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On the discretization of Dirac equations in the framework of Finite Element Systems

POR

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To Jimena and Mario

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>
> *Taisen Deshimaru*

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CHAPTER 1

Introduction

Dirac operators arise in numerous areas of science. Notably, in quantum mechanics, general relativity, mathematical physics, mathematical analysis (see [ER07] for a review, see also [Tay23, Chapter 10]) and differential geometry; see [Fri00] for a detailed account of Dirac operators in Riemannian geometry. The original and most well–known appearance of such an operator is in the Dirac equation proposed by P.A.M. Dirac in 1928, which was conceived as his solution to the relativization of Schrödinger's equation in flat space-time (see [Dir28]). Later on, this was generalized to curved space-time—and more generally, to curved manifolds—and higher dimensions. Properly doing so was not immediate; it took the development of the theories of Clifford algebras and principal fibre bundles to overcome several difficulties. A Dirac operator, in general, solves the problem of finding a square root of a given Laplacian, a question that was raised by P.A.M Dirac himself. In this way, a Dirac operator is always a first order operator which is expected to preserve analytic properties of a certain Laplacian operator. By nature, Dirac operators acting on sections of a bundle over a manifold are self-adjoint, elliptic operators, which are not positive definite. From the numerical point of view, this presents difficulties, as the standard theory of Galerkin discretizations is greatly benefitted from this property.

Two of the most used examples of Laplacians are the Hodge-Laplace and Laplace-Beltrami operators, both acting on forms. The corresponding square roots are given by the Hodge-Dirac and Dirac-Kähler operators, respectively. These are the objects of study in this work. One should also mention the Dirac operator associated with the spin bundle of a manifold, which is arguably the most relevant one. However, its treatment requires the advanced technology of spin geometry. This leans towards the study of spin bundles and spinor-valued fields (see [Fri00]), which complicates the analysis from the numerical viewpoint, as the theory of Galerkin discretizations is best understood in the language of differential forms. This, together with further reasons discussed later, leads us to focus on the Hodge-Dirac and Dirac-Kähler operators.

This work consists of two parts. We address the numerical discretization of the two aforementioned operators of Dirac type, which are closely related, and are equally relevant both in physics and mathematics. We work within the framework of differential forms and finite elements, or more precisely, in the framework of finite element systems (FES), as coined by Snorre H. Christiansen in [Chr09]. See Section 2.6.2 and the references therein for a brief review of the FES framework.

In Part I, we study the numerical approximation of eigenmodes of the Hodge-Dirac operator on a generic torus. In doing so, we follow closely the approach made by Christiansen in [Chr18], where he approaches this problem from the standpoint of nonconforming finite element methods. In [Chr18], a general theory for the discretization of a class of eigenvalue problems is developed; special care is taken to allow for compact perturbations of the operator, such as those arising from the incorporation of an electromagnetic field into the original Dirac equation. At this point it is worth remarking that the general theory of Babuška and Osborn for variational eigenvalue problems does not apply directly , as it is mostly focused on positive definite operators. In addition, estimates on Fractional Sobolev spaces are derived, which allow him to obtain almost optimal convergence rates in terms of the discretization parameter. As mentioned in [Chr18], the technique used to discretize the Hodge-Dirac operator was learned by

Christiansen at a conference given by Ari Stern in 2013. In turn, in [LS16], Stern and Leopardi studied the discretization of what they called the abstract Hodge-Dirac operator. They adopt the framework of finite element exterior calculus (FEEC), mostly developed by Arnold, Falk and Winther in [AFW06], with the exception that they do not use Hilbert complexes explicitly, but instead note that it is enough to consider unbounded, densely defined nilpotent operators from a Hilbert space to itself. There, they focus on the source problem for the Hodge-Dirac operator instead of the eigenvalue problems. They motivate the problem by establishing connections between Clifford Analysis and Exterior Calculus and between the original Dirac operator and the Hodge-Dirac operator. They provide a solvability and stability analysis, as well as *a priori* error estimates and a two-dimensional numerical example using Whitney forms. Connections between the source problem for the Hodge Laplacian and that for the Hodge-Dirac operator are also discussed, showing that many results for the latter imply those studied for the Hodge Laplacian in [AFW06] .

Now, regarding our work; we address the discretization of the eigenvalue problem for the Hodge-Dirac operator in the setting of conforming Galerkin discretizations, also in the torus. The conforming approach leads to simplifications such as dispensing with Fractional Sobolev spaces. The motivation for studying conforming approximations stems from the introduction of H^1-conforming finite element systems for differential forms, done by Christiansen and Hu in [CH18]. We begin by motivating the study of the Hodge-Dirac eigenvalue problem, starting from the Dirac equation on flat Minkowski time-space $\mathbb{R}\oplus\mathbb{R}^3$. This is done by making an identification of this space with the quaternions $\mathbb{H}$, and then looking for the energy states associated with the Dirac operator. The solutions of the latter become relevant when one seeks to write the solution of the time-dependent Dirac equation as a series of stationary solutions, which are exactly the ones given by the eigenvalue problem. This is the main motivation for studying the eigenproblem for the Hodge-Dirac operator. Next, as in [Chr18], we abstract matters by studying the discretization of the eigenmode problem associated with a self-adjoint Fredholm operator, there we provide sufficient conditions for eigenmode convergence and stability of compact perturbations. This is followed by an application to the Hodge-Dirac eigenmode problem on a torus, which requires to develop analytic tools called commuting smoothed projections, which connect the continuous and discrete spaces and enjoy enhanced continuity properties with respect to standard interpolators. With these objects at hand, we are able to derive stability properties of various operators and spaces involved in the discretization. We finish Part I by showing that all the conditions required by the abstract theory are verified in our specific setting. For greater reusability, this is done by requiring the discrete spaces to satisfy some generic assumptions, instead of staying with a particular choice.

In Part II, we are concerned with the stable discretization of a particular instance of the Dirac-Kähler operator. As mentioned earlier, the Dirac-Kähler equation may be written in the language of differential forms, and it can be understood as the natural geometric realization of the Dirac equation. Nevertheless, under appropriate conditions, this equation also bears physical relevance. Indeed, in [Kä62] (in German), Kähler showed that the Dirac equation in four-dimensional space-time can be written nicely using differential forms. Indeed, it can be shown that the Dirac-Kähler equation in four-dimensional space-time is equivalent to a set of four Dirac equations, which, in particular, makes possible to study fermions with spin 1/2. For a review on the Dirac-Kähler equation from this viewpoint, we refer to [Kru02]. In this work we take the geometric approach, in that we study the discretization of the Dirac-Kähler operator in the context of an abstract compact manifold with boundary. Moreover, we are particularly interested on the operator arising from restricting the Dirac-Kähler operator to the even part of the bundle of differential forms, which, to us, will be both of topological and numerical interest.

In the field of differential geometry, it is a well–known fact that, for a compact Riemannian manifold $\mathcal{S}$, the *Euler-Poincaré characteristic*, denoted $\chi(\mathcal{S})$, may be computed in terms of differential forms or, more precisely, harmonic forms. This is a direct consequence of the de Rham theorem, which states that the singular cohomology groups are isomorphic to the de Rham cohomology groups. Although we shall provide some details for these facts, we refer to [BT82, Lee13, Pet16] for a complete treatment of singular cohomology, de Rham cohomology, and the subsequent implications of their relation for the Euler-Poincaré characteristic. In addition, as we shall see later, one can reformulate this as saying that the Euler-Poincaré characteristic of a compact manifold is the Fredholm index of a specific instance of Dirac-

Kähler operator, which we call the *graded Dirac-Kähler*. This interpretation may be related to the celebrated index theorem of Atiyah and Singer, which relates the analytical (Fredholm) and topological indices of an elliptic operator acting between sections of vector bundles. In fact, the combination of both results yields the *Gauss-Bonnet-Chern* formula. The Euler-Poincaré characteristic is a topological invariant and, as such, we would like to preserve it when discretizing equations. In the context of *structure preserving discretizations*, one is interested in inheriting as many topological, geometric and analytic properties as possible.

Specifically, in Part II, we focus on proving that, under suitable assumptions, the Euler-Poincaré characteristic "is preserved", in the sense that, for an appropriate notion of *discrete Fredholm index*, the discretization of the aforementioned Dirac-Kähler operator inherits the continuous Fredholm index. In other words, the discrete Fredholm index of the graded Dirac-Kähler is also the Euler-Poincaré characteristic of $\mathcal{S}$.

In [Chr04], Christiansen introduced a notion of discrete left semi-Fredholm operator to address the numerical discretization of the *electric field integral equation*. There, an entire section is dedicated to motivate this concept and deduce its main properties. In this work we build on that for the aforementioned purposes. We introduce the notion of discrete Fredholm operator and discrete Fredholm index, two concepts that allows us to have the correct language for saying that the index of the graded Dirac-Kähler operator is preserved at the discrete level. This fact will follow as a consequence of a further abstract characterization of discrete Fredholmness, which we develop in the case in which the gap (harmonic gap in the case of differential forms) converges in appropriate norms.

The application of the theory is done in the setting of generic, orientable Riemannian manifolds with boundary, and, as in Part I, the analysis is carried out by imposing generic assumptions on the finite element systems.

We also provide a connection (in the linguistic sense) between the Dirac-Kähler operator and another operator of Dirac type acting on sections of a Clifford bundle, which we call the *Clifford-Dirac* operator. Indeed, we shall see that they correspond to each other via a natural isomorphism of operators on sections of bundles.

We end this chapter by giving a brief outline of this work:

In Chapter 2, we give a review of most of the concepts needed to follow the rest of the work. We include the necessary material on Riemannian geometry, Exterior Algebra, Exterior Calculus, Homological Algebra, Category Theory, Finite Element Systems, and finish by giving proofs for some results in Functional Analysis, which are mostly related to various properties of operators and bilinear forms.

Starting with Part I, in Chapter 3 we motivate our framework, starting from the time-dependent Dirac equation and arriving at a compact perturbation of the eigenmode problem for the Hodge-Dirac operator. This is followed by the development of the aforementioned theory for the discretization of symmetric Fredholm operators, this is the subject of Chapter 4. Chapter 5 is concerned with approximation properties for H^1-conforming FES, in particular, we construct smoothed projections and study the implications for the stability of relevant projectors and norms. We finish Part I by using the properties of Chapter 5 to verify that the conditions of the abstract theory are verified in for the eigenmode problem associated with a compact perturbation of the Hodge-Dirac operator in a generic torus. Moreover, we derive optimal convergence rates for the eigenmodes of the discrete problems. This is done in Chapter 6.

We start Part II by reviewing some algebraic constructions such as the tensor and exterior powers of a vector space, and the Clifford algebra associated with a quadratic space. Then, we draw a connection between the Clifford algebra and the exterior algebra of an inner product space, in the form of a natural isomorphism, which is then transferred to the corresponding vector bundles over a manifold. In particular, this allows one to recover the Dirac-Kähler operator starting from the Dirac operator on the Clifford bundle, thus motivating the study of the Dirac-Kähler. This is the object of Chapter 8. Next, in Chapter 9, we propose a discrete Fredholm theory and study the discrete index of discrete Fredholm operators as well as their stability properties. We finish by showing how, under suitable assumptions, this theory may be applied to the graded Dirac-Kähler in a generic compact manifold with boundary, this is done in Chapter 10.

CHAPTER 2

Preliminaries

In this chapter, with the objective of making this work as self-contained as possible, we briefly introduce the concepts needed to follow the subsequent chapters. In doing so, we try to be precise and concise, and hence we do not expect thoroughness of the exposition at all.

For complete expositions we refer to [Tu17], [dC92], [Lan99], [Hat01], and [GH19]. For a brief exposition that covers most of the necessary material see [Chr09].

2.1 Notation

Throughout the document, as customary, the symbols $\preceq$, $\succeq$ and $\approx$ will be used when deducing uniform estimates on quantities which depend on a parameter. Moreover, when presentation is improved, we omit the symbol '$\circ$' when denoting composition of maps.

To say that we have a map f going from X to Y, we sometimes write $X \to Y$. This is particularly useful in two cases: when we want to specify the topologies with respect to which f is continuous and when we want to specify a property holding on a subspace V of X. For instance, to say that $f : X \to Y$ is compact from V to Y we write: f is compact $V \to Y$. The symbol $\hookrightarrow$ is often used to denote an inclusion map between two spaces. We say that the inclusion $X \hookrightarrow Y$ of topological spaces is dense, if the closure of X in the Y-topology is Y. A bilinear form which is induced by an isomorphism will be said to be **invertible**.

As customary, when no confusion arises, we use the Einstein summation convention. That is, when we have a summation of the form $\sum_i \alpha_i v^i$, with repeated lower and upper indices, we drop the sum symbol and simply write $\alpha_i v^i$.

2.2 Riemannian Geometry

Given a Hausdorff, second countable topological space $\mathcal{S}$, we say it is a **smooth manifold** if it is locally diffeomorphic to Euclidean space $\mathbb{R}^n$, in which case, we call n the dimension of $\mathcal{S}$. More precisely, there is a family $\{(U_\alpha, \mathbf{x}_\alpha)\}_\alpha$ of open sets $U_\alpha \subset \mathbb{R}^n$ and homeomorphisms $\mathbf{x}_\alpha : U_\alpha \to \mathbf{x}_\alpha(U_\alpha) \subset \mathcal{S}$, called an atlas, which satisfies:

i. $\{\mathbf{x}_\alpha(U_\alpha)\}_\alpha$ is a covering of $\mathcal{S}$.

ii. If $\mathcal{U}_{\alpha\beta} := \mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) \neq \emptyset$, then the transition maps

$$\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha : \mathbf{x}_\alpha^{-1}(\mathcal{U}_{\alpha\beta}) \to \mathbf{x}_\beta^{-1}(\mathcal{U}_{\alpha\beta}) \quad \text{and} \quad \mathbf{x}_\alpha^{-1} \circ \mathbf{x}_\beta : \mathbf{x}_\beta^{-1}(\mathcal{U}_{\alpha\beta}) \to \mathbf{x}_\alpha^{-1}(\mathcal{U}_{\alpha\beta})$$

are *diffeomorphisms*.

We say that a mapping $\varphi : \mathcal{S} \to \mathcal{S}'$ between manifolds is differentiable if its **local expression** is differentiable in the usual sense. That is, for every $x \in \mathcal{S}$, the function $\mathbf{y}^{-1} \circ \varphi \circ \mathbf{x}$ is differentiable at $\mathbf{x}^{-1}(x)$, for some (and therefore any) system of coordinates $\mathbf{x}$ in $\mathcal{S}$ and $\mathbf{y}$ in $\mathcal{S}'$ about x and $\varphi(x)$, respectively.

Given $x \in \mathcal{S}$ and a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to \mathcal{S}$ satisfying $\alpha(0) = x$, we define the velocity v_x of α at the instant $t = 0$ as the object characterized by the action $v_x f = (f \circ \alpha)'(0)$, for every $f : \mathcal{S} \to \mathbb{R}$ differentiable at x. The set of velocities v_x of all the curves α passing through x is called **tangent space of $\mathcal{S}$ at** x, and it is denoted $T_x\mathcal{S}$; its elements are called

tangent vectors. Given $x \in \mathcal{S}$, when endowed with the pointwise operations, $T_x\mathcal{S}$ is a vector space of dimension n. The disjoint union of all tangent spaces is called **tangent bundle** and it is denoted $T\mathcal{S}$. A parametrization $\mathbf{x}$ of $\mathcal{S}$ around a point x induces a basis of $T_x\mathcal{S}$, given by $\partial_i(x)f = \left.\dfrac{\partial(f \circ \mathbf{x})}{\partial x_i}\right|_{\mathbf{x}^{-1}(x)}$, for every i and $f : \mathcal{S} \to \mathbb{R}$. If a mapping $\varphi : \mathcal{S} \to \mathcal{S}'$ between manifolds is differentiable, it induces, at each $x \in \mathcal{S}$, a linear mapping $D_x\varphi : T_x\mathcal{S} \to T_{\varphi(x)}\mathcal{S}'$ called the **differential** or **tangent map** of φ at x, which acts on tangent vectors by: $D_x\varphi v = (\varphi \circ \alpha)'(0)$, for any curve $\alpha : (-\varepsilon, \varepsilon) \to \mathcal{S}$ satisfying $\alpha(0) = x$ and $\alpha'(0) = v$.

A **vector field** X on $\mathcal{S}$ is an assignment of a tangent vector $X_x \in T_x\mathcal{S}$ to each $x \in \mathcal{S}$. A **frame** is a family of vector fields which is a basis at each point. As expected, a parametrization gives the local frame $\{\partial_i\}_i$. A vector field X is said to be smooth if the assignment $x \mapsto X_x$ is smooth, or equivalently, if its coefficient functions in a frame are smooth $\mathcal{S} \to \mathbb{R}$. The set of all smooth vector fields on $\mathcal{S}$ is denoted by $\mathfrak{X}(\mathcal{S})$ or $\Gamma(T\mathcal{S})$, the latter of which will make more sense later in this discussion. When given the pointwise operations, $\mathfrak{X}(\mathcal{S})$ is a module over the ring $\mathrm{C}^\infty(\mathcal{S})$ of smooth functions in $\mathcal{S}$.

A **Riemannian metric** is an assignment of a positive definite bilinear form $\langle\cdot,\cdot\rangle_x$ to each tangent space $T_x\mathcal{S}$ in such a way that $x \mapsto \langle\cdot,\cdot\rangle_x$ is differentiable. This is, $x \mapsto \langle X_x, Y_x\rangle_x$ is smooth $\mathcal{S} \to \mathbb{R}$, for every $X, Y \in \mathfrak{X}(\mathcal{S})$. A **Riemannian manifold** is a smooth manifold which carries a Riemannian metric.

Every Riemannian manifold may be given a mapping $\nabla : \mathfrak{X}(\mathcal{S}) \times \mathfrak{X}(\mathcal{S}) \to \mathfrak{X}(\mathcal{S})$ called the **Levi-Civita connection**, which is entirely characterized by the axioms:

i) ∇ is $\mathrm{C}^\infty(\mathcal{S})$-linear in its first entry,

ii) ∇ is additive in its second entry,

iii) ∇ satisfies the Leibniz rule in its second entry: $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$,

iv) ∇ is **compatible with the metric**: $X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$,

v) ∇ is **torsion-free**: $\nabla_X Y - \nabla_Y X - [X, Y] = 0$,

for every $X, Y, Z \in \mathfrak{X}(\mathcal{S})$, where $[X, Y] = XY - YX$ is the **Lie bracket** or **commutator**.

A map which only satisfies i)-iii) is called an affine connection.

Vector bundles

Given a positive integer r, loosely speaking, a vector bundle is a family of vector spaces which, locally, looks like $\mathcal{U} \times \mathbb{R}^r$, for $\mathcal{U}$ an open set of $\mathcal{S}$. More precisely, a **vector bundle** of rank r is a smooth surjection $\pi : E \to \mathcal{S}$ such that, for every $x \in \mathcal{S}$, there hold:

i) the fiber $E_x := \pi^{-1}(x)$ is an r-dimensional vector space, and

ii) there is a neighborhood $\mathcal{U}$ and a **fiber-preserving** diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^r$, which restricts to a linear isomorphism $\phi_{\mathcal{U}}|_{E_x} \colon E_x \to \{x\} \times \mathbb{R}^r$ along fibers.

In this situation, E is called the **total space**, $\mathcal{S}$ the **base space**, and E_x the **fiber above** x in E. One usually says that E is a vector bundle over $\mathcal{S}$. Since we shall only deal with vector bundles (not principal or fiber bundles) in this document, we sometimes simply call E a bundle. An open set $\mathcal{U}$ satisfying the second condition is called a **trivializing open subset**, and the corresponding mapping a **trivialization** of $\pi^{-1}(\mathcal{U})$. A **trivializing open cover** is a cover made of trivializing open subsets.

A **bundle morphism** between bundles $\pi : E \to \mathcal{S}$ and $\pi' : E' \to \mathcal{S}'$ is a pair of smooth maps $\varphi : E \to E'$ and $\underline{\varphi} : \mathcal{S} \to \mathcal{S}'$ such that $\underline{\varphi} \circ \pi = \pi' \circ \varphi$ and φ restricts to linear maps $\varphi_x : E_x \to E'_{\underline{\varphi}(x)}$, for each $x \in \mathcal{S}$.

A **section** of E over an open set $\mathcal{U}$ is a map $s : \mathcal{U} \to E$ such that $\pi \circ s = \mathsf{id}_{\mathcal{U}}$. This is, for each $x \in \mathcal{U}$, a section s gives an element of the fiber E_x. The set of all smooth sections of E over $\mathcal{U}$ is denoted by $\Gamma(\mathcal{U}, E)$, if $\mathcal{U}$ is the entire manifold, we write $\Gamma(E)$. The set of smooth sections $\Gamma(E)$ constitutes both an (infinite-dimensional) $\mathbb{R}$-vector space and an r-dimensional module over the ring of smooth functions $\mathrm{C}^\infty(\mathcal{S})$.

A **connection** on a vector bundle E over $\mathcal{S}$ is a map $\nabla : \mathfrak{X}(\mathcal{S}) \times \Gamma(E) \to \Gamma(E)$ such that, for $X \in \mathfrak{X}(\mathcal{S})$ and $s \in \Gamma(E)$, there holds:

(i) $\nabla_X s$ is C^∞-linear in X and $\mathbb{R}$-linear in s, and

(ii) ∇ satisfies a Leibniz in its second entry:

$$\nabla_x(fs) = (Xf)s + f\nabla_X s, \qquad \text{for} \quad f \in \mathrm{C}^\infty(\mathcal{S})\,.$$

We finish the discussion on bundles by giving the canonical example of a bundle—the tangent bundle—thus illustrating why all these definitions make sense. To this end, put $T\mathcal{S} = \bigsqcup_{x\in\mathcal{S}} T_x\mathcal{S}$ and notice that there is a natural projection $\pi : T\mathcal{S} \to \mathcal{S}$ which, for each $x \in \mathcal{S}$, sends $v \in T_x\mathcal{S}$ to x. By considering local charts along with the induced bases of tangent spaces, one may construct trivializations, which allows us to conclude that $\pi : T\mathcal{S} \to \mathcal{S}$ is a vector bundle. We observe that, a smooth map $f : \mathcal{S} \to \mathcal{S}'$ together with its differential $Df : T\mathcal{S} \to T\mathcal{S}'$ is a bundle morphism. Indeed, for $x \in \mathcal{S}$, we have that $D_x f : T_x\mathcal{S} \to T_{f(x)}\mathcal{S}'$ is a linear map and the commutation property is immediate. A section $s \in \Gamma(T\mathcal{S})$ is a map sending $x \in \mathcal{S}$ to $s_x \in T_x\mathcal{S}$, i.e., the sections of the tangent bundle are precisely the vector fields, which explains the previous notation. Finally, we notice that a connection on a smooth manifold is simply a connection on its tangent bundle.

In Part II, we shall provide (and make use of) some additional examples of vector bundles: the tensor bundle, exterior bundle and Clifford bundle of a smooth manifold $\mathcal{S}$ (see also Section 2.3.2). In addition, we shall see how a unique connection in these bundles may be singled out when starting with a connection on the tangent bundle $T\mathcal{S}$.

2.3 Exterior Algebra and Calculus

2.3.1 Exterior Algebra

Tensor product and contraction

Let us step out of manifolds for a moment, and consider a vector space V of dimension n. Given a positive integer k, a (covariant) **k-tensor** is a mapping $V^k \to \mathbb{R}$ which is linear in each entry. The set of all such mappings is denoted by $\mathcal{L}^k(V)$. The **tensor product** of $T \in \mathcal{L}^k(V)$ and $T' \in \mathcal{L}^l(V)$ is a $k + l$ tensor, whose action is given by

$$(T \otimes T')(v_1, \ldots, v_{k+l}) = T(v_1, \ldots, v_k)\, T'(v_{k+1}, \ldots, v_{k+l})\,,$$

for every $v_1, \ldots, v_{k+l} \in V$.

In addition, given $v \in V$ and $T \in \mathcal{L}^k(V)$, the **contraction** of T by v is a $(k-1)$-tensor defined by

$$(v \lrcorner T)(v_1, \dots, v_{k-1}) := T(v, v_1, \dots, v_{k-1}) \,,$$

for $v_1, \dots, v_{k-1} \in V$. This operation is also called the interior product of v and T.

Alternating forms

A k-tensor is said to be **alternating** if it changes sign whenever two of its arguments are swapped. The set of all such tensors is denoted $\mathsf{Alt}^k(V)$, and its elements are sometimes called **algebraic k-forms** or **k-covectors**. An alternative way of defining this is as follows. Let S_r be the group of permutations on r letters. The permutation group S_k acts naturally on $\mathcal{L}^k(V)$ by

$$(\sigma \cdot T)(v_1, \dots, v_k) = T(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Then, a k-tensor T is alternating if and only if $\sigma \cdot T = \mathsf{sgn}(\sigma) T$, for every $\sigma \in S_k$. Given $T \in \mathcal{L}^k(V)$, one defines the **alternating operator**

$$\mathsf{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathsf{sgn}(\sigma) \sigma \cdot T$$

and easily verifies that $\mathsf{Alt}(T) \in \mathsf{Alt}^k(V)$.

Exterior product

Given $T \in \mathsf{Alt}^k(V)$ and $T' \in \mathsf{Alt}^l(V)$, their **wedge/exterior product** is given by

$$T \wedge T' = \frac{(k+l)!}{k!\, l!} \mathsf{Alt}(T \otimes T') \in \mathsf{Alt}^{k+l}(V).$$

The reader should be cautioned that there are different conventions on the extra factors on $\mathsf{Alt}(T)$ and $T \wedge T'$.

Given a basis $\{v_i\}_i$ of V with dual basis $\{v^i\}_i$ of the algebraic dual V^*, it is checked that the set of all the elements $v^I := v^{i_1} \wedge \dots \wedge v^{i_k}$, with $I = (i_1, \dots, i_k) \in \mathbb{N}^k$ a strictly increasing multi-index, constitutes a basis of $\mathsf{Alt}^k(V)$. As a consequence, $\dim \mathsf{Alt}^k(V) = \binom{n}{k}$. This coincides (up to canonical isomorphism) with the **kth exterior power** of V^*; we shall identify them.

The **exterior/Grassman algebra** of V^* is identified with

$$\mathsf{Alt}^{\bullet}(V) = \bigoplus_{k=0}^{\infty} \mathsf{Alt}^k(V) = \bigoplus_{k=0}^{n} \mathsf{Alt}^k(V),$$

where we decree $\mathsf{Alt}^0(V) = \mathcal{L}^0(V) = \mathbb{R}$, and notice that $\mathsf{Alt}^k(V) = \{0\}$ for $k > n$.

Pullback

A linear map between vector spaces $A : V \to W$ induces a linear map $A^* : \mathsf{Alt}^k(W) \to \mathsf{Alt}^k(V)$ given by $A^*T(v_1, \ldots, v_k) = T(Av_1, \ldots, Av_k)$, for $v_i \in V$, called the **pullback** of T under A. The assignment of pullbacks is a *contravariant functor* (see Section 2.5), in particular, $(BA)^* = A^*B^*$, for $A \in \mathcal{L}(V, W)$ and $B \in \mathcal{L}(W, U)$..

Inner product of algebraic forms

If V carries an inner product $\langle \cdot, \cdot \rangle$, $\mathsf{Alt}^k(V)$ can be given an inner product in the following fashion: define the product of 1-forms to be the one deduced from the *Riesz isomorphism*. Then, given $\{v^i\}_i$ a basis of V^*, we define

$$\left\langle v^I, v^J \right\rangle := \det \left[\left(\langle v^{i_r}, v^{j_s} \rangle \right)_{rs} \right],$$

for multi-indices $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$, and extend by bilinearity to $\mathsf{Alt}^k(V)$. It is verified that this yields an inner product which respects the orthonormality. Namely, if the given basis is orthonormal, then so is $\{v^I\}_I$ as a basis of $\mathsf{Alt}^k(V)$. This inner product is extended to the entire algebra by requiring that forms of different degrees are orthogonal.

2.3.2 Exterior Calculus

Differential forms.

Given a smooth Riemannian manifold $(\mathcal{S}, g)$, a differential k-form ω is an assignment of an alternating k-tensor $\omega_x \in \mathsf{Alt}^k(T_x\mathcal{S})$ to each $x \in \mathcal{S}$. In the language of *bundles*, a differential k-form is a section of the vector bundle $\mathsf{Alt}^k(TS) = \bigsqcup_{x\in\mathcal{S}} \mathsf{Alt}^k(T_x\mathcal{S})$. In analogy with vector fields—which are sections of the tangent bundle—the space of differential k-forms is a module of rank $\binom{n}{k}$ over the ring of scalar functions in $\mathcal{S}$.

All the operations on algebraic k-forms carry over in a natural way to differential k-forms. Namely, pullback, tensor product, exterior product, scalar product and contraction.

Locally, a differential k-form can be written as

$$\omega_x = \sum_{1\leq i_1<\cdots<i_k\leq n} \omega_{i_1\cdots i_k}(x)\mathsf{d}x^{i_1} \wedge \cdots \wedge \mathsf{d}x^{i_k},$$

where $\{\mathsf{d}x^j\}_j$ is the dual frame associated with the frame $\{\partial_j\}_j$ induced by a parametrization around x, and $\omega_{i_1\cdots i_k}$ are local functions on $\mathcal{S}$.

The exposition if often restricted to those differential k-forms whose coefficients are smooth, but given that we make use of forms enjoying different levels of regularity, we introduce *ad-hoc* notation: $\mathcal{F}\Lambda^k(\mathcal{S})$ will denote the space of differential k-forms whose coefficients, in some frame, are in $\mathcal{F}$, where $\mathcal{F}$ is a space of functions $\mathcal{S} \to \mathbb{R}$. Several choices of $\mathcal{F}$ will be relevant, most of which we shall discuss in the next section. Here, the space of all measurable differential k-forms (with no regularity requirements) is denoted $\Lambda^k(\mathcal{S})$, and the one consisting of smooth differential k-forms is denoted $\mathrm{C}^\infty\Lambda^k(\mathcal{S})$.

There is a map $\mathsf{d} : \mathrm{C}^\infty\Lambda^k(\mathcal{S}) \to \mathrm{C}^\infty\Lambda^{k+1}(\mathcal{S})$, for each $k \in \{0, \ldots, n-1\}$, called the **exterior derivative**, which, for $f \in \mathrm{C}^\infty\Lambda^0(\mathcal{S})$, is defined by $\mathsf{d}f(X) = Xf$, for all $X \in \mathfrak{X}(\mathcal{S})$, and extended to differential forms of all degrees by requiring that it respects pullbacks, satisfies a Leibniz rule for the wedge product and $\mathsf{d} \circ \mathsf{d} = 0$. In particular, we have

$$\mathsf{d}(\phi^*\omega) = \phi^*(\mathsf{d}\omega),$$

for a smooth map $\phi : \mathcal{S} \to \mathcal{S}'$ between manifolds and $\omega \in \mathrm{C}^\infty\Lambda^k(\mathcal{S}')$, and

$$\mathsf{d}(\omega \wedge \eta) = \mathsf{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathsf{d}\eta,$$

for $\omega \in \mathrm{C}^\infty\Lambda^k(\mathcal{S})$ and $\eta \in \mathrm{C}^\infty\Lambda^l(\mathcal{S})$.

Integration of forms of top degree $\omega \in \mathrm{C}^\infty\Lambda^n(\mathcal{S})$ is well–defined, and the inner product on $\Lambda^\bullet(\mathcal{S})$ can be integrated to obtain an L^2-inner product $(\cdot,\cdot)$, whose induced norm is denoted by $|\cdot|$. More generally, for a subset $\mathcal{U}$ of $\mathcal{S}$, we denote $|\cdot|_{\mathcal{U}}$ the L^2-norm on $\mathcal{U}$. We drop this subscript only when the integration domain is the entire manifold.

The pointwise norm allows one to define the Lebesgue spaces

$$\mathrm{L}^p\Lambda^k(\mathcal{S}) = \left\{\omega \in \Lambda^k(\mathcal{S}) \mid \int_{\mathcal{S}} |\omega|^p \,\mathsf{vol} < +\infty\right\},$$

for $p \geq 1$, which are endowed with the usual norms. Despite denoting the pointwise norm and the L^2-norm of k-forms by the same symbol, we hope that the context makes clear which one is alluded.

There is an isomorphism $\star : \mathrm{C}^\infty\Lambda^k(\mathcal{S}) \to \mathrm{C}^\infty\Lambda^{n-k}(\mathcal{S})$, called **Hodge star**, defined by

$$\omega \wedge \star\eta = \langle \omega, \eta \rangle \,\mathsf{vol},$$

for all $\omega \in \mathrm{C}^\infty\Lambda^k(\mathcal{S})$, where $\mathsf{vol} \in \mathrm{C}^\infty\Lambda^n(\mathcal{S})$ is given by the metric and orientation of $\mathcal{S}$, and it is called the **volume form**. Using this, we can define a map $\mathsf{d}^\star : \mathrm{C}^\infty\Lambda^k(\mathcal{S}) \to \mathrm{C}^\infty\Lambda^{k-1}(\mathcal{S})$, called coderivative, characterized by

$$\mathsf{d}^\star\omega = (-1)^{n(k+1)+1} \star \mathsf{d} \star \omega,$$

which is the formal adjoint of d. Namely, it satisfies

$$(\mathsf{d}\omega, \eta) = \int_{\mathcal{S}} \langle \mathsf{d}\omega, \eta \rangle \,\mathsf{vol} = \int_{\mathcal{S}} \langle \omega, \mathsf{d}^\star\eta \rangle \,\mathsf{vol} = (\omega, \mathsf{d}^\star\eta)\,,$$

for $\omega \in \mathrm{C}^\infty\Lambda^k(\mathcal{S})$, $\eta \in \mathrm{C}^\infty_c\Lambda^{k+1}(\mathcal{S})$.

The Stokes theorem (see, for instance, [AMR88, Theorem 7.2.8]) states that, for $u \in \mathrm{C}^\infty\Lambda^{n-1}(\mathcal{S})$ with compact support, there holds

$$\int_{\mathcal{S}} \mathsf{d}u = \int_{\partial\mathcal{S}} \mathsf{Tr}\, u,$$

where Tr denotes pullback by the inclusion $\partial\mathcal{S} \hookrightarrow \mathcal{S}$. It can be combined with the Leibniz rule for the exterior product and the definition of the L^2 inner product, to get

$$(\mathsf{d}\omega, \eta) = (\omega, \mathsf{d}^\star\eta) + \int_{\partial\mathcal{S}} \mathsf{Tr}\,\omega \wedge \mathsf{Tr}(\star\eta)\,, \tag{2.3.1}$$

for $\omega \in \mathrm{C}^\infty\Lambda^k(\mathcal{S})$ and $\eta \in \mathrm{C}^\infty\Lambda^{k+1}(\mathcal{S})$ with compact support.

The Levi-Civita connection ∇ is uniquely extended to $\mathrm{C}^\infty\Lambda^\bullet(\mathcal{S})$ by requiring that it satisfies a Leibniz rule for the tensor product. This gives a map $\nabla : \mathfrak{X}(\mathcal{S}) \times \Lambda^k(\mathcal{S}) \to \Lambda^k(\mathcal{S})$; we usually regard $\nabla\omega$ as a $(k+1)$-tensor field in $\Lambda^1(\mathcal{S})\otimes\Lambda^k(\mathcal{S})$. In this way, applying r times the connection gives the rth **total covariant derivative** ∇^r, which maps k-tensor fields to $k+r$-tensor that are alternating in the last k indices. This is more naturally described in the language of mixed tensor powers, i.e., using covariant and contravariant indices, but since we will make little to no use of this, it is not worth introducing it in detail. We refer to the aforementioned expositions for more details. See also the related discussion we make in Section 8.2.1.

On a compact manifold, using the Levi-Civita connection, one may intrinsically define the m-th Sobolev space of differential forms as

$$\mathrm{W}^{m,p}\Lambda^k(\mathcal{S}) = \left\{\omega \in \mathrm{L}^p\Lambda^k(\mathcal{S}) \mid \forall\, 1 \le l \le m \quad : \quad |\nabla^l\omega| \in \mathrm{L}^p(\mathcal{S})\right\}.$$

As customary, we use the notation $\mathrm{H}^m\Lambda^k(\mathcal{S}) = \mathrm{W}^{m,2}\Lambda^k(\mathcal{S})$. In addition, it is useful to define the spaces

$$\mathrm{H}\Lambda^k(\mathcal{S}) = \left\{\omega \in \mathrm{L}^2\Lambda^k(\mathcal{S}) \mid \mathsf{d}\omega \in \mathrm{L}^2\Lambda^{k+1}(\mathcal{S})\right\}.$$

For $k = 0$ and $k = n$, they coincide with $\mathrm{H}^1(\mathcal{S})$ and $\mathrm{L}^2\Lambda^n(\mathcal{S})$, respectively, and, for $1 \le k \le n-1$, they sit strictly between $\mathrm{H}^1\Lambda^k(\mathcal{S})$ and $\mathrm{L}^2\Lambda^k(\mathcal{S})$. These are the generalization of the spaces

H(div) and H(curl) to differential forms. Lastly, we define the space

$$\mathrm{H}^{\star}\Lambda^{k}(\mathcal{S}) = \left\{\omega \in \mathrm{L}^{2}\Lambda^{k}(\mathcal{S}) \mid \mathsf{d}^{\star}u \in \mathrm{L}^{2}\Lambda^{k-1}(\mathcal{S})\right\} = \star\,\mathrm{H}\Lambda^{n-k}(\mathcal{S}),$$

where we used Hodge duality for the second characterization.

2.4 Homological Algebra

A **$\mathbb{Z}$-graded vector space** V is the direct sum of a sequence of vector spaces V_k, indexed by the integers. That is

$$A = \bigoplus_{k=-\infty}^{\infty} A_k.$$

If V_k is nontrivial for only finitely many indices, we say that the graded vector space is finite; such will be the case throughout this entire work. Given graded vector spaces V and W, a **graded map** $f : V \to W$ **of degree** p is a sequence of linear maps $f_k : V_k \to W_{k+p}$, for each level k. In the case in which V and W are finite, we write $f = f_0 + f_1 + \cdots + f_n$. A **cochain complex** is a pair (V, d), where A is a graded space and $d : A \to A$ is a graded map of degree $+1$ which satisfies $d \circ d = 0$; we sometimes refer to the last property as **nilpotency**. The operator d is usually called the differential of the complex. By convention, cochain complexes are indexed with superscripts $(A^{\bullet}, d^{\bullet})$ and are usually represented as follows:

$$\cdots \longrightarrow V^{k-1} \xrightarrow{d^{k-1}} V^{k} \xrightarrow{d^{k}} V^{k+1} \xrightarrow{d^{k+1}} \cdots .$$

Sometimes, if no confusion arises, the labels of the arrows may be dropped. The nilpotency condition says that, at every level k, we have $\mathsf{Im}\, d^{k-1} \subset \mathsf{ker}\, d^{k}$; therefore, a complex has well–defined **cohomology spaces**, which are given by the quotients $\mathsf{ker}\, d^{k} / \mathsf{Im}\, d^{k-1}$. Regarding language, the elements of $\mathsf{ker}\, d^{k}$ are called *k-cocycles* and those of $\mathsf{Im}\, d^{k-1}$ are called *k-coboundaries*. A complex is a particular case of a **sequence**, which is just a collection of spaces together with arrows:

$$\cdots \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow \cdots .$$

We say that this sequence is **exact at** B when $\mathsf{Im}\, a = \mathsf{ker}\, b$. Moreover, the entire sequence is exact if this holds at each index. For a complex, exactness is equivalent to trivial cohomology at each level.

The canonical example of a (cochain) complex is the **de Rham complex** associated with a manifold $\mathcal{S}$, which is given by $(\mathrm{C}^\infty\Lambda^\bullet(\mathcal{S}), \mathsf{d}^\bullet)$. In contractible open sets, the *Poincaré Lemma* establishes the exactness of the de Rham complex at each positive degree $k \geq 1$.

2.5 Category Theory

As pointed out in [CH18,CH23], the framework of finite element systems is more naturally stated in categorical language. For this reason, we introduce some elementary notions of Category theory. Standard references for the subject are [ML98] and [Awo10].

A **category** $\mathfrak{C}$ is formed by a collection $\mathsf{Obj}(\mathfrak{C})$ of objects and a collection $\mathsf{Mor}(\mathfrak{C})$ of arrows or morphisms, subject to the following axioms:

i) To every morphism $f \in \mathsf{Mor}(\mathfrak{C})$, there are assigned two objects $\mathsf{Dom}(f) \in \mathsf{Obj}(\mathfrak{C})$ and $\mathsf{Cod}(f) \in \mathsf{Obj}(\mathfrak{C})$, which are called its **domain** and **codomain**, respectively. This situation is denoted $f\colon \mathsf{Dom}(f) \to \mathsf{Cod}(f)$.

ii) For every object $A \in \mathsf{Obj}(\mathfrak{C})$, there is a morphism $1_A : A \to A$, called the **identity morphism**.

iii) Given $f, g \in \mathsf{Mor}(\mathfrak{C})$ such that $\mathsf{Cod}(f) = \mathsf{Dom}(g)$, there is a morphism $g \circ f\colon \mathsf{Dom}(f) \to \mathsf{Cod}(g)$, called the composite of f and g.

iv) Composition is associative: for $f, g, h \in \mathsf{Mor}(\mathfrak{C})$ such that $\mathsf{Cod}(f) = \mathsf{Dom}(g)$ and $\mathsf{Cod}(g) = \mathsf{Dom}(h)$, there holds

$$h \circ (g \circ f) = (h \circ g) \circ f\,.$$

v) The identities work as a neutral element: for $f \in \mathsf{Mor}(\mathfrak{C})$, there holds

$$f \circ 1_{\mathsf{Dom}(f)} = f = 1_{\mathsf{Cod}(f)} \circ f\,.$$

The axioms regarding morphisms make more sense when we think about them as functions. Despite in many examples they are functions, this is not always the case.

As a first example of a category, given a field $\mathbb{K}$, we consider the category $\mathsf{Vec}_{\mathbb{K}}$, whose collection of objects is formed by all the vector spaces over $\mathbb{K}$, and its morphisms are the $\mathbb{K}$-linear maps between $\mathbb{K}$-vector spaces. In this case, morphisms are functions, so there is a natural notion of domain and codomain. Moreover, the identity in a vector space is linear, so for $V \in \mathsf{Obj}(\mathsf{Vec}_{\mathbb{K}})$, we may choose $1_V = \mathsf{id}_V$. The usual composition of functions is associative, invariant under composition with the identity, and respects linearity. Thus, we have a category. The same exercise can be done for $\mathsf{TVec}_{\mathbb{K}}$, the category whose objects are topological $\mathbb{K}$-vector spaces, and whose morphisms are $\mathbb{K}$-linear and continuous maps.

In general, one should think about morphisms as *structure preserving maps*, for instance, the morphisms in the category Grp of groups are group homomorphisms: maps which respect the multiplication of the groups. Likewise, for Top, the category of topological spaces, the morphisms are continuous maps.

In category theory, the interest lies in the objects themselves and not in something 'inside' the objects. Namely, if we are in the category of groups, we are not interested in elements of the groups but rather in the global behavior of groups in their own right. Moreover, rather than studying the interaction between elements of different groups, connected by functions, one is interested in the interaction between objects of different categories, connected by *functors*. Given categories $\mathfrak{B}$ and $\mathfrak{C}$, a **functor** $\mathfrak{f}$ from $\mathfrak{B}$ to $\mathfrak{C}$, denoted $\mathfrak{f} : \mathfrak{B} \to \mathfrak{C}$, is an assignment of:

i) an object $\mathfrak{f}(B) \in \mathsf{Obj}(\mathfrak{C})$ to each object $B \in \mathsf{Obj}(\mathfrak{B})$, and

ii) a morphism $\mathfrak{f}(g) \in \mathsf{Mor}(\mathfrak{C})$ to each morphism $g \in \mathsf{Mor}(\mathfrak{B})$, satisfying

iii) $\mathsf{Dom}(\mathfrak{f}(g)) = \mathfrak{f}(\mathsf{Dom}(g))$ and $\mathsf{Cod}(\mathfrak{f}(g)) = \mathfrak{f}(\mathsf{Cod}(g))$,

iv) $\mathfrak{f}(1_B) = 1_{\mathfrak{f}(B)}$, for $B \in \mathfrak{B}$, and

v) $\mathfrak{f}(g \circ h) = \mathfrak{f}(g) \circ \mathfrak{f}(h)$, for $g, h \in \mathsf{Mor}(\mathfrak{B})$, with $\mathsf{Cod}(h) = \mathsf{Dom}(g)$.

One also says that $\mathfrak{f}$ is a **covariant functor** and, when items iii) and v) above are modified by 'reversing the arrows', we say that $\mathfrak{f}$ is a **contravariant functor**.

The name ‘functor’ is due to what is seen in the definition, a hybrid between a function and an operator, for it acts both on objects and morphisms.

The simplest functors one can imagine are the so-called **forgetful functors**. One such functor sends a category of sets with additional structure, to the category of sets, thus forgetting the additional structure. For instance, one can define $\mathfrak{f} : \mathsf{Grp} \to \mathsf{Set}$ as $\mathfrak{f}((G, \cdot)) = G$, the underlying set of the group, and $\mathfrak{f}(\varphi) = \varphi$, as a map of sets. This can be done for any category of this type, e.g., $\mathsf{Vec}_{\mathbb{K}}$, Top, $\mathsf{TVec}_{\mathbb{K}}$.

Examples of functors abound in all areas of mathematics. For instance, if Top_* is the category of pointed topological spaces, i.e., spaces with a distinguished point, the construction of the first fundamental group is a functor $\mathsf{Top}_* \to \mathsf{Grp}$; it assigns to each pointed topological space (X, x_0) its first fundamental group $\pi_1(X, x_0)$, and to each continuous function $f : X \to Y$ such that $f(x_0) = y_0$, the induced function on classes $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, which is always well–defined.

We already saw other examples of functors in this exposition. Smooth manifolds form a category, denoted $\mathsf{DiffMan}$, whose morphisms are smooth maps. Denote by $\mathsf{DiffMan}_*$ the category of pointed smooth manifolds, and observe that the assignment of the tangent spaces is a functor $\mathsf{DiffMan}_* \to \mathsf{Vec}_{\mathbb{R}}$; it sends a pointed manifold $(\mathcal{S}, x)$ to the vector space $T_x\mathcal{S}$, and a smooth map $f : \mathcal{S} \to \mathcal{S}'$ such that $f(x) = x'$, to its differential $D_x f : T_x\mathcal{S} \to T_{x'}\mathcal{S}'$, a linear map between vector spaces. According to the chain rule, this functor is covariant.

Another example is the pullback on differential forms. For a smooth manifold $\mathcal{S}$, one assigns the vector space $\mathrm{C}^\infty\Lambda^k(\mathcal{S})$, and for a smooth map $f : \mathcal{S} \to \mathcal{S}'$, one assigns the pullback on k-forms $f^*\colon \mathrm{C}^\infty\Lambda^k(\mathcal{S}') \to \mathrm{C}^\infty\Lambda^k(\mathcal{S})$. This is a contravariant functor $\mathsf{DiffMan} \to \mathsf{Vec}_{\mathbb{R}}$.

Lastly, for cochain complexes, the **cohomology functor** sends the complex $(A^\bullet, d^\bullet)$ to its graded cohomology $\bigoplus_k \mathsf{ker}\, d^k / \mathsf{Im}\, d^{k-1}$, this is a functor from the category $\mathsf{Ch}^\bullet(\mathsf{Vec}_{\mathbb{R}})$ of cochain complexes to the category of graded vector spaces.

2.6 Cellular complexes and Finite Element Systems

2.6.1 Cellular complexes

A key aspect of discretizing equations is concerned with the discretization of the domain in which they are posed. In finite elements for polygonal/simplicial domains, the natural discretization is a *triangulation*, which, as we shall see, can be understood as a *simplicial complex*. Other type of discretizations arise in the context of polytopal methods, e.g., virtual element methods. An object that encompasses all these discretizations, and so allows for flexibility, is that of a *cellular complex*, which we borrow from *algebraic topology*. The following discussion is based on [CMKO11] and [CR16].

Given a positive integer k, we denote by $\mathbb{B}^k$ and $\mathbb{S}^{k-1}$ the closed unit ball and sphere in $\mathbb{R}^k$, respectively. Furthermore, we decree $\mathbb{B}^0 = \{0\}$ and $\mathbb{S}^{-1} = \emptyset$. Given a compact smooth manifold (or more generally a metric space) $\mathcal{S}$, for us, a **cell** T in $\mathcal{S}$ will be a closed subset for which there is a bi-Lipschitz transformation $\mathbb{B}^k \to \mathcal{S}$, for some nonnegative integer k. The number k is called the **dimension** of the cell. The relative boundary of a cell T is denoted by ∂T and it constitutes the image of $\mathbb{S}^{k-1}$ under the chosen transformation. The interior of T is denoted by $\mathsf{int}(T) = T \setminus \partial T$. It can be checked that ∂T does not depend on the chosen map.

A **cellular complex** $\mathcal{T}$ is a finite set of cells, which satisfies:

i. Different cells have disjoint interiors.

ii. The boundary of any cell in $\mathcal{T}$ is a union of cells in $\mathcal{T}$.

iii. The union of all the cells in $\mathcal{T}$ is $\mathcal{S}$.

From these axioms, it can be deduced that the intersection of two cells in $\mathcal{T}$ is a union of cells in $\mathcal{T}$ (see [CMKO11]).

The **unit simplex** of dimension k, denoted Δ_k, is the convex hull of the set formed by the origin and the k elements of the canonical basis of $\mathbb{R}^k$. Equivalently, as a subset of $\mathbb{R}^{k+1}$, it is

$$\Delta_k = \left\{ (x_0, \dots, x_k) \mid \sum_{i=0}^{k} x_i = 1 \quad \text{and} \quad x_i \geq 0 \right\}.$$

A **simplicial complex** is a cellular complex in which the intersection of pairs of distinct cells is again a cell, and the boundary of any cell splits into cells just as the boundary of Δ_k.

Given a cell T in a cellular complex $(\mathcal{T}, \mathcal{S})$, a cell T' is a **subcell** of T if it belongs to $\mathcal{T}$ and it is a subset of T. In this situation, we denote $T' \lhd T$. We say that a set $\mathcal{T}'$ of cells in $\mathcal{T}$ is a **subcomplex** of $\mathcal{T}$ if it is a cellular complex on a closed subset of $\mathcal{S}$, and $\mathcal{T}' \subseteq \mathcal{T}$. For instance, a cell T is naturally equipped with the subcomplex (see Figure 2.1)

$$\mathsf{Cl}(T) := \{T' \in \mathcal{T} \mid T' \lhd T\}\,.$$

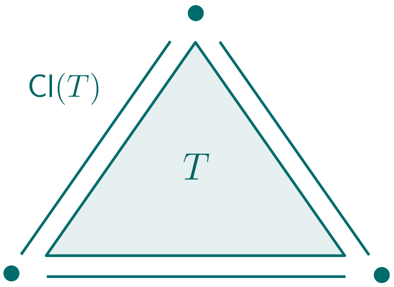

Figure 2.1: Simplicial complex carried by a 2-simplex.

The boundary ∂T of a cell carries another subcomplex of T, given by

$$\{T' \in \mathcal{T} \mid T' \lhd T \;\text{ and }\; T' \neq T\}\,.$$

Whenever the context is clear, we regard ∂T as this complex.

Given a cellular complex $\mathcal{T}$, we denote by $\mathcal{T}^k$ the set of k-dimensional cells in $\mathcal{T}$, and by $\mathcal{T}^{(k)}$ the set of cells of dimension up to k. So that

$$\mathcal{T}^{(k)} = \bigsqcup_{l=0}^{k} \mathcal{T}^l.$$

This set is called the k-th skeleton of $\mathcal{T}$.

Notice that a cell of dimension k in a complex $\mathcal{T}$ has the structure of a k-dimensional compact manifold with boundary. In what follows, we assume that every cell has been oriented. The **relative orientation** or **incidence number** $\epsilon(T, T')$ of two cells $T, T' \in \mathcal{T}$ is defined as follows: the boundary of a cell $T \in \mathcal{T}^1$ consists of two points, denote them by v, v', and assume that the given orientation is from v to v' (see Figure 2.2), then $\epsilon(T, v) = -1$ and $\epsilon(T, v') = +1$. For cells T of arbitrary dimension $k > 1$, we set, for $T' \in \mathcal{T}^{k-1} \setminus \{T\}$, $\epsilon(T, T') = 1$ if T' is outward oriented with respect to T, and $\epsilon(T, T') = -1$ otherwise. For pairs T, T' of cells of codimension different to 1, we put $\epsilon(T, T') = 0$. This yields a real matrix called the **incidence matrix**, which is indexed by $\mathcal{T} \times \mathcal{T}$. Observe that, by definition of the incidence number, this matrix is triangular: if the entry (T, T') is nonzero, then the entry (T', T) is zero.

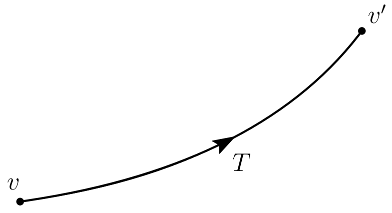

Figure 2.2: One-dimensional cell T oriented from v to v'.

For each k, we let $\mathcal{C}^k(\mathcal{T})$ be the set of maps $\mathcal{T}^k \to \mathbb{R}$; such maps are called k**-cochains**. The **coboundary** operator $\delta^\bullet : \mathcal{C}^k(\mathcal{T}) \to \mathcal{C}^{k+1}(\mathcal{T})$ is defined by

$$(\delta c)_T = \sum_{T' \in \mathcal{T}^k} \epsilon(T, T') c_{T'}$$

for all $c \in \mathcal{C}^k(\mathcal{T})$ and $T \in \mathcal{T}^{k+1}$, where subscript denotes evaluation in a cell. This notation is justified by the following observation:

Addition and scalar multiplication are well–defined in $\mathcal{C}^k(\mathcal{T})$. Moreover, since a k-cochain assigns a number to each k-dimensional cell, we have $\mathcal{C}^k(\mathcal{T}) \equiv \mathbb{R}^{|\mathcal{T}^k|}$, thereby it is a finite-dimensional vector space, whose canonical basis may be indexed by $\mathcal{T}^k$. Bearing this in mind, the coboundary operator may be regarded as the linear transformation whose canonical matrix is the incidence matrix ϵ, indexed in the same order. This matrix is *sparse*, as the incidence numbers can only be nonzero when a cell has codimension one in another cell.

It is proved that the pair $(\mathcal{C}^\bullet(\mathcal{T}), \delta^\bullet)$ defines a cochain complex, so there are well–defined cohomology spaces $H^k(\mathcal{C}^\bullet(\mathcal{T}))$. This is also valid for any cellular subcomplex of $\mathcal{T}$.

2.6.2 Finite element systems

Although the spirit of the FES framework was already present in Christiansen's article [Chr07], it was not formally introduced until 2008 in [Chr08]. It has been further developed in [CMKO11, CG16, CR16, CH18, CH23]. FES presents a systematical approach to discretizing complexes of vector spaces, which allows one to leverage the combinatorial structure of cellular complexes. This enables, for instance, inductive constructions of objects, proceeding from lower-dimensional cells to top-dimensional cells. We review its main tools in the following.

First, we observe that, since a cellular complex $\mathcal{T}$ is a poset with respect to the partial order given by the inclusion of cells, it forms a category whose objects are the cells, and whose morphisms are the inclusions. Here, differing slightly from [CH23], we introduce the notion of a **FES functor**: a contravariant functor $\mathfrak{f} : \mathcal{T} \to \mathsf{Ch}^\bullet(\mathsf{Vec})$, whose image spaces are finite-dimensional. This is, a functor that assigns, to each cell $T \in \mathcal{T}$, a cochain complex, and to each inclusion of cells, a map between vector spaces, which reverses the arrows. One also may say that $\mathfrak{f}$ is a presheaf with values in $\mathsf{Ch}^\bullet(\mathsf{Vec})$, this point of view will be convenient to understand some notions.

We define a **finite element system** to be the image of a FES functor. Namely, if we denote

$$(\mathcal{A}^\bullet(T), d_T^\bullet) := \mathfrak{f}(T) \quad \text{and} \quad \mathsf{r}_{T'T} := \mathfrak{f}(T' \hookrightarrow T), \text{ for } T' \lhd T,$$

for each $T \in \mathcal{T}$, the triple $(\mathcal{A}, d, \mathsf{r})$ is a finite element system. The spaces $\mathcal{A}^k(T)$ are called **differential elements** or **local spaces**, d_T is the differential of the complex, and $\mathsf{r}_{T'T}$ are called **restriction maps**.

If we replace the requirement that the complex induced by the FES functor is finite-dimensional, by requiring only that it is closed (with respect to some enclosing space), we get an **element system** (cf. [CR16]). Most of the treatment carries over to these systems.

Remark 2.6.1. Two relevant properties that are encoded in this definition are

$$\mathsf{r}_{T'T}\, d_T = d_{T'}\, \mathsf{r}_{T'T} \quad \text{and} \quad \mathsf{r}_{T''T} = \mathsf{r}_{T''T'}\, \mathsf{r}_{T'T},$$

for $T'', T', T \in \mathcal{T}$ such that $T'' \lhd T' \lhd T$. The first property comes from the fact that the maps $\mathsf{r}_{T'T}$ are morphisms of complexes, which means that they are linear and commute with the differential. The second one is due to the contravariance of the functor (cf. Section 2.5). The commuting property is natural from the point of view of mixed finite element methods and, more precisely, of finite element exterior calculus. The property of restrictions essentially says that there is a unique way of restricting to cells of lower dimensions. ◇

Remark 2.6.2. In practice, one is mostly interested in discretizing a finite number of indices. For instance, when discretizing differential forms, we are interested only on the indices up to $\dim T$, and so we may consider a FES functor that gives us trivial vector spaces for the remaining ones. In this situation, the vector spaces $\mathcal{A}^k(T)$ may be, for instance, subspaces of $\mathrm{H}^1\Lambda^k(T)$, then we may choose: $d_T = \mathsf{d}|_{\mathcal{A}(T)}$, the restriction of the exterior derivative, and $r_{T'T} = \mathsf{Tr}$, where the right-hand side denotes the extension of the canonical restriction map, which in this case is pullback on forms by the inclusion $T' \hookrightarrow T$. This example illustrates very well the need of every part of the definition. ◇

Now, using the local spaces, we define **glued spaces**: for a cellular subcomplex $\mathcal{T}'$ of $\mathcal{T}$, we set, for each k,

$$\mathcal{A}^k(\mathcal{T}') = \left\{ (u_T)_T \in \bigoplus_{T \in \mathcal{T}'} \mathcal{A}^k(T) \mid \forall T' \lhd T \in \mathcal{T}' \ : \ \mathsf{r}_{T'T}\, u_T = u_{T'} \right\}.$$

Intuitively, the glued spaces consist of elements defined piecewise, such that their restriction to subcells is well–behaved, which is very similar to traditional finite elements, with the difference being that the gluing is not given in terms of DOFs. We call $\mathcal{A}^k(\mathcal{T})$ the **global spaces**.

Remark 2.6.3. This definition is natural from the point of view of category theory. Indeed, we first remark that a FES functor constitutes a projective system in $\mathsf{Ch}^\bullet(\mathsf{Vec})$, indexed by $\mathcal{T}$. Then, a glued space is identified with the projective limit (cf. [KS06, Section 2.1]):

$$\mathcal{A}^k(\mathcal{T}') \equiv \varprojlim_{T \in \mathcal{T}'} \mathcal{A}^k(T).$$

◇

It is important to remark that, for better or for worse, a FES differs from Ciarlet's definition of finite element in two main aspects:

i) In a FES, there are spaces attached to each cell in the cellular complex, whereas in a standard finite element space, only cells of maximal dimension carry a space. And so the notion of restriction is not intrinsic to finite elements.

ii) In FES, degrees of freedom are not a primary object of interest. Indeed, it may be argued that including DOFs in the definition of finite element obscures the relationship between DOFs and the resulting global space. For instance, *a priori*, its is not clear when two different finite elements yield the same global space, whereas FES does not rely on degrees of freedom to define global spaces. Nevertheless, the FES framework does not forget about DOFs and, indeed, allows one to deduce necessary and sufficient conditions—of combinatorial nature—for the existence of unisolvent degrees of freedom; the so-called softness and compatibility conditions of a FES.

The restriction maps induce an isomorphism $\mathsf{r} : \mathcal{A}^k(T) \to \mathcal{A}^k(\mathsf{Cl}(T))$, which sends u_T to $(\mathsf{r}_{T'T}\, u_T)_{T'\in\mathsf{Cl}(T)}$. Thereby, from now on, we identify the local space $\mathcal{A}^k(T)$ with the glued space $\mathcal{A}^k(\mathsf{Cl}(T))$. Likewise, the restriction maps also induce a map

$$\mathsf{r} : \mathcal{A}^k(T) \to \mathcal{A}^k(\partial T), \quad (u_{T'})_{T'\in\mathsf{Cl}(T)} \longmapsto (\mathsf{r}_{T'T}\, u_T)_{T'\in\partial T} = (u_{T'})_{T'\in\partial T},$$

whose kernel is denoted by $\mathcal{A}^k_0(T)$. By definition, elements of $\mathcal{A}^k_0(T)$ are those which are zero when restricted to the boundary. Since we considered the boundary of a point to be empty, for zero dimensional cells we have $\mathcal{A}^k_0(T) = \mathcal{A}^k(T)$.

We say that our FES **admits extensions** on $T \in \mathcal{T}$, if the induced map $\mathsf{r} : \mathcal{A}^\bullet(T) \to \mathcal{A}^\bullet(\partial T)$ is a surjection. We say that the FES is **soft** or **admits extensions** on $\mathcal{T}$ if it admits extensions on every cell. We have a first result characterizing this:

Proposition 2.6.1. *A FES $(\mathcal{A}, d, \mathsf{r})$ admits extensions if and only if, for any cellular complexes $\mathcal{T}', \mathcal{T}''$ such that $\mathcal{T}'' \subset \mathcal{T}' \subset \mathcal{T}$, the restriction $\mathcal{A}^\bullet(\mathcal{T}') \to \mathcal{A}^\bullet(\mathcal{T}'')$ is surjective.*

Proof. This is Proposition 2 in [CH23, Section 3]. □

The reader familiarized with sheaf theory will notice that this amounts to say that the property of admitting extensions corresponds to the notion of *soft sheaves*.

In [CH23], a notion of discrete vector bundle was introduced. Although it is an intuitive construction from the geometric point of view, we believe that, for the purposes of this document, there are no benefits in doing this abstraction. For this reason, we henceforth assume that all the finite element systems considered consist of differential forms:

More precisely, following [CR16] and [CMKO11], for all k and $T \in \mathcal{T}$, we denote

$$\mathfrak{A}^k(T) = \left\{ u \in \mathrm{L}^2\Lambda^k(T) \mid \forall T' \lhd T \mid \mathsf{i}^*_{T'T} u \in \mathrm{H}\Lambda^k(T') \right\},$$

and henceforth assume that the FES functor $\mathfrak{f}$ is $\mathfrak{A}$-conforming. Namely, for all $T \in \mathcal{T}$,

$$\mathcal{A}^\bullet(T) \subset \mathfrak{A}^\bullet(T) \ , \quad d_T = \mathsf{d}|_{\mathcal{A}(T)} \quad \text{and} \quad \mathsf{r}_{T'T} = \mathsf{i}^*_{T'T}, \quad \text{for } T' \lhd T.$$

$(\mathcal{A}, d, \mathsf{r})$ is said to be **exact** on $T \in \mathcal{T}$ if $(\mathcal{A}^\bullet(T), d^\bullet)$ is exact at every positive level and $\ker d^0_T = \mathbb{R}$. A FES is **locally exact** if it is exact at each cell in $\mathcal{T}$. Recall that cells have the topology of a ball, which implies that they have trivial cohomology at each positive degree, and that their zeroth cohomology space is $\mathbb{R}$. In this sense, the condition of local exactness guarantees that the local spaces have the right cohomology.

We say that a FES is **compatible** if it is both soft and locally exact.

Proposition 2.6.2. $(\mathfrak{A}^\bullet(T), \mathsf{d}^\bullet, \mathsf{i}^*)$ *constitutes a compatible element system.*

Proof. See [CR16, Proposition 2.4] □

We also have:

Proposition 2.6.3. *Suppose that* $(\mathcal{A}^\bullet, d^\bullet, \mathsf{r})$ *is soft. Then, it is compatible if and only if:*

i) *for each* $T \in \mathcal{T}$*, the sequence* $(\mathcal{A}^\bullet_0(T), d_T)$ *has nontrivial cohomology only at degree* $k = \dim T$*, and*

ii) *integration induces an isomorphism* $H^0(A^\bullet_0(T)) \to \mathbb{R}$.

Proof. See [CH23, Section 3, Theorem 3]. □

We remark that integration induces a map on cohomology because of Stokes theorem.

The next result provides a dimension count for FES:

Proposition 2.6.4. *For all k and $T \in \mathcal{T}$, there holds*

$$\dim \mathcal{A}^k(\mathcal{T}) \leq \sum_{T \in \mathcal{T}} \dim A_0^k(T),$$

and equality holds if and only if $\mathcal{A}^k$ admits extensions on each $T \in \mathcal{T}$.

Proof. See [CR16, Proposition 2.1]. □

A **system of degrees of freedom** is a choice of a subspace $\mathcal{Z}^k(T)$ of $\mathfrak{A}^k(T)^\star$ for each k and $T \in \mathcal{T}$. One such a choice is made, one may define maps

$$\Phi^k(T) : \mathcal{A}^k(T) \to \bigoplus_{T' \lhd T} \mathcal{Z}^k(T')^\star \quad \text{by} \quad \Phi^k(T)u = \Big(\langle \cdot, \mathsf{r}_{T'T}\, u \rangle \Big)_{T' \lhd T},$$

for all $u \in A^k(T)$. Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between each $A^k(T')$ and its dual. We say that the system $\mathcal{Z}$ is **unisolvent** on $\mathcal{A}$ if $\Phi^k(T)$ is an isomorphism for each k and $T \in \mathcal{T}$. These maps are sometimes called mirrors (cf. [CMKO11, Section 5.2]).

Proposition 2.6.5. *Let $(\mathcal{A}, d, \mathsf{r})$ be a FES on a cellular complex $\mathcal{T}$, and let $\mathcal{Z}$ denote a system of degrees of freedom for $\mathcal{A}$. Assume that:*

(i) for each $T \in \mathcal{T}$, the canonical map $\mathcal{A}_0^k(T) \to \mathcal{Z}^k(T)^\star$ is injective, and

(ii) that $\bar{T} \in \mathcal{T}$ satisfies

$$\dim \mathcal{A}^k(\bar{T}) \geq \sum_{T' \lhd \bar{T}} \dim \mathcal{Z}^k(T')\,. \tag{2.6.1}$$

Then, $\mathcal{Z}$ is unisolvent on $\mathcal{A}$ on $\mathsf{Cl}(\bar{T})$, $\mathcal{A}$ is soft on $\mathsf{Cl}(T)$ and equality holds in (2.6.1).

Proof. See [CH23, Section 3.2, Proposition 5] and [CR16, Proposition 2.5]. □

An **interpolator** $\mathcal{I}$ for a FES $\mathcal{A}$ is a collection of projectors $\mathcal{I}^k(T) : \mathfrak{A}(T) \to \mathcal{A}^k(T)$, for each nonnegative integer k and $T \in \mathcal{T}$, which commute with restriction to subcells. That is,

$$\mathsf{r}_{T'T}\,\mathcal{I}^k(T) = \mathcal{I}^k(T')\,\mathsf{r}_{T'T} \qquad \forall\, T' \lhd T.$$

This defines a global operator which is characterized by the relation

$$(\mathcal{I}(u))|_T = \mathcal{I}(T)(u|_T) \qquad \forall\, T \in \mathcal{T},$$

for $u \in \mathcal{A}^k(\mathcal{T})$. This is very similar to how one obtains a global interpolator in traditional finite elements, starting from local interpolators deduced from DOFs.

A unisolvent system of degrees of freedom $\mathcal{Z}$ defines an interpolator by requiring that

$$\Phi^k(T)\mathcal{I}^k(T)u = \Phi^k(T)u,$$

for all $u \in \mathcal{A}^k(\mathcal{T})$. Namely, the interpolator is required to preserve degrees of freedom. We call this the interpolator associated with the system of degrees of freedom $\mathcal{Z}$.

The following constitutes the aforementioned relationship between existence of unisolvent DOFs and softness of FES:

Proposition 2.6.6. *The following statements are equivalent:*

i) $\mathcal{A}$ *admits extensions,*

ii) $\mathcal{A}$ *has a unisolvent system of degrees of freedom,*

iii) $\mathcal{A}$ *can be equipped with an interpolator.*

Proof. This is Proposition 5.37 in [CMKO11]. □

An important feature of interpolators, both inside the FES framework and in traditional mixed finite elements, is the commuting property of interpolants, i.e., they are cochain maps linking the continuous and discrete spaces. For instance, the well–known Raviart-Thomas spaces paired with piecewise polynomials carry commuting interpolators. Namely, the following diagram commutes

$$\begin{tikzcd} \mathrm{H}(\mathsf{div}) \arrow[r, "\mathsf{div}"] \arrow[d, "\Pi_k"] & \mathrm{L}^2 \arrow[d, "\mathcal{P}_k"] \\ \mathrm{RT}_k \arrow[r, "\mathsf{div}"] & \mathrm{P}_k \end{tikzcd} ,$$

this is fundamental to prove stability of discretizations, in the sense of Babuška and Brezzi.

In this regard, we have the following result:

Proposition 2.6.7. *If $\mathcal{A}$ is a compatible FES and $\mathcal{Z}$ is a unisolvent system of degrees of freedom, the associated interpolator commutes with the exterior derivative if and only if*

$$l \circ \mathsf{d} \in \bigoplus_{T' \lhd T} \mathcal{Z}^{k-1}(T'),$$

for all $l \in \mathcal{Z}^k(T)$.

Proof. See [CMKO11, Proposition 5.41]. □

The following construction corresponds to the so-called **harmonic degrees of freedom**: equip each space $\mathfrak{A}^k(T)$ with a continuous scalar product a, which could be, for instance, the L^2-products on differential forms. Next, for a FES $\mathcal{A}$, define the following system of degrees of freedom:

$$\mathcal{Z}^{\dim T}(T) = \left\{ a(\cdot, v) \mid v \in \mathsf{d}\mathcal{A}_0^{\dim T - 1}(T) \right\} \oplus \mathsf{span}\left\{ \int_T \cdot \right\} ,$$

and, for $k < \dim T$,

$$\mathcal{Z}^k(T) = \left\{ a(\cdot, v) \mid v \in \mathsf{d}\mathcal{A}_0^{k-1}(T) \right\} \oplus \left\{ a(\mathsf{d}\cdot, v) \mid v \in \mathsf{d}\mathcal{A}_0^{k}(T) \right\} ,$$

for all $T \in \mathcal{T}$.

The following is the announced connection between unisolvent DOFs and compatibility of FES:

Proposition 2.6.8. *The following statements are equivalent:*

i) *$\mathcal{A}$ is compatible,*

ii) *harmonic degrees of freedom are unisolvent on* $\mathcal{A}$*,*

iii) $\mathcal{A}$ *has a unisolvent system of degrees of freedom.*

iv) $\mathcal{A}$ *can be equipped with a commuting interpolator.*

Proof. This corresponds to Proposition 2.8 in [CR16]. □

2.7 Functional Analysis

We devote this section to give results that shall prove useful later; most of which, if not all, are well–known in the literature, and many are simple consequences of the *great theorems of functional analysis*. This makes the task of finding references difficult, as particular cases of these facts are widely used in literature. We provide elementary proofs for every result, aiming to stay within the self-contained nature of this work. We specially highlight Lemma 2.7.2 and Proposition 2.7.5, which were not known to the author, and seem to have value themselves. For instance, Lemma 2.7.2 may be helpful to strengthen discrete stability, as seen in Lemma 6.1.1 and Proposition 10.2.3.

It is worth mentioning that many of the results that we present here are abstractions of the techniques repeatedly employed in the actual work. One could even say that a big part of the effort of the work is contained in this section, in the sense that most of the principles and ideas behind the given results are collected here. We believe that, by abstracting and including them here, the legibility of the ideas is better preserved, thereby improving presentation. Lastly, we remark that every result presented in this section is used in some part of the work. Therefore, if desired, a reader may skip it and come back when needed.

Linear and bounded operators

Proposition 2.7.1. *Given Banach spaces* $(Y, \| \, \|_Y)$ *and* $(Z, \| \, \|_Z)$*, assume we are given a sequence* $(T_n)_n$ *of linear and bounded operators* $Y \to Z$*, and let* X *be a normed vector space with dense inclusion in* Y*. Then,* $(T_n)_n$ *is pointwise convergent* $Y \to Z$ *if and only if* $(T_n|_X)_n$ *is pointwise convergent, and* $(T_n)_n$ *is uniformly bounded in operator norm* $Y \to Z$*.*

Proof. The first implication is direct. Indeed, if $(T_n)_n$ is pointwise convergent in X, its restriction to any subset is as well. Moreover, that the sequence is uniformly bounded $Y \to Z$ follows from an application of the well–known *uniform boundedness principle*.

To prove the converse, denote by T the pointwise limit of $(T_n|_X)_n$. For $u \in X$, we have that

$$\|Tu\|_Z = \left\| \lim_{n\to\infty} T_n u \right\|_Z = \lim_{n\to\infty} \|T_n u\|_Z \leq \|u\|_Y \limsup_{n\to\infty} \|T_n\|_{Y\to Z} \leq M \|u\|_Y \,,$$

where $M = \sup_n \|T_n\|_{Y \to Z}$. Moreover, T is clearly linear, and hence $T \in \mathcal{L}(X, Z)$. Since X is dense in Y, T admits a unique linear and bounded extension to Y. We denote this extension by $\tilde{T} \in \mathcal{L}(Y, Z)$.

Now, given $u \in Y$ and $\tilde{u} \in X$, we have

$$\begin{aligned} \left\|\tilde{T}u - T_n u\right\|_Z &\leq \|\tilde{T}u - T\tilde{u}\|_Z + \|T\tilde{u} - T_n\tilde{u}\|_Z + \|T_n\tilde{u} - T_n u\|_Z \\ &\leq C\,\|u - \tilde{u}\|_Y + \|T\tilde{u} - T_n\tilde{u}\|_Z\,, \end{aligned} \tag{2.7.1}$$

with $C = \max\left\{\|\tilde{T}\|, M\right\}$. According to density, fixed $\varepsilon > 0$, we may choose $\tilde{u}$ such that

$$\|u - \tilde{u}\|_Y < \frac{\varepsilon}{2C}.$$

Moreover, by pointwise convergence of $(T_n)_n$ in X, there is $N \in \mathbb{N}$, such that, for all $n \geq N$, there holds

$$\|T\tilde{u} - T_n\tilde{u}\| < \frac{\varepsilon}{2}.$$

These last two bounds together with (2.7.1) give pointwise convergence of $(T_n)_n$ to $\tilde{T}$, thus yielding the desired result. □

We are particularly interested in the *if* part of the previous proposition, which is combined with the following result to obtain convergence in operator norm in the case in which a dense and compact inclusion is at hand.

Proposition 2.7.2. *Let X, Y, Z be Banach spaces, and let $(T_n)_n$ be a sequence of linear and bounded operators $Y \to Z$, and let $K : X \to Y$ be compact. Suppose that $(T_n)_n$ is pointwise convergent on Y and uniformly bounded in operator norm $Y \to Z$. Then, $(T_nK)_n$ converges in operator norm $X \to Z$.*

Proof. Denote by T the pointwise limit of $(T_n)_n$, by reasoning as in the previous proof, we have that $T \in \mathcal{L}(Y, Z)$. Now, by letting $B = B_X(0, 1)$ denote the open unit ball in X, we notice that, $K(B)$ is precompact in Y, for K is a compact operator. Therefore, for every $\delta > 0$, there are finite sets $I \subset \mathbb{N}$ and $\{y_k\}_{k \in I} \subset \overline{K(B)}$ such that

$$K(B) \subset \bigcup_{k \in I} B_Y(y_k, \delta).$$

Thus, for every $\delta > 0$ and $x \in B$, there is $k \in I$ such that

$$\|Kx - y_k\|_Y < \delta. \tag{2.7.2}$$

It is clear that TK and T_nK are continuous $X \to Z$ and that pointwise convergence is preserved. We shall prove that this pointwise convergence is uniform in B.

Given $\epsilon > 0$ and $k \in I$, pointwise convergence implies that there is $N_k \in \mathbb{N}$, such that

$$\forall\, n \geq N_k \;\; : \;\; \|T_n y_k - T y_k\|_Z < \frac{\epsilon}{2}.$$

Moreover, since I is finite, we may choose $N = \max\{N_k\}_{k \in I}$, to get that, for all $\epsilon > 0$, there is $N \in \mathbb{N}$, such that

$$\forall\, n \geq N \;\; : \;\; \|T_n y_k - T y_k\|_Z < \frac{\epsilon}{2}, \tag{2.7.3}$$

for all $k \in I$.

Then, for all $n \geq N$ and $x \in B$, there is $k \in I$, such that

$$\begin{aligned} \|T_n Kx - TKx\|_Z &\leq \|T_n Kx - T_n y_k\|_Z + \|T_n y_k - T y_k\|_Z + \|T y_k - TKx\|_Z \\ &\preceq \|Kx - y_k\|_Y + \|T y_k - T_n y_k\|_Z \\ &< \epsilon, \end{aligned}$$

where the second bound uses the uniform boundedness, and the third one comes from (2.7.2) and (2.7.3). This is, by definition, uniform convergence in B.

Finally, we write

$$\|T_n K - TK\|_{X \to Z} = \sup_{x \in X} \frac{\|T_n Kx - TKx\|_Z}{\|x\|_X} = \sup_{x \in B} \|T_n Kx - TKx\|_Z < \epsilon,$$

which gives the desired result. □

Definition 2.7.1. Given two closed subspaces E and F of a normed vector space $(X, \| \ \|)$, the gap and symmetrized gap between E and F are defined, respectively, as (see [Kat76, Section 4.2.1])

$$\delta(E,F) = \sup_{x\in E} \frac{\mathsf{dist}(x,F)}{\|x\|}, \qquad \text{and} \qquad \hat{\delta}(E,F) = \max\left\{\delta(E,F), \delta(F,E)\right\}.$$

◇

Proposition 2.7.3. *Given two closed subspaces E and F of a Hilbert space X with norm $\|\cdot\|$, there hold*

$$\delta(E,F) = \|(I-P_F)P_E\|, \qquad \delta(F,E) = \|(I-P_E)P_F\|_, \tag{2.7.4}$$

and

$$\hat{\delta}(E,F) = \|P_E - P_F\|, \tag{2.7.5}$$

where P_E and P_F are the orthogonal projectors onto E and F, respectively.

Proof. First, we prove (2.7.4). By symmetry, we may focus on proving the first inequality. We know that, for $u \in E$, the orthogonal projection $P_F u$, gives the best approximation of u by elements of F. This amounts to

$$\|u - P_F u\| = \mathsf{dist}(u,F). \tag{2.7.6}$$

By projecting an arbitrary element $x \in X$ onto E, we write $\|P_E x - P_F P_E x\| = \mathsf{dist}(P_E x, F)$.

Using this in the definition of the gap, we obtain a lower bound

$$\delta(E,F) = \sup_{v\in E} \frac{\mathsf{dist}(v,F)}{\|v\|} = \sup_{x\in X} \frac{\|(I-P_F)P_E x\|}{\|P_E x\|} \geq \|(I-P_F)P_E\|, \tag{2.7.7}$$

where the last inequality comes from the fact that an orthogonal projection has norm 1. Conversely, we notice that, for $v \in E$, there holds

$$\frac{\|(I-P_F)P_E v\|}{\|P_E v\|} = \frac{\|(I-P_F)P_E v\|}{\|v\|} \leq \sup_{x\in X} \frac{\|(I-P_F)P_E x\|}{\|x\|},$$

whence, according to the first equality in (2.7.7), we get

$$\delta(E,F) = \sup_{v\in E} \frac{\|(I-P_F)v\|}{\|v\|} \le \|(I-P_F)P_E\|.$$

This proves (2.7.4).

For the second part, we start by giving a lower bound. Bearing in mind (2.7.6), we have

$$\|P_E - P_F\| \ge \sup_{x\in E} \frac{\|x - P_F x\|}{\|x\|} = \delta(E,F).$$

Exchanging the roles of E and F, we get $\|P_E - P_F\| \ge \delta(F,E)$, and consequently

$$\|P_E - P_F\| \ge \hat{\delta}(E,F).$$

We employ (2.7.4) to prove the converse inequality. First, since projections are self-adjoint, we have

$$\langle P_E(I-P_F)u, v\rangle = \langle u, (I-P_F)P_E v\rangle \qquad \forall\, v \in X,$$

therefore

$$\|P_E(I-P_F)\| = \|(I-P_F)P_E\| \,. \tag{2.7.8}$$

Then, by orthogonality, idempotency, the previous identity, and (2.7.4), we may write

$$\begin{aligned}
\|P_E u - P_F u\|^2 &= \|P_E(P_E u - P_F u) + (I-P_E)(P_E u - P_F u)\|^2 \\
&= \|P_E(I-P_F)u\|^2 + \|(I-P_E)P_F u\|^2 \\
&\le \|P_E(I-P_F)\|^2 \, \|(I-P_F)u\|^2 + \|(I-P_E)P_F\|^2 \, \|P_F u\|^2 \\
&\le \max\Big\{\|(I-P_F)P_E\|, \|(I-P_E)P_F\|\Big\}^2 \Big(\|(I-P_F)u\|^2 + \|P_F u\|^2\Big)\,,
\end{aligned}$$

for all $u \in X$, where in the last inequality we used (2.7.8). This gives the desired estimate, and allows us to conclude. □

Bilinear forms

Proposition 2.7.4. *Let X and Y be Hilbert spaces, and let $a : X \times Y \to \mathbb{R}$ denote a bounded bilinear form. Consider V and W closed subspaces of X and Y, respectively, such that:*

i) *for all $v \in V \setminus \{0\}$, there holds*

$$\sup_{w \in W} a(v, w) > 0,$$

ii) *there holds*

$$\inf_{w \in W} \sup_{v \in V} \frac{a(v, w)}{\|v\|_X \|w\|_Y} \succeq 1.$$

Then, a induces a projector $P : X \to V$ characterized by

$$a(Px, w) = a(x, w) \qquad \forall\, w \in W.$$

Proof. Denote by $A : X \to Y$ the unique linear and bounded operator satisfying

$$\langle Ax, y \rangle = a(x, y) \qquad \forall\, (x, y) \in X \times Y.$$

Then, if $P_W : Y \to W$ is the orthogonal projection and $\mathfrak{i}_V : V \to X$ is the inclusion, we have that $P_W A\, \mathfrak{i}_V$ is the unique operator $V \to W$ satisfying

$$\langle P_W A\, \mathfrak{i}_V v, w \rangle = a(v, w) \qquad \forall\, (v, w) \in V \times W.$$

To see that the operator P is well–defined, we must prove that there exists a unique element in V satisfying the desired identity. For existence, we notice that the identity characterizing P may be equivalently rewritten as

$$P_W A\, \mathfrak{i}_V P x = P_W A x,$$

for $x \in X$. Since $P_W A x$ belongs to W, a sufficient condition for such $Px \in V$ to exist is that $P_W A\, \mathfrak{i}_V$ is surjective $V \to W$. We know that this is equivalent to say that the Hilbert adjoint $(P_W A\, \mathfrak{i}_V)^\star$ is injective and has closed range. This is exactly our second hypothesis, as can be seen from

$$\|(P_W A\, \mathsf{i}_V)^\star w\|_X = \sup_{v\in V} \frac{\langle v, (P_W A\, \mathsf{i}_V)^\star w\rangle}{\|v\|_X} = \sup_{v\in V} \frac{a(v,w)}{\|v\|_X},$$

for all $w \in W$.

According to the previous discussion, uniqueness is equivalent to injectivity of $P_W A\, \mathsf{i}_V$, which is to say that, for all $v \in V \setminus \{0\}$, there holds $P_W A\, \mathsf{i}_V v \neq 0$, this is directly equivalent to our first hypothesis, the proof thus concludes. $\square$

As announced, we prove an interesting result regarding *inf-sup conditions*. We shall see in chapters 6 and 10 that the following lemmas are key to obtain stability of our discretization in appropriate norms.

Lemma 2.7.1. *Let $A : X \to Y$ be a linear and bounded operator between Hilbert spaces whose range is closed. Then*

$$\inf_{x\in \mathsf{ker}\, A^\perp} \sup_{y \in \mathsf{Im}\, A} \frac{\langle Ax, y\rangle}{\|x\|_X \|y\|_Y} = \inf_{x\in \mathsf{ker}\, A^\perp} \sup_{y \in Y} \frac{\langle Ax, y\rangle}{\|x\|_X \|y\|_Y} = \frac{1}{\left\|\underline{A}^{-1}\right\|}, \tag{2.7.9}$$

where $\underline{A} = PA\,\mathsf{i} \colon (\mathsf{ker}\, A)^\perp \to \mathsf{Im}\, A$, with $\mathsf{i} : (\mathsf{ker}\, A)^\perp \to X$ the canonical inclusion and $P : Y \to \mathsf{Im}\, A$ the orthogonal projector.

Proof. We first notice that $\underline{\mathrm{A}}$ is injective. Indeed, we have

$$\mathsf{ker}\, \underline{\mathrm{A}} = \left\{x \in \mathsf{ker}\, A^\perp \mid P(Ax) = 0\right\} = \left\{x \in \mathsf{ker}\, A^\perp \mid Ax = 0\right\} = \{0\}.$$

Then, since P restricts to the identity in $\mathsf{Im}\, A$, $PA\,\mathsf{i}$ is surjective. Thus, recalling that A has closed range, *Banach's bounded inverse theorem* ensures that $\underline{\mathrm{A}}$ is an isomorphism of Hilbert spaces. Moreover, by construction, $\underline{\mathrm{A}}^{-1}A$ restricts to the identity in $\mathsf{ker}\, A^\perp$. Therefore, by definition of operator norm, we have

$$\|\underline{\mathrm{A}}^{-1}\| = \sup_{y\in \mathsf{Im}\, A} \frac{\|\underline{\mathrm{A}}^{-1} y\|_X}{\|y\|_Y} = \sup_{x \in \mathsf{ker}\, A^\perp} \frac{\|x\|_X}{\|Ax\|_Y},$$

or equivalently,

$$\inf_{x\in\mathsf{ker}\,A^{\perp}} \frac{\|Ax\|_Y}{\|x\|_X} = \frac{1}{\|\underline{\mathrm{A}}^{-1}\|}.$$

This proves the second equality in (2.7.9). The first equality in (2.7.9) is well–known and follows from the orthogonal decomposition $Y = \mathsf{Im}\,A \oplus \mathsf{ker}\,A^{\star}$. □

The previous lemma elucidates that, for a sequence of operators, the fact that one has a *uniform inf-sup condition* in the entire space, is equivalent to invertibility of the sequence, with the inverses being uniformly bounded in operator norm.

The following consequence of the previous result shall prove useful:

Lemma 2.7.2. *A linear and bounded operator $A : X \to Y$ that has closed range satisfies*

$$\inf_{x\in\mathsf{ker}\,A^{\perp}} \sup_{y\in\mathsf{Im}\,A} \frac{\langle Ax, y\rangle}{\|x\|_X\,\|y\|_Y} = \inf_{y\in\mathsf{Im}\,A} \sup_{x\in\mathsf{ker}\,A^{\perp}} \frac{\langle Ax, y\rangle}{\|x\|_X\,\|y\|_Y}.$$

Proof. It follows immediately from the previous lemma, by recalling that an operator and its adjoint have equal norms. □

We shall repeatedly use this result to swap the order of spaces when working with inf-sup conditions.

Dual space induced by an isomorphism.

The following concerns the construction of an instance of the so–called *spaces of negative norm* or *extrapolated spaces*. Here, we adapt a construction due to *Ju. M. Berezanskiĭ* in [Ber68, Ch. 3] so as to suit our purposes. The main difference is that we do not use the (adjoint of the) inclusion map to *pivot* the spaces, but rather the appropriate restriction of an isomorphism. The author is not aware of any existing references for this specific construction, so all the details are provided.

Consider Hilbert spaces $(H_0, \langle\ ,\ \rangle_0)$ and $(H_2, \langle\ ,\ \rangle_2)$ with continuous inclusion $H_2 \hookrightarrow H_0$, and let $A_0 : H_2 \to H_0$ denote an isomorphism of Hilbert spaces. Denote the induced continuous bilinear form by $a : H_2 \times H_0 \to \mathbb{R}$. The objective is then to, given an intermediate space H_1, define a space $H_{1,a}$, which allows us to "extend" a to an invertible continuous bilinear form in $H_{1,a} \times H_1$, and that inherits compactness properties. In addition, it is desirable that this space identifies in a *nice* way with the *true* dual space of H_1, in a sense to be made precise later. Lastly, we let $\mathfrak{i}_{k,l}$ denote the inclusion map $H_k \hookrightarrow H_l$, for $2 \geq k \geq l \geq 0$; which we require to be continuous and, moreover, we require that $\mathfrak{i}_{1,0}$ is dense (but still strict).

Since $\mathfrak{i}_{1,0}$ and a are continuous, given $u \in H_2$, the map $v \mapsto a(u, v)$ is continuous in the H_1-norm. This, according to the *Riesz representation theorem*, gives a linear and bounded map $A_1 : H_2 \to H_1$ characterized by

$$a(u, v) = \langle A_1 u, v \rangle_1 \qquad \forall\, (u, v) \in H_2 \times H_1.$$

We shall define the sought space as the completion of H_2 in a suitable topology, defined in terms of A_1. We then extend A_1 by continuity to get the desired "extension" of A_0.

Let us define a new scalar product in H_2 by requiring that A_1 is an *isometry*: for $u, v \in H_2$, we put

$$\langle u, v \rangle_{1,a} := \langle A_1 u, A_1 v \rangle_1 \,. \tag{2.7.10}$$

To see that it is a scalar product, we just have homogeneity: given $u \in H_2$ such that $\langle u, u \rangle_{1,a} = 0$, we have that $A_1 u = 0$, and therefore

$$\langle A_0 u, v \rangle_0 = a(u, v) = 0 \qquad \forall\, v \in H_1.$$

Since H_1 is dense in H_0 and A_0 is invertible, we get $u = 0$. We remark that this also proves injectivity of A_1.

We denote by $H_{1,a}$ the completion of H_2 under the induced norm $\|\cdot\|_{1,a}$. First, we check that this completion is nontrivial—in that H_2 is not complete with respect to $\| \cdot \|_{1,a}$— and thereby H_2 is strictly embedded in $H_{1,a}$. The following constitutes the main part of the proof for this fact:

Lemma 2.7.3. $(H_2, \| \cdot \|_{1,a})$ *is complete if and only* A_1 *has closed range.*

Proof. Assume that H_2 is complete with respect to $\|\cdot\|_{1,a}$, and consider a sequence $(A_1 u_k)_k$ convergent to z in H_1. In particular, this implies that it is Cauchy, thereby

$$\|u_k - u_\ell\|_{1,a} = \|A_1 u_k - A_1 u_\ell\|_1 \xrightarrow{k,\ell} 0.$$

Therefore, by assumption, there is $u \in H_2$ such that $\|u_k - u\|_{1,a} \xrightarrow{k} 0$, which, according to the definition of $\|\cdot\|_{1,a}$ and the fact that a normed space is Hausdorff, implies that $A_1 u = z$, hence A_1 has closed range.

Conversely, if A_1 has closed range, we have $\|u\|_2 \preceq \|A_1 u\|_1 = \|u\|_{1,a}$, for all $u \in H_2$. Then, a Cauchy sequence $(u_k)_k$ in H_2 with respect to $\|\cdot\|_{1,a}$ is also Cauchy with respect to $\|\cdot\|_2$. Since H_2 is complete, there exists $u \in H_2$ such that $\|u_k - u\|_2 \xrightarrow{k} 0$. From the continuity of A_1 and the definition of $\|\cdot\|_{1,a}$, we get that $(u_k)_k$ converges to u in $\|\cdot\|_{1,a}$. Hence H_2 is complete with this norm. $\square$

Now, we shall see that A_1 cannot have closed range. To this end, notice that

$$\langle u, A_0^\star v\rangle_2 = \langle A_0 u, v\rangle_0 = a(u,v) = \langle A_1 u, v\rangle_1 = \langle u, A_1^\star v\rangle_2\,,$$

for all $(u,v) \in H_2 \times H_1$. There follows that

$$A_1^\star = A_0^\star \circ \mathfrak{i}_{1,0}. \tag{2.7.11}$$

Since A_1 is injective, it has closed range if and only if $A_1^\star$ is surjective. From the fact that A_0 is bijective and (2.7.11), we get that ${A_1}^\star$ is surjective if and only if $H_1 = H_0$, which is not the case, as we assumed a strict inclusion. Thus, H_2 is a proper subspace of $H_{1,a}$. This fact ensures that the foregoing discussion is not trivial.

In this situation, as previously announced, A_1 extends to a continuous linear map $\mathcal{A}_1 : H_{1,a} \to H_1$, in the standard way, and we may pass to the limit in (2.7.10) to get

$$\langle u, v\rangle_{1,a} = \langle \mathcal{A}_1 u, \mathcal{A}_1 v\rangle_1 \qquad \forall\, u, v \in H_{1,a}. \tag{2.7.12}$$

We then extend a to $a' : H_{1,a} \times H_1 \to \mathbb{R}$ by putting, for $u \in H_{1,a}$,

$$a'(u,v) = \langle \mathcal{A}_1 u, v\rangle_1 .$$

It is clear from the definitions of A_1 and $\mathcal{A}_1$ that a and a' coincide in $H_2 \times H_1$; this is the "extension" property we were looking for. Moreover, by construction, we have

$$|a'(u,v)| \leq \|\mathcal{A}_1 u\|_1 \|v\|_1 = \|u\|_{1,a} \|v\|_1 , \tag{2.7.13}$$

for $(u,v) \in H_{1,a} \times H_1$.

We claim that a' is invertible, which is equivalent to say that $\mathcal{A}_1$ is an isomorphism. In fact, from (2.7.12), we see that $\mathcal{A}_1$ is injective and has closed range. To see that it is surjective, we assume otherwise; then, we have a nontrivial orthogonal decomposition $H_1 = \mathsf{Im}\,\mathcal{A}_1 \oplus \mathsf{Im}\,\mathcal{A}_1^{\perp}$. An element v in the second space, in particular, satisfies

$$0 = \langle v, \mathcal{A}_1 u\rangle_1 = a'(u,v) = a(u,v) = \langle v, A_0 u\rangle_0 ,$$

for all $u \in H_2$. Since A_0 is bijective, this gives $v = 0$, which contradicts the nontrivial decomposition. We conclude that $\mathcal{A}_1$ is an isomorphism.

We remark that (2.7.13), together with the fact that $\mathcal{A}_1$ is invertible, implies that a' may be regarded as a non-degenerate duality pairing between $H_{1,a}$ and H_1. Furthermore, from (2.7.13), we also see that every $u \in H_{1,a}$ induces a functional $v \mapsto a'(u,v)$ which lies in the *true* dual space $H_1^\star$. Conversely, by *Riesz representation theorem*, a functional $f \in H_1^\star$ admits a unique $u_f \in H_1$, such that $f(v) = \langle u_f, v\rangle_1$. Since $\mathcal{A}_1$ invertible there is a unique $z_f \in H_{1,a}$, such that $\mathcal{A}_1 z_f = u_f$, and so,

$$a'(z_f, v) = f(v),$$

for all $v \in H_1$. Moreover, we have that $\|z_f\|_{1,a} = \|u_f\|_1 = \|f\|$. Thus, we have described an isometric identification (isomorphism) between $H_{1,a}$ and $H_1^\star$.

Lastly, we shall prove that, if $\mathfrak{i}_{1,0}$ is compact, then $\mathfrak{i}_{2;1,a} : H_2 \hookrightarrow H_{1,a}$ is compact as well.

From (2.7.11), we deduce that

$$A_1 = ({A_0}^\star \circ \mathfrak{i}_{1,0})^\star = \mathfrak{i}_{1,0}^\star \circ A_0.$$

We notice that $\mathfrak{i}_{1,0}^\star$, being the adjoint of a compact operator, is compact as well. So, the previous identity shows compactness of $A_1 : H_2 \to H_1$. We shall see that this is equivalent to say that the inclusion $\mathfrak{i}_{2;1,a} : H_2 \to H_{1,a}$ is compact. In fact, according to *Banach-Alaoglu theorem*, a bounded sequence $(u_n)_n$ in H_2 has a (not reindexed) weakly convergent subsequence. Then, A_1 upgrades this to strong convergence of $(A_1 u_n)_n$ in H_1, it follows that

$$\|u_k - u_\ell\|_{1,a} = \|A_1 u_k - A_1 u_\ell\|_1 \ ,$$

which, in virtue of the completeness of $H_{1,a}$, gives convergence of $(u_n)_n$ in $H_{1,a}$. This is the definition of a compact operator.

We remark that one also could have started by directly defining $A_1 = \mathfrak{i}_{1,0}^\star \circ A_0$, in which case everything follows in an analogous way. We chose to present it in this way because it feels less terse and more intuitive.

We summarize this discussion in the following result, for later reference:

Proposition 2.7.5. *Consider a sequence of continuous inclusions $H_2 \hookrightarrow H_1 \hookrightarrow H_0$ of Hilbert spaces, with the second one being dense. Let $A_0 : H_2 \to H_0$ be an isomorphism with induced bilinear form $a : H_2 \times H_0 \to \mathbb{R}$. Then, there exists a Hilbert space $H_{1,a}$ and an isomorphism $\mathcal{A}_1 : H_{1,a} \to H_1$, with induced bilinear form $a' : H_{1,a} \times H_1 \to \mathbb{R}$, such that*

i) *for all $(u, v) \in H_2 \times H_1$, there holds $a'(u, v) = a(u, v)$,*

ii) *$H_{1,a}$ and $H_1^\star$ are isometrically isomorphic, and*

iii) *if the inclusion $H_1 \hookrightarrow H_0$ is compact, then $H_2 \hookrightarrow H_{1,a}$ is compact as well.*

Part I

Discretization of Hodge-Dirac eigenmode problems

CHAPTER 3

Model Problem

3.1 The Dirac equation

In this section, we follow the exposition in [Chr18] on the interpretation of the Dirac equation in Minkowski space, with exponential decay in time, as a self-adjoint eigenvalue problem.

We begin by defining the 2×2 complex **Pauli matrices**:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In terms of the Pauli matrices, we define the 4×4 **Dirac matrices**:

$$\gamma^0 = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{bmatrix}, \quad \gamma^k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix},$$

for $k \in \{1, 2, 3\}$. Our starting point is the Dirac equation with minimal coupling to an electromagnetic gauge potential $A = (A_0, A_1, A_2, A_3)$, with $A_\mu \in \mathbb{C}^{4\times 4}$, $\mu \in \{0, 1, 2, 3\}$, which seeks a field $\psi : \mathbb{R} \oplus \mathbb{R}^3 \to \mathbb{C}^4$ satisfying

$$\mathrm{i}\hbar\gamma^\mu(\partial_\mu + \mathrm{i}A_\mu)\psi = mc\psi\,, \tag{3.1.1}$$

where $\hbar$, m and c denote the *reduced Planck constant*, mass and *light velocity*, respectively, and summation is implied. From now on, we choose units so that c and $\hbar$ have numerical value 1. In which case the Dirac equation reads

$$\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}A_\mu)\psi = m\psi\,. \tag{3.1.2}$$

A solution to this equation is called a spinor-valued field on time-space. In the remaining of this section, we assume that we have homogeneous Dirichlet conditions, or that we are in the whole Minkowski space $\mathbb{R} \oplus \mathbb{R}^3$, it shall not make a difference in the foregoing discussion.

We identify the quaternions with the space of complex matrices:

$$\mathbb{H} := \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \ : \ a, b \in \mathbb{C} \right\}.$$

$\mathbb{H}$ constitutes a vector space over $\mathbb{R}$ and a real associative algebra with respect to matrix multiplication. Moreover, by construction, all nonzero elements in $\mathbb{H}$ are invertible.

The real elements are identified with multiples of the identity matrix, and the imaginary elements with those spanned by the matrices $J_k = -\mathrm{i}\sigma_k$. It is then verified that

$$\mathbb{H} = \mathbb{R}\mathrm{I} \oplus \mathbb{R}J_1 \oplus \mathbb{R}J_2 \oplus \mathbb{R}J_3\,.$$

Given $x \in \mathbb{R}^3$, we denote $x \cdot J = \sum_{k=1}^{3} x_k J_k$. It is direct to see that $\mathbb{R} \oplus \mathbb{R}^3 \cong \mathbb{H}$, with an isomorphism being $\Xi : \mathbb{R} \oplus \mathbb{R}^3 \to \mathbb{H}$, $(x_0, x) \mapsto \Xi(x_0, x) = x_0\mathrm{I} + x \cdot J$. Via this identification, we may transport matrix operations on $\mathbb{H}$ to operations on Minkowski space. The matrices J_k

are verified to satisfy the *Clifford relations*:

$$J_kJ_l + J_kJ_l = 0, \quad \text{for } k \neq l, \qquad \text{and} \qquad J_k^2 = -\mathrm{I}\,,$$

and the cyclic relation

$$J_1J_2 = J_3, \quad J_2J_3 = J_1, \quad J_3J_1 = J_2\,.$$

Bearing this in mind, given $X = \Xi(x_0, x)$, $Y = \Xi(y_0, y)$ in $\mathbb{H}$, their product is proved to be

$$XY = (x_0y_0 - x \cdot y)I + (x_0y + y_0x + x \times y) \cdot J\,.$$

This product is transported back to $\mathbb{R} \oplus \mathbb{R}^3$ by requiring that Ξ is an algebra homomorphism:

$$\Xi((x_0, x)(y_0, y)) = \Xi(x_0, x)\Xi(y_0, y)\,,$$

or equivalently

$$(x_0, x)(y_0, y) = (x_0y_0 - x \cdot y, x_0y + y_0x + x \times y).$$

Conjugation takes the familiar form $\overline{(x_0, x)} = (x_0, -x)$, which can be seen as the one induced from hermitian conjugation of matrices, denoted X^H. The scalar product in $\mathbb{H}$ is $X \cdot \mathrm{Y} = \mathfrak{Re}(X^HY)$ which, via identification, allows us to recover the usual Euclidean inner product in $\mathbb{R} \oplus \mathbb{R}^3$. Namely, $(x_0, x) \cdot (y_0, y) = x_0y_0 + x \cdot y$.

Now, we intend to interpret the Dirac operator in terms of $\mathbb{H}$ and its operations. We denote $J \cdot \nabla = \sum_{k=1}^{3} J_k\partial_k$, which is easily verified to act on fields $\mathbb{R}^3 \to \mathbb{H}$ as

$$J \cdot \nabla X = -\operatorname{\mathsf{div}} \underline{\mathrm{X}}\, I + (\operatorname{\mathsf{grad}} X^{(0)} + \operatorname{\mathsf{curl}} \underline{\mathrm{X}}) \cdot J\,,$$

for $X(x) = X^{(0)}(x) + \underline{\mathrm{X}} \cdot J$, $\underline{\mathrm{X}} \in \mathbb{R}^3$. From here, we see that $J \cdot \nabla$ is a symmetric operator in $\mathrm{L}^2(\mathbb{R}^3; \mathbb{H})$. Defining $\sigma \cdot \nabla$ analogously, we have the identity $J \cdot \nabla = -i\sigma \cdot \nabla$.

Now, by reexpressing the electromagnetic potential as a combination of an electric scalar potential $V : \mathbb{R} \oplus \mathbb{R}^3 \to \mathbb{R}$ and a magnetic vector potential $A : \mathbb{R} \oplus \mathbb{R}^3 \to \mathbb{R}^3$, we may rewrite

(3.1.2) as:

$$\begin{bmatrix} \partial_0 + \mathrm{i}V & 0 \\ 0 & -\partial_0 - \mathrm{i}V \end{bmatrix} \psi + \begin{bmatrix} 0 & \sigma \cdot (\nabla + \mathrm{i}A) \\ -\sigma \cdot (\nabla + \mathrm{i}A) & 0 \end{bmatrix} \psi = -\mathrm{i}m\psi,$$

or equivalently,

$$\partial_0 \psi = -\begin{bmatrix} 0 & \sigma \cdot (\nabla + \mathrm{i}A) \\ \sigma \cdot (\nabla + \mathrm{i}A) & 0 \end{bmatrix} \psi - \mathrm{i}m \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \psi - \mathrm{i}V\psi. \tag{3.1.3}$$

We shall only be concerned with the so-called eigenstates or stationary solutions, as a general solution may then be written as a series involving them. This ammount to assume that the dependence of ψ on time is given by $t \mapsto \exp(-\mathrm{i}\lambda t)$, with $\lambda \in \mathbb{R}$ to be determined, in which case the equation reads

$$\lambda \psi = \begin{bmatrix} 0 & J \cdot (\nabla + \mathrm{i}A) \\ J \cdot (\nabla + \mathrm{i}A) & 0 \end{bmatrix} \psi + m \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \psi + V\psi. \tag{3.1.4}$$

It is not hard to see that this can be regarded as a self-adjoint eigenvalue problem.

3.2 Abstract Hodge-Dirac equation in Hilbert complexes

We recall the identification of the exterior algebra with scalars and vectors:

$$\mathsf{Alt}^{\bullet}\mathbb{R}^3 \equiv \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R},$$

and define a map $\Theta : \mathsf{Alt}^{\bullet}\mathbb{R}^3 \to \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R} \oplus \mathbb{R}^3$ by $(s, u, v, t) \mapsto (s, v, t, u)$, which is essentially a reordering of the exterior algebra, putting first the forms of even degree (0 and 2), and then those of odd degree. By using Ξ, we can let $J \cdot \nabla$ act on $\mathbb{R} \oplus \mathbb{R}^3$-valued fields:

$$(f, g) \mapsto \Xi^{-1}(J \cdot \nabla)\Xi(f, g) = (-\operatorname{\mathsf{div}} g, \operatorname{\mathsf{grad}} f + \operatorname{\mathsf{curl}} g).$$

This motivates the definition of an operator $\mathcal{D} : \mathrm{C}^\infty\Lambda^\bullet(\mathbb{R}^3) \to \mathrm{C}^\infty\Lambda^\bullet(\mathbb{R}^3)$ as

$$\mathcal{D} = \Theta^{-1} \begin{bmatrix} \Xi^{-1}(J\cdot\nabla)\Xi & 0 \\ 0 & \Xi^{-1}(J\cdot\nabla)\Xi \end{bmatrix} \Theta,$$

which satisfies the mapping property

$$\begin{bmatrix} s \\ u \\ v \\ t \end{bmatrix} \longmapsto \begin{bmatrix} -\operatorname{\mathsf{div}} u \\ \operatorname{\mathsf{grad}} s + \operatorname{\mathsf{curl}} v \\ \operatorname{\mathsf{grad}} t + \operatorname{\mathsf{curl}} u \\ -\operatorname{\mathsf{div}} v \end{bmatrix}.$$

Defining the slightly unusual (notice a difference of sign) exterior derivative

$$\mathsf{d} = \mathsf{grad} \oplus \mathsf{curl} \oplus -\,\mathsf{div},$$

we are able to write $\mathcal{D} = \mathsf{d} + \mathsf{d}^\star$, where $\mathsf{d}^\star$ stands for the formal adjoint of d. One readily sees that $\mathcal{D}$ is a square root of the corresponding Hodge-Laplace operator, i.e., $\mathcal{D}^2 = \Delta$. For this reason, $\mathcal{D}$ is referred to as a **Hodge-Dirac operator** (cf. [LS16]).

In this situation, the ideal scenario would be to regard the Dirac equation (3.1.4) as the eigenvalue problem: find $(\lambda, \psi) \in \mathbb{R} \times \mathrm{C}^\infty\Lambda^\bullet(\mathbb{R}^3)$ satisfying

$$\mathcal{D}\psi + C\psi = \lambda\psi, \tag{3.2.1}$$

where C constitutes the remaining terms on the right-hand side of (3.1.4). Namely

$$C = \Theta^{-1} \begin{bmatrix} m + V & \Xi^{-1}(J\cdot \mathrm{i}A)\Xi \\ \Xi^{-1}(J\cdot \mathrm{i}A)\Xi & -m + V \end{bmatrix} \Theta.$$

Nevertheless, this is not well–defined, because the term $J \cdot \mathrm{i}A$ involving the magnetic potential maps $\mathbb{H}$ into $\mathrm{i}\mathbb{H}$, and therefore, composition with Ξ^{-1} is not possible.

Being unable to cast the true Dirac equation into the form (3.2.1) is one of the main disadvantages of this approach. However, an immediate advantage is that one may readily abstract it to an equation formulated in a generic Hilbert complex, in the following way: given a Hilbert complex $(O^\bullet, d^\bullet)$ with domain complex $(Y^\bullet, d^\bullet)$, the associated abstract Hodge-Dirac operator is given by $\mathcal{D} = d + d^\star$. The problem then seeks $(\lambda, \psi) \in \mathbb{R} \times Y^\bullet$ satisfying

$$\mathcal{D}\psi + C\psi = \lambda\psi, \tag{3.2.2}$$

where $C : O^\bullet \to O^\bullet$ is referred to as a perturbation.

In the remaining of Part I, we derive a framework to approach discretization of this type of self-adjoint eigenvalue problems, which is then followed by its application to the Hodge-Dirac equation (3.2.1), within the framework of finite element systems.

CHAPTER 4

An abstract theory for the discretization of symmetric Fredholm operators

In the present section, we deduce sufficient conditions to ensure eigenmode convergence of the discrete eigenvalue problems. We pay special attention to including the case of unsigned operators and low order perturbations, the latter representing, for instance, the electromagnetic field in the model problem (3.2.2). We follow closely the second theory developed in [Chr18, Section 6], although with the crucial difference that we adopt the setting of conforming methods, whereas in [Chr18] a non-conforming approach is considered. As a result, the development of the theory is simplified, and some results are strengthened. No reference to the exterior derivative or an abstract analogue is made, thereby allowing for greater reusability outside our particular context. We remark that the development of H^1-conforming methods for our model problem is justified by the somewhat recent construction of complexes of discrete differential forms with improved regularity made in [CH18].

4.1 Eigenmode convergence

Consider Hilbert spaces $(\mathrm{Y}, \langle\cdot,\cdot\rangle_{\mathrm{Y}})$ and $(\mathrm{O}, \langle\cdot,\cdot\rangle_{\mathrm{O}})$, with norms denoted by $\|\cdot\|_{\mathrm{Y}}$ and $|\cdot|_{\mathrm{O}}$. Assume that $\mathrm{Y} \hookrightarrow \mathrm{O}$ is dense and compact, and that we are given a linear and bounded operator $A : \mathrm{Y} \to \mathrm{O}$ which is Fredholm and symmetric, of Fredholm index zero. This operator induces a symmetric bounded bilinear form a, given by $a(u,v) = \langle Au, v\rangle_{\mathrm{O}}$, for all $(u,v) \in \mathrm{Y}\times\mathrm{Y}$, which can be extended to both $\mathrm{Y} \times \mathrm{O}$ and $\mathrm{O} \times \mathrm{Y}$. In this section, unless otherwise stated, orthogonality will always be understood with respect to the inner product in O.

In this section, we are interested in the discretization of the eigenvalue problem for A, which seeks $(\lambda, u) \in \mathbb{R} \times \mathrm{Y}$ such that

$$a(u,v) = \lambda \langle u, v\rangle_{\mathrm{O}} \qquad \forall\, v \in \mathrm{Y}. \tag{4.1.1}$$

Now, assume we are given a sequence of finite-dimensional subspaces Y_n of Y; we are then interested in the discrete eigenvalue problems: find $(\lambda, u) \in \mathbb{R} \times \mathrm{Y}_n$ such that

$$a(u,v) = \lambda \langle u, v\rangle_{\mathrm{O}} \qquad \forall\, v \in \mathrm{Y}_n. \tag{4.1.2}$$

We proceed to deduce sufficient conditions to ensure that eigenmode convergence is achieved.

As customary, the stability will be expressed in terms of an *inf-sup condition*, of the form

$$\inf_{u\in\mathrm{Y}_n} \sup_{u\in\mathrm{Y}_n} \frac{|a(u,v)|}{|u|_{\mathrm{O}}\, \|v\|_{\mathrm{Y}}} \succeq 1. \tag{4.1.3}$$

Let us assume for a moment that A is injective; since its Fredholm index is zero, this implies that A is invertible. Denote its inverse by $K : \mathrm{O} \to \mathrm{Y}$, and notice that it satisfies

$$a(Ku, v) = \langle AKu, v\rangle_{\mathrm{O}} = \langle u, v\rangle_{\mathrm{O}} \qquad \forall\, u, v \in \mathrm{O}.$$

We denote by $\mathfrak{i}_n : \mathrm{Y}_n \to \mathrm{O}$ the inclusion map and by $\pi_n : \mathrm{O} \to \mathrm{Y}_n$ the O-orthogonal projector.

Then, $A_n = \pi_n A\, \mathfrak{i}_n$, is the unique operator $\mathrm{Y}_n \to \mathrm{Y}_n$ that satisfies

$$a(u,v) = \langle A_n u, v\rangle_{\mathrm{O}} \qquad \forall\, u,v \in \mathrm{Y}_n\,.$$

According to the symmetry of A_n, the discrete inf-sup condition translates into its bijectivity. Thereby, we are able to define operators $K_n : \mathrm{O} \to \mathrm{Y}_n$ by $K_n = A_n^{-1}\pi_n$, which have the property:

$$a(K_n u, v) = \langle A_n K_n u, v\rangle_{\mathrm{O}} = \langle u, v\rangle_{\mathrm{O}} \qquad \forall\, (u,v) \in \mathrm{O}\times \mathrm{Y}_n.$$

Remark 4.1.1. We remark that $K : \mathrm{O} \to \mathrm{O}$ is a self-adjoint compact operator, for the inclusion $\mathrm{Y} \hookrightarrow \mathrm{O}$ is compact. Furhtermore, K_n is clearly self-adjoint and compact as well. Therefore, the classical theory of Babuška & Osborn applies (see, for instance, [BO91, Section 8]). More precisely, eigenmode convergence is guaranteed if K_n converges to K in operator norm $\mathrm{O} \to \mathrm{O}$. ◇

Definition 4.1.1. We say that $(\mathrm{Y}_n)_n$ is O-**approximating** if, for every $v \in \mathrm{Y}$, there is a sequence $(v_n)_n$, with $v_n \in \mathrm{Y}_n$, such that $|v_n - v|_{\mathrm{O}} \xrightarrow{n} 0$. If the same property holds with the Y-norm, we say that $(\mathrm{Y}_n)_n$ is Y-**approximating**. ◇

Under a mild assumption on the finite-dimensional subspaces, eigenmode convergence is guaranteed.

Proposition 4.1.1. *Assume that A is injective, the discrete inf-sup condition* (4.1.3) *holds and* $(\mathrm{Y}_n)_n$ *is* O-*approximating. Then, there holds*

$$\|K_n - K\|_{\mathrm{O}\to\mathrm{O}} \xrightarrow{n} 0.$$

Proof. Let Z be an intermediate Hilbert space between Y and O, satisfying the hypotheses of Proposition 2.7.5, with local notation

$$H_2 = \mathrm{Y}, \quad H_1 = \mathrm{Z}, \quad H_0 = \mathrm{O} \quad \text{and} \quad A_0 = A,$$

then, according to this result, there is a Hilbert space Z' and an invertible bounded bilinear form $a' : \mathrm{Z}' \times \mathrm{Z} \to \mathbb{R}$ such that

$$\mathfrak{i}_{\mathrm{Y},\mathrm{Z}'} : \mathrm{Y} \hookrightarrow \mathrm{Z}' \text{ is compact} \qquad \text{and} \qquad a'(u,v) = a(u,v) \qquad \forall\,(u,v) \in \mathrm{Y} \times \mathrm{Z}.$$

We define an operator $P_n : \mathrm{Z}' \to \mathrm{Y}_n$ by requiring that, for $u \in \mathrm{Z}'$, there holds

$$a(P_n u, v) = a'(u,v), \qquad \forall\, v \in \mathrm{Y}_n.$$

Notice that, according to Proposition 2.7.4, this is well–defined. We check that the operators P_n are bounded, and furthermore, they constitute a uniformly bounded sequence in operator norm $\mathrm{Z}' \to \mathrm{O}$. In fact, given $u \in \mathrm{Z}'$, the discrete inf-sup condition yields

$$|P_n u|_{\mathrm{O}} \preceq \sup_{v \in \mathrm{Y}_n} \frac{a(P_n u, v)}{\|v\|_{\mathrm{Y}}} = \sup_{v \in \mathrm{Y}_n} \frac{a'(u,v)}{\|v\|_{\mathrm{Y}}} \preceq \|u\|_{\mathrm{Z}'},$$

where in the last inequality we used that the inclusion $\mathfrak{i}_{\mathrm{Y},\mathrm{Z}'}$ is continuous. Hence, we have that $\|P_n\|_{\mathrm{Z}' \to \mathrm{O}} \preceq 1$.

In turn, fixed $u \in \mathrm{Y}$ and a sequence $(u_n)_n \in (\mathrm{Y}_n)_n$ with $|u_n - u|_{\mathrm{O}} \xrightarrow{n} 0$, we can argue in a similar way to get

$$|P_n u - u_n|_{\mathrm{O}} \preceq \sup_{v \in \mathrm{Y}_n} \frac{a(u - u_n, v)}{\|v\|_{\mathrm{Y}}} \preceq |u - u_n|_{\mathrm{O}},$$

so $(P_n|_{\mathrm{Y}})_n$ is pointwise convergent to the identity $\mathrm{Y} \to \mathrm{O}$. Since Y is dense in Z', by using Proposition 2.7.1 we get that $(P_n)_n$ is pointwise convergent $\mathrm{Z}' \to \mathrm{O}$.

In turn, since a' restricts to a in $\mathrm{Y} \times \mathrm{Z}$, from the relation defining P_n, we deduce that $A_n P_n \mathfrak{i}_{\mathrm{Y},\mathrm{Z}'} = \pi_n A$, or equivalently, $K_n = P_n \mathfrak{i}_{\mathrm{Y},\mathrm{Z}'} K$. Then, since $\mathfrak{i}_{\mathrm{Y},\mathrm{Z}'} K : \mathrm{O} \to \mathrm{Z}'$ is compact, Proposition 2.7.2 gives convergence in operator norm $\|K_n - K\|_{\mathrm{O} \to \mathrm{O}} \xrightarrow{n} 0$, as desired. □

4.2 Perturbations

In the previous section, we studied eigenmode convergence for the eigenvalue problem (4.1.2). Convergence of the *solution operators* was proved in the case in which A is injective and stability is achieved. Nevertheless, as mentioned earlier, we are interested in low order perturbations of

A, which take the form $A+C$, where $C : \mathrm{O} \to \mathrm{O}$ is a linear, bounded and symmetric operator, typically representing the electromagnetic potential in the model problem. We shall see that injectivity of A is not as critical as that of $A+C$; indeed it suffices to use that the kernel of A is finite-dimensional. Moreover, since Fredholmness is preserved under compact (or sufficiently small) perturbations, injectivity of $A+C$ may be achieved by adding suitable multiples of the identity. This does not damage the eigenvalue problem, as adding multiples of the identity only shifts the eigenvalues both at the continuous and discrete level.

Definition 4.2.1. Recall Definition 2.7.1, and denote by G and G_n the kernels of A and A_n, respectively. We say that we have **Y-convergence of the gap** if there holds

$$\hat{\delta}(G_n, G) \xrightarrow{n} 0, \tag{4.2.1}$$

in the Y-norm. ◇

We denote by $c : \mathrm{O} \times \mathrm{O} \to \mathbb{R}$ the bilinear form induced by C. This is,

$$c(u,v) = \langle Cu, v \rangle_{\mathrm{O}}\,.$$

As before, we put $C_n = \pi_n C\, \mathfrak{i}_n$, which is the unique operator $\mathrm{Y}_n \to \mathrm{Y}_n$ satisfying

$$\langle C_n u, v \rangle_{\mathrm{O}} = c(u,v) \qquad \forall\, u, v \in \mathrm{Y}_n.$$

From now on, we denote $\widetilde{\mathrm{Y}}_n = \mathrm{Y}_n \cap G_n^{\perp}$. It is clear that A_n is injective in $\widetilde{\mathrm{Y}}_n$. In the following result, we show that stability may be preserved, and even upgraded, under suitable perturbations.

Proposition 4.2.1. *Suppose that we have* Y*-convergence of the gap and that* $(\mathrm{Y}_n)_n$ *is* O*-approximating. Furthermore, assume we have discrete stability in the form*

$$\inf_{u \in \widetilde{\mathrm{Y}}_n} \sup_{v \in \widetilde{\mathrm{Y}}_n} \frac{a(u,v)}{\|u\|_{\mathrm{Y}}\, |v|_{\mathrm{O}}} \succeq 1. \tag{4.2.2}$$

If $A + C : Y \to O$ is injective, then

$$\inf_{u \in \mathrm{Y}_n} \sup_{v \in \mathrm{Y}_n} \frac{a(u,v) + c(u,v)}{\|u\|_{\mathrm{Y}} \, |v|_{\mathrm{O}}} \succeq 1. \tag{4.2.3}$$

Proof. Arguing by contradiction, we supose that (4.2.3) does not hold. Then, there are sequences $(n_k)_k$ and $(u_k)_k$ in $(\mathrm{Y}_{n_k})_k$ such that

$$\|u_k\|_{\mathrm{Y}} = 1 \quad \text{and} \quad \lim_{k \to \infty} \sup_{v \in \mathrm{Y}_{n_k}} \frac{a(u_k,v) + c(u_k,v)}{|v|_{\mathrm{O}}} = 0.$$

In particular, since $(\mathrm{Y}_n)_n$ is O-approximating, for every $v \in \mathrm{Y}$, there is $(v_k)_k$ in $(\mathrm{Y}_{n_k})_k$ such that $|v_k - v|_{\mathrm{O}} \xrightarrow{k} 0$. Then,

$$\frac{1}{M} \lim_{k \to \infty} |a(u_k, v_k) + c(u_k, v_k)| \leq \lim_{k \to \infty} \sup_{v \in \mathrm{Y}_{n_k}} \frac{a(u_k,v) + c(u_k,v)}{|v|_{\mathrm{O}}} = 0,$$

where $M = \sup_k |v_k|_{\mathrm{O}}$. Moreover, we have

$$|(a+c)(u_k, v_k - v)| \preceq \|u_k\|_{\mathrm{Y}} \, |v_k - v|_{\mathrm{O}} = |v_k - v|_{\mathrm{O}} \xrightarrow{k} 0,$$

from where we deduce that, for all $v \in \mathrm{Y}$, there holds

$$\lim_{k \to \infty} a(u_k, v) + c(u_k, v) = 0.$$

Since Y is a Hilbert space, up to passing to subsequences, we may further assume that $(u_k)_k$ converges weakly in Y to some $u \in \mathrm{Y}$, in which case, we have that

$$a(u,v) + c(u,v) = 0 \qquad \forall \, v \in \mathrm{Y},$$

and therefore $u = 0$. Since the inclusion $\mathrm{Y} \hookrightarrow \mathrm{O}$ is compact, we have that $C : \mathrm{Y} \to \mathrm{O}$ is compact, and hence

$$Cu_k \xrightarrow{k} Cu = 0 \quad \text{in} \quad \mathrm{O},$$

Now, writing $a(u_k, v) = (a(u_k, v) + c(u_k, v)) - c(u_k, v)$, bounding and using the previously computed limits, we obtain

$$\lim_{k\to\infty} \sup_{v\in \mathrm{Y}_{n_k}} \frac{a(u_k, v)}{|v|_{\mathrm{O}}} = 0. \tag{4.2.4}$$

We define the Y-orthogonal projectors $P_k : \mathrm{Y} \to G_{n_k}$ and $P : \mathrm{Y} \to G$ which, according to the Y-convergence of the gap and Proposition 2.7.3, satisfy

$$\|P_k - P\|_{\mathrm{Y}\to\mathrm{Y}} = \hat{\delta}(G_{n_k}, G) \xrightarrow{k} 0.$$

Then, weak convergence and boundedness of $(u_k)_k$ gives

$$\|P_k u_k - P u_k\|_{\mathrm{Y}} \le \|P_k - P\|_{\mathrm{Y}\to\mathrm{Y}}, \qquad \text{and} \qquad P u_k \rightharpoonup P u \quad \text{in} \quad \mathrm{Y}.$$

Furthermore, the latter is strong convergence, as G is finite-dimensional. Thus, we get that $(P_k u_k)_k$ converges to $Pu = 0$ strongly in Y. Setting $\tilde{u}_k = (I - P_k)u_k \in \mathrm{Y}_{n_k} \cap G_{n_k}^{\perp_{\mathrm{Y}}}$, we write

$$\|\tilde{u}_k\|_{\mathrm{Y}}^2 = \|u_k\|_{\mathrm{Y}}^2 - \|P_k u_k\|_{\mathrm{Y}}^2,$$

which implies that $\|\tilde{u}_k\|_{\mathrm{Y}} \xrightarrow{k} 1$. Furthermore, from this convergence, given $\epsilon > 0$, there is $\bar{k} \in \mathbb{N}$, such that $\forall\, k \ge \bar{k}$, there holds

$$\|\tilde{u}_k\|_{\mathrm{Y}} > 1 - \epsilon.$$

Finally, we notice that $a(u_k, v) = a(\tilde{u}_k, v)$ for all $v \in \mathrm{Y}_{n_k}$, so we can use (4.2.4) to get

$$\lim_{k\to\infty} \sup_{v\in \mathrm{Y}_{n_k}} \frac{a(\tilde{u}_k, v)}{\|\tilde{u}_k\|_{\mathrm{Y}} |v|_{\mathrm{O}}} \le \frac{1}{1-\epsilon} \lim_{k\to\infty} \sup_{v\in \mathrm{Y}_{n_k}} \frac{a(u_k, v)}{|v|_{\mathrm{O}}} = 0.$$

This violates the uniform inf-sup condition

$$\inf_{u\in \mathrm{Y}_n \cap G_n^{\perp_{\mathrm{Y}}}} \sup_{v\in \mathrm{Y}_{n_k}} \frac{a(u, v)}{\|u\|_{\mathrm{Y}} |v|_{\mathrm{O}}} \succeq 1,$$

which, in turn, violates (4.2.2). This last implication is proved, in a more general form, in Proposition 9.2.1. □

CHAPTER 5

Stability and Approximation properties for H^1-conforming FES

This chapter is devoted to the derivation of various discrete estimates, which will be key to the verification of the hypotheses required by the abstract theory developed in Chapter 4. We start by fixing notation and defining our specific setting. This is followed by the construction of the so-called *smoothed projections* and the deduction of error estimates associated with projection and interpolation onto the discrete spaces. We finish the chapter by proving a stable equivalence of norms, relating a 'discrete H^1-norm' with the continuous one.

5.1 Setting

Continuous setting

For a positive integer n, consider a compact and flat Riemannian n-manifold without boundary $\mathcal{S}$, and put

$$\mathbf{O} = \mathrm{L}^2\Lambda^\bullet(\mathcal{S}), \quad \mathbf{X} = \mathrm{H}\Lambda^\bullet(\mathcal{S}) \quad \text{and} \quad \mathbf{Y} = \mathrm{H}^1\Lambda^\bullet(\mathcal{S})\,.$$

We maintain the notation introduced in Section 2.3 for the inner product and norm in $\mathbf{O}$. We assume that $\mathbf{X}$ carries the graph norm $\|\cdot\|_{\mathbf{X}}$ induced by the exterior derivative, and that the norm in $\mathbf{Y}$, denoted $\|\cdot\|_{\mathbf{Y}}$, is the usual one. In $\mathcal{S}$, according to the *Weitzenböck formula* (see, e.g., [Pet16, Theorem 9.4.1] and [GQ20, Theorem 8.26]) relating the Bochner Laplacian $\nabla^\star\nabla$ with the Hodge Laplacian Δ, we have

$$\nabla^\star\nabla = \Delta = \mathsf{d}\mathsf{d}^\star + \mathsf{d}^\star\mathsf{d}\,,$$

where ∇ is the Levi-Civita connection and $\nabla^\star$ its formal adjoint. Integration by parts yields

$$|\nabla u|^2 = |\mathsf{d}u|^2 + |\mathsf{d}^\star u|^2\,,$$

which amounts to say that a differential form belongs to $\mathbf{Y}$ if and only if both its exterior derivative and coderivative are in $\mathbf{O}$. Moreover, this means that the domain of the Hodge-Dirac operator $\boldsymbol{\mathcal{D}} = \mathsf{d} + \mathsf{d}^\star$ is $\mathbf{Y}$.

The canonical example of manifolds with the aforementioned properties are flat tori, we briefly describe their construction in what follows. Given a basis $\{v_i\}_i$ of $\mathbb{R}^n$, the associated full-rank **lattice** $\mathbb{L}$ in $\mathbb{R}^n$ is the additive group of the form

$$\mathbb{L} = \bigoplus_{i=1}^{n} \mathbb{Z}v_i.$$

That is, $\mathbb{L}$ is the $\mathbb{Z}$-span of a basis of $\mathbb{R}^n$, which may be pictured as a grid in Euclidean space, whose vertices are given by all the $\mathbb{Z}$-linear combinations of the chosen basis. A lattice induces an action on $\mathbb{R}^n$ via translations which, in turn, induces an identification of points in space.

One can picture this as the grid collapsing into a single cell: a so-called *fundamental domain* (see Figure 5.1). A **torus** is then the induced quotient group

$$\mathcal{S} = \mathbb{R}^n \big/ \mathbb{L},$$

endowed with the quotient topology. This results in a compact smooth manifold without boundary. Furthermore, if $\mathcal{S}$ is given the ambient (Riemannian) metric, we get a flat Riemannian manifold without boundary. The canonical projection is denoted by $\pi : \mathbb{R}^n \to \mathcal{S}$.

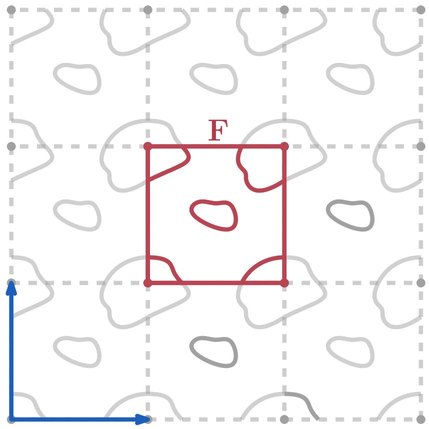

Figure 5.1: Grid representing the lattice spanned by the canonical basis (in blue) of $\mathbb{R}^2$ and a fundamental domain (in red). The identification of points under the quotient is also illustrated.

Remark 5.1.1. We say that the example of flat tori is canonical because, according to *Bieberbach's first theorem*, ***any*** such a manifold is the quotient of a torus by a finite group, see for instance [Cha86, Theorem 5.3]. Despite this, we try to resort to particular features of tori as little as possible. Hopefully, this point of view can help in generalizations, for instance, to manifolds with curvature or boundary. ◇

We continue by setting the notation

$$\begin{aligned} \mathbf{G} &= \{u \in \mathbf{Y} \mid \mathsf{d}u = 0 \ \text{ and } \ \mathsf{d}^\star u = 0\}\,, \\ \mathbf{W} &= \{w \in \mathbf{X} \mid \mathsf{d}w = 0\}\,, \\ \mathbf{V} &= \{v \in \mathbf{X} \mid \forall\, w \in \mathbf{W} \ \ (v, w) = 0\}\ . \end{aligned} \tag{5.1.1}$$

We claim that the $\mathbf{O}$-completion $\overline{\mathbf{V}}$ of $\mathbf{V}$ is $\mathbf{W}^{\perp\mathbf{o}}$. In fact, the inclusion $\overline{\mathbf{V}} \subset \mathbf{W}^{\perp\mathbf{o}}$ is direct to prove. To see the converse inclusion, we take $\bar{v} \in \mathbf{W}^{\perp\mathbf{o}}$ and find a sequence in $\mathbf{V}$

converging to $\bar{v}$ in the $\mathbf{O}$-norm. According to the density of $\mathbf{X}$ in $\mathbf{O}$, there is a sequence $(x_n)_n$ in $\mathbf{X}$, converging to $\bar{v}$ in $\mathbf{O}$. Denote by $\mathcal{P} : \mathbf{O} \to \mathbf{W}^{\perp_{\mathbf{O}}}$ the $\mathbf{O}$-orthogonal projector, and set $v_n = \mathcal{P}x_n$. We have that $x_n = v_n + (I - \mathcal{P})x_n$, and since $I - \mathcal{P}$ is the $\mathbf{O}$-orthogonal projector onto $\mathbf{W} \subset \mathbf{X}$, we conclude that $v_n \in \mathbf{X}$, hence $v_n \in \mathbf{V}$. It remains to check convergence to $\bar{v}$, which is immediate, because $\mathcal{P}\bar{v} = \bar{v}$ and $\mathcal{P}$ is continuous, thus proving the claim.

We have Hodge decompositions for $\mathbf{X}$ and $\mathbf{O}$:

$$\mathbf{X} = \mathbf{V} \oplus \mathbf{W} = \mathbf{V} \oplus \mathsf{d}\mathbf{X} \oplus \mathbf{G}, \qquad \mathbf{O} = \overline{\mathbf{V}} \oplus \mathbf{W} = \overline{\mathbf{V}} \oplus \mathsf{d}\mathbf{X} \oplus \mathbf{G}\,,$$

which are orthogonal in the corresponding scalar products.

We remark that $u \in \mathbf{X}$ if and only if $\mathcal{P}u \in \mathbf{X}$, in which case we have $\mathsf{d}u = \mathsf{d}\mathcal{P}u$. Furthermore, if $u \in \mathbf{V}$, then $\mathcal{P}u = u$. Furthermore, we note that any element $v \in \mathbf{V}$ satisfies

$$(v, \mathsf{d}z) = 0 \qquad \forall\, z \in \mathbf{X}\,,$$

which, since $\mathbf{X}$ is the domain of d inside $\mathbf{O}$, by definition of the formal adjoint $\mathsf{d}^\star$, gives $\mathsf{d}^\star v = 0$.

Associated with a Hodge decomposition, there is always a Poincaré inequality (see [Arn18, Section 4.2.3]):

Proposition 5.1.1. *There is a positive constant $\mathsf{C_P}$ that satisfies*

$$\|v\|_{\mathbf{X}} \leq \mathsf{C_P}\, \|\mathsf{d}v\|_{\mathbf{X}}\ ,$$

for $v \in \mathbf{V}$.

Proof. Recall that, since $\mathcal{S}$ is compact, d has closed range in $\mathbf{O}$, and notice that it is a continuous map $\mathbf{X} \to \mathbf{X}$, whose restriction to $\mathbf{V}$ is injective and has the same range, and hence it is an isomorphism onto ints range. By *Banach's bounded inverse theorem*, the result holds with constant $\mathsf{C_P} = \|(\mathsf{d}|_{\mathbf{V}})^{-1}\|$. □

Discrete setting

Consider a family $\mathcal{H}$ of positive parameters accumulating at 0, and a family of cellular complexes $(\mathcal{T}_h)_h$ on $\mathcal{S}$ parametrized by $\mathcal{H}$. Putting $h_T = \mathsf{diam}\, T$, for $T \in \mathcal{T}_h$, we assume that the cellular complex were constructed to satisfy $h = \max_{T\in\mathcal{T}_h} h_T$, for all $h \in \mathcal{H}$. We also introduce the notation $|T| = \mathsf{vol}(T)$, measured with the corresponding volume element. In addition, for each $h \in \mathcal{H}$, consider an H^1-conforming *FES functor* $\mathfrak{f}_h$, and set

$$(\mathcal{A}_h^\bullet(T), \mathsf{d}_T^\bullet) = \mathfrak{f}_h(T)\,.$$

We remark that H^1-conformity, in particular, implies that the differentials associated with the complexes $\mathcal{A}_h^\bullet(T)$ are the corresponding restrictions of the exterior derivative.

We henceforth denote the induced global spaces by $\mathbf{Y}_h := \mathcal{A}_h^\bullet(\mathcal{T}_h) = \varprojlim_{T\in\mathcal{T}_h} \mathcal{A}_h^\bullet(T)$.

By construction, the exterior derivative induces maps $\mathsf{d} : \mathbf{Y}_h \to \mathbf{Y}_h$, which we ocasionally call d_h to emphasize its domain and codomain. However, this need not be the case for the coderivative, which maps $\mathbf{Y}_h$ into $\mathbf{O}$; we solve this problem by defining the operator $\mathsf{d}_h^\star : \mathbf{Y}_h \to \mathbf{Y}_h$ as the $\mathbf{O}$-adjoint of d in $\mathbf{Y}_h$. If we let $\mathfrak{i}_h : \mathbf{Y}_h \to \mathbf{O}$ denote the inclusion map and $\pi_h : \mathbf{O} \to \mathbf{Y}_h$ denote the $\mathbf{O}$-orthogonal projector, we have $\mathsf{d}_h^\star = \pi_h\, \mathsf{d}^\star\, \mathfrak{i}_h$. It is direct to check that $\mathsf{d}_h^\star \circ \mathsf{d}_h^\star = 0$, and so we also have a complex structure. The discrete Hodge-Dirac operator $\boldsymbol{\mathcal{D}}_h : \mathbf{Y}_h \to \mathbf{Y}_h$ is given by $\boldsymbol{\mathcal{D}}_h = \mathsf{d} + \mathsf{d}_h^\star$, which coincides with the $\mathbf{O}$-orthogonal projection of $\boldsymbol{\mathcal{D}}$ onto $\mathbf{Y}_h$.

By analogy with the continuous setting, we put

$$\begin{aligned}
\mathbf{G}_h &= \{u \in \mathbf{Y}_h \mid \mathsf{d}u = 0 \quad \text{and} \quad \mathsf{d}_h^\star u = 0\} = \ker \boldsymbol{\mathcal{D}}_h\,,\\
\mathbf{W}_h &= \{w \in \mathbf{Y}_h \mid \mathsf{d}w = 0\} = \ker \mathsf{d}_h = \mathsf{d}\mathbf{Y}_h \oplus \mathbf{G}_h\,,\\
\mathbf{V}_h &= \{v \in \mathbf{Y}_h \mid \forall\, w \in \mathbf{W}_h \mid (v, w) = 0\} = \mathbf{W}_h^\perp \cap \mathbf{Y}_h = \mathsf{d}_h^\star \mathbf{Y}_h\,.
\end{aligned}$$

A *discrete Hodge decomposition* is obtained:

$$\mathbf{Y}_h = \mathbf{V}_h \oplus \mathsf{d}\mathbf{Y}_h \oplus \mathbf{G}_h = \mathsf{d}_h^\star \mathbf{Y}_h \oplus \mathsf{d}\mathbf{Y}_h \oplus \mathbf{G}_h,$$

which is **O**-orthogonal and **X**-orthogonal. This Hodge decomposition is stable, in the sense that uniform Poincaré inequalities hold for d_h and $\mathsf{d}_h^\star$. We delay their statement until next section, as their proofs make use of stable commuting projections.

5.2 Smoothed projections

Assume that the finite element subspaces carry interpolators $\mathcal{I}_h$ (see Section 2.6.2) which commute with the exterior derivative (cf. Proposition 2.6.7). Further assumptions will be required further into this section, but this is enough for now. Typically, $\mathcal{I}_h$ is the interpolator deduced from a choice of a unisolvent system of degrees of freedom. In general, these interpolators may only be defined for regular functions. Furthermore, even if $\mathcal{I}_h$ is defined in the finite element subspaces, it generally fails to satisfy the desired continuity properties. To overcome this issue, several techniques involving averaging/regularization of interpolators have been proposed throughout the years, see for instance [Chr07, AFW06, Sch08, CW08, CMKO11]. Our situation is less involved: we are in a manifold without boundary, which is particularly convenient for smoothing techniques, as they are not completely local, and thereby usually require extension operators in order to smooth near the boundary.

We begin this section by describing some convenient facts that follow from being in a torus, and introduce some notation to ease the task of constructing smoothed projections.

5.2.1 Lifting cells and finite element systems from $\mathcal{S}$ to $\mathbb{R}^n$

Hereon, assume we have fixed a fundamental domain **F** for $\mathcal{S}$. A torus $\mathcal{S} = \mathbb{R}^n \big/ \mathbb{L}$ is a compact topological group, which implies that integration is intrinsically defined with respect to its Haar measure μ; the pushforward of the Lebesgue measure in $\mathbb{R}^n$, restricted to **F**. This allows us to write

$$\int_{\mathcal{S}} f \, d\mu = \int_{\mathbf{F}} \widetilde{f} dx,$$

for $f \in \mathrm{L}^1(\mathcal{S})$. Here, $\widetilde{f}$ denotes the periodic lifting of f to $\mathbb{R}^n$. This relation will be used repeatedly to relate Lebesgue norms in the torus with Lebesgue norms in $\mathbb{R}^n$.

Locally, the canonical projection $\pi : \mathbb{R}^n \to \mathcal{S}$ is an isometric diffeomorphism, because $\mathcal{S}$ carries the quotient topology. Given a set $U \subset \mathcal{S}$, we say it can be **lifted** to $\mathbb{R}^n$ if there exists a set $\widetilde{U}$ of $\mathbb{R}^n$ such that $\pi|_{\widetilde{U}}$ is injective and $\pi(\widetilde{U}) = U$, or equivalently, $\widetilde{U} \cap (\widetilde{U} + \mathbb{L}) = \emptyset$. Intuitively, this means that $\widetilde{U}$ 'fits' in $\mathbf{F}$, up to translation. For instance, it is clear that a *toast-shaped* set in Figure 5.1 does not intersect any of its integer translations and, despite not being contained in a fundamental domain, a translation is enough to put it inside one. As seen in the figure, this is not too restrictive, as a set can be as large as the interior of $\mathbf{F}$ and still satisfy this condition.

Since the torus does not have boundary, there are not cells touching it. Namely, for $T \in \mathcal{T}_h$, the macroelement

$$\mathcal{M}(T) := \{T' \in \mathcal{T}_h \mid T' \cap T \neq \emptyset\}$$

always contains a neighborhood of T and, *a fortiori*, when the lift of $\mathcal{M}(T)$ exists, it contains a neighborhood of $\widetilde{T}$.

We summarize this by saying that, with enough care, one can establish estimates on cells in $\mathcal{S}$ by working with their lifts to $\mathbb{R}^n$.

The previous discussion motivates the following assumption:

Assumption 5.2.1. *For each $h \in \mathcal{H}$ and every cell $T \in \mathcal{T}_h$, the macroelement $\mathcal{M}(T)$ can be lifted to $\mathbb{R}^n$; this lift is denoted by $\widetilde{\mathcal{M}}(\widetilde{T})$. Furthermore, we assume that all the lifts are chosen so that they intersect the same fixed fundamental domain $\mathbf{F}$, which is certainly possible by the periodic nature of π. This implies that they are confined to a fixed compact set of $\mathbb{R}^n$.* ◇

We stress that the previous assumption is just made for rigourosity. Indeed, this will always be the case, because the diameters of cells tend to zero, when the parameter h does.

We emphasize that $\widetilde{\mathcal{M}}(\widetilde{T})$ behaves as a macroelement in $\mathbb{R}^n$, as the process of lifting is isometric. Indeed, we have

$$\widetilde{\mathcal{M}}(\widetilde{T}) = \bigcup_{T' \cap T \neq \emptyset} \widetilde{T}'.$$

It will also be convenient to introduce the notation $\widetilde{\mathcal{T}}_h = \left\{\widetilde{T} \mid T \in \mathcal{T}_h\right\}$.

Remark 5.2.1. When $U \subset \mathcal{S}$ can be lifted to $\widetilde{U} \subset \mathbb{R}^n$, a k-tensor field u on U can be pulled back to a k-tensor field $\widetilde{u} := (\pi|_{\widetilde{U}})^* u$ acting on $\widetilde{U}$. As a consequence of Assumption 5.2.1, this can be done for all the elements of $\mathcal{A}_h^k(T)$, so we may denote

$$\widetilde{\mathcal{A}}_h^k(\widetilde{T}) := (\pi|_{\widetilde{T}})^*(\mathcal{A}_h^k(T)).$$

This pullback preserves regularity and integrability, because π is smooth and locally isometric, which, in particular, implies that the pulled back elements $\widetilde{\mathcal{A}}_h^k(\widetilde{T})$ are subspaces of $\mathrm{H}^1\Lambda^k(\widetilde{T})$.

Notice that the cells included in $\mathcal{M}(T)$ form a cell complex. Since each of these cells carries a local space, we can consider the associated glued space, denoted $\mathcal{A}_h^k(\mathcal{M}(T))$. This space is also pulled back to $\mathbb{R}^n$, and we denote $\widetilde{\mathcal{A}}_h^k(\widetilde{\mathcal{M}}(\widetilde{T})) = \pi^*(\mathcal{A}_h^k(\mathcal{M}(T)))$.

In addition, for each $\widetilde{T} \in \widetilde{\mathcal{T}}_h$, we define an operator $\widetilde{\mathcal{I}}_h(T) \colon \mathrm{H}^1\Lambda^\bullet(\widetilde{T}) \to \widetilde{\mathcal{A}}_h^\bullet(\widetilde{T})$ by requiring that

$$\widetilde{\mathcal{I}}_h(\widetilde{T})(\pi|_{\widetilde{T}})^* = (\pi|_{\widetilde{T}})^* \, \mathcal{I}_h(T)\,.$$

◇

In the remaining of this section q denotes a fixed real number in $[1, \infty)$. Most of the foregoing construction is valid for a generic Lebesgue index q, however, since our discrete spaces are only H^1-conforming, when deriving estimates involving them, we are restricted to use $q = 2$. If the FES were $\mathrm{W}^{1,q}$-conforming, everything would carry out in the same way.

We give a scaling lemma for k-tensor fields in Euclidean space:

Lemma 5.2.1. *For a d-dimensional cell T on $\mathbb{R}^d$ and $\delta > 0$, define $\phi : \mathbb{R}^d \to \mathbb{R}^d$ by $\phi(x) = \delta x + \kappa_T$, where κ_T is a point in T, and set $\hat{T} = \phi^{-1}(T)$. Then*

$$\|\phi^* u\|_{\mathrm{L}^q(\hat{T})} = \delta^{k-d/q} \|u\|_{\mathrm{L}^q(T)}\,, \tag{5.2.1}$$

for all $u \in \mathrm{L}^q(T, \mathcal{L}^k(\mathbb{R}^n))$.

Proof. First, we observe that, for all $x \in \hat{T}$, there holds

$$\forall\, \xi \in \mathbb{R}^d \ : \ (D_x\phi)\xi = \delta\xi\,, \qquad \text{and} \qquad |\det D_x\phi| = \delta^d\,,$$

where we have identified $T_x\hat{T} \equiv \mathbb{R}^d \equiv T_{\phi(x)}T$. Then, for u in $\mathrm{L}^q(T; \mathcal{L}^k(\mathbb{R}^n))$, we have

$$(\phi^* u)[x](\xi_1, \cdots, \xi_k) = u[\phi(x)](D_x\phi\xi_1, \ldots, D_x\phi\xi_k) = \delta^k\, u[\phi(x)](\xi_1, \ldots, \xi_k)\,,$$

whence

$$\phi^* u = \delta^k\, u \circ \phi\,.$$

Now, recall the change of variables formula which, for $f : T \to \mathbb{R}$, states that

$$\int_T f = \int_{\hat{T}} |\det D\phi| f \circ \phi.$$

Choosing $f = |u|^q$, we have

$$f \circ \phi = |u \circ \phi|^q = \delta^{-kq} |\phi^* u|^q,$$

whence

$$\|u\|^q_{\mathrm{L}^q(T)} = \int_T |u|^q = \delta^{d-kq} \int_{\hat{T}} |\phi^* u|^q = \delta^{d-kq} \|\phi^* u\|^q_{\mathrm{L}^q(\hat{T})},$$

which was the desired estimate. □

Scaling by $\phi^{-1} : T \to \hat{T}$ appearing in Lemma 5.2.1 is often referred to as **scaling by dilation and translation**, as there is no rotation involved (cf. Figure 5.2). One usually uses this scaling with $\delta = h_T$ or $\delta = h$, which yields a reference cell of diameter at most 1.

Assumption 5.2.2. *Following [CMKO11, Section 5.3] (see also Remark on p. 64 in [AFW06]), we henceforth assume that the cell complexes and differential elements enjoy the following property: when their lifts to* $\mathbb{R}^n$ *are pulled back by dilation and translation to reference elements of diameter bounded by 1, they belong to compact families of cells with continuously parametrized differential elements. In particular, we assume that the cellular complexes are sufficiently regular, in that*

i) *for all* $0 \leq k \leq n$ *and* $T \in \mathcal{T}_h^k$, *there holds* $h_T^k \preceq |T|$,

ii) *there holds* $h_T \preceq \operatorname{\mathsf{diam}} B_T$, *where* $B_T \subset T$ *is the largest ball such that* T *is star-shaped with respect to* B_T.

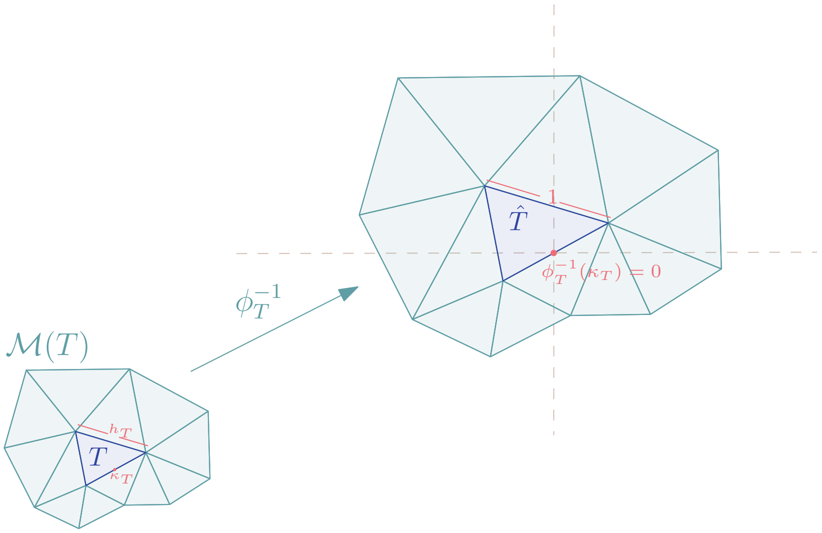

Figure 5.2: T is scaled by dilation and translation, via ϕ_T^{-1}, to $\hat{T}$, a cell of diameter 1.

iii) *the number of cells in the macroelements corresponding to top-dimensional cells is uniformly bounded by above, independently of the parameter* h.

◇

This is the case, for instance, when we have a quasi-uniform or shape-regular family of simplicial complexes together with an affine family of finite elements (cf. [AFW06, Section 5.3] and [CW08, Section 4.3], respectively).

It will also be useful to have an **inverse inequality**:

Lemma 5.2.2. *Given* $u \in \mathbf{Y}_h$, *for each cell* $T \in \mathcal{T}_h^n$, *there holds*

$$|\nabla u|_T \preceq h^{-1} |u|_T.$$

Furthermore, globally

$$|\nabla u| \preceq h^{-1} |u|.$$

Proof. We shall use the scaling lemma on lifted cells. Given $T \in \mathcal{T}_h^n$ with lift $\widetilde{T}$. Moreover, given $u \in \mathcal{A}_h^k(T)$, denote by $\widetilde{u} \in \widetilde{\mathcal{A}}_h^k(T)$ its lift to $\widetilde{T}$. Since $\pi|_{\widetilde{T}}$ is an isometry, by the change of variables formula

$$|\nabla u|_T = |\nabla \widetilde{u}|_{\widetilde{T}} \quad \text{and} \quad |u|_T = |\widetilde{u}|_{\widetilde{T}}, \tag{5.2.2}$$

thereby it suffices to prove the estimate for $\widetilde{T}$.

Consider $\phi = \phi_{\widetilde{T}} : \mathbb{R}^n \to \mathbb{R}^n$ given by $\phi(x) = hx + \kappa_{\widetilde{T}}$, for $\kappa_{\widetilde{T}} \in \widetilde{T}$. Set $\hat{T} = \phi^{-1}(\widetilde{T})$ and $\hat{u} = \phi^* \widetilde{u}$. Then, by equivalence of norms in finite dimensions, we have

$$|\nabla \hat{u}|_{\hat{T}} \leq \|\hat{u}\|_{\mathrm{H}^1(\hat{T})} \leq C(\hat{T}) |\hat{u}|_{\hat{T}}. \tag{5.2.3}$$

Since the differential elements are continuously parametrized (cf. Assumption 5.2.2), if one takes $C(\hat{T})$ to be the smallest constant satisfying the second inequality in (5.2.3), one obtains continuous dependence on cells. Moreover, notice that the mappings ϕ_T map cells T to cells $\hat{T}$ of diameter bounded above by 1, which are anchored at the origin, because $\phi^{-1}(\kappa_T) = 0 \in \hat{T}$ (see Figure 5.2). This, together with Assumption 5.2.2, ensures that the family of reference elements is compact. Thus, since we have a continuous function on a compact set, by the extreme value theorem, we may choose a uniform constant C, independent of $\hat{T}$, such that

$$|\nabla \hat{u}|_{\hat{T}} \leq C |\hat{u}|_{\hat{T}}. \tag{5.2.4}$$

Then, since ϕ is affine, we get that $\phi^* \nabla = \nabla \phi^*$. Therefore, we may use scaling on both sides of (5.2.4), to obtain

$$|\nabla \widetilde{u}|_{\widetilde{T}} \leq C h^{-1} |\widetilde{u}|_{\widetilde{T}}. \tag{5.2.5}$$

One concludes by looking at (5.2.2) and summing over $T \in \mathcal{T}_h^n$. □

Remark 5.2.2. The technique employed in this proof is sometimes called a *compactness argument*. We remark that this is only possible when Assumption 5.2.2 is verified. See, for instance, proof of Theorem 5.3 and remark in p. 64 of [AFW06]. ◇

Remark 5.2.3. From (5.2.5), it is seen that this result remains valid for lifts of cells to $\mathbb{R}^n$. ◇

5.2.2 The construction of smoothed projections

We devote this section to the construction of smoothed projections. In [CMKO11, Section 5.3], Christiansen constructed space-dependent regularizers and suitable extension operators which yield a smoothed projection when composed with a given interpolator. The resulting operators have the desired stability properties, preserve polynomials up to a given degree and commute with the exterior derivative. Our approach is inspired by the approach followed there. However, we use standard convolution, which allows us to obtain more detailed arguments and cleaner estimates, often with explicit constants. As we mentioned earlier, this is possible because the torus does not have boundary, and therefore convolution is always supported in $\mathcal{S}$. We pay special attention to ensuring that everything carries over to the torus.

First, we construct regularizers in $\mathbb{R}^n$, and show that they enjoy stronger stability properties with respect to $\widetilde{\mathcal{I}}_h$ and commute with the exterior derivative; this analysis is done locally. Then, we transfer all the estimates to the torus, and construct the global smoothed projectors.

For the remaining of Part I, p denotes a fixed nonnegative integer. We henceforth assume that the elements $\widetilde{\mathcal{A}}_h^k(T)$, for all $T \in \widetilde{\mathcal{T}}_h$, contains polynomials of degree up to p. Furthermore, we assume that the interpolator $\widetilde{\mathcal{I}}_h$ is the one given by the following result:

Proposition 5.2.1. *In the afore described setting, there is a sequence of commuting interpolators $\widetilde{\mathcal{I}}_h$ satisfying, for all $T \in \widetilde{\mathcal{T}}_h$,*

$$|u - \widetilde{\mathcal{I}}_h u|_T \preceq \sum_{T' \lhd T} h_T^{\ell + (\dim T - \dim T')/2} |\nabla^\ell u|_{T'} \,,$$

for u sufficiently regular and $0 \le \ell \le p+1$.

Proof. This corresponds to Proposition 5.51 in [CMKO11]. □

Remark 5.2.4. The last result may only be used with differential forms which are ℓ times continuously differentiable, as the right-hand side involves evaluations on lower-dimensional cells, including points. Alternatively, one can use it for differential forms which have H^s-regularity, with $s - n/2 > 0$, for traces to be well–defined on subcells of all dimensions. However, by first using the regularizer described in the following, and then this proposition, we bypass this restriction (see Proposition 5.2.8).

◇

The regularizers

We shall use smoothing by convolution. To this end, consider a mollifier ψ in $\mathbb{R}^n$. Namely, $\psi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp} \psi \subset \mathbb{B}^n \qquad \text{and} \qquad \int_{\mathbb{R}^n} \psi = 1.$$

Notice that we are not assuming that $\psi \geq 0$. For $x \in \mathbb{R}^n$ and $\delta > 0$, denote by $\mathbb{B}(x, \delta)$ the ball with center x and radius δ, and by $\mathcal{V}^\delta(U)$ the δ-neighborhood of a subset U of $\mathbb{R}^n$. Moreover, define the usual scaling $\psi_\delta := \delta^{-n}\psi(\cdot/\delta)$. The function ψ_δ inherits the properties of ψ and satisfies $\operatorname{supp} \psi_\delta \subset \mathbb{B}(0, \delta)$. In addition, for $y \in \mathbb{R}^n$, we define the translation $\tau_y : \mathbb{R}^n \to \mathbb{R}^n$ as $\tau_y(x) = x - y$.

In this situation, given a locally integrable (in the sense of Bochner) k-tensor field u in $\mathbb{R}^n$, we define

$$\mathcal{R}^\delta u := \int_{\mathbb{R}^n} \psi_\delta(y)\, \tau_y^* u \, dy = \int_{\mathbb{B}(0,\delta)} \psi_\delta(y)\, \tau_y^* u \, dy \, .$$

It will also be convenient to denote $\mathcal{R} := \mathcal{R}^\delta$, $\mathcal{R}_h^\delta := \mathcal{R}^{\delta h}$ and $\mathcal{V}_h^\delta := \mathcal{V}^{\delta h}$, where $h \in \mathcal{H}$.

We observe that $D_x \tau_y = \mathsf{id}_{\mathbb{R}^n}$, which yields $(\tau_y^* u)[x] = u[x - y]$, for all $x \in \mathbb{R}^n$. This allows us to see that $\mathcal{R}^\delta$ is almost local, in the sense that, when y varies on a small neighborhood, say $\mathbb{B}(0, \delta)$, the evaluation $(\mathcal{R}^\delta u)[x]$ only depends on the values of u on $\mathcal{V}^\delta(x)$. In the following result, this dependence is clarified and a first estimate is established:

Proposition 5.2.2. *Given $\delta > 0$, $x \in \mathcal{R}^n$ and nonnegative integers r, ℓ such that $0 \leq r \leq \ell$. For u in $\mathrm{W}^{\ell - r, 1}(\mathcal{V}^\delta(x); \mathcal{L}^k(\mathbb{R}^n))$, there holds*

$$\|(\nabla^\ell R^\delta u)[x]\| \leq \|\nabla^r \psi_\delta\|_{\mathrm{L}^\infty(\mathbb{B}(0,\delta))} \|\nabla^{\ell - r} u\|_{\mathrm{L}^1(\mathcal{V}^\delta(x))}, \tag{5.2.6}$$

where the norm in the left-hand side is the pointwise tensor norm.

Proof. The key part of the proof is discovering that our operator is intimately related with scalar convolution. Indeed, in the setting of the statement, we have

$$\mathcal{R}^\delta u = \int_{\mathbb{B}(0,\delta)} \psi_\delta(y) (\tau_y^* u)_{i_1 \cdots i_k} \mathsf{d}x^{i_1} \otimes \cdots \otimes \mathsf{d}x^{i_k} \, dy,$$

where summation is implied and $1 \leq i_s \leq n$ are integers. By the previous comment on pullback by translation, we have

$$(\tau_y^* u)_{i_1 \cdots i_k}(x) = u_{i_1 \cdots i_k}(x - y).$$

Therefore, we recognize scalar convolution inside the previous integral. Namely,

$$\mathcal{R}^\delta u = \psi_\delta * u_{i_1 \cdots i_k} \, \mathsf{d}x^{i_1} \otimes \cdots \otimes \mathsf{d}x^{i_k}.$$

This will help us to differentiate. Since we are in $\mathbb{R}^n$, covariant differentiation reduces to standard differentiation, and so we have

$$(\nabla \mathcal{R}^\delta u)_{j i_1 \cdots i_k} = \partial_j (\psi_\delta * u_{i_1 \cdots i_k}).$$

By the usual properties of scalar convolution, we may choose whether to differentiate the mollifier or the coefficients of u. Say we differentiate the mollifier for all $j \in \{1, \ldots, n\}$, then

$$\nabla \mathcal{R}^\delta u = (\partial_j \psi_\delta) * u_{i_1 \cdots i_k} \mathsf{d}x^j \otimes \mathsf{d}x^{i_1} \otimes \cdots \otimes \mathsf{d}x^{i_k}, \tag{5.2.7}$$

where summation is implied over $1 \leq j \leq n$ and all multi-indices indices of length k. From here, we easily get

$$\nabla \mathcal{R}^\delta u = \int_{\mathbb{B}(0,\delta)} (\nabla \psi_\delta)[y] \otimes \tau_y^* u \, dy.$$

Observe that, if we instead differentiate u, we obtain

$$\nabla \mathcal{R}^\delta u = \int_{\mathbb{B}(0,\delta)} \psi_\delta(y) \otimes \nabla \tau_y^* u \, dy = \int_{\mathbb{B}(0,\delta)} \psi_\delta(y) \otimes \tau_y^* (\nabla u) \, dy = \mathcal{R}^\delta \nabla u, \tag{5.2.8}$$

where we used that pullback by τ_y commutes with ∇. Since this reasoning is valid for a tensor of any degree, by replacing u with ∇u repeatedly, identities (5.2.7) and (5.2.8) allow us to deduce that

$$\nabla \mathcal{R}^\delta u = \int_{\mathbb{B}(0,\delta)} (\nabla^r \psi_\delta)[y] \otimes \tau_y^* (\nabla^{\ell - r} u) \, dy$$

for all $0 \leq r \leq \ell$.

All what is left now is bounding. We proceed as follows: first evaluate at x, then apply the pointwise tensor norm followed by the triangle inequality for the Bochner integral and Hölder's inequality. This yields

$$\begin{aligned} \|(\nabla \mathcal{R}^{\delta} u)[x]\| &\leq \int_{\mathbb{B}(0,\delta)} \|(\nabla^{r}\psi_{\delta})[y] \otimes (\nabla^{\ell-r} u)[x-y]\| \\ &\leq \|\nabla^{r}\psi_{\delta}\|_{\mathrm{L}^{\infty}(\mathbb{B}(0,\delta))} \|\tau_{x}(\nabla^{\ell-r}u)\|_{\mathrm{L}^{1}(\mathbb{B}(0,\delta))}. \end{aligned} \tag{5.2.9}$$

In the last estimate, after using Hölder's inequality we used that the L^1-norm and the ball $\mathbb{B}(0,\delta)$ are invariant under reflection. Moreover, there holds

$$\|\tau_{x} v\|_{\mathrm{L}^{1}(\mathbb{B}(0,\delta))} = \|v\|_{\mathrm{L}^{1}(\mathbb{B}(x,\delta))},$$

for any tensor field in $\mathrm{L}^1(\mathbb{B}(x,\delta); \mathcal{L}^{\ell-r+k}(\mathbb{R}^n))$. This, according to (5.2.9) gives the desired estimate. □

We state a simple consequence of this just for the record:

Corollary 5.2.1. *Given a subset U of $\mathbb{R}^n$ and $\delta > 0$. The regularizer $\mathcal{R}^{\delta}$ maps $\mathrm{L}^1(\mathcal{V}^{\delta}(U); \mathcal{L}^k(\mathbb{R}^n))$ into $\mathrm{C}^{\infty}(U; \mathcal{L}^k(\mathbb{R}^n))$. Furthermore, for every nonnegative integer ℓ, $\mathcal{R}^{\delta}$ is linear and bounded from $\mathrm{L}^1(\mathcal{V}^{\delta}(U); \mathcal{L}^k(\mathbb{R}^n))$ into $\mathrm{W}^{\ell,\infty}(U, \mathcal{L}^k(\mathbb{R}^n))$ and satisfies*

$$\|\nabla^{m} \mathcal{R}^{\delta} u\|_{\mathrm{L}^{\infty}(U)} \leq \|\nabla^{m}\psi_{\delta}\|_{\mathrm{L}^{\infty}(\mathbb{B}(0,\delta))} \|u\|_{\mathrm{L}^{1}(\mathcal{V}^{\delta}(U))} \tag{5.2.10}$$

for every $0 \leq m \leq \ell$.

Proof. Fix ℓ and u as in the statement. Choosing $r = \ell$ in Proposition 5.2.2, we get that $R^{\delta}u \in \mathrm{W}^{\ell,\infty}(U; \mathcal{L}^k(\mathbb{R}^n))$ and the estimate (5.2.10) follows from (5.2.6). The smoothness is clear from the previous proposition, but it also can be seen to follow from the Sobolev embedding, using that $\mathcal{R}^{\delta}u \in \mathrm{W}^{\ell,\infty}(U; \mathcal{L}^k(\mathbb{R}^n))$ for each ℓ. □

Similarly to the previous results, we can get a cleaner estimate for norms:

Proposition 5.2.3. *For $\delta > 0$, a subset U of $\mathbb{R}^n$, and $u \in \mathrm{W}^{\ell,q}\Lambda^k(\mathcal{V}^\delta(U))$, there holds*

$$\|\nabla^\ell \mathcal{R}^\delta u\|_{\mathrm{L}^q(U)} \leq \|\nabla^\ell u\|_{\mathrm{L}^q(\mathcal{V}^\delta(U))} .$$

Proof. In Proposition 5.2.2, we saw that $\mathcal{R}^\delta$ is precisely componentwise mollification. Using this, one can prove through a direct computation that Young's inequality for scalar convolution extends to $\mathcal{R}^\delta$. We may use this to write

$$\|\mathcal{R}^\delta v\|_{\mathrm{L}^q(U)} \leq \|\psi_\delta\|_{\mathrm{L}^1(\mathbb{B}(0,\delta))}\|v\|_{\mathrm{L}^q(\mathcal{V}^\delta(U))} = \|v\|_{\mathrm{L}^q(\mathcal{V}^\delta(U))} . \tag{5.2.11}$$

In turn, we also showed that, for $u \in \mathrm{W}^{\ell,q}\Lambda^k(U)$, we have

$$\mathcal{R}^\delta \nabla^\ell u = \nabla^\ell \mathcal{R}^\delta u \,.$$

Plugging this into (5.2.11), we obtain the desired result. □

Now, we show that these regularizers commute with d, as in the case of scalar convolution:

Proposition 5.2.4. *Fix $\delta > 0$ and $x \in \mathbb{R}^n$. Then, for all $u \in \mathrm{H}\Lambda^k(\mathcal{V}^\delta(x))$, there holds*

$$(\mathsf{d}\mathcal{R}^\delta u)[x] = (\mathcal{R}^\delta \mathsf{d}u)[x].$$

Proof. Given u as in the statement, differentiating under the integral sign and using that d commutes with pullbacks, we obtain

$$\mathsf{d}\mathcal{R}^\delta u = \mathsf{d}\int_{\mathbb{B}(0,\delta)} \psi_\delta(y)\tau_y^* u \, dy = \int_{\mathbb{B}(0,\delta)} \psi_\delta(y)\mathsf{d}\tau_y^* u \, dy = \int_{\mathbb{B}(0,\delta)} \psi_\delta(y)\tau_y^* \mathsf{d}u \, dy = \mathcal{R}^\delta \mathsf{d}u.$$

□

Local estimates on cells

We shall use these results to obtain local estimates on cells. To this end, it will be convenient to introduce the following notation: for each cell $T \in \widetilde{\mathcal{T}}_h$ and $h \in \mathcal{H}$, consider $\phi = \phi_T(x) = hx + \kappa_T$, for $\kappa_T \in T$, and put

$$\hat{T} = \phi^{-1}(T), \quad \widehat{\mathcal{A}}^k(\hat{T}) := \phi^*(\widetilde{\mathcal{A}}_h^k(T)), \quad \widehat{\mathcal{M}}(\hat{T}) = \phi^{-1}(\widetilde{\mathcal{M}}(T)) \quad \text{and} \quad \hat{\mathcal{I}}(\hat{T})\phi^* = \phi^*\widetilde{\mathcal{I}}_h(T)\,,$$

they act as differential elements, macroelements and interpolators with respect to the reference cell $\hat{T}$.

First, we show that the regularizers transform well under dilation and translation, this result will be repeatedly used when using scaling:

Proposition 5.2.5. *Fix a parameter h and $T \in \widetilde{\mathcal{T}}_h$. Then, for every $\epsilon > 0$ there holds*

$$\phi_T^* \mathcal{R}_h^\epsilon = \mathcal{R}^\epsilon \phi_T^*\,.$$

Proof. Let u be a fixed element of $\mathrm{L}^q(\mathcal{V}_h^\epsilon(T); \mathcal{L}^k(\mathbb{R}^n))$ and put $\hat{u} = \phi^* u$.

First, we notice that

$$\phi^* \mathcal{R}_h^\epsilon u = \int_{\mathbb{B}(0,\epsilon h)} \psi_{\epsilon h}(y) \phi^* \tau_y^* u \, dy = \int_{\mathbb{B}(0,\epsilon h)} \psi_{\epsilon h}(y) \phi^* \tau_y^* \phi_* \hat{u} \, dy\,,$$

where $\phi_* = (\phi^{-1})^*$ is the pushforward. By contravariance of the pullback, understanding the integrand in the right-hand side reduces to understanding $\phi^{-1}\tau_y\phi$. For $x \in \hat{T}$, we have,

$$(\phi^{-1}\tau_y\phi)(x) = (\phi(x) - y - \kappa_T)/h = (hx - y)/h = \tau_{y/h}(x)\,,$$

whence

$$\phi^* \mathcal{R}_h^\epsilon u = \int_{\mathbb{B}(0,\epsilon h)} \psi_{\epsilon h}(y) \tau_{y/h}^* \hat{u} \, dy.$$

Now, we intend to use change of variables with the map $y \mapsto hy$, which maps $\mathbb{B}(0,\epsilon)$ onto $\mathbb{B}(0,\epsilon h)$. To this end, we notice that

$$|\det D_y(y \mapsto hy)|\, \psi_{\epsilon h}(hy) = h^n \,(\epsilon h)^{-n} \psi(hy/\epsilon h) = \psi_\epsilon(y)\,,$$

whence we obtain

$$\phi^*\mathcal{R}_h^\epsilon u = \int_{\mathbb{B}(0,\epsilon)} \psi_\epsilon(y)\tau_y^* \hat{u}\, dy = \mathcal{R}^\epsilon \hat{u} = \mathcal{R}^\epsilon \phi^* u,$$

as desired. □

Orders of approximation will follow mainly from the following result:

Proposition 5.2.6. *Fix $\epsilon > 0$ and nonnegative integers r and ℓ such that $r \leq \ell$. There is $\boldsymbol{\alpha}(\epsilon, r) > 0$, depending only on ϵ and r, such that, for all $T \in \widetilde{\mathcal{T}}_h$,*

$$h^{r+(n-\dim T)/q} \|\nabla^\ell \mathcal{R}_h^\epsilon u\|_{\mathrm{L}^q(T)} \preceq \boldsymbol{\alpha}(\epsilon, r) \|\nabla^{\ell-r} u\|_{\mathrm{L}^q(\mathcal{V}_h^\epsilon(T))} ,$$

for all $u \in \mathrm{W}^{\ell-r,q}(\mathcal{V}_h^\epsilon(T); \mathcal{L}^k(\mathbb{R}^n))$.

Proof. We first remark that, by definition of ϕ, a direct computation shows that

$$\mathcal{V}^\epsilon(\hat{T}) = \phi^{-1}(\mathcal{V}_h^\epsilon(T)).$$

We will proceed as follows: first, we use (5.2.6) to obtain estimates on reference cells, and then we use Proposition 5.2.5 in combination with scaling (cf. Lemma 5.2.1) to obtain powers of h.

Similarly to Corollary 5.2.1, from estimate (5.2.6), one deduces that

$$\|\nabla^\ell \mathcal{R}^\epsilon \hat{u}\|_{\mathrm{L}^\infty(\hat{T})} \leq \|\nabla^r \psi_\epsilon\|_{\mathrm{L}^\infty(\mathbb{B}(0,\epsilon))} \|\nabla^{\ell-r} \hat{u}\|_{\mathrm{L}^1(\mathcal{V}^\epsilon(\hat{T}))}.$$

Then, by using Hölder's inequality, we obtain

$$\|\nabla^\ell \mathcal{R}^\epsilon \hat{u}\|_{\mathrm{L}^q(\hat{T})} \leq C(\epsilon; \hat{T}) \|\nabla^r \psi_\epsilon\|_{\mathrm{L}^\infty(\mathbb{B}(0,\epsilon))} \|\nabla^{\ell-r} \hat{u}\|_{\mathrm{L}^q(\mathcal{V}^\epsilon(\hat{T}))} ,$$

where $C(\epsilon, \hat{T}) = |\hat{T}|^{1/q} |\mathcal{V}^\epsilon(\hat{T})|^{1-1/q}$.Since we are interested in the behavior when $\epsilon \to 0$, we may assume that ϵ varies in a bounded range, say $(0, 1)$. Therefore, we may bound

$$C(\epsilon, \hat{T}) \leq |\mathbb{B}(0, 1)|,$$

where we used that $\mathsf{diam}\,\hat{T} = h_T/h \le 1$. This allows us to write

$$\|\nabla^\ell \mathcal{R}^\epsilon \hat{u}\|_{\mathrm{L}^q(\hat{T})} \preceq \|\nabla^r \psi_\epsilon\|_{\mathrm{L}^\infty(\mathbb{B}(0,\epsilon))} \|\nabla^{\ell-r} \hat{u}\|_{\mathrm{L}^q(\mathcal{V}^\epsilon(\hat{T}))}\,. \tag{5.2.12}$$

Now, notice that, for scaling purposes, $\mathcal{V}^\epsilon(\hat{T})$ is an n-dimensional cell, independently of the dimension of $\hat{T}$. Therefore, by using Lemma 5.2.1, we obtain

$$\|\nabla^{\ell-r} \hat{u}\|_{\mathrm{L}^q(\mathcal{V}^\epsilon(\hat{T}))} = \|\phi^* \nabla^{\ell-r} u\|_{\mathrm{L}^q(\mathcal{V}^\epsilon(\hat{T}))} = h^{\ell-r+k-n/q} \|\nabla^{\ell-r} u\|_{\mathrm{L}^q(\mathcal{V}^\epsilon_h(T))}. \tag{5.2.13}$$

In turn, using Proposition 5.2.5 and Lemma 5.2.1, we have

$$\|\nabla^\ell \mathcal{R}^\epsilon \hat{u}\|_{\mathrm{L}^q(\hat{T})} = \|\phi^* \nabla^\ell \mathcal{R}^\epsilon_h u\|_{\mathrm{L}^q(\hat{T})} = h^{\ell+k-\dim T/q} \|\nabla^\ell \mathcal{R}^\epsilon_h u\|_{\mathrm{L}^q(T)}. \tag{5.2.14}$$

Finally, replacing (5.2.13) and (5.2.14) into (5.2.12) and canceling out powers of h, we obtain

$$h^{-\dim T/q} \|\nabla^\ell \mathcal{R}^\epsilon_h u\|_{\mathrm{L}^q(T)} \preceq \boldsymbol{\alpha}(\epsilon, r) h^{-r-n/q} \|\nabla^{\ell-r} u\|_{\mathrm{L}^q(\mathcal{V}^{\epsilon h}_h(T))},$$

which proves the desired estimate, with $\boldsymbol{\alpha}(\epsilon, r) := \|\nabla^r \psi_\epsilon\|_{\mathrm{L}^\infty(\mathbb{B}(0,\epsilon))}$. □

Remark 5.2.5. To get optimal orders of approximation we only need the previous estimate for $r = 0$. In this case, alternatively, one may simply use Proposition 5.2.3. We chose to give this proof because it produces a more general estimate, which could be useful, for instance, for inverse estimates with respect to h. We also notice that, from (5.2.12), one can get an estimate with the left and right-hand sides having different Lebesgue norms, which also might be useful. The downside of this approach is that the bound depends on ϵ. ◇

According to the last remark, we also have the following result, which, although requires more regularity on u, gives a bound that does not depend on the regularization parameter.

Proposition 5.2.7. *Fix $\epsilon > 0$ and a nonnegative integer ℓ. Then, for all $T \in \widetilde{\mathcal{T}}_h$, there holds*

$$h^{r+(n-\dim T)/q} \|\nabla^\ell \mathcal{R}^\epsilon_h u\|_{\mathrm{L}^q(T)} \le \|\nabla^\ell u\|_{\mathrm{L}^q(\mathcal{V}^\epsilon(T))},$$

for $u \in \mathrm{W}^\ell(\mathcal{V}^\epsilon_h(T); \mathcal{L}^k(\mathbb{R}^n))$.

Proof. See Remark 5.2.5. $\square$

The last result, combined with Proposition 5.2.1, gives optimal orders of approximation for regularized forms. As remarked previously, since we are using the discrete spaces, we are forced to take $q = 2$.

Proposition 5.2.8. *Fix $\epsilon > 0$ and a nonnegative integer $\ell \leq p+1$. Then, for all $T \in \widetilde{\mathcal{T}}_h$ and $u \in \mathrm{H}^\ell\Lambda^k(\mathcal{V}_h^\epsilon(T))$, there holds*

$$|\mathcal{R}_h^\epsilon u - \widetilde{\mathcal{I}}_h \mathcal{R}_h^\epsilon u|_T \preceq h^\ell \sum_{T' \lhd T} |\nabla^\ell u|_{\mathcal{V}_h^\epsilon(T')}. \tag{5.2.15}$$

Proof. Fix $T \in \widetilde{\mathcal{T}}_h^n$ and a nonnegative integer ℓ. Given $u \in \mathrm{H}^\ell(\mathcal{V}_h^\epsilon(T))$ we use Proposition 5.2.1, and subsequently Proposition 5.2.7 to get

$$|\mathcal{R}_h^\epsilon u - \widetilde{\mathcal{I}}_h \mathcal{R}_h^\epsilon u|_T \preceq \sum_{T' \lhd T} h^{\ell + (n - \dim T')/2} |\nabla^\ell \mathcal{R}_h^\epsilon u|_{T'} \leq \sum_{T' \lhd T} h^\ell |\nabla^\ell u|_{\mathcal{V}_h^\epsilon(T')}\,.$$

$\square$

The following lemma regarding scalar convolution will be useful:

Lemma 5.2.3. *Let $\delta > 0$ and U a subset of $\mathbb{R}^n$. Then, for any scalar function $f \in \mathrm{W}^{1,q}(\mathcal{V}^\delta(U))$, there holds*

$$\|\psi_\delta * f - f\|_{\mathrm{L}^q(U)} \leq \delta \|\operatorname{\mathsf{grad}} f\|_{\mathrm{L}^q(\mathcal{V}^\delta(U))}.$$

Proof. Since ψ_δ integrates to 1, we have

$$\psi_\delta * f - f = \int_{\mathbb{B}(0,\delta)} \psi_\delta(y)(f\tau_y - f)\, dy. \tag{5.2.16}$$

In turn, for $x \in U$, there holds

$$(f\tau_y - f)(x) = f(x-y) - f(x) = -\int_0^1 y \cdot \mathsf{grad}(f)(x - ty)\, dt\,,$$

whence

$$|(f\tau_y - f)(x)| \leq |y| \int_0^1 |\,\mathsf{grad}(f)(x-ty)|\, dt. \tag{5.2.17}$$

Passing to the L^q-norm in (5.2.16) and then using (5.2.17), we obtain

$$\begin{aligned}\|\psi_\delta * f - f\|_{\mathrm{L}^q(U)} &\leq \int_{\mathbb{B}(0,\delta)} \psi_\delta(y)\|f\tau_y - f\|_{\mathrm{L}^q(U)}\,dy\\ &\leq \int_{\mathbb{B}(0,\delta)} \psi_\delta(y)|y|\int_0^1 \|\operatorname{\mathsf{grad}}(f)(\cdot - ty)\|_{\mathrm{L}^q(U)}\,dt\,dy\\ &\leq \int_{\mathbb{B}(0,\delta)} \psi_\delta(y)|y|\|\operatorname{\mathsf{grad}}(f)\|_{\mathrm{L}^q(\mathcal{V}^\delta(U))}\,dy\\ &\leq \delta\|\operatorname{\mathsf{grad}}(f)\|_{\mathrm{L}^q(\mathcal{V}^\delta(U))},\end{aligned}$$

where in the third inequality we used the behavior of L^q-norms under translation. □

This estimate can be easily transferred to differential k-forms:

Lemma 5.2.4. *Consider a subset U of $\mathbb{R}^n$ and $\delta > 0$. Then, for all $u \in \mathrm{H}^1\Lambda^k(\mathcal{V}^\delta(U))$, there holds*

$$|u - \mathcal{R}^\delta u|_U \leq \delta\|\nabla u\|_{\mathcal{V}^\delta(U)}.$$

Proof. Consider u as in the statement. First, we observe that the pointwise norm of u, satisfies

$$|u[x]|^2 = \langle u[x], u[x]\rangle = v_{i_1\cdots i_k}(x)u_{j_1\cdots j_k}(x)\left\langle \mathsf{d}x^{i_1}\cdots\mathsf{d}x^{i_k}, \mathsf{d}x^{j_1}\cdots\mathsf{d}x^{j_k}\right\rangle = \sum_{1\leq i_1<\cdots<i_k\leq n} u_{i_1\cdots i_k}(x)^2,$$

where summation is over all the strictly increasing multi-indices of length k, and we used that the canonical basis of $\mathbb{R}^n$ is orthonormal. Integrating this identity over U, we deduce

$$|u|_U = \|[u]\|_U, \tag{5.2.18}$$

where $[u] : \mathcal{V}^\delta(U) \to \mathbb{R}^{\binom{n}{k}}$ is the vector-valued function formed by the coefficients, and $\|\cdot\|_U$ denotes its L^2-norm in U. Similarly, we have

$$|\mathcal{R}^\delta u|_U = \|\,[\mathcal{R}^\delta u]\,\|_U\,. \tag{5.2.19}$$

Now, recalling from the proof of Proposition 5.2.2 that $(\mathcal{R}^\delta u)_{i_1\cdots i_k} = \psi_\delta * u_{i_1\cdots i_k}$, we use

Lemma 5.2.3 to write

$$\|\psi_\delta * u_{i_1\cdots i_k} - u_{i_1\cdots i_k}\|_U \leq \delta \|\,\mathsf{grad}(u_{i_1\cdots i_k})\|_{\mathcal{V}^\delta(U)}.$$

In turn, in the proof of Proposition 5.2.2 we saw that $(\nabla u)_{ji_1\ldots i_k} = \partial_j u_{i_1\cdots i_k}$. Therefore, summing over all the increasing multi-indices, the previous estimate, together with (5.2.18) and (5.2.19), gives

$$|\mathcal{R}^\delta u - u|_T \leq \delta |\nabla u|_{\mathcal{V}^\delta(T)}, \tag{5.2.20}$$

the desired estimate. □

From now on, in order to work with forms in the discrete spaces, we further restrict the possible range of ϵ:

Assumption 5.2.3. *ϵ is small enough so that for all $T \in \widetilde{\mathcal{T}}_h^n$, there holds $\mathcal{V}_h^\epsilon(T) \subset \widetilde{\mathcal{M}}(T)$. In the following, we call the interval of such values the* **admissible range***.* ◇

Proposition 5.2.9. *Fix ϵ in the admissible range. Then, for all $T \in \widetilde{\mathcal{T}}_h$, there holds*

$$|\mathcal{R}_h^\epsilon u - \widetilde{\mathcal{I}}_h \mathcal{R}_h^\epsilon u|_T \preceq h^\ell |\nabla^\ell u|_{\widetilde{\mathcal{M}(T)}},$$

for $u \in \mathrm{H}^\ell(\widetilde{\mathcal{M}}(T))$.

Proof. Fix T in $\mathcal{T}_h^n$ and recall that, since ϵ is in the admissible range, we have $\mathcal{V}_h^\epsilon(T) \subset \widetilde{\mathcal{M}}(T)$. Therefore, for $u \in \mathrm{H}^\ell(\widetilde{\mathcal{M}}(T))$, using Proposition 5.2.8, we get

$$|\mathcal{R}_h^\epsilon u - \widetilde{\mathcal{I}}_h \mathcal{R}_h^\epsilon u|_T \preceq h^\ell \sum_{T' \lhd T} |\nabla^\ell u|_{\widetilde{\mathcal{M}(T')}}\,.$$

In turn, the number of cells in a macroelement is uniformly bounded (cf. Assumption 5.2.2) by above. So, using that $\widetilde{\mathcal{M}}(T') \subset \widetilde{\mathcal{M}}(T)$, we obtain

$$|\mathcal{R}_h^\epsilon u - \widetilde{\mathcal{I}}_h \mathcal{R}_h^\epsilon u|_{\widetilde{T}} \preceq h^\ell |\nabla^\ell u|_{\widetilde{\mathcal{M}(T)}}.$$

□

It will be very helpful to have a mollifier that preserves polynomials of degree up to p:

Assumption 5.2.4. *The mollifier ψ is chosen so as to satisfy*

$$\int_{\mathbb{B}(0,1)} \psi(y) f(y)\, dy = f(0)\,, \tag{5.2.21}$$

for every polynomial f of degree p in $\mathbb{R}^n$. ◇

This ensures that the regularizers preserve polynomial differential forms, as showed in the following result:

Proposition 5.2.10. *The regularizers preserve polynomial differential forms of degree up to p.*

Proof. Fix $x \in \mathbb{R}^n$ and $\delta > 0$, and consider a differential k-form u which is polynomial of degree at most p in $\mathcal{V}^\delta(x)$. Recall that the regularizer smooths componentwise. More precisely, we have $(\mathcal{R}^\delta u)_{i_1 \cdots i_k} = \psi_\delta * u_{i_1 \cdots i_k}$.

By assumption, all the coefficients of u are polynomials of degree up to p. In general, for a polynomial f in $\mathcal{V}^\delta(x)$, we have that $y \mapsto f(x-y)$ is polynomial of the same degree, and its evaluation at $y = 0$ is $f(x)$. Therefore, by the assumption, we have

$$(\psi_\delta * f)(x) = \int_{\mathbb{B}(0,\delta)} \psi_\delta(y) f(x-y)\, dy = f(x)\,.$$

Applying this to $f = u_{i_1 \cdots i_k}$ for each multi-index, and reconstructing $\mathcal{R}^\delta u$, we get

$$(\mathcal{R}^\delta u)[x] = (\psi_\delta * u_{i_1 \cdots i_k})(x) \mathsf{d}x^{i_1} \wedge \cdots \wedge \mathsf{d}x^{i_k} = u_{i_1 \cdots i_k}(x) \mathsf{d}x^{i_1} \wedge \cdots \wedge \mathsf{d}x^{i_k} = u[x]\,,$$

as desired. □

Remark 5.2.6. Mollifiers satisfying Assumption 5.2.4 may be constructing by a general procedure. Given a standard mollifier

$$\bar{\psi} \in C_c^\infty(\mathbb{R}^n)\,, \qquad \bar{\psi} \geq 0, \qquad \operatorname{\mathsf{supp}} \bar{\psi} \subset \mathbb{B}^n \qquad \text{and} \qquad \int_{\mathbb{R}^n} \bar{\psi} = 1\,.,$$

One seeks a polynomial P of degree at most p such that $\psi := P\bar{\psi}$ satisfies Assumption 5.2.4. Using the monomial basis, we may write $P(x) = c_I x^I$, where $x^I = x_1^{i_1} \cdots x_n^{i_n}$, which allows us to rewrite condition (5.2.21) as:

$$\sum_I c_I \int_{\mathbb{B}(0,1)} \bar{\psi}(x) x^I x^J = \delta_{0,J} ,$$

for every multi-index J of length n and size up to p. Since $\bar{\psi}$ is nonnegative and not identically zero, the left-hand side is defined by a symmetric, positive definite matrix. Therefore , the coefficients c_I exist and are uniquely determined by the choice of $\bar{\psi}$. ◇

In what follows, given a domain U in $\mathbb{R}^n$, we denote

$$\rho_{\mathsf{max}} = \sup \{\rho \mid \ U \text{ is star-shaped with respect to a ball of radius } \rho\} \ ,$$

and

$$\gamma(U) = \mathsf{diam}\, U/\rho_{\mathsf{max}} \, .$$

Moreover, if U is star-shaped with respect to a ball $B \subset U$, or f a scalar function in $\mathrm{W}^{\ell,q}(U)$, we let $T^\ell f$ denote the Taylor polynomial of order ℓ averaged over B, f. We recall that $T^\ell f$ is a polynomial of degree at most $\ell - 1$. See [BS08, Section 4.1] for details about this topic. For a differential k-form $u = u_{i_1 \cdots i_k} \mathsf{d}x^{i_1} \wedge \cdots \wedge \mathsf{d}x^{i_k} \in \mathrm{L}^1\Lambda^k(U)$, we denote

$$\boldsymbol{T}^\ell u = (T^\ell u_{i_1} \cdots i_k)\, \mathsf{d}x^{i_1} \wedge \cdots \wedge \mathsf{d}x^{i_k}.$$

The standard estimates for averaged Taylor polynomials can be transferred to differential forms. We have the following result, which is a variant of Bramble-Hilbert for differential forms:

Lemma 5.2.5. *Let U be a domain in $\mathbb{R}^n$, which is star-shaped with respect to a ball $B \subset U$ of radius $\rho > 1/2\rho_{\mathsf{max}}$. Then, there is $C_{\ell,n,\gamma} > 0$ such that*

$$|u - \boldsymbol{T}^\ell u|_U \leq C_{\ell,n,\gamma(U)}\, \mathsf{diam}(U)^\ell |\nabla^\ell u|_U \, ,$$

for $u \in \mathrm{H}^\ell\Lambda^k(U)$.

Proof. This is a simple application of Lemma 4.3.8 in [BS08]. Indeed, fixed u, from our definition of $\boldsymbol{T}^\ell u$, we may apply it componentwise to obtain

$$|u-\boldsymbol{T}^\ell u|_U^2 = \sum_{1\le i_1<\cdots i_k\le n} |u_{i_1\cdots i_k} - T^\ell u_{i_1\cdots i_k}|_U^2 \le C^2_{\ell,n,\gamma(U)}\,\mathsf{diam}(U)^{2\ell} \sum_{1\le i_1<\cdots i_k\le n} |\nabla^\ell u_{i_1\cdots i_k}|_U^2 .$$

There holds that $\nabla^\ell u = (\nabla^\ell u_{i_1\cdots i_k}) \otimes \mathsf{d}x^{i_1} \wedge \cdots \wedge \mathsf{d}x^{i_k}$, which gives

$$|u-\boldsymbol{T}^\ell u|_U \le C_{\ell,n,\gamma(U)}\,\mathsf{diam}(U)^\ell |\nabla^\ell u|_U\,,$$

as desired. ☐

Proposition 5.2.11. *Consider ϵ in the admissible range. Then, for every $T \in \widetilde{\mathcal{T}}_h^n$ and $u \in \mathrm{H}^\ell\Lambda^k(\widetilde{\mathcal{M}}(T))$, there holds*

$$|u-\mathcal{R}_h^\epsilon u|_T \preceq h^\ell |\nabla^\ell u|_{\widetilde{\mathcal{M}}(T)} .$$

Proof. Firs, observe that, for $u \in \mathrm{H}^\ell\Lambda^k(\widetilde{\mathcal{M}}(T))$, using Proposition 5.2.3, we get

$$|\mathcal{R}_h^\epsilon u|_T \le |u|_{\widetilde{\mathcal{M}}(T)} .$$

Then, since the regularizers preserve polynomials (see Proposition 5.2.10), we may write

$$|u-\mathcal{R}_h^\epsilon u|_T \le |u-\boldsymbol{T}^\ell u|_T + |\mathcal{R}_h^\epsilon \boldsymbol{T}^\ell u - \mathcal{R}_h^\epsilon u|_T \preceq |u-\boldsymbol{T}^\ell u|_{\widetilde{\mathcal{M}}(T)} .$$

Now, using Lemma 5.2.5, we get

$$|u-\mathcal{R}_h^\epsilon u|_T \preceq C_{\ell,n,\gamma(T)} h^\ell |\nabla^\ell u|_{\widetilde{\mathcal{M}}(T)}\,,$$

where we used that the number of cells in a macroelement is uniformly bounded. We finish by noticing that, by Assumption 5.2.2, we have $\gamma(T) \preceq 1$ (cf. [BS08, Remark 4.4.17, i)]). The result follows. ☐

The following results concern the last local ingredient for smoothed projections. Notice that they only involve estimates on top-dimensional cells, which is due to the lack of regularity of

$\mathbf{Y}_h$ in subcells.

Proposition 5.2.12. *Consider* $\delta > 0$ *and* U *an open subset of* $\mathbb{R}^n$*. Suppose we are given a finite-dimensional subspace* $\mathbf{A}$ *of* $\mathrm{W}^{\ell,q}\Lambda^k(\mathcal{V}^\delta(U))$*. Then, for all* $\epsilon' > 0$*, there exists a threshold* $\bar{\epsilon} \in (0,\delta)$*, such that for all* $\epsilon \in (0,\bar{\epsilon})$*, there holds*

$$\|\mathcal{R}^\epsilon u - u\|_{\mathrm{W}^{\ell,q}(U)} \leq \epsilon' |u|_{\mathrm{L}^q(\mathcal{V}^\delta(U))}\,,$$

for all $u \in \mathbf{A}$ *.*

Proof. Since $\mathbf{A}$ is finite-dimensional, the unit ball

$$\mathbf{B} = \Big\{ u \in \mathbf{A} \mid \|u\|_{\mathrm{L}^q(\mathcal{V}^\delta(U))} < 1 \Big\}$$

is precompact in the topology of $\mathrm{W}^{\ell,q}\Lambda^k(\mathcal{V}^\delta(U))$. This, together with the fact that $\mathrm{C}^\infty\Lambda^k(\overline{\mathcal{V}^\delta(U)})$ is dense in $\mathrm{W}^{\ell,q}\Lambda^k(\mathcal{V}^\delta(U))$, ensures that, for all $\xi > 0$, there is a positive integer N and a finite subset $\{u_i\}_{i=1}^N$ of $\mathrm{C}^\infty\Lambda^k(\overline{\mathcal{V}^\delta(U)})$ such that, for every $u \in \bar{\mathbf{B}}$ there is $j \in \{1,\dots,N\}$ satisfying

$$\|u_j - u\|_{\mathrm{W}^{\ell,q}(\mathcal{V}^\delta(U))} < \xi\,. \tag{5.2.22}$$

In particular, fixed $\epsilon' > 0$, there is a set $\{u_j\}_{j=1}^N$ satisfying (5.2.22) with $\xi = \epsilon'/r$.

In turn, by convergence of mollification, we may choose $\epsilon \in (0,\delta)$ so small that

$$\|\mathcal{R}^\epsilon u_j - u_j\|_{\mathrm{W}^{\ell,q}(U)} < \epsilon'/3, \tag{5.2.23}$$

for all $j \in \{1,\dots,N\}$. Moreover, for $u \in \bar{\mathbf{B}}$, using Proposition 5.2.3 and (5.2.22), we obtain

$$\|\mathcal{R}^\epsilon(u - u_j)\|_{\mathrm{W}^{\ell,q}(U)} \leq \|u - u_j\|_{\mathrm{W}^{\ell,q}(\mathcal{V}^\epsilon(U))} < \epsilon'/3\,, \tag{5.2.24}$$

for some $j \in \{1,\dots,N\}$. We then write

$$\|\mathcal{R}^\epsilon u - u\|_{\mathrm{W}^{\ell,q}(U)} \leq \|\mathcal{R}^\epsilon(u - u_j)\|_{\mathrm{W}^{\ell,q}(U)} + \|\mathcal{R}^\epsilon u_j - u_j\|_{\mathrm{W}^{\ell,q}(U)} + \|u_j - u\|_{\mathrm{W}^{\ell,q}(U)}.$$

Now, using (5.2.23) together with (5.2.24), we obtain

$$\|\mathcal{R}^\epsilon u - u\|_{\mathrm{W}^{\ell,q}(U)} < \epsilon' .$$

For nonzero $u \in \mathbf{A}$, we notice that $u/\|u\|_{\mathrm{L}^q(\mathcal{V}^\delta(U))}$ belongs to $\bar{\mathbf{B}}$, so we may use the previous estimate to conclude the desired result. $\square$

We need a last assumption on the interpolators:

Assumption 5.2.5. *The interpolators are uniformly bounded* $\mathrm{H}^1\Lambda^\bullet(T) \to \mathrm{L}^2\Lambda^\bullet(T)$*. Namely, there is* $\boldsymbol{\beta} > 0$ *such that*

$$\|\widetilde{\mathcal{I}}_h(T)\|_{\mathrm{H}^1(T)\to\mathrm{L}^2(T)} \leq \boldsymbol{\beta},$$

for all $T \in \widetilde{\mathcal{T}}_h$ *and any parameter* h. $\diamond$

Proposition 5.2.13. *For* $\epsilon' > 0$*, there exists a threshold* $\bar{\epsilon} \in (0,1)$ *in the admissible range, such that for all* $\epsilon \in (0,\bar{\epsilon})$ *and* $T \in \widetilde{\mathcal{T}}_h^n$*, there holds*

$$|u - \widetilde{\mathcal{I}}_h \mathcal{R}_h^\epsilon u|_T \leq \epsilon' |u|_{\widetilde{\mathcal{M}(T)}} ,$$

for all $u \in \mathcal{A}_h^k(\widetilde{\mathcal{M}}(T))$ *.*

Proof. By Proposition (5.2.12), ϵ' and δ in the admissible range, there is $\epsilon > 0$ such that

$$\|\mathcal{R}^\epsilon \hat{u} - \hat{u}\|_{\mathrm{H}^1(\hat{T})} \leq \frac{\epsilon'}{\boldsymbol{\beta}} |\hat{u}|_{\widehat{\mathcal{M}(\hat{T})}} ,$$

for all $\hat{u} \in \widehat{\mathcal{A}}^k(\widehat{\mathcal{M}}(\hat{T}))$.

Now, observing that $\|\hat{\mathcal{I}}(\hat{T})\|_{\mathrm{H}^1(\hat{T})\to\mathrm{L}^2(\hat{T})} = \|\widetilde{\mathcal{I}}_h(T)\|_{\mathrm{H}^1(T)\to\mathrm{L}^2(T)}$, from the last assumption we get

$$|\hat{\mathcal{I}}\mathcal{R}^\epsilon \hat{u} - \hat{u}|_{\hat{T}} \leq \|\hat{\mathcal{I}}(\hat{T})\|_{\mathrm{H}^1(\hat{T})\to\mathrm{L}^2(\hat{T})} \|\mathcal{R}^\epsilon \hat{u} - \hat{u}\|_{\mathrm{H}^1(\hat{T})} \leq \boldsymbol{\beta} \|\mathcal{R}^\epsilon \hat{u} - \hat{u}\|_{\mathrm{H}^1(\hat{T})}.$$

Then, combining both estimates, we obtain

$$|\hat{\mathcal{I}}\mathcal{R}^\epsilon \hat{u} - \hat{u}|_{\hat{T}} \leq \epsilon' |\hat{u}|_{\widehat{\mathcal{M}(\hat{T})}} .$$

As in previous proofs, scaling back to T finishes the proof. □

Now, given $\epsilon > 0$ and a paremeter h, for all $T \in \widetilde{\mathcal{T}}_h$, we define an operator

$$\mathcal{Q}_h^\epsilon(T) := \widetilde{\mathcal{I}}_h(T)\mathcal{R}_h^\epsilon : \mathrm{L}^1\Lambda^k(\widetilde{\mathcal{M}}(T)) \to \widetilde{\mathcal{A}}_h^k(T)\,.$$

The following result translates all the previous results into their analogues for $\mathcal{Q}_h^\epsilon$.

Proposition 5.2.14. *Given $\epsilon' > 0$, fix $\epsilon \in (0, \bar{\epsilon})$. Then, for $T \in \widetilde{\mathcal{T}}_h^n$ the following estimates hold:*

i) *for all $u \in \mathrm{H}^\ell\Lambda^k(\widetilde{\mathcal{M}}(T))$, we have*

$$|u - \mathcal{Q}_h^\epsilon u|_T \preceq h^\ell |\nabla^\ell u|_{\widetilde{\mathcal{M}}(T)}\,. \tag{5.2.25}$$

ii) *for all $u \in \widetilde{\mathcal{A}}_h^k(\widetilde{\mathcal{M}}(T))$, we have*

$$|u - \mathcal{Q}_h^\epsilon u|_T \leq \epsilon' |u|_{\widetilde{\mathcal{M}}(T)}. \tag{5.2.26}$$

iii) *for all $x \in \mathbb{R}^n$ and $u \in \mathrm{H}\Lambda^k(\mathcal{V}_h^\epsilon(x))$, there holds*

$$(\mathcal{Q}_h^\epsilon \mathsf{d}u)[x] = (\mathsf{d}\mathcal{Q}_h^\epsilon u)[x]. \tag{5.2.27}$$

Proof. For the first item, for $u \in \mathrm{H}^\ell(\widetilde{\mathcal{M}}(T))$, we write

$$|u - \mathcal{Q}_h^\epsilon u|_T \leq |\mathcal{R}_h^\epsilon u - \mathcal{Q}_h^\epsilon u|_T + |u - \mathcal{R}_h^\epsilon u|_T.$$

The first estimate then follows directly from propositions 5.2.9 and 5.2.11.The second estimate is exactly the one given by Proposition 5.2.13. Lastly, the third item follows directly from Proposition 5.2.4, together with the fact that $\widetilde{\mathcal{I}}_h$ commutes with d. □

The operators $\mathcal{Q}_h^\epsilon$ have almost all the desired properties: they have higher continuity, commute with the exterior derivative, and enjoy optimal approximation properties. However, they lack a fundamental property; they are not projections, as they involve a smoothing step. To solve this problem, one needs to pass to the global setting. In our context, this amounts to, first, passing from $\mathbb{R}^n$ to the torus, assembling a global operator, and then correcting the projection issue.

Estimates on the torus

As done previously for the interpolators, we define counterparts for $\mathcal{R}_h^\epsilon$ and $\mathcal{Q}_h^\epsilon$ acting on forms in the torus. This is done via pushforward; namely, for ϵ in the admissible range, $h \in \mathcal{H}$, and $T \in \mathcal{T}_h$ with lift $\widetilde{T}$, we define:

$$\mathcal{R}_{\mathcal{S},h}^\epsilon = \mathcal{R}_{\mathcal{S},h}^\epsilon(T) \colon \mathrm{L}^1\Lambda^k(\mathcal{M}(T)) \to \mathrm{H}^1\Lambda^k(T) \quad \text{by requiring} \quad \pi^*\mathcal{R}_{\mathcal{S},h}^\epsilon = \mathcal{R}_h^\epsilon \pi^*,$$

and

$$\mathcal{Q}_{\mathcal{S},h}^\epsilon = \mathcal{Q}_{\mathcal{S},h}^\epsilon(T) \colon \mathrm{L}^1\Lambda^k(\mathcal{M}(T)) \to \mathcal{A}_h^k(T) \quad \text{by requiring} \quad \pi^*\mathcal{Q}_{\mathcal{S},h}^\epsilon = \mathcal{Q}_h^\epsilon \pi^*.$$

Here, we have omitted the restrictions of π to $\widetilde{T}$ and $\widetilde{\mathcal{M}}(\widetilde{T})$, as it should be clear from the domain and codomain of the operators which domain of π is considered for pulling back.

Notice that, by definition, we have

$$\mathcal{Q}_{\mathcal{S},h}^\epsilon = \pi_* \mathcal{Q}_h^\epsilon \pi^* = \pi^{-*}\widetilde{\mathcal{I}}_h \mathcal{R}_h^\epsilon \pi^* = \pi_* \widetilde{\mathcal{I}}_h \pi^* \mathcal{R}_{\mathcal{S},h}^\epsilon = \mathcal{I}_h \mathcal{R}_{\mathcal{S},h}^\epsilon,$$

where we omitted the dependence on T to simplify the notation. Since the interpolators $\mathcal{I}_h$ glue together to give a global operator, this identity ensures that $\mathcal{Q}_{\mathcal{S},h}^\epsilon$ does as well. Namely, there is an induced operator

$$\mathcal{Q}_{\mathcal{S},h}^\epsilon \colon \mathrm{L}^1\Lambda^k(\mathcal{S}) \to \mathbf{Y}_h.$$

The following result states the local estimates in the torus, which are simple consequences of the previous discussion. In the following, we shall denote by $\mathcal{V}_{\mathcal{S},h}^\epsilon$ the (ϵh)-neighborhood of a set in the torus.

Proposition 5.2.15. *Fix $\epsilon' \in (0,1)$ and $\epsilon \in (0,\bar{\epsilon})$. Then, for every cell $T \in \mathcal{T}_h^n$, there hold:*

i) *for all* $u \in \mathrm{H}^{\ell}\Lambda^k(\mathcal{M}(T))$*, we have*

$$|u - \mathcal{Q}^{\epsilon}_{\mathcal{S},h}u|_T \preceq h^{\ell}|\nabla^{\ell}u|_{\mathcal{M}(T)}.$$

ii) *for all* $u \in \mathcal{A}^k_h(\mathcal{M}(T))$*, we have*

$$|u - \mathcal{Q}^{\epsilon}_{\mathcal{S},h}u| \leq \epsilon'|u|_{\mathcal{M}(T)}.$$

iii) *for all* $x \in \mathcal{S}$ *and* $u \in \mathrm{H}\Lambda^k(\mathcal{V}^{\epsilon}_{\mathcal{S},h}(x))$*, we have*

$$(\mathcal{Q}^{\epsilon}_{\mathcal{S},h}\mathsf{d}u)[x] = (\mathsf{d}\mathcal{Q}^{\epsilon}_{\mathcal{S},h}u)[x].$$

Proof. The statements follow directly from Proposition 5.2.14, using that $\mathcal{Q}^{\epsilon}_{\mathcal{S},h}$ and $\widetilde{\mathcal{M}}(\widetilde{T})$ were defined via pullback, recalling that π is a local isometry, and that d commutes with pullbacks. These facts were used several times in this section, so we omit the details. □

The following result is concerned with the corresponding global estimates. There, C denotes the constant appearing in the following bound:

$$\sum_{T \in \mathcal{T}^n_h} |u|_{\mathcal{M}(T)} \leq C|u| \qquad \forall\, u \in \mathrm{L}^2\Lambda^k(\mathcal{S})\,,$$

which holds because the cell complexes are assumed to be sufficiently regular (cf. Assumption 5.2.2).

Proposition 5.2.16. *Given* $\epsilon' \in (0,1)$*, we take* $\epsilon \in (0, \bar{\epsilon}/C)$*. Then, we have the following estimates:*

(i) for all $u \in \mathrm{H}^{\ell}\Lambda^k(\mathcal{S})$*, there holds*

$$|u - \mathcal{Q}^{\epsilon}_{\mathcal{S},h}u| \preceq h^{\ell}|\nabla^{\ell}u|\,, \quad \textit{and} \tag{5.2.28}$$

(ii) for all $u \in \mathbf{Y}_h$*, there holds*

$$|u - \mathcal{Q}^{\epsilon}_{\mathcal{S},h} u| \leq \epsilon' |u| \,.$$

Proof. They follow from Proposition 5.2.15 by summing over all cells $T \in \mathcal{T}_h^n$, bearing in mind the previous comment. □

The smoothed projections

We observe that the second estimate in the last proposition yields invertibility of $\mathcal{Q}^{\epsilon}_{\mathcal{S},h}|_{\mathbf{Y}_h}$: $\mathbf{Y}_h \to \mathbf{Y}_h$, as can be proved by a standard argument using the Neumann series of an operator (see, e.g., [Gat24, Exercise 3.10]). Furthermore, the same argument gives a bound on the norm of the inverse:

$$\|(\mathcal{Q}^{\epsilon}_{\mathcal{S},h}|_{\mathbf{Y}^k_h})^{-1}\|_{\mathrm{O}\to\mathrm{O}} \leq \frac{1}{1 - \|(I - \mathcal{Q}^{\epsilon}_{\mathcal{S},h})|_{\mathbf{Y}^k_h}\|} \leq \frac{1}{1-\epsilon'}.$$

From now on, for any parameter h, we fix $\epsilon' = 1/2$ and ϵ sufficiently small, so that

$$\|(\mathcal{Q}^{\epsilon}_{\mathcal{S},h}|_{\mathbf{Y}_h})^{-1}\| \leq 2 \,.$$

The smoothed projection is then defined by

$$\mathbf{\Pi}^k_h = (\mathcal{Q}^{\epsilon}_{\mathcal{S},h}|_{\mathbf{Y}^k_h})^{-1} \mathcal{Q}^{\epsilon}_{\mathcal{S},h} \,.$$

By construction, $\mathbf{\Pi}_h$ is a projector onto $\mathbf{Y}_h$. It remains to be checked that we did not spoil the good properties of $\mathcal{Q}^{\epsilon}_{\mathcal{S},h}$. This is the object of the following proposition:

Proposition 5.2.17. *The following properties hold:*

i) $\mathbf{\Pi}_h$ *is a projection onto* $\mathbf{Y}_h$ *which is stable* $\mathbf{O} \to \mathbf{O}$*, and whose restriction to* $\mathbf{X}$ *commutes with the exterior derivative.*

ii) *For all* $0 \leq \ell \leq p+1$ *and* $u \in \mathrm{H}^{\ell}\Lambda^{\bullet}(S)$*, there holds*

$$|u - \mathbf{\Pi}_h u| \preceq h^{\ell} |\nabla^{\ell} u| \,. \tag{5.2.29}$$

Proof. First, we observe that stability follows from the previous discussion. More precisely

$$\|\boldsymbol{\Pi}_h^k\|_{\mathbf{O}\to\mathbf{O}} = \|(\mathcal{Q}^\epsilon_{\mathcal{S},h}|_{\mathbf{Y}_h^k})^{-1}\mathcal{Q}^\epsilon_{\mathcal{S},h}\|_{\mathbf{O}\to\mathbf{O}} \leq 2\|\mathcal{Q}^\epsilon_{\mathcal{S},h}|_{\mathbf{Y}_h^k}\|_{\mathbf{O}\to\mathbf{O}} \leq 3\,.$$

Next, we see that it commutes with the exterior derivative because, since $\mathcal{Q}^\epsilon_{\mathcal{S},h}$ does, we have

$$\mathsf{d}(\mathcal{Q}^\epsilon_{\mathcal{S},h}|_{\mathbf{Y}_h^k})^{-1} = (\mathcal{Q}^\epsilon_{\mathcal{S},h}|_{\mathbf{Y}_h^k})^{-1}\mathsf{d},$$

which implies the desired commutativity.

To prove the second item, for $u \in \mathrm{H}^\ell\Lambda^\bullet(\mathcal{S})$, $0 \leq \ell \leq p+1$, we write

$$|u - \boldsymbol{\Pi}_h u| \leq |u - \mathcal{Q}^\epsilon_{\mathcal{S},h}u| + |\boldsymbol{\Pi}_h u - \mathcal{Q}^\epsilon_{\mathcal{S},h}u|. \tag{5.2.30}$$

We take care of the second term. Using that $\mathcal{Q}^\epsilon_{\mathcal{S},h}|_{\mathbf{Y}_h}$ is invertible and uniformly bounded $\mathbf{O} \to \mathbf{O}$, we get

$$\begin{aligned}|\boldsymbol{\Pi}_h u - \mathcal{Q}^\epsilon_{\mathcal{S},h}u| &= |(I - (\mathcal{Q}^\epsilon_{\mathcal{S},h}|_{\mathbf{Y}_h})^{-1})\mathcal{Q}^\epsilon_{\mathcal{S},h}u| \\ &\preceq |\mathcal{Q}^\epsilon_{\mathcal{S},h}(I - (\mathcal{Q}^\epsilon_{\mathcal{S},h}|_{\mathbf{Y}_h})^{-1})\mathcal{Q}^\epsilon_{\mathcal{S},h}u| \\ &= |\mathcal{Q}^\epsilon_{\mathcal{S},h}(I - \mathcal{Q}^\epsilon_{\mathcal{S},h})u|\,,\end{aligned}$$

whence $|\boldsymbol{\Pi}_h u - \mathcal{Q}^\epsilon_{\mathcal{S},h}u| \preceq |u - \mathcal{Q}^\epsilon_{\mathcal{S},h}u|$. Therefore, putting this into (5.2.30) and then using estimate (5.2.28), we get

$$|u - \boldsymbol{\Pi}_h u| \preceq |u - \mathcal{Q}^\epsilon_{\mathcal{S},h}u| \preceq h^\ell|\nabla^\ell u|,$$

the desired estimate. □

For later reference, we state rates of approximation in the $\mathbf{Y}$-norm: To this end, we need to assume that another instance of Proposition 5.2.1 holds:

Assumption 5.2.6. *The chosen interpolator satisfies:*

$$|\nabla u - \nabla\widetilde{\mathcal{I}}_h u|_T \preceq \sum_{T' \lhd T} h_T^{\ell+(\dim T - \dim T')/2}|\nabla^{\ell+1}u|_{T'},$$

for u sufficiently regular and $0 \leq \ell \leq p$ ◇

Proposition 5.2.18. *Given $0 \leq \ell \leq p$, for $u \in H^{\ell+1}\Lambda^\bullet(\mathcal{S})$, there holds*

$$\|u - \mathbf{\Pi}_h u\|_{\mathbf{Y}} \preceq h^\ell |\nabla^{\ell+1} u|.$$

Proof. Notice that Assumption 5.2.6 guarantees that an analogous of Proposition 5.2.9 for the H^1-seminorm may be obtained. Namely, for $u \in \mathrm{H}^{\ell+1}\Lambda^\bullet(\widetilde{\mathcal{M}}(T))$, there holds

$$|\nabla \mathcal{R}_h^\epsilon u - \nabla \widetilde{\mathcal{I}}_h \mathcal{R}_h^\epsilon u| \preceq h^\ell |\nabla^{\ell+1} u|_{\widetilde{\mathcal{M}}(T)}\,.$$

On the other hand, since $\mathcal{R}$ commutes with ∇, Proposition 5.2.11 gives

$$|\nabla u - \nabla \mathcal{R}_h^\epsilon u|_T \preceq h^\ell |\nabla^{\ell+1} u|_{\widetilde{\mathcal{M}}(T)}\,.$$

Together, these two estimates give

$$|\nabla u - \nabla \mathcal{Q}_h^\epsilon u| \preceq h^\ell |\nabla^{\ell+1} u|_{\widetilde{\mathcal{M}}(T)}\,.$$

Bearing this in mind, and following a similar procedure to that leading to (5.2.29), one is able to get the desired property. □

To finish this section, we show that stability in $\mathbf{O}$ implies stability in $\mathbf{X}$:

Lemma 5.2.6. *The smoothed projection $\mathbf{\Pi}_h : \mathbf{X} \to \mathbf{Y}_h$ is stable $\mathbf{X} \to \mathbf{X}$.*

Proof. Given $u \in \mathbf{X}$, we have

$$\|\mathbf{\Pi}_h u\|_{\mathbf{X}} \approx |\mathbf{\Pi}_h u| + |\mathsf{d}\mathbf{\Pi}_h u| \leq \|\mathbf{\Pi}_h\|_{\mathbf{O}\to\mathbf{O}}(|u| + |\mathsf{d}u|) \preceq \|u\|_{\mathbf{X}}.$$

□

5.3 Projection Error and Stable Equivalences of Norms

In this section, we focus on approximation properties, mainly on estimating projection error, which shall play an important role in the following section; in particular, to handle the fact that $\mathbf{V}_h \not\subset \mathbf{V}$. The main objective of this section is to prove that $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{D}}_h$ induce equivalent graph norms, this constitutes the last result of the section.

Recall from Section 5.1 that $\mathcal{P}$ is the O-orthogonal projector onto $\overline{\mathbf{V}}$. We have the following result:

Proposition 5.3.1. *For $v \in \mathbf{V}_h$, there holds*

$$|v - \mathcal{P}v| \leq |\mathcal{P}v - \mathbf{\Pi}_h \mathcal{P}v|. \tag{5.3.1}$$

Proof. Given $v \in \mathbf{V}_h$, we can write

$$\mathcal{P}v - \mathbf{\Pi}_h \mathcal{P}v = (v - \mathbf{\Pi}_h \mathcal{P}v) - (v - \mathcal{P}v).$$

Since $v - \mathcal{P}v \in \mathbf{W}$ and $\mathbf{\Pi}_h$ is a commuting projection, we have that $v - \mathbf{\Pi}_h \mathcal{P}v \in \mathbf{W}_h \subset \mathbf{W}$. There follows that

$$v - \mathbf{\Pi}_h \mathcal{P}v \perp v \quad \text{and} \quad v - \mathbf{\Pi}_h \mathcal{P}v \perp \mathcal{P}v.$$

This orthogonality gives

$$|v - \mathcal{P}v| \leq |(v - \mathcal{P}v) - (v - \mathbf{\Pi}_h \mathcal{P}v)| = |\mathcal{P}v - \mathbf{\Pi}_h \mathcal{P}v|,$$

as desired. □

Lemma 5.3.1. *For $v \in \mathbf{V}_h$, there holds*

$$\|v - \mathcal{P}v\|_{\mathbf{X}} \preceq h|\mathsf{d}v|$$

Proof. Since $\mathsf{d}^\star$ vanishes on $\mathbf{V}$ and $\mathcal{P}$ preserves elements in $\mathbf{V}$, by using the previous estimate together with Proposition 5.2.17, with $\ell = 1$, we get

$$\|v - \mathcal{P}v\|_{\mathbf{X}} = |v - \mathcal{P}v| \leq |\mathcal{P}v - \mathbf{\Pi}_h \mathcal{P}v| \preceq h|\nabla \mathcal{P}v| = h|\mathsf{d}\mathcal{P}v| = h|\mathsf{d}v|,$$

for $v \in \mathbf{V}_h$. ☐

Now, we prove the already announced *discrete Friedrichs-Poincaré inequality*, which constitutes one of the most important implications of having stable commuting projections. The proof presented here is modelled in the one given in [Arn18]. It is worth mentioning that this result may still be obtained when the projection does not commute with the exterior derivative, provided that the relevant *commutator* is still controlled; we refer to [Chr07] for details about this construction.

Lemma 5.3.2. *Given $v \in \mathbf{V}_h$, there holds $\|v\|_{\mathbf{X}} \preceq |\mathsf{d}v|$.*

Proof. First, since $\mathbf{V}_h \subset \mathbf{X}$, given $v \in \mathbf{V}_h$, we notice that $\mathcal{P}v \in \mathbf{V}$ and $\mathsf{d}v = \mathsf{d}\mathcal{P}v$. Therefore, in virtue of the (continuous) Friedrichs-Poincaré inequality, we have

$$\|\mathcal{P}v\|_{\mathbf{X}} \preceq |\mathsf{d}\mathcal{P}z| = |\mathsf{d}v|.$$

In turn, since $\mathbf{\Pi}_h$ is a commuting projection, we have

$$\mathsf{d}\mathbf{\Pi}_h \mathcal{P}v = \mathbf{\Pi}_h \mathsf{d}\mathcal{P}v = \mathbf{\Pi}_h \mathsf{d}v = \mathsf{d}v,$$

which amounts to say that $v - \mathbf{\Pi}_h \mathcal{P}v \in \mathbf{W}_h$; hence must be $\mathbf{X}$-orthogonal to v. There follows that

$$\|v\|_{\mathbf{X}} \leq \|v - (v - \mathbf{\Pi}_h \mathcal{P}v)\|_{\mathbf{X}} = \|\mathbf{\Pi}_h \mathcal{P}v\|_{\mathbf{X}} \preceq \|\mathcal{P}v\|_{\mathbf{X}},$$

where we have used that $\mathbf{\Pi}_h : \mathbf{X} \to \mathbf{Y}_h$ is stable $\mathbf{X} \to \mathbf{X}$ (cf. Lemma 5.2.6). This, in virtue of the first estimate, yields the desired result. ☐

The following constitute two simple consequences of the discrete Poincaré inequality:

Lemma 5.3.3. *Given $u \in \mathsf{d}\mathbf{Y}_h$, there holds $\|u\|_{\mathbf{X}} \preceq |\mathsf{d}_h^\star u|$.*

Proof. We have

$$\|u\|_{\mathbf{X}} = |u| = \sup_{w\in\mathbf{Y}_h} \frac{(u, \mathsf{d}w)}{|\mathsf{d}w|} = \sup_{w\in\mathbf{V}_h} \frac{(\mathsf{d}_h^\star u, w)}{|\mathsf{d}w|} \preceq \sup_{w\in\mathbf{V}_h} \frac{(\mathsf{d}_h^\star u, w)}{|w|} = |\mathsf{d}_h^\star u|,$$

where the second equality comes from the orthogonality of the *discrete Hodge decomposition* and the inequality comes from the discrete Poincaré inequality. □

From Lemmas 5.3.2 and 5.3.3 one deduces the following result:

Lemma 5.3.4. *Given $u \in \mathbf{V}_h \oplus \mathsf{d}\mathbf{Y}_h$, there holds $\|u\|_{\mathbf{X}} \preceq |\boldsymbol{\mathcal{D}}_h u|$. In particular, $|\boldsymbol{\mathcal{D}}_h \cdot|$ is a norm.*

Proof. It follows from the discrete Poincaré inequality and Lemma 5.3.3, by recalling that $\mathsf{d}\mathbf{Y}_h$ and $\mathbf{V}_h$ are $\mathbf{X}$-orthogonal. □

Next, we give a simple proof of the fact that the continuous and discrete cohomology spaces coincide. We only deal with the particular case in which $\mathcal{S}$ is a torus.

Proposition 5.3.2. *Assuming that $\mathcal{S} = \mathbb{R}^n/\mathbb{L}$, there holds that $\mathbf{G} = \mathbf{G}_h$.*

Proof. First, we prove they are isomorphic. The decompositions

$$\mathbf{W} = \mathsf{d}X \oplus \mathbf{G} \quad \text{and} \quad \mathbf{W}_h = \mathsf{d}\mathbf{Y}_h \oplus \mathbf{G}_h$$

give isomorphisms $\mathbf{G} \cong \mathbf{W}/\mathsf{d}\mathbf{X}$ and $\mathbf{G}_h \cong \mathbf{W}_h/\mathsf{d}\mathbf{Y}_h$.

We shall prove that the induced map $\widetilde{\Pi}_h : \mathbf{W}/\mathsf{d}\mathbf{X} \to \mathbf{W}_h/\mathsf{d}\mathbf{Y}_h$ is injective. We need to show that if, $\mathbf{\Pi}_h u \in \mathsf{d}\mathbf{Y}_h$, for $u \in \mathbf{W}$, then $u \in \mathsf{d}\mathbf{X}$.

We use the continuous Hodge decomposition to write $u = z + g$, with $z \in \mathsf{d}\mathbf{X}$ and $g \in \mathbf{G}$. We recall that, on the torus, we have $\mathbf{G} = \bigoplus_{k=0}^{n} \mathsf{Alt}^k(\mathbb{R}^n)$. Thereby, since $\mathbf{Y}_h$ contains the constant forms, $\mathbf{\Pi}_h$ preserves them, and so we have $\mathbf{\Pi}_h(g) = g$, for all $g \in \mathbf{G}$.

Using this along with the fact that $\mathbf{\Pi}_h$ is a cochain map and the assumption $\mathbf{\Pi}_h u \in \mathsf{d}\mathbf{Y}_h$, we obtain

$$g = \mathbf{\Pi}_h g = \mathbf{\Pi}_h u - \mathbf{\Pi}_h z \in \mathsf{d}\mathbf{Y}_h \subset \mathsf{d}\mathbf{X}.$$

Therefore $g = 0$ and, *a fortiori*, $u = z \in \mathsf{d}\mathbf{X}$. This proves that $\mathbf{W}/\mathsf{d}\mathbf{X}$ is embedded in $\mathbf{W}_h/\mathsf{d}\mathbf{Y}_h$ via $\widetilde{\Pi}_h$. In particular, we have $\dim \mathbf{G} \leq \dim \mathbf{G}_h$.

Lastly, we observe that $\mathsf{d}g = 0$ and $\mathsf{d}_h^\star g = \pi_h \mathsf{d}^\star g = 0$, for all $g \in \mathbf{G}$. This gives $\mathbf{G}_h \subset \mathbf{G}$, and therefore proves that they are equal. □

We shall need an energy projector $\mathcal{P}_h : \mathbf{X} \to \mathbf{V}_h$, defined by requiring that, for $u \in \mathbf{X}$, it satisfies

$$(\mathsf{d}\mathcal{P}_h u, \mathsf{d}v) = (\mathsf{d}u, \mathsf{d}v) \qquad \forall\, v \in \mathbf{V}_h.$$

According to Proposition 2.7.4 and the discrete Hodge decomposition, this is well–defined.

Lemma 5.3.5. *$\mathcal{P}_h$ is stable $\mathbf{X} \to \mathbf{X}$, and there holds*

$$|v - \mathcal{P}_h v| \preceq h|\mathsf{d}v|,$$

for all $v \in \mathbf{V}$.

Proof. According to the discrete Poincaré inequality (see Lemma 5.3.2), we may write

$$\|\mathcal{P}_h u\|_{\mathbf{X}} \preceq \frac{(\mathsf{d}\mathcal{P}_h u, \mathsf{d}\mathcal{P}_h u)}{\|\mathcal{P}_h u\|_{\mathbf{X}}} \leq \sup_{v \in \mathbf{V}_h} \frac{(\mathsf{d}u, \mathsf{d}v)}{\|v\|_{\mathbf{X}}} \leq |\mathsf{d}u| \leq \|u\|_{\mathbf{X}} \qquad \forall\, u \in \mathbf{X},$$

hence $\mathcal{P}_h$ is stable $\mathbf{X} \to \mathbf{X}$. We remark that this also proves $|\mathsf{d}\mathcal{P}_h u| \preceq |\mathsf{d}u|$, for $u \in \mathbf{X}$.

In turn, using Lemma 5.3.1, we obtain

$$|\mathcal{P}\mathcal{P}_h u - \mathcal{P}_h u| \preceq h|\mathsf{d}\mathcal{P}_h u| \preceq h|\mathsf{d}u| \qquad \forall\, u \in \mathbf{X}.$$

Then, using triangle inequality together with the previous estimate, we obtain

$$|u - \mathcal{P}_h u| \preceq |u - \mathcal{P}\mathcal{P}_h u| + h|\mathsf{d}u| \qquad \forall\, u \in \mathbf{V}\,. \tag{5.3.2}$$

To bound the first term we remark that, since $\mathcal{S}$ is a compact manifold without boundary, we have elliptic regularity for the Hodge Laplacian, in the form

$$\inf_{u\in\mathbf{V}} \sup_{v\in\mathbf{V}\cap\mathrm{H}^2\Lambda(\mathcal{S})} \frac{(\mathsf{d}u,\mathsf{d}v)}{|u|\|v\|_{\mathrm{H}^2}} \succeq 1.$$

Observe that, for $u \in \mathbf{V}$, we have $\mathsf{d}\mathcal{P}\mathcal{P}_h u = \mathsf{d}\mathcal{P}_h u$ and $u - \mathcal{P}\mathcal{P}_h u \in \mathbf{V}$. Then, with the help of the inf-sup condition, we write

$$|u - \mathcal{P}\mathcal{P}_h u| \preceq \sup_{v\in\mathbf{V}\cap\mathrm{H}^2\Lambda(\mathcal{S})} \frac{(\mathsf{d}(u-\mathcal{P}_h u),\mathsf{d}v)}{\|v\|_{\mathrm{H}^2}} = \sup_{v\in\mathbf{V}\cap\mathrm{H}^2\Lambda(\mathcal{S})} \frac{(\mathsf{d}(u-\mathcal{P}_h u),\mathsf{d}(v-\widetilde{v}))}{\|v\|_{\mathrm{H}^2}},$$

for all $\widetilde{v} \in \mathbf{V}_h$.

In particular, letting $\mathcal{Q}_h : \mathbf{O} \to \mathbf{W}_h$ denote the $\mathbf{O}$-orthogonal projector, for any $v \in \mathbf{V}$ we may consider $\widetilde{v} = (I - \mathcal{Q}_h)\mathbf{\Pi}_h v \in \mathbf{V}_h$, which satisfies $\mathsf{d}\widetilde{v} = \mathsf{d}\mathbf{\Pi}_h v = \mathbf{\Pi}_h \mathsf{d}v$. We use this in the previous estimate, followed by Proposition 5.2.17 with $\ell = 1$, to obtain

$$|u - \mathcal{P}\mathcal{P}_h u| \preceq \sup_{v\in\mathbf{V}\cap\mathrm{H}^2\Lambda(\mathcal{S})} \frac{(\mathsf{d}(u-\mathcal{P}_h u),\mathsf{d}v - \mathbf{\Pi}_h\mathsf{d}v))}{\|v\|_{\mathrm{H}^2}} \preceq h \sup_{v\in\mathbf{V}\cap\mathrm{H}^2\Lambda(\mathcal{S})} \frac{|\mathsf{d}(u-\mathcal{P}_h u)||\nabla\mathsf{d}v|}{\|v\|_{\mathrm{H}^2}},$$

whence

$$|u - \mathcal{P}\mathcal{P}_h u| \preceq h|\mathsf{d}u - \mathsf{d}\mathcal{P}_h u| \preceq h|\mathsf{d}u|.$$

The proof ends by looking at (5.3.2). □

We finish this section by showing that the continuous and discrete Hodge-Dirac operators induce equivalent graph norms:

Proposition 5.3.3. *There holds*

$$|\boldsymbol{\mathcal{D}}u| \approx |\boldsymbol{\mathcal{D}}_h u| \qquad \forall\, u \in \mathbf{Y}_h.$$

Proof. The inequality $|\boldsymbol{\mathcal{D}}_h u| \preceq |\boldsymbol{\mathcal{D}}u|$ is immediate because orthogonal projectors have norm 1. It remains to show that $|\boldsymbol{\mathcal{D}}u| \preceq |\boldsymbol{\mathcal{D}}_h u|$; we point out that it suffices to address the discrepancy between $\mathsf{d}^\star$ and $\mathsf{d}_h^\star$, so we essentially need to prove that $|\mathsf{d}^\star u| \preceq |\boldsymbol{\mathcal{D}}_h u|$. To this end, given

$u \in \mathbf{Y}_h$, we consider its discrete Hodge decomposition

$$u = \mathsf{d}z + v + g, \qquad z \in \mathbf{Y}_h,\ v \in \mathbf{V}_h,\ g \in \mathbf{G}_h\,,$$

and estimate each term separately. First, we write

$$|\mathsf{d}^\star \mathsf{d}z| = \sup_{\utilde{v} \in X} \frac{(\mathsf{d}^\star \mathsf{d}z, \utilde{v})}{|\utilde{v}|} = \sup_{\utilde{v} \in \mathbf{V}} \frac{(\mathsf{d}v, \mathsf{d}\utilde{z})}{|\utilde{v}|} = \sup_{\utilde{v} \in \mathbf{V}} \frac{(\mathsf{d}z, \mathsf{d}\mathcal{P}_h \utilde{v})}{|\utilde{v}|}, \tag{5.3.3}$$

where the second equality comes from noticing that $\mathsf{d}^\star \mathsf{d}z \in \overline{\mathbf{V}}$ and that the function being maximized is continuous.

Now, for $\utilde{v} \in \mathbf{V}$, we may use triangle inequality in Lemma 5.3.5, followed by the inverse inequality (see Lemma 5.2.2), to get

$$|\mathcal{P}_h \utilde{v}| \preceq |\utilde{v}| + h|\mathsf{d}\utilde{v}| = |\utilde{v}| + h|\nabla \utilde{v}| \preceq |\utilde{v}|. \tag{5.3.4}$$

Using (5.3.4) to bound the right-most term in (5.3.3), we get

$$|\mathsf{d}^\star \mathsf{d}z| \preceq \sup_{\utilde{v} \in \mathbf{V}} \frac{(\mathsf{d}z, \mathsf{d}\mathcal{P}_h \utilde{v})}{|\mathcal{P}_h \utilde{v}|} \preceq \sup_{\utilde{v} \in \mathbf{Y}_h} \frac{(\mathsf{d}z, \mathsf{d}\utilde{v})}{|\utilde{v}|} = |\mathsf{d}_h^\star \mathsf{d}z|. \tag{5.3.5}$$

For the second term, we recall that $\mathcal{P}$ maps all of $\mathbf{X}$ into $\mathbf{X}$, which implies that

$$\mathcal{P}v \in \mathbf{V} \quad \text{ and } \quad \mathsf{d}\mathcal{P}v = \mathsf{d}v.$$

Bearing this in mind, we write

$$|\mathsf{d}^\star v|^2 = |\mathsf{d}^\star (v - \mathcal{P}v)|^2 = |\mathsf{d}^\star (v - \mathcal{P}v)|^2 + |\mathsf{d}(v - \mathcal{P}v)|^2 = |\nabla (v - \mathcal{P}v)|^2.$$

Next, we use the inverse inequality (see Lemma 5.2.2) followed by Lemma 5.3.1 to get

$$|\mathsf{d}^\star v| \preceq h^{-1}|v - \mathcal{P}v| \preceq |\mathsf{d}v|. \tag{5.3.6}$$

In third place, to address the discrete harmonic part, we rely on the fact that $\mathcal{S}$ is a torus and use Proposition 5.3.2 to get $|\mathsf{d}^\star g| = 0$.

According to the initial observation, we conclude by using triangle inequality and gathering the estimates (5.3.5), (5.3.6):

$$|\mathsf{d}^\star u| = |\mathsf{d}^\star(\mathsf{d}z + v + g)| \leq |\mathsf{d}^\star \mathsf{d}z| + |\mathsf{d}^\star v| + |\mathsf{d}^\star g| \preceq |\mathsf{d}v| + |\mathsf{d}_h^\star \mathsf{d}z| \approx |\boldsymbol{\mathcal{D}}_h u|.$$

□

Although the last part of Proposition 5.3.3 relies on the fact that $\mathbf{G} = \mathbf{G}_h$, it is worth mentioning that, in our application in Chapter 6, this property is only required in $\mathbf{V}_h \oplus \mathsf{d}\mathbf{Y}_h$.

CHAPTER 6

Application to the Hodge-Dirac eigenproblem in the torus

In this chapter, we apply the framework developed in Chapter 4 to a particular case of the abstract problem (see (3.2.2)). Let $\mathcal{S} = \mathbb{R}^n/\mathbb{L}$, as before. We continue to use the notation introduced in the previous section

$$\mathbf{O} = \mathrm{L}^2\Lambda^\bullet(\mathcal{S}) \quad \text{and} \quad \mathbf{Y} = \mathrm{H}^1\Lambda^\bullet(\mathcal{S}),$$

which also coincides with the setting in Chapter 4. Denote by $\mathbf{a} : \mathbf{Y} \times \mathbf{O} \to \mathbb{R}$ the bilinear form induced by $\boldsymbol{\mathcal{D}}$. Namely

$$\mathbf{a}(u, v) = (\boldsymbol{\mathcal{D}}u, v) \qquad \forall\, u \in \mathbf{Y},\ v \in \mathbf{O}.$$

The discrete operator $\boldsymbol{\mathcal{D}}_h$ also induces a bilinear form $\mathbf{Y}_h \times \mathbf{Y}_h \to \mathbb{R}$ which, as customary, we again denote by $\mathbf{a}$.

The discrete bilinear form coincides with the restriction of $\mathbf{a}$ to $\mathbf{Y}_h \times \mathbf{Y}_h$. Namely

$$(\boldsymbol{\mathcal{D}}_h u, v) = (\boldsymbol{\mathcal{D}} u, v) = \mathbf{a}(u, v) \qquad \forall\, u, v \in \mathbf{Y}_h .$$

Lemma 6.0.1. *$\boldsymbol{\mathcal{D}} : \mathbf{Y} \to \mathbf{O}$ is a symmetric Fredholm operator and has index zero.*

Proof. We omit a detailed proof of this, for it is a well–known fact in the study of elliptic operators. Nevertheless, we remark that this hinges on the fact that an elliptic operator on a compact manifold without boundary is Fredholm. That the index is zero follows from the fact that it is symmetric. ☐

6.1 Stability

We start by proving that a weak inf-sup condition holds. As we saw previously, this is crucial to obtaining eigenmode convergence; it will be a direct consequence of the previous analysis.

Putting $\widetilde{\mathbf{Y}}_h = \mathbf{Y}_h \cap \mathbf{G}_h^{\perp} = \mathbf{V}_h \oplus \mathsf{d}\mathbf{Y}_h$, we have

Lemma 6.1.1. *There holds*

$$\inf_{u \in \widetilde{\mathbf{Y}}_h} \sup_{v \in \widetilde{\mathbf{Y}}_h} \frac{|\mathbf{a}(u, v)|}{|u||\boldsymbol{\mathcal{D}}_h v|} = 1 \qquad \forall\, u \in \mathbf{Y}_h \tag{6.1.1}$$

Proof. First, we observe that the discrete Hodge decomposition implies $\boldsymbol{\mathcal{D}}_h(\widetilde{\mathbf{Y}}_h) \subset \widetilde{\mathbf{Y}}_h$. Thereby, for every $v \in \widetilde{\mathbf{Y}}_h$, $u = \boldsymbol{\mathcal{D}}_h v \in \widetilde{\mathbf{Y}}_h$. It follows that

$$\inf_{v \in \widetilde{\mathbf{Y}}_h} \sup_{u \in \widetilde{\mathbf{Y}}_h} \frac{(\boldsymbol{\mathcal{D}}_h u, v)}{|u||\boldsymbol{\mathcal{D}}_h v|} \geq 1 \,.$$

Moreover, symmetry and Cauchy-Schwarz inequality prove the converse inequality, and so

$$\inf_{v \in \widetilde{\mathbf{Y}}_h} \sup_{u \in \widetilde{\mathbf{Y}}_h} \frac{(\boldsymbol{\mathcal{D}}_h u, v)}{|u||\boldsymbol{\mathcal{D}}_h v|} = 1 \,.$$

Furthermore, since $|\boldsymbol{\mathcal{D}}_h \cdot |$ is a norm in $\widetilde{\mathbf{Y}}_h$ and the range of $\boldsymbol{\mathcal{D}}_h$ is closed, we have access to

Lemma 2.7.2; yielding (6.1.1). ☐

By using the results in Chapter 5, we can immediately upgrade this result to stability with the full $\mathbf{Y}$-norm.

Lemma 6.1.2. *There holds*

$$\inf_{u\in\widetilde{\mathbf{Y}}_h}\sup_{v\in\widetilde{\mathbf{Y}}_h}\frac{|\mathbf{a}(u,v)|}{|u|\,\|v\|_{\mathbf{Y}}}\succeq 1. \tag{6.1.2}$$

Proof. From Proposition 5.3.3 and Lemma 5.3.4, it follows that

$$\|v\|_{\mathbf{Y}}\preceq|\boldsymbol{\mathcal{D}}_h v|\qquad\forall\, v\in\widetilde{\mathbf{Y}}_h\,.$$

Using this to bound (6.1.1) gives (6.1.2). ☐

6.2 Perturbations

As in Section 4.2, let $\boldsymbol{\mathcal{C}}:\mathbf{O}\to\mathbf{O}$ denote a bounded self-adjoint operator, which could represent the electromagnetic field in the Hodge-Dirac equation. Denote by $\mathbf{c}:\mathbf{O}\times\mathbf{O}\to\mathbb{R}$ the induced bilinear form. Choose $\mu\in\mathbb{R}$ such that $\boldsymbol{\mathcal{D}}+\boldsymbol{\mathcal{C}}+\mu I$ is injective, where I is the identity.

In this situation, we recall that we are interested in the eigenvalue problem that seeks $(\lambda,u)\in\mathbb{R}\times\mathbf{Y}$ such that

$$\mathbf{a}(u,v)+\mathbf{c}(u,v)=\lambda(u,v)\qquad\forall\, v\in\mathbf{Y}, \tag{6.2.1}$$

and its Galerkin discretization, which seeks $(\lambda_h,u_h)\in\mathbb{R}\times\mathbf{Y}_h$ such that

$$\mathbf{a}(u_h,v)+\mathbf{c}(u_h,v)=\lambda_h(u,v)\qquad\forall\, v\in\mathbf{Y}_h. \tag{6.2.2}$$

In the remaining of this section, we verify that our problem satisfies the conditions deduced in sections 4.1 and 4.2

Lemma 6.2.1. *The family $(\mathbf{Y}_h)_h$ is $\mathbf{O}$-approximating, in the sense of Definition 4.1.1.*

Proof. This follows from the rates of approximation given by propositions 5.2.17 and 5.2.18, by a standard density argument. ☐

Lemma 6.2.2. *We have* $\mathbf{Y}$*-convergence of the gap between* $\mathbf{G}_h$ *and* $\mathbf{G}$*, in the sense of Definition 4.2.1.*

Proof. In Proposition 5.3.2, we proved that they coincide. ☐

With these results at hand, and owing to propositions 4.1.1 and 4.2.1, we state the main result of this chapter:

Theorem 6.2.3. *There holds eigenmode convergence. That is, the discrete eigenmodes of* (6.2.1) *converge to those of* (6.2.2).

Proof. First, observe that $\boldsymbol{\mathcal{C}} + \mu I : \mathbf{Y} \to \mathbf{O}$ is compact. Then, as Fredholmness is stable under compact perturbations, it follows that $\boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{C}} + \mu I$ is Fredholm (cf. Lemma 6.0.1). Moreover, since $\boldsymbol{\mathcal{C}}$ is symmetric and the Fredholm index is invariant under compact perturbations, $\boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{C}} + \mu I$ is a symmetric and injective Fredholm operator of index zero, therefore invertible.

Regarding the discrete setting, Lemmas 6.2.1, 6.2.2 and 6.1.2 ensure that we have acces to Proposition 4.2.1. According to this and Lemma 2.7.2, we have discrete stability in the form:

$$\inf_{u \in \mathbf{Y}_h} \sup_{v \in \mathbf{Y}_h} \frac{((\boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{C}} + \mu I)u, v)}{|u| \, \|v\|_{\mathbf{Y}}} \succeq 1. \tag{6.2.3}$$

The previous discussion collects all the assumptions of Proposition 4.1.1; eigenmode convergence follows. ☐

6.3 Rates of convergence

We finish Part I by giving explicit rates of convergence of the eigenmodes of (6.2.1) to those of (6.2.2). To simplify notation, we let $b : \mathbf{Y} \times \mathbf{O} \to \mathbb{R}$ denote the bilinear form induced by $\boldsymbol{\mathcal{D}} + C + \mu I$, which satisfies

$$\mathbf{b}(u, v) = ((\boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{C}} + \mu I)u, v) \qquad \forall\, u \in \mathbf{Y},\, v \in \mathbf{O}.$$

In addition, we define the Galerkin projection $\mathbf{R}_h : \mathbf{Y} \to \mathbf{Y}_h$, by requiring that, for $u \in \mathbf{Y}$, there holds

$$\mathbf{b}(\mathbf{R}_h u, v) = \mathbf{b}(u, v) \qquad \forall\, v \in \mathbf{Y}_h.$$

According to the proof of Theorem 6.2.3, $\mathbf{R}_h$ is well–defined. Moreover, it is uniformly stable $\mathbf{Y} \to \mathbf{Y}$. Indeed, given $u \in \mathbf{Y}$, discrete stability (see (6.2.3)) and Lemma 2.7.2 give

$$\|\mathbf{R}_h u\|_{\mathbf{Y}} \preceq \sup_{v \in \mathbf{Y}_h} \frac{\mathbf{b}(\mathbf{R}_h u, v)}{|v|} = \sup_{v \in \mathbf{Y}_h} \frac{\mathbf{b}(u, v)}{|v|} \preceq \|u\|_{\mathbf{Y}}\,.$$

By combining this with the approximation properties of $\mathbf{\Pi}_h$, we get the desired optimal rates of convergence. This is formulated in the following proposition:

Proposition 6.3.1. *Let p be the nonnegative integer fixed in Section 5.2. Then, for $0 \leq \ell \leq p$, there holds*

$$\|u - \mathbf{R}_h u\|_{\mathbf{Y}} \preceq h^{\ell} |\nabla^{\ell+1} u|, \tag{6.3.1}$$

for $u \in \mathrm{H}^{\ell+1}\Lambda^{\bullet}(\mathcal{S})$.

Proof. Given $u \in \mathrm{H}^{\ell+1}\Lambda^{\bullet}(\mathcal{S})$, we have

$$\begin{aligned}
\|u - \mathbf{R}_h u\|_{\mathbf{Y}} &\leq \|u - \mathbf{\Pi}_h u\|_{\mathbf{Y}} + \|\mathbf{\Pi}_h u - \mathbf{R}_h u\|_{\mathbf{Y}} \\
&= \|u - \mathbf{\Pi}_h u\|_{\mathbf{Y}} + \|\mathbf{R}_h(\mathbf{\Pi}_h u - u)\|_{\mathbf{Y}} \\
&\leq (1 + \|\mathbf{R}_h\|)\, \|u - \mathbf{\Pi}_h u\|_{\mathbf{Y}} \\
&\preceq h^{\ell} |\nabla^{\ell+1} u|\,,
\end{aligned}$$

where in the last estimate we used the uniform stability of $\mathbf{R}_h$ and Proposition 5.2.18. □

Remark 6.3.1. Rates of convergence for the Galerkin projection yield rates for the convergence of eigenfunctions and eigenvalues by the standard methods of theory of Babuška and Osborn. We refer to [BO89, BO91] for details. ◇

Part II

Discrete Fredholm theory and the Dirac-Kähler operator

CHAPTER 7

Overview

As mentioned in Chapter 1, in this part of the work, we shall see how the Euler-Poincaré characteristic, a topological invariant of a manifold, may be preserved at the discrete level. For this, building on the theory presented in [Chr04], we develop a notion of discrete Fredholmness and discrete Fredholm index, which allows us to give the correct interpretation to asymptotic stability on certain subspaces of approximating sequences of spaces. We also identify a strong connection between the Dirac-Kähler operator acting on differential forms and the *Clifford-Dirac* operator acting on Clifford fields.

We begin this journey by giving the algebraic point of view for constructing various relevant algebras over a vector space, and the corresponding induced bundles on manifolds. Moreover, we show the somewhat less standard fact that, in some sense, there is a natural isomorphism between the Clifford bundle and the exterior bundle associated with the cotangent bundle, which leads to a way of transforming the corresponding Dirac operators from one to the other, this is inspired, to some extent, by the treatment in [Gra78]; this is the subject of Chapter 8. Next, in Chapter 9 we give a review of some discrete Fredholm properties relevant to our context, followed by some new notions and properties required for our objectives. This includes a characterization of discrete Fredholmness in terms of suitable stable inf-sup conditions. Lastly, in Chapter 10, we apply this theory to the case of the graded Dirac-Kähler in a smooth Riemannian manifold with boundary, and show that, under some assumptions, the *graded Dirac-Kähler operator* is discrete Fredholm. A consequence will be that the continuous Fredholm index is preserved at the discrete level.

CHAPTER 8

From Clifford fields to differential forms

8.1 A short review on Clifford algebras

In this section, we present the main properties of Clifford algebras, a theory intimately related with operators of Dirac type. To this end, it will be necessary to first be familiarized with the algebraic construction of tensor and exterior powers of a vector space. We shall discuss these topics in the following.

8.1.1 Algebraic construction of the tensor algebra

Given vector spaces V and W over a field $\mathbb{K}$, denote by $\mathsf{Free}(V \times W)$ the **Free vector space** generated by $V \times W$, this is, the vector space formed by all the (finite) formal $\mathbb{K}$-linear combinations of pairs (v, w) with $v \in V$ and $w \in W$. Then, consider the vector subspace S of $\mathsf{Free}(V \times W)$ spanned by all the elements of the form

$$(v + v', w) - (v, w) - (v', w), \qquad (v, w + w') - (v, w) - (v, w'),$$

$$k(v,w)-(kv,w), \qquad\qquad k(v,w)-(v,kw)\,,$$

for $v, v' \in V,\ w, w' \in W$ and $k \in \mathbb{K}$. Then, the **tensor product** of V and W over $\mathbb{K}$ is

$$V \otimes_{\mathbb{K}} W = \frac{\mathsf{Free}(V \times W)}{S}.$$

When the field is clear from context, we drop the subscript and simply write $V \otimes W$. The class corresponding to (v, w) is denoted $v \otimes w$ and called the tensor product of v and w. Such elements are called decomposable. It is easy to check that the construction of S makes the map $\otimes : V \times W \to V \otimes W$ bilinear. It is called the canonical map between $V \times W$ and $V \otimes W$. Moreover, by construction, every element of $V \otimes W$ is a finite sum of decomposable elements. However, notice that such a decomposition is not unique, for $\otimes$ is bilinear. Thus, in general, we have a canonical spanning set but not a basis. Later in the discussion we shall see that the situation improves in the finite-dimensional case.

It can be proved that the tensor product satisfies the so-called universal mapping property for bilinear maps: for any bilinear map $f : V \times W \to Z$, with Z a $\mathbb{K}$-vector space there exists a unique linear map $\widetilde{f} : V \otimes W \to Z$ such that $f = \widetilde{f} \circ \otimes$; we often say that $\widetilde{f}$ is the lift of f to $V \otimes W$.

It turns out that the tensor product is uniquely determined (up to natural isomorphism) by the universal mapping property. More precisely, given a vector space T and a bilinear mapping $\phi : V \times W \to T$, if (T, ϕ) satisfies the universal mapping property for bilinear maps, then $T \cong V \otimes W$ as vector spaces. The natural isomorphisms are given by the corresponding lifts $\widetilde{\phi} : V \otimes W \to T$ and $\widetilde{\otimes} : T \to V \otimes W$.

If V and W are finite-dimensional, with bases $\{v_i\}_i$ and $\{w_j\}_j$, respectively. Then, the set $\{v_i \otimes w_j\}_{ij}$ is a basis of $V \otimes W$. A consequence of this is that $\dim(V \otimes W) = \dim(V)\dim(W)$. In the same situation, denoting by $V^\star$ and $W^\star$ the corresponding algebraic duals, one proves that $V^\star \otimes W^\star \cong (V \otimes W)^\star$ with an isomorphism being the map that sends $\alpha \otimes \beta \in V^\star \otimes W^\star$ to the map $V \otimes W \ni v \otimes w \mapsto \alpha(v)\beta(w)$.

As a consequence of the previous discussion, one can prove that, given additional vector spaces V' and W', and linear maps $f : V \to V'$, $g : W \to W'$, there is a unique linear map $f \otimes g : V \otimes W \to V' \otimes W'$ such that $(f \otimes g)(v \otimes w) = f(v) \otimes g(w)$. This is sometimes called the Kronecker product.

In summary the tensor product can be understood as an object which sends pairs (V, W) of vector spaces to a vector space $V \otimes W$, and pairs of linear maps (f, g) to a linear map $f \otimes g$. This allows us to understand this construction in the language of categories: the tensor product is a covariant functor from the category of pairs of vector spaces—whose morphisms are pairs of linear maps—to the category of vector spaces.

The described construction extends in a straightforward way to an arbitrary (finite) number of factors, by considering the corresponding free vector space and then quotienting by the subspace encoding the desired multilinearity property. This time, the tensor product is determined (up to isomorphism) by the universal mapping property for multilinear maps (of the given number of entries). This construction is associative, which means that $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ are naturally isomorphic to $U \otimes V \otimes W$, with the isomorphisms sending $(u \otimes v) \otimes w$ and $u \otimes (v \otimes w)$ to $u \otimes v \otimes w$, respectively, for $u \in U$, $v \in V$ and $w \in W$.

We are in a position to define the tensor algebra. The kth tensor power of V with itself is defined as

$$T^k(V) := \underbrace{V \otimes \cdots \otimes V}_{k}, \quad \text{for } k \geq 1, \quad \text{and} \quad T^0(V) := \mathbb{K}\,.$$

Then, we set $T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$.

The tensor product is associative and distributive with respect to addition, so there is a well-defined multiplication map $T^k(V) \times T^l(V) \to T^{k+l}(V)$, which sends (a, b) to $a \otimes b$, for $a \in T^k(V)$ and $b \in T^l(V)$. This is extended by bilinearity to a multiplication map $T(V) \times T(V) \to T(V)$. This makes $T(V)$ into an associative graded $\mathbb{K}$-algebra, the **tensor algebra of** V.

8.1.2 The exterior algebra

Given a $\mathbb{K}$-vector space V, denote by V^k the kth Cartesian power of V with itself. In the same way that the tensor product solves the universal mapping problem for multilinear maps, the exterior algebra solves the universal mapping problem for alternating multilinear maps. We obtain it by quotienting the tensor algebra in a convenient way. More precisely, consider the two-sided ideal I of $T(V)$ (additively and multiplicatively) generated by all the elements of the form $v \otimes v$, for $v \in V$. The **exterior algebra** of V is the quotient

$$\bigwedge(V) = \frac{T(V)}{I}.$$

The image of $v_1 \otimes \cdots \otimes v_k$ inside the exterior algebra is denoted by $v_1 \wedge \cdots v_k$. The induced operation is called the wedge product. The exterior algebra inherits a grading from the tensor algebra. More precisely, we set the kth exterior power $\bigwedge^k(V)$ of V as the image of $T^k(V)$ inside $\bigwedge(V)$; it is naturally isomorphic, as a vector space, to $T^k(V)/I^k$, where $I^k = I(V) \cap T^k(V)$.

One occasionally denotes by $\pi : T(V) \to \bigwedge(V)$ the projection map, so that $\wedge = \pi \circ \otimes$, which elucidates that $\wedge$ is bilinear and $v \wedge v = 0$. More generally, one has a general anti-commutativity law $u \wedge v = (-1)^{kl} v \wedge u$, for $u \in \bigwedge^k(V)$ and $v \in \bigwedge^l(V)$. Similarly to the tensor product, if V has dimension n and a basis is given by $\{v_i\}_i$, the set

$$\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a basis of $\bigwedge^k(V)$. In particular, this says that the dimension of the kth exterior power is $\binom{n}{k}$.

There is a natural map $\wedge : V^k \to \bigwedge^k(V)$, which sends $(v_1, \ldots, v_k)$ to $v_1 \wedge \cdots \wedge v_k$; the kth exterior power, together with this map, satisfy the universal mapping property for alternating k-linear maps. More precisely, for every alternating k-linear map $f : V^k \to Z$, there exists a unique linear map $\widetilde{f} : \bigwedge^k(V) \to Z$, called the lift of f to $\bigwedge^k(V)$, satisfying $f = \widetilde{f} \circ \wedge$. We remark that this fact is proved by, first lifting the map to $T^k(V)$, and then realizing that it descends to the quotient, because the lift of f to $T^k(V)$ vanishes on I^k, by construction.

Similarly to the tensor product, one can prove that the space of alternating k-linear forms $\mathsf{Alt}^k(V)$ is naturally isomorphic to $(\bigwedge^k(V))^\star$. Moreover, the space is finite-dimensional, the linear map that sends $\alpha_1 \wedge \cdots \wedge \alpha_k \in \bigwedge^k(V^\star)$ to the map $v_1 \wedge \cdots \wedge v_k \mapsto \det(\alpha_i(v_j))$ in $(\bigwedge^k(V))^\star$ is an isomorphism. Therefore, in this situation, $\mathsf{Alt}^k(V)$ is isomorphic to $\bigwedge^k(V^\star)$. In this way, the constructions made for alternating k-tensors in Section 2.3 carry over to the latter.

8.1.3 Clifford algebras

In this section, V denotes a $\mathbb{K}$-vector space and B denotes a bilinear form on $V \times V$. The bilinear form induces a quadratic form $Q : V \to \mathbb{K}$, given by $Q(v) = B(v, v)$, for $v \in V$. A Clifford algebra for (V, Q) is a pair $(C\ell(V, Q), \jmath)$ with $C\ell(V, Q)$ a unital associative $\mathbb{K}$-algebra, and $\jmath : V \to C\ell(V, Q)$ a linear map satisfying $\jmath(v)^2 = Q(v)1$, for $v \in V$, which satisfies the following: any pair (A, u) satisfying the previous two properties, there exists a unique algebra homomorphism $\widetilde{U} : C\ell(Q) \to A$ such that $u = \widetilde{U} \circ j$, equivalently, the diagram

$$\begin{tikzcd}
V \arrow[r, "u"] \arrow[d, "\jmath"'] & A \\
C\ell(V,Q) \arrow[ru, "\tilde{U}"'] &
\end{tikzcd} \tag{8.1.1}$$

commutes.

For any (V, Q), there exists a Clifford algebra. One proves this as follows: let K be the two-sided ideal of $T(V)$ generated by elements of the form

$$v \otimes v - Q(v)1,$$

for $v \in V$. Then, set $C\ell(V, Q) = T(V)/K$ and $\jmath : V \to C\ell(V, Q)$ the map that sends $v \in V \subset T(V)$ to its image inside the quotient $C\ell(V, Q)$. As before, the quotient induces a product $\cdot$ on $C\ell(V, Q)$, given by the image of $\otimes$ inside $C\ell(V, Q)$. Since $v \otimes v - Q(v) \in K$, for $v \in V$, it follows that $\jmath(v)^2 = Q(v)1$, as desired. The verification of the last property is more involved, therefore we omit it. However, we remark that it is proved by considering another pair (A, u) and lifting u to the tensor algebra, one then discovers that it descends to the quotient $C\ell(V, Q)$, by definition of the ideal K.

As the constructions in the previous sections, $C\ell(V, Q)$ thus defined satisfies a universal mapping property, sometimes called the universal property of Clifford algebras. Furthermore, it is uniquely determined by this universal property, up to natural isomorphism. Therefore, from now on, we refer to $C\ell(V, Q)$ as **the** Clifford algebra associated with (V, Q).

Notice that the ideal K does not have any elements of degree 1, therefore the quotient leaves V invariant. This is $V \cong \jmath(V)$, justifying that sometimes, for $v \in V$, we write v instead of $\jmath(v)$. The property that V generates the tensor algebra is preserved, namely $\jmath(V)$ generates the Clifford algebra.

$C\ell(V,Q)$ may be equipped with an algebra endomorphism $\beta : C\ell(V,Q) \to C\ell(V,Q)$ which is also an involution, i.e., $\beta \circ \beta = \mathsf{id}_{C\ell(V,Q)}$. Moreover β is uniquely determined by the property $\beta(\jmath(v)) = -v$, for $v \in V$. According to this, β has exactly two eigenspaces:

$$C\ell^0(V,Q) = \{x \in C\ell(V,Q) \mid \beta(x) = x\} \quad \text{and} \quad C\ell^1(V,Q) = \{x \in C\ell(V,Q) \mid \beta(x) = -x\}\,,$$

which realize a direct sum $C\ell(V,Q)) = C\ell^0(V,Q) \oplus C\ell^1(V,Q)$. This gives a $\mathbb{Z}_2$-grading to the Clifford algebra.

When V has dimension n and B is non-degenerate, Lagrange's theorem ensures that there is a canonical basis, i.e., a basis in which the matrix associated to B is diagonal. Using this fact one proves that the dimension of the Clifford algebra is 2^n. Moreover, a basis $\{v_i\}_i$ of V multiplicatively generates the entire Clifford algebra and satisfies the **Clifford relations**:

$$v_i v_j + v_j v_i = B(v_i, v_j) + B(v_j, v_i). \tag{8.1.2}$$

A particular case of this is the relation satisfied by the famous γ matrices in Minkowski space, which form a representation of the Clifford algebra. A basis for $C\ell(V,Q)$ as a vector space is given by the unit 1 together with all the products of the form

$$v_{i_1} \cdots v_{i_r}, \quad \text{with} \;\; 1 \leq i_1 < \cdots < i_r \leq n \;\; \text{and} \;\; r \leq n\,. \tag{8.1.3}$$

Although much more can be said about Clifford algebras, this shall be more than enough for our purpose. We finish this section by remarking that the given basis resembles the basis of the exterior algebra. In fact, we notice that $C\ell(V,Q) \cong \bigwedge(V)$ as vector spaces, for they have the same dimension. Moreover, (8.1.2) is still valid if the form is degenerate, and so we can take $B \equiv 0$, and we recover the anti-commutativity of the exterior algebra. Thereby, they coincide, and so one may regard the exterior algebra as a very particular case of a Clifford algebra. Beyond this, the two algebras are intimately related; in the following section we shall see that they are (almost) naturally isomorphic. And thus one can regard Clifford elements as multivectors or, more generally, Clifford fields as differential forms. All this will be made precise in the forthcoming discussion.

8.2 From the Clifford algebra to the Exterior algebra

8.2.1 The exterior and Clifford bundles

Exterior bundle

Let $\mathcal{S}$ denote an n-dimensional smooth manifold and consider a smooth vector bundle $\pi : E \to \mathcal{S}$ on M of rank r. The process of collecting the kth exterior power $\bigwedge^k(E_x)$ at each point $x \in \mathcal{S}$ gives raise to a vector bundle. More precisely, denote

$$\bigwedge^k(E) = \bigsqcup_{x\in\mathcal{S}} \bigwedge^k(E_x).$$

Since E is a smooth vector bundle, $\bigwedge^k(E)$ is a vector bundle of rank $\binom{r}{k}$, called the kth exterior power bundle of E. It is clear what the projection map $\hat{\pi}$ should be. Local trivializations may be constructed as follows: for $x \in \mathcal{S}$, choose a local trivialization $\phi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{R}^r$ of E, with $\mathcal{U}$ an open neighborhood of x in $\mathcal{S}$, and define $\Phi_{\mathcal{U}} : \hat{\pi}^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{R}^{\binom{r}{k}}$ as the fiberwise linear map which satisfies, for every $x \in \mathcal{S}$ and $e_1, \ldots, e_k \in E_x$,

$$\Phi_{\mathcal{U}}(e_1 \wedge \cdots \wedge e_k) = (x, \phi_{\mathcal{U}}(e_1), \ldots, \phi_{\mathcal{U}}(e_k))\,.$$

It can be checked that the smoothness of $\phi_{\mathcal{U}}$ is inherited by $\Phi_{\mathcal{U}}$.

Remark 8.2.1. Here, we chose to follow the more intuitive construction, but it is worth mentioning that the resulting vector bundle coincides with the fiber product between the frame bundle associated with E and the kth exterior power of $\mathbb{R}^r$, through the usual representation from $\mathsf{GL}(n)$ into the group of automorphisms of $\bigwedge^k(\mathbb{R}^r)$. ◇

In addition, if we sum over k, we get

$$\bigwedge(E) = \bigoplus_{k=0}^{r} \bigwedge^{k}(E)\,,$$

which is called the **exterior bundle** of E.

We are particularly interested in the cases $E = T\mathcal{S}$ and $E = T^\star\mathcal{S}$. In the latter case, we know that the sections of $\bigwedge^k E$ correspond to differential k-forms. Namely $\Gamma(\bigwedge^k T^\star\mathcal{S}) = \Lambda^k(\mathcal{S})$.

Remark 8.2.2. This construction is also valid for tensor powers of E. The kth tensor power of E is denoted $T^k(E)$, and the tensor bundle is denoted $T(E) = \bigoplus_{k=0}^{\infty} T^k(E)$. When $E = T^\star\mathcal{S}$ the sections of $T^k(E)$ are simply k-tensor fields, because they assign, to each $x \in \mathcal{S}$, a k-tensor in $\mathcal{L}^k(T_x\mathcal{S})$. ◇

A connection ∇ on $\mathcal{S}$ induces a unique connection $\widetilde{\nabla}$ on the tensor bundle of $T^\star\mathcal{S}$, by requiring that:

(i) for all $X, Y \in \Gamma(T\mathcal{S})$ and $\alpha \in \Gamma(T^\star\mathcal{S})$, there holds

$$X\left\langle \alpha, Y\right\rangle = \left\langle \widetilde{\nabla}_X \alpha, Y\right\rangle + \left\langle \alpha, \nabla_X Y\right\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical duality pairing.

(ii) for $X \in \Gamma(T\mathcal{S})$ and $\alpha_1, \cdots, \alpha_k \in \Gamma(T^\star\mathcal{S})$, there holds

$$\widetilde{\nabla}_X(\alpha_1 \otimes \cdots \otimes \alpha_k) = \sum_{i=1}^{k} \alpha_1 \otimes \cdots \otimes \widetilde{\nabla}_X \alpha_i \otimes \cdots \otimes \alpha_k,$$

a Leibniz rule for the tensor product.

It is customary to denote $\widetilde{\nabla}$ by ∇, as it is compatible with the connection of $\mathcal{S}$ and thus no confusion should arise.

Since the wedge product is the antisymmetrization of the tensor product, from the Leibniz rule for the latter, one can deduce a Leibniz rule for the wedge product, which reads

$$\nabla_X(\omega \wedge \eta) = \nabla_X(\omega) \wedge \eta + \omega \wedge \nabla_X \eta\,,$$

for $X \in \mathfrak{X}(\mathcal{S})$, and $\omega, \eta \in \mathrm{C}^\infty\Lambda^\bullet(T\mathcal{S})$.

In a completely analogous way to the previous constructions, we may construct the **Clifford bundle** associated with E. Namely, we put

$$C\ell(E) = \bigsqcup_{x \in \mathcal{S}} C\ell(E_x).$$

The operations on the Clifford algebra such as the Clifford product pass to the Clifford bundle. Moreover, notice that, since $T_x\mathcal{S}$ has a canonical embedding into $C\ell(T_x\mathcal{S})$, for $x \in \mathcal{S}$, there holds $\Gamma(T\mathcal{S}) \subset \Gamma(C\ell(T\mathcal{S}, g))$. Here, sections of $C\ell(T\mathcal{S})$ will be called **Clifford fields**.

A connection on $\mathcal{S}$ also induces a connection $\nabla^{C\ell}$ on the Clifford bundle, this time by requiring that it is compatible on vector fields and respects the Clifford product:

(i) for $X, Y \in \Gamma(T\mathcal{S})$, there holds $\nabla^{C\ell}_X Y = \nabla_X Y$, and

(ii) for $X \in \Gamma(T\mathcal{S})$ and $\psi, \varphi \in \Gamma(C\ell(T\mathcal{S}))$

$$\nabla_X(\psi \cdot \varphi) = (\nabla_X \psi) \cdot \varphi + \psi \cdot \nabla_X \varphi,$$

a Leibniz rule for the Clifford product.

8.2.2 A natural isomorphism

In this section we shall show how the exterior algebra of $V^\star$ may be regarded as a Clifford algebra for an inner-product space (V, g).

The algebra of alternating forms as a Clifford algebra

The following construction is less standard, though we make no claim of originality, as many similar treatments appear in the literature from various viewpoints.

First, given an n-dimensional inner product space (V, g), we shall define a Clifford-like product on the exterior algebra of $V^\star$: for $\alpha \in V^\star$, define

$$c(\alpha) : \mathsf{Alt}^\bullet(V) \to \mathsf{Alt}^\bullet(V) \quad \text{by} \quad c(\alpha)(\omega) = \alpha^\sharp \lrcorner\, \omega + \alpha \wedge \omega,$$

where $\sharp$ is the sharp map or Riesz isomorphism, defined by the relation $\alpha(v) = g(\alpha^\sharp, v)$, for $v \in V$.

Now, we use the induced map c to define a product $\vee : \mathsf{Alt}^\bullet(V) \times \mathsf{Alt}^\bullet(V) \to \mathsf{Alt}^\bullet(V)$ by requiring linearity in its first argument and

$$(\alpha_1 \wedge \cdots \wedge \alpha_k) \vee \omega := (c(\alpha_1) \circ \cdots \circ c(\alpha_k))(\omega)\,,$$

for $\alpha_i \in V^\star$, $i \in \{1, \ldots, k\}$ and $\omega \in \mathsf{Alt}^\bullet(V)$. We decree $\vee$-multiplication by scalars to be usual scalar multiplication.

We endow $V^\star$ with the usual inner product: $g(\alpha, \beta) := g(\alpha^\sharp, \beta^\sharp)$, for all $\alpha, \beta \in V^\star$.

Lemma 8.2.1. *Let $\alpha_1, \ldots, \alpha_k \in V^\star$ be pairwise orthogonal. Then,*

$$\alpha_1 \vee \cdots \vee \alpha_k = \alpha_1 \wedge \cdots \wedge \alpha_k\,.$$

Proof. We proceed by induction on k. For $k = 2$, we have

$$\alpha_1 \vee \alpha_2 = {\alpha_1}^\sharp \lrcorner\, \alpha_2 + \alpha_1 \wedge \alpha_2 = g(\alpha_1, \alpha_2) + \alpha_2 \wedge \alpha_2 = \alpha_1 \wedge \alpha_2\,.$$

Now, let us assume this holds for k and prove it for $k+1$. Associativity gives

$$\alpha_1 \vee \cdots \vee \alpha_{k+1} = \alpha_1 \vee (\alpha_2 \wedge \cdots \wedge \alpha_{k+1}) = \alpha_1 \lrcorner\, (\alpha_2 \wedge \cdots \wedge \alpha_{k+1}) + \alpha_1 \wedge \cdots \wedge \alpha_{k+1}\,.$$

The contraction of a wedge-product satisfies the following formula

$$\alpha_1 \lrcorner\, (\alpha_2 \wedge \cdots \wedge \alpha_{k+1}) = \sum_{j=2}^{k+1} (-1)^{j-1} \alpha_j(\alpha_1)\, \alpha_2 \wedge \cdots \wedge \widehat{\alpha_j} \wedge \cdots \wedge \alpha_{k+1}\,,$$

where $\widehat{}$ indicates a suppressed term. Therefore, by orthogonality

$$\alpha_1 \lrcorner (\alpha_2 \wedge \cdots \wedge \alpha_{k+1}) = 0\,.$$

This finishes the inductive step and proves the desired identity. □

Denote by $\imath : V \hookrightarrow (\mathsf{Alt}^\bullet(V), \vee)$ the embedding $v \mapsto v^\flat \in V^\star \subset \mathsf{Alt}^\bullet(V)$, where $\flat := \sharp^{-1}$ is the flat isomorphism. Then, we have the following result:

Proposition 8.2.1. *$C\ell(V,g)$ and $(\mathsf{Alt}^\bullet(V), \vee)$ are naturally isomorphic as algebras.*

Proof. We shall use the universal mapping property of Clifford algebras.

We begin by noticing that, since composition is associative, $\vee$-multiplication is as well. This makes $(\mathsf{Alt}^\bullet(V), \vee)$ into a unital associative algebra.

In turn, $\imath$ is linear and there holds

$$v^\flat \vee v^\flat = v \lrcorner v^\flat + v^\flat \wedge v^\flat = g(v,v),$$

for all $v \in V$, which amounts to say that $\imath(v)^2 = g(v,v)$. Thereby, we have checked that the first two defining properties of Clifford algebras (see Section 8.1.3) hold.

Now, we make use of the universal property of the Clifford algebra $C\ell(V,g)$. Namely, by the previous discussion, the inclusion $\imath$ lifts to a unique algebra homomorphism

$$\Theta : C\ell(V,g) \to (\mathsf{Alt}^\bullet(V), \vee), \quad \text{satisfying } \; \Theta(\jmath(v)) = \imath(v) = v^\flat, \quad \text{for } v \in V\,,$$

where $\jmath$ is the canonical inclusion $V \hookrightarrow C\ell(V,g)$. We claim that Θ is an isomorphism of algebras. To see this, consider an orthonormal basis $\{v_i\}_i$ of V, and denote by $\{v^i\}_i$ the dual basis, which satisfies $v^i = {v_i}^\flat$, and recall that

$$\bigsqcup_{k=1}^{n} \{v_{i_1} \cdots v_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}\,,$$

together with the unit 1 constitutes a basis of $C\ell(V,g)$ (cf. (8.1.3)). Here and in what follows we identify $\jmath(\alpha_{i_s})$ with α_{i_s}, $1 \leq s \leq k$, for notational convenience.

Since Θ is an algebra homomorphism, given a positive integer k and $1 \leq i_1 < \cdots < i_k \leq n$, we have

$$\Theta(v_{i_1} \cdots v_{i_k}) = \Theta(v_{i_1}) \vee \cdots \vee \Theta(v_{i_k}) = v^{i_1} \vee \cdots \vee v^{i_k}.$$

Moreover, since the basis is orthonormal, Lemma 8.2.1 gives

$$\Theta(v_{i_1} \cdots v_{i_k}) = v^{i_1} \wedge \cdots \wedge v^{i_k}\,.$$

Namely, Θ maps a basis of $C\ell(V, g)$ to a basis of $(\mathsf{Alt}^\bullet(V), \vee)$, hence an algebra isomorphism. $\square$

In this way, we have showed that $(\mathsf{Alt}^\bullet(V), \vee)$ is a Clifford algebra. The triple $(\mathsf{Alt}^\bullet(V), \wedge, \vee)$ is sometimes called a Kähler-Atiyah algebra and, as expected, it brings together the Clifford and exterior algebras. For more details about this and implications for spinors, we refer to [Gra78].

From Clifford fields to differential forms

Given a smooth Riemannian manifold $(\mathcal{S}, g)$, the $\vee$-product we defined carries over pointwise to the bundle of differential forms, just as the Clifford and wedge products do. Moreover, by Proposition 8.2.1, at every point $x \in \mathcal{S}$, we have an algebra isomorphism

$$\Theta_x : C\ell(T_x\mathcal{S}, g_x) \to (\mathsf{Alt}^\bullet(T_x\mathcal{S}), \vee)\,.$$

We have the following result:

Proposition 8.2.2. *The induced mapping* $\Theta : C\ell(T\mathcal{S}, g) \to (\mathsf{Alt}^\bullet(T\mathcal{S}), \vee)$ *is a smooth isomorphism.*

Proof. Around a point $x \in \mathcal{S}$, choose a local, orthonormal smooth frame, $\{e_i\}_i$ for $T\mathcal{S}$. Denote by $\{e^i\}_i$ the dual frame. The vector fields e_i induce a local frame for $C\ell(T\mathcal{S}, g)$, given by the Clifford products

$$\bigsqcup_{k=1}^{n} \{e_{i_1} \cdots e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\} \tag{8.2.1}$$

and the unit $x \mapsto 1$. Similarly, the covector fields e^i induce a frame for $\mathsf{Alt}^\bullet(T\mathcal{S})$

$$\bigsqcup_{k=1}^{n} \left\{ e^{i_1} \wedge \cdots \wedge e^{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n \right\} \tag{8.2.2}$$

together with the unit $x \mapsto 1$. According to the proof of Proposition 8.2.1, at each point in the chosen neighborhood, Θ maps (8.2.1) into (8.2.2) in a one-to-one fashion. Thereby it is smooth. $\square$

A smooth isomorphism between total spaces of bundles induces an isomorphism on smooth sections

$$\widetilde{\Theta} : \Gamma(C\ell(T\mathcal{S}, g)) \to \mathrm{C}^\infty\Lambda^\bullet(\mathcal{S}), \quad \psi \mapsto \Theta \circ \psi\,,$$

for all $\psi \in \Gamma(C\ell(T\mathcal{S}, g))$.

It can be easily checked that, just as Θ, $\widetilde{\Theta}$ respects the products. Namely,

$$\widetilde{\Theta}(\psi \cdot \varphi) = \widetilde{\Theta}(\psi) \vee \widetilde{\Theta}(\varphi) \quad \text{for} \quad \psi, \varphi \in \Gamma(C\ell(T\mathcal{S}, g))\,.$$

Now, we shall study its interaction with the connections. We have the following result in this direction:

Proposition 8.2.3. *There holds* $\widetilde{\Theta}\, \nabla^{C\ell} = \nabla \widetilde{\Theta}$.

Proof. For vector fields this is immediate from the compatiblity of both connections on vector fields. Namely

$$\widetilde{\Theta}(\nabla^{C\ell}\psi) = \widetilde{\Theta}(\nabla\psi) = (\nabla\psi)^\flat = \nabla(\psi^\flat) = \nabla(\widetilde{\Theta}(\psi))\,,$$

for $\psi \in \Gamma(T\mathcal{S}) \hookrightarrow \Gamma(C\ell(T\mathcal{S}, g))$.

Since vector fields generate the Clifford bundle and both connections satisfy a Leibniz rule for the products, this can be extended to any Clifford field by an inductive step. By reasons of economy, we omit further details. $\square$

Dirac operators

The Clifford bundle carries an operator of Dirac type, which we will call the **Clifford-Dirac operator**. It is defined as:

$$\mathcal{D}^{C\ell} : \Gamma(C\ell(T\mathcal{S}, g)) \to \Gamma(C\ell(T\mathcal{S}, g)), \qquad \psi \mapsto \sum_i e_i \cdot \nabla^{C\ell}_{e_i} \psi,$$

for any local, orthonormal smooth frame $\{e_i\}_i$ for $T\mathcal{S}$. It can be proved that this definition does not depend on the choice of local frame.

We devote the remaining of this section to show how this operator, acting on Clifford fields, transforms into the *Dirac-Kähler operator*, acting on differential forms.

First, we observe that, since $\widetilde{\Theta}$ is an isomorphism, we may use it to define an operator on differential forms. Namely, we may define

$$\mathcal{D} : \mathrm{C}^\infty\Lambda^\bullet(\mathcal{S}) \to \mathrm{C}^\infty\Lambda^\bullet(\mathcal{S}), \quad \text{as} \quad \mathcal{D} := \widetilde{\Theta}\, \mathcal{D}^{C\ell}\, \widetilde{\Theta}^{-1}.$$

We have the following result:

Proposition 8.2.4. *$\mathcal{D} = \mathsf{d} - \mathsf{d}^\star$, the **Dirac-Kähler operator**.*

Proof. By the previous proposition, for $\omega \in \mathrm{C}^\infty\Lambda^k(\mathcal{S})$, we have

$$\mathcal{D}\omega = \sum_i \widetilde{\Theta}(e_i \cdot \nabla^{C\ell}_{e_i} (\widetilde{\Theta}^{-1}\omega)) = \sum_i {e_i}^\flat \vee (\widetilde{\Theta}\nabla^{C\ell}_{e_i} \widetilde{\Theta}^{-1})(\omega) = \sum_i {e_i}^\flat \vee \nabla_{e_i}\omega,$$

whence

$$\mathcal{D}\omega = \sum_i {e_i}^\flat \wedge \nabla_{e_i}\omega + \sum_i e_i \lrcorner \nabla_{e_i}\omega.$$

It is a direct computation to deduce, from the identity (see [Pet16, Section 2.2.2.2])

$$(\mathsf{d}\omega)(X_0, \ldots, X_k) = \sum_i (-1)^i (\nabla_{X_i}\omega)(X_0, \ldots, \widehat{X_i}, \ldots, X_k)\,,$$

that

$$\mathsf{d}\omega = \sum_i {e_i}^\flat \wedge \nabla_{e_i}\omega \,. \tag{8.2.3}$$

In turn, using (8.2.3) and the fact that $\mathsf{d}^\star = \star^{-1}\,\mathsf{d}\,\star$, it is similarly proved that

$$\mathsf{d}^\star\omega = -\sum_i e_i \lrcorner \nabla_{e_i}\omega \,.$$

Therefore $\mathcal{D} = \mathsf{d} - \mathsf{d}^\star$. For details on these computations, we refer to [Gra78, Section 5, F)]. □

In this way, we have showed that the Clifford bundle identifies in a natural way with a Kähler-Atiyah bundle and, moreover, the Clifford-Dirac operator corresponds to the Dirac-Kähler operator acting on differential forms. We hope this serves as a motivation for studying the properties of the latter.

CHAPTER 9

Discrete Fredholm theory

In this chapter, we build on the theory developed in [Chr04] to arrive at a notion of discrete Fredholmness and discrete Fredholm index. Then, we study stability properties of discrete Fredholm operators, and their close relation to stable inf-sup conditions. We postulate that the concepts here proposed are natural from the point of view of preserving analytic properties such as the continuous Fredholm index and the invariance under compact perturbations of the traditional Fredholm theory.

9.1 Discrete Fredholm operators

We begin by recalling that a linear and bounded operator $A : X \to Y$ between Banach spaces is said to be **left semi-Fredholm** if it has closed range and finite-dimensional kernel. The set of all such operators, denoted $\mathsf{LSF}(X, Y)$, is stable under translation by compact operators.

To motivate the foregoing discussion, we state the following characterization:

Proposition 9.1.1. *Given a linear and bounded operator $A : X \to Y$ between Banach spaces,* *the following conditions are equivalent:*

i) *A has closed range and finite-dimensional kernel,*

ii) *A has finite-dimensional kernel, and on any supplementary space M of $\operatorname{ker} A$, there holds*

$$\inf_{x\in M}\sup_{l\in Y^\star}\frac{l(Ax)}{\|x\|\|l\|}>0\,,$$

iii) *there is a closed, finite-codimensional subspace M of X satisfying*

$$\inf_{x\in M}\sup_{l\in Y^\star}\frac{l(Ax)}{\|x\|\|l\|}>0\,.$$

To walk towards a discrete notion of left semi-Fredholmness, we shall focus on the third property given by this result.

Given a Banach space Y, we say that a family $(Y_h)_h$ of closed subspaces of Y is **approximating** if, for every $y \in Y$, there exists a sequence $(y_h)_h$ in $(Y_h)_h$ converging to y. Here and in what follows, the parameters h are positive real numbers accumulating at zero.

The following result corresponds to Lemma 1.5 in [Chr04].

Proposition 9.1.2. *Consider a linear and bounded operator $A : X \to Y^\star$ between Banach spaces, and a closed subspace M of X. Given sequences $(X_h)_h$ and $(Y_h)_h$ of closed subspaces of X and Y, respectively. If*

$$\liminf_h \inf_{x\in X_h\cap M}\sup_{y\in Y_h}\frac{A(x)(y)}{\|x\|\|y\|}>0,$$

then $\operatorname{ker} A\cap M=\{0\}$. Furthermore, if M is finite-codimensional, then $A\in \mathsf{LSF}(X,Y)$.

Remark 9.1.1. Since in this chapter we work in the setting of Banach spaces, we consider operators mapping into a dual space. In practice, when the target space is Hilbert, this is not necessary, as we may use the inner product to pair Y with itself. Moreover, all the results in this section carry over to the Hilbertian setting by using the Riesz isomorphism. ◇

Lemma 9.1.1. *Consider a Banach space X and closed subspaces M and U such that $X = M\oplus U$, where U is finite-dimensional, and an approximating family $(X_h)_h$ of closed subspaces of X. Denote by $P : X \to X$ the projector with range M and kernel U. Then,* *(i) there is a family $(P_h)_h$ of projectors $P_h : X \to X$ with range M, which converges in operator norm to P and leaves X_h invariant, for sufficiently small h.*

(ii) For a sequence $(P_h)_h$ satisfying the previous properties, for h sufficiently small, we have

$$\ker P_h \subset X_h \quad \text{and} \quad X_h = (X_h \cap M) \oplus \ker P_h .$$

Proof. For any basis $\{e^i\}_i$ of U, by the approximation property of $(X_h)_h$, we can find elements $\{e^i_h\}$ in X_h such that $\|e^i - e^i_h\| \xrightarrow{h} 0$. Then, setting $U_h = \mathsf{span}\,\{e^i_h\}$, for h smaller than a threshold h_0, we have $X = M \oplus U_h$. For $h > h_0$, define $P_h = P$, and for $h \leq h_0$, define $P_h : X \to X$ as the projector with range P and kernel U_h. The sequence $(P_h)_h$ has the desired properties. For more details we refer to [Chr04, Lemma 1.6]. □

Proposition 9.1.3. *Let X and Y be reflexive Banach spaces, and consider a left semi-Fredholm operator $A : X \to Y^\star$ with kernel $U := \ker A$. Moreover, let $(X_h)_h$ and $(Y_h)_h$ be approximating sequences of closed subspaces of X and Y, respectively.*

Given a supplementary space M of U in X. Suppose we are given a closed subspace M' of M such that M/M' is finite-dimensional. If

$$\liminf_h \inf_{x \in X_h \cap M'} \sup_{y \in Y_h} \frac{A(x)(y)}{\|x\|\|y\|} > 0 \,,$$

then

$$\liminf_h \inf_{x \in X_h \cap M} \sup_{y \in Y_h} \frac{A(x)(y)}{\|x\|\|y\|} > 0 \,.$$

Proof. This is Theorem 1.8 in [Chr04]. □

According to this result, if we want a stable inf-sup condition on $X_h \cap M$, it suffices to prove it for any smaller subspace M' that is finite-codimensional in M.

Proposition 9.1.4. *Let X and Y be reflexive Banach spaces, and consider a left semi-Fredholm operator $A : X \to Y^\star$ with kernel $U := \ker A$. Moreover, let $(X_h)_h$ and $(Y_h)_h$ be approximating sequences of closed subspaces of X and Y, respectively.*

Consider two supplementary spaces M and M' of U in X. Assume that

$$\liminf_h \inf_{x\in X_h\cap M} \sup_{y\in Y_h} \frac{A(x)(y)}{\|x\|\|y\|} > 0\,,$$

then

$$\liminf_h \inf_{x\in X_h\cap M'} \sup_{y\in Y_h} \frac{A(x)(y)}{\|x\|\|y\|} > 0\,.$$

Proof. This is Theorem 1.9 in [Chr04]. □

The previous results motivate the following definition:

Definition 9.1.1. Let X and Y be reflexive Banach spaces. Given a linear and bounded operator $A : X \to Y^\star$ and sequences $(X_h)_h$ and $(Y_h)_h$ of closed, approximating subspaces of X and Y, respectively. We say that A is **discrete left semi-Fredholm** on $(X_h \times Y_h)_h$ if there is a closed, finite-codimensional subspace M of X such that

$$\liminf_h \inf_{x\in X_h\cap M} \sup_{y\in Y_h} \frac{A(x)(y)}{\|x\|\|y\|} > 0. \tag{9.1.1}$$

Although we shall see that the property of being discrete LSF does not depend on a choice of M, if we want to make a particular choice we say that A is discrete LSF on $(X_h \times Y_h)_h$ with respect to M. ◇

Corollary 9.1.1. *If A is discrete LSF on $(X_h \times Y_h)_h$ with respect to $\bar{M}$, then it is so with respect to any supplementary space of $\ker A$ in X.*

Proof. Consider $\bar{M}$ as in the statement. Notice that, by Proposition 9.1.2, $\bar{M} \cap \ker A = \{0\}$. Therefore, $\bar{M} \oplus \ker A$ is a closed, finite-codimensional subspace of X. If V is a supplementary space, we get

$$X = V \oplus \bar{M} \oplus \ker A,$$

whence $V \oplus M$ is a supplementary space of $\ker A$.

Since, in addition, V is finite-dimensional, Proposition 9.1.3 implies that the inf-sup condition (9.1.1) extends from $((X_h \cap \bar{M}) \times Y_h)_h$ to $((X_h \cap (V \oplus M)) \times Y_h)_h$. Thus, we have showed that A is discrete LSF with respect to a supplementary space of $\ker A$ and, therefore, by Proposition 9.1.4, it is so with respect to any other supplementary space of $\ker A$. $\square$

Now, we give a few remarks which will indicate the natural definition for discrete right semi-Fredholmness. First, we have a simple lemma:

Lemma 9.1.2. *Given a closed subspace V of a normed space X. There holds*

$$V^\circ \cong (X/V)^\star ,$$

where $V^\circ = \{f \in X^\star \mid f|_V \equiv 0\}$ is the annihilator.

Proof. Denote by $\pi : X \to X/V$ the quotient map. For $f \in V^\circ$ and $x \in X$, we notice that $f(x+V)$ is a singleton. Therefore, $\widetilde{f} : X/V \to \mathbb{R}$ defined by $\widetilde{f} \circ \pi = f$ is well-defined. Linearity of $\widetilde{f}$ is immediate, and boundedness comes from

$$|\widetilde{f}(\pi(x))| = |f(x+v)| \leq \|f\| \|x+v\| \qquad \forall\, v \in V$$

whence

$$|\widetilde{f}(\pi(x))| \leq \|f\| \inf_{v \in V} \|x+v\| = \|f\| \|\pi(x)\|_{X/V} .$$

This induces a map $\Phi : V^\circ \to (X/V)^\star$ given by $f \mapsto \Phi(f) = \widetilde{f}$. Φ is injective by definition. Indeed, if $\Phi(f) = 0$, then $f = \widetilde{f} \circ \pi \equiv 0$. Surjectivity follows from the fact that, for any $G \in (X/V)^\star$, there holds $\Phi(G \circ \pi) = G$. $\square$

We recall that $A : X \to Y$ is said to be **right semi-Fredholm** if it has finite-codimensional range. In fact, one usually requires the range to be closed but, as showed in [Mül07, Lemma 16.2], a finite-codimensional range is always closed (see also [Hör07, Lemma 19.1.1]).

We have the following characterization

Proposition 9.1.5. *For a linear and bounded operator $A : X \to Y$ between normed vector spaces, the following conditions are equivalent:*

i) *A has finite-codimensional range*

ii) $A^* : Y^\star \to X^\star$ *has closed range and finite-dimensional kernel.*

Proof. If A has finite-codimensional range, then $\mathsf{Im}(A)$ is closed. By duality in Banach spaces [Gat24, Theorem 3.2] and Lemma 9.1.2, there holds

$$\mathsf{ker}(A^*) = \mathsf{Im}(A)^\circ \equiv (Y/\,\mathsf{Im}(A))^\star. \tag{9.1.2}$$

Since the right-most space is finite-dimensional, so is $\mathsf{ker}(A^*)$. Conversely, if $\mathsf{ker}(A^*)$ is finite-dimensional and $\mathsf{Im}(A)$ is closed (9.1.2) allows to conclude that $\mathsf{Im}(A)$ is finite-codimensional. □

Now, given an operator $A : X \to Y^\star$, we define its transpose $A^{\mathsf{t}} \colon Y \to X^\star$ as the unique linear and bounded operator characterized by the relation

$$A^{\mathsf{t}}(y)(x) = A(x)(y) \qquad \forall\, x \in X,\ y \in Y.$$

Denoting $A^* : Y^{\star\star} \to X^\star$ the Banach adjoint of A, it is direct to check that $A^{\mathsf{t}} = A^* \mathcal{J}_Y$, where $\mathcal{J}_Y : Y \to Y^{\star\star}$ is the canonical injection. According to reflexivity, this identity and the previous proposition, A is right semi-Fredholm if and only if A^{t} is left semi-Fredholm. This motivates the following definition:

Definition 9.1.2. Given an operator $A : X \to Y^\star$ between reflexive Banach spaces, consider sequences $(X_h)_h$ and $(Y_h)_h$ of approximating subspaces of X and Y, respectively. A is said to be **discrete right semi-Fredholm** on $(X_h \times Y_h)_h$ if A^{t} is discrete right semi-Fredholm on $(Y_h \times X_h)_h$. Namely, there exists a closed, finite-codimensional subspace N of Y such that

$$\liminf_h \inf_{y \in Y_h \cap N} \sup_{x \in X_h} \frac{A(x)(y)}{\|x\|\|y\|} > 0\,.$$

Moreover, we say that A is **discrete Fredholm** on $(X_h \times Y_h)_h$ if it is both discrete LSF and discrete RSF. ◇

By virtue of this definition, we may obtain any property related to discrete right semi-Fredholmness by reasoning in analogy with discrete left semi-Fredholmness of the transpose.

Corollary 9.1.2. *In the setting of the previous definition, if A is discrete Fredholm on $(X_h \times Y_h)_h$, then A is Fredholm.*

Proof. It follows from applying Proposition 9.1.2 to A and A^{t}. ☐

Compact perturbations of discrete Fredholm operators

Proposition 9.1.6. *Let X and Y be reflexive Banach spaces, and let $A : X \to Y^\star$ be a left semi-Fredholm operator and $B : X \to Y^\star$ be compact. Consider families $(X_h)_h$ and $(Y_h)_h$ of closed subspaces of X and Y, respectively. Suppose that $(Y_h)_h$ is approximating and that M is a closed subspace of X which satisfies*

$$\liminf_h \inf_{x \in X_h \cap M} \sup_{y \in Y_h} \frac{A(x)(y)}{\|x\|\|y\|} > 0.$$

Let $N' = \mathsf{ker}((A+B)|_M)$, and suppose there is a closed subspace M' of M such that $M = M' \oplus N'$. Then

$$\liminf_h \inf_{x \in X_h \cap M'} \sup_{y \in Y_h} \frac{(A+B)(x)(y)}{\|x\|\|y\|} > 0.$$

Proof. This is Theorem 1.12 in [Chr04]. ☐

Corollary 9.1.3. *In the aforementioned setting, if A is discrete left semi-Fredholm on $(X_h \times Y_h)_h$, then $A + B$ is discrete left semi-Fredholm as well.*

Corollary 9.1.4. *Assume that A is discrete left semi-Fredholm on $(X_h \times Y_h)_h$ and B is compact. If $A + B$ is injective, then $A + B$ is discrete left semi-Fredholm with respect to X. Namely*

$$\liminf_h \inf_{x \in X_h} \sup_{y \in Y_h} \frac{(A+B)(x)(x)}{\|x\|\|y\|} > 0\,.$$

Proof. This follows from the previous result and Corollary 9.1.1. ☐

Compare this with Proposition 4.2.1 in Part I.

Discrete Fredholm index

We finish this section by introducing a notion of discrete Fredholm index, which will help us with language in Chapter 10. To this end we denote $\mathsf{ind}\, D = \dim(\mathsf{ker}\, A) - \dim(Y^\star / \mathsf{Im}(A))$.

Definition 9.1.3. In the aforementioned setting, for an operator $A : X \to Y^\star$ that is discrete Fredholm on $(X_h \times Y_h)_h$, we define the **discrete Fredholm index** of A as

$$\mathsf{ind}_{\mathsf{d}}(A) := \inf_{M} \dim(X/M) - \inf_{N} \dim(Y/N),$$

where the infimums are taken over all the possible subspaces $M \subset X$ and $N \subset Y$ with respect to which A is discrete Fredholm on $(X_h \times Y_h)_h$. ◇

This is well-defined. Indeed, in the following result we prove that this definition is rather trivial once we understand discrete Fredholmness.

Proposition 9.1.7. *Let $A : X \to Y^\star$ be discrete Fredholm on $(X_h \times Y_h)_h$. Then*

$$\mathsf{ind}_{\mathsf{d}}(A) = \mathsf{ind}(A).$$

Proof. First, we recall that discrete Fredholmness implies Fredholmness (see Corollary 9.1.2). Then, from Corollary 9.1.1, we get that A is discrete Fredholm on $(X_h \times Y_h)_h$ with respect to any pair (M, N) of supplementary spaces of $(\mathsf{ker}\, A, \mathsf{ker}\, A^{\mathsf{t}})$. Therefore, we have that

$$\inf_{M'} \dim(X/M') \leq \dim(\mathsf{ker}\, A) \quad \text{and} \quad \inf_{N'} \dim(X/N') \leq \dim(\mathsf{ker}\, A^{\mathsf{t}}).$$

Now, we prove the converse inequalities. We know that every admissible subspace M' in the first infimum satisfies

$$\dim(X/M') < +\infty \quad \text{and} \quad M' \cap \mathsf{ker}\, A = \{0\}.$$

Denote by $\pi : X \to X/M'$ the canonical projection, and observe that $\pi|_{\mathsf{ker}\, A}$ is injective, because otherwise $M' \cap \mathsf{ker}\, A$ would be nontrivial. Therefore, we have

$$\dim(\mathsf{ker}\, A) = \dim(\pi(\mathsf{ker}\, A)) \leq \dim(\pi(X)) \leq \dim(X/M').$$

It follows that $\dim(\mathsf{ker}\, A) \leq \inf_{M'} \dim(X/M')$. The same reasoning works for the second infimum, and so we obtain $\dim(\mathsf{ker}\, A^{\mathsf{t}}) \leq \inf_{N'} \dim(Y/N')$. In summary, we have proved

$$\mathsf{ind}_{\mathsf{d}}(A) = \dim(\mathsf{ker}\, A) - \dim(\mathsf{ker}\, A^{\mathsf{t}}) = \mathsf{ind}(A),$$

where in the last inequality we used that

$$\dim(\mathsf{ker}\, A^{\mathsf{t}}) = \dim(\mathsf{ker}\, A^{*}) = \dim(Y^{\star}/\, \mathsf{Im}\, A)\,.$$

□

Remark 9.1.2. We can interpret the previous proposition in two ways: the first one is that we gave a trivial definition. In the sense that, every time the index is well-defined, it is already determined by the continuous one and so there is no need to compute it. However, a second point of view could be that this is a sign that the notion of discrete Fredholmness we defined is the correct one: it is strong enough to always preserve the Fredholm index. ◇

9.2 Stability properties of discrete Fredholm operators

In this section, X and Y denote reflexive Banach spaces and $A : X \to Y^{\star}$ denotes a linear and bounded operator with finite-dimensional kernel G. In addition, let $(X_h)_h$ and $(Y_h)_h$ denote sequences of approximating subspaces of X and Y, respectively.

We let $A_h : X_h \to Y_h^{\star}$ denote the operator characterized by

$$A_h(x)(y) = A(x)(y) \qquad \forall\, x \in X_h,\ y \in Y_h,$$

and denote by G_h its kernel.

To ease the reading, we shall say that A_h satisfies a uniform or stable inf-sup condition on

$(X_h \times Y_h)_h$ if, for all h, there holds

$$\inf_{x\in X_h} \sup_{y\in Y_h} \frac{A_h(x)(y)}{\|x\|\|y\|} \succeq 1\,.$$

Moreover, if such condition only holds for h smaller than some threshold, we say that an asymptotic inf-sup condition on $(X_h \times Y_h)_h$ holds. This amounts to having

$$\liminf_{h} \inf_{x\in X_h} \sup_{y\in Y_h} \frac{A_h(x)(y)}{\|x\|\|y\|} > 0\,.$$

We have two preliminary results:

Lemma 9.2.1. *Let U be a finite-dimensional subspace of X. Consider supplementary spaces M and M' of U in X. Then, the projector $P' : X \to X$ with range M' and kernel U induces an isomorphism $P'|_M$: $M \to M'$, whose inverse is $P|_{M'}$, where $P : X \to X$ is the projection with range M and kernel U.*

Proof. Given $m \in M$ such that $P'm = 0$, since $\mathsf{ker}\, P' = U$, we have that $m \in M \cap U = \{0\}$, hence $P'|_M$ is injective.

Now, given $m' \in M'$, there exist $m \in M$ and $u \in U$ such that

$$m' = m + u,$$

whence $P'm = P'(m + u) = P'm' = m'$, as P' is a projector onto M'. This proves surjectivity.

Lastly, we check that P and P' are inverses of each other. Indeed, given $m' \in M'$, we have

$$m' = Pm' + (m' - Pm').$$

Then, since $m' - Pm' \in U$, we have

$$m' = P'm' = P'Pm' + P'(m' - Pm') = P'Pm'.$$

This shows that $P'|_M P|_{M'} = \mathsf{id}_{M'}$, which concludes the proof. ☐

Proposition 9.2.1. *For every parameter h, consider supplementary spaces M_h and M'_h of G_h in X_h. Denote by $P'_h : X_h \to X_h$ the projector with range M'_h and kernel G_h. Assume that the sequence $(P'_h)_h$ is uniformly bounded in operator norm $X \to X$, and that A_h satisfies a uniform inf-sup condition in $(M_h \times Y_h)_h$, then A_h satisfies a uniform inf-sup condition in $(M'_h \times Y_h)_h$.*

Proof. First, observe that, since $\ker P'_h = G_h$, we have $A_h P'_h = A_h$. Furthermore, in the previous lemma we learned that

$$P'_h|_{M_h} : \; M_h \to M'_h$$

is an isomorphism.

In turn, from the uniform boundedness, there follows that

$$\|P'_h x\| \preceq \|x\| \qquad \forall\, x \in X_h.$$

Bearing in mind the previous comments, we may use P'_h to write

$$\inf_{x\in M'_h} \sup_{y\in Y_h} \frac{A_h(x)(y)}{\|x\|\|y\|} = \inf_{z\in M_h} \sup_{y\in Y_h} \frac{A_h(z)(y)}{\|P'_h z\|\|y\|} \succeq \inf_{z\in M_h} \sup_{y\in Y_h} \frac{A_h(z)(y)}{\|z\|\|y\|} \succeq 1\,,$$

where the last bound follows from the assumed uniform inf-sup condition. □

It can be seen from the proof that this proposition also holds for asymptotic inf-sup conditions.

An important consequence of this result will be that, loosely speaking, every stable LSF operator is discrete LSF. Before making this precise, we need an assumption regarding the kernels and a last preliminary result.

Assumption 9.2.1. *The symmetrized gap $\hat{\delta}(G_h, G)$ converges to zero, in the X-norm.* ◇

Hereafter, according to our purposes, we further assume that X is a Hilbert space.

Lemma 9.2.2. *Given a supplementary space M of G in X, denote by $P : X \to X$ the projector with range M and kernel G. Then, there exists a uniformly bounded sequence of projectors $(P_h)_h$*

from X to X, which realizes a decomposition

$$X_h = (X_h \cap M) \oplus G_h \,,$$

for h sufficiently small.

Proof. Recall that the symmetrized gap satisfies

$$\hat{\delta}(G_h, G) = \|P_G - P_{G_h}\| \,,$$

where, for a subspace U of X, P_U denotes the orthogonal projector onto U.

Given a basis $\{g^i\}_i$ of G, we define $g^i_h = P_{G_h} g^i$ and notice that

$$\|g^i - g^i_h\| = \|(P_G - P_{G_h})g^i\| \leq \|g^i\| \|P_G - P_{G_h}\| = \|g^i\| \hat{\delta}(G_h, G) \,.$$

From Assumption 9.2.1, we obtain $\|g^i - g^i_h\| \xrightarrow{h} 0$, for all $i \in \{1, \ldots, \dim G\}$. Moreover, we observe that

$$\|(I - P_{G_h})|_G\| \leq \|P_G - P_{G_h}\| = \hat{\delta}(G_h, G) \,,$$

which, according to the gap convergence, implies that, for h small, $P_{G_h}|_G$ is invertible and therefore $\{g^i_h\}_i$ is a basis of G_h.

Notice that this is the proof of Lemma 9.1.1 is constructive. Moreover, the kernel of the induced projectors is given by the choice of a sequence converging to $\{g^i\}_i$. Since here we chose specific kernels for the induced projectors, i.e., the discrete kernels G_h, we may use Lemma 9.1.1 to obtain projectors with the desired properties. □

We are finally in a position to state the main result of this section:

Theorem 9.2.3. *Assume that A_h satisfies an asymptotic inf-sup condition on $(\bar{M}_h \times Y_h)_h$, where $\bar{M}_h$ is a supplementary space of G_h in X_h. Then, A is discrete LSF on $(X_h \times Y_h)_h$ with respect to the orthogonal of G. Conversely, if A is discrete LSF, then A_h satisfies an asymptotic stable inf-sup condition on $((X_h \cap G_h^{\perp}) \times Y_h)_h$ and, a fortiori, on any compatible supplementary space of G_h in X_h, in the sense of Proposition 9.2.1.*

Proof. Let M denote the orthogonal of G, and denote by $P : X \to X$ the orthogonal projector onto M. Since M is finite-codimensional in X, we use Lemma 9.2.2 to obtain a uniformly bounded sequence of projectors $P_h : X \to X$ with range $X_h \cap M$ and kernel G_h, for h small enough. Next, assume the asymptotic inf-sup condition on $(\bar{M}_h \times Y_h)_h$. This gathers all the assumptions of Proposition 9.2.1 with local notation

$$P'_h = P_h, \quad M'_h = X_h \cap M, \quad \text{and} \quad M_h = \bar{M}_h\,,$$

which gives that A satisfies an asymptotic inf-sup condition on $((X_h \cap M) \times Y_h)_h$. This is the definition of discrete left semi-Fredholmness.

For the converse, we assume that A is discrete semi-Fredholm on $(X_h \times Y_h)_h$. Observe that, by Corollary 9.1.1, we may assume that A is discrete semi-Fredholm with respect to M. In this situation, we denote by $Q_h : X_h \to X_h$ the orthogonal projector onto $X_h \cap G_h^\perp$, and notice that the sequence $(Q_h)_h$ is uniformly bounded, because orthogonal projectors have norm 1. Therefore, using Proposition 9.2.1 with

$$M_h = X_h \cap M, \quad M'_h = X_h \cap G_h^\perp \quad \text{and} \quad P'_h = Q_h\,,$$

we obtain the desired result. □

CHAPTER 10

Discrete Fredholm properties of the graded Dirac-Kähler operator

In this chapter, we apply the theory developed in Chapter 9 to show that the graded Dirac-Kähler operator on a compact Riemannian manifold is discrete Fredholm, and, as a consequence, its discrete Fredholm index coincides with the continuous one.

10.1 Setting

Continuous setting

Hereon, $\mathcal{S}$ denotes an oriented, compact smooth Riemannian manifold with boundary. Denote $\mathbf{O}^\bullet = \mathrm{L}^2\Lambda^\bullet(\mathcal{S})$ and consider the exterior derivative as an unbounded operator $\mathsf{d} : \mathbf{O}^\bullet \to \mathbf{O}^\bullet$ with domain

$$\mathbf{X}^\bullet := \mathring{\mathrm{H}}\Lambda^\bullet(\mathcal{S}) = \{u \in \mathrm{H}\Lambda^\bullet(\mathcal{S}) \mid \mathsf{Tr}\, u = 0\} .$$

We equip this space with the graph norm induced by d. Namely, for $u \in \mathbf{X}$, we define

$$\|u\|_{\mathbf{X}}^2 = |u|^2 + |\mathsf{d}u|^2 ,$$

which, since d is closed, makes $\mathbf{X}$ into a Hilbert space.

In the following, we briefly recall the construction of $\mathsf{Tr}^k : \mathrm{H}\Lambda^k(\mathcal{S}) \to \mathrm{H}^{-1/2}\Lambda^k(\partial\mathcal{S})$. A standard reference is Section 2.2 [AFW06]. For smooth differential k-forms, the boundary trace operator $\mathsf{Tr}\colon \mathrm{C}^\infty\Lambda^k(\mathcal{S}) \to \mathrm{C}^\infty\Lambda^k(\partial\mathcal{S})$ is defined as pullback on forms along the inclusion $\imath_{\partial S} : \partial\mathcal{S} \to \mathcal{S}$, i.e., $\mathsf{Tr}\, u = \imath_{\partial\mathcal{S}}^* u$. Similarly to the standard theory of Sobolev spaces, Tr extends by continuity to a surjective, linear and bounded operator $\mathsf{Tr}\colon \mathrm{H}^1\Lambda^k(\mathcal{S}) \to \mathrm{H}^{1/2}\Lambda^k(\mathcal{S})$. Next, to define a trace in $\mathrm{H}\Lambda^k(\mathcal{S})$, one may proceed analogously to how the normal trace of $\mathrm{H}(\mathsf{div})$ is obtained. Namely, for $\psi \in \mathrm{H}^{1/2}\Lambda^k(\partial\mathcal{S})$, one notices that its Hodge dual $\star_{\partial\mathcal{S}}\,\psi$ belongs to $\mathrm{H}^{1/2}\Lambda^{n-k-1}(\partial\mathcal{S})$, hence there is $v \in \mathrm{H}^1\Lambda^{k+1}(\mathcal{S})$ such that $\mathsf{Tr}\star v = \star_{\partial\mathcal{S}}\psi$ and

$$\|v\|_{\mathrm{H}^1(\mathcal{S})} \leq \|\psi\|_{\mathrm{H}^{1/2}(\partial\mathcal{S})}.$$

Here, we used that, as in the Euclidean setting, the best boundedness constant can be proved to be 1 (compare eq. (4.101) in [Gat24]). Thus, for $u \in \mathrm{C}^\infty\Lambda^k(\mathcal{S})$, by the integration by parts formula (2.3.1), we write

$$(\mathsf{Tr}\, u, \psi)_{\partial\mathcal{S}} = \int_{\partial\mathcal{S}} \mathsf{Tr}\, u \wedge \mathsf{Tr} \star v = (\mathsf{d}u, v) - (u, \mathsf{d}^\star v) , \tag{10.1.1}$$

whence

$$|(\mathsf{Tr}\, u, \psi)| \leq |\mathsf{d}u||v| + |u||\mathsf{d}^\star v| \;\leq \|u\|_{\mathrm{H}\Lambda(\mathcal{S})}\|v\|_{\mathrm{H}^1\Lambda(\mathcal{S})} .$$

Here, we used that $\mathrm{H}^1\Lambda(\mathcal{S})$ is continuously embedded in $\mathrm{H}\Lambda(\mathcal{S})$, because d is the alternating projection of ∇. From Lemma 3.10 in [Arn18] and the discussion thereafter, there is a linear and bounded operator $\mathsf{Tr} : \mathrm{H}\Lambda^k(\mathcal{S}) \to \mathrm{H}^{-1/2}\Lambda^k(\partial\mathcal{S})$ such that, for $u \in \mathrm{H}\Lambda^k(\mathcal{S})$, there holds

$$\left\langle \mathsf{Tr}\, u, \star_{\partial\mathcal{S}}^{-1} \mathsf{Tr} \star v \right\rangle = (\mathsf{d}u, v) - (u, \mathsf{d}^\star v) \qquad \forall\, v \in \mathrm{H}^1\Lambda^k(\mathcal{S}),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathrm{H}^{-1/2}\Lambda^k(\mathcal{S})$ and $\mathrm{H}^{1/2}\Lambda^k(\mathcal{S})$.

Having cleared boundary traces in Sobolev spaces of differential forms, we continue to describe our setting.

Bearing in mind (10.1.1), we notice that, when $\mathbf{X}$ is the domain of d, the domain of the coderivative as an unbounded operator $\mathsf{d}^\star\colon \mathbf{O}\to\mathbf{O}$ is given by $\mathbf{X}_\star := \mathrm{H}^\star\Lambda(\mathcal{S})$. It is a Hilbert space when endowed with the graph norm

$$\|u\|^2_{\mathbf{X}_\star} = |u|^2 + |\mathsf{d}^\star u|^2, \qquad \forall\, u\in\mathbf{X}_\star\,.$$

Having defined the domains of d and $\mathsf{d}^\star$, we are in a position to define the Dirac-Kähler operator as the unbounded operator $\mathcal{D}\colon \mathbf{O}\to\mathbf{O}$ with domain $\mathbf{Y} := \mathbf{X}\cap\mathbf{X}_\star$, given by $\mathcal{D} = \mathsf{d}-\mathsf{d}^\star$. We notice from (10.1.1) that there holds

$$(\mathcal{D}u, v) + (u, \mathcal{D}v) = 0 \qquad \forall\, u, v\in\mathbf{Y}\,.$$

Thus, this choice of domain implies that $\mathcal{D}$ is a densely defined, antisymmetric operator. This, together with the fact that the exterior derivative is closed, implies that $\mathcal{D}$ is closed as well. Thus, $\mathbf{Y}$ is a Hilbert space with the graph norm

$$\|u\|^2_{\mathbf{Y}} = |u|^2 + |\mathsf{d}u|^2 + |\mathsf{d}^\star u|^2 = |u|^2 + |\mathcal{D}u|^2\,.$$

With this choice of boundary conditions, **Gaffney's inequality** (see [Sch95, Corollary 2.1.6]) states that

$$\|u\|_{\mathrm{H}^1(\mathcal{S})} \preceq |u| + |\mathsf{d}u| + |\mathsf{d}^\star u| \qquad \forall\, u\in\mathbf{Y}\,.$$

We emphasize that this is possible beacuse $\mathcal{S}$ has smooth boundary . A consequence of this inequality is that $\mathbf{Y}$ is continuously embedded in

$$\mathring{\mathrm{H}}^1\Lambda(\mathcal{S}) := \left\{u\in\mathrm{H}^1\Lambda^k(\mathcal{S}) \;\middle|\; \mathsf{Tr}\, u = 0\right\}\,,$$

but we already knew that the converse inclusion holds, hence $\mathbf{Y} = \mathring{\mathrm{H}}^1\Lambda(\mathcal{S})$.

Recall that the bundle of differential forms has a $\mathbb{Z}_2$-grading given by the forms of even and odd degree. In what follows, the subscripts e and o in a subspace of $\Lambda^\bullet(\mathcal{S})$ denote its even and odd parts, respectively. In particular, we have

$$\mathbf{O} = \mathbf{O}_\mathsf{e} \oplus \mathbf{O}_\mathsf{o}, \qquad \mathbf{X} = \mathbf{X}_\mathsf{e} \oplus \mathbf{X}_\mathsf{o}, \qquad \mathbf{X}_\star = \mathbf{X}_{\star,\mathsf{e}} \oplus \mathbf{X}_{\star,\mathsf{o}} \quad \text{and} \quad \mathbf{Y} = \mathbf{Y}_\mathsf{e} \oplus \mathbf{Y}_\mathsf{o}\,.$$

Observe that, by nature, the Dirac-Kähler operator maps one part of the bundle into the other one. In this sense, we can say that its restriction to the even part of $\mathbf{Y}$ is a $\mathbb{Z}_2$-graded operator of degree 1. This is, we have $\mathcal{D}_\mathsf{e} : \mathbf{Y}_\mathsf{e} \to \mathbf{O}_\mathsf{o}$. We refer to $\mathcal{D}_\mathsf{e}$ as the **graded Dirac-Kähler** operator, which, as we announced earlier, shall be our subject of study.

Likewise, we have $\mathcal{D}_\mathsf{o} : \mathbf{Y}_\mathsf{o} \to \mathbf{O}_\mathsf{e}$ for the restriction to the odd part of the bundle. According to the previous relations, $\mathcal{D}_\mathsf{e}$ and $\mathcal{D}_\mathsf{o}$ satisfy the relation:

$$(\mathcal{D}_\mathsf{e} u, v) + (u, \mathcal{D}_\mathsf{o} v) = 0 \qquad \forall\, u \in \mathbf{Y}_\mathsf{e},\ v \in \mathbf{Y}_\mathsf{o}\,.$$

This amounts to say that the adjoint of $\mathcal{D}_\mathsf{e}$ is $-\mathcal{D}_\mathsf{o}$. However, instead of the latter, we will usually work with $\mathcal{D}_\mathsf{o}$, because a difference of sign will not change the kernel nor the study of inf-sup conditions.

Lastly, we shall denote the graded cohomology group by $\mathbf{G}$. This is

$$\mathbf{G} = \ker \mathcal{D} = \{u \in \mathbf{Y} \mid \mathsf{d}u = 0,\ \mathsf{d}^\star u = 0\}\ .$$

We have $\mathbf{G} = \mathbf{G}_\mathsf{e} \oplus \mathbf{G}_\mathsf{o}$.

Discrete setting

Now, consider a family $\mathcal{H}$ of real positive parameters accumulating at 0, and construct a family of cellular complexes $(\mathcal{T}_h)_h$ in $\mathcal{S}$ to satisfy $h = \max_{T \in \mathcal{T}_h} h_T$, where $h_T := \operatorname{\mathsf{diam}} T$. Then, consider a $\mathbf{Y}$-conforming FES functor $\mathfrak{A}_h$. Moreover, for all $h \in \mathcal{H}$ and $T \in \mathcal{T}_h$, denote

$$(\mathcal{A}_h(T), \mathsf{d}) := \mathfrak{A}_h(T).$$

Here, we assumed that $\mathfrak{A}_h$ was constructed so that the differential associated with the local complexes is the restriction of the exterior derivative. The global spaces are denoted by

$$\mathbf{Y}_h^k := \varprojlim_{T \in \mathcal{T}_h} \mathcal{A}_h^k(T).$$

We recall that this construction implies that the exterior derivatives induces maps $\mathsf{d} \colon \mathbf{Y}_h \to \mathbf{Y}_h$, and that, the restriction of $\mathsf{d}^\star$ to $\mathbf{Y}_h$ does not enjoy this property. For this reason, we consider $\mathsf{d}_h^\star : \mathbf{Y}_h \to \mathbf{Y}_h$ as the unique map satisfying

$$(\mathsf{d}u, v) = (u, \mathsf{d}_h^\star v) \qquad \forall\, u, v \in \mathbf{Y}_h\,.$$

Then, we define the discrete Dirac-Kähler operator $\mathcal{D}_h : \mathbf{Y}_h \to \mathbf{Y}_h$ as $\mathcal{D}_h = \mathsf{d} - \mathsf{d}_h^\star$. We remark that it satisfies

$$(\mathcal{D}_h u, v) = (\mathcal{D} u, v) \qquad \forall\, u, v \in \mathbf{Y}_h\,.$$

Henceforth, we shall denote the discrete graded cohomology by $\mathbf{G}_h$. Namely

$$\mathbf{G}_h = \ker \mathcal{D}_h = \{u \in \mathcal{D}_h \mid \mathsf{d}u = 0,\ \mathsf{d}_h^\star u = 0\}\ .$$

We observe that, just as in the continuous setting, we have decompositions

$$\mathbf{Y}_h = \mathbf{Y}_{\mathsf{e},h} \oplus \mathbf{Y}_{\mathsf{o},h}, \quad \mathbf{G}_h = \mathbf{G}_{\mathsf{e},h} \oplus \mathbf{G}_{\mathsf{o},h} \quad \text{and} \quad \mathcal{D}_h = \mathcal{D}_{\mathsf{e},h} \oplus \mathcal{D}_{\mathsf{o},h}\,.$$

At the discrete level, it is also verified that

$$(\mathcal{D}_{\mathsf{e},h} u, v) + (u, \mathcal{D}_{\mathsf{o},h} v) = 0 \qquad \forall\, u \in \mathbf{Y}_{\mathsf{e},h},\ v \in \mathbf{Y}_{\mathsf{o},h}\,.$$

10.2 Fredholm properties of $\mathcal{D}_{\mathsf{e}}$

We begin this section by relating the Fredholm index of the graded Dirac-Kähler operator to the Euler-Poincaré characteristic. Although simple, this relation is a beautiful example of the interplay between analysis, geometry and topology. After this, we pass to the discrete level and prove that, under suitable assumptions, the theory we developed in Chapter 9 applies. Thereby showing that the method of discretization is, to some extent, structure preserving.

10.2.1 Fredholm index and Euler-Poincaré characteristic

Since $\mathcal{S}$ is compact, its de Rham cohomology is finite-dimensional. Since the cohomology spaces identify with the kernel of $\mathcal{D}$, we have that $\mathbf{G}$ is finite-dimensional. Now, since $\mathcal{D}_{\mathsf{e}}$ and $-\mathcal{D}_{\mathsf{o}}$ are adjoint of each other, by standard relations between range and kernels of unbounded operators, we have

$$\mathcal{D}_{\mathsf{o}}(\mathbf{Y}_{\mathsf{o}})^{\perp} = \mathbf{G}_{\mathsf{e}} \quad \text{and} \quad \mathcal{D}_{\mathsf{e}}(\mathbf{Y}_{\mathsf{e}})^{\perp} = \mathbf{G}_{\mathsf{o}}\,, \tag{10.2.1}$$

where the orthogonal is taken in $\mathbf{O}$.

From here, in particular, we infer that the range of $\mathcal{D}_{\mathsf{e}}$ is finite-codimensional in $\mathbf{O}_{\mathsf{o}}$, hence it is Fredholm. Let us now compute its Fredholm index, which, according to (10.2.1) corresponds to

$$\operatorname{\mathsf{ind}} \mathcal{D}_{\mathsf{e}} = \dim(\operatorname{\mathsf{ker}} \mathcal{D}_{\mathsf{e}}) - \dim(\mathbf{O}_{\mathsf{o}}/\operatorname{\mathsf{Im}} \mathcal{D}_{\mathsf{e}}) = \dim \mathbf{G}_{\mathsf{e}} - \dim \mathbf{G}_{\mathsf{o}}\,.$$

Recalling that $\mathbf{G} = \mathbf{G}_{\mathsf{e}} \oplus \mathbf{G}_{\mathsf{o}}$, we can rewrite this as an alternating sum over the harmonic spaces, namely

$$\operatorname{\mathsf{ind}} \mathcal{D}_{\mathsf{e}} = \sum_{k=0}^{n} (-1)^k \dim \mathbf{G}^k\,.$$

In turn, the **Euler-Poincaré characteristic** can be defined as

$$\chi(\mathcal{S}) = \sum_{k=0}^{n} (-1)^k b_k\,,$$

where b_k is the kth Betti number of $\mathcal{S}$. Therefore, by virtue of the de Rham theorem, we can conclude that

$$\operatorname{\mathsf{ind}}(\mathcal{D}_{\mathsf{e}}) = \chi(\mathcal{S})\,.$$

A simple example of this is the torus. We saw in Section 5.3 that the cohomology spaces of an n-torus are given by the constant forms $\mathbf{G}^k = \mathsf{Alt}^k(\mathbb{R}^n)$. Therefore, we may use Newton's binomial theorem to write

$$(1-1)^n = \sum_{k=0}^{n}(-1)^k 1^{n-k}\binom{n}{k} = \sum_{k=0}^{n}(-1)^k \dim \mathbf{G}^k = \chi(\mathcal{S})\,.$$

This computation says both that in this case $\mathcal{D}_{\mathsf{e}}$ is a Fredholm operator of index zero, and that the Euler-Poincaré characteristic of the torus is 0. Two different points of view of a same fact.

10.2.2 Discrete Fredholm properties of $\mathcal{D}_{\mathsf{e}}$

The main object of this section is to establish the discrete Fredholmness of the graded Dirac-Kähler operator on $(\mathbf{Y}_{\mathsf{e},h} \times \mathbf{Y}_{\mathsf{o},h})_h$. To this end, by definition, we need to find closed, finite-codimensional subspaces of $\mathbf{Y}_{\mathsf{e}}$ and $\mathbf{O}_{\mathsf{o}}$ such that assymptotic inf-sup conditions, in the sense of Section 9.2, hold.

Indeed, it follows from (10.2.1) that we have orthogonal decompositions

$$\mathbf{Y}_{\mathsf{e}} = (\mathbf{Y}_{\mathsf{e}} \cap \mathbf{G}_{\mathsf{e}}^{\perp}) \oplus \mathbf{G}_{\mathsf{e}} \quad \text{and} \quad \mathbf{O}_{\mathsf{o}} = \mathbf{G}_{\mathsf{o}}^{\perp} \oplus \mathbf{G}_{\mathsf{o}} = \mathcal{D}_{\mathsf{e}}(\mathbf{Y}_{\mathsf{e}}) \oplus \mathbf{G}_{\mathsf{o}}\,.$$

Therefore, since $\mathbf{G}_{\mathsf{e}}$ and $\mathbf{G}_{\mathsf{o}}$ are finite-dimensional, natural choices of such subspaces are given by

$$\mathcal{M} := \mathbf{Y}_{\mathsf{e}} \cap \mathbf{G}_{\mathsf{e}}^{\perp} \quad \text{and} \quad \mathcal{N} := \mathbf{G}_{\mathsf{o}}^{\perp}\,. \tag{10.2.2}$$

We shall bear this in mind throughout this section. The strategy will be to use Proposition 9.2.1, so we must walk towards a stable inf-sup condition on suitable subspaces of $\mathbf{Y}_{\mathsf{e},h} \times \mathbf{Y}_{\mathsf{o},h}$.

In the following, we assume that the family of FES $(\mathbf{Y}_h)_h$ carries interpolators $\mathcal{I}_h \colon \mathbf{Y} \to \mathbf{Y}_h$, deduced from a choice of a unisolvent system degrees of freedom, or another standard technique (cf. Section 2.6.2).

Furthermore, we make an assumption regarding smoothed projections:

Assumption 10.2.1. *We have smoothed projections $\mathbf{\Pi}_h : \mathbf{O} \to \mathbf{Y}_h$ satisfying,*

i) $\mathbf{\Pi}_h|_{\mathbf{X}}$ *commutes with the exterior derivative,*

ii) $\mathbf{\Pi}_h$ *is stable* $\mathbf{O} \to \mathbf{O}$*, and*

iii) $\mathbf{\Pi}_h$ *has optimal approximation properties in* $\mathbf{O}$ *and* $\mathbf{Y}$*. Namely, if* $u \in \mathrm{H}^\ell\Lambda^k(\mathcal{S})$*, we have*

$$|u - \mathbf{\Pi}_h u| \preceq h^{\ell+1}|\nabla^{\ell+1}u|$$

and

$$\|u - \mathbf{\Pi}_h u\|_{\mathbf{Y}} \leq h^{\ell}|\nabla^{\ell+1}u|$$

for ℓ *any nonnegative integer in a certain range.*

◇

Remark 10.2.1. In section 5.2, we proved that this assumption holds when $\mathcal{S}$ is a generic flat torus. There, we also mentioned early work in which smoothed projections enjoying these properties are constructed, most of them in the context of Euclidean domains. In contrast, the existence of smoothed projections in arbitrary manifolds is not trivial nor guaranteed at all. In [Lic23], they construct smoothed projections on compact Riemannian manifolds satisfying the first two requirements of Assumption 10.2.1. However, as remarked there at the end of Section 7, the third property is not readily accessible, as there is no intrinsic notion of polynomials on a manifold and, therefore, the standard derivation of approximation orders as a combination of polynomial-preserving operators and Bramble-Hilbert lemma is not available. We also mention that some work in this direction was done in [GJ24], in the setting of manifolds with Regge metrics, which are Riemannian manifolds whose metric is piecewise smooth with respect to some partition, with some continuity imposed at interfaces. However, in that article, they also assume the existence of commuting projections. ◇

A consequence of this assumption is that there hold stable Poincaré inequalities. To state this, we first notice that there is a complex structure:

$$\begin{tikzcd}
\mathbf{Y}_{\mathsf{o},h} \arrow[r, shift left, "\mathsf{d}_{\mathsf{o},h}"] & \mathbf{Y}_{\mathsf{e},h} \arrow[l, shift left, "\mathsf{d}^\star_{\mathsf{o},h}"] \arrow[r, shift left, "\mathsf{d}_{\mathsf{e},h}"] & \mathbf{Y}_{\mathsf{o},h} \arrow[l, shift left, "\mathsf{d}^\star_{\mathsf{e},h}"]
\end{tikzcd} ,$$

from which we obtain $\mathbf{Y}_{\mathsf{e},h} = \ker \mathsf{d}_{\mathsf{e},h} \oplus \mathsf{d}^\star_{\mathsf{e},h}(\mathbf{Y}_{\mathsf{o},h})$ and $\ker \mathsf{d}_{\mathsf{e},h} = \mathbf{G}_{\mathsf{e},h} \oplus \mathsf{d}_{\mathsf{o},h}(\mathbf{Y}_{\mathsf{o},h})$. Therefore, we have discrete Hodge decompositions:

$$\mathbf{Y}_{\mathsf{e},h} = \mathsf{d}^\star_{\mathsf{e},h}(\mathbf{Y}_{\mathsf{o},h}) \oplus \mathsf{d}_{\mathsf{o},h}(\mathbf{Y}_{\mathsf{o},h}) \oplus \mathbf{G}_{\mathsf{e},h} , \tag{10.2.3}$$

which are orthogonal in $\mathbf{O}_{\mathsf{e}}$. From now on, we set the notation $\mathbf{V}_{\mathsf{e},h} := \mathsf{d}^\star_{\mathsf{e},h}(\mathbf{Y}_{\mathsf{o},h})$ and $\mathbf{Z}_{\mathsf{e},h} =$

$\mathsf{d}_{\mathsf{o},h}(\mathbf{Y}_{\mathsf{o},h})$, so that we have

$$\mathbf{Y}_{\mathsf{e},h} = \mathbf{V}_{\mathsf{e},h} \oplus \mathbf{Z}_{\mathsf{e},h} \oplus \mathbf{G}_{\mathsf{e},h} \,.$$

Proposition 10.2.1. *The following estimates hold:*

i) *for* $v \in \mathbf{V}_{\mathsf{e},h}$, *there holds* $\|v\|_{\mathbf{X}_{\mathsf{e}}} \preceq |\mathsf{d}_{\mathsf{o},h} v|\,,$

ii) *for* $z \in \mathbf{Z}_{\mathsf{e},h}$, *there holds* $\|z\|_{\mathbf{X}_{\mathsf{e}}} \preceq |\mathsf{d}^{\star}_{\mathsf{e},h} z|\,,$ *and*

iii) *for* $u \in \mathbf{V}_{\mathsf{e},h} \oplus \mathbf{Z}_{\mathsf{e},h}$, *there holds* $\|u\|_{\mathbf{X}_{\mathsf{e}}} \preceq |\mathcal{D}_{\mathsf{e},h} u|\,.$

Proof. As remarked in Section 5.3, a stable Poincaré inequality is a direct consequence of having a complex structure and smoothed projections. Indeed, the present setting is such that the proof of Lemma 5.3.2 works verbatim to prove item i) here. The two remaining estimates are simple corollaries of i) (cf. lemmas 5.3.3 and 5.3.4). The reason why everything still works despite having the graded version of d_h and the discrete Dirac-Kähler $\mathcal{D}_h$ is that the first two estimates are valid degreewise. The last one results from summing i) and ii) for each even degree, and using the discrete Hodge decomposition. □

Notice that the last estimate in the previous lemma holds exactly in the orthogonal complement of the kernel of $\mathcal{D}_{\mathsf{e},h}$. For notation convenience, we also denote $\widetilde{\mathbf{Y}}_{\mathsf{e},h} := \mathbf{V}_{\mathsf{e},h} \oplus \mathbf{Z}_{\mathsf{e},h}$.

Assumption 10.2.2. *For all* $u \in \widetilde{\mathbf{Y}}_{\mathsf{e},h}$, *there holds* $|\mathcal{D}_{\mathsf{e}} u| \preceq |\mathcal{D}_{\mathsf{e},h} u|\,.$ ◇

Remark 10.2.2. This assumption is verified in a generic torus (cf. Proposition 5.3.3). It seems very likely that this also holds in the current, more general setting. However, for reasons of economy and conciseness, we choose to not include this in the present manuscript. ◇

A direct consequence of the previous discussion is the stability of the graded Dirac-Kähler operator on $(\widetilde{\mathbf{Y}}_{\mathsf{e},h} \times \mathbf{Y}_{\mathsf{o},h})_h$:

Proposition 10.2.2. *There holds*

$$\inf_{u \in \widetilde{\mathbf{Y}}_{\mathsf{e},h}} \sup_{v \in \mathbf{Y}_{\mathsf{o},h}} \frac{(\mathcal{D}_{\mathsf{e}} u, v)}{\|u\|_{\mathbf{Y}_{\mathsf{e}}} |v|} \succeq 1\,.$$

Proof. We observe that, for every $u \in \widetilde{\mathbf{Y}}_{\mathsf{o},h}$, we may take $v = \mathcal{D}_{\mathsf{e},h} u$ to obtain

$$\inf_{u \in \widetilde{\mathbf{Y}}_{\mathsf{e},h}} \sup_{v \in \mathbf{Y}_{\mathsf{o},h}} \frac{(\mathcal{D}_{\mathsf{e}} u, v)}{\|u\|_{\mathbf{Y}_{\mathsf{e}}} |v|} \geq \inf_{u \in \widetilde{\mathbf{Y}}_{\mathsf{o},h}} \frac{|\mathcal{D}_{\mathsf{e},h} u|}{\|u\|_{\mathbf{Y}_{\mathsf{e}}}} .$$

By Cauchy-Schwarz inequality, the converse inequality also hold, thereby

$$\inf_{u \in \widetilde{\mathbf{Y}}_{\mathsf{e},h}} \sup_{v \in \mathbf{Y}_{\mathsf{o},h}} \frac{(\mathcal{D}_{\mathsf{e}} u, v)}{\|u\|_{\mathbf{Y}_{\mathsf{e}}} |v|} = \inf_{u \in \widetilde{\mathbf{Y}}_{\mathsf{o},h}} \frac{|\mathcal{D}_{\mathsf{e},h} u|}{\|u\|_{\mathbf{Y}_{\mathsf{e}}}} . \tag{10.2.4}$$

Then, by item iii) in Proposition 10.2.1 and Assumption 10.2.2, we have

$$|\mathcal{D}_{\mathsf{e},h} u| \succeq |u| + |\mathcal{D}_{\mathsf{e}} u| \approx \|u\|_{\mathbf{Y}_{\mathsf{e}}} ,$$

for $u \in \widetilde{\mathbf{Y}}_{\mathsf{e},h}$. This, together with (10.2.4), gives the desired estimate. □

Now, by analogy with (10.2.3), after setting $\mathbf{V}_{\mathsf{o},h} := \mathsf{d}^{\star}_{\mathsf{o},h}(\mathbf{Y}_{\mathsf{e},h})$ and $\mathbf{Z}_{\mathsf{o},h} := \mathsf{d}_{\mathsf{e},h}(\mathbf{Y}_{\mathsf{e},h})$, we obtain an $\mathbf{O}_{\mathsf{o}}$-orthogonal discrete Hodge decomposition

$$\mathbf{Y}_{\mathsf{o},h} = \mathbf{V}_{\mathsf{o},h} \oplus \mathbf{Z}_{\mathsf{o},h} \oplus \mathbf{G}_{\mathsf{o},h} .$$

Moreover, we also denote $\widetilde{\mathbf{Y}}_{\mathsf{o},h} := \mathbf{V}_{\mathsf{o},h} \oplus \mathbf{Z}_{\mathsf{o},h}$. Bearing in mind this notation, we are in a position to prove a stable inf-sup condition swapping the spaces, which, if we recall Chapter 9, amounts to prove an inf-sup condition for the transpose operator.

Proposition 10.2.3. *There holds*

$$\inf_{v \in \widetilde{\mathbf{Y}}_{\mathsf{o},h}} \sup_{u \in \mathbf{Y}_{\mathsf{e},h}} \frac{(\mathcal{D}_{\mathsf{e}} u, v)}{\|u\|_{\mathbf{Y}_{\mathsf{e}}} |v|} \succeq 1$$

Proof. We notice that, since $\mathcal{D}_{\mathsf{e},h}$ and $-\mathcal{D}_{\mathsf{o},h}$ are adjoints of each other, we have

$$\widetilde{\mathbf{Y}}_{\mathsf{o},h} = \mathbf{Y}_{\mathsf{o},h} \cap \mathbf{G}^{\perp}_{\mathsf{o},h} = \mathcal{D}_{\mathsf{e},h}(\mathbf{Y}_{\mathsf{e},h}) .$$

Therefore, using this and Lemma 2.7.2, we obtain

$$\inf_{v\in\widetilde{\mathbf{Y}}_{\mathsf{o},h}}\sup_{u\in\mathbf{Y}_{\mathsf{e},h}}\frac{(\mathcal{D}_{\mathsf{e}}u,v)}{\|u\|_{\mathbf{Y}_{\mathsf{e}}}|v|}=\inf_{v\in\widetilde{\mathbf{Y}}_{\mathsf{o},h}}\sup_{u\in\widetilde{\mathbf{Y}}_{\mathsf{e},h}}\frac{(\mathcal{D}_{\mathsf{e}}u,v)}{\|u\|_{\mathbf{Y}_{\mathsf{e}}}|v|}=\inf_{u\in\widetilde{\mathbf{Y}}_{\mathsf{e},h}}\sup_{v\in\widetilde{\mathbf{Y}}_{\mathsf{o},h}}\frac{(\mathcal{D}_{\mathsf{e}}u,v)}{\|u\|_{\mathbf{Y}_{\mathsf{e}}}|v|}=\inf_{u\in\widetilde{\mathbf{Y}}_{\mathsf{e},h}}\sup_{v\in\mathbf{Y}_{\mathsf{o},h}}\frac{(\mathcal{D}_{\mathsf{e}}u,v)}{\|u\|_{\mathbf{Y}_{\mathsf{e}}}|v|}\,.$$

This, according to Proposition 10.2.2, concludes the proof. □

So far, we have obtained stable inf-sup conditions for both $\mathcal{D}_{\mathsf{e},h}$ and $\mathcal{D}_{\mathsf{o},h}$, or, said differently, for $\mathcal{D}_{\mathsf{e}}$ and its transpose. According to the theory developed in Chapter 9, if we are able to find stable projections connecting these spaces to the discrete parts of $\mathcal{M}$ and $\mathcal{N}$ (see (10.2.2)), discrete Fredholmness will follow. We begin by verifying the assumptions on the discrete spaces.

Lemma 10.2.1. *The spaces $\mathbf{Y}_h$ are approximating as subspaces of $\mathbf{Y}$ and $\mathbf{O}$. This is*

i) *for every $u\in\mathbf{Y}$, there exists a sequence $(u_h)_h$ in $(\mathbf{Y}_h)_h$ such that*

$$\|u_h-u\|_{\mathbf{Y}}\xrightarrow{h}0,\qquad \textit{and}$$

ii) *for every $v\in\mathbf{O}$, there exists a sequence $(v_h)_h$ in $(\mathbf{Y}_h)_h$ such that*

$$|v_h-v|\xrightarrow{h}0\,.$$

Proof. According to the rates of approximation given by Assumption 10.2.1, a standard density argument allows us to prove that $\mathbf{Y}_h$ approximates $\mathbf{Y}$ and $\mathbf{O}$, in the corresponding norms. Further details are omitted. □

Assumption 10.2.3. *We have gap convergence*

$$\hat{\delta}(\mathbf{G}_{\mathsf{e},h},\mathbf{G}_{\mathsf{e}})\xrightarrow{h}0\quad \textit{and}\quad \hat{\delta}(\mathbf{G}_{\mathsf{o},h},\mathbf{G}_{\mathsf{o}})\xrightarrow{h}0,$$

in the $\mathbf{Y}_{\mathsf{e}}$-norm and the $\mathbf{O}_{\mathsf{o}}$-norm, respectively. ◇

Now, we just have to apply the theory of Section 9.2.

Proposition 10.2.4. *$\mathcal{D}_{\mathsf{e}}$ is discrete left semi-Fredholm on $(\mathbf{Y}_{\mathsf{e},h}\times\mathbf{Y}_{\mathsf{o},h})_h$ with respect to $\mathcal{M}$.*

Proof. By virtue of Assumption 10.2.3 and Proposition 10.2.2, we may use Theorem 9.2.3, with local notation

$$A_h = \mathcal{D}_{\mathsf{e},h}, \quad \bar{M}_h = \widetilde{\mathbf{Y}}_{\mathsf{e},h}, \quad Y_h = \mathbf{Y}_{\mathsf{o},h}, \quad G_h = \mathbf{G}_{\mathsf{e},h}, \quad X_h = \mathbf{Y}_{\mathsf{e},h} \quad \text{and} \quad G = \mathbf{G}_{\mathsf{e}}\,,$$

which asserts exactly the desired property. □

Proposition 10.2.5. *$\mathcal{D}_{\mathsf{e}}$ is discrete right semi-Fredholm on $(\mathbf{Y}_{\mathsf{e},h} \times \mathbf{Y}_{\mathsf{o},h})_h$ with respect to $\mathcal{N}$.*

Proof. We recall that, by definition, $\mathcal{D}_{\mathsf{e}}$ is discrete right semi-Fredholm if its transpose is left semi-Fredholm. For this, according to Theorem 9.2.3, it suffices to have an asymptotic inf-sup condition for $\mathcal{D}_{\mathsf{o}}$ on $(\widetilde{\mathbf{Y}}_{\mathsf{o},h} \times \mathbf{Y}_{\mathsf{e},h})_h$. We observe that this proposition is at our disposal because we have gap convergence in the $\mathbf{O}_{\mathsf{o}}$-norm (see Assumption 10.2.3). Moreover, according to the inf-sup condition given by Proposition 10.2.3, we use Theorem 9.2.3 with local notation

$$A_h = \mathcal{D}_{\mathsf{o},h}, \quad \bar{M}_h = \widetilde{\mathbf{Y}}_{\mathsf{o},h}, \quad Y_h = \mathbf{Y}_{\mathsf{e},h}, \quad G_h = \mathbf{G}_{\mathsf{o},h}, \quad X_h = \mathbf{Y}_{\mathsf{o},h} \quad \text{and} \quad G = \mathbf{G}_{\mathsf{o}},$$

to obtain the desired result. □

Corollary 10.2.1. *$\mathcal{D}_{\mathsf{e}}$ is discrete Fredholm on $(\mathbf{Y}_{\mathsf{e},h} \times \mathbf{Y}_{\mathsf{o},h})_h$.*

Corollary 10.2.2. *The discrete Fredholm index of $\mathcal{D}_{\mathsf{e}}$ is the Euler-Poincaré characteristic of the underlying manifold. Namely*

$$\mathsf{ind}_{\mathsf{d}}(\mathcal{D}_{\mathsf{e}}) = \chi(\mathcal{S})\,.$$

CHAPTER 11

Conclusions and future work

11.1 Conclusions.

In this thesis, on the one hand, motivated by the time-dependent Dirac equation, we successfully studied a spurious-free, H^1-conforming discretization of the eigenmode problem associated with compact perturbations of the Hodge-Dirac operator in a torus. Namely, we proved discrete stability under compact perturbations, eigenmode convergence, and derived optimal rates of convergence. Furthermore, as remarked, we believe that the abstract theory here developed may have value on its own. The main advantages of the approach adopted here are:

i) The H^1-conforming setting helps one to avoid several technical complications in Chapter 5, which allows us to elucidate the key parts for the treatment of the problem. This then becomes useful in Part II.

ii) We can dispense with duality arguments such as Aubin-Nitsche tricks to derive estimates, as the method is already H^1-conforming. As a result, and in contrast with [Chr18], we easily get optimal rates of convergence by the standard techniques.

On the other hand, taking [Chr04] as a starting point, we proposed a discrete Fredholm theory, which proves to be closely linked to stability properties such as discrete inf-sup conditions. Some favorable points of this theory are:

i) Discrete Fredholmness implies stable inf-sup conditions on supplementary spaces of discrete kernels, which are relevant in finite element methods. The converse also holds.

ii) The class of discrete Fredholm operators is invariant under compact perturbations, just as in the continuous Fredholm theory. This makes the theory suitable to simplify the numerical analysis of perturbed problems.

We also proved that the graded Dirac-Kähler in a compact manifold with boundary fits into this framework, and thereby preserves the Euler-Poincaré characteristic of the underlying manifold. From the analysis, it can also be seen that the ungraded version of the Dirac-Kähler also fits in the framework. This, together with the aforementioned invariance under compact perturbations, yields inf-sup conditions for problems of the form studied in Section 4.2.

11.2 Future work.

Several natural continuations of the work done in this thesis can be identified. We mention some interesting directions:

i) Implementation of Part I.

    a) The fact that we work in a torus and we use H^1-conforming elements makes implementation difficult. However, the recent implementation of `formoniq`—a FEEC-oriented finite element library which relies on coordinate-free simplicial complexes in arbitrary dimension—bridges the gap on the first point; we refer to [Wir25]. Using this library to implement the eigenmode problem on a flat torus is entirely feasible (L. Wirth, personal communication, 2025).

  b) The main difficulties in the analysis arise from having nontrivial cohomology groups. Thereby, once suitable boundary conditions are imposed, the analysis may be carried out in any simply connected, open domain of $\mathbb{R}^n$, provided that Assumption 10.2.1 holds. Implementation in this case is greatly simplified.

ii) Study the preservation of various geometric, analytic and topological properties in the framework of Discrete Fredholm theory. For instance, it may be useful to combine the theory of discrete vector bundles developed in [CH23] with the Discrete Fredholm theory developed here. Hopefully, this path could lead to discrete counterparts of the theorem of Atiyah & Singer.

iii) Study of the unsteady problem associated with operators of Dirac type, which is challenging due to non-definiteness, but it gives a more accurate representation of the true Dirac equation.

iv) Extension of the analysis made in Part I to arbitrary compact manifolds with boundary (see Remark 10.2.2). In addition, symmetric spaces—quotients of Lie groups by Lie subgroups—seem like a natural class of manifolds to which our techniques may generalize.

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