24). In the near horizon region, this solution also obeys the fall-off in (2.32). Note that the metric (2.34) is diffeomorphic to a BTZ geometry, but as we will show in the next section, it carries nontrivial charges associated to improper (large) gauge transformations [57], and consequently describes a different physical state.

It is worth emphasizing that there is a one-to-one map between three–dimensional geometries described by eq. (2.34), and solutions of the members of the Gardner hierarchy. In this sense, we can say that this integrable system was "fully geometrized" in terms of certain three–dimensional spacetimes which are locally of constant curvature.

2.4 Black holes

2.4.1 Regularity conditions and thermodynamics

Euclidean black holes solutions are obtained by requiring regularity of the Euclidean geometries associated to the family of metrics in (2.34). This fixes the inverse of left and right temperatures $\beta_\pm = T_\pm^{-1}$ in terms of the fields $\zeta_\pm$ according to

$$\beta_\pm = \frac{2\pi}{\zeta_\pm}. \tag{2.35}$$

These conditions can also be obtained by requiring that the holonomy around the thermal cycle for the gauge connections (2.4) be trivial.

A direct consequence of eq. (2.35) is that the chemical potentials $\zeta_\pm$ are now constants, and hence from the field equations (2.5), the regular Euclidean solutions are characterized by $\dot{\mathcal{J}}_\pm = 0$, i.e., by static solutions of the members of the Gardner hierarchy. In sum, in order to obtain an explicit black hole solution, the following equations must be solved

$$\partial_\phi R^\pm_{(k)} = 0, \tag{2.36}$$

restricted to the conditions

$$T_\pm = \frac{2}{\kappa} R^\pm_{(k)}. \tag{2.37}$$

The Bekenstein–Hawking entropy can be directly obtained from the near horizon expansion (2.32), and gives

$$S = \frac{A}{4G} = \frac{\kappa}{2}\oint d\phi \left(\mathcal{J}_+ + \mathcal{J}_-\right). \tag{2.38}$$

As expected, the first law is automatically fulfilled. Indeed, using (2.37) one obtains

$$\beta_+ \delta H^+_{(k)} + \beta_- \delta H^-_{(k)} = \oint d\phi \left(\beta_+ R^+_{(k)} \delta\mathcal{J}_+ + \beta_- R^-_{(k)} \delta\mathcal{J}_-\right) = \delta\left[\frac{\kappa}{2}\oint d\phi \left(\mathcal{J}_+ + \mathcal{J}_-\right)\right] = \delta S.$$

Here, $\beta_{\pm}$ turn out to be the conjugates to the left and right energies $H^{\pm}_{(k)}$. The inverse temperature, conjugate to the energy E in eq. (2.8), is expressed in terms of the left and right temperatures according to $T^{-1} = \frac{1}{2}\left(T_{+}^{-1} + T_{-}^{-1}\right)$.

The previous analysis was performed in a rather abstract form without using an explicit solution to eq. (2.36), which in general are very hard to find. A simple solution corresponds to $\mathcal{J}_{\pm} = const.$, which describes a BTZ configuration. In this case the Hamiltonians take the form

$$H^{\pm}_{(k)} = \sum_{n=k+2}^{2k+2} \alpha^{\pm}_{n} \mathcal{J}^{n}_{\pm},$$

where $\alpha^{\pm}_{n}$ are constant coefficients which are not specified in general, but whose values can be determined once the corresponding Hamiltonians are explicitly computed through the recursion relation for the generalized Gelfand-Dickey polynomials.

Some simplifications occur when we turn off either a or b (mKdV and KdV cases), that we discuss next.

2.4.2 mKdV case ($a = 0$)

When $a = 0$, the metric associated to the black hole solution with $\mathcal{J}_{\pm} = const.$ can be written as

$$\begin{aligned} ds^2 &= dr^2 + \tfrac{\ell^2}{4}\cosh^2(r/\ell)\left[4\pi^2\left(\tfrac{\pi\kappa}{2\sigma_{(k)}(k+1)}\right)^{2k+1}\left(\mathcal{J}_{+}^{2k+1} - \mathcal{J}_{-}^{2k+1}\right)dt + \left(\mathcal{J}_{+} + \mathcal{J}_{-}\right)d\phi\right]^2 \\ &\quad -\tfrac{\ell^2}{4}\sinh^2(r/\ell)\left[4\pi^2\left(\tfrac{\pi\kappa}{2\sigma_{(k)}(k+1)}\right)^{2k+1}\left(\mathcal{J}_{+}^{2k+1} + \mathcal{J}_{-}^{2k+1}\right)dt + \left(\mathcal{J}_{+} - \mathcal{J}_{-}\right)d\phi\right]^2, \end{aligned} \tag{2.39}$$

where the constant $\sigma_{(k)}$, given by

$$\sigma_{(k)} := \left(\frac{\pi\kappa}{2k+2}\right)^{\frac{k+1}{k+\frac{1}{2}}}\left(\frac{\sqrt{\pi}}{\kappa 2^{k-2} b^{k}}\frac{\Gamma(k+2)}{\Gamma\left(k+\frac{1}{2}\right)}\right)^{\frac{1}{2k+1}},$$

will play the role of the anisotropic Stefan–Boltzmann constant of the system. The metric can be written in Schwarzschild–like coordinates using the following coordinate transformation

$$r = \frac{\ell}{2}\log\left(\frac{4\sqrt{\left(\bar{r}^2 - \frac{\ell^2}{4}\left(\mathcal{J}_+ - \mathcal{J}_-\right)^2\right)\left(\bar{r}^2 - \frac{\ell^2}{4}\left(\mathcal{J}_+ + \mathcal{J}_-\right)^2\right)} - \ell^2\left(\mathcal{J}_+^2 + \mathcal{J}_-^2\right) + 4\bar{r}^2}{2\ell^2\mathcal{J}_+\mathcal{J}_-}\right). \tag{2.40}$$

It coincides with the metric of a BTZ black hole in a rotating frame, with outer and inner horizons located at $\bar{r}_\pm = \frac{\ell}{2}\left(\mathcal{J}_+ \pm \mathcal{J}_-\right)$.

With the choice $a = 0$, the expression for the left and right energies written in terms of the constants $\mathcal{J}_\pm$ becomes simpler than the one in the general case. Indeed, it can be written in a closed form as

$$H^{\pm}_{(k)} = \sigma_{(k)}^{-z}\left(\frac{\pi\kappa}{(z+1)}\mathcal{J}_\pm\right)^{z+1}, \tag{2.41}$$

where $z = 2k+1$ is the dynamical exponent of the Lifshitz scale symmetry (2.27) of the k-th element of the mKdV hierarchy.

Using eqs. (2.37) and (2.41), the left and right energies $H^{\pm}_{(k)}$ can be expressed in terms of the left and right temperatures $T_\pm$, acquiring the form dictated by the Stefan–Boltzmann law for a two–dimensional system with anisotropic Lifshitz scaling [58]

$$H^{\pm}_{(k)} = \sigma_{(k)} T_\pm^{1+\frac{1}{z}}.$$

The Bekenstein–Hawking entropy, given by eq. (2.38), can be expressed in terms of the left and right energies $H^{\pm}_{(k)}$ as follows

$$S = (1+z)\,\sigma_{(k)}^{\frac{z}{1+z}}\left(\left(H^{+}_{(k)}\right)^{\frac{1}{z+1}} + \left(H^{-}_{(k)}\right)^{\frac{1}{z+1}}\right). \tag{2.42}$$

Note that the dependence of the entropy in terms of the left/right energies is consistent with the Lifshitz scaling of the k–th element of the mKdV hierarchy.

Power partitions and microstate counting

The dependence of the entropy in terms of the left/right energies in eq. (2.42) might be understood from a microscopic point of view if, following [32], we assume that there exists a two–dimensional field theory with Lifshitz scaling, defined on a circle, whose dispersion relation for very high energies takes the form

$$E_n^{\pm} = \varepsilon_{(z)}^{\pm} n^z , \tag{2.43}$$

where n is a non–negative integer, and $\varepsilon_{(z)}^{\pm}$ denote the characteristic energy of the left/right modes. The problem of computing the entropy in the microcanonical ensemble is then equivalent to compute the power partitions of given integers $N_{\pm} = E^{\pm}/\varepsilon_{(z)}^{\pm}$. Here $E^{\pm}$ are the left/right energies given by

$$E^{\pm} = \varepsilon_{(z)}^{\pm} \sum_i n_i^z .$$

This problem was solved long ago by Hardy and Ramanujan in [59], where at the end of their paper they conjecture that the asymptotic growth of power partitions is given by

$$p_z\left(N_{\pm}\right) \approx \exp\left[(1+z)\left(\frac{\Gamma\left(1+\frac{1}{z}\right)\zeta\left(1+\frac{1}{z}\right)}{z}\right)^{\frac{z}{1+z}} N_{\pm}^{\frac{1}{1+z}}\right] ,$$

result that was proven later by Wright in 1934 [60].

The (left/right) entropies then reads

$$S^{\pm} = \log\left[p_z\left(N_{\pm}\right)\right] = (1+z)\left(\frac{\Gamma\left(1+\frac{1}{z}\right)\zeta\left(1+\frac{1}{z}\right)}{z}\right)^{\frac{z}{1+z}} \left(\frac{E^{\pm}}{\varepsilon_{(z)}^{\pm}}\right)^{\frac{1}{1+z}} . \tag{2.44}$$

This expression precisely coincides with the entropy of the black hole in eq. (2.42), provided $E^{\pm} = H_{(k)}^{\pm}$, and

$$\varepsilon_{(z)}^{\pm} = \left(\frac{\Gamma\left(1+\frac{1}{z}\right)\zeta\left(1+\frac{1}{z}\right)}{\sigma_{(k)} z}\right)^{z} .$$

Note that AdS spacetime is not contained within the spectrum of our boundary conditions, and consequently one can naively think that the anisotropic extension of Cardy formula of refs. [6, 58] cannot be used to reproduce the entropy of the black hole (2.42). However, there is a known case [58], where the anisotropic extension of Cardy formula can still be used in spite of the fact that ground state, given by a gravitational soliton, does not fit within the boundary conditions that accommodate the Lifshitz black hole. This approach is based on the use of an anisotropic extension of modular invariance that relates the Euclidean black hole and its corresponding soliton, which turn out to be diffeomorphic. It would be interesting to explore in the future whether this approach could be applied to the BTZ black holes in the context of our boundary conditions.

2.4.3 KdV case ($b = 0$)

When $b = 0$, the metric associated to the black hole solution with $\mathcal{J}_{\pm} = const.$ takes the form

$$\begin{aligned} ds^2 &= dr^2 + \tfrac{\ell^2}{4}\cosh^2(r/\ell)\left[4\pi^2\left(\tfrac{\pi\kappa}{\bar{\sigma}_{(k)}(k+2)}\right)^{k+1}\left(\mathcal{J}_+^{k+1} - \mathcal{J}_-^{k+1}\right)dt + \left(\mathcal{J}_+ + \mathcal{J}_-\right)d\phi\right]^2 \\ &\quad -\tfrac{\ell^2}{4}\sinh^2(r/\ell)\left[4\pi^2\left(\tfrac{\pi\kappa}{\bar{\sigma}_{(k)}(k+2)}\right)^{k+1}\left(\mathcal{J}_+^{k+1} + \mathcal{J}_-^{k+1}\right)dt + \left(\mathcal{J}_+ - \mathcal{J}_-\right)d\phi\right]^2, \end{aligned} \tag{2.45}$$

where

$$\bar{\sigma}_{(k)} = \left(\frac{\pi\kappa}{k+2}\right)^{\frac{k+2}{k+1}}\left(\frac{\sqrt{\pi}}{\kappa\,(2a)^k}\frac{\Gamma(k+3)}{\Gamma\left(k+\frac{3}{2}\right)}\right)^{\frac{1}{k+1}}.$$

Using the change of coordinates (2.40), the metric (2.45) can be written in Schwarzschild–like form, and coincides with the one of a BTZ black hole with outer and inner horizons located at $\bar{r}_{\pm} = \frac{\ell}{2}\left(\mathcal{J}_+ \pm \mathcal{J}_-\right)$.

The left and right energies $H_{(k)}^{\pm}$ can then be expressed in terms of the constants $\mathcal{J}_{\pm}$

according to

$$H^{\pm}_{(k)} = \bar{\sigma}_{(k)}^{-\frac{z+1}{2}} \left(\frac{2\pi\kappa}{z+3}\mathcal{J}_{\pm}\right)^{\frac{z+3}{2}}, \tag{2.46}$$

where $z = 2k+1$ is the dynamical exponent associated to the Lifshitz symmetry (2.28) of the k-th member of the KdV hierarchy.

The expression for the left/right energies $H^{\pm}_{(k)}$ in terms of the left/right temperatures $T_{\pm}$, is then given by

$$H^{\pm}_{(k)} = \bar{\sigma}_{(k)} T_{\pm}^{\frac{z+3}{z+1}}.$$

In spite of the fact that the k-th equation of the KdV hierarchy is invariant under Lifshitz scaling with dynamical exponent z, the power in the temperature is not the one expected for a two–dimensional theory with this symmetry. Furthermore, this is inherited to the expression for the entropy written in terms of the left/right energies $H^{\pm}_{(k)}$

$$S = \left(\frac{z+3}{2}\right) \bar{\sigma}_{(k)}^{\frac{z+1}{z+3}} \left(\left(H^{+}_{(k)}\right)^{\frac{2}{z+3}} + \left(H^{-}_{(k)}\right)^{\frac{2}{z+3}}\right),$$

which is not of the expected form (2.44).

Remarkably, if instead of the Hamiltonians $H^{\pm}_{(k)}$, one uses the Hamiltonians $H^{\pm}_{(2k)}$, this naive incompatibility with the Lifshitz symmetry disappears. Indeed, the relation between $H^{\pm}_{(2k)}$ and the left/right temperatures takes the form

$$H^{\pm}_{(2k)} = \bar{\sigma}_{(2k)} T_{\pm}^{1+\frac{1}{z}},$$

while the expression for the entropy in terms of the extensive quantities $H^{\pm}_{(2k)}$ reads

$$S = (1+z)\, \bar{\sigma}_{(2k)}^{\frac{z}{1+z}} \left(\left(H^{+}_{(2k)}\right)^{\frac{1}{z+1}} + \left(H^{-}_{(2k)}\right)^{\frac{1}{z+1}}\right).$$

The entropy then takes the expected Hardy–Ramanujan form (2.44), with the characteristic energy of the dispersion relation given by

$$\varepsilon^{\pm}_{(z)} = \left(\frac{\Gamma\left(1+\frac{1}{z}\right)\zeta\left(1+\frac{1}{z}\right)}{\bar{\sigma}_{(2k)} z} \right)^{z}.$$

Black hole with nonconstants $\mathcal{J}_{\pm}$

In the particular case when $b = 0$ and $k = 1$, it is possible to find explicit nonconstants solutions to eq. (2.36) that characterize a regular Euclidean black hole. These are static solutions of the left/right KdV equations, which take the form of periodic cnoidal waves. The solutions are then given by

$$\mathcal{J}_{\pm} = -\frac{8K^2(m_{\pm})}{3a\pi^2}\left(1 - 2m_{\pm} + 3m_{\pm}\mathrm{cn}^2\left(\frac{K(m_{\pm})}{\pi}\phi | m^{\pm}\right)\right), \tag{2.47}$$

where $m_{\pm}$ are constants in the range $0 \leq m_{\pm} < 1$, cn denotes the Jacobi elliptic cosine function, and $K(m)$ is the complete elliptic integral of the first kind defined as

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{2.48}$$

The regularity conditions (2.37) fix the left and right temperatures $T_{\pm}$ in terms of the constant $m_{\pm}$ according to

$$T_{\pm} = \frac{16\left(m_{\pm}^2 - m_{\pm} + 1\right)}{3\pi^5 a \left(m_{\pm} - 1\right)^2} K\left(\frac{m_{\pm}}{m_{\pm} - 1}\right)^4.$$

Chapter 3

Boundary conditions for higher spin gravity on AdS$_3$ and the modified Gelfand-Dickey hierarchy

Here we discuss the generalization of the results described in the previous chapter to spin-N gravity. In this case, the corresponding set of integrable systems is given by the N-th modified Gelfand-Dickey hierarchy, which for $N = 2$ reduces to the modified KdV hierarchy, while for $N = 3$ yields the modified Boussinesq hierarchy. Indeed, in ref. [64] a connection between spin-3 gravity and a "Boussinesq equation in the light-cone" was pointed out. In refs. [65, 66], some particular boundary conditions for higher spin gravity with gauge group $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$ were associated to a generalized KdV hierarchy that includes the Boussinesq equation. On the other hand, in ref. [6] a very precise link with the Boussinesq hierarchy was described for spin-3 gravity, as well as for its extension including fields with arbitrary higher spins. The analysis was based on boundary conditions defined in the highest weight gauge with a particular choice of chemical potentials, generalizing the results obtained for the KdV hierarchy in pure gravity.

The Boussinesq equation (1.25) was first introduced by Joseph Boussinesq in 1872 in the context of the study of the propagation of one-dimensional long waves in shallow water moving in both directions [68]. Long after, in 1974, it was realized that the equation was integrable, and that belongs to a hierarchy of differential equations [69]. Most of the important properties of the Boussinesq equation, including its infinite set of commuting conserved charges, can be easily derived when a potential equation, called "modified Boussinesq" (mBoussinesq), is introduced (see e.g [70]). Both equations are then related by an appropriate generalization of the Miura transformation (1.30) [71–73].

In this chapter we show that the asymptotic dynamics of spin-3 gravity on AdS_3 endowed with a special class of boundary conditions, is precisely described by the members of the mBoussinesq hierarchy. In this framework, the gauge fields are defined in the "diagonal gauge," where the excitations go along the generators of the Cartan subalgebra of $sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R})$ [67]. The link with the integrable system is then obtained by choosing the chemical potentials as precise functionals of the dynamical fields, in a way consistent with the action principle. Hence, the entire integrable structure of the mBoussinesq hierarchy, i.e., the phase space, the fundamental Poisson brackets given by two independent $\hat{u}(1)$ current algebras, and the infinite set of Hamiltonians in involution, are obtained from the asymptotic structure of the higher spin theory in the bulk. Furthermore, the relation with the Boussinesq hierarchy previously found in ref. [6] is inherited from our analysis once the asymptotic conditions are re-expressed in the highest weight gauge along the lines of ref. [67]. Thus, the Miura map is recovered from a purely geometric construction in the higher spin theory. Black hole solutions that fit within our boundary conditions, the Hamiltonian reduction at the boundary, and the generalization to higher spin gravity with gauge group $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ are also discussed.

3.1 Review of the modified Boussinesq hierarchy

The mBoussinesq equation is the first member of the mBoussinesq hierarchy and is given by the following set of differential equations

$$\begin{aligned} \dot{\mathcal{J}} &= \lambda_1 \mathcal{J}' - \lambda_2 \left(2\left(\mathcal{J}\mathcal{U}\right)' + \mathcal{U}''\right), \\ \dot{\mathcal{U}} &= \lambda_1 \mathcal{U}' + \lambda_2 \left(\mathcal{U}^{2\prime} - \mathcal{J}^{2\prime} + \mathcal{J}''\right). \end{aligned} \tag{3.1}$$

Here dots and primes denote derivatives with respect to the time t and the angle ϕ respectively, and λ_1, λ_2 are arbitrary constants associated to the two different flows of the hierarchy [72]. The case with $\lambda_1 = 0$ and $\lambda_2 = 1$ is known as the mBoussinesq equation, while the case with $\lambda_1 = 1$ and $\lambda_2 = 0$ describes two independent chiral fields.

As was described in chapter 1, the dynamics of the above equations may be described using the Hamiltonian formalism. If the Poisson brackets of two arbitrary functional F and G is given by

$$\{F, G\} = \frac{4\pi}{\hat{\kappa}} \int d\phi \left(\frac{\delta F}{\delta \mathcal{J}} \partial_\phi \frac{\delta G}{\delta \mathcal{J}} + \frac{\delta F}{\delta \mathcal{U}} \partial_\phi \frac{\delta G}{\delta \mathcal{U}} \right), \tag{3.2}$$

together with the Hamiltonian

$$H_{(1)} = \frac{\hat{\kappa}}{4\pi} \int d\phi \left\{ \frac{\lambda_1}{2} \left(\mathcal{J}^2 + \mathcal{U}^2\right) + \lambda_2 \left(\frac{1}{3}\mathcal{U}^3 - \mathcal{J}^2\mathcal{U} - \mathcal{J}\mathcal{U}' \right) \right\}, \tag{3.3}$$

then eq. (3.1) can be rewritten as

$$\dot{\mathcal{J}} = \left\{\mathcal{J}, H_{(1)}\right\}, \qquad \dot{\mathcal{U}} = \left\{\mathcal{U}, H_{(1)}\right\}.$$

Note that the arbitrary constant $\hat{\kappa}$ does not appear in the differential equations (3.1), however it is useful to introduce it in (3.2) and (3.3) for later convenience.

Alternatively, if we define the operator

$$\mathcal{D} := \frac{4\pi}{\hat{\kappa}} \begin{pmatrix} \partial_\phi & 0 \\ 0 & \partial_\phi \end{pmatrix}, \tag{3.4}$$

the equations in (3.1) can be re-written in vector form as follows

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{U}} \end{pmatrix} = \mathcal{D} \begin{pmatrix} \frac{\delta H_{(1)}}{\delta \mathcal{J}} \\ \frac{\delta H_{(1)}}{\delta \mathcal{U}} \end{pmatrix}.$$

The operator $\mathcal{D}$ in (3.4) defines the symplectic structure in eq. (3.2).

It is worth to emphasize that one of the key points in the relation of this integrable system with higher spin gravity comes from the fact that, according to eq. (3.2), the fundamental Poisson brackets are described by two independent $\hat{u}(1)$ current algebras

$$\begin{aligned} \left\{\mathcal{J}(\phi), \mathcal{J}(\phi')\right\} &= \frac{4\pi}{\hat{\kappa}} \partial_\phi \delta(\phi - \phi'), \\ \left\{\mathcal{U}(\phi), \mathcal{U}(\phi')\right\} &= \frac{4\pi}{\hat{\kappa}} \partial_\phi \delta(\phi - \phi'). \end{aligned} \tag{3.5}$$

As we will show below, once appropriate boundary conditions are imposed, this Poisson bracket algebra is obtained from the Dirac brackets in the higher spin theory.

The integrability of (3.1) and the existence of a hierarchy of equations, rely on the fact that this system is actually bi-Hamiltonian. Indeed, there exists an alternative symplectic structure characterized by the non-local operator given from the eq. (1.35)

$$\mathcal{D}_{(2)} = \mathcal{D} M^\dagger \mathcal{O} M \mathcal{D}. \tag{3.6}$$

Here,

$$M=\begin{pmatrix}\mathcal{J}+\partial_\phi & \mathcal{U}\\ -2\mathcal{J}\mathcal{U}-\frac{1}{2}\mathcal{U}\partial_\phi-\frac{3}{2}\mathcal{U}' & \mathcal{U}^2-\mathcal{J}^2-\frac{1}{2}\mathcal{J}'-\frac{3}{2}\mathcal{J}\partial_\phi-\frac{1}{2}\partial_\phi^2\end{pmatrix}, \tag{3.7}$$

and $M^\dagger$ is the formal adjoint of M (see e.g. [74]). The operator $\mathcal{O}$ corresponds to the inverse of the first Poisson structure associated to the Boussinesq hierarchy and is given by

$$\mathcal{O}=\frac{2\hat{\kappa}}{\pi}\begin{pmatrix}0 & \partial_\phi^{-1}\\ \partial_\phi^{-1} & 0\end{pmatrix}.$$

Consequently, the Poisson bracket of two arbitrary functionals F and G associated with the operator $\mathcal{D}_{(2)}$ is

$$\{F,G\}_2=\int d\phi\begin{pmatrix}\frac{\delta F}{\delta\mathcal{J}} & \frac{\delta F}{\delta\mathcal{U}}\end{pmatrix}\mathcal{D}_{(2)}\begin{pmatrix}\frac{\delta G}{\delta\mathcal{J}}\\ \frac{\delta G}{\delta\mathcal{U}}\end{pmatrix}. \tag{3.8}$$

The explicit components of $\mathcal{D}_{(2)}$ are exhibited in appendix C.

The modified Boussinesq equations (3.1) can then be recovered using the Poisson bracket (3.8), together with the Hamiltonian[1]

$$H_{(0)}=\frac{\hat{\kappa}}{4\pi}\int d\phi\left(\lambda_1\mathcal{J}+\lambda_2\mathcal{U}\right).$$

This system possess an infinite number of conserved charges in involution that can be constructed from the recursion relation (1.31)

$$R_{(n+1)}=\mathcal{D}^{-1}\mathcal{D}_{(2)}R_{(n)}, \tag{3.9}$$

where

$$R_{(n)}=\begin{pmatrix}\frac{\delta H_{(n)}}{\delta\mathcal{J}}\\ \frac{\delta H_{(n)}}{\delta\mathcal{U}}\end{pmatrix}, \tag{3.10}$$

[1]The coefficients λ_1 and λ_2 in eq. (3.1) are determined by the integration constants obtained by the action of $\mathcal{D}_{(2)}$. In the case of higher members of the hierarchy, the subsequent integration constants may be consistently set to zero as they contribute nothing new (see e.g [72]).

are the Gelfand-Dickey polynomials associated to the hierarchy. The conserved quantities $H_{(n)}$, with n being a nonnegative integer, are generically decomposed into two flows proportional to the constants λ_1 and λ_2 respectively

$$H_{(n)} = \sum_{I=1}^{2} \lambda_I H^I_{(n)}. \tag{3.11}$$

Then one can prove that the $H^I_{(k)}$ are in involution with both Poisson brackets, i.e.,

$$\left\{H^I_{(n)}, H^J_{(m)}\right\} = \left\{H^I_{(n)}, H^J_{(m)}\right\}_2 = 0.$$

Furthermore, if we one uses the conserved quantities $H^I_{(k)}$ as new Hamiltonians, it is then possible to define a hierarchy of integrable equations labelled by the nonnegative integer k of the form

$$\begin{aligned} \dot{\mathcal{J}} &= \left\{\mathcal{J}, H_{(k)}\right\} = \left\{\mathcal{J}, H_{(k-1)}\right\}_2 , \\ \dot{\mathcal{U}} &= \left\{\mathcal{U}, H_{(k)}\right\} = \left\{\mathcal{U}, H_{(k-1)}\right\}_2 . \end{aligned} \tag{3.12}$$

The equations associated to each flow, labelled by the index $I = 1, 2$, have different scaling properties. Under a Lifshitz scaling transformation with dynamical exponent z

$$t \to \varepsilon^z t\,, \qquad \phi \to \varepsilon\phi\,, \qquad \mathcal{J} \to \varepsilon^{-1}\mathcal{J}\,, \qquad \mathcal{U} \to \varepsilon^{-1}\mathcal{U},$$

the flow with $I = 1$ is invariant for $z = 3k - 2$, while the flow with $I = 2$ is invariant for $z = 3k - 1$.

As explained above, the mBoussinesq equation is a “potential equation” for the Boussinesq one. Indeed, if $\mathcal{U}$ and $\mathcal{J}$ obey the mBoussinesq equation, then the fields $\mathcal{L}$ and $\mathcal{W}$ defined by the Miura transformation as

$$\mathcal{L} = \frac{1}{2}\mathcal{J}^2 + \frac{1}{2}\mathcal{U}^2 + \mathcal{J}',$$
$$\mathcal{W} = \frac{1}{3}\mathcal{U}^3 - \mathcal{J}^2\mathcal{U} - \frac{1}{2}\mathcal{U}\mathcal{J}' - \frac{3}{2}\mathcal{J}\mathcal{U}' - \frac{1}{2}\mathcal{U}'', \tag{3.13}$$

obey the Boussinesq equation given by[2]

$$\dot{\mathcal{L}} = 2\mathcal{W}',$$
$$\dot{\mathcal{W}} = 2\mathcal{L}^{2\prime} - \frac{1}{2}\mathcal{L}'''. \tag{3.14}$$

Combining both equations and eliminating the field $\mathcal{W}$, one finds that the field $\mathcal{L}$ must satisfy the "Good" Boussinesq equation.

The entire Boussinesq hierarchy, including the infinite set of charges in involution, is obtained from the mBoussinesq one by using the Miura transformation (see appendix E for more details on the Boussinesq hierarchy). It is worth noting that the Miura transformation (3.13) coincides the twisted Sugawara construction of the stress tensor and a spin-3 current in terms of two independent $U(1)$ currents in a two-dimensional CFT. Hence, using the Poisson brackets (3.5) and the Miura map (3.13), one can show that the (first) Poisson brackets for $\mathcal{L}$ and $\mathcal{W}$ are precisely given by the classical W_3 algebra.

[2]Eqs. (1.24) are recovered from eqs. (3.14) by applying the following rescaling $\mathcal{L} \to -\frac{1}{2}\mathcal{L}$, $\mathcal{W} \to -\frac{\sqrt{3}}{2}\mathcal{W}$, $t \to \frac{1}{\sqrt{3}}t$.

3.2 Modified Boussinesq hierarchy from spin-3 gravity on AdS$_3$

3.2.1 Chern-Simons formulation of spin-3 gravity on AdS$_3$

Higher spin gravity in 3D has the very special property that, in contrast with their higher dimensional counterparts [75–77], its spectrum can be consistently truncated to a finite number of higher spin fields [61, 62, 78, 79]. One of the simplest cases corresponds to a spin-two field non-minimally coupled to a spin-three field, that may be described by a Chern-Simons action for the gauge group $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$,

$$I = I_{CS}\left[A^{+}\right] - I_{CS}\left[A^{-}\right], \tag{3.15}$$

where

$$I_{CS}[A] = \frac{\kappa_3}{4\pi}\int_{\mathcal{M}} \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right). \tag{3.16}$$

Here, the level is given by $\kappa_3 = \kappa/4 = l/16G$, where l and G correspond to the AdS radius and the three-dimensional Newton constant respectively, and the trace is in the fundamental representation of the $sl(3,\mathbb{R})$ algebra in the principal embedding (see appendix F). The field equations are then given by the vanishing of the field strength

$$F^{\pm} = dA^{\pm} + A^{\pm 2} = 0. \tag{3.17}$$

The metric and the spin-three field are reconstructed in terms of a generalized dreibein $e := \frac{l}{2}\left(A^{+} - A^{-}\right)$ according to

$$g_{\mu\nu} = \frac{1}{2}\mathrm{tr}\left(e_{\mu}e_{\nu}\right), \qquad \varphi_{\mu\nu\rho} = \frac{1}{3!}\mathrm{tr}\left(e_{(\mu}e_{\nu}e_{\rho)}\right).$$

3.2.2 Asymptotic behavior of the fields. Diagonal gauge

Following refs. [47, 78, 79], it is convenient to perform the analysis of the asymptotic symmetries of spin-three gravity in terms of an auxiliary connection depending only on t and ϕ. For simplicity, and without loss of generality, hereafter we will consider only the "plus copy," and hence the superscript "+" will be omitted. The gauge field A is then written as

$$A = b^{-1}\left(d + a\right)b, \tag{3.18}$$

where $a = a_t dt + a_\phi d\phi$ is the auxiliary connection, and $b = b\left(r\right)$ is a gauge group element which captures the whole the radial dependence of the gauge connection. The asymptotic analysis will be insensitive to the precise form of $b\left(r\right)$.

We will consider asymptotic conditions in the "diagonal gauge," i.e., where all the permissible excitations in the auxiliary connection go along the generators of the Cartan subalgebra of $sl\left(3,\mathbb{R}\right)$ [67]. Then, it takes the form

$$a = \left(\mathcal{J}d\phi + \zeta dt\right)L_0 + \frac{\sqrt{3}}{2}\left(\mathcal{U}d\phi + \zeta_{\mathcal{U}}dt\right)W_0. \tag{3.19}$$

The fields $\mathcal{J}$ and $\mathcal{U}$ belong to the spatial components of the auxiliary connection, and hence they are identified as the dynamical fields. On the other hand, ζ and $\zeta_{\mathcal{U}}$ are defined along the temporal components, and therefore they correspond to the boundary values of the Lagrange multipliers. In ref. [67] the same asymptotic form for the auxiliary connection was used, with the replacement $\mathcal{J}_{(3)} \to \frac{\sqrt{3}}{2}\mathcal{U}$ and $\zeta_{(3)} \to \frac{\sqrt{3}}{2}\zeta_{\mathcal{U}}$. However, the boundary conditions will be different. In [67] it was assumed that ζ and $\zeta_{(3)}$ are kept fixed at the boundary, while here, as we will show in the next subsection, they will acquire a precise functional dependence on the dynamical fields $\mathcal{J}, \mathcal{U}$ and their spatial derivatives.

3.2.3 Boundary conditions for spin-3 gravity and the modified Boussinesq hierarchy

A fundamental requirement in the study of the asymptotic structure of spacetime is that the boundary conditions must be compatible with the action principle. In the canonical formalism one has to add an appropriate boundary term B_∞ to the canonical action in order to guarantee that the action principle attains an extremum [42]

$$I_{can}\left[A\right] = -\frac{\kappa}{16\pi}\int dtd^2x\epsilon^{ij}\left\langle A_i\dot{A}_j - A_tF_{ij}\right\rangle + B_\infty. \tag{3.20}$$

Following [67], for the action (3.20) and the asymptotic conditions (3.18), (3.19), the variation of the boundary term is given by

$$\delta B_\infty = -\frac{\kappa}{4\pi}\int dtd\phi\left(\zeta\delta\mathcal{J} + \zeta_\mathcal{U}\delta\mathcal{U}\right). \tag{3.21}$$

In the absence of ingoing or outgoing radiation, as is the case in three-dimensional higher spin gravity, the boundary term B_∞ has to be integrable in a functional sense, i.e., one must be able to "take the delta outside" in (3.21). The precise way in which ζ and $\zeta_\mathcal{U}$ are fixed at the boundary is what defines the boundary conditions. Thus, following [6], in order to make contact with the Boussinesq hierarchy in a way consistent with the action principle, we choose the Lagrange multipliers as

$$\zeta = \frac{4\pi}{\kappa}\frac{\delta H_{(k)}}{\delta\mathcal{J}}\,, \qquad \zeta_\mathcal{U} = \frac{4\pi}{\kappa}\frac{\delta H_{(k)}}{\delta\mathcal{U}}, \tag{3.22}$$

where $H_{(k)}$ is the Hamiltonian associated to the k-th element of the mBoussinesq hierarchy. With this choice, the boundary term can be readily integrated, and yields

$$B_\infty = -\int dtH_{(k)}. \tag{3.23}$$

Thus, the Hamiltonian of the gravitational theory in the reduced phase space precisely matches the one of the integrable system.

On the other hand, the field equations in the higher spin theory given by the vanishing of the field strength become

$$\dot{\mathcal{J}} = \zeta' \,, \qquad \dot{\mathcal{U}} = \zeta'_{\mathcal{U}} \,, \tag{3.24}$$

which, by virtue of (3.22), precisely coincide with the differential equations associated to the k-th element of the mBoussinesq hierarchy in eq. (3.12), provided the constant $\hat{\kappa}$ in eqs. (3.2) and (3.3) is assumed to depend on the cosmological and Newton constants according to $\hat{\kappa} = \kappa$. Then,

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{U}} \end{pmatrix} = \mathcal{D} \begin{pmatrix} \frac{\delta H_{(k)}}{\delta \mathcal{J}} \\ \frac{\delta H_{(k)}}{\delta \mathcal{U}} \end{pmatrix} = \begin{pmatrix} \{\mathcal{J}, H_{(k)}\} \\ \{\mathcal{U}, H_{(k)}\} \end{pmatrix} . \tag{3.25}$$

3.2.4 Asymptotic symmetries and conserved charges

The asymptotic symmetries are determined by set of gauge transformations that preserve the asymptotic form of the gauge connection, with non-vanishing associated charges. The form of the the auxiliary connection in eq. (3.19) is preserved by gauge transformations $\delta a = d\lambda + [a, \lambda]$, with parameter

$$\lambda = \eta L_0 + \frac{\sqrt{3}}{2} \eta_{\mathcal{U}} W_0 .$$

There could be some additional terms in the non-diagonal components of the gauge parameter λ, but they are pure gauge in the sense that there are no generators associated to them, so they can be consistently set to zero.

The preservation of the angular components of the auxiliary connection gives the transformation law of the dynamical fields

$$\delta\mathcal{J} = \eta' \,, \qquad \delta\mathcal{U} = \eta'_{\mathcal{U}}, \tag{3.26}$$

while that the preservation of the temporal components provides the transformation law of the Lagrange multipliers

$$\delta\zeta = \dot{\eta} \,, \qquad \delta\zeta_{\mathcal{U}} = \dot{\eta}_{\mathcal{U}}. \tag{3.27}$$

The variation of the conserved charges can be computed using the Regge-Teitelboim method [42], and they are given by the following surface integral

$$\delta Q\left[\eta, \eta_{\mathcal{U}}\right] = \frac{\kappa}{4\pi}\int d\phi\left(\eta\delta\mathcal{J} + \eta_{\mathcal{U}}\delta\mathcal{U}\right). \tag{3.28}$$

The Dirac brackets of the dynamical fields $\mathcal{J}$ and $\mathcal{U}$ induced by the asymptotic conditions may be obtained from the relation $\delta_Y Q\left[X\right] = \left\{Q\left[X\right], Q\left[Y\right]\right\}$, and is given by two independent $\hat{u}\left(1\right)$ current algebras

$$\begin{aligned}
\left\{\mathcal{J}\left(\phi\right), \mathcal{J}\left(\phi'\right)\right\}^{\star} &= \frac{4\pi}{\kappa}\partial_\phi\delta\left(\phi - \phi'\right), \\
\left\{\mathcal{U}\left(\phi\right), \mathcal{U}\left(\phi'\right)\right\}^{\star} &= \frac{4\pi}{\kappa}\partial_\phi\delta\left(\phi - \phi'\right),
\end{aligned} \tag{3.29}$$

expression that coincides with the first Poisson bracket of the mBoussinesq hierarchy given by eq. (3.5). Furthermore, the infinite set of commuting charges of the hierarchy is obtained from the surface integral (3.28) as follows: if we take into account that due to eq. (3.22) the Lagrange multipliers are field dependent, then the consistency with their transformation law (3.27) implies the following differential equation that must be obeyed by η and $\eta_{\mathcal{U}}$

$$\begin{pmatrix} \dot{\eta}\left(t,\theta\right) \\ \dot{\eta}_{\mathcal{U}}\left(t,\theta\right) \end{pmatrix} = \int d\phi \begin{pmatrix} \frac{\delta^2 H_{(k)}}{\delta\mathcal{J}(t,\theta)\delta\mathcal{J}(t,\phi)} & \frac{\delta^2 H_{(k)}}{\delta\mathcal{U}(t,\theta)\delta\mathcal{J}(t,\phi)} \\ \frac{\delta^2 H_{(k)}}{\delta\mathcal{J}(t,\theta)\delta\mathcal{U}(t,\phi)} & \frac{\delta^2 H_{(k)}}{\delta\mathcal{U}(t,\theta)\delta\mathcal{U}(t,\phi)} \end{pmatrix} \mathcal{D} \begin{pmatrix} \eta \\ \eta_{\mathcal{U}} \end{pmatrix}.$$

By virtue of the integrability of the system, the most general solution of this equation, under the assumption that η and $\eta_{\mathcal{U}}$ depend locally on $\mathcal{J}, \mathcal{U}$ and their spatial derivatives, is given by

$$\begin{pmatrix} \eta \\ \eta_{\mathcal{U}} \end{pmatrix} = \frac{4\pi}{\kappa} \sum_{n=0}^{\infty} \alpha_{(n)} \begin{pmatrix} \frac{\delta H_{(n)}}{\delta\mathcal{J}} \\ \frac{\delta H_{(n)}}{\delta\mathcal{U}} \end{pmatrix}, \tag{3.30}$$

where the $\alpha_{(n)}$ are arbitrary constants. Therefore, replacing the solution (3.30) in (3.28) one can integrate the charge in the functional sense (taking the delta outside), obtaining

$$Q = \sum_{n=0}^{\infty} \alpha_{(n)} H_{(n)}.$$

Thus, the conserved charges in the higher spin theory are precisely given by a linear combination of the Hamiltonians of the mBoussinesq hierarchy. Indeed, using the transformation law (3.26), one can show the Hamiltonians $H^I_{(n)}$ are in involution with respect to the Dirac bracket (3.29)

$$\left\{ H^I_{(m)}, H^J_{(n)} \right\}^\star = 0,$$

as expected.

In sum, all the relevant properties of the integrable mBoussinesq hierarchy described in section 3.1 are derived from spin-3 gravity endowed with the boundary conditions defined in eqs. (3.18), (3.19) and (3.22). Thus, the reduced phase space of spin-3 gravity and its boundary dynamics are equivalent to the ones of the mBoussinesq hierarchy. In particular, this provides an explicit one-to-one map between solutions of the integrable system at the boundary and solutions of the higher spin gravity theory in the bulk.

3.2.5 Highest weight gauge, Miura map and the Boussinesq hierarchy

Asymptotic conditions for spin-3 gravity on AdS_3 in the highest weight gauge were first given in refs. [78, 79], where it was shown that the asymptotic symmetries are spanned by two copies of the classical W_3 algebra with the Brown-Henneaux central charge. In what follows we consider the generalization introduced in refs. [12, 13], where the most general form of the Lagrange multipliers a_t, compatible with the W_3 symmetry, is allowed. This generalization has the important property that it accommodates black holes carrying non-trivial higher spin charges.

In this subsection we show that, with a particular gauge transformation, the auxiliary connection in the diagonal gauge (3.19) can be mapped to an auxiliary connection in the highest weight gauge, such that the Miura transformation in eq. (3.13), that relates the mBoussinesq with the Boussinesq hierarchies, is recovered from a purely geometric construction in the higher spin theory. The analysis is very close to the one in ref. [67], with the main difference that now the Lagrange multipliers ζ and $\zeta_{\mathcal{U}}$ depend on the dynamical fields according to eq. (3.22).

The angular components of the auxiliary connection in the highest weight gauge $\hat{a}$ is assumed to be of the form

$$\hat{a}_\varphi = L_1 - \frac{1}{2}\mathcal{L}L_{-1} - \frac{1}{4\sqrt{3}}\mathcal{W}W_{-2}. \tag{3.31}$$

Following [12, 13], the most general form of $\hat{a}_t$ which is compatible with the field equations is

$$\begin{aligned}\hat{a}_t = {}& \mu L_1 - \frac{\sqrt{3}}{2}\nu W_2 - \mu' L_0 + \frac{\sqrt{3}}{2}\nu' W_1 + \frac{1}{2}\left(\mu'' - \mu\mathcal{L} - 2\mathcal{W}\nu\right) L_{-1} \\ & - \frac{\sqrt{3}}{48}\left(4\mathcal{W}\mu - 7\mathcal{L}'\nu' - 2\nu\mathcal{L}'' - 8\mathcal{L}\nu'' + 6\mathcal{L}^2\nu + \nu''''\right) W_{-2} \\ & - \frac{\sqrt{3}}{4}\left(\nu'' - 2\mathcal{L}\nu\right) W_0 + \frac{\sqrt{3}}{12}\left(\nu''' - 2\nu\mathcal{L}' - 5\mathcal{L}\nu'\right) W_{-1}\ . \end{aligned} \tag{3.32}$$

It is possible to find a gauge group element $g = g^{(1)} g^{(2)}$, such that the auxiliary connection in the diagonal gauge a is mapped to the auxiliary connection in the highest weight gauge $\hat{a}$, by a gauge transformation of the form $\hat{a} = g^{-1}(d+a)\,g$, with

$$
\begin{aligned}
g^{(1)} &= \exp\left[x L_1 + y W_1 + z W_2\right], \\
g^{(2)} &= \exp\left[-\tfrac{1}{2}\mathcal{J} L_{-1} - \tfrac{\sqrt{3}}{6}\mathcal{U} W_{-1} + \tfrac{\sqrt{3}}{12}\left(\mathcal{J}\mathcal{U} + \tfrac{1}{2}\mathcal{U}'\right) \mathrm{W}_{-2}\right].
\end{aligned}
$$

Here, the functions x, y, z are restricted to obey the following differential equations

$$
\begin{aligned}
x' &= 1 + x\,\mathcal{J} + \sqrt{3} y\,\mathcal{U}, \\
y' &= y\,\mathcal{J} + \sqrt{3} x\,\mathcal{U}, \\
z' &= -\tfrac{1}{2} y + 2z\,\mathcal{J}.
\end{aligned}
$$

The fields $\mathcal{L}$ and $\mathcal{W}$ are then related to the fields $\mathcal{J}$ and $\mathcal{U}$ precisely by the Miura transformation (3.13)

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\mathcal{J}^2 + \frac{1}{2}\mathcal{U}^2 + \mathcal{J}', \\
\mathcal{W} &= \frac{1}{3}\mathcal{U}^3 - \mathcal{J}^2\mathcal{U} - \frac{1}{2}\mathcal{U}\mathcal{J}' - \frac{3}{2}\mathcal{J}\mathcal{U}' - \frac{1}{2}\mathcal{U}''.
\end{aligned} \tag{3.33}
$$

The Lagrange multipliers in the highest weight gauge, given by μ and ν, are related to the variables in the diagonal gauge through the following equations

$$
\begin{aligned}
\zeta &= \mathcal{J}\mu - \mu' - 2\left(\mathcal{J}\mathcal{U} + \frac{1}{2}\mathcal{U}'\right)\nu + \frac{1}{2}\mathcal{U}\nu', \\
\zeta_{\mathcal{U}} &= \mathcal{U}\mu - \left(\mathcal{J}^2 - \mathcal{U}^2 - \mathcal{J}'\right)\nu + \frac{3}{2}\mathcal{J}\nu' - \frac{1}{2}\nu''.
\end{aligned} \tag{3.34}
$$

The complete Boussinesq hierarchy is then obtained from the mBoussinesq one by virtue of the relations (3.33) and (3.34). Indeed, from (3.34), one can prove that the chemical potentials in the highest weight gauge take the form

$$\mu = \frac{4\pi}{\kappa}\frac{\delta H^{\text{Bsq}}_{(k)}}{\delta\mathcal{L}}, \qquad \nu = \frac{4\pi}{\kappa}\frac{\delta H^{\text{Bsq}}_{(k)}}{\delta\mathcal{W}},$$

where $H^{\text{Bsq}}_{(k)}$ corresponds the k-th Hamiltonian of the Boussinesq hierarchy (see appendix E for more details on the Boussinesq hierarchy). For example, for the first member given by $k = 1$, one has

$$H^{\text{Bsq}}_{(1)} = \frac{\kappa}{4\pi}\int d\phi\left(\lambda_1\mathcal{L} + \lambda_2\mathcal{W}\right),$$

and hence the chemical potentials in the highest weight gauge become

$$\mu = \lambda_1, \qquad \nu = \lambda_2.$$

In the particular case with $\lambda_1 = 1$ and $\lambda_2 = 0$ (first flow), the equations of motion are given by two chiral movers, and the asymptotic conditions in eqs. (3.31), (3.32) reduce to the ones in refs. [78,79] but written in terms of the composite fields $\mathcal{L}$ and $\mathcal{W}$ according to (3.13). On the other hand, for the second flow with $\lambda_1 = 0$ and $\lambda_2 = 1$, the field equation in the bulk become equivalent to the Boussinesq equation in (3.14), in agreement with the result found in ref. [6].

3.3 Higher spin Black holes

The line element is no longer gauge invariant in higher spin gravity, since it generically changes under the action of a higher spin gauge transformation. Therefore, the spacetime geometry and the causal structure cannot be directly used to define black holes. In refs. [80,81], a new notion of higher spin black hole was introduced in the Euclidean formulation of the theory, by requiring trivial holonomies for the gauge connection around a thermal cycle $\mathcal{C}$, i.e.,

$$\mathcal{H}_{\mathcal{C}} = \mathcal{P}e^{\int_{\mathcal{C}} a^{\pm}} = 1. \tag{3.35}$$

Here we have restored the $\pm$ superscript to denote the plus/minus copy of the gauge field. If we assume that the Euclidean time is in the range $0 \leq t_E < 1$ then, for time-independent configurations in (3.19), the regularity condition (3.35) imposes the following restrictions on $\zeta^{\pm}$ and $\zeta^{\pm}_{\mathcal{U}}$

$$\zeta^{\pm} = \pi \left(2n + m\right) , \qquad \zeta^{\pm}_{\mathcal{U}} = \sqrt{3}\pi m, \tag{3.36}$$

with m and n being integers. Static configurations that obey (3.36) are regular Euclidean solutions and consequently we call them "black holes."

According to ref. [67], the entropy takes the form

$$S = \frac{\kappa}{4} \int d\phi \left(\left(2n + m\right) \left(\mathcal{J}^{+} + \mathcal{J}^{-}\right) + \sqrt{3}m \left(\mathcal{U}^{+} + \mathcal{U}^{-}\right) \right) , \tag{3.37}$$

which, by virtue of (3.22), (3.36) and (3.11), obeys the following first law

$$\delta S = \sum_{I=1}^{2} \left(\lambda_I^{+} \delta H^{I+}_{(k)} + \lambda_I^{-} \delta H^{I-}_{(k)} \right) ,$$

where $H^{I\pm}_{(k)}$ are the left/right k-th Hamiltonian of the mBoussinesq hierarchy associated to the flows labelled by $I = 1, 2$. Note that the constants $\lambda_I^{\pm}$ correspond to the chemical potentials conjugate to the extensive quantities $H^{I\pm}_{(k)}$.

When the integers n and m acquire the values $n = 1$, $m = 0$, we obtain a branch which is connected with the pure gravitational sector and the BTZ black hole. In that case, the entropy acquires the simple expression

$$S = \frac{\kappa}{2} \int d\phi \left(\mathcal{J}^{+} + \mathcal{J}^{-}\right) . \tag{3.38}$$

As was pointed out in [67], for constants $\mathcal{J}^{\pm}$ and $\mathcal{U}^{\pm}$, the entropy for this branch acquires the expected form found in [12, 13] for a higher spin black hole, once it is written in terms of the charges of the W-algebra.

In sum, black holes that fit within our boundary conditions in eqs. (3.19) and (3.22) are identified with static solutions of the k-th element of the mBoussinesq hierarchy. It is worth noting that it is of fundamental importance to consider both flows to admit generic black hole configurations without restricting the possible space of solutions. To illustrate some of their properties, we will study the particular cases with $k = 0, 1, 2$ in the branch connected with the BTZ black hole. For simplicity we consider only the plus copy.

Case with $k = 0$. The general solution of the field equations (3.24) that obeys (3.36) is given by two arbitrary functions of ϕ, i.e., $\mathcal{J} = \mathcal{J}(\phi)$ and $\mathcal{U} = \mathcal{U}(\phi)$. This case corresponds to the "higher spin black flower" described in ref. [67], carrying an infinite set of $\hat{u}(1)$ soft hairy charges. The particular configuration with constant $\mathcal{J}$ and vanishing $\mathcal{U}$ in the branch with $n = 1$, $m = 0$, corresponds to the BTZ black hole embedded within this set of boundary conditions.

Case with $k = 1$. The choice of boundary conditions (3.22), together with the regularity conditions, imply the following differential equations

$$\begin{aligned} \lambda_1 \mathcal{J} - \lambda_2 \left(\mathcal{U}' + 2\mathcal{J}\mathcal{U}\right) &= 2\pi, \\ \lambda_1 \mathcal{U} + \lambda_2 \left(\mathcal{J}' - \mathcal{J}^2 + \mathcal{U}^2\right) &= 0. \end{aligned} \tag{3.39}$$

These equations relate the chemical potentials λ_1 and λ_2 with the fields $\mathcal{J}$ and $\mathcal{U}$. In particular, for constants $\mathcal{J}$ and $\mathcal{U}$ we obtain

$$\lambda_1 = \frac{2\pi\left(\mathcal{J}^2 - \mathcal{U}^2\right)}{\mathcal{J}\left(\mathcal{J}^2 - 3\mathcal{U}^2\right)}, \qquad \lambda_2 = \frac{2\pi\mathcal{U}}{\mathcal{J}\left(\mathcal{J}^2 - 3\mathcal{U}^2\right)}. \tag{3.40}$$

The regularity conditions for the BTZ solution in the pure gravity sector is recovered when $\mathcal{U} = 0$, as expected. On the other hand, the configurations with $\mathcal{J}\left(\mathcal{J}^2 - 3\mathcal{U}^2\right) = 0$ possess non-trivial holonomies along the thermal cycle. Hence, one might identify them with extremal configurations, along the lines of ref. [82] (see also [83]). Indeed, when $\mathcal{U}^{\pm} = 0$, solutions with vanishing $\mathcal{J}^+$ or $\mathcal{J}^-$ correspond to extremal BTZ black holes.

Case with $k=2$. For constants $\mathcal{J}$ and $\mathcal{U}$, the regularity conditions (3.36) fix the chemical potentials λ_I according to

$$\begin{aligned}
\lambda_1 &= \frac{3\pi\mathcal{U}\left(15\mathcal{J}^4-10\mathcal{J}^2\mathcal{U}^2+7\mathcal{U}^4\right)}{2\mathcal{J}\left(9\mathcal{J}^8-72\mathcal{J}^6\mathcal{U}^2+210\mathcal{J}^4\mathcal{U}^4-224\mathcal{J}^2\mathcal{U}^6-3\mathcal{U}^8\right)}, \\
\lambda_2 &= \frac{3\pi\left(3\mathcal{J}^4+6\mathcal{J}^2\mathcal{U}^2-5\mathcal{U}^4\right)}{2\mathcal{J}\left(9\mathcal{J}^8-72\mathcal{J}^6\mathcal{U}^2+210\mathcal{J}^4\mathcal{U}^4-224\mathcal{J}^2\mathcal{U}^6-3\mathcal{U}^8\right)}.
\end{aligned} \tag{3.41}$$

When $\mathcal{U}=0$, the auxiliary connection (3.19) reduces to the one that describes the BTZ geometry [5]. However, in contrast to the case with $k=1$, the chemical potential λ_1 associated to the first flow now vanishes, and hence, the information of the BTZ black hole is completely encoded in the second flow.

Note that for the cases with $k=1$ and $k=2$ described above, it is of fundamental importance to take into account both flows to have black holes characterized by two independent constants $\mathcal{J}^{\pm}$ and $\mathcal{U}^{\pm}$ for each copy. If one of the chemical potentials $\lambda_1^{\pm}$ or $\lambda_2^{\pm}$ is set to zero, then eqs. (3.40) and (3.41) would imply non-trivial restrictions in the values of $\mathcal{J}^{\pm}$ and $\mathcal{U}^{\pm}$, truncating in this way the spectrum of allowed black hole configurations.

3.4 Hamiltonian reduction and boundary dynamics

In this section we will discuss the Hamiltonian reduction and the boundary dynamics of the Chern-Simons action that describes spin-3 gravity endowed with the boundary conditions associated to the mBoussinesq hierarchy. In the case of pure gravity with Brown-Henneaux asymptotic conditions, the boundary dynamics is described by two left and right chiral bosons which, by virtue of a Bäcklund transformation, turn out to be equivalent to a Liouville theory [47, 84]. The analysis was done by performing a Hamiltonian reduction of the Wess-Zumino-Witten (WZW) theory at the boundary [85–88].

Here we follow an approach similar to the one proposed in ref. [89] for KdV-type boundary conditions in pure gravity, where instead of passing through the WZW theory, the boundary conditions are implemented directly in the Hamiltonian action.

Let us consider the Hamiltonian action with the appropriate boundary term in eq. (3.20). The constraints $F_{ij} = 0$ are locally solved by expressing the spatial components of the gauge connection in terms of a group element G as $A_i = G^{-1}\partial_i G$, provided that there are no holes in the spatial section. After replacing the solution of the constraints into the action (3.20), the following decomposition is obtained

$$I_{can}\left[A\right] = I_1 + I_2 + B_\infty, \tag{3.42}$$

where

$$I_1 = \frac{\kappa}{16\pi}\int dtdrd\phi\epsilon^{ij}\left\langle\partial_t\left(G^{-1}\right)\partial_i GG^{-1}\partial_j G\right\rangle, \tag{3.43}$$

$$I_2 = -\frac{\kappa}{16\pi}\int d\phi dt\left\langle\partial_t G\partial_\phi\left(G^{-1}\right)\right\rangle. \tag{3.44}$$

The Wess-Zumino term I_1 reduces to a boundary term that vanishes provided the group element is decomposed as $G = g\left(t,\phi\right)b\left(r\right)$ near the boundary (see appendix G for a detailed proof). Here $b\left(r\right)$ is the group element that depends on the radial coordinate in eq. (3.18), while $g\left(t,\phi\right)$ is such that the auxiliary connection a in (3.19) is written as $a_i = g^{-1}\partial_i g$. With this decomposition the term I_2 becomes

$$I_2 = -\frac{\kappa}{16\pi}\int d\phi dt\left\langle\dot{g}\partial_\phi g^{-1}\right\rangle. \tag{3.45}$$

Since the auxiliary connection in (3.19) is a diagonal matrix, we can write the group element g as follows

$$g = \exp\left[\sqrt{\frac{8\pi}{\kappa}}\varphi L_0 + \sqrt{\frac{6\pi}{\kappa}}\psi W_0\right], \tag{3.46}$$

where φ and ψ are functions that depend only on t and ϕ. In what follows we will assume that these fields are periodic in the angle, and consequently possible contributions coming from non-trivial holonomies around ϕ are not considered in this analysis.

Consistency with the asymptotic form of the auxiliary connection (3.19) then implies

$$\mathcal{J}=\sqrt{\frac{8\pi}{\kappa}}\varphi'\,,\qquad\qquad \mathcal{U}=\sqrt{\frac{8\pi}{\kappa}}\psi'. \tag{3.47}$$

Replacing (3.46) in (3.45) we find

$$I_2 \;=\; \int d\phi dt\left(\varphi'\dot{\varphi}+\psi'\dot{\psi}\right). \tag{3.48}$$

Thus, if we use the expression (3.23) for the boundary term B_∞, we finally obtain the following reduced action at the boundary

$$I_{(k)} \;=\; \int dt\left[\int d\phi\left(\varphi'\dot{\varphi}+\psi'\dot{\psi}\right)-H_{(k)}\right]. \tag{3.49}$$

This action describes the dynamics of the fields φ and ψ, whose interactions are described by the k-th Hamiltonian of the mBoussinesq hierarchy. The members of the mBoussinesq hierarchy are then recovered from the equations of motion derived from the action (3.49), provided we identify the fields according to (3.47). In this sense, the field equations coming from (3.49) define "potential equations" for the ones of the mBoussinesq hierarchy.

The action (3.49) is invariant under the following transformations

$$\delta\varphi=\sqrt{\frac{\kappa}{8\pi}}\eta+f\left(t\right)\,,\qquad\qquad \delta\psi=\sqrt{\frac{\kappa}{8\pi}}\eta_{\mathcal{U}}+f_{\mathcal{U}}\left(t\right),$$

where the parameter η and $\eta_{\mathcal{U}}$, given by (3.30), are associated to the infinite charges in involution of the integrable system. Indeed, all the Hamiltonians of the hierarchy may be obtained by a direct application of Noether theorem. On the other hand, the arbitrary functions of the time $f(t)$ and $f_{\mathcal{U}}(t)$ define gauge symmetries of (3.49) that allow to gauge away the zero modes of these fields.

Furthermore, the action (3.49) has an additional Lifschitz scaling symmetry in the particular case when only one of the two flows is considered. For the first flow, with $\lambda_1 = 1$ and $\lambda_2 = 0$, the dynamical exponent is $z = 3k - 2$, while for the second flow, with $\lambda_1 = 0$ and $\lambda_2 = 1$, is $z = 3k - 1$, as expected from the invariance properties of the mBoussinesq hierarchy described in section 3.1. For the I-th flow of the k-th element of the hierarchy, the generators of the Lifshitz algebra are given by

$$H = H^{I}_{(k)}\,, \qquad P = H^{1}_{(1)}\,, \qquad D = -\frac{\kappa}{4\pi}\int d\phi\left(\frac{1}{2}\phi\left(\mathcal{J}^2 + \mathcal{U}^2\right)\right) - ztH^{I}_{(k)}.$$

Here, H is the generators of translations in time, P of translations in space, and D of anisotropic dilatations. Using the the Dirac brackets (3.29) it is straightforward to show that they close in the Lifshitz algebra

$$\{P,H\}^{\star} = 0, \qquad \{D,P\}^{\star} = P, \qquad \{D,H\}^{\star} = zH,$$

where z is the dynamical exponent associated to the corresponding flow.

Let us consider as an example the case with $k = 1$. The action then takes the following form

$$I_{(1)} = \int d\phi dt\left[\varphi'\dot{\varphi} + \psi'\dot{\psi} - \lambda_1\left(\varphi'^2 + \psi'^2\right) + 2\lambda_2\left(\varphi'\psi'' + \sqrt{\frac{8\pi}{\kappa}}\left(\varphi'^2\psi' - \frac{1}{3}\psi'^3\right)\right)\right].$$

For the flow with $\lambda_1 = 1$ and $\lambda_2 = 0$ we recover the Floreanini-Jackiw action for two free chiral bosons [90]. On the other hand, for the flow with $\lambda_1 = 0$ and $\lambda_2 = 1$, the action contains non-trivial interacting term and is invariant under Lifshitz transformations with dynamical exponent $z = 2$, as expected.

3.5 Generalized Gibbs ensemble

The existence of an infinite number of commuting charges in the mBoussinesq hierarchy opens the possibility of study more general thermodynamic ensembles that generically might include all possible charges. They are called "Generalized Gibbs Ensemble" (GGE).

In the case of a two-dimensional conformal field theory, a GGE is constructed with the infinite set of charges in involution of the KdV hierarchy, which are obtained as composite operators in terms of the stress tensor [14–16] (see refs. [6, 17–29] for recent results on GGE).

In our context, the Hamiltonians of the mBoussinesq hierarchy can be used to construct the GGE of a two-dimensional CFT with spin-3 currents. As discussed in subsection 3.2.5, the Miura transformation (3.13) maps the Hamiltonians of the mBoussinesq hierarchy into the Hamiltonians of the Boussinesq one, which depend on the stress tensor $\mathcal{L}$ and the spin-3 current $\mathcal{W}$. These Hamiltonians belong to the enveloping algebra of the W_3-algebra, and consequently define an infinite set of commuting charges that are composite operators in terms of $\mathcal{L}$ and $\mathcal{W}$. In this sense, the relation between the mBoussinesq hierarchy and spin-3 gravity discussed in this article, provides a natural holographic bulk dual description of this type of GGE.

This may be implemented as follows. Instead of considering one particular $H_{(k)}$ as the Hamiltonian of the dynamical system, we deal with a linear combination of them, i.e.,

$$H_{GGE} = \sum_{n=1}^{\infty} \sum_{I=1}^{2} \gamma_n \left(\lambda_I H^I_{(n)} \right) . \tag{3.50}$$

If we want to interpret this general Hamiltonian as the one of a CFT$_2$ given by $H^1_{(1)}$, deformed by (multitrace) deformations that include spin-3 currents, then we must identify the inverse (right) temperature as $\beta_+ = T_+^{-1} = \alpha_1 \lambda_1$, and the additional chemical potentials as $\mu_{n,I} := T_+ \alpha_n \lambda_I$. However, this is not the only possibility. Any $H^I_{(k)}$ could be considered as the "undeformed Hamiltonian," allowing new branches that generically change the phase structure of the theory [91].

Black holes are described by static configurations of the dynamical system with Hamiltonian (3.50). The regularity condition (3.36) then takes the form

$$\sum_{n=1}^{\infty}\sum_{I=1}^{2}\gamma_n\lambda_I\frac{\delta H_{(n)}^{I}}{\delta\mathcal{J}}=\frac{\kappa}{2}\,,\qquad\qquad \sum_{n=1}^{\infty}\sum_{I=1}^{2}\gamma_n\lambda_I\frac{\delta H_{(n)}^{I}}{\delta\mathcal{U}}=0.$$

These equations guarantee that the Euclidean action principle attains an extremum, and hence they fully characterize the thermodynamics of the GGE.

On the other hand, the boundary dynamics obtained by the Hamiltonian reduction is easily generalized to the case when the Hamiltonian is given by (3.50). Indeed, the boundary action now becomes

$$I_{GGE} \;=\; \int dt\left[\int d\phi\left(\varphi'\dot{\varphi}+\psi'\dot{\psi}\right)-H_{GGE}\right]. \tag{3.51}$$

spin-N gravity and modified Gelfand-Dickey hierarchies

3.6 Higher spin gravity with gauge group $SL\left(N,\mathbb{R}\right)\times SL\left(N,\mathbb{R}\right)$ and modified Gelfand-Dickey hierarchies

Our results can be generalized to the case of three-dimensional higher spin gravity with gauge group $SL\left(N,\mathbb{R}\right)\times SL\left(N,\mathbb{R}\right)$, where the corresponding integrable systems are the called "modified Generalized KdV hierarchies," or "modified Gelfand-Dickey (mGD) hierarchies" (see e.g. chapter 4 of ref. [92]). The link with higher spin gravity is based on the zero curvature formulation of these integrable systems [73].

Let us consider asymptotic conditions for spin-N gravity described by the following auxiliary gauge connection valued on the $sl\left(N,\mathbb{R}\right)$ algebra

$$a=\left(\mathcal{J}d\phi+\zeta dt\right)L_0+\sum_{s=3}^{N}\sigma_s\left(\mathcal{U}^{(s)}d\phi+\zeta_{\mathcal{U}}^{(s)}dt\right)W_0, \tag{3.52}$$

with σ_s given by

$$\sigma_s = \left(\frac{(2s-1)!\,(2s-2)!}{48(s-1)!^4} \frac{1}{\prod_{i=2}^{s-1}(N^2-i^2)} \right)^{\frac{1}{2}} .$$

The Chern-Simons action takes the same form as in (3.16), but replacing $\kappa_3 \to \kappa_N = 3l/(2N(N^2-1)G)$. Hence, the variation of the boundary term of the canonical Chern-Simons action becomes

$$\delta B_\infty = -\frac{\kappa}{4\pi} \int dt d\phi \left(\zeta \delta \mathcal{J} + \sum_{s=3}^{N} \zeta_{\mathcal{U}}^{(s)} \delta \mathcal{U}^{(s)} \right) . \tag{3.53}$$

To make contact with the mGD hierarchies, we choose the Lagrange multipliers as follows

$$\zeta = \frac{4\pi}{\kappa} \frac{\delta H^{\mathrm{mGD}}_{(k,N)}}{\delta \mathcal{J}}, \qquad \zeta_{\mathcal{U}}^{(s)} = \frac{4\pi}{\kappa} \frac{\delta H^{\mathrm{mGD}}_{(k,N)}}{\delta \mathcal{U}^{(s)}}, \tag{3.54}$$

where $H^{\mathrm{mGD}}_{(k,N)}$ corresponds to the k-th Hamiltonian of the N-th hierarchy [73]. With this choice of boundary conditions, the boundary term of the canonical Chern-Simons action integrates as

$$B_\infty = -\int dt H_{(k,N)}.$$

As expected, the Hamiltonian of the higher spin theory coincides with one of the hierarchy.

The Dirac brackets are described by $N-1$ $\hat{u}(1)$ current algebras

$$\begin{aligned} \left\{ \mathcal{J}(\phi), \mathcal{J}(\phi') \right\}^\star &= \frac{4\pi}{\kappa} \partial_\phi \delta(\phi - \phi') , \\ \left\{ \mathcal{U}^{(s)}(\phi), \mathcal{U}^{(s')}(\phi') \right\}^\star &= \frac{4\pi}{\kappa} \partial_\phi \delta(\phi - \phi') \, \delta^{s,s'} , \end{aligned} \tag{3.55}$$

matching the first Poisson structure of the mGD hierarchies. Moreover, their members are obtained from the equations of motion of the higher spin theory with the boundary conditions (3.52), (3.54), and are given by

$$\dot{\mathcal{J}} = \frac{4\pi}{\kappa} \partial_\phi \left(\frac{\delta H^{\mathrm{mGD}}_{(k,N)}}{\delta \mathcal{J}} \right) , \qquad \dot{\mathcal{U}}^{(s)} = \frac{4\pi}{\kappa} \partial_\phi \left(\frac{\delta H^{\mathrm{mGD}}_{(k,N)}}{\delta \mathcal{U}^{(s)}} \right) .$$

The mGD hierarchies are related to the called "Generalized KdV hierarchies," or "Gelfand-Dickey (GD) hierarchies" by an appropriate generalization of the Miura transformation. One of the two Poisson brackets of the GD hierarchies is described by the W_N-algebra, whose generators are composite in terms of the $\hat{u}(1)$ currents of the mGD hierarchies. Hence, according to the Hamiltonian reduction in [73], the generalized Miura transformation should emerge geometrically from our boundary conditions once they are expressed in the highest weight gauge, as in the case for $N = 3$ described in subsection 3.2.5. Note that since generically the expression for the stress tensor of the W_N-algebra in terms of $\hat{u}(1)$ currents is the one of a twisted Sugawara construction, there is a particular flow in which the currents are chiral. This case precisely corresponds to one of the proposals in ref. [9], for describing gravitational duals of averaged CFT's on the Narain lattice [93, 94] (see [95] for an alternative proposal for a possible gravitational dual).

Conclusions

In this thesis, a new set of boundary conditions for pure gravity on AdS_3 was proposed, with the very special property that the boundary dynamics of the gravitational theory is described by an integrable hierarchy of differential equations called the Gardner hierarchy [8]. These boundary conditions can be easily described using the diagonal gauge in the Chern-Simons formulation of General Relativity, and are closely related with the soft hairy ones in ref. [5]. The main difference is that now the chemical potentials are chosen to depend on the dynamical fields in very precise way, instead of being fixed at infinity. This different choice of chemical potentials produces a drastic change in the boundary dynamics of the theory, and therefore, the whole integrable structure of the hierarchy can be recovered from the gravitational theory, including its infinite number of conserved charges in involution. Black hole solutions which fit within these boundary conditions were also found, and they were shown to be described by static configurations associated to the corresponding member of the Gardner hierarchy.

These results generalize the previous ones presented in ref. [6] along new directions, reinforcing the deep relationship between three-dimensional gravity with certain special boundary conditions and two-dimensional integrable systems. Indeed, as it was shown in ref. [35] and explained in detail in chapter 3, it is also possible to further extend these results to the case of higher spin gravity on AdS_3 with gauge group $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$, where the corresponding integrable system is now given by the modified Gelfand-Dickey hierarchy.

Remarkably, this construction allows to establish a one-to-one map between three-dimensional (higher-spin) geometries that solve the bulk gravitational field equations, and solutions of the corresponding two-dimensional integrable system. Our results open the possibility of studying certain problems in black hole physics by virtue of the powerful tools of integrable systems. Conversely, different phenomena in the two-dimensional integrable system could be understood in terms of (higher-spin) geometry, that can be seen as a kind of geometrization of the integrable system. In summary, this new link between three-dimensional gravitational theories in the bulk and two-dimensional integrable systems on the boundary provides a new kind of duality between these two seemingly different physical systems. This field of research is still in an early stage, and one would expect that it should be possible to provide new insights into some of the open problems in (higher-spin) gravity, as well as in integrable systems.

Appendix A

Operator P in KdV Lax pair

The operator P can be obtained directly from the square root of the operator L in eq. (1.14). Let us propose the following ansatz

$$L^{\frac{1}{2}} = \partial_\phi + f_0 + f_1\partial_\phi^{-1} + f_2\partial_\phi^{-2} + \dots, \tag{A.1}$$

which is a pseudo-differential operator, where ∂_ϕ^{-1} stands for an operator that satisfies

$$\partial_\phi\partial_\phi^{-1} = \partial_\phi^{-1}\partial_\phi = 1,$$

and the general Leibniz rule

$$\partial_\phi^{-1}h = h\partial_\phi^{-1} - h'\partial_\phi^{-2} + h''\partial_\phi^{-3} - \dots.$$

The coefficients f_n in eq. (A.1) are functions depending on the field $\mathcal{L}$ and its spatial derivatives. They can be obtained recursively through the condition

$$L = L^{\frac{1}{2}} \cdot L^{\frac{1}{2}} = \partial_\phi^2 + 2f_0\partial_\phi + \left(f_0^2 + 2f_1 + f_0'\right) + \left(f_1' + 2f_0f_1 + 2f_2\right)\partial_\phi^{-1} + \dots$$

where L is defined in eq. (1.14), so that we obtain

$$L^{\frac{1}{2}} = \partial_\phi + \mathcal{L}\partial_\phi^{-1} - \frac{1}{2}\mathcal{L}'\partial_\phi^{-2} + \ldots \tag{A.2}$$

where the lower orders in eq. (A.2) are needed in order to calculate the odd powers of this root operator. The first one is given by

$$L^{\frac{3}{2}} = L^{\frac{1}{2}} \cdot L = \partial_\phi^3 + 3\mathcal{L}\partial_\phi + \frac{3}{2}\mathcal{L}' + \left(\frac{3}{2}\mathcal{L}^2 + \frac{1}{4}\mathcal{L}''\right)\partial_\phi^{-1} + (\ldots)\,\partial_\phi^{-2} + \ldots, \tag{A.3}$$

Finally, the operator P is defined from eq. (A.3) as

$$P = \left(L^{\frac{3}{2}}\right)_{\geq 0} = \partial_\phi^3 + \frac{3}{2}\mathcal{L}\partial_\phi + \frac{3}{4}\mathcal{L}',$$

where $(\ldots)_{\geq 0}$ correspond to the purely differential part of the pseudo-differential operator.

Appendix B

Gelfand-Dickey polynomials and Hamiltonians of the Gardner hierarchy

In this appendix, we explicitly display the first Hamiltonians $H_{(k)}$ associated to the Gardner integrable system, as well as their corresponding generalized Gelfand-Dickey polynomials $R_{(k)}$. The first Hamiltonians $H_{(k)}$ are given by

$$
\begin{aligned}
\frac{4\pi}{\hat{\kappa}} H_{(-1)} &= \int d\phi \left(a^{-1} \mathcal{J} \right), \\
\frac{4\pi}{\hat{\kappa}} H_{(0)} &= \int d\phi \left(\frac{1}{2} \mathcal{J}^2 \right), \\
\frac{4\pi}{\hat{\kappa}} H_{(1)} &= \int d\phi \left(\frac{1}{2} a \mathcal{J}^3 + \frac{1}{4} b \mathcal{J}^4 + \mathcal{J}'^2 \right), \\
\frac{4\pi}{\hat{\kappa}} H_{(2)} &= \int d\phi \left(\frac{5}{8} a^2 \mathcal{J}^4 - \frac{5}{2} a \mathcal{J}^2 \mathcal{J}'' + \frac{1}{4} b^2 \mathcal{J}^6 - \frac{5}{3} b \mathcal{J}^3 \mathcal{J}'' + \frac{3}{4} ab \mathcal{J}^5 + 2 \mathcal{J}''^2 \right), \\
\frac{4\pi}{\hat{\kappa}} H_{(3)} &= \int d\phi \left(\frac{7}{8} a^3 \mathcal{J}^5 - \frac{35}{6} a^2 \mathcal{J}^3 \mathcal{J}'' + 7 a \mathcal{J}^2 \mathcal{J}'''' + \frac{5}{16} b^3 \mathcal{J}^8 + \frac{35}{2} b^2 \mathcal{J}'^2 \mathcal{J}^4 \right. \\
&\quad \left. + \frac{7}{3} b \left(2 \mathcal{J}^3 \mathcal{J}'''' + \mathcal{J}'^4 \right) + \frac{7}{4} a^2 b \mathcal{J}^6 + \frac{5}{4} a b^2 \mathcal{J}^7 + 35 ab \mathcal{J}'^2 \mathcal{J}^3 - 4 \mathcal{J} \mathcal{J}'''''' \right).
\end{aligned}
\tag{B.1}
$$

Note that the Hamiltonians of the Gardner hierarchy cannot be written as the sum of the Hamiltonians of KdV ($b=0$) and mKdV ($a=0$), because there are cross terms. The case $H_{(-1)}$, that it is obtained by extending the hierarchy backwards, is special because contains a^{-1}, and consequently it cannot be defined in the mKdV hierarchy.

The generalized Gelfand-Dickey polynomials $R_{(k)}$ are obtained using eq. (2.20), and take the form

$$\begin{aligned}
\frac{4\pi}{\hat{\kappa}}R_{(-1)} &= \frac{1}{a}, \\
\frac{4\pi}{\hat{\kappa}}R_{(0)} &= \mathcal{J}, \\
\frac{4\pi}{\hat{\kappa}}R_{(1)} &= \left(\frac{3}{2}a\mathcal{J}^2 + b\mathcal{J}^3 - 2\mathcal{J}''\right), \\
\frac{4\pi}{\hat{\kappa}}R_{(2)} &= \left(\frac{5}{2}a^2\mathcal{J}^3 - 5a\left(\mathcal{J}'^2 + 2\mathcal{J}\mathcal{J}''\right) + \frac{3}{2}b^2\mathcal{J}^5\right. \\
&\quad \left. -10b\left(\mathcal{J}\mathcal{J}'^2 + \mathcal{J}^2\mathcal{J}''\right) + \frac{15}{4}ab\mathcal{J}^4 + 4\mathcal{J}''''\right), \\
\frac{4\pi}{\hat{\kappa}}R_{(3)} &= \left(\frac{35}{8}a^3\mathcal{J}^4 - 35a^2\left(\mathcal{J}\mathcal{J}'^2 + \mathcal{J}^2\mathcal{J}''\right) + 7a\left(4\mathcal{J}\mathcal{J}'''' + 6\mathcal{J}''^2 + 8\mathcal{J}'\mathcal{J}'''\right)\right. \\
&\quad +\frac{5}{2}b^3\mathcal{J}^7 + 7b\left(4\mathcal{J}^2\mathcal{J}'''' + 12\mathcal{J}\mathcal{J}''^2 + 16\mathcal{J}\mathcal{J}'\mathcal{J}''' + 20\mathcal{J}'^2\mathcal{J}''\right) - 8\mathcal{J}'''''' \\
&\quad \left. -35b^2\left(\mathcal{J}^4\mathcal{J}'' + 2\mathcal{J}^3\mathcal{J}'^2\right) + \frac{21}{2}a^2b\mathcal{J}^5 + \frac{35}{4}ab^2\mathcal{J}^6 - 35ab\left(2\mathcal{J}^3\mathcal{J}'' + 3\mathcal{J}^2\mathcal{J}'^2\right)\right).
\end{aligned} \tag{B.2}$$

Appendix C

Second Hamiltonian structure of the modified Boussinesq hierarchy

The second Hamiltonian structure of the mBoussinesq hierarchy is defined by the operator $\mathcal{D}_{(2)}$ in (3.6), whose explicit components are given by

$$
\begin{aligned}
\frac{\hat{\kappa}}{4\pi}\mathcal{D}^{11}_{(2)} &= -4\Big(2\left(\mathcal{U}''+2\left(\mathcal{J}\mathcal{U}\right)'\right)\partial_\phi^{-1}\left(\mathcal{J}\partial_\phi+\partial_\phi^2\right)+\mathcal{J}'\partial_\phi^{-1}\left(\left(4\mathcal{J}\mathcal{U}+3\mathcal{U}'\right)\partial_\phi+\mathcal{U}\partial_\phi^2\right)\\
&\quad+\left(8\mathcal{J}^2\mathcal{U}-5\mathcal{U}\mathcal{J}'-3\mathcal{U}''\right)\partial_\phi-3\mathcal{U}'\partial_\phi^2-2\mathcal{U}\partial_\phi^3\Big)\,,\\
\frac{\hat{\kappa}}{4\pi}\mathcal{D}^{12}_{(2)} &= -4\Big(2\left(\mathcal{U}''+2\left(\mathcal{J}\mathcal{U}\right)'\right)\partial_\phi^{-1}\left(\mathcal{U}\partial_\phi\right)+\mathcal{J}'\partial_\phi^{-1}\left(\left(2\mathcal{J}^2-2\mathcal{U}^2+\mathcal{J}'\right)\partial_\phi+3\mathcal{J}\partial_\phi^2+\partial_\phi^3\right)\\
&\quad+\left(2\mathcal{J}^3+2\mathcal{J}\mathcal{U}^2+2\mathcal{U}^{2\prime}-3\mathcal{J}\mathcal{J}'-\mathcal{J}''\right)\partial_\phi+\left(\mathcal{J}^2+\mathcal{U}^2-4\mathcal{J}'\right)\partial_\phi^2-2\mathcal{J}\partial_\phi^3-\partial_\phi^4\Big)\,,\\
\frac{\hat{\kappa}}{4\pi}\mathcal{D}^{21}_{(2)} &= -4\Big(2\left(\mathcal{J}^{2\prime}-\mathcal{U}^{2\prime}-\mathcal{J}''\right)\partial_\phi^{-1}\left(\mathcal{J}\partial_\phi+\partial_\phi^2\right)+\mathcal{U}'\partial_\phi^{-1}\left(\left(4\mathcal{J}\mathcal{U}+3\mathcal{U}'\right)\partial_\phi+\mathcal{U}\partial_\phi^2\right)\\
&\quad+\left(2\mathcal{J}^3+2\mathcal{J}\mathcal{U}^2-4\mathcal{J}^{2\prime}+3\mathcal{U}\mathcal{U}'+\mathcal{J}''\right)\partial_\phi-\left(3\mathcal{J}'+\mathcal{J}^2+\mathcal{U}^2\right)\partial_\phi^2-2\mathcal{J}\partial_\phi^3+\partial_\phi^4\Big)\,,\\
\frac{\hat{\kappa}}{4\pi}\mathcal{D}^{22}_{(2)} &= -4\Big(-2\left(\mathcal{J}''+\mathcal{U}^{2\prime}-\mathcal{J}^{2\prime}\right)\partial_\phi^{-1}\left(\mathcal{U}\partial_\phi\right)+\mathcal{U}'\partial_\phi^{-1}\left(\left(2\mathcal{J}^2-2\mathcal{U}^2+\mathcal{J}'\right)\partial_\phi+3\mathcal{J}\partial_\phi^2+\partial_\phi^3\right)\\
&\quad+\left(-4\mathcal{U}^3+4\mathcal{J}^2\mathcal{U}-3\mathcal{J}\mathcal{U}'-4\mathcal{J}'\mathcal{U}+\mathcal{U}''\right)\partial_\phi+2\mathcal{U}'\partial_\phi^2+2\mathcal{U}\partial_\phi^3\Big)\,.
\end{aligned}
$$

Appendix D

Gelfand-Dickey polynomials and Hamiltonians of the modified Boussinesq hierarchy

In this appendix we exhibit the first Gelfand-Dickey polynomials and Hamiltonians of the mBoussinesq hierarchy.

The Gelfand-Dickey polynomials can be explicitly constructed using the recurrence relation (3.9). The first of them are given by

$$
\begin{aligned}
\frac{4\pi}{\hat{\kappa}}R_{\mathcal{J}}^{(0)} &= \lambda_1 , \\
\frac{4\pi}{\hat{\kappa}}R_{\mathcal{J}}^{(1)} &= \lambda_1\mathcal{J} - \lambda_2\left(\mathcal{U}' + 2\mathcal{J}\mathcal{U}\right) , \\
\frac{4\pi}{\hat{\kappa}}R_{\mathcal{J}}^{(2)} &= 4\lambda_1\left(-4\mathcal{J}^3\mathcal{U} - \frac{4}{3}\mathcal{J}\mathcal{U}^3 - 2\mathcal{J}^2\mathcal{U}' - 2\mathcal{U}^2\mathcal{U}' + 2\mathcal{J}'\mathcal{U}' + 2\mathcal{J}''\mathcal{U} + 2\mathcal{J}\mathcal{U}'' \right. \\
&\quad \left. + \mathcal{U}'''\right) + 4\lambda_2\left(\mathcal{J}^5 + 10\mathcal{J}^3\mathcal{U}^2 - \frac{5}{3}\mathcal{J}\mathcal{U}^4 - 5\mathcal{J}\mathcal{J}'^2 + 5\mathcal{J}^2\mathcal{U}^{2\prime} - \frac{10}{12}\mathcal{U}^{4\prime} \right. \\
&\quad \left. - 5\mathcal{J}'\mathcal{U}^{2\prime} + 5\mathcal{J}\mathcal{U}'^2 - 5\mathcal{J}^2\mathcal{J}'' - 5\mathcal{J}''\mathcal{U}^2 - 5\mathcal{J}'\mathcal{J}'' + 5\mathcal{U}'\mathcal{U}'' + \mathcal{J}''''\right) ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{4\pi}{\hat{\kappa}} R^{(0)}_{\mathcal{U}} &= \lambda_2 , \\
\frac{4\pi}{\hat{\kappa}} R^{(1)}_{\mathcal{U}} &= \lambda_1 \mathcal{U} + \lambda_2 \left(\mathcal{J}' - \mathcal{J}^2 + \mathcal{U}^2 \right) , \\
\frac{4\pi}{\hat{\kappa}} R^{(2)}_{\mathcal{U}} &= 4\lambda_1 \Big(-\mathcal{J}^4 - 2\mathcal{J}^2\mathcal{U}^2 + \frac{5}{3}\mathcal{U}^4 + \frac{2}{3}\mathcal{J}^{3\prime} + 2\mathcal{U}^2\mathcal{J}' + \mathcal{J}'^2 - \mathcal{U}'^2 + 2\mathcal{J}\mathcal{J}'' \\
&\quad -2\mathcal{U}\mathcal{U}'' - \mathcal{J}''' \Big) + 4\lambda_2 \Big(5\mathcal{J}^4\mathcal{U} - \frac{10}{3}\mathcal{J}^2\mathcal{U}^3 + \frac{7}{3}\mathcal{U}^5 - \frac{10}{3}\mathcal{J}^{3\prime}\mathcal{U} + \frac{10}{3}\mathcal{J}'\mathcal{U}^3 \\
&\quad +5\mathcal{U}\mathcal{J}'^2 - 5\mathcal{J}^{2\prime}\mathcal{U}' - 5\mathcal{U}\mathcal{U}'^2 + 5\mathcal{J}''\mathcal{U}' - 5\mathcal{J}^2\mathcal{U}'' - 5\mathcal{U}^2\mathcal{U}'' + 5\mathcal{J}'\mathcal{U}'' + \mathcal{U}'''' \Big) .
\end{aligned}
$$

The corresponding Hamiltonians can then be obtained using eq. (3.10). Thus,

$$
\begin{aligned}
\frac{4\pi}{\hat{\kappa}} H_{(0)} &= \int d\phi \left(\lambda_1 \mathcal{J} + \lambda_2 \mathcal{U} \right) , \\
\frac{4\pi}{\hat{\kappa}} H_{(1)} &= \int d\phi \left\{ \frac{\lambda_1}{2} \left(\mathcal{J}^2 + \mathcal{U}^2 \right) + \lambda_2 \left(\frac{1}{3}\mathcal{U}^3 - \mathcal{J}^2\mathcal{U} - \mathcal{J}\mathcal{U}' \right) \right\} , \\
\frac{4\pi}{\hat{\kappa}} H_{(2)} &= \int d\phi \Big\{ \frac{4\lambda_1}{3} \Big(\mathcal{U}^5 - 3\mathcal{J}^4\mathcal{U} - 2\mathcal{J}^2\mathcal{U}^3 - 3\mathcal{J}'^2\mathcal{U} - 2\mathcal{J}^3\mathcal{U}' - 2\mathcal{J}\mathcal{U}^{3\prime} \\
&\quad +3\mathcal{J}^2\mathcal{U}'' - \frac{3}{2}\mathcal{U}^2\mathcal{U}'' + 3\mathcal{J}\mathcal{U}''' \Big) + \frac{2\lambda_2}{3} \Big(\mathcal{J}^6 + 15\mathcal{J}^4\mathcal{U}^2 - 5\mathcal{J}^2\mathcal{U}^4 + \frac{7}{3}\mathcal{U}^6 \\
&\quad +15\mathcal{J}^2\mathcal{J}'^2 + 15\mathcal{U}^2\mathcal{J}'^2 + 10\mathcal{J}^3\mathcal{U}^{2\prime} - 5\mathcal{J}\mathcal{U}^{4\prime} + 15\mathcal{J}^2\mathcal{U}'^2 + 15\mathcal{U}^2\mathcal{U}'^2 \\
&\quad -10\mathcal{J}\mathcal{J}'\mathcal{J}'' + 30\mathcal{J}\mathcal{U}'\mathcal{U}'' + 3\mathcal{J}\mathcal{J}'''' + 3\mathcal{U}\mathcal{U}'''' \Big) \Big\} .
\end{aligned}
$$

Appendix E

Boussinesq hierarchy

The Boussinesq hierarchy is an integrable bi-Hamiltonian system which possesses two different Poisson brackets defined by the following operators

$$\mathcal{D}^{\mathrm{Bsq}}_{(1)} = \frac{\pi}{2\hat{\kappa}} \begin{pmatrix} 0 & \partial_\phi \\ \partial_\phi & 0 \end{pmatrix}, \tag{E.1}$$

$$\mathcal{D}^{\mathrm{Bsq}}_{(2)} = \frac{4\pi}{\hat{\kappa}} \begin{pmatrix} 2\mathcal{L}\partial_\phi + \mathcal{L}' - \partial^3_\phi & 3\mathcal{W}\partial_\phi + 2\mathcal{W}' \\ 3\mathcal{W}\partial_\phi + \mathcal{W}' & -\frac{1}{2}\mathcal{L}''' + 2\mathcal{L}^{2\prime} - \frac{9}{4}\left(\mathcal{L}'' - \frac{16}{9}\mathcal{L}^2\right)\partial_\phi - \frac{15}{4}\mathcal{L}'\partial^2_\phi - \frac{5}{2}\mathcal{L}\partial^3_\phi + \frac{1}{4}\partial^5_\phi \end{pmatrix}.$$

The Poisson bracket associated to the operator $\mathcal{D}^{\mathrm{Bsq}}_{(2)}$ is given by the classical W_3-algebra.

The infinite Hamiltonians in involution can be obtained using the following recursion relation

$$\mathcal{D}^{\mathrm{Bsq}}_{(1)} \begin{pmatrix} R_{\mathcal{L}} \\ R_{\mathcal{W}} \end{pmatrix}_{(k+1)} = \mathcal{D}^{\mathrm{Bsq}}_{(2)} \begin{pmatrix} R_{\mathcal{L}} \\ R_{\mathcal{W}} \end{pmatrix}_{(k)} .$$

Here the corresponding Gelfand-Dickey polynomials are defined through

$$\begin{pmatrix} R_{\mathcal{L}} \\ R_{\mathcal{W}} \end{pmatrix}_{(k)} = \begin{pmatrix} \frac{\delta H^{\mathrm{Bsq}}_{(k)}}{\delta \mathcal{L}} \\ \frac{\delta H^{\mathrm{Bsq}}_{(k)}}{\delta \mathcal{W}} \end{pmatrix},$$

where the first Hamiltonian is given by

$$H^{\mathrm{Bsq}}_{(1)} = \frac{\hat{\kappa}}{4\pi}\int d\phi \left(\lambda_1 \mathcal{L} + \lambda_2 \mathcal{W}\right).$$

Therefore, the members of the hierarchy can be written as follows

$$\begin{pmatrix} \dot{\mathcal{L}} \\ \dot{\mathcal{W}} \end{pmatrix}_{(k)} = \mathcal{D}^{\mathrm{Bsq}}_{(1)} \begin{pmatrix} R_{\mathcal{L}} \\ R_{\mathcal{W}} \end{pmatrix}_{(k+1)} = \mathcal{D}^{\mathrm{Bsq}}_{(2)} \begin{pmatrix} R_{\mathcal{L}} \\ R_{\mathcal{W}} \end{pmatrix}_{(k)} . \tag{E.2}$$

As explained in section 3.1, the Boussinesq and the mBoussinesq hierarchies are related by the Miura transformation (3.13), that can be rewritten in the following vector form

$$\begin{pmatrix} \mathcal{L} \\ \mathcal{W} \end{pmatrix} = F\left[\mathcal{J},\mathcal{U}\right],$$

for a functional F defined through (3.13). Taking the derivative with respect to the time one obtains

$$\begin{pmatrix} \dot{\mathcal{L}} \\ \dot{\mathcal{W}} \end{pmatrix} = M \begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{U}} \end{pmatrix}, \tag{E.3}$$

where $M = M\left[\mathcal{J},\mathcal{U}\right]$ correspond to the Fréchet derivative of F with respect to $\mathcal{J}$ and $\mathcal{U}$ [74], and is precisely given by the matrix M in (3.7), i.e.,

$$M = \begin{pmatrix} \mathcal{J} + \partial_\phi & \mathcal{U} \\ -2\mathcal{J}\mathcal{U} - \frac{1}{2}\mathcal{U}\partial_\phi - \frac{3}{2}\mathcal{U}' & \mathcal{U}^2 - \mathcal{J}^2 - \frac{1}{2}\mathcal{J}' - \frac{3}{2}\mathcal{J}\partial_\phi - \frac{1}{2}\partial_\phi^2 \end{pmatrix}.$$

If one takes into account its formal adjoint

$$M^\dagger = \begin{pmatrix} \mathcal{J} - \partial_\phi & -2\mathcal{J}\mathcal{U} + \frac{1}{2}\mathcal{U}\partial_\phi - \mathcal{U}' \\ \mathcal{U} & \mathcal{U}^2 - \mathcal{J}^2 + \mathcal{J}' + \frac{3}{2}\mathcal{J}\partial_\phi - \frac{1}{2}\partial_\phi^2 \end{pmatrix},$$

the Gelfand-Dickey polynomials of both hierarchies are then related by

$$\begin{pmatrix} R_{\mathcal{J}} \\ R_{\mathcal{U}} \end{pmatrix} = M^{\dagger} \begin{pmatrix} R_{\mathcal{L}} \\ R_{\mathcal{W}} \end{pmatrix}. \tag{E.4}$$

Taking into account eqs. (E.3), (3.25) and (E.4) we can write

$$\begin{pmatrix} \dot{\mathcal{L}} \\ \dot{\mathcal{W}} \end{pmatrix}_{(k)} = M\mathcal{D} \begin{pmatrix} R_{\mathcal{J}} \\ R_{\mathcal{U}} \end{pmatrix}_{(k)} = M\mathcal{D}M^{\dagger} \begin{pmatrix} R_{\mathcal{L}} \\ R_{\mathcal{W}} \end{pmatrix}_{(k)} = \mathcal{D}_{(2)}^{\text{Bsq}} \begin{pmatrix} R_{\mathcal{L}} \\ R_{\mathcal{W}} \end{pmatrix}_{(k)},$$

which imply that the second Poisson structure for the Boussinesq hierarchy can be expressed in terms of the first Poisson structure of the mBoussinesq hierarchy according to

$$\mathcal{D}_{(2)}^{\text{Bsq}} = M\mathcal{D}M^{\dagger}.$$

Appendix F

Fundamental representation of the principal embedding of $sl(2,\mathbb{R})$ within $sl(N,\mathbb{R})$

In the principal embedding of the $sl(2,\mathbb{R})$ algebra within the $sl(N,\mathbb{R})$ algebra the generators can be written in the basis $\left\{L_i, W_m^{(s)}\right\}$, with $i=-1,0,1$, $s=3,4,\ldots$ and $m=-s+1,\ldots,s-1$. In the fundamental representation of $sl(N,\mathbb{R})$, the generators may be represented by the following $N\times N$ matrices

$$
\begin{aligned}
(L_1)_{jk} &= -\sqrt{j\,(N-j)}\delta_{j+1,k},\\
(L_{-1})_{jk} &= \sqrt{k\,(N-k)}\delta_{j,k+1},\\
(L_0)_{jk} &= \frac{1}{2}\,(N+1-2j)\,\delta_{j,k},\\
W_m^{(s)} &= 2\,(-1)^{s-m-1}\,\frac{(s+m-1)!}{(2s-2)!}\underbrace{\left[L_{-1},\left[L_{-1},\cdots\left[L_{-1},(L_1)^{s-1}\right]\cdots\right]\right]}_{s-m-1\text{ terms}},\\
&= 2\,(-1)^{s-m-1}\,\frac{(s+m-1)!}{(2s-2)!}\left(\mathrm{ad}_{L_{-1}}\right)^{s-m-1}(L_1)^{s-1}.
\end{aligned}
$$

with $j,k=1,\ldots,N$, and where $\mathrm{ad_x}(\mathrm{Y}) := [\mathrm{X},\mathrm{Y}]$. From the commutation relations

$$[L_i, L_j] = (i-j)\, L_{i+j},$$
$$\left[L_i, W_m^{(s)}\right] = ((s-1)\, i - m)\, W_{i+m}^{(s)},$$

it can be seen that the L_i generators close in a $sl(2,\mathbb{R})$ subalgebra, while the generators $W_m^{(s)}$ transform in a spin-s representation under $sl(2,\mathbb{R})$.

$sl(3,\mathbb{R})$ generators

The generators of $sl(3,\mathbb{R})$ algebra are given by the following 3×3 matrices

$$L_{-1} = \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix},$$

and the spin–3 generators $W_m^{(3)} = W_m$

$$W_{-2} = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W_{-1} = \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad W_0 = \frac{2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$W_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

Appendix G

Wess-Zumino term

Here we show that for our boundary conditions in (3.18), (3.19), (3.22), the Wess-Zumino term

$$I_1 = \frac{\kappa}{16\pi}\int dtdrd\phi\epsilon^{ij}\left\langle \partial_t\left(G^{-1}\right)\partial_i G G^{-1}\partial_j G\right\rangle,$$

in (3.43) vanishes.

Let us perform the following Gauss decomposition of the group element

$$G = e^{TL_1+MW_1+QW_2}e^{\Phi L_0+\Phi_W W_0}e^{XL_{-1}+YW_{-1}+ZW_{-2}}, \tag{G.1}$$

where all the functions that appear in (G.1) generically depend on t, r and ϕ. Then, if we replace (G.1) in I_1, one can show that it reduces to a boundary term of the form

$$\begin{aligned} I_1 = {} & \frac{\kappa}{16\pi}\int d\phi dt\Big[2e^{\Phi+2\Phi_W}\left(\left(X'+Y'\right)\left(\dot{M}+\dot{T}\right)-\left(\dot{X}+\dot{Y}\right)\left(M'+T'\right)\right) \\ & -2e^{\Phi-2\Phi_W}\left(\left(X'-Y'\right)\left(\dot{M}-\dot{T}\right)-\left(\dot{X}-\dot{Y}\right)\left(M'-T'\right)\right) \\ & +e^{2\Phi}\left(8\left(XY'-YX'\right)\dot{Q}-8\left(X\dot{Y}-Y\dot{X}\right)Q'+8\left(T\dot{M}-M\dot{T}\right)Z'-8\left(TM'-MT'\right)\dot{Z}\right. \\ & \left.+4\left(X\dot{Y}-Y\dot{X}\right)\left(TM'-MT'\right)-4\left(XY'-YX'\right)\left(T\dot{M}-M\dot{T}\right)+16\left(Q'\dot{Z}-Z'\dot{Q}\right)\right)\Big]. \end{aligned} \tag{G.2}$$

Now, it is useful to perform the following decomposition in the asymptotic region which is compatible with (3.18)

$$G = g\left(t,\phi\right) b\left(r\right),$$

with

$$g\left(t,\phi\right) = \exp\left[\sqrt{\frac{8\pi}{\kappa}}\varphi L_0 + \sqrt{\frac{6\pi}{\kappa}}\psi W_0\right],$$

and where $b\left(r\right)$ is an arbitrary gauge group element depending on the radial coordinate that generically can be decomposed as

$$b\left(r\right) = b_{(+)} b_{(0)} b_{(-)},$$

with

$$b_{(+)} \;=\; e^{\left(b_1 L_1 + \bar{b}_1 W_1 + \bar{b}_2 W_2\right)}, \qquad b_{(0)} = e^{\left(b_0 L_0 + \bar{b}_0 W_0\right)}, \qquad b_{(-)} = e^{\left(b_{-1} L_{-1} + \bar{b}_{-1} W_{-1} + \bar{b}_{-2} W_{-2}\right)}.$$

Consistency with (G.1) then implies the following conditions

$$\Phi = b_0\left(r\right) + \sqrt{\frac{8\pi}{\kappa}}\varphi\left(t,\phi\right), \qquad \Phi_W = \bar{b}_0\left(r\right) + \sqrt{\frac{6\pi}{\kappa}}\psi\left(t,\phi\right),$$

$$X = b_{-1}\left(r\right), \qquad Y = \bar{b}_{-1}\left(r\right), \qquad Z = \bar{b}_{-2}\left(r\right),$$

$$Q = e^{-2\sqrt{\frac{8\pi}{\kappa}}\varphi}\bar{b}_{-2}\left(r\right), \qquad M = e^{-\sqrt{\frac{8\pi}{\kappa}}\varphi}\left(\bar{b}_1\left(r\right)\cosh\left(2\sqrt{\frac{6\pi}{\kappa}}\psi\right) - b_1\left(r\right)\sinh\left(2\sqrt{\frac{6\pi}{\kappa}}\psi\right)\right),$$

$$T = e^{-\sqrt{\frac{8\pi}{\kappa}}\varphi}\left(b_1\left(r\right)\cosh\left(2\sqrt{\frac{6\pi}{\kappa}}\psi\right) - \bar{b}_1\left(r\right)\sinh\left(2\sqrt{\frac{6\pi}{\kappa}}\psi\right)\right).$$

Note that since, X, Y and Z depend only on the radial coordinate, then the WZ term in eq. (G.2) identically vanishes.

Bibliography

[1] J. D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun. Math. Phys.* **104** (1986) 207.

[2] G. Compère, W. Song and A. Strominger, *New Boundary Conditions for AdS_3*, *JHEP* **1305** (2013) 152 [1303.2662].

[3] C. Troessaert, *Enhanced asymptotic symmetry algebra of AdS_3*, *JHEP* **08** (2013) 044 [1303.3296].

[4] S. G. Avery, R. R. Poojary and N. V. Suryanarayana, *An sl(2,$\mathbb{R}$) current algebra from AdS_3 gravity*, *JHEP* **01** (2014) 144 [1304.4252].

[5] H. Afshar, S. Detournay, D. Grumiller, W. Merbis, A. Pérez, D. Tempo and R. Troncoso, *Soft Heisenberg hair on black holes in three dimensions*, *Phys. Rev.* **D93** (2016) 101503 [1603.04824].

[6] A. Pérez, D. Tempo and R. Troncoso, *Boundary conditions for General Relativity on AdS_3 and the KdV hierarchy*, *JHEP* **06** (2016) 103 [1605.04490].

[7] D. Grumiller and M. Riegler, *Most general AdS_3 boundary conditions*, *JHEP* **10** (2016) 023 [1608.01308].

[8] E. Ojeda and A. Pérez, *Boundary conditions for General Relativity in three-dimensional spacetimes, integrable systems and the KdV/mKdV hierarchies*, *JHEP* **08** (2019) 079 [1906.11226].

[9] A. Pérez and R. Troncoso, *Gravitational dual of averaged free CFT's over the Narain lattice*, *JHEP* **11** (2020) 015 [2006.08216].

[10] A. Strominger, *Black hole entropy from near-horizon microstates*, *JHEP* **02** (1998) 009 [9712251].

[11] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[12] M. Henneaux, A. Pérez, D. Tempo and R. Troncoso, *Chemical potentials in three-dimensional higher spin anti-de Sitter gravity*, *JHEP* **1312** (2013) 048 [1309.4362].

[13] C. Bunster, M. Henneaux, A. Pérez, D. Tempo and R. Troncoso, *Generalized Black Holes in Three-dimensional Spacetime*, *JHEP* **1405** (2014) 031 [1404.3305].

[14] R. Sasaki and I. Yamanaka, *Virasoro Algebra, Vertex Operators, Quantum Sine-Gordon and Solvable Quantum Field Theories*, *Adv. Stud. Pure Math.* **16** (1988) 271.

[15] T. Eguchi and S.-K. Yang, *Deformations of Conformal Field Theories and Soliton Equations*, *Phys. Lett.* **B224** (1989) 373.

[16] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz*, *Commun. Math. Phys.* **177** (1996) 381 [hep-th/9412229].

[17] P. Calabrese, F. H. L. Essler and M. Fagotti, *Quantum Quench in the Transverse Field Ising Chain*, *Phys. Rev. Lett.* **106** (2011) 227203 [1104.0154].

[18] S. Sotiriadis and P. Calabrese, *Validity of the GGE for quantum quenches from interacting to noninteracting models*, *J. Stat. Mech.* **1407** (2014) P07024 [1403.7431].

[19] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler and T. Prosen, *Complete generalized gibbs ensembles in an interacting theory*, *Phys. Rev. Lett.* **115** (2015) 157201.

[20] L. Vidmar and M. Rigol, *Generalized gibbs ensemble in integrable lattice models*, *Journal of Statistical Mechanics: Theory and Experiment* **2016** (2016) 064007.

[21] J. de Boer and D. Engelhardt, *Remarks on thermalization in 2D CFT*, *Phys. Rev.* **D94** (2016) 126019 [1604.05327].

[22] B. Pozsgay, E. Vernier and M. A. Werner, *On generalized gibbs ensembles with an infinite set of conserved charges*, *Journal of Statistical Mechanics: Theory and Experiment* **2017** (2017) 093103.

[23] A. Dymarsky and K. Pavlenko, *Generalized Gibbs Ensemble of 2d CFTs at large central charge in the thermodynamic limit*, *JHEP* **01** (2019) 098 [1810.11025].

[24] A. Maloney, G. S. Ng, S. F. Ross and I. Tsiares, *Thermal Correlation Functions of KdV Charges in 2D CFT*, *JHEP* **02** (2019) 044 [1810.11053].

[25] A. Maloney, G. S. Ng, S. F. Ross and I. Tsiares, *Generalized Gibbs Ensemble and the Statistics of KdV Charges in 2D CFT*, *JHEP* **03** (2019) 075 [1810.11054].

[26] A. Dymarsky and K. Pavlenko, *Exact generalized partition function of 2D CFTs at large central charge*, *JHEP* **05** (2019) 077 [1812.05108].

[27] E. M. Brehm and D. Das, *Korteweg–de Vries characters in large central charge CFTs*, *Phys. Rev. D* **101** (2020) 086025 [1901.10354].

[28] A. Dymarsky and K. Pavlenko, *Generalized Eigenstate Thermalization Hypothesis in 2D Conformal Field Theories*, Phys. Rev. Lett. **123** (2019) 111602 [1903.03559].

[29] A. Dymarsky and S. Sugishita, *KdV-charged black holes*, JHEP **05** (2020) 041 [2002.08368].

[30] O. Fuentealba, J. Matulich, A. Pérez, M. Pino, P. Rodríguez, D. Tempo and R. Troncoso, *Integrable systems with BMS_3 Poisson structure and the dynamics of locally flat spacetimes*, JHEP **01** (2018) 148 [1711.02646].

[31] R. Hirota and J. Satsuma, *Soliton Solutions of a Coupled Korteweg-de Vries Equation*, Phys. Lett. **A85** (1981) 407.

[32] D. Melnikov, F. Novaes, A. Pérez and R. Troncoso, *Lifshitz Scaling, Microstate Counting from Number Theory and Black Hole Entropy*, JHEP **06** (2019) 054 [1808.04034].

[33] H. McKean, *Boussinesq's equation on the circle*, Physica D: Nonlinear Phenomena **3** (1981) 294 .

[34] A. Das, *Integrable Models*. World Scientific, 1989, 10.1142/0858.

[35] E. Ojeda and A. Pérez, *Integrable systems and the boundary dynamics of higher spin gravity on AdS_3*, JHEP **11** (2020) 089 [2009.07829].

[36] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *Method for solving the Korteweg-deVries equation*, Phys. Rev. Lett. **19** (1967) 1095.

[37] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Communications on Pure and Applied Mathematics **21** (1968) 467.

[38] M. Blaszak, *Multi-Hamiltonian Theory of Dynamical Systems*. Springer, 1998.

[39] R. M. Miura, *Korteweg-de Vries Equation and Generalizations. I. A Remarkable Explicit Nonlinear Transformation*, J. Math. Phys. **9** (1968) 1202.

[40] H. Afshar, D. Grumiller, W. Merbis, A. Pérez, D. Tempo and R. Troncoso, *Soft hairy horizons in three spacetime dimensions*, Phys. Rev. **D95** (2017) 106005 [1611.09783].

[41] W. Ames and C. Rogers, *Nonlinear Equations in the Applied Sciences*, vol. 185. Mathematics in Science and Engineering, 1992.

[42] T. Regge and C. Teitelboim, *Role of surface integrals in the Hamiltonian formulation of General Relativity*, Ann. Phys. **88** (1974) 286.

[43] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, Phys. Rev. Lett. **69** (1992) 1849 [hep-th/9204099].

[44] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, Phys. Rev. **D48** (1993) 1506 [gr-qc/9302012].

[45] A. Achucarro and P. K. Townsend, *A Chern-Simons action for three-dimensional Anti-de Sitter supergravity theories*, *Phys. Lett.* **B180** (1986) 89.

[46] E. Witten, *(2+1)-dimensional gravity as an exactly soluble system*, *Nucl. Phys.* **B311** (1988) 46.

[47] O. Coussaert, M. Henneaux and P. van Driel, *The Asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant*, *Class.Quant.Grav.* **12** (1995) 2961 [gr-qc/9506019].

[48] D. Grumiller, A. Pérez, M. M. Sheikh-Jabbari, R. Troncoso and C. Zwikel, *Spacetime structure near generic horizons and soft hair*, *Phys. Rev. Lett.* **124** (2020) 041601 [1908.09833].

[49] G. Barnich and C. Troessaert, *BMS charge algebra*, *JHEP* **1112** (2011) 105 [1106.0213].

[50] C. Bunster, A. Gomberoff and A. Pérez, *Regge-Teitelboim analysis of the symmetries of electromagnetic and gravitational fields on asymptotically null spacelike surfaces*, (2018), 1805.03728.

[51] C. Bunster, A. Gomberoff and A. Pérez, *Bondi-metzner-sachs invariance and electric-magnetic duality*, *Phys. Rev. D* **101** (2020) 044003 [1905.07514].

[52] S. W. Hawking, M. J. Perry and A. Strominger, *Soft Hair on Black Holes*, *Phys. Rev. Lett.* **116** (2016) 231301 [1601.00921].

[53] S. W. Hawking, M. J. Perry and A. Strominger, *Superrotation Charge and Supertranslation Hair on Black Holes*, 1611.09175.

[54] G. Barnich, C. Troessaert, D. Tempo and R. Troncoso, *Asymptotically locally flat spacetimes and dynamical nonspherically-symmetric black holes in three dimensions*, *Phys. Rev.* **D93** (2016) 084001 [1512.05410].

[55] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *Massive Gravity in Three Dimensions*, *Phys. Rev. Lett.* **102** (2009) 201301 [0901.1766].

[56] E. A. Bergshoeff, O. Hohm and P. K. Townsend, *More on Massive 3D Gravity*, *Phys. Rev.* **D79** (2009) 124042 [0905.1259].

[57] R. Benguria, P. Cordero and C. Teitelboim, *Aspects of the Hamiltonian dynamics of interacting gravitational gauge and Higgs fields with applications to spherical symmetry*, *Nucl. Phys.* **B122** (1977) 61.

[58] H. A. Gonzalez, D. Tempo and R. Troncoso, *Field theories with anisotropic scaling in 2D, solitons and the microscopic entropy of asymptotically Lifshitz black holes*, *JHEP* **11** (2011) 066 [1107.3647].

[59] G. H. Hardy and S. Ramanujan, *Asymptotic formulaæ in combinatory analysis*, *Proceedings of the London Mathematical Society* **s2-17** (1918) 75.

[60] E. M. Wright, *Asymptotic partition formulae. iii. partitions into k-th powers*, *Acta Mathematica* **63** (1934) 143.

[61] M. Blencowe, *A consistent interacting massless higher spin field theory in d = (2+1)*, *Class.Quant.Grav.* **6** (1989) 443.

[62] E. Bergshoeff, M. Blencowe and K. Stelle, *Area preserving diffeomorphisms and higher spin algebra*, *Commun.Math.Phys.* **128** (1990) 213.

[63] M. A. Vasiliev, *Higher spin gauge theories in four-dimensions, three-dimensions, and two-dimensions*, *Int. J. Mod. Phys.* **D5** (1996) 763 [hep-th/9611024].

[64] G. Compère and W. Song, *$\mathcal{W}$ symmetry and integrability of higher spin black holes*, *JHEP* **1309** (2013) 144 [1306.0014].

[65] M. Gutperle and Y. Li, *Higher Spin Lifshitz Theory and Integrable Systems*, *Phys. Rev. D* **91** (2015) 046012 [1412.7085].

[66] M. Beccaria, M. Gutperle, Y. Li and G. Macorini, *Higher spin Lifshitz theories and the Korteweg-de Vries hierarchy*, *Phys. Rev. D* **92** (2015) 085005 [1504.06555].

[67] D. Grumiller, A. Pérez, S. Prohazka, D. Tempo and R. Troncoso, *Higher Spin Black Holes with Soft Hair*, *JHEP* **10** (2016) 119 [1607.05360].

[68] J. Boussinesq, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond.*, *Journal de Mathématiques Pures et Appliquées* (1872) 55.

[69] V. E. Zakharov, *On stochastization of one-dimensional chains of nonlinear oscillations*, *Sov. Phys. JETP* **38** (1974) 108.

[70] D. J. Kaup, *A Higher-Order Water-Wave Equation and the Method for Solving It*, *Progress of Theoretical Physics* **54** (1975) 396 [https://academic.oup.com/ptp/article-pdf/54/2/396/5181677/54-2-396.pdf].

[71] R. Hirota and J. Satsuma, *Nonlinear Evolution Equations Generated from the Bäcklund Trans-formation for the Boussinesq Equation*, *Progress of Theoretical Physics* **57** (1977) 797 [https://academic.oup.com/ptp/article-pdf/57/3/797/5467478/57-3-797.pdf].

[72] A. P. Fordy and J. Gibbons, *Factorization of operators.II*, *Journal of Mathematical Physics* **22** (1981) 1170.

[73] V. Drinfeld and V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, *J. Sov. Math.* **30** (1984) 1975.

[74] P. Mathieu and W. Oevel, *The $W_3^{(2)}$ conformal algebra and the Boussinesq hierarchy*, *Modern Physics Letters A* **06** (1991) 2397.

[75] E. Fradkin and M. A. Vasiliev, *On the Gravitational Interaction of Massless Higher Spin Fields*, *Phys.Lett.* **B189** (1987) 89.

[76] M. A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions*, *Phys.Lett.* **B243** (1990) 378.

[77] M. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)*, *Phys.Lett.* **B567** (2003) 139 [hep-th/0304049].

[78] M. Henneaux and S.-J. Rey, *Nonlinear $W_{infinity}$ as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity*, *JHEP* **1012** (2010) 007 [1008.4579].

[79] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **1011** (2010) 007 [1008.4744].

[80] M. Gutperle and P. Kraus, *Higher Spin Black Holes*, *JHEP* **1105** (2011) 022 [1103.4304].

[81] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, *Spacetime Geometry in Higher Spin Gravity*, *JHEP* **1110** (2011) 053 [1106.4788].

[82] M. Henneaux, A. Perez, D. Tempo and R. Troncoso, *Hypersymmetry bounds and three-dimensional higher-spin black holes*, *JHEP* **08** (2015) 021 [1506.01847].

[83] M. Banados, A. Castro, A. Faraggi and J. I. Jottar, *Extremal Higher Spin Black Holes*, *JHEP* **04** (2016) 077 [1512.00073].

[84] M. Henneaux, L. Maoz and A. Schwimmer, *Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity*, *Annals Phys.* **282** (2000) 31 [hep-th/9910013].

[85] P. Forgacs, A. Wipf, J. Balog, L. Feher and L. O'Raifeartaigh, *Liouville and Toda Theories as Conformally Reduced WZNW Theories*, *Phys. Lett. B* **227** (1989) 214.

[86] A. Alekseev and S. L. Shatashvili, *Path Integral Quantization of the Coadjoint Orbits of the Virasoro Group and 2D Gravity*, *Nucl. Phys. B* **323** (1989) 719.

[87] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351.

[88] S. Elitzur, G. W. Moore, A. Schwimmer and N. Seiberg, *Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory*, *Nucl. Phys.* **B326** (1989) 108.

[89] H. A. González, J. Matulich, M. Pino and R. Troncoso, *Revisiting the asymptotic dynamics of General Relativity on AdS_3*, *JHEP* **12** (2018) 115 [1809.02749].

[90] R. Floreanini and R. Jackiw, *Selfdual Fields as Charge Density Solitons*, *Phys. Rev. Lett.* **59** (1987) 1873.

[91] A. Pérez, D. Tempo and R. Troncoso, *Work in progress.*

[92] L. A. Dickey, *Soliton Equations and Hamiltonian Systems.* World Scientific, 2nd ed., 2003, 10.1142/5108.

[93] A. Maloney and E. Witten, *Averaging Over Narain Moduli Space*, *JHEP* **10** (2020) 187 [2006.04855].

[94] N. Afkhami-Jeddi, H. Cohn, T. Hartman and A. Tajdini, *Free partition functions and an averaged holographic duality*, 2006.04839.

[95] J. Cotler and K. Jensen, *AdS_3 gravity and random CFT*, 2006.08648.

[96] M. Ammon, D. Grumiller, S. Prohazka, M. Riegler and R. Wutte, *Higher-Spin Flat Space Cosmologies with Soft Hair*, *JHEP* **05** (2017) 031 [1703.02594].

[97] F. A. Smirnov and A. B. Zamolodchikov, *On space of integrable quantum field theories*, *Nucl. Phys.* **B915** (2017) 363 [1608.05499].

[98] A. B. Zamolodchikov, *Expectation value of composite field T anti-T in two-dimensional quantum field theory*, hep-th/0401146.

[99] A. Cavaglia, S. Negro, I. M. Szécsényi and R. Tateo, *$T\bar{T}$-deformed 2D Quantum Field Theories*, *JHEP* **10** (2016) 112 [1608.05534].

[100] J. Cardy, *The $T\overline{T}$ deformation of quantum field theory as random geometry*, *JHEP* **10** (2018) 186 [1801.06895].

[101] O. Aharony and T. Vaknin, *The TT^* deformation at large central charge*, *JHEP* **05** (2018) 166 [1803.00100].

[102] S. Datta and Y. Jiang, *$T\bar{T}$ deformed partition functions*, *JHEP* **08** (2018) 106 [1806.07426].

[103] O. Aharony, S. Datta, A. Giveon, Y. Jiang and D. Kutasov, *Modular invariance and uniqueness of $T\bar{T}$ deformed CFT*, *JHEP* **01** (2019) 086 [1808.02492].

[104] M. Guica, *An integrable Lorentz-breaking deformation of two-dimensional CFTs*, *SciPost Phys.* **5** (2018) 048 [1710.08415].

[105] S. Chakraborty, A. Giveon and D. Kutasov, *$J\overline{T}$ deformed CFT_2 and string theory*, *JHEP* **10** (2018) 057 [1806.09667].

[106] O. Aharony, S. Datta, A. Giveon, Y. Jiang and D. Kutasov, *Modular covariance and uniqueness of $J\bar{T}$ deformed CFTs*, *JHEP* **01** (2019) 085 [1808.08978].

[107] M. Guica, *On correlation functions in $J\bar{T}$-deformed CFTs*, *J. Phys.* **A52** (2019) 184003 [1902.01434].

[108] L. McGough, M. Mezei and H. Verlinde, *Moving the CFT into the bulk with $T\overline{T}$*, *JHEP* **04** (2018) 010 [1611.03470].

[109] A. Giveon, N. Itzhaki and D. Kutasov, $\mathrm{T}\overline{\mathrm{T}}$ *and LST*, *JHEP* **07** (2017) 122 [1701.05576].

[110] P. Kraus, J. Liu and D. Marolf, *Cutoff AdS_3 versus the $T\overline{T}$ deformation*, *JHEP* **07** (2018) 027 [1801.02714].

[111] A. Bzowski and M. Guica, *The holographic interpretation of $J\bar{T}$-deformed CFTs*, *JHEP* **01** (2019) 198 [1803.09753].

[112] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, *Nonlinear Fluid Dynamics from Gravity*, *JHEP* **02** (2008) 045 [0712.2456].

[113] M. Haack and A. Yarom, *Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT*, *JHEP* **10** (2008) 063 [0806.4602].

[114] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, *Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions*, *JHEP* **12** (2008) 116 [0809.4272].

[115] V. E. Hubeny, S. Minwalla and M. Rangamani, *The fluid/gravity correspondence*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: String theory and its Applications: From meV to the Planck Scale*, pp. 348–383, 2012, 1107.5780.

[116] A. Campoleoni, L. Ciambelli, C. Marteau, P. M. Petropoulos and K. Siampos, *Two-dimensional fluids and their holographic duals*, *Nucl. Phys. B* **946** (2019) 114692 [1812.04019].

[117] M. Henneaux, W. Merbis and A. Ranjbar, *Asymptotic dynamics of AdS_3 gravity with two asymptotic regions*, *JHEP* **03** (2020) 064 [1912.09465].

[118] H. Afshar, A. Bagchi, R. Fareghbal, D. Grumiller and J. Rosseel, *Higher spin theory in 3-dimensional flat space*, *Phys.Rev.Lett.* **111** (2013) 121603 [1307.4768].

[119] H. A. Gonzalez, J. Matulich, M. Pino and R. Troncoso, *Asymptotically flat spacetimes in three-dimensional higher spin gravity*, *JHEP* **1309** (2013) 016 [1307.5651].

[120] M. Gary, D. Grumiller, M. Riegler and J. Rosseel, *Flat space (higher spin) gravity with chemical potentials*, *JHEP* **1501** (2015) 152 [1411.3728].

[121] J. Matulich, A. Pérez, D. Tempo and R. Troncoso, *Higher spin extension of cosmological spacetimes in 3D: asymptotically flat behaviour with chemical potentials and thermodynamics*, *JHEP* **05** (2015) 025 [1412.1464].