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Fourier p-capacity and characteristic function on step by step constructed fractals

Tesis para optar al grado de Magíster en Matemática

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A Ylenia Caro, mi madre.

Prefacio

Quisiera declarar que la motivación para el estudio de los objetos matemáticos de interés (capacidades Fourier) nació de un artículo no publicado de mi profesor guía Vicente Vergara, en el que se demuestran varias propiedades presentadas en este trabajo (principalmente de la Fourier p-capacity), y de las constantes reuniones que se realizaron para decidir el camino de la investigación. A lo largo del año 2025 se consideraron muchos posibles caminos de estudio, como lo son estudiar la capacidad Fourier del producto y proyección entre dos conjuntos, condiciones para asegurar capacidad positiva, capacidad de la suma de dos conjuntos, y problemas de restricción de Fourier. Todo lo anterior fueron principalmente recomendaciones de caminos dados por mi profesor guía, y en la mayoría no se tuvo un avance significativo hasta que se propuso investigar bajo qué condiciones la Fourier (p, k)*-capacity es una capacidad de Choquet, además de investigar cuando se pueden obtener generalizaciones de resultados obtenidos para la Fourier p-capacity.*

Debido a que fue un año intenso y con muchos intentos fallidos, quisiera agradecer a quienes me acompañaron en el proceso.

Quiero expresar mi más profundo agradecimiento a mi familia por su amor y apoyo incondicional a lo largo de estos años, pues realmente quedé impresionado por todo el apoyo que me han dado, incluso muchas veces sin haberlo solicitado. Debido a esto y mucho más, gracias. Los amo.

Por supuesto, agradezco enormemente a mi pareja por su constante apoyo y compañía en los momentos buenos y malos. Su existencia y amor en mi vida han sido definitivamente cruciales en el desarrollo de mi vida académica y no académica.

Agradezco inmensamente también a mi director de tesis Vicente Vergara por contagiarme su pasión por el área de estudio en cada conversación que tuvimos, por ayudarme y darme ideas para resolver problemas en mis demostraciones, y por permitirme investigar en un área que me apasiona, pero que no es tan estudiada en el país.

Agradezco también a los amigos que me han acompañado durante estos años, pues pasar tiempo jugando en línea junto a ellos, muchas veces me dio una dosis importante de entretenimiento, chisme y compañía.

Por último, y no menos importante, agradezco a la Kyra, pues a pesar de su indiferencia, me ha demostrado destellos de cariño que alegraron muchos momentos.

Concepción, Chile *Rodrigo León Caro*

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Chapter 1

Introduction

In the study of abstract sets in Euclidean spaces, the most natural geometric quantity of size for these sets is their volume, defined by the Hausdorff measure. This measure historically changed the whole of mathematical analysis, making it possible to prove the completeness of L^p spaces and establish strong integration theorems (Dominated and Monotone Convergence theorems), something the Riemann integral was never able to do. But the Hausdorff measure is geometrically crude: it cannot separate sets of measure zero and considers countable points and highly non-trivial fractal objects (like the Cantor set) on an equal footing.

This insensitivity to geometry led to a measurement problem for fractal sets, which was resolved by the introduction of more precise tools. Constantin Carathéodory Carathéodory (1914) and Felix Hausdorff Hausdorff (1918) (see also Edgar (2019) for a historical translation) extended the notion of spatial measurement for fractional dimensions, enabling the mathematical community to derive meaningful geometric information from irregular sets (see Mandelbrot (1983), Falconer (2013)). Shortly thereafter, another key instrument was developed from potential theory: the idea of *capacity*, fully formalized in an axiomatic way by Gustave Choquet in Choquet (1954).

Capacities, including the classical Riesz, Bessel and Sobolev capacities, allow for a deep quantification of the “size” of a set in terms of how well it can accommodate a distribution of mass or charge with finite energy (extensive historical information on this subject can be found in Doob et al. (1984)). These operators succeed precisely where the Hausdorff measure fails, providing a strict refinement of properties holding “almost everywhere” to properties holding “quasi-everywhere” (see Section 2.7 and chapter 11 in Adams and Hedberg (2012), chapter 7 & 9 in Heinonen et al. (2018), and the concept of $(1, p, w)$-quasieverywhere in Kilpeläinen (1994)). Moreover, they create deep structural connections with the Hausdorff dimension of the relevant sets (see Landkof (1972), Adams and Hedberg (2012)).

Conventionally, such capacities are introduced and analyzed entirely in the spatial domain. However, the classical spatial energy of a measure can be calculated also as a weighted integral in the frequency domain by applying Parseval’s identity together with the Fourier transform. The previous two-fold interpretation leads to a very natural change of viewpoint: it is no longer only about transporting spatial energies, but it becomes crucially important to ask to what extent one can define and investigate capacities *directly* in the Fourier domain. By imposing an integrability condition on the Fourier transform of test functions, we move the analytic attention from spatial convolutions to frequency weights and open a new avenue to analyze the fine structure of sets using harmonic analysis.

Despite the natural elegance of defining capacities via the Fourier transform, this frequency-based approach introduces deep analytical challenges. The main object studied in this thesis is the Fourier (p, k)-capacity, $c_{p,k}(K)$. While classical spatial capacities (like the Sobolev capacity) are well suited to pointwise operations such as truncation or maximum of two admissible functions, the Fourier capacity is dictated by frequency weights and is non-local in nature. Such non-local effects are largely absent from the present theory. Pointwise spatial truncations do not in general preserve L^p control of frequency weights when $p \neq 2$. As a result, basic properties of classical potential theory, which are overlooked in most classical treatments (e.g., verification that it satisfies Choquet's axioms, *strong subadditivity* and others) become highly nontrivial obstructions. In the absence of strong subadditivity, it is nearly impossible to control the capacity's behavior under intersection and union of sets, and this deprives it of one of the key properties that make it a useful measure-theoretic tool. In addition, a major open problem is to determine the geometric meaning of these Fourier capacities. While it is well understood how classical capacities relate to the Hausdorff dimension, the precise dimensional thresholds for the positivity of the Fourier (p, k)-capacity on complex fractal structures remain largely unexplored. To figure out how these capacities "perceive" very irregular sets (for example, self-similar fractals or fat Cantor sets), one needs to prove profound relations between the L^p integrability of the Fourier transform and the existence of certain measures with structure supported on the set. Resolving this requires moving beyond standard potential theory and incorporating advanced tools from harmonic analysis, specifically Fourier restriction theorems and Frostman's lemma.

The general aim of this thesis is to fill in the above gap by a systematic study of the Fourier (p, k)-capacity and to obtain sharp geometric estimates of it, since many of the properties that form the foundation of Fourier p-capacity were proven by my advisor Vicente Vergara (some basic properties, as well as the bounds established in Theorem 6.1.16). It should be noted that this work is a partial extension, not an independent reconstruction of my advisor's complete program, since the original work examined properties of the Fourier (p, k)-capacity from the perspective of potential theory.

To fill the gap mentioned above, we make **three main contributions**.

First, we correct and optimize proofs made by my advisor, and we prove that $c_{p,k}(K)$ enjoys several basic properties as a potential-theoretic quantity. We establish its transformation properties under affine maps and study its partial Choquet capacity nature. A portion of this study is devoted to the issue of strong subadditivity: we obtain a definitive proof for the linear case ($p = 2$), and we capture the analytical obstructions for the non-linear case ($p \neq 2$), the status of which is presently an open problem. Furthermore, this capacity is connected to probability theory through weights controlled by negative definite functions (see Jacob (2005)), which in turn connect to the potential theory of Lévy and Markov processes (see van den Berg and Forst (2012), Sato (1999)). Second, and our main result of theoretical interest, is a simple direct estimate between the Fourier (p, k)-capacity and spatial Frostman measures (see 6.3.10):

Theorem: Let $K \subset \mathbb{R}^d$ be a compact set and μ be its Frostman measure (see 3.1.9) with

$$\mu(B(x,r)) \lesssim r^{\gamma}, \quad \forall x \in \mathbb{R}^d,\ r > 0 \tag{1.0.1}$$

for some $0 < \gamma \leqslant d$. Let $\beta > 0$ be the Fourier dimension of μ. Then, for all $p \geqslant \frac{4d-4\gamma+2\beta}{\beta}$, and non-constant k, then there exists a constant λ depending on μ such that:

$$c_{p,k}(K) \lesssim \lambda^{-1}\sqrt{\mu(K)}.$$

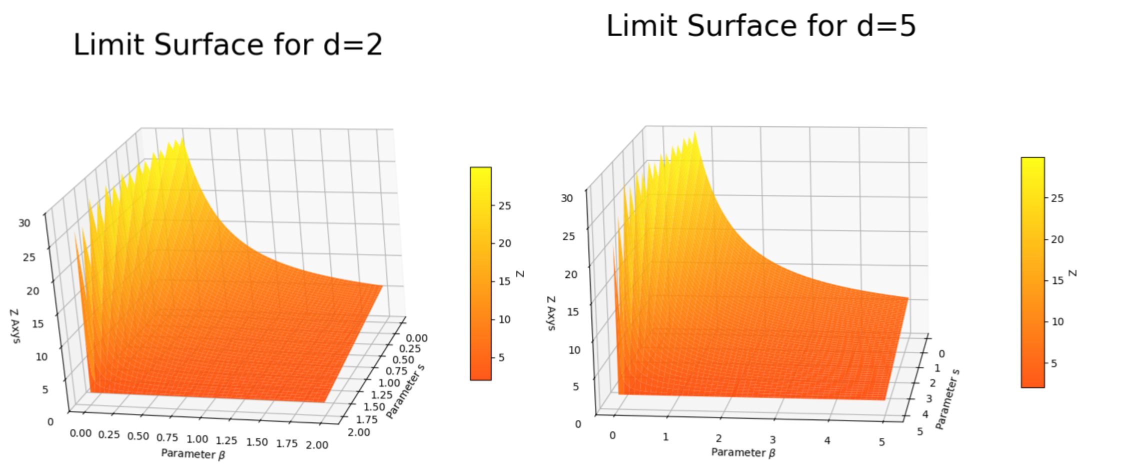
Applying the Mockenhaupt-Mitsis-Bak-Seeger Fourier restriction theorem (see Hambrook and

Łaba (2013)), we prove that if a compactly supported measure satisfies a certain growth condition, then the Fourier p-capacity acts as a lower bound for that measure. This theorem effectively recasts abstract restriction estimates as concrete capacity estimates, and implies a direct relationship between the Fourier (p,k)-capacity and the Hausdorff measure:

Corollary: Let $K \subset \mathbb{R}^d$ be a compact set with Hausdorff dimension equal to s and such that $\mathcal{H}^s|_K(B(x,r)) \lesssim r^s$. Let $\beta > 0$ be the Fourier dimension of $\mathcal{H}^s|_K$, and k a non-constant weight function. Then

$$\forall p \geqslant \frac{4d - 4s + 2\beta}{\beta} \Longrightarrow c_{p,k}(K) \lesssim \sqrt{\mathcal{H}^s(K)}.$$

This corollary corresponds to a general version of Theorem 6.1.16 (even though the case $k = 1$ is excluded), and leads to a direct conclusion: the case $k = 1$ implies a direct relationship between the Fourier capacity and the volume of the compact set, while adding a non-constant weight k allows us to compare the Fourier capacity with the Hausdorff measure, which usually captures more information than the Lebesgue measure. Recent research suggests that the critical value in the Mockenhaupt-Mitsis-Bak-Seeger theorem is sharp (see the preprint Fraser et al. (2025)), which implies that the dimensional bound obtained in the main theorem is optimal; therefore, our result is valid for the following values of p given a fixed dimension d:



These surfaces correspond to $z = \frac{4d-4s+2\beta}{\beta}$ which will allow us to visualize the lowest values that p can take depending on the values of s and β.

Finally, tools were obtained that allow the behavior of Fourier capacity to be studied on irregular sets, such as self-similar sets and fat Cantor sets, and which could lead to more robust results in future research. This application not only validates our general theorems but also highlights the sensitivity of Fourier capacities to the fine structure of fractals that possess positive Hausdorff measure (fat Cantor sets), or thinner sets such as self-similar sets.

The remainder of this thesis is organized as follows:

In **Chapter 3**, we establish the geometric preliminaries, focusing on the Hausdorff measure and dimension. We introduce essential tools such as Frostman's lemma and review the construction and dimensional properties of self-similar sets.

In **Chapter 4**, we introduce the analytical tools required for our frequency-based approach. We review the Fourier transform and tempered distributions, and then study the structure of Hörmander spaces $(B_{p,k})$.

In **Chapter 5**, we review the classical theory of capacities. We present the axiomatic framework of Choquet capacities, discuss the concept of polar sets, and provide classical examples such as the Riesz, Hausdorff, Sobolev, and Analytic capacities to give historical and technical context.

In **Chapter 6**, we present the core objects of our study and our main results. We formally introduce the Fourier p-capacity, the Fourier (p, k)-capacity, and we mention the connection between this capacity and probability theory. We prove their foundational properties as partial Choquet capacities (restricted to open and compact sets), explore their behavior under affine transformations, and establish our main theoretical bounds linking these capacities to spatial Frostman measures using the Mockenhaupt-Mitsis-Bak-Seeger Fourier restriction theorem. Furthermore, we apply these results to compute capacity thresholds for specific fractal structures, including self-similar sets and multidimensional fat Cantor sets.

Finally, in **Chapter 7**, we summarize our conclusions and discuss the significant open problems that remain, such as the proof of strong subadditivity for the non-linear cases ($p \neq 2$) and the potential for deep probabilistic interpretations of these capacities in future research.

Chapter 2

Problem description

The initial motivation for the research came from a need to analyze whether it would be possible to recover fine geometric information, such as the Hausdorff dimension and measure, of fractal subsets of $\mathbb{R}^d$ by using only tools related to Fourier analysis. The obvious analogue in this context is Maz'ya's *Fourier p-capacity*, introduced in Maz'ya (2018), which for a compact set K is given by:

$$c_{p,\mathcal{F}}(K) := \inf\{\|k\hat{u}\|_p : u \in C_0^\infty(\mathbb{R}^d), u \geqslant 1 \text{ on } K\}.$$

In other words, when applied to this object, potential theory turned out to be very useful. The Fourier p-capacity turned out to be a powerful mathematical object after its monotonicity, regularity, and subadditivity were proven. In particular, the Hilbert space case ($p = 2$) is a very special one: applying Plancherel's theorem, one gets a nice and precise geometric interpretation; the Fourier 2-capacity of a compact set is the square root of its Lebesgue volume. This was a significant fact, since it established that a spectral operator defined purely in terms of frequencies could also detect, in an unexpected manner, some direct spatial geometrical features. Although the above properties were proved by my advisor, the case $p = 2$ was used to prove strong subadditivity for that specific case, which increases its value as a mathematical object, since with that property, it follows that $c_{2,\mathcal{F}}$ is a capacity in the Choquet sense for arbitrary sets.

Initial exploration of the unweighted Fourier p-capacity (regarding the capacity of a Cartesian product between two compact sets, the projection of a compact set onto a subspace, the sum of two sets, or the construction of a compact set with positive capacity) revealed intrinsic analytical rigidities. Standard harmonic analysis tools proved insufficient to capture the fine geometric structure of arbitrary compact sets, primarily due to the non-local nature of the Fourier transform which breaks standard spatial truncation techniques for $p \neq 2$.

Nevertheless, as the investigation focused on extremely irregular fractal sets, the classical unweighted Fourier capacity was revealing its limitations. Clearly, the standard L^p integrability of the Fourier transform does not have enough "degrees of freedom" to capture the "thin" nature of fractal sets. Early efforts to prove direct bounds using standard harmonic analysis methods (like dyadic decomposition, Blaschke-product approximations, Bochner's theorem) turned out not to give sharp enough geometric statements for arbitrary sets. The unweighted capacity is just too rigid to detect the subtle frequency decay induced by sets of Hausdorff measure zero.

This geometric rigidity demanded a paradigm shift: to properly measure a fractal in the frequency domain, it is not enough to just evaluate the L^p norm of admissible functions; one must also understand how this capacity behaves under specific integrability thresholds and, more importantly, identify the analytical barriers that prevent standard tools from functioning correctly.

The Fourier transform is a profoundly non-local problem, and this is at the heart of the difficulty here. In classical potential theory, spatial energy based capacities (like the Sobolev or Riesz capacities) are well suited to pointwise manipulations. The proof of basic structural features, such as strong subadditivity, uses in an essential way, spatial truncations. In particular, it uses the fact that maximum and minimum of two admissible test functions are themselves admissible, and their energies can be cleanly bounded.

Nevertheless, if we view capacity purely as a function of the L^p integrability of the frequency weights, then all sense of spatial locality is lost. Pointwise truncations in the spatial domain do not preserve, in general, L^p bounds in the frequency domain. The only familiar exception is the Hilbert space case ($p = 2$), in which Plancherel's theorem gives a perfect isometric link. When $p \neq 2$, this absence of local control implies a significant theoretical barrier. This makes the proof of strong subadditivity an extremely challenging open problem, and it makes the interaction of the capacity with the union (and intersection) of general sets very complicated.

To get around this rigidity and isolate the dimension of highly irregular sets, one needs to bring heavier machinery from harmonic analysis. This means that the main domain of the study must be limited to the range $p \geqslant 2$. In this region, the analytical terrain becomes more "friendly," and one can take advantage of strong estimates, such as the Hausdorff-Young inequality and, even more importantly, Fourier restriction theorems. As a result, the main question changes: to show that a fractal set is indeed captured by the Fourier capacity, one must understand when the Fourier transform of a spatial measure supported on the set (e.g., a Frostman measure) decays fast enough to lie in $L^{p'}$. This of course brings into play the restriction theory of Mockenhaupt, Mitsis, Bak and Seeger, providing the ultimate analytical backdrop for this work.

After realizing that the unweighted capacity is too rigid, the problem naturally developed from a single application of a basic Fourier-analytic operator to a construction with the exact number of degrees of freedom to identify finitely many fractal structures. The now natural framework for this structural resolution exploits ideas from classical function spaces, weighted Sobolev spaces (see Kufner (1985), Turesson (2000)), Bessel potentials (see Stein (1970), Aronszajn and Smith (1961)) and generalized Hörmander spaces (see Hormander (1983)). In all of these theories, the introduction of a tempered weight function $k(\xi)$ in the domain of frequencies penalizes or favors certain decay rates in frequencies and can be thought of as an analytical "microscope" which adjusts the regularity of the space.

By twisting the definition of capacity with a tempered weight, we obtain the *Fourier* (p, k)*-capacity*:

$$c_{p,k}(K) := \inf\{\|k\hat{u}\|_p : u \in C_0^\infty(\mathbb{R}^d), u \geqslant 1 \text{ on } K\}.$$

This generalization is not simply a formal extension of mathematical notions; it is an essential geometric one. The weight $k(\xi)$ is precisely what one needs to regulate how the capacity will "measure" high-frequency oscillations generated by very irregular sets. It should be noted that this class of capacities is a generalization of the Fourier p-capacity, since the weight $k(\xi) \equiv 1$ is considered valid. Although this notation is similar to the (p, l)-capacity studied in Maz'ya and Khavin (1972), it should be noted that in Maz'ya's definition, the parameter l is a natural number, whereas for the Fourier (p, k)-capacity, the weight $k(\xi)$ is a function with controlled growth and decay.

As a result, the main mathematical problem of this thesis is focused on a precise analytical question: what are the suitable classes of tempered weights $k(\xi)$ and the admissible exponents $p \geqslant 2$ for which the Fourier (p,k)-capacity can be bounded from above by a spatial Frostman measure of a given set? Stating and solving this problem would imply that adding a tempered weight is precisely the key to enabling the Fourier capacity to capture some geometric information for the most irregular sets (such as self-similar sets) via tools that are entirely frequency-based.

Considering the above, specific sets that may be the subject of future study with respect to the range $1 < p < 2$ are explored, using a generalization of Lebedev's result (see Lebedev (2013)), with the aim of not closing the door to interesting results in this range of values. The construction given for the specific sets mentioned above is based on the construction given in Liu and Pego (2024).

The last aspect of the problem considered in this thesis comes from the need to find structurally relevant weights. Although classical tempered weights (for instance, those used to define Bessel potentials) provide sufficient analytical handling, we are looking for a setting with a richer theoretical meaning. For this purpose, we suggest looking at weights dominated by continuous negative definite functions, $\psi \in CN(\mathbb{R}^d)$. From an analytic point of view, as will be proved in this thesis, these weights form a proper subclass of classical tempered weights. Mathematically, this is a very nice fact as it ensures that the associated Fourier capacity will have the strong stability, inclusion and regularity properties of classical Hörmander spaces. Nevertheless, the real reason for this particular choice of weight is not to define a bigger or more "exotic" Banach space, but to make a deep structural connection between harmonic analysis and probability theory. Continuous negative definite functions are also intimately connected (via the Lévy-Khintchine representation, Schoenberg's theorem and Bochner's theorem) to convolution semigroups and characteristic exponents of infinitely divisible distributions. Thus the ultimate challenge that this thesis attempts to meet and the window that it wants to keep open for future research is the re-interpretation of the Fourier (p,k)-capacity. By these special weights, what we do is to go beyond seeing the capacity as simply a deterministic geometric device, and regard it as a potential-theoretic quantity closely related to the hitting probabilities of Lévy and Markov processes.

The details of the contributions in this thesis are as follows:

- **By my advisor:** Finiteness of capacity, monotonicity, definitions for more general sets and their monotonicity, $c_{p,\mathcal{F}}(\varnothing) = 0$, invariance under isometry, contraction and dilation by λ, definition of Fourier (p,k)-capacity.
- **Jointly reviewed results:** Subadditivity for compact sets.
- **Corrections:** Regularity and subadditivity on arbitrary sets, countable subadditivity, and Theorem 6.1.16(i).
- **By this work:** Regularity for compact sets, strong subadditivity for $c_{2,\mathcal{F}}$. Theorems 6.1.14, 6.1.16(ii), 6.1.20, 6.2.4, 6.2.6, 6.3.6, 6.3.7, 6.3.8, 6.3.9, and 6.3.10 (and their corollaries). Lemma 6.3.2.
- **Literature adaptation:** Lemmas 6.1.7 and 6.1.8. Theorem 6.2.1.

Chapter 3

Preliminaries

3.1 Hausdorff measure and Dimension

The concept of dimension in geometry has been present since its beginnings, either explicitly or implicitly. In Euclidean geometry there was already an implicit notion of dimension by defining a point, a segment, and a plane as follows: a point is something that has no parts, the ends of a line are points, the ends of a surface are lines (see Heath et al. (1956)). It is now well-known that the dimensions of such elements are 0, 1, and 2, respectively, since geometric and algebraic definitions confirm this. The latter is noteworthy, since in linear algebra the dimension can be defined through the number of vectors in a basis of the space (see Axler (2024), Friedberg et al. (2013), Hoffmann and Kunze (1971), and Lang (2012)), and both notions are functional for "good" geometric sets, such as those of classical geometry (points, segments, geometric figures, geometric bodies, etc.). It is now clear that the definition of topological dimension is useful for these objects, and it is defined as follows:

- 0 if it is totally disconnected (like two separated points).
- 1 if each point has arbitrarily small neighborhoods with boundary of dimension 0.
- 2 if each point has arbitrarily small neighborhoods with boundary of dimension 1. etc.

It was in 1883 when Cantor presented the well-known Cantor Set, which is a set with a structure that replicates at arbitrarily small scales, and highlighted the need for a concept that helps to describe sets of similar structures. Over the years, more similar sets appeared; for example, the Peano curve in 1890 that fills the plane, or the Koch curve in 1904, which broke with the comfort of working with differentiable curves, as it has no tangent line at any point. An important physical example is Brownian motion (which is, in fact, a particular case of the Lévy processes mentioned in the introduction), first seen in 1827, and explained in 1905 by Albert Einstein in Einstein (1905), describing the random motion of particles suspended in a liquid or gaseous medium.

In 1917, the mathematician Gaston Julia worked with complex numbers, iterating them through holomorphic functions, and thus obtaining a family of "monster" sets that led the mathematician Benoît Mandelbrot to create one of the most famous sets in modern mathematics, called the Mandelbrot set in his honor. Mandelbrot is a great reference in our context, because his book Mandelbrot (1983) gives a general and natural view on totally irregular sets, which considers all the above examples and many more. He decided to call the above sets fractals, which he defines as follows: "A fractal is by definition a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension". Of course, to understand this, we need to understand the concept of Hausdorff-Besicovitch dimension, or simply Hausdorff dimension. In order to understand the above and the results of this research, it is necessary to understand how the Hausdorff measure works, since **one of the main results of this research is deeply linked to the Hausdorff measure and dimension.**

3.1.1 Hausdorff measure

The Hausdorff measure is defined as follows:

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F),$$

where $\mathcal{H}^s_\delta(F) := \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : U_i \text{ is a } \delta\text{-covering}\right\}$, $|U| := \sup\{|x-y| : x, y \in U\}$, and a δ-covering is a cover such that $0 \leqslant |U_i| \leqslant \delta$, and $F \subseteq \bigcup_{i \geqslant 1} U_i$.

Properties 3.1.2. $\mathcal{H}^s$ is a metric outer measure, and the restriction of $\mathcal{H}^s$ to the Borel sets is a measure. In addition, the Hausdorff measure satisfies the following relations:

a). If $E \subset \mathbb{R}^d$ is a Borel set, $\mathcal{H}^0(E)$ counts the number of points in E.

b). If $E \subset \mathbb{R}$ is a Borel set, $\mathcal{H}^1(E) = m_1(E)$, where m_1 is the Hausdorff measure on $\mathbb{R}$.

c). If $E \subset \mathbb{R}^d$ is a Borel set, then there exists c_d such that

$$c_d \mathcal{H}^d(E) = m_d(E).$$

d). $\mathcal{H}^s(F + h) = \mathcal{H}^s(F)$ for all $h \in \mathbb{R}^d$, and $\mathcal{H}^s(rF) = \mathcal{H}^s(F)$ where r is a rotation in $\mathbb{R}^d$.

e). Let $\lambda > 0$. Then, for all $s \geqslant 0$:

$$\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F).$$

Remarks 3.1.3. The notation $m(E)$ for Hausdorff measure will be more used, because the dimension is not always that important.

3.1.4 Hausdorff Dimension

Let $t, s \in \mathbb{R}$ be such that $0 \leqslant s < t$, and let $\{U_i\}$ be a δ-covering of a set $F \subseteq \mathbb{R}^d$. Note the following inequalities:

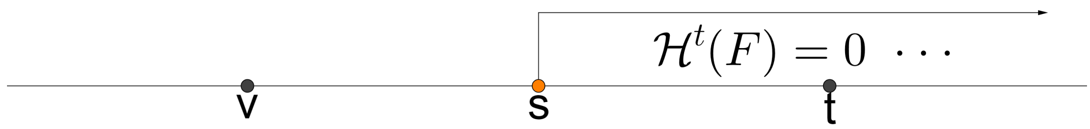
$$\sum_{i=1}^{\infty} |U_i|^t \leqslant \sum_{i=1}^{\infty} |U_i|^{t-s} |U_i|^s \leqslant \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s.$$

Taking the infimum over all coverings, we have:

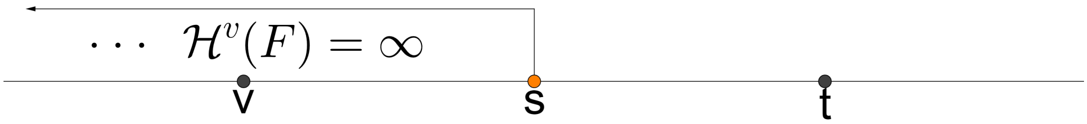
$$\mathcal{H}^t_\delta(F) \leqslant \delta^{t-s} \mathcal{H}^s_\delta(F).$$

From this relation, the following properties can be concluded:
Let $0 \leqslant v < s < t$, then $\mathcal{H}^t_\delta(F) \leqslant \delta^{t-s} \mathcal{H}^s_\delta(F)$ and $\mathcal{H}^s_\delta(F) \leqslant \delta^{s-v} \mathcal{H}^v_\delta(F)$. Thus:

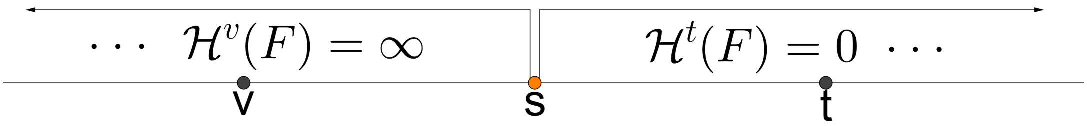
- If $\mathcal{H}^s(F) < \infty \Longrightarrow \mathcal{H}^t(F) = 0$



- If $\mathcal{H}^s(F) > 0 \Longrightarrow \mathcal{H}^v(F) = \infty$



- From the above, if $0 < \mathcal{H}^s(F) < \infty$



From this we can conclude that there is a specific value for each set at which the Hausdorff measure can give a result other than zero and infinity. We call this value the Hausdorff Dimension.

Definition 3.1.5. Let $F \subseteq \mathbb{R}^d$. The Hausdorff dimension of F is the number:

$$\dim_H(F) = \inf\{s \geqslant 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geqslant 0 : \mathcal{H}^s(F) = \infty\}.$$

Note that we must use the convention $\sup \varnothing = 0$; otherwise, if the Hausdorff dimension were $s = 0$, there would be no value t such that $\mathcal{H}^t(F) = \infty$, and the definition using the supremum would not be well-defined.

Now, we can write the Hausdorff measure of a set F in the following way:

$$\mathcal{H}^s(F) = \begin{cases} \infty & if \quad 0 \leqslant s < \dim_H(F) \\ a & if \quad s = \dim_H(F) \\ 0 & if \quad s > \dim_H(F) \end{cases}$$

where $0 \leqslant a \leqslant \infty$.

Remarks 3.1.6. In the context of self-similar sets, it is common for their Hausdorff measure to be positive. However, it is possible to construct s-dimensional sets such that their s-dimensional Hausdorff measure is equal to 0 or infinity.

Next, we will look at some properties satisfied by the Hausdorff dimension.

Proposition 3.1.7. *The Hausdorff dimension satisfies the following properties:*

a). If $E \subseteq F$, then $\dim_H(E) \leqslant \dim_H(F)$ (Monotonicity).

b). If $\{F_i\}$ is a sequence of subsets of $\mathbb{R}^d$, then:

$$\dim_H\left(\bigcup_{i=1}^{\infty} F_i\right) = \sup\{\dim_H(F_i)\} \qquad \textit{(Countable stability).}$$

c). If $F \subseteq \mathbb{R}^d$ is countable, then $\dim_H(F) = 0$ (Dimension of countable sets).

d). If $F \subseteq \mathbb{R}^d$ is an open set, then $\dim_H(F) = d$ (Dimension of open sets).

In this context, the Hausdorff Dimension becomes a particularly complex object, as it breaks with the notion of counting vectors in space, yet it gives us information about the "roughness" and "irregularity" of our set. This will become clearer in the section on self-similar fractals. It is clear that it must be treated with care despite its good properties seen above. However, mathematicians have made many advances in recent years, studying complex problems such as: Falconer's distance set problem (see Guth et al. (2020), Du et al. (2021)), Marstrand-type projections and theorems (see Käenmäki et al. (2025), Orponen and Shmerkin (2023)), Fourier dimension and Salem sets or measures (see Fraser and Hambrook (2020), Chen and Seeger (2017)).

The above properties and definitions allow the reader to have an idea of the relationship between measure and Hausdorff dimension, which will be **extremely important in our results section**.

The following lemma will be useful later on, and throughout history, it has been celebrated as a very important result, as it allows us to connect Hausdorff measure with capacity theory, as well as providing an important insight into the implications of the Hausdorff measure of a set. For more details, you can see Theorem 8.8 and the following in Mattila (1999).

Lemma 3.1.8. *(Frostman Lemma) Let B be a Borel set in $\mathbb{R}^d$. Then $\mathcal{H}^s(B) > 0$ if and only if there exists a Radon measure supported in B with $0 < \mu(B) < \infty$ such that*

$$\mu(B(x,r)) \lesssim r^s \qquad \forall x \in \mathbb{R}^d \text{ and } r > 0. \tag{3.1.1}$$

Definition 3.1.9. For a set B with $0 < \mathcal{H}^s(B) \leqslant \infty$, we will say that μ is its associated **Frostman Measure** if it satisfies the above conditions.

Remarks 3.1.10. The existence of a measure that satisfies (3.1.1) implies that $\dim_H(B) \geqslant s$. The case $\mu(B(x,r)) \lesssim r^\gamma$ with $\gamma > \dim_H(B)$ is impossible. Due to the above, the Hausdorff dimension can be considered as:

$$\dim_H(B) = \sup\{s \in [0,d] : \mu(B(x,r)) \lesssim r^s \text{ with } \mu \text{ a Frostman measure of B}\}. \tag{3.1.2}$$

There is a similar condition that is often called Ahlfors-David regularity, or AD-regularity in short:

Definition 3.1.11. A Borel measure μ on $\mathbb{R}^d$ is said to be AD-regular if

$$\frac{r^s}{A} \leqslant \mu(B(x,r)) \leqslant Ar^s$$

for all $x \in \operatorname{supp}\mu$ and $0 < r \leqslant |\operatorname{supp}\mu|$. An $\mathcal{H}^s$-measurable set K is called AD-regular, if $0 < \mathcal{H}^s(K) < \infty$, and the restriction $\mu := \mathcal{H}^s|_K$ of $\mathcal{H}^s$ to K is AD-regular (see Theorem 1 Section 1.2 in Jonsson (1984)).

3.2 Self-similar sets

Definition 3.2.1. An IFS (iterated function system) is a set $\{f_1, f_2, \ldots, f_m\}$ of similitudes, that is, functions $f_i : \mathbb{R}^d \to \mathbb{R}^d$ such that $f_i(x) = r_i \cdot O_i x + t_i$ $\forall i = 1, 2, \ldots, m$ and $m \geqslant 2$, where $0 < r_i < 1$ is called the contraction ratio, and O_i is an orthogonal matrix.

Hutchinson has proved in Hutchinson (1981) that if an IFS satisfies the Open Set Condition, then its attractor has good dimensional properties (see Theorem 1, Section 5.3):

Theorem 3.2.2. *Let $\{f_1, f_2, ..., f_m\}$ be an IFS that satisfies the Open Set Condition with contraction ratios r_i, and F its attractor, then $\dim_H F = \dim_B F = s$, where s is given by*

$$\sum_{i=1}^{m} r_i^s = 1.$$

That is, $0 < \mathcal{H}^s(F) < \infty$.

Fundamentally, the proof of the above theorem consists of demonstrating that the mapping $S = \bigcup_{i=1}^m f_i$ is a contraction. Then, the Banach fixed point theorem allows us to conclude.

Remarks 3.2.3. $\dim_B(K)$ is the Box Counting Dimension or Minkowski Dimension of K.

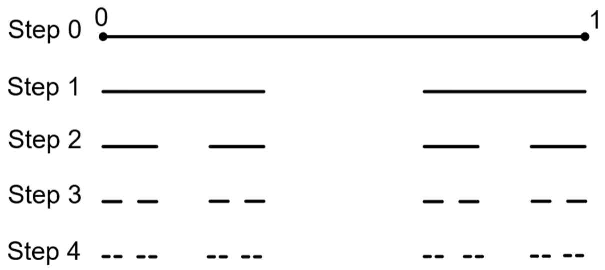
There is a lot of research on these type of sets, with notable authors including John Hutchinson and Robert Strichartz (already mentioned), Kenneth Falconer (see Falconer (2013) and Falconer (1985)), Pertti Mattila (see Martín and Mattila (2000) and Martín and Mattila (1993)), Tuomas Orponen (see Orponen (2012) and Fraser et al. (2014)), and Jonathan Fraser (see Fraser and Kolossváry (2023) and Fraser et al. (2015)).

Example 3.2.4.

- **Cantor set:** Let $\{f_1, f_2\}$ be an IFS, with $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, and

$$f_1(x) = \frac{x}{3}, \qquad f_2(x) = \frac{x}{3} + \frac{2}{3}.$$

Then an easy way to see what its attractor looks like is by applying these similitudes to a compact set in $\mathbb{R}$, for example, the interval $[0, 1]$:



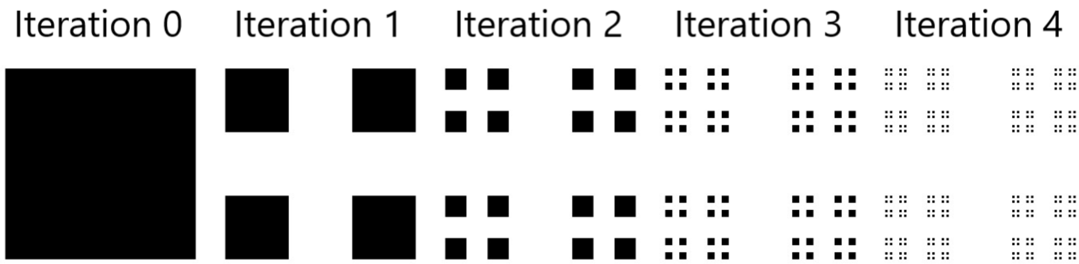
The attractor (which is obtained by repeating the above process infinitely) is called Cantor Ternary Set (denoted by $\mathcal{C}$), and it has Hausdorff dimension equal to $\log(2)/\log(3)$.

- **Cantor dust:** Let $\{f_1, f_2, f_3, f_4\}$ be an IFS, with $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, 3, 4$ and

$$f_1(x, y) = \left(\frac{x}{3}, \frac{y}{3}\right), \qquad f_2(x, y) = \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{3}\right)$$

$$f_3(x, y) = \left(\frac{x}{3}, \frac{y}{3} + \frac{2}{3}\right), \qquad f_4(x, y) = \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{3} + \frac{2}{3}\right).$$

Then its attractor (which can be obtained by applying the similitudes to the unit square $[0, 1] \times [0, 1]$) is called 2D Cantor dust (and denoted by $\mathcal{C}^2$), and it has Hausdorff dimension equal to $\log(4)/\log(3)$.

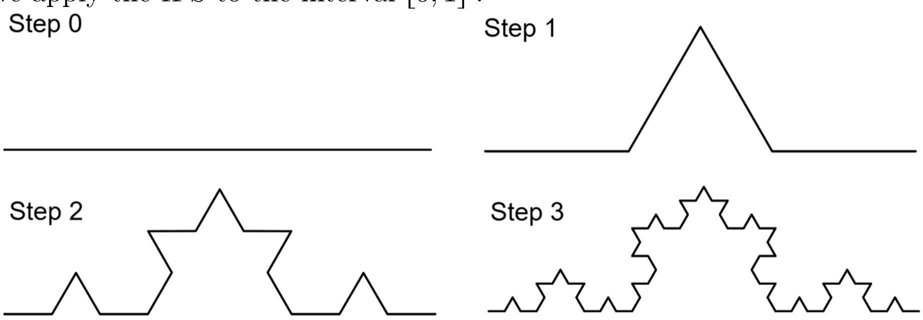


- **Koch curve:** Let $\{f_1, ..., f_4\}$ be an IFS with $f_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, 3, 4$, and

$$f_1(x) = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} x, \qquad f_2(x) = \begin{pmatrix} 1/6 & -\sqrt{3}/6 \\ \sqrt{3}/6 & 1/6 \end{pmatrix} x + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$$

$$f_3(x) = \begin{pmatrix} 1/6 & \sqrt{3}/6 \\ -\sqrt{3}/6 & 1/6 \end{pmatrix} x + \begin{pmatrix} 1/2 \\ \sqrt{3}/6 \end{pmatrix}, \qquad f_4(x) = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} x + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}.$$

Then we apply the IFS to the interval $[0, 1]$:



The attractor is called the Koch curve, and it has Hausdorff dimension equal to $\log(4)/\log(3)$.

The introduction of these objects primarily serves two purposes: to prove that there are objects with fractional Hausdorff dimension that can be easily constructed as examples, and to provide an understanding of the structure of these types of sets, since later on we will give the general construction of sets called **fat Cantor sets** (see 6.2.6).

Chapter 4

Fourier transform and spaces of interest

One tool that has proven useful is the Fourier transform, and in general, the tools provided by harmonic analysis (as shown by Stein and Shakarchi in Stein and Shakarchi (2011), and Fraser et al. (2019)).

The objects of study (belonging to harmonic analysis) seem to have a strong relation with the Hausdorff dimension of the associated set, perhaps even collecting more geometric information than Hausdorff measure.

A huge amount of literature has emerged in recent decades on harmonic analysis (see Zygmund (2002), Adams and Hedberg (2012), Maz'ya and Saposnikova (2013), Landkof (1972)), and in particular on the properties, applications, and studies in different contexts of the Fourier transform (and Fourier series). In 1822, the French mathematician Jean-Baptiste Joseph Fourier published in his book Théorie analytique de la chaleur (Analytical Theory of Heat) (see Fourier (1878)) a method for representing heat diffusion using infinite sums of trigonometric functions, which gave rise to the theory of Fourier Series and Transforms, although the exact mathematical formalization and some aspects of his work were refined by other mathematicians, such as Augustin-Louis Cauchy (see Cauchy (1827)), converging on the current definition of the Fourier transform of a function f:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx.$$

It is worth noting that in the literature, the term $2\pi i$ changes to i.
One of the main applications of this transform is solving partial differential equations (PDEs) (see Evans (2022) and Folland (1995)).
An interesting difference between working with functions in common Euclidean space and their Fourier transforms is that the space in which the variable moves in the latter (which we denote by ξ) corresponds to a frequency space, which also has useful applications, such as the decomposition of a mixed wave into each of the waves that compose it (a very common application in physics).
The Fourier transform is of vital importance in this research. This object has different definitions, depending on the space in which one wishes to work. For example, for integrable functions, we have the following definition:

Definition 4.0.1. Let $f \in L^1(\mathbb{R}^d)$. We define the function $\mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx,$$

which is called the *Fourier transform* of f.

However, in the context of harmonic analysis, it is useful to define it for commonly used spaces,

such as Schwartz Space and its dual, the tempered distribution space.
For the introduction of Schwartz functions, we will use the document van Zuijlen (2022) as a reference, but there is a lot of literature on this space of functions, and the Fourier transform. See, for example, Grafakos et al. (2008), Mitrea (2013) and Folland (2009) for an extensive analysis on the definition, properties, and connection with the Fourier transform. Also, Wolff (2003) is a good option for a more geometric view, Rudin (1991) for a more topological view, Reed and Simon (1972) for its properties on operator theory, Hormander (1983) for connections with PDE theory, or Treves (1967) for an analysis between the Schwartz space and its dual.
Now, we say that a function $f : \mathbb{R}^d \to \mathbb{F}$ is of *rapid decay* if

$$\lim_{|x|\to\infty} P(x)f(x) = 0,$$

for all polynomials P, where $\lim_{|x|\to\infty} g(x) = a$ means that for all $\varepsilon > 0$ there exists an $R > 0$ such that for all $x \in \mathbb{R}^d$ with $|x| > R$, $|g(x) - a| < \varepsilon$. Using this concept, a smooth function φ is called a *Schwartz function* if the function and all its derivatives are of rapid decay. We write $\mathcal{S}$ for the space of Schwartz functions.

Definition 4.0.2. (Fourier transform) Let $\varphi \in \mathcal{S}$. We define the function $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ as

$$\mathcal{F}(\varphi)(\xi) = \hat{\varphi}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}\varphi(x)dx,$$

that is called the *Fourier transform* of φ.

Proposition 4.0.3. *Given $f, g \in \mathcal{S}(\mathbb{R}^d)$, $y \in \mathbb{R}^d$, and $b \in \mathbb{C}$, we have:*

(1) $\|\hat{f}\|_\infty \leqslant \|f\|_1$.

(2) $\widehat{f+g} = \hat{f} + \hat{g}$.

(3) $\widehat{bf} = b\hat{f}$.

(4) The map $\wedge : \phi \to \hat{\phi}$ *is continuous.*

(5) $(D^\alpha\phi)^\wedge(\xi) = \xi^\alpha\hat{\phi}(\xi)$, *for all* α.

(6) $(x_j\phi)^\wedge = -2\pi i D^j\hat{\phi}$.

(7) $(e^{2\pi i x\cdot y}f(x))^\wedge(\xi) = \tau^y(\hat{f})(\xi)$.

(8) $\widehat{\tau^y f}(\xi) = e^{-2\pi i y\cdot\xi}\hat{f}(\xi)$.

(9) $\widehat{f * g} = \hat{f}\hat{g}$.

(10) $\widehat{f \circ A}(\xi) = \hat{f}(A\xi)$, *where A is an orthogonal matrix and ξ is a column vector.*

(11) The transform $\wedge$ *is an isomorphism from* $\mathcal{S}$ *to* $\mathcal{S}$.

where $(\tau^y f)(x) = f(x - y)$ corresponds to a translation.

Example 4.0.4. The Fourier transform can be defined for measures by

$$\hat{\mu}(\xi) = \int e^{-2\pi i\xi\cdot y}d\mu(y).$$

Here, Strichartz has studied the asymptotic behavior for the case of self-similar measures (see Strichartz (1990), Strichartz (1993a), and Strichartz (1993b)).

Theorem 4.0.5. *(Mockenhaupt-Mitsis-Bak-Seeger Theorem) Let μ be a compactly supported* *positive measure on* $\mathbb{R}^d$ *such that for some* $\alpha, \beta \in (0, d)$ *we have:*

$$\mu(B(x,r)) \leqslant C_1 r^\alpha \quad \text{for all } x \in \mathbb{R}^d \text{ and } r > 0,$$
$$|\widehat{\mu}(\xi)| \leqslant C_2(1+|\xi|)^{-\beta/2} \text{ for all } \xi \in \mathbb{R}^d.$$

Then for all $p \geqslant p_{d,\alpha,\beta} := \dfrac{2(2d-2\alpha+\beta)}{\beta}$*, there is a constant* $C(p) > 0$ *such that for all* $f \in L^2(d\mu)$*:*

$$\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^d)} \leqslant C(p)\|f\|_{L^2(d\mu)}.$$

This theorem has proven to be the key to demonstrating the main theorem of this research, and the fact that it has recently been shown (preliminarily) that the value of $p_{n,\alpha,\beta}$ is optimal gives the result greater robustness (see Fraser et al. (2025)).

The following results are very strong tools in our context (see propositions 2.2.14 and 2.2.16 in Grafakos et al. (2008)).

Theorem 4.0.6. *(Plancherel's identity) If* f *belongs to both* L^1 *and* L^2*, then* $\hat{f}$ *also belongs to* L^2 *and*

$$\|f\|_2 = \|\hat{f}\|_2.$$

Theorem 4.0.7. *(Hausdorff-Young inequality) Let* p, p' *be conjugates, and* $p \in [2, \infty]$*. Then for any* $f \in L^{p'}(\mathbb{R}^d)$ *we have:*

$$\|\hat{f}\|_p \leqslant \|f\|_{p'}.$$

Remarks 4.0.8. There is a sharper version of the Hausdorff Young inequality (see Beckner (1975)).

Theorem 4.0.9. *(Riemann-Lebesgue lemma) Let* $f \in L^1(\mathbb{R}^d)$ *be an integrable function. Then* $\hat{f}$ *vanishes at infinity:* $|\hat{f}(\xi)| \to 0$ *as* $|\xi| \to \infty$.

Because the Fourier transform of an integrable function is continuous (see Theorem 7.5 in Rudin (1991)), the Fourier transform $\hat{f}$ is a continuous function vanishing at infinity. The Fourier transform maps $L^1(\mathbb{R}^d)$ to the space of continuous functions vanishing at infinity.

Definition 4.0.10. (Fourier dimension of a measure) The Fourier dimension of a finite Borel measure μ on $\mathbb{R}^d$ is defined as

$$\dim_F \mu := \sup\{s \in [0,d] : \hat{\mu}(\xi) \lesssim |\xi|^{-s/2}\}.$$

4.1 Distributions and Tempered Distributions

To provide a rigorous framework for the analytical methods used in this work, we briefly review the theory of distributions, with a particular focus on tempered distributions. These objects have been extensively studied, yielding quite useful results in different areas (see Mitrea (2013), Hormander (1963), Treves (2016)).

4.1.1 The Space of Distributions

Let $\Omega \subseteq \mathbb{R}^d$ be an open set. We denote by $\mathcal{D}(\Omega)$ (or $C_0^\infty(\Omega)$) the space of smooth functions with compact support in Ω. A sequence $\{\phi_j\} \subset \mathcal{D}(\Omega)$ converges to ϕ if there exists a compact set $K \subset \Omega$ containing the supports of all ϕ_j, and all derivatives $\partial^\alpha \phi_j$ converge uniformly to $\partial^\alpha \phi$.

Definition 4.1.2. (Distribution) A distribution is a continuous linear functional on $\mathcal{D}(\Omega)$. The set of all such functionals is denoted by $\mathcal{D}'(\Omega)$.

Example 4.1.3.

- If μ is a locally finite (i.e., satisfies $\mu(K) < \infty$ for every compact $K \subset \mathbb{R}^d$) Borel measure, then

$$\mu : \mathcal{D} \to \mathbb{R}, \quad \mu(\phi) := \int_{\mathbb{R}^d} \phi d\mu, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d),$$

  is a distribution. In our context, it is also useful to consider measures as distributions.
- For each $f \in L^1_{loc}(\mathbb{R}^d)$ define the functional $u_f : \mathcal{D}(\mathbb{R}^d) \to \mathbb{R}$ by

$$u_f(\phi) := \int_{\mathbb{R}^d} f(x)\phi(x)dx, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d).$$

  Then u_f is linear, and defines a distribution (see Mitrea (2013) for the details).
- An important distribution is the Dirac distribution δ defined by

$$\delta(\phi) := \phi(0), \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d).$$

4.1.4 Tempered Distributions

While general distributions are powerful, they do not behave well with the Fourier transform.

Definition 4.1.5 (Tempered Distribution)**.** The space of tempered distributions, denoted by $\mathcal{S}'(\mathbb{R}^d)$, is the set of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$.

Tempered distributions are particularly useful because they allow for a closed-form definition of the Fourier transform.

Lemma 4.1.6. *The space of tempered distributions satisfies good properties:*

- ***Fourier transform:*** *For any $T \in \mathcal{S}'$, the Fourier transform $\hat{T}$ is defined by $\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle$ for all $\phi \in \mathcal{S}$. The Fourier transform is an isomorphism from $\mathcal{S}'$ to itself.*
- ***Differentiation:*** *Every tempered distribution is infinitely differentiable, where $D^\alpha T$ is defined by $\langle D^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle$.*
- ***Inclusion:*** *We have the following chain of continuous inclusions:*

$$\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d). \tag{4.1.1}$$

Tempered distributions are essential for solving linear partial differential equations via the Fourier transform and for the study of Sobolev spaces $H^s(\mathbb{R}^d)$, which are defined using the growth of the Fourier transform of tempered distributions (see Bahouri (2011), Duistermaat and Kolk (2010)).

4.2 Hörmander Spaces

The study of general linear partial differential operators with constant coefficients led Lars Hörmander in the early 1960s to introduce a broad class of function spaces that generalize the classical Sobolev spaces. While Sobolev spaces $H^s(\mathbb{R}^d)$ are characterized by their behavior under the Fourier transform with respect to polynomial weights, Hörmander's spaces allow for much more refined control over the regularity and growth of distributions (see Hormander (1963) and Hormander (1983)).

Definition 4.2.1. ($L^p_m(\mathbb{R}^d)$ spaces) $L^p_m = \{u \in L^p : D^\alpha u \in L^p, \forall \alpha \text{ with } |\alpha| \leqslant m\}$, $1 < p < \infty$, $m \in \mathbb{Z}^+$.

The space L^p_m equipped with the norm $\|u\|_{p,m} = (\sum_{|\alpha|\leqslant m} \|D^\alpha u\|_p^2)^{1/2}$ is called a **Sobolev space**. A property of these spaces is: if $0 \leqslant j \leqslant m$, then $L^p_m \subset L^p_j \subset L^p$ and the embeddings are continuous. Also, the spaces L^p_m are Banach spaces for all $p \geqslant 1$, $m \in \mathbb{Z}^+$, and are isomorphic to each other for $m \geqslant 0, 1 < p < \infty$.

Now, let L^2_m be the space of all tempered distributions $u \in \mathcal{S}'$ such that $D^\alpha u \in L^2(\mathbb{R}^d)$, $0 \leqslant |\alpha| \leqslant m$. We have the inclusions $L^2_m \subset L^2 \subset \mathcal{S}'$, and if $u = u(x) \in L^2_m$ then

$$\xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^d) \quad \text{for } 0 \leqslant |\alpha| \leqslant m,$$

and vice versa.

For the next theorem, see Proposition 31.6 in Treves (1967).

Theorem 4.2.2. *$u(x) \in L^2_m$ if and only if $(1 + |\xi|^2)^{m/2}\hat{u} \in L^2$.*

Due to the above theorem, the space L^2_m is also usually defined as follows:

Definition 4.2.3. $L^2_m = \{u \in L^2 : (1 + |\xi|^2)^{m/2}\hat{u} \in L^2\}$.

With the above definition, the space L^2_m has the following norm:

$$\|u\|_{L^2_m} = \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^m |\hat{u}|^2 d\xi\right)^{1/2}. \tag{4.2.1}$$

4.2.4 Tempered weight Functions

To define these spaces, we first recall the notion of a *tempered weight function*. A positive function $k : \mathbb{R}^d \to (0, \infty)$ is called a tempered weight if there exist constants $C > 0$ and $N > 0$ such that:

$$k(\xi + \eta) \leqslant (1 + C|\xi|)^N k(\eta), \quad \xi, \eta \in \mathbb{R}^d.$$

Common examples include the polynomial weights $k_s(\xi) = (1 + |\xi|^2)^{s/2}$, which recover the standard Sobolev scale.

We can obtain an important consequence of the definition by replacing η by $\xi + \eta$, and replacing ξ by $-\xi$:

$$(1 + C|\xi|)^{-N} k(\eta) \leqslant k(\xi + \eta) \leqslant (1 + C|\xi|)^N k(\eta), \quad \xi, \eta \in \mathbb{R}^d. \tag{4.2.2}$$

This means that we have control over the decay and growth of these weights.

The fact that $k(\xi) = (1 + |\xi|^2)^{\alpha/2}$ is a tempered weight follows from the estimates:

$$1 + |\xi + \eta|^2 \leqslant 1 + |\xi|^2 + 2|\xi||\eta| + |\eta|^2 \leqslant (1 + |\xi|)^2(1 + |\eta|^2)$$

$$k(\xi + \eta) \leqslant (1 + |\xi|)^\alpha (1 + |\eta|^2)^{\alpha/2} = (1 + |\xi|)^\alpha k(\eta).$$

Lemma 4.2.5. *If k_1 and k_2 are tempered weights, then k_1+k_2, k_1k_2, $\sup(k_1, k_2)$ and $\inf(k_1, k_2)$ are also tempered weights. If $s \in \mathbb{R}$, then k^s is a tempered weight.*

4.2.6 Definition of $B_{p,k}$ Spaces

Let $\mathcal{S}'(\mathbb{R}^d)$ be the space of tempered distributions, and $\mathcal{E}'(\mathbb{R}^d)$ the space of distributions with compact support. For $1 \leqslant p \leqslant \infty$ and a tempered weight function k, the Hörmander space $B_{p,k}$ is defined as:

$$B_{p,k} = \{u \in \mathcal{S}'(\mathbb{R}^d) : \hat{u} \in L^1_{loc}(\mathbb{R}^d) \text{ and } |u|_{p,k} < \infty\},$$

where the norm is given by:

$$|u|_{p,k} = \left(\int_{\mathbb{R}^d} |k(\xi)\hat{u}(\xi)|^p d\xi \right)^{1/p},$$

with the usual modification for $p = \infty$. These spaces are Banach spaces, and if $p = 2$ and $k_\alpha = (1 + |\xi|^2)^{\alpha/2}$, then B_{2,k_α} is precisely the Sobolev space $H^\alpha(\mathbb{R}^d)$ (see Hormander (1963), Hormander (1983),Mikhailets and Murach (2014)). For the next properties, see Hormander (1963) from Theorem 2.2.1 to Theorem 2.2.5.

Properties 4.2.7. a). $B_{p,k}$ is a Banach space and we have in the topological sense

$$C_0^\infty \subset \mathcal{S} \subset B_{p,k} \subset \mathcal{S}', \tag{4.2.3}$$

and C_0^∞ is dense in $B_{p,k}$ if $p < \infty$.

b). If k_1, k_2 are tempered weights, and

$$k_2(\xi) \lesssim k_1(\xi), \qquad \forall \xi \in \mathbb{R}^d,$$

it follows that $B_{p,k_1} \subset B_{p,k_2}$ and $B_{p,k_1} \bigcap B_{p,k_2} = B_{p,k_1+k_2}$.

c). If p, p' are conjugates, then $B_{p',1/k}$ is the dual space of $B_{p,k}$.

d). If $u \in B_{p,k}$ and $\phi \in \mathcal{S}$, it follows that $\phi u \in B_{p,k}$ and that

$$|\phi u|_{p,k} \leqslant |\phi|_{1,M_k} |u|_{p,k}, \tag{4.2.4}$$

where $M_k(\xi) = \sup_\eta k(\xi + \eta)k(\eta)$ (this means that M_k is the smallest function such that $k(\xi + \eta) \leqslant M_k(\xi)k(\eta)$).

These results are mainly intended to show something of the structure of these spaces (for more details, see Hormander (1983)), since the definition is not at all arbitrary, but rather generalizes important spaces such as Potential spaces, maintaining strong and useful properties. In particular, Theorem 4.2.4 will be especially useful for the main theorem of this research.

Considering the above, the introduction of these spaces is crucial to understanding the context in which our main object of study develops, as **these spaces are key to defining Fourier (p,k)-capacity**.

4.3 Positive and Negative definite functions

While positive definite functions have a strong connection to finite measures through Bochner's theorem, for us the natural objects of interest (in the same sense that shadows are natural objects for light) are **negative definite functions**. These functions are so intimately related to convolution semigroups that they yield a direct structural connection to the characteristic exponents of infinitely divisible distributions and Lévy processes.

With this class of functions, we have not only the means to define the Fourier (p,k)-capacity with weights controlled by negative definite functions, but also **to anchor our frequency approach in a solid probabilistic framework for subsequent investigations** (for more details, see Jacob (2005) and Jacob (2002)).

We will denote the space of negative definite functions by $N(\mathbb{R}^d)$.

Definition 4.3.1. A function $\psi : \mathbb{R}^d \to \mathbb{C}$ belongs to the class $N\left(\mathbb{R}^d\right)$ if for any choice of $k \in \mathbb{N}$ and vectors $\xi^1, \ldots, \xi^k \in \mathbb{R}^d$ the matrix

$$\left(\psi\left(\xi^j\right) + \overline{\psi\left(\xi^l\right)} - \psi\left(\xi^j - \xi^l\right)\right)_{j,l=1,\ldots,k} ,$$

is positive Hermitian (i.e., has positive eigenvalues). Furthermore we set

$$CN\left(\mathbb{R}^d\right) := N\left(\mathbb{R}^d\right) \cap \mathrm{C}\left(\mathbb{R}^d\right),$$

for the continuous negative definite functions.

For examples and constructions of negative definite functions, see Jacob's book from Lemma 3.6.7 onwards in Jacob (2005).

Now, the following two lemmas will be useful for understanding the types of bounds that can be used over negative definite functions (see Lemma 3.6.22 in Jacob (2005)):

Lemma 4.3.2. *For any locally bounded negative definite function $\psi \in N(\mathbb{R}^d)$ there exists a constant $c_\psi > 0$ such that for all $\xi \in \mathbb{R}^d$:*

$$|\psi(\xi)| \leqslant c_\psi(1 + |\xi|^2).$$

In addition to the above upper bound, the family of negative definite functions satisfies several other useful inequalities within their respective areas of study (see Lemma 3.6.21, Lemma 3.2.23, corollary 3.6.24, and Lemma 3.6.25 in Jacob (2005)). Another useful result is the following (see corollary 3.6.17 in Jacob (2005)).

Lemma 4.3.3. *A function $\psi \in CN(\mathbb{R}^d)$ is continuous and negative definite if and only if*

$$\psi(0) \geqslant 0 \quad \text{and} \quad \xi \mapsto e^{-t\psi(\xi)}, t > 0, \text{ is continuous and positive definite.}$$

It is worth mentioning that by combining Bochner's theorem (see Theorem 3.5.7 in Jacob (2005)), and Lemma 4.3.3, we obtain that given a function $\psi \in CN(\mathbb{R}^d)$, then the functions $e^{-t\psi(\xi)}$ correspond to the Fourier transforms of a family of probability measures $(\mu_t)_{t\geqslant 0}$, which forms a convolution semigroup (see Definition 3.6.1 in Jacob (2005)). Specifically, if $(\mu_t)_{t\geqslant 0}$ is a convolution semigroup of bounded Borel measures on $\mathbb{R}^d$, the Fourier transform of each measure yields $\widehat{\mu}_t(\xi) = e^{-t\psi(\xi)}$, where the exponential $e^{-t\psi(\xi)}$ is a continuous positive definite function for every $t > 0$, and the exponent $\psi : \mathbb{R}^d \to \mathbb{C}$ is a continuous negative definite function. By looking at Table 3.9.19 in Jacob (2005) we now have several examples of negative definite functions.

Chapter 5

Capacities

In addition to the above theory, it is necessary to define what is meant by a *Capacity*, **since the main object of study is a capacity**. The notion of capacity emerged in analysis thanks to Gustave Choquet in Choquet (1954), to measure how "large" a set is from a more refined perspective than the Lebesgue measure and even the Hausdorff measure. Capacities have emerged that even provide information about a set's ability to withstand electrical charge, but the concept can be generalized to other contexts. There is a wide range of literature on this subject, as many types of capacities can be defined (see Xu (1995), Mattila (1984), Mattila (1985), Khoshnevisan (1999),Ponce (2016), Khoshnevisan and Shi (1999)), but the basic requirements a set function must meet to be considered a capacity correspond to the following definition:

Definition 5.0.1. Let $\mathscr{F}$ be a paving of X, i.e., a collection of subsets of X such that $\varnothing \in \mathscr{F}$, and which is closed under the operations of finite unions and finite intersections. An $\mathscr{F}$-capacity is an increasing set function $I : \mathcal{P}(X) \to \mathbb{R}_+$ satisfying the following:

- If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of subsets of X then $I(A_n) \to I(\bigcup_m A_m)$ as $n \to \infty$.
- If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of subsets of X such that $A_n \in \mathscr{F}$ for each n, then $I(A_n) \to I(\bigcap_m A_m)$ as $n \to \infty$.

Here, the condition of being increasing means that $I(A) \leqslant I(B)$ whenever $A \subseteq B$. The definition provided above corresponds to the classical notion of an **$\mathscr{F}$-capacity in the sense of Choquet**. Unlike standard measures, capacities are not required to be additive; instead, they are characterized by their behavior under monotonic limits.

However, a highly desirable property for robust capacities is a specialized form of subadditivity. An $\mathscr{F}$-capacity I is called **strongly subadditive** (or an alternating capacity of order 2) if for any subsets $A, B \subseteq X$:

$$I(A \cup B) + I(A \cap B) \leqslant I(A) + I(B).$$

This property is crucial in nonlinear potential theory, as it allows capacities to interact consistently with lattice operations such as the maximum and minimum of admissible functions.

In most practical settings within $\mathbb{R}^d$, the paving $\mathscr{F}$ is taken to be the collection of compact subsets $\mathcal{K}$. Under this choice, the capacity I is said to be *Borel regular* if it can be approximated from below by compact sets, a property known as *capacitability.*

Theorem 5.0.2 (Choquet's Capacitability Choquet (1954))**.** *Let I be a Choquet capacity on a Hausdorff space X relative to the paving of compact sets $\mathcal{K}$. Then every Borel set (and more generally, every Analytic or Suslin set) $B \subseteq X$ is I-capacitable, meaning:*

$$I(B) = \sup\{I(K) : K \subseteq B, K \in \mathcal{K}\}.$$

This theorem is a bridge between topology and measure theory. It implies that to understand the capacity of any "measurable" set, it is sufficient to know the capacity of its compact subsets. This is vital in geometric measure theory when dealing with fractal sets that are defined as limits of compact constructions.

Beyond the axiomatic structure, capacities play a definitive role in identifying exceptionally "thin" sets that classical measures overlook.

Definition 5.0.3. A set $E \subset \mathbb{R}^d$ is called **polar** with respect to a capacity I if $I(E) = 0$. If a property holds everywhere except on a polar set, we say it holds **quasi-everywhere** (q.e.).

The concept of "quasi-everywhere" is a strict refinement of the standard "almost everywhere" (a.e.) from Hausdorff measure theory, as sets of Hausdorff measure zero can still have strictly positive capacity.

Furthermore, capacities provide deep geometric insights. The positivity of a capacity is intrinsically linked to the Hausdorff dimension of the set. For instance, in classical Riesz or Sobolev capacities, if a set E is dimensionally too "thin," it becomes invisible to the capacity operator (i.e., its capacity is zero). Thus, $I(E) > 0$ strictly implies a lower bound on $\dim_H(E)$.

Finally, classical capacities often admit two equivalent dual definitions. While functional definitions compute capacity by minimizing a norm (like an L^p energy) over a space of admissible test functions (a variational inf problem), the dual perspective computes capacity by maximizing the total mass of a Radon measure μ supported on the set, subject to an energy bound (a variational sup problem). This duality naturally links the analytical capacity of a set to the existence of structured measures supported on it, motivating the use of tools like Frostman's lemma in harmonic analysis.

Despite the important difference between the concept of capacity and measure (because there are no additivity or subadditivity conditions imposed on the capacity generally), every finite measure μ can be assigned a capacity of the form:

$$\mu^*(S) = \inf\{\mu(A) : A \in \mathscr{F}, A \supset S\}.$$

5.1 Examples

To fully appreciate the scope of Choquet capacities and their applications in modern analysis, it is essential to review the classical examples. These capacities not only identify sets of "size zero" across different mathematical contexts (such as potential theory, partial differential equations, and complex analysis) but they also provide the historical and technical foundation for the Fourier capacities studied in this work.

Riesz Capacity ($C_{\alpha,p}$ and C_α).

The Riesz capacity is the cornerstone of classical potential theory, originally inspired by electrostatic capacity (see Landkof (1972), Adams and Hedberg (2012)). It is defined using the Riesz kernel $K_\alpha(x) = |x|^{\alpha-d}$ in $\mathbb{R}^d$ (where $0 < \alpha < d$).

Definition 5.1.1. In the non-linear setting, the Riesz (p, α)-capacity of a compact set K is defined via the minimization of the L^p norm of functions whose Riesz potential dominates 1 on K:

$$C_{\alpha,p}(K) = \inf\{\|f\|_p^p : f \in L^p(\mathbb{R}^d), f \geqslant 0, K_\alpha * f \geqslant 1 \text{ on } K\}.$$

In the classical linear setting ($p = 2$), it has an equivalent dual definition based on the Riesz energy of a measure. For a Radon measure μ supported on K, its α-energy is $I_\alpha(\mu) = \iint |x-y|^{\alpha-d} d\mu(x) d\mu(y)$. The classical Riesz capacity is:

$$C_\alpha(K) = \sup\{\mu(K) : \mu \text{ is a Radon measure on } K, I_\alpha(\mu) \leqslant 1\}.$$

- **Utility and Fourier Connection:** It characterizes polar sets (those that cannot support a charge with finite energy, see 5.0.3). Due to Parseval's identity, the energy can be rewritten in the frequency domain as $I_\alpha(\mu) = c \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi$. This explicit connection between spatial energy and the decay of the Fourier transform is the exact historical precursor to the Fourier capacities defined in this thesis.

Hausdorff Capacity ($\mathcal{H}^s_\infty$).

Unlike the Hausdorff measure ($\mathcal{H}^s$), which involves a limit as the diameter of the covering goes to zero ($\delta \to 0$), the Hausdorff capacity evaluates covers of arbitrary size (see Section 5 Adams and Hedberg (2012)).

Definition 5.1.2. For a set $F \subseteq \mathbb{R}^d$ and $s \geqslant 0$, the Hausdorff capacity is defined as:

$$\mathcal{H}^s_\infty(F) = \inf\left\{\sum_{i=1}^\infty r_i^s : F \subseteq \bigcup_{i=1}^\infty B(x_i, r_i)\right\}.$$

- **Utility:** Because it avoids the limit $\delta \to 0$, it satisfies the axioms of a Choquet capacity, particularly the convergence of increasing sequences of sets, which the Hausdorff measure fails to do. It is vital in geometric measure theory because it shares the same null sets as the Hausdorff measure: $\mathcal{H}^s(F) = 0$ if and only if $\mathcal{H}^s_\infty(F) = 0$. Frostman's lemma provides a deep duality between this capacity and the existence of measures satisfying the growth condition $\mu(B(x,r)) \lesssim r^s$.

Sobolev Capacity ($Cap_{k,p}$).

This capacity is intrinsically linked to the Sobolev spaces $W^{k,p}(\mathbb{R}^d)$, replacing the Riesz kernel with the Bessel potential to improve behavior at infinity (see Heinonen et al. (2018), Evans (2018)).

Definition 5.1.3. The capacity of a set A with respect to the Sobolev space $W^{k,p}(\mathbb{R}^d)$ is defined as:

$$Cap_{k,p}(A) = \inf\{\|u\|^p_{W^{k,p}} : u \in W^{k,p}(\mathbb{R}^d), u \geqslant 1 \text{ in an open neighborhood of } A\}.$$

- **Utility:** It is used to study the "fine" properties of Sobolev functions and removable singularities for elliptic PDEs. While standard L^p functions are defined only almost everywhere, Sobolev functions admit "quasicontinuous" representatives that are precisely defined everywhere except on sets of zero Sobolev capacity. Furthermore, it establishes a critical dimensional threshold: if $Cap_{k,p}(A) = 0$, then its Hausdorff dimension satisfies $\dim_H(A) \leqslant d - kp$.

Analytic Capacity (γ).

Introduced by Ahlfors (see Ahlfors (1947)), this capacity leaves the realm of real potentials to study sets in the complex plane $\mathbb{C}$ regarding bounded holomorphic functions (see Tolsa (2014)).

Definition 5.1.4. For a compact set $K \subset \mathbb{C}$, the analytic capacity is defined as:

$$\gamma(K) = \sup\{|f'(\infty)| : f : \mathbb{C} \backslash K \to \mathbb{C} \text{ is analytic }, \|f\|_\infty \leqslant 1, f(\infty) = 0\}.$$

where $f'(\infty) = \lim_{z\to\infty} zf(z)$.

- **Utility:** Its primary application is the Painlevé problem regarding removable singularities. A compact set K is “removable” for bounded holomorphic functions if and only if $\gamma(K) = 0$. This capacity has been extensively studied, and equivalent definitions that do not depend on analytical functions have even been found (see Verdera (2007)). Furthermore, it has been found that it is related to similar capacities (which use a kind of Riesz kernel), which in turn are comparable to the Hausdorff measure (see Prat Baiget (2003)).

Chapter 6

Properties of Fourier Capacities and Main Results

6.1 Fourier p-capacity

Let $C_0^\infty(\mathbb{R}^d)$ be the space of infinitely differentiable functions with compact support in $\mathbb{R}^d$, and $\mathcal{K}$ the space of compact sets in $\mathbb{R}^d$. We define the **Fourier p-capacity** of $K \in \mathcal{K}$ as follows:

$$c_{p,\mathcal{F}}(K) := \inf\{\|\hat{u}\|_p : u \in C_0^\infty(\mathbb{R}^d), u \geqslant 1 \text{ on } K\}.$$

This is an object mentioned by Maz'ya in Maz'ya (2018), and one of the main objectives is to obtain some properties related to this new object, and to characterize the behavior of specific compact sets under this capacity.

It is clear that both the infimum and the p-norm in the definition are useful tools when attempting to obtain nice properties for sets in general, for example, when making a proof, both objects can provide useful inequalities by definition.

Since the Fourier p-capacity works with sets, it would be interesting to analyze whether it satisfies the properties of a measure, an outer measure or a capacity in the sense of Choquet (see Choquet (1954)). Examples of useful properties would be the following:

a). (Monotonicity) Let $K_1, K_2 \in \mathcal{K}$. If $K_1 \subset K_2$ then $c_{p,\mathcal{F}}(K_1) \leqslant c_{p,\mathcal{F}}(K_2)$.

b). (Regularity) Let $K \in \mathcal{K}$. For each $a > c_{p,\mathcal{F}}(K)$, there exists $U \in \mathcal{O}$ such that $K \subset U$ and for all $C \in \mathcal{K}$ such that $C \subset U$ we have $c_{p,\mathcal{F}}(C) < a$.

c). (Subadditivity)

$$c_{p,\mathcal{F}}(K_1 \cup K_2) \leqslant c_{p,\mathcal{F}}(K_1) + c_{p,\mathcal{F}}(K_2), \quad \forall K_1, K_2 \in \mathcal{K}.$$

The subadditivity might be a powerful tool if we want to work with self-similar sets, or some type of set that has a similar construction (where the union of sets is an important factor in its construction).

Focusing on self-similar sets, and drawing inspiration from properties that satisfy certain measures (such as the Hausdorff measure), it would also be interesting to investigate how the Fourier p-capacity of a compact K behaves when a similarity is applied to it. This can be divided into investigating its behavior under isometries and under translations. In particular, a desirable property would be the *translation invariance:*

$$c_{p,\mathcal{F}}(K + a) = c_{p,\mathcal{F}}(K) \ \ \forall K \in \mathcal{K}, \ \ \forall a \in \mathbb{R}^d.$$

In order to prove these properties, various tools have been provided in the previous sections.

Lemma 6.1.1. *For any $K \in \mathcal{K}$, $c_{p,\mathcal{F}}(K)$ is finite.*

Proof: We proceed by contradiction. First, we establish the monotonicity property. Suppose $K_1 \subset K_2$, where $K_1, K_2 \in \mathcal{K}$. Let $u \in C_0^\infty(\mathbb{R}^d)$ be such that $u \geqslant 1$ on K_2, and in particular, $u \geqslant 1$ on K_1. This implies the inclusion:

$$\{\|\hat{u}\|_p : u \in C_0^\infty(\mathbb{R}^d), u \geqslant 1, \text{ on } K_2\} \subseteq \{\|\hat{u}\|_p : u \in C_0^\infty(\mathbb{R}^d), u \geqslant 1, \text{ on } K_1\},$$

since being greater than or equal to 1 in K_1 is a weaker constraint than being greater than or equal to 1 in K_2. Then, taking the infimum on both sides, we obtain $c_{p,\mathcal{F}}(K_1) \leqslant c_{p,\mathcal{F}}(K_2)$. Now, suppose there exists a compact set $K \in \mathcal{K}$ such that $c_{p,\mathcal{F}}(K) = \infty$. This implies that for any function $u \in C_0^\infty(\mathbb{R}^d)$ with $u \geqslant 1$ on K, we must have $\|\hat{u}\|_p = \infty$. However, since K is compact, it is bounded, and there exists an $R \geqslant 0$ such that $K \subset B_R$, where B_R denotes the closed ball with center 0 and radius R. On the other hand, there exists a bump function $\phi \in C_0^\infty(\mathbb{R}^d)$ with support in B_{R+1} such that $\phi(x) = 1$ for $x \in B_R$. Since $\phi \in C_0^\infty(\mathbb{R}^d)$, $\hat{\phi}$ is a Schwartz function (see 4.1.1 and item 11 in 4.0.3), which means it is rapidly decreasing; in particular, for any $N > d/p$, there exists a constant C_N such that

$$|\hat{\phi}(\xi)| \leqslant C_N(1 + |\xi|)^{-N} \quad \text{for all } \xi \in \mathbb{R}^d.$$

In particular,

$$\|\hat{\phi}\|_p \leqslant \int_{\mathbb{R}^d} C_N(1 + |\xi|)^{-Np} < \infty,$$

for any $p \geqslant 1$. Therefore, by the monotonicity property, we conclude

$$c_{p,\mathcal{F}}(K) \leqslant c_{p,\mathcal{F}}(B_R).$$

However, this leads to a contradiction, since we have established earlier that $c_{p,\mathcal{F}}(B_R) < \infty$ while assuming $c_{p,\mathcal{F}}(K) = \infty$. Hence,

$$c_{p,\mathcal{F}}(K) < \infty.$$

This completes the argument. ∎

Lemma 6.1.2. *The following statements hold.*

(C1) (Monotonicity) Let $K_1, K_2 \in \mathcal{K}$. If $K_1 \subset K_2$ then $c_{p,\mathcal{F}}(K_1) \leqslant c_{p,\mathcal{F}}(K_2)$.

(C2) (Regularity) Let $K \in \mathcal{K}$. For each $a > c_{p,\mathcal{F}}(K)$, there exists $U \in \mathcal{O}$ such that $K \subset U$ and for all $C \in \mathcal{K}$ such that $C \subset U$ we have $c_{p,\mathcal{F}}(C) < a$.

(C3) (Subadditivity)

$$c_{p,\mathcal{F}}(K_1 \cup K_2) \leqslant c_{p,\mathcal{F}}(K_1) + c_{p,\mathcal{F}}(K_2), \quad \forall K_1, K_2 \in \mathcal{K}.$$

Proof: (C1) It was already proved in the lemma above.
(C2) Let $a > c_{p,\mathcal{F}}(K)$. By the definition of Fourier p-capacity, there exists $u \in C_0^\infty\left(\mathbb{R}^d\right)$ such that $u \geqslant 1$ on K and $\|\hat{u}\|_p < a$. Since the inequality is strict, we can choose an $\epsilon \in (0, 1)$ such that

$$\frac{1}{1-\epsilon}\|\hat{u}\|_p < a.$$

Define $U = \{x \in \mathbb{R}^d : u(x) > 1 - \epsilon\}$. This is an open set containing K.

Now, let $C \in \mathcal{K}$ be such that $C \subset U$. By the definition of U, we know that $u(x) > 1 - \epsilon$ for all $x \in C$.

Define $v(x) = \dfrac{u(x)}{1-\epsilon}$. It is easy to see that this new function is admissible for C. Furthermore, this function satisfies:

$$\|\hat{v}\|_p = \left\| \frac{1}{1-\epsilon}\hat{u} \right\|_p = \frac{1}{1-\epsilon}\|\hat{u}\|_p < a.$$

The above estimate allows us to conclude the regularity property $(C2)$:

$$c_{p,\mathcal{F}}(C) \leqslant \|\hat{v}\|_p < a.$$

(C3) By the definition of infimum, there exist functions $u_1, u_2 \in C_0^\infty\left(\mathbb{R}^d\right)$ such that $u_i \geqslant 1$ on K_i, and

$$\|\hat{u}_i\|_p < c_{p,\mathcal{F}}\left(K_i\right) + \epsilon, \qquad i = 1, 2.$$

It is trivial that if $u_i \geqslant 0$, $i = 1, 2$, then $u_1 + u_2$ is an admissible function in $K_1 \bigcup K_2$, and

$$\|\widehat{u_1 + u_2}\|_p = \|\hat{u}_1 + \hat{u}_2\|_p \leqslant \|\hat{u}_1\|_p + \|\hat{u}_2\|_p.$$

Taking the infimum and letting $\epsilon \to 0$, allows us to conclude the subadditivity with only non-negative functions.

However, standard test functions in the definition of the capacity are not required to be non-negative. To extend the subadditivity to the general case, we utilize cut-off functions with controlled Fourier norm.

Let u_1, u_2 be the above functions. Since both are continuous and $u_i \geqslant 1$ on the compact set K_i for all $i = 1, 2$, there exist open neighborhoods V_i of K_i such that $u_i(x) > 0$ for all $x \in V_i$. We now introduce smooth cut-off functions to eliminate the negative parts of u_i without significantly increasing their norms (for more details, see 6.1.3). There exist functions $\nu_i \in C_0^\infty$ with support in V_i such that $\nu_i(x) = 1$ for all $x \in K_i$, $\nu_i(x) \geqslant 0$ for all x, and $\|\hat{\nu}_i\|_1 \leqslant 1 + \epsilon$.

Now, we define the new functions:

$$w_i(x) = u_i(x)\nu_i(x), \qquad i = 1, 2.$$

Now, these functions are admissible and are non-negative. Now, we have the following estimate using Young's inequality:

$$\|\hat{w}_i\|_p = \|\hat{u}_i * \hat{\nu}_i\|_p \leqslant \|\hat{u}_i\|_p\|\hat{\nu}_i\|_1 < (c_{p,\mathcal{F}}(K_i) + \epsilon)(1 + \epsilon).$$

As before, we use the function $w_1 + w_2$ for the compact set $K_1 \bigcup K_2$, and then:

$$\|\hat{w}_1 + \hat{w}_2\|_p \leqslant \|\hat{w}_1\|_p + \|\hat{w}_2\|_p \leqslant (c_{p,\mathcal{F}}(K_1) + \epsilon)(1 + \epsilon) + (c_{p,\mathcal{F}}(K_2) + \epsilon)(1 + \epsilon).$$

Taking the infimum and letting $\epsilon \to 0$, we conclude the subadditivity. ■

Remarks 6.1.3. The existence of the cut-off function is a non-trivial problem, and in fact, it has a lot of theory behind it, for example Leptin's theorem, Urysohn's theorem, regularity of Algebras, amenability of locally compact abelian groups, approximation identities, local units, Gelfand theory, etc. However, we will only mention the existence of this, as too much preliminary information would be needed to explain the existence of these functions (you can see Kaniuth (2009), Reiter and Stegeman (2000), and Runde (2004)).

We extend the Fourier p-capacity to the open subsets of $\mathbb{R}^d$.

Definition 6.1.4. (The Fourier p-capacity on open sets). Let $U \subseteq \mathbb{R}^d$ be an open set.

$$c_{p,\mathcal{F}}(U) := \sup \{c_{p,\mathcal{F}}(K) : K \subset U, K \in \mathcal{K}\}.$$

Note that although $c_{p,\mathcal{F}}(K) < \infty$ by Lemma 6.1.1, $c_{p,\mathcal{F}}(U)$ could be infinity.
More generally, we extend the Fourier p-capacity to arbitrary subsets of $\mathbb{R}^d$ as follow:

Definition 6.1.5. (The Fourier p-capacity on arbitrary sets). Let $E \subseteq \mathbb{R}^d$ be an arbitrary set.

$$c_{p,\mathcal{F}}(E) := \inf \{c_{p,\mathcal{F}}(U) : E \subset U, U \in \mathcal{O}\}.$$

Theorem 6.1.6. *The following statements hold.*

(G1) (Monotonicity) Let $E_1, E_2 \subseteq \mathbb{R}^d$. If $E_1 \subset E_2$ then $c_{p,\mathcal{F}}(E_1) \leqslant c_{p,\mathcal{F}}(E_2)$.

(G2) (Regularity) For every $K \in \mathcal{K}$, $c_{p,\mathcal{F}}(K) = \inf \{c_{p,\mathcal{F}}(U) : K \subset U, U \in \mathcal{O}\}$.

(G3) (Subadditivity)

$$c_{p,\mathcal{F}}(E_1 \bigcup E_2) \leqslant c_{p,\mathcal{F}}(E_1) + c_{p,\mathcal{F}}(E_2), \forall E_1, E_2 \subseteq \mathbb{R}^d.$$

Proof: (G1) First we show the monotonicity for open sets: Let $U \subset V$ be open sets. For any compact set $K \subset U$, we also have $K \subset V$. Therefore,

$$c_{p,\mathcal{F}}(U) = \sup \{c_{p,\mathcal{F}}(K) : K \subset U, K \in \mathcal{K}\} \leqslant \sup \{c_{p,\mathcal{F}}(K) : K \subset V, K \in \mathcal{K}\} = c_{p,\mathcal{F}}(V).$$

Now, let $E_1 \subset E_2 \subseteq \mathbb{R}^d$. For any open set U containing E_2, we also have $E_1 \subset U$. Therefore,

$$c_{p,\mathcal{F}}(E_1) = \inf \{c_{p,\mathcal{F}}(U) : E_1 \subset U, U \in \mathcal{O}\} \leqslant \inf \{c_{p,\mathcal{F}}(U) : E_2 \subset U, U \in \mathcal{O}\} = c_{p,\mathcal{F}}(E_2).$$

(G2) By the definition of $c_{p,\mathcal{F}}(U)$, for each $K \in \mathcal{K}$ and all $U \in \mathcal{O}$ such that $K \subset U$, we have

$$c_{p,\mathcal{F}}(K) \leqslant c_{p,\mathcal{F}}(U).$$

For the reverse inequality, for each $\epsilon > 0$, we use the regularity property on compact sets (C2) with $a = c_{p,\mathcal{F}}(K) + \epsilon$, there exists $U \in \mathcal{O}$ such that $\forall C \in \mathcal{K}$ with $C \subset U$ we obtain:

$$c_{p,\mathcal{F}}(K) > c_{p,\mathcal{F}}(C) - \epsilon.$$

Taking the supremum over all admissible compact set C, we arrive at $c_{p,\mathcal{F}}(U)$, and letting $\epsilon \to 0$ we conclude $c_{p,\mathcal{F}}(K) = \inf \{c_{p,\mathcal{F}}(U) : K \subset U, U \in \mathcal{O}\}$.
(G3) As in the proof of (G1), we first establish (G3) for open sets and then we deduce the general case. Let $U_1, U_2 \in \mathcal{O}$. By the Subadditivity property on compact set (C3), we have for

all $K_1, K_2 \in \mathcal{K}$:

$$c_{p,\mathcal{F}}\left(K_1 \bigcup K_2\right) \leqslant c_{p,\mathcal{F}}\left(K_1\right) + c_{p,\mathcal{F}}\left(K_2\right), \quad \forall K_1, K_2 \in \mathcal{K}.$$

Considering the compact sets such that $K_1 \subset U_1$ and $K_2 \subset U_2$, we obtain, by the definition of $c_{p,\mathcal{F}}$ on open sets (i.e., applying supremum), the property (G3) on open sets, that is, for all $U_1, U_2 \in \mathcal{O}$, we have:

$$c_{p,\mathcal{F}}\left(U_1 \bigcup U_2\right) \leqslant c_{p,\mathcal{F}}\left(U_1\right) + c_{p,\mathcal{F}}\left(U_2\right).$$

Now the general case. By definition, for any set $E \subset \mathbb{R}^d$, the Fourier p-capacity is given by

$$c_{p,\mathcal{F}}(E) = \inf\left\{c_{p,\mathcal{F}}(U) : E \subset U, U \in \mathcal{O}\right\}.$$

For each $i = 1, 2$, let $\epsilon > 0$ be arbitrary. By the definition of $c_{p,\mathcal{F}}\left(E_i\right)$, there exist open sets $U_i \supset E_i$ such that:

$$c_{p,\mathcal{F}}\left(U_i\right) \leqslant c_{p,\mathcal{F}}\left(E_i\right) + \epsilon.$$

Note that $E_1 \bigcup E_2 \subset U_1 \bigcup U_2$. Therefore, by the subadditivity property on open sets, and monotonicity, it follows

$$\begin{aligned} c_{p,\mathcal{F}}\left(E_1 \bigcup E_2\right) &\leqslant c_{p,\mathcal{F}}\left(U_1 \bigcup U_2\right) \\ &\leqslant c_{p,\mathcal{F}}\left(U_1\right) + c_{p,\mathcal{F}}\left(U_2\right) \\ &\leqslant c_{p,\mathcal{F}}\left(E_1\right) + 2\epsilon + c_{p,\mathcal{F}}\left(E_2\right). \end{aligned}$$

As this holds for all $\epsilon > 0$, we conclude that

$$c_{p,\mathcal{F}}\left(E_1 \bigcup E_2\right) \leqslant c_{p,\mathcal{F}}\left(E_1\right) + c_{p,\mathcal{F}}\left(E_2\right).$$

■

The proofs of the following two properties are basically set-theoretic arguments similar to those done in Willem (2023) (properties 7.1.7 and 7.1.8).

Lemma 6.1.7. *Let (K_n) be a decreasing sequence in $\mathcal{K}$. Then:*

$$c_{p,\mathcal{F}}\left(\bigcap_{n=1}^{\infty} K_n\right) = \lim_{n\to\infty} c_{p,\mathcal{F}}\left(K_n\right).$$

Proof: Let $K = \bigcap_{n=1}^{\infty} K_n$ and $U \in \mathcal{O}, U \supset K$. By compactness, there exists m such that $K_m \subset U$. We obtain, by monotonicity, $c_{p,\mathcal{F}}(K) \leqslant \lim_{n\to\infty} c_{p,\mathcal{F}}\left(K_n\right) \leqslant c_{p,\mathcal{F}}(U)$. It suffices then to take the infimum with respect to U. ■

Lemma 6.1.8. *Let (U_n) be an increasing sequence in $\mathcal{O}$. Then:*

$$c_{p,\mathcal{F}}\left(\bigcup_{n=1}^{\infty} U_n\right) = \lim_{n\to\infty} c_{p,\mathcal{F}}\left(U_n\right).$$

Proof: Let $U = \bigcup_{n=1}^{\infty} U_n$ and $K \in \mathcal{K}, K \subset U$. By compactness, there exists m such that $K \subset U_m$. We obtain by monotonicity $c_{p,\mathcal{F}}(K) \leqslant \lim_{n\to\infty} c_{p,\mathcal{F}}\left(U_n\right) \leqslant c_{p,\mathcal{F}}(U)$. It then suffices to take the supremum over K. ■

Remarks 6.1.9. These properties are highly important, but the discussion about them will take place after analyzing all the capacities of this research (see 6.3.5).

Theorem 6.1.10. *The Fourier p-capacity is an outer measure on* $\mathbb{R}^d$. *That is,* $c_{p,\mathcal{F}}$ *satisfies the following properties:*

(i) $c_{p,\mathcal{F}}(\varnothing) = 0$.
(ii) For any sequence of sets $(E_i)_{i=1}^{\infty}$ *in* $\mathbb{R}^d$*:*

$$c_{p,\mathcal{F}}\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} c_{p,\mathcal{F}}\left(E_i\right).$$

Proof: (i) We show that for any $\epsilon > 0$, we can find a function $u \in C_0^{\infty}\left(\mathbb{R}^d\right)$ such that $u \geqslant 1$ on $\varnothing$ and $\|\hat{u}\|_p < \epsilon$. Let $\phi \in C_0^{\infty}\left(\mathbb{R}^d\right)$ be a smooth (bump) function with compact support such that $\phi(0) = 1$. For any $\epsilon > 0$, define $u_\epsilon(x) = \epsilon\phi(x)$. Then, $u_\epsilon \in C_0^{\infty}\left(\mathbb{R}^d\right)$ since $\phi \in C_0^{\infty}\left(\mathbb{R}^d\right)$. Moreover, $u_\epsilon \geqslant 1$ on $\varnothing$ vacuously, as this condition is satisfied for any function. By the linearity and scaling properties of the Fourier transform, we obtain $\hat{u}_\epsilon(\xi) = \epsilon\hat{\phi}(\xi)$. Therefore,

$$\left\|\hat{u}_\epsilon\right\|_p = \epsilon\|\hat{\phi}\|_p.$$

Note that $\|\hat{\phi}\|_p$ is finite because ϕ is a Schwartz function, and the Fourier transform of a Schwartz function is also a Schwartz function, which is in $L_p\left(\mathbb{R}^d\right)$ for all $1 \leqslant p \leqslant \infty$.

Now, for any $\delta > 0$, we choose $\epsilon = \delta/\|\hat{\phi}\|_p$, which gives

$$\left\|\hat{u}_\epsilon\right\|_p = \epsilon\|\hat{\phi}\|_p = \delta.$$

Since $\delta > 0$ was arbitrary, we conclude:

$$c_{p,\mathcal{F}}(\varnothing) = \inf\left\{\|\hat{u}\|_p : u \in C_0^{\infty}\left(\mathbb{R}^d\right), u \geqslant 1 \text{ on } \varnothing\right\} = 0.$$

This completes the proof.
(ii) Let $(E_i)_{i=1}^{\infty}$ be an arbitrary sequence of sets in $\mathbb{R}^d$. Let $\epsilon > 0$ be arbitrary. By the definition of infimum, for each E_i, there exists an open set U_i such that:

$$E_i \subset U_i \quad \text{and} \quad c_{p,\mathcal{F}}\left(U_i\right) < c_{p,\mathcal{F}}\left(E_i\right) + \frac{\epsilon}{2^i}.$$

Let $U = \bigcup_{i=1}^{\infty} U_i$. Clearly, U is open and $\bigcup_{i=1}^{\infty} E_i \subset U$. For open sets, we have

$$c_{p,\mathcal{F}}(U) = \sup\left\{c_{p,\mathcal{F}}(K) : K \subset U, K \text{ is compact}\right\}.$$

Let K be an arbitrary compact set contained in U. Since K is compact and $\{U_i\}_{i=1}^{\infty}$ forms an open cover of K, there exists a finite subcover. That is, there exists an N such that:

$$K \subset \bigcup_{i=1}^{N} U_i.$$

By the subadditivity and monotonicity of $c_{p,\mathcal{F}}$ for arbitrary sets (6.1.6), we have:

$$c_{p,\mathcal{F}}(K) \leqslant \sum_{i=1}^{N} c_{p,\mathcal{F}}\left(U_i \cap K\right) \leqslant \sum_{i=1}^{N} c_{p,\mathcal{F}}\left(U_i\right) < \sum_{i=1}^{N}\left(c_{p,\mathcal{F}}\left(E_i\right)+\frac{\epsilon}{2^i}\right) \leqslant \sum_{i=1}^{\infty}\left[c_{p,\mathcal{F}}\left(E_i\right)\right]+\epsilon.$$

Since this holds for all compact $K \subset U$, we obtain:

$$c_{p,\mathcal{F}}(U) \leqslant \sum_{i=1}^{\infty} c_{p,\mathcal{F}}\left(E_i\right)+\epsilon.$$

Therefore,

$$c_{p,\mathcal{F}}\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant c_{p,\mathcal{F}}(U) \leqslant \sum_{i=1}^{\infty} c_{p,\mathcal{F}}\left(E_i\right)+\epsilon.$$

As $\epsilon>0$ is arbitrary, we conclude:

$$c_{p,\mathcal{F}}\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} c_{p,\mathcal{F}}\left(E_i\right).$$

This completes the proof of subadditivity for the Fourier p-capacity. ■

Lemma 6.1.11. *The Fourier p-capacity is invariant under translations. That is,*

$$c_{p,\mathcal{F}}(E)=c_{p,\mathcal{F}}(E+a),$$

where $E+a:=\{x+a \mid x \in E\}$ for any set $E \subseteq \mathbb{R}^d$ and $a \in \mathbb{R}^d$.

Proof: Note that it is enough to prove the property on compact sets. Let $u \in C_0^{\infty}\left(\mathbb{R}^d\right)$ such that $u \geqslant 1$ on E. Define $v(x)=u(x-a)$. Then $v \in C_0^{\infty}\left(\mathbb{R}^d\right)$ and $v(x) \geqslant 1$ for all $x \in E+a$. By the properties of the Fourier transform, we have $\hat{v}(\xi)=e^{-2\pi i a \cdot \xi} \hat{u}(\xi)$.
Taking the L^p norm on both sides,

$$\|\hat{v}\|_p=\left(\int_{\mathbb{R}^d}\left|e^{-2\pi i a \cdot \xi} \hat{u}(\xi)\right|^p d\xi\right)^{1/p}=\left(\int_{\mathbb{R}^d}|\hat{u}(\xi)|^p d\xi\right)^{1/p}=\|\hat{u}\|_p.$$

This holds because $\left|e^{-2\pi i a \cdot \xi}\right|=1$ for all ξ. Therefore, for every u satisfying the conditions for E, there exists a v satisfying the conditions for $E+a$, and $\|\hat{v}\|_p=\|\hat{u}\|_p$. This implies

$$c_{p,\mathcal{F}}(E+a) \leqslant c_{p,\mathcal{F}}(E).$$

Similarly, applying the same argument with the translation by $-a$, we obtain

$$c_{p,\mathcal{F}}(E)=c_{p,\mathcal{F}}((E+a)-a) \leqslant c_{p,\mathcal{F}}(E+a). \Longrightarrow c_{p,\mathcal{F}}(E)=c_{p,\mathcal{F}}(E+a).$$

Thus, the Fourier p-capacity is invariant under translations, as required. ■

Theorem 6.1.12. *Let $K \subset \mathbb{R}^d$ be a compact set and let $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an affine isometry.*

Then

$$c_{p,\mathcal{F}}(f(K)) = c_{p,\mathcal{F}}(K).$$

Moreover, for any $\lambda > 0$ we have:

$$c_{p,\mathcal{F}}(\lambda f(K)) = \lambda^{d\left(1-\frac{1}{p}\right)} c_{p,\mathcal{F}}(K).$$

Proof: Since f is an affine isometry, there exist an orthogonal matrix A and a vector $b \in \mathbb{R}^d$ such that $f(x) = Ax + b$. Let $\epsilon > 0$ be arbitrary. By the definition of $c_{p,\mathcal{F}}(K)$, there exists a function $u \in C_0^\infty\left(\mathbb{R}^d\right)$ such that $u \geqslant 1$ on K and $\|\hat{u}\|_p \leqslant c_{p,\mathcal{F}}(K) + \epsilon$. Define $v(x) = u\left(f^{-1}(x)\right) = u\left(A^{-1}(x-b)\right)$. Note that $v \in C_0^\infty\left(\mathbb{R}^d\right)$ and $v \geqslant 1$ on $f(K)$.

Now, using properties of the Fourier transform of v, and using the change of variables $x = Ay+b$ and $|\det A| = 1$, we obtain:

$$\hat{v}(\xi) = e^{-2\pi i b\cdot\xi}\hat{u}\left(A^T\xi\right),$$

where A^T is the transpose matrix of A. Furthermore, $|\hat{v}(\xi)| = \left|\hat{u}\left(A^T\xi\right)\right|$. Since A^T is also an orthogonal matrix, the change of variables $\eta = A^T\xi$ gives:

$$\|\hat{v}\|_p^p = \int_{\mathbb{R}^d} |\hat{v}(\xi)|^p d\xi = \int_{\mathbb{R}^d} |\hat{u}(\eta)|^p d\eta = \|\hat{u}\|_p^p.$$

Thus, $\|\hat{v}\|_p = \|\hat{u}\|_p \leqslant c_{p,\mathcal{F}}(K) + \epsilon$. Since v is admissible for $f(K)$, we have $c_{p,\mathcal{F}}(f(K)) \leqslant c_{p,\mathcal{F}}(K)+\epsilon$. By applying the same argument to f^{-1} and $f(K)$, we get $c_{p,\mathcal{F}}(K) \leqslant c_{p,\mathcal{F}}(f(K))+\epsilon$. As $\epsilon > 0$ was arbitrary, we conclude that $c_{p,\mathcal{F}}(f(K)) = c_{p,\mathcal{F}}(K)$.

Now, let $\lambda > 0$. Applying the result above to the set λK, we have $c_{p,\mathcal{F}}(A\lambda K + \lambda b) = c_{p,\mathcal{F}}(\lambda K)$. Let u be an admissible function for the set K, that is, $u \in C_0^\infty\left(\mathbb{R}^d\right)$ and $u \geqslant 1$ on K. Then the function $v(x) := u(x/\lambda)$ is admissible for λK as well. By changing variables, we obtain $\hat{v}(\xi) = \lambda^d\hat{u}(\lambda\xi)$, and:

$$\|\hat{v}\|_p = \lambda^{d\left(1-\frac{1}{p}\right)}\|\hat{u}\|_p.$$

Since every admissible function u for K defines an admissible function v for λK and vice versa, we complete the proof. ■

Remarks 6.1.13. The dilation/contraction effect on K remains invariant under $c_{p,\mathcal{F}}$ when $p = 1$. If $\lambda > 1$, the dilation effect increases as p increases. Meanwhile, for $0 < \lambda < 1$, the contraction effect decreases as p increases.

Theorem 6.1.14. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a contraction with contraction ratio $r < 1$ and $K \in \mathcal{K}$ with int$(K) \neq \varnothing$. Then there exists $m \in \mathbb{N}$ such that:*

$$c_{p,\mathcal{F}}(f^m(K)) \leqslant c_{p,\mathcal{F}}(K).$$

Proof: Let $B(c,\delta)$ be the largest ball inside K.
Claim: If $m \geqslant \log_r \dfrac{\delta}{|K|}$, then the theorem holds true.
Let us note the following inequalities:

$$m \geqslant \log_r \frac{\delta}{|K|} \Rightarrow r^m \leqslant \frac{\delta}{|K|} \Rightarrow r^m|K| \leqslant \delta.$$

Then there exists $a \in \mathbb{R}^d$ such that:

$$f^m(K) + a \subset B(c, \delta).$$

Finally, by monotonicity and translation invariance: $c_{p,\mathcal{F}}(f^m(K)) \leqslant c_{p,\mathcal{F}}(B(c,\delta)) \leqslant c_{p,\mathcal{F}}(K)$. ■

Remarks 6.1.15. Here we need the hypothesis $int(K) \neq \varnothing$ since $f^m(K)$ might not fit within K for any $m \in \mathbb{N}$ (for example, if K were a Fat cantor set, it would be difficult to fit $f(K)$ inside K). Of course, the above hypothesis is restrictive at least in the context of fractals. Due to theorem 6.1.12, we know how capacity behaves under isometries and scaling contractions, however, this new theorem could be applied to fractals generated by contractions that are not similarities. The previous application would allow control over the capacity of intermediate sets in the construction of the attractor.

To conclude this section, we provide estimates of Fourier capacity for $p \geqslant 2$.

Theorem 6.1.16. *Let K be a compact set in $\mathbb{R}^d$, and let $p, p' \geqslant 1$ be conjugate exponents with $p \geqslant 2$. The following statements hold:*

(i) The following upper bound holds:

$$c_{p,\mathcal{F}}(K) \leqslant \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{d/2} m(K)^{1-1/p}.$$

In particular, for $p = 2$, $c_{2,\mathcal{F}}(K) = m(K)^{1/2}$.

(ii) Let χ_K be the characteristic function of K. Then there exists a constant $C_K \geqslant 0$ depending on K such that:

$$c_{p,\mathcal{F}}(K) \leqslant \|\hat{\chi}_K\|_p \leqslant C_K + c_{p,\mathcal{F}}(K).$$

Proof: Part (i): By the Hausdorff-Young inequality (see 4.0.7, and Beckner (1975) for its sharper version), for $p \geqslant 2$ we have the following bound on the Fourier transform of the characteristic function χ_K of the set K:

$$\|\hat{\chi}_K\|_p \leqslant \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{d/2} m(K)^{1-1/p}$$

This establishes an upper bound for the Fourier transform of χ_K, which depends on the volume of the set K and the exponent p.

To prove the desired result, it is enough to establish the inequality:

$$c_{p,\mathcal{F}}(K) \leqslant \|\hat{\chi}_K\|_p .$$

Let $\epsilon > 0$. We define an auxiliary function ϕ_ϵ that equals 1 on a small neighborhood of K to ensure admissibility after mollification:

$$\phi_\epsilon(x) := \begin{cases} 1 & \text{if } d(x,K) \leqslant \epsilon/2, \\ e^{-\frac{d(x,K)-\epsilon/2}{\epsilon-d(x,K)}} & \text{if } \epsilon/2 < d(x,K) < \epsilon, \\ 0 & \text{if } d(x,K) \geqslant \epsilon. \end{cases}$$

Note that for each $\epsilon > 0$, the support of ϕ_ϵ is contained in the ϵ-neighborhood of K. As $\epsilon \to 0$, the function $\phi_\epsilon(x)$ converges pointwise to the characteristic function $\chi_K(x)$ for almost every $x \in \mathbb{R}^d$.

Next, let η_δ be a standard mollifier supported in the ball $B(0,\delta)$. For any $0 < \delta < \epsilon/2$, define the convolution $v_{\epsilon,\delta} := \phi_\epsilon * \eta_\delta$.

By construction, $\phi_\epsilon = 1$ on the entire $\epsilon/2$-neighborhood of K. By choosing the mollifier's radius such that $\delta < \epsilon/2$, we ensure that for any $x \in K$, the integration domain $B(x,\delta)$ is strictly contained within this neighborhood. Consequently, if $x \in K$:

$$v_{\epsilon,\delta}(x) = \int_{B(0,\delta)} \phi_\epsilon(x-y)\eta_\delta(y)dy = \int_{B(x,\delta)} \phi_\epsilon(z)\eta_\delta(x-z)dz = 1,$$

making it strictly admissible for the Fourier capacity. Moreover, it is known that

$$\|v_{\epsilon,\delta} - \phi_\epsilon\|_p \to 0 \quad \text{as } \delta \to 0 \text{ for all } p \geqslant 1.$$

Thus, for any $\epsilon > 0, v_{\epsilon,\delta}$ is a smooth approximation of ϕ_ϵ.
Now, by applying the Hausdorff-Young inequality with $p \geqslant 2$ and letting $\delta \to 0$, we obtain the following estimate for the Fourier transform of $v_{\epsilon,\delta}$:

$$\begin{aligned} \|\hat{v}_{\epsilon,\delta} - \hat{\chi}_K\|_p &\leqslant \left\|\hat{v}_{\epsilon,\delta} - \hat{\phi}_\epsilon\right\|_p + \left\|\hat{\phi}_\epsilon - \hat{\chi}_K\right\|_p \leqslant C\,\|v_{\epsilon,\delta} - \phi_\epsilon\|_{p'} + C\,\|\phi_\epsilon - \chi_K\|_{p'} \\ &\xrightarrow{\delta\to 0} C\,\|\phi_\epsilon - \chi_K\|_{p'} = \left(\int_{0<d(x,K)<\epsilon/2} 1dx + \int_{\epsilon/2<d(x,K)<\epsilon} e^{-\frac{d(x,K)-\epsilon/2}{\epsilon-d(x,K)}p'}dx\right)^{1/p'} \end{aligned}$$

The last integral tends to zero as $\epsilon \to 0$ because the support of ϕ_ϵ shrinks to K, and the values of the integrand decay outside this shrinking support. Hence, we conclude that the integral vanishes as $\epsilon \to 0$.

Thus, since $v_{\epsilon,\delta}$ is admissible for all $\epsilon, \delta > 0$, we conclude that:

$$c_{p,\mathcal{F}}(K) \leqslant \|\hat{v}_{\epsilon,\delta}\|_p \leqslant \|\hat{v}_{\epsilon,\delta} - \hat{\chi}_K\|_p + \|\hat{\chi}_K\|_p.$$

Finally, by letting both ϵ and δ tend to zero, we obtain the desired inequality:

$$c_{p,\mathcal{F}}(K) \leqslant \|\hat{\chi}_K\|_p.$$

For the case $p = 2$, we can apply the Plancherel identity (see 4.0.6), which gives the exact value of the Fourier capacity:

$$c_{2,\mathcal{F}}(K) = \inf\left\{|u|_2 : u \in C_0^\infty\left(\mathbb{R}^d\right), u \geqslant 1 \text{ on } K\right\} = m(K)^{1/2} = |\hat{\chi}_K|_2.$$

This completes the proof of part (i).
Part (ii): We now prove the inequality:

$$c_{p,\mathcal{F}}(K) \leqslant \|\hat{\chi}_K\|_p \leqslant C_K + c_{p,\mathcal{F}}(K).$$

For this, note that for each $\epsilon > 0$, there exists an admissible function u_ϵ such that:

$$\|\hat{u}_\epsilon\|_p \leqslant c_{p,\mathcal{F}}(K) + \epsilon$$

Then, applying the triangle inequality and Hausdorff-Young's inequality, we obtain:

$$\|\hat{\chi}_K\|_p \leqslant \|\hat{\chi}_K - \hat{u}_\epsilon\|_p + \|\hat{u}_\epsilon\|_p \leqslant \|\chi_K - u_\epsilon\|_{p'} + c_{p,\mathcal{F}}(K) + \epsilon$$

Since this holds for all $\epsilon > 0$, we can take the limit inferior as $\epsilon \to 0$ on both sides (to ensure the existence of the limit):

$$\|\hat{\chi}_K\|_p \leqslant \liminf_{\epsilon \to 0} \|u_\epsilon - \chi_K\|_{p'} + c_{p,\mathcal{F}}(K).$$

Defining

$$C_K := \liminf_{\epsilon \to 0} \|u_\epsilon - \chi_K\|_{p'},$$

we conclude the proof of the theorem. ■

Remarks 6.1.17. a). In particular, for $p \geqslant 2$, the Fourier capacity of a finite set is zero. Indeed, let $A = \{x_1, \ldots, x_N\}$ be a set of N distinct points in $\mathbb{R}^d$. Then:

$$c_{p,\mathcal{F}}(A) \leqslant \sum_{i=1}^{N} c_{p,\mathcal{F}}(\{x_i\}) = N \cdot c_{p,\mathcal{F}}(\{0\}).$$

by the invariance of translations. Now, since the L_p-norm of the Fourier transform of $\chi_{\{0\}}$ is zero, we conclude that the capacity of A is zero.

b). In the proof of (ii) in 6.1.16, the constant C_K indicates that there may exist compact sets K such that $C_K = 0$. Consequently, $c_{p,\mathcal{F}}(K) = \|\hat{\chi}_K\|_p$, which is true for the case $p = 2$, and holds for all compact sets K by part (i).

Corollary 6.1.18. *$c_{2,\mathcal{F}}$ has the strong subadditive property:*

$$c_{2,\mathcal{F}}(K_1 \cup K_2) + c_{2,\mathcal{F}}(K_1 \cap K_2) \leqslant c_{2,\mathcal{F}}(K_1) + c_{2,\mathcal{F}}(K_2), \quad \forall K_1, K_2 \in \mathcal{K}.$$

Proof: Since $c_{2,\mathcal{F}}(K) = m(K)^{1/2}$, we want to prove that:

$$\sqrt{m(K_1 \cup K_2)} + \sqrt{m(K_1 \cap K_2)} \leqslant \sqrt{m(K_1)} + \sqrt{m(K_2)}$$

Let $x = m(K_1)$, $y = m(K_2)$, and $z = m(K_1 \cap K_2)$. Since $K_1 \cup K_2$ is measurable, then:

$$m(K_1 \cup K_2) = m(K_1) + m(K_2) - m(K_1 \cap K_2).$$

Therefore, the inequality we wanted to prove becomes the following:

$$\sqrt{x+y-z}+\sqrt{z} \leqslant \sqrt{x}+\sqrt{y}$$

Squaring both sides (since both are positive):

$$(\sqrt{x+y-z}+\sqrt{z})^2 \leqslant (\sqrt{x}+\sqrt{y})^2.$$

$$(x+y-z)+z+2\sqrt{z(x+y-z)} \leqslant x+y+2\sqrt{xy}$$
$$x+y+2\sqrt{xz+yz-z^2} \leqslant x+y+2\sqrt{xy}.$$

Canceling $x+y$ and dividing by 2:

$$\sqrt{xz+yz-z^2} \leqslant \sqrt{xy}.$$

Squaring both sides again:

$$xz+yz-z^2 \leqslant xy \Longrightarrow 0 \leqslant xy-xz-yz+z^2$$

$$\Longrightarrow 0 \leqslant x(y-z)-z(y-z) \Longrightarrow 0 \leqslant (x-z)(y-z).$$

Since $z = m(K_1 \bigcap K_2)$, it is evident that $z \leqslant x$ and $z \leqslant y$; therefore, $0 \leqslant (x-z)(y-z)$, which proves the strong subadditivity. ■

Remarks 6.1.19. The strong subadditivity property is successfully verified for the case $p = 2$, primarily due to the explicit characterization $c_{2,\mathcal{F}}(K) = m(K)^{1/2}$ derived from the Plancherel theorem (see Theorem 4.0.6). This implies that **the Fourier 2-capacity is a Choquet capacity**. However, extending this result to the general case $p \neq 2$ presents significant theoretical obstacles.
In classical nonlinear potential theory (e.g., for capacities associated with Sobolev spaces $W^{1,p}$), strong subadditivity is typically deduced from the pointwise maximum $w_1 = \max(u, v)$ and minimum $w_2 = \min(u, v)$ functions that remain admissible and satisfy strict energy inequalities, specifically, the key inequality is the following:

$$\|w_1\|_p + \|w_2\|_p \leqslant \|v\|_p + \|u\|_p \ ,$$

that is satisfied almost everywhere. In the context of Fourier p-capacity, the "energy" is measured by the L^p norm of the Fourier transform, therefore, the spatial truncations $\max(u, v)$ and $\min(u, v)$ do not preserve the L^p-control of the Fourier transform in a predictable manner (and they do not satisfy the above inequality), preventing the application of standard truncation techniques to prove strong subadditivity; therefore, it remains an open problem.

Theorem 6.1.20. *Let $K \subset \mathbb{R}^d$ be a self-similar set with similarity dimension s. Then:*

$$p > \frac{1}{1-s/d} \Longrightarrow c_{p,\mathcal{F}}(K) = 0.$$

Proof: Let $\{f_1, f_2, ..., f_n\}$ be the IFS that generates K, and $\{r_1, r_2, ..., r_n\}$ the corresponding contraction ratios. By subadditivity, we have:

$$K = \bigcup_{i=1}^{n} f(K) \Longrightarrow c_{p,\mathcal{F}}(K) \leqslant \sum_{i=1}^{n} c_{p,\mathcal{F}}(f(K)).$$

Now, using the property of dilations and contractions (see 6.1.12), we have:

$$c_{p,\mathcal{F}}(K) \leqslant \sum_{i=1}^{n} c_{p,\mathcal{F}}(f(K)) \leqslant r_1^{d(1-1/p)} c_{p,\mathcal{F}}(K) + r_2^{d(1-1/p)} c_{p,\mathcal{F}}(K) + \cdots + r_n^{d(1-1/p)} c_{p,\mathcal{F}}(K)$$
$$= c_{p,\mathcal{F}}(K) \left(\sum_{i=1}^{n} r_i^{d(1-1/p)} \right).$$

The above inequalities, tell us that if $\sum_{i=1}^{n} r_i^{d(1-1/p)} < 1$, then $c_{p,\mathcal{F}}(K) = 0$. Now, since s is the exact number such that $\sum_{i=1}^{n} r_i^s = 1$, and the contraction ratios are less than 1, then any exponent larger than s will reduce the sum. The above reasoning leads us to conclude that the values of p that imply $c_{p,\mathcal{F}}(K) = 0$ are:

$$d(1 - 1/p) > s \implies 1 - 1/p > s/d \implies 1 - s/d > 1/p \implies p > \frac{1}{1 - s/d}.$$

■

Remarks 6.1.21.

- If the set K is generated by an IFS that satisfies the Open Set Condition, then its Hausdorff dimension is equal to the similarity dimension.
- Although no examples of sets with positive capacity have been found, the theorem allows us to intuit that the capacity for self-similar sets is space-sensitive, in the sense that:
  - A dimensionally small fractal, in a dimensionally large space, has zero capacity for almost all values of p ($p > \frac{1}{1-s/d} \approx 1$). However, the value $p = 1$ will always be an option for a positive capacity.
  - A dimensionally large fractal has zero capacity for fewer and fewer values of p ($p > \frac{1}{1-s/d} = N$ with N a large number).

Although the similarity dimension of the set K is s, this does not imply that its Hausdorff dimension is also equal to s. For that, the IFS must satisfy the Open Set Condition. Therefore, the following corollary is mentioned:

Corollary 6.1.22. *Let $\{f_1, f_2, ..., f_n\}$ be an IFS that satisfies the Open Set Condition, and K be its associated self-similar set with Hausdorff dimension equal to s. Then:*

$$p > \frac{1}{1 - s/d} \Longrightarrow c_{p,\mathcal{F}}(K) = 0.$$

Proof: Since the IFS satisfies the Open Set Condition, the similarity dimension of K is equal to its Hausdorff dimension. Thus, to prove the result, it is sufficient to apply the theorem above. ■

6.2 Integrability of the Fourier transform of characteristic functions

Regarding the analysis of the Fourier transform for the characteristic function of a compact set, Lebedev mentions in his article Lebedev (2013) that only the case $1 < p < 2$ is interesting, and introduces the space $A_p(\mathbb{R}^d)$, which corresponds to the tempered distributions f on $\mathbb{R}^d$ such that $\hat{f} \in L^p(\mathbb{R}^d)$. The principal question here is: When is $\chi_K \in A_p(\mathbb{R}^d)$? Basic cases are mentioned in Lebedev's article, such as a cube and a polytope (finite union of simplices), since in both cases, $\chi_K \in A_p(\mathbb{R}^d)$, $\forall p > 1$. A more interesting bound for p appears if we consider the case of the ball, since it is mentioned that using the well-known asymptotics for Bessel functions, one can verify that if $B \subset \mathbb{R}^d$ is a ball, then $\chi_B \in A_p(\mathbb{R}^d)$ if and only if $p > 2d/(d+1)$. The above result can even be applied in a general case of domains with C^2-smooth boundary, with the same critical value $2d/(d+1)$.
A more general result can be obtained following the method of Lebedev in Lebedev (2013) (Theorem 1):

Theorem 6.2.1. *Let $E \subseteq \mathbb{R}^d$, $d \geqslant 1$, be a set of positive measure and α the upper Minkowski dimension of its boundary ∂E. Then $\chi_E \in A_p(\mathbb{R}^d)$, $\forall p > 2d/(2d-\alpha)$.*

Proof: We proceed similarly to Lebedev's proof, proving that $\chi_E \in L^2_s(\mathbb{R}^d)$ (since for $2d/(d+2s) < p < 2$, we have $L^2_s(\mathbb{R}^d) \subset A_p(\mathbb{R}^d)$), and using that the norm $\|\cdot\|_{L^2_s(\mathbb{R}^d)}$ (see 4.2.1) is equivalent to the norm

$$\|f\| = \|f\|_2 + \left(\int_{\mathbb{R}^d} \frac{1}{|t|^{n+2s}} \left(\int_{\mathbb{R}^d} |f(x+t) - f(x)|^2 dx\right) dt\right)^{1/2}.$$

when $0 < s < 1$.
Note now that for each $t \in \mathbb{R}^d$, the symmetric difference:

$$((E-t)\backslash E) \cup (E\backslash(E-t)),$$

of the sets $E - t$ and E is contained in the (closed) $|t|$-neighborhood of the boundary ∂E of E, so its Hausdorff measure is at most $c|t|^{d-\alpha}$ by the definition of Minkowski dimension (see proposition 3.2 in Falconer (2013)). It is also clear that the measure of this symmetric difference is at most $2m(E)$. Thus,

$$\int_{\mathbb{R}^d} |\chi_E(x+t) - \chi_E(x)|^2 dx \leqslant \min(c|t|^{d-\alpha}, 2m(E)), \qquad t \in \mathbb{R}^d,$$

and then:

$$\int_{\mathbb{R}^d} \frac{1}{|t|^{d+2s}} \left(\int_{\mathbb{R}^d} |\chi_E(x+t) - \chi_E(x)|^2 dx\right) dt \leqslant \int_{\mathbb{R}^d} \frac{\min(c|t|^{d-\alpha}, 2m(E))}{|t|^{d+2s}} dt.$$

If $c|t|^{d-\alpha} \geqslant 2m(E) \Leftrightarrow |t| \geqslant (2m(E)/c)^{1/(d-\alpha)}$, and we set $r = (2m(E)/c)^{1/(d-\alpha)}$, then $\min(c|t|^{d-\alpha}, 2m(E)) = 2m(E)$, and consequently:

$$\int_{\mathbb{R}^d\backslash B(0,r)} \frac{\min(c|t|^{d-\alpha}, 2m(E))}{|t|^{d+2s}} dt = 2m(E) \int_{\mathbb{R}^d\backslash B(0,r)} \frac{1}{|t|^{d+2s}} dt < \infty,$$

since $d + 2s > d$.
On the other hand, if $c|t|^{d-\alpha} \leqslant 2m(E) \iff |t| \leqslant (2m(E)/c)^{1/(d-\alpha)}$, then $\min(c|t|^{d-\alpha}, 2m(E)) = c|t|^{d-\alpha}$, and consequently:

$$\int_{B(0,r)} \frac{\min(c|t|^{d-\alpha}, 2m(E))}{|t|^{d+2s}} dt = \int_{B(0,r)} \frac{c|t|^{d-\alpha}}{|t|^{d+2s}} dt = \int_{B(0,r)} \frac{c}{|t|^{2s+\alpha}} dt < \infty,$$

where if $s < \frac{d-\alpha}{2} \Leftrightarrow 2s < d-\alpha \Leftrightarrow 2s+\alpha < d$, the integral converges.
Note that for $s < \frac{d-\alpha}{2}$ the usage of the equivalence of norms is justified since $\alpha \geqslant d-1$.

Indeed, let us verify that if a set $E \subseteq \mathbb{R}^d$ is bounded and has positive measure, then $\overline{\dim}_M \partial E \geqslant d-1$ (following Lebedev's proof). Assuming that $E \backslash \partial E \neq \varnothing$ (otherwise there is nothing to prove), we fix a point $x_0 \in E\backslash\partial E$. There exists an open ball B centered at x_0 that does not contain points of the boundary ∂E and moreover lies at positive distance from ∂E. Denote by S the boundary sphere of the ball B. Define a map $\theta : \mathbb{R}^d \backslash B \to S$ as follows. Take a point $x \in \mathbb{R}^d\backslash B$ and consider the ray that passes through x and has its origin at x_0. Denote by $\theta(x)$ the point of intersection of this ray with the sphere S. Clearly the map θ is Lipschitz (moreover it is a contraction, i.e., $|\theta(x_1) - \theta(x_2)| \leqslant |x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}^d\backslash B$). It is easy to see that the image of the boundary of the set E under the map θ is the whole sphere S (since x_0 is an interior point). At the same time it is known (see chapter 7 in Mattila (1999)) that Lipschitz maps do not increase the dimension of a set. Thus,

$$d - 1 = \overline{\dim}_M S = \overline{\dim}_M \theta(\partial E) \leqslant \overline{\dim}_M \partial E.$$

Since $\alpha \geqslant d-1$, the range of values $s < \frac{d-\alpha}{2}$ implies that $\chi_E \in A_p(\mathbb{R}^d)$ for all $p > 2d/(2d-\alpha)$.

The theorem is proved since the above integrals are finite. ■

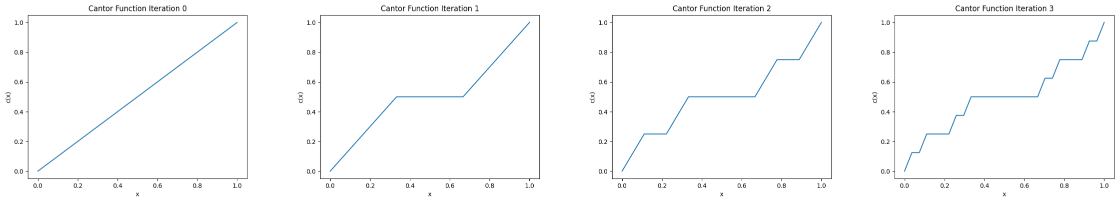
Remarks 6.2.2. It is useful to recall the following convergence criteria for integrals:

- $\displaystyle\int_{B(0,1)} \frac{1}{|x|^p} dx < \infty \iff p < d.$
- $\displaystyle\int_{\mathbb{R}^d\backslash B(0,1)} \frac{1}{|x|^p} dx < \infty \iff p > d.$

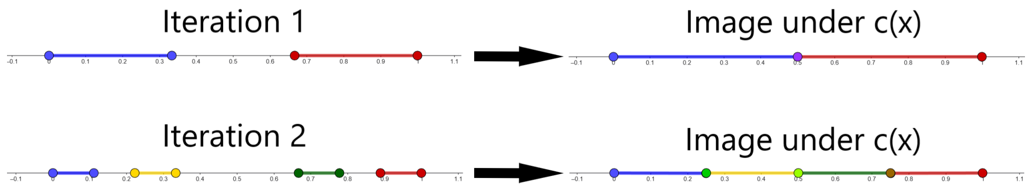
In order to apply this theorem, we construct fat Cantor sets in any dimension, as examples of "bad sets" from which we can draw conclusions about the integrability of the Fourier transform of their characteristic function. However, the fact that we do not have the Hausdorff-Young inequality as a tool for the range $p < 2$ prevents us from obtaining results similar to the Theorem 6.1.16.

Fat Cantor sets (sets similar to the Cantor Set, but with positive Hausdorff measure) can be constructed similarly to the Cantor Ternary Set, but they are not self-similar sets. In order to provide a general construction of these sets, the Cantor function will be introduced.
We will begin with the most intuitive definition of how this function (and its corresponding graph) is generated, which is an iterative construction. We start with the identity function $f_0(x) = x$ on the interval $[0, 1]$. Then, we take the open third in the middle of the interval (that is, the open interval $(1/3, 2/3)$), and make the function constant equal to the intermediate value of the image $f_1(x) = 1/2$. In the two remaining segments $[0, 1/3]$ and $[2/3, 1]$, we repeat the process. We take their middle thirds and assign the intermediate values ($1/4$ and $3/4$, respectively). By repeating this process infinitely, we obtain the graph of the Cantor function

(also known as the devil's staircase because of its strange properties).



It is not difficult to see that this function homogenizes the iterations of the construction of the Cantor Set, and in fact, this property is crucial, since later on, when giving a general construction of the fat Cantor sets, the Cantor Function works, in part, because of this property.



The colored dots are, in fact, points corresponding to the final Cantor set, and it can be seen that there are pairs of points that are mapped into the same image.

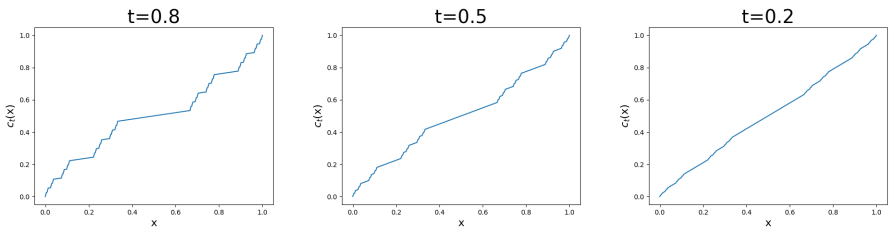
It is well-known that this function maps numbers written in base 3 to binary numbers (in a non-injective manner), and in fact, the formal definition comes from there. Only its properties will be mentioned, since the objective is not to conduct research on the function itself, but rather on its usefulness (for more details, see Dovgoshey et al. (2006)).

Properties 6.2.3. The function $c(x)$ is continuous, increasing, and maps the Cantor Set $\mathcal{C}$ onto $[0, 1]$.

We will not prove the above, but the three properties can be deduced from the above intuitions. Now we have all the ingredients to take a closer look at the construction of fat Cantor sets, specifically, the one given by Liu and Pego in the article Liu and Pego (2024).
They defined the following function: For any $t(0, 1)$, the function $c_t : [0, 1] \to [0, 1]$ is defined as

$$c_t(x) = (1 - t)x + tc(x).$$

It is not difficult to see that this function is continuous (algebra of continuous functions), strictly increasing (the term $(1 - t)x$ is strictly increasing, and $tc(x)$ is increasing; therefore, the sum is strictly increasing), and bijective (even if $c(x)$ is not injective, the addition of the strictly increasing term $(1 - t)x$ ensures injectivity).



Due to the above properties, the set $c_t(\mathcal{C})$, like $\mathcal{C}$, is closed and nowhere dense, and for any component interval I of $\mathcal{C}^c$, its image $c_t(I)$ is an open interval with length $m(c_t(I)) = (1-t)m(I)$ (where m is Hausdorff measure). In fact, since I is an interval (a, b) inside $\mathcal{C}^c$, then $c(a) = c(b)$

and therefore:

$$m(c_t(I)) = c_t(b) - c_t(a) = (1-t)b + tc(b) - ((1-t)a + tc(a)) = (1-t)(b-a) = (1-t)m(I).$$

Now, by additivity of Hausdorff measure:

$$m(c_t(\mathcal{C}^c)) = (1-t)\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n} = (1-t)\frac{1}{2}\sum_{n=1}^{\infty}\frac{2^n}{3^n} = \frac{(1-t)}{2}\cdot\frac{2/3}{1-2/3} = (1-t).$$

Then

$$m(c_t(\mathcal{C})) = m(c_t(\mathcal{C}^c)^c) = t.$$

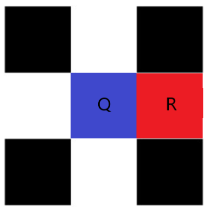
Therefore, the set $c_t(\mathcal{C})$ is a fat Cantor set with Hausdorff measure t.
The above is simply a detailed explanation of what Liu and Pego did in their article, but as part of this research, a generalization of this method for any integer dimension was obtained.

Theorem 6.2.4. *Let c_t be defined as above and $t \in (0,1)$. Then, the function $\tilde{c}_t : \mathbb{R}^2 \to \mathbb{R}^2$ defined as:*

$$\tilde{c}_t(x,y) = (c_t(x), c_t(y)),$$

maps the 2D Cantor dust into a 2-dimensional fat Cantor set with Hausdorff measure t^2.

Proof: Let Q be the square with vertices $(1/3, 1/3)$ and $(2/3, 2/3)$, and R the square with vertices $(2/3, 1/3)$ and $(1, 2/3)$ as shown in the figure:



Also, let $\mathcal{C}_i^2$ be the i-th iteration in the construction of the 2D Cantor dust. Then $\tilde{c}_t(Q)$ is a square, and

$$m(\tilde{c}_t(Q)) = [c_t(2/3) - c_t(1/3)]\cdot[c_t(2/3) - c_t(1/3)] = (1-t)^2\cdot\frac{1}{9} = (1-t)^2 m(Q).$$

The same does **not** hold for the other four squares that are subtracted from the square $[0,1]\times[0,1]$ in the first iteration of the Cantor dust. Let us see what happens with the red square R:

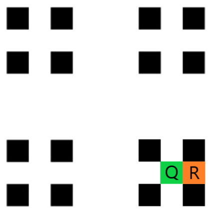
$$\begin{aligned}
m(\tilde{c}_t(R)) &= [c_t(1) - c_t(2/3)]\cdot[c_t(2/3) - c_t(1/3)]\\
&= [(1-t) + tc(1) - ((1-t)\cdot 2/3 + tc(2/3))]\cdot[(1-t)\cdot 2/3 + tc(2/3) - ((1-t)\cdot 1/3 + tc(1/3))]\\
&= [(1-t)\cdot(1-2/3) + t(c(1) - c(2/3))]\cdot[(1-t)\cdot(2/3-1/3) + t(c(2/3) - c(1/3))]\\
&= [(1-t)\cdot 1/3 + t/2]\cdot(1-t)\cdot 1/3.
\end{aligned}$$

It is important to mention that the term $c(2/3) - c(1/3)$ is equal to 0, and this occurs because this square is at the same height as square Q.
Also, it is easy to check that this is the area of the image of the other three remaining squares,

therefore:

$$\begin{aligned} m(\tilde{c}_t(\mathcal{C}_1^2)^c) &= m(\tilde{c}_t(Q)) + 4 \cdot m(\tilde{c}_t(R)) \\ &= (1-t)^2 \cdot \frac{1}{9} + 4 \cdot [(1-t) \cdot 1/3 + t/2] \cdot (1-t) \cdot 1/3 \\ &= 5(1-t)^2 \cdot \frac{1}{9} + \frac{2}{3} t(1-t). \end{aligned}$$

Now, by the construction of the 2D Cantor dust, these calculations will be similar in the following iterations. Let us see what happens in the second iteration, and then we can generalize:
Let Q be now the square of vertices $(7/9, 1/9)$ and $(8/9, 2/9)$ (the green one), and R the square of vertices $(8/9, 1/9)$ and $(1, 2/9)$ (the orange one) as shown in the figure:



Now, let us calculate the area of $\tilde{c}_t(Q)$ and $\tilde{c}_t(R)$:
The area of $\tilde{c}_t(Q)$ is easy to calculate, since it is similar to the previous calculation of $\tilde{c}_t(Q)$ (iteration 1), so we get:

$$m(\tilde{c}_t(Q)) = (1-t)^2 \frac{1}{9^2}.$$

Now, the area of $\tilde{c}_t(R)$ can be calculated as in iteration 1:

$$\begin{aligned} & m(\tilde{c}_t(R)) = [c_t(1) - c_t(8/9)] \cdot [c_t(2/9) - c_t(1/9)] \\ & = [(1-t) + tc(1) - ((1-t) \cdot 8/9 + tc(8/9))] \cdot [(1-t) \cdot 2/9 + tc(2/9) - ((1-t) \cdot 1/9 + tc(1/9))] \\ & = [(1-t) \cdot (1 - 8/9) + t(c(1) - c(8/9))] \cdot [(1-t) \cdot (2/9 - 1/9) + t(c(2/9) - c(1/9))] \\ & = [(1-t) \cdot 1/9 + t/4] \cdot (1-t) \cdot 1/9. \end{aligned}$$

Since there are sixteen squares in total with the same area as R (including R itself), and four squares with the same area as Q, then:

$$\begin{aligned} m(\tilde{c}_t(\mathcal{C}_2^2)^c) &= m(\tilde{c}_t(\mathcal{C}_1^2)^c) + 4 \cdot (1-t)^2 \cdot \frac{1}{9^2} + 16 \cdot [(1-t) \cdot 1/9 + t/4] \cdot (1-t) \cdot 1/9 \\ &= \frac{5}{9}(1-t)^2 + \frac{2}{3} t(1-t) + \frac{20}{9^2}(1-t)^2 + \frac{4}{9} t(1-t) \\ &= \frac{5}{4} \cdot \frac{4}{9}(1-t)^2 + \frac{2}{3} t(1-t) + \frac{5}{4}\left(\frac{4}{9}\right)^2 (1-t)^2 + \left(\frac{2}{3}\right)^2 t(1-t). \end{aligned}$$

It is not difficult to see the pattern here. Using the same method for future iterations, we will arrive at the formula for an iteration m:

$$m(\tilde{c}_t(\mathcal{C}_m^2)^c) = \sum_{n=1}^{m} \left[\frac{5}{4} \left(\frac{4}{9}\right)^n (1-t)^2 + \left(\frac{2}{3}\right)^n t(1-t) \right].$$

Then, we can compute the area of $\tilde{c}_t(\mathcal{C}^2)^c$:

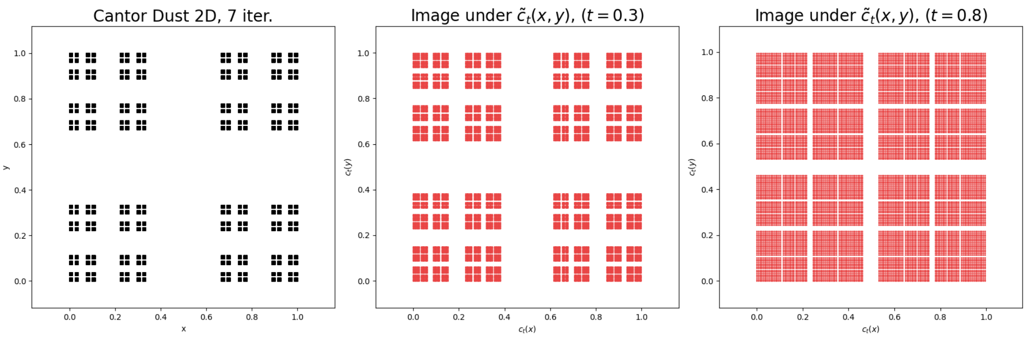
$$m(\tilde{c}_t(\mathcal{C}^2)^c) = \sum_{n=1}^{\infty} \left[\frac{5}{4} \left(\frac{4}{9} \right)^n (1-t)^2 + \left(\frac{2}{3} \right)^n t(1-t) \right] = (1-t)^2 + 2t(1-t) = 1 - t^2.$$

Now, by the same argument given by Liu and Pego, we have:

$$m(\tilde{c}_t(\mathcal{C}^2)) = 1 - m(\tilde{c}_t(\mathcal{C}^2)^c) = 1 - (1 - t^2) = t^2.$$

■

Example 6.2.5. Here are some examples of different images under the function $\tilde{c}_t$:



Here one can see that the images of the squares R mentioned in the proof are not squares but rectangles.

Although the above proof was lengthy, the theorem for arbitrary integer dimensions is actually simple due to the good properties of the functions used:

Theorem 6.2.6. *The function $\tilde{c}_t : \mathbb{R}^d \to \mathbb{R}^d$ defined as*

$$\tilde{c}_t(x_1, x_2, ..., x_d) = (c_t(x_1), c_t(x_2), ..., c_t(x_d)),$$

maps the d-dimensional Cantor dust, into a d-dimensional fat Cantor set with Hausdorff measure t^d.

Proof: The d-dimensional Cantor dust is defined as:

$$\mathcal{C}^d = \underbrace{\mathcal{C} \times \mathcal{C} \times \cdots \times \mathcal{C}}_{d \text{ times}}.$$

Since $\tilde{c}_t$ acts independently on each coordinate, the image of the product is the product of the images:

$$\tilde{c}_t(\mathcal{C}^d) = \tilde{c}_t(\mathcal{C} \times \cdots \times \mathcal{C}) = c_t(\mathcal{C}) \times c_t(\mathcal{C}) \times \cdots \times c_t(\mathcal{C}).$$

Then

$$m(\tilde{c}_t(\mathcal{C}^d)) = m(c_t(\mathcal{C}) \times c_t(\mathcal{C}) \times \cdots \times c_t(\mathcal{C})) = \prod_{i=1}^{d} m(c_t(\mathcal{C})) = t^d.$$

■

Remarks 6.2.7. It is not difficult to see that as the dimension increases, the measure of these fat Cantor sets decreases to 0. However, since the measure of the d-dimensional fat Cantor set depends solely on the measure of the 1-dimensional fat Cantor set, then by making a small change in the power of t, such as using $t^{1/d}$ in the definition of c_t, we can have the measure of the d-dimensional fat Cantor set be exactly t.

The construction of these new d-dimensional sets will be useful as examples for the theorem 6.2.1, where the positivity of the measure could allow us to obtain relationships between the capacity and the Fourier transform of the characteristic function of these sets.

6.3 Fourier (p, k)-capacity

A generalization of the Fourier p-capacity will now be presented, along with basic properties that prove that it is a capacity in the Choquet sense.
In the following sections, $k(\xi)$ will be a tempered weight function (which we already mentioned in 4.2.4), that is, there exist positive constants C, N such that:

$$k(\xi + \eta) \leqslant (1 + C|\xi|)^N k(\eta), \quad \xi, \eta \in \mathbb{R}^d.$$

Definition 6.3.1. Let $K \in \mathcal{K}$, and define the set $E(K) = \{\phi \in C_0^\infty(\mathbb{R}^d) : \phi(x) \geqslant 1 \text{ on } K\}$. We define the Fourier (p, k)-capacity of K as the quantity:

$$c_{p,k}(K) := \inf\{|\phi|_{p,k} : \phi \in E(K)\}.$$

where $|\phi|_{p,k} = \left(\int_{\mathbb{R}^d} k(\xi)^p |\hat{\phi}(\xi)|^p d\xi\right)^{1/p}$.

Of course, a good starting point is to analyze when the properties that we have already shown Fourier p-capacity to satisfy are fulfilled. The following lemma is a compilation of properties that are fulfilled by the Fourier (p, k)-capacity, and share exactly the same proof:

Lemma 6.3.2. *a). For any $K \in \mathcal{K}$, $c_{p,k}(K)$ is finite.*

b). $c_{p,k}(\varnothing) = 0$.

c). The Fourier (p, k)-capacity is invariant under translations. That is,

$$c_{p,k}(K) = c_{p,k}(K + a), \quad a \in \mathbb{R}^d.$$

d). (Monotonicity) Let $K_1, K_2 \in \mathcal{K}$. If $K_1 \subset K_2$ then $c_{p,k}(K_1) \leqslant c_{p,k}(K_2)$.

e). (Regularity) Let $K \in \mathcal{K}$. For each $a > c_{p,k}(K)$, there exists $U \in \mathcal{O}$ such that $K \subset U$ and for all $C \in \mathcal{K}$ such that $C \subset U$ we have $c_{p,k}(C) < a$.

f). (Subadditivity)

$$c_{p,k}(K_1 \cup K_2) \leqslant c_{p,k}(K_1) + c_{p,k}(K_2), \quad \forall K_1, K_2 \in \mathcal{K}.$$

Proof: The proofs for a) and b) are the same as those for the Fourier p-capacity (see 6.1.1 and 6.1.10 respectively), but using the inclusions 4.2.3 in both (ensuring that $|\phi|_{p,k} < \infty$), and $\epsilon = \delta/|\phi|_{p,k}$ in b).

The other proofs are the same as for the Fourier p-capacity (see 6.1.11; and (C1), (C2), (C3) in 6.1.2 respectively). ■

Of course, the fact that many properties are so directly inherited from Fourier p-capacity is quite good, and gives hope that we are studying an object with good structure and properties. Also, it is clear that not all properties should be inherited so easily, since the presence of tempered weights strongly influences its behavior under rescaling and isometries. However, before analyzing these cases, we will define this capacity for open and arbitrary sets, just as we did for the Fourier p-capacity:

Definition 6.3.3. Let U be an open set and E an arbitrary set. Then their Fourier (p,k)-capacity is defined as:

$$c_{p,k}(U) := \sup\{c_{p,k}(K) : K \subset U, K \in \mathcal{K}\},$$

and

$$c_{p,k}(E) := \inf\{c_{p,k}(U) : E \subset U, U \in \mathcal{O}\},$$

respectively.

Since these definitions are analogous with the definitions for the Fourier p-capacity, the properties of 6.1.6, 6.1.7, and 6.1.8 are inherited by the Fourier (p,k)-capacity, that is:

Lemma 6.3.4. *The following statements hold.*

a). (Monotonicity) Let $E_1, E_2 \subseteq \mathbb{R}^d$. If $E_1 \subset E_2$, then $c_{p,k}(E_1) \leqslant c_{p,k}(E_2)$.

b). (Regularity) For every $K \in \mathcal{K}$,

$$c_{p,k}(K) = \inf\{c_{p,k}(U) : K \subset U, U \in \mathcal{O}\}.$$

c). (Subadditivity)

$$c_{p,k}(E_1 \cup E_2) \leqslant c_{p,k}(E_1) + c_{p,k}(E_2), \forall E_1, E_2 \subseteq \mathbb{R}^d.$$

d). Let (K_n) be a decreasing sequence in $\mathcal{K}$. Then:

$$c_{p,k}\left(\bigcap_{n=1}^{\infty} K_n\right) = \lim_{n\to\infty} c_{p,k}(K_n).$$

e). Let (U_n) be an increasing sequence in $\mathcal{O}$. Then:

$$c_{p,k}\left(\bigcup_{n=1}^{\infty} U_n\right) = \lim_{n\to\infty} c_{p,k}(U_n).$$

Remarks 6.3.5.

- The fact that this new capacity inherits subadditivity, monotonicity, and translation invariance implies that theorems 6.1.14 and 6.1.20 are also inherited.
- It is important to address the scope of the Choquet axioms satisfied by the Fourier capacities studied in this work. While the Fourier (p,k)-capacity (and, analogously,

the Fourier p-capacity) satisfies monotonicity and subadditivity for arbitrary sets, the convergence of increasing sequences has only been established for sequences of open sets. Extending this limit property to arbitrary subsets of $\mathbb{R}^d$ remains an open problem for $p \neq 2$. In classical nonlinear potential theory, this extension heavily relies on the strong subadditivity property, which is typically proven using the spatial truncations $\max(u, v)$ and $\min(u, v)$. However, as discussed in Remark 4.2.16, the Fourier transform does not preserve the required L^p-control under these lattice operations. While strong subadditivity holds for the Hilbert space case $p = 2$ via Plancherel's identity, the general case is hindered by the non-local nature of the Fourier operator.

Nevertheless, this limitation on arbitrary sets does not restrict the analytical utility of the capacity. The established properties on the pavings of compact $\mathcal{K}$ and open $\mathcal{O}$ sets are mathematically sufficient for rigorous applications. By virtue of Choquet's capacitability theorem (Theorem 5.0.2), the regularity and convergence properties guaranteed for $\mathcal{K}$ and $\mathcal{O}$ ensure that every Borel and Analytic set is capacitable. Consequently, the capacities defined herein provide a robust framework for studying the fine properties of topologically structured sets (such as fractals, regular domains, and supports of Radon measures) without requiring full compliance with the axioms on arbitrary sets.

Now, similar to the Fourier p-capacity, this generalized capacity behaves well under affine isometries, except that we do not have as exact control as in the case $k \equiv 1$.

Theorem 6.3.6. *Let $K \subset \mathbb{R}^d$ be a compact set, and $f : \mathbb{R}^d \to \mathbb{R}^d$ an affine isometry such that $f(x) = Ax + b$. Then:*

$$c_{p,k}(f(K)) = c_{p,k_A}(K).$$

Proof: Let $u \in C_0^\infty(\mathbb{R}^d)$ be an admissible function for K, then there exists $\epsilon > 0$ such that

$$|u|_{p,k_A} \leqslant c_{p,k_A}(K) + \epsilon.$$

Let $v \in C_0^\infty(\mathbb{R}^d)$ be defined as $v(x) = u(f^{-1}(x)) = u(A^{-1}(x - b))$. Now we have an admissible function for $f(K)$. Also, using properties of the Fourier transform of v we have: $\hat{v}(\xi) = e^{-2\pi i b\cdot\xi}\hat{u}\left(A^T\xi\right)$, where A^T is the transpose matrix of A. Furthermore, $|\hat{v}(\xi)| = \left|\hat{u}\left(A^T\xi\right)\right|$. Since A^T is also an orthogonal matrix, the change of variables $\eta = A^T\xi$ gives:

$$|v|_{p,k}^p = \int_{\mathbb{R}^d} |k(\xi)\hat{v}(\xi)|^p d\xi = \int_{\mathbb{R}^d} \left|k(\xi)\hat{u}(A^T\xi)\right|^p d\xi = \int_{\mathbb{R}^d} |k(A\eta)\hat{u}(\eta)|^p d\eta = |u|_{p,k_A}^p.$$

Thus, $|v|_{p,k} \leqslant c_{p,k_A}(K) + \epsilon$. Since v is admissible for $f(K)$, we have:

$$c_{p,k}(f(K)) \leqslant c_{p,k_A}(K) + \epsilon. \tag{6.3.1}$$

Now, let $w_1 \in C_0^\infty(\mathbb{R}^d)$ be an admissible function for $f(K)$ such that there exists $\epsilon' > 0$: $|w_1|_{p,k} \leqslant c_{p,k}(f(K)) + \epsilon'$. If we define $w_2(x) = w_1(f(x))$, then $w_2(x)$ is an admissible function for K, and then $c_{p,k}(K) \leqslant |w_2|_{p,k}$.

With the above definition, we have the following relations:

$$\begin{aligned}\hat{w}_2(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} w_2(x)dx &= \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} w_1(Ax+b)dx \\ &= \int_{\mathbb{R}^d} e^{-2\pi i A^{-1}y\cdot\xi} e^{2\pi i A^{-1}b\cdot\xi} w_1(y)dy \quad \text{using } x = A^{-1}(y-b) \\ &= e^{2\pi i A^{-1}b\cdot\xi}\int_{\mathbb{R}^d} e^{-2\pi i y\cdot A\xi} w_1(y)dy \quad (*) \\ &= e^{2\pi i A^{-1}b\cdot\xi}\hat{w}_1(A\xi).\end{aligned}$$

Note that in the equality (*) the property $Ax\cdot y = x\cdot A^T y = x\cdot A^{-1}y$ was used. Now, we have the key relation of this change of variable:

$$\hat{w}_2(\xi) = e^{2\pi i A^{-1}b\cdot\xi}\hat{w}_1(A\xi) \Rightarrow |\hat{w}_2(\xi)| = |\hat{w}_1(A\xi)|. \tag{6.3.2}$$

Using this relation, we can obtain the other inequality we are looking for:

$$\begin{aligned}|w_2|^p_{p,k_A} = \int_{\mathbb{R}^d} |k(A\xi)\hat{w}_2(\xi)|^p d\xi &= \int_{\mathbb{R}^d} |k(A\xi)\hat{w}_1(A\xi)|^p\, d\xi \qquad \text{using 6.3.2} \\ &= \int_{\mathbb{R}^d} |k(\eta)\hat{w}_1(\eta)|^p\, d\eta \qquad \text{using } \eta = A\xi \\ &= |w_1|^p_{p,k}.\end{aligned}$$

By construction, we conclude:

$$c_{p,k_A}(K) \leqslant |w_2|_{p,k_A} = |w_1|_{p,k} \leqslant c_{p,k}(f(K)) + \epsilon'$$

$$\Longrightarrow c_{p,k_A}(K) \leqslant c_{p,k}(f(K)) + \epsilon'. \tag{6.3.3}$$

Therefore, by making $\epsilon, \epsilon' \to 0$ and combining 6.3.1 and 6.3.3, we obtain the desired result. ■

Theorem 6.3.7. *Let $K \subset \mathbb{R}^d$ be a compact set, and $f : \mathbb{R}^d \to \mathbb{R}^d$ an affine isometry such that $f(x) = Ax + b$. If the tempered weight function satisfies the property $k_A(\xi) \asymp_c k(\xi)$, then:*

$$c_{p,k}(f(K)) \asymp_c c_{p,k}(K).$$

Proof: Let $u \in C_0^\infty(\mathbb{R}^d)$ be an admissible function for K, then there exists $\epsilon > 0$ such that $|u|_{p,k} \leqslant c_{p,k}(K) + \epsilon$. Now, using properties of the Fourier transform of v and changing variables with $x = Ay + b$ and $|\det A| = 1$, we obtain:

$$\hat{v}(\xi) = e^{-2\pi i b\cdot\xi}\hat{u}\left(A^T\xi\right),$$

where A^T is the transpose matrix of A. Furthermore, $|\hat{v}(\xi)| = \left|\hat{u}\left(A^T\xi\right)\right|$. Since A^T is also an orthogonal matrix, the change of variables $\eta = A^T\xi$ gives:

$$|v|^p_{p,k} = \int_{\mathbb{R}^d} |k(\xi)\hat{v}(\xi)|^p d\xi = \int_{\mathbb{R}^d} |k(A\eta)\hat{u}(\eta)|^p\, d\xi \leqslant \int_{\mathbb{R}^d} |c\cdot k(\eta)\hat{u}(\eta)|^p d\eta = c^p |u|^p_{p,k}.$$

Thus, $|v|_{p,k} \leqslant c_{p,k}(K) + \epsilon$. Since v is admissible for $f(K)$, we have:

$$c_{p,k}(f(K)) \leqslant c\cdot c_{p,k}(K) + \epsilon. \tag{6.3.4}$$

Now, let $w_1 \in C_0^\infty(\mathbb{R}^d)$ be an admissible function for $f(K)$ such that there exists $\epsilon' > 0$: $|w_1|_{p,k} \leqslant c_{p,k}(f(K)) + \epsilon'$. If we define $w_2(x) = w_1(f(x))$, then $w_2(x)$ is an admissible function for K, and then $c_{p,k}(K) \leqslant |w_2|_{p,k}$.
Similar to previous calculations, we have the following relations:

$$\hat{w}_2(\xi) = e^{-2\pi i A^{-1}b\cdot\xi}\hat{w}_1(A\xi) \Rightarrow |\hat{w}_2(\xi)| = |\hat{w}_1(A\xi)|. \tag{6.3.5}$$

Using this relation, we can obtain the other inequality we are looking for:

$$\begin{aligned}
|w_2|_{p,k}^p = \int_{\mathbb{R}^d} |k(\xi)\hat{w}_2(\xi)|^p d\xi &= \int_{\mathbb{R}^d} \left|\frac{k(\xi)}{k(A\xi)}k(A\xi)\hat{w}_2(\xi)\right|^p d\xi \\
&\leqslant \int_{\mathbb{R}^d} |c\cdot k(A\xi)\hat{w}_2(\xi)|^p \, d\xi \\
&= c^p \int_{\mathbb{R}^d} |k(A\xi)\hat{w}_1(A\xi)|^p \, d\xi \qquad \text{using } 6.3.5 \\
&= c^p \int_{\mathbb{R}^d} |k(\eta)\hat{w}_1(\eta)|^p \, d\eta, \qquad \text{using } \eta = A\xi \\
&= c^p |w_1|_{p,k}^p.
\end{aligned}$$

By construction, we conclude:

$$\begin{aligned}
c_{p,k}(K) \leqslant |w_2|_{p,k} \leqslant c|w_1|_{p,k} \leqslant c\cdot c_{p,k}(f(K)) + c\cdot\epsilon' \\
\Longrightarrow c^{-1}c_{p,k}(K) \leqslant c_{p,k}(f(K)) + \epsilon'.
\end{aligned} \tag{6.3.6}$$

Therefore, by making $\epsilon, \epsilon' \to 0$ and combining 6.3.4 and 6.3.6, we obtain the desired result. ■

Similar to the last two theorems, there are two others (which follow similar arguments) that tell us how this capacity behaves under contractions and dilations:

Theorem 6.3.8. *Let $K \subset \mathbb{R}^d$ be a compact set, and $\lambda > 0$ be a constant. Then:*

$$c_{p,k}(\lambda K) = \lambda^{d(1-1/p)} c_{p,k_{I/\lambda}}(K).$$

Here I is the corresponding identity matrix, and $k_{I/\lambda}(\xi) = k(\xi/\lambda)$.

Proof: Let u be an admissible function for the set K, that is, $u \in C_0^\infty\left(\mathbb{R}^d\right)$ and $u \geqslant 1$ on K. Then the function $v(x) := u(x/\lambda)$ is admissible for λK as well. By changing variables, we obtain $\hat{v}(\xi) = \lambda^d \hat{u}(\lambda\xi)$ and:

$$\begin{aligned}
|v|_{p,k}^p = \int_{\mathbb{R}^d} |k(\xi)\hat{v}(\xi)|^p d\xi &= \int_{\mathbb{R}^d} |k(\xi)\lambda^d \hat{u}(\lambda\xi)|^p d\xi \\
&= \int_{\mathbb{R}^d} |k(\eta/\lambda)\lambda^d\hat{u}(\eta)|^p \lambda^{-d} d\eta, \qquad \text{using } \eta = \lambda\xi \\
&= \lambda^{dp-d}\int_{\mathbb{R}^d} |k(\eta/\lambda)\hat{u}(\eta)|^p d\eta \\
&= \lambda^{dp-d}|u|_{p,k_{I/\lambda}}^p.
\end{aligned}$$

Then

$$|v|_{p,k} = \lambda^{d(1-1/p)}|\hat{u}|_{p,k_{I/\lambda}}.$$

Since every admissible function u for K defines an admissible function v for λK and vice versa, this completes the proof. ■

Theorem 6.3.9. *Let $K \subset \mathbb{R}^d$ be a compact set, and $\lambda > 0$ be a constant. If the tempered weight function satisfies the property $k_{I/\lambda}(\xi) \asymp_c k(\xi)$, then*

$$c_{p,k}(\lambda K) \asymp_c \lambda^{d(1-1/p)} c_{p,k}(K).$$

Proof: Let u be an admissible function for the set K, that is, $u \in C_0^\infty(\mathbb{R}^d)$ and $u \geqslant 1$ on K. Then the function $v(x) := u(x/\lambda)$ is admissible for λK as well. By changing variables, we obtain $\hat{v}(\xi) = \lambda^d \hat{u}(\lambda\xi)$ and:

$$\begin{aligned}
|v|_{p,k}^p = \int_{\mathbb{R}^d} |k(\xi)\hat{v}(\xi)|^p d\xi &= \int_{\mathbb{R}^d} |k(\xi)\lambda^d \hat{u}(\lambda\xi)|^p d\xi \\
&= \int_{\mathbb{R}^d} |k(\eta/\lambda)\lambda^d \hat{u}(\eta)|^p \lambda^{-d} d\eta, \qquad \text{using } \eta = \lambda\xi \\
&= \lambda^{dp-d} \int_{\mathbb{R}^d} |k(\eta/\lambda)\hat{u}(\eta)|^p d\eta \\
&= \lambda^{dp-d} \int_{\mathbb{R}^d} \left| \frac{k(\eta/\lambda)}{k(\eta)} k(\eta)\hat{u}(\eta) \right|^p d\eta \\
&\asymp_c \lambda^{dp-d} |u|_{p,k}^p.
\end{aligned}$$

Then:

$$|v|_{p,k} \asymp_c \lambda^{d(1-1/p)} |u|_{p,k}.$$

Since every admissible function u for K defines an admissible function v for λK and vice versa, we complete the proof. ■

In the following part of this section, the tempered weight will satisfy that there exists an $\alpha \geqslant 0$ such that

$$k(\xi) \leqslant (1 + |\xi|^2)^{\alpha/2} k(0),$$

and G_α corresponds to the respective Bessel kernel (see Section 3 in Stein (1970)), this is:

$$\hat{G}_\alpha(\xi) = (2\pi)^{-d/2}(1 + |\xi|^2)^{-\alpha/2}.$$

The following theorem represents the most significant contribution made in this research, due to its generality and its direct corollary. It is important to mention that the conditions of the theorem are not really restrictive, since the theorem is applicable to many sets, as is its corollary linking the Hausdorff measure with the Fourier (p, k)-capacity (which was one of the main objectives from the beginning).

Theorem 6.3.10. *Let $K \subset \mathbb{R}^d$ be a compact set and μ be its Frostman measure (see 3.1.9) with*

$$\mu(B(x, r)) \lesssim r^\gamma, \quad \forall x \in \mathbb{R}^d,\ r > 0, \tag{6.3.7}$$

for some $0 < \gamma \leqslant d$. Let $\beta > 0$ be the Fourier dimension of μ. Then, for all $p \geqslant \frac{4d-4\gamma+2\beta}{\beta}$, and non-constant k, there exists a constant λ depending on μ such that:

$$c_{p,k}(K) \lesssim \lambda^{-1}\sqrt{\mu(K)}.$$

Proof: We consider a smooth approximation with a nonnegative mollifier function $\rho \in C_0^\infty(\mathbb{R}^d)$ with $\|\rho\|_1 = 1$ and $\mathrm{supp}(\rho) \subset B(0,1)$. For $\epsilon > 0$, define $\rho_\epsilon(x) := \epsilon^{-d}\rho(x/\epsilon)$. Now, we define $w := (G_\alpha * \mu) * \rho_\epsilon$, which is a function in $B_{p,k}$ due to the following inequalities:

$$|k \cdot \hat{w}| \lesssim |(1 + |\xi|^2)^{\alpha/2} \cdot \hat{w}| = |\hat{\mu} \cdot \hat{\rho}_\epsilon| \leqslant |\hat{\mu}|. \tag{6.3.8}$$

Here, $\|\hat{\mu}\|_p < \infty$ is an important fact. The justification is as follows:
Since β is the Fourier dimension of μ, then $|\hat{\mu}(\xi)| \lesssim |\xi|^{-\beta/2}$. This inequality is useful for high frequencies, because for low frequencies, if we normalize the measure μ, then $\hat{\mu}(\xi) \approx 1$ ($|\hat{\mu}(0)| = 1$). Then, it is true that $|\hat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\beta/2}$. On the other hand, since μ is a Frostman measure, we have $\mu(B(x,r)) \lesssim r^\gamma$.
Considering the above properties, we have all of the hypotheses required to apply the Mockenhaupt-Mitsis-Bak-Seeger theorem (see 4.0.5 or Hambrook and Łaba (2013)). Then:

$$\|\hat{\mu}\|_p \leqslant C(p)\sqrt{\mu(K)}, \qquad \forall p \geqslant \frac{4d - 4\gamma + 2\beta}{\beta}. \tag{6.3.9}$$

Now, we have that $w \in B_{p,k}$, and also, $w \in C^\infty(\mathbb{R}^d)$ due to the properties of convolution with a mollifier. Since we still need a function with compact support and greater than or equal to 1 on K, we define $u := \dfrac{\phi \cdot w}{\lambda}$, where ϕ is a bump function such that $\phi(x) = 1$ for all $x \in K$ and $\mathrm{supp}(\phi) \subset K_\delta$ (here, $K_\delta := \{x \in \mathbb{R}^d : d(x, K) \leqslant \delta\}$), and $\lambda := \inf_{x \in K} w(x)$ (this λ ensures that the property $u \geqslant 1$ on K is satisfied). Note that $\lambda > 0$ since $G_\alpha > 0$ (see Stein (1970) equation 26), and $\mu(K) > 0$. With all of the above, we have constructed an admissible function.
Now, by 4.2.4, and inequality 6.3.8, we have:

$$|u|_{p,k} = \left|\frac{\phi \cdot w}{\lambda}\right|_{p,k} \leqslant \frac{1}{\lambda}|\phi|_{1,M_k}|w|_{p,k} \leqslant \frac{1}{\lambda}|\phi|_{1,M_k}\|\hat{\mu}\|_p.$$

By inequality 6.3.9, we conclude that:

$$c_{p,k}(K) \lesssim \lambda^{-1}\|\hat{\mu}\|_p \lesssim C(p)\lambda^{-1}\sqrt{\mu(K)}, \qquad \forall p \geqslant \frac{4d - 4\gamma + 2\beta}{\beta}.$$

■

While these dimensional bounds establish a solid baseline, the non-local nature of the Fourier transform leaves several structural questions open, which we discuss in the concluding chapter. For now, a strong corollary of the previous theorem will be stated.

Corollary 6.3.11. *Let $K \subset \mathbb{R}^d$ be a compact set with Hausdorff dimension equal to s and such that $\mathcal{H}^s|_K(B(x,r)) \lesssim r^s$. Let $\beta > 0$ be the Fourier dimension of $\mathcal{H}^s|_K$, and k a non-constant weight function. Then:*

$$\forall p \geqslant \frac{4d - 4s + 2\beta}{\beta} \Longrightarrow c_{p,k}(K) \lesssim \sqrt{\mathcal{H}^s(K)}.$$

Proof: Apply Theorem 6.3.10 with $\mu = \mathcal{H}^s|_K$ and $\gamma = s$. ■

Remarks 6.3.12.
- Although this theorem is similar to 6.1.16, the fact that exact equality is not guaranteed for any value of p means that the same strong subadditivity proof given

for $p = 2$ in section 6.1 cannot be used here. Therefore, strong subadditivity cannot be guaranteed for the Fourier (p, k)-capacity in any case.

- While the strict positivity of G_α and the compactness of K guarantee that $\lambda > 0$, the quantitative usefulness of this bound heavily depends on λ not being arbitrarily small. If K contains highly sparse regions, λ may approach zero (since most of the mass of the measure μ would be very far away), causing the upper bound to diverge. Ensuring that λ is uniformly bounded away from zero likely requires geometric thickness conditions on K, such as Ahlfors-David regularity. We leave the full exploration of this technical subtlety for **future research**.
  Indeed, to illustrate this, suppose K is γ-dimensional AD-regular with constant $A > 0$. For any $x \in K$ and a fixed radius $0 < r \leqslant \operatorname{diam}(K)$, we have:
  $$\begin{aligned}(G_\alpha * \mu)(x) = \int_K G_\alpha(x-y), d\mu(y) &\geqslant \left(\inf_{y\in B(x,r)} G_\alpha(x-y)\right)\int_{K\cap B(x,r)} d\mu(y)\\ &= \left(\inf_{y\in B(x,r)} G_\alpha(x-y)\right)\mu(B(x,r))\\ &\geqslant \left(\inf_{y\in B(x,r)} G_\alpha(x-y)\right)\frac{r^\gamma}{A}.\end{aligned}$$
  Here, we use the fact that G_α is a strictly decreasing radial function. This implies that the infimum on $B(x, r)$ is attained at the boundary. That is, $\inf_{y\in B(x,r)} G_\alpha(x-y) = G_\alpha(z_r)$ for any vector $z_r \in \mathbb{R}^d$ such that $|z_r| = r$. Thus,
  $$(G_\alpha * \mu)(x) \geqslant G_\alpha(z_r)\frac{r^\gamma}{A}.$$
  Since r, γ, A, and the radial profile $G_\alpha(z_r)$ are fixed strictly positive constants independent of x, the potential is uniformly bounded from below. Because convolution with the positive mollifier ρ_ϵ preserves this lower bound, the constant $\lambda = \inf_{x\in K} w(x)$ does not tend to zero.
- If K is γ-dimensional AD-regular, then $\mathcal{H}^\gamma|_K(B(x,r)) \lesssim r^\gamma$, $\forall x \in \mathbb{R}^d$ and $\forall r > 0$ (see 3.1.11). In fact, since $\mathcal{H}^\gamma|_K$ is a finite measure, the inequality is obvious for any big $r > 1$, and the AD-regularity gives us the inequality for $0 < r \leqslant 1$.
- Every smooth manifold and Lipschitz surface is AD-regular on $\mathbb{R}^d$. If a set A is s-dimensional AD-regular, then its Hausdorff dimension is equal to s (see Pan and Xiong (2020)). Also, every self-similar set satisfying the Open Set Condition is AD-regular (see corollary 6.4.4 in Fraser (2020) for this result, and more examples of AD-regular sets).
- It is not difficult to see that the bound $\frac{4d-4\gamma+2\beta}{\beta}$ is greater than or equal to 2 for every γ, β (both are less than or equal to d).
- The bound $\frac{4d-4\gamma+2\beta}{\beta}$ is suggested to be optimal for the inequality to hold (see the preprint Fraser et al. (2025)), and as a corollary, the bound is optimal in the sense of the range of values of p, since the Hausdorff dimension is the supremum of the values of γ (as mentioned in 3.1.2).

As a final part of this research, we consider it worth mentioning that the Fourier (p, k)-capacity can be defined using even more specific weight functions, which, of course, implies that all the above properties are satisfied. The objective of defining the Fourier (p, k)-capacity through these weights would not be to create a generalization, but rather to build a bridge between probability theory and harmonic analysis.

If we define the weight functions k such that $k(\xi + \eta) \leqslant (1 + |\psi(\xi)|)^N k(\eta)$ with ψ a locally bounded negative definite function, we can still use the same definition of Fourier (p, k)-capacity:

$$c_{p,k}(K) := \inf\{|\phi|_{p,k} : \phi \in E(K)\}.$$

The fact that this capacity has more specific control over weight k is what could generate links between the areas of study mentioned above. However, from the point of view of our theory, this capacity corresponds to a Fourier (p, k)-capacity with tempered weight, since using 4.3.2 we have:

$$\begin{aligned} k(\xi + \eta) \leqslant (1 + c_\psi(1 + |\xi|^2))^N k(\eta) = (1 + c_\psi + c_\psi|\xi|^2)^N k(\eta) &\leqslant (1 + c_\psi)(1 + |\xi|^2)k(\eta) \\ &\leqslant (1 + c_\psi)(1 + |\xi|)^2 k(\eta). \end{aligned}$$

Therefore, our weight is tempered, since we have

$$k(\xi + \eta) \leqslant A(1 + |\xi|)^M k(\eta)$$

with $A = (1 + c_\psi)^N$, and $M = 2N$.

Of course, studying this object in future research could build a direct bridge to Bochner's theorem and the theory of convolution semigroups, connecting with probability theory.

Chapter 7

Conclusion, comments and future research

In this thesis, we have mainly explored a new object and a few properties related to it, some quite basic and others quite revealing, connecting an operator that belongs purely to Fourier analysis with geometric properties such as Frostman measures and Hausdorff measure. Studying Fourier p-capacity and its generalizations allowed us to lay the groundwork for further theory in future research, perhaps in the direction taken by other classical capacities, using theory from the main areas in which this concept operates.

While several solid results were obtained, it must also be recognized that current Fourier analysis tools have not allowed much further progress, as non-trivial questions arise when studying these Fourier capacities, leaving numerous problems open in this regard.

a). **The strong subadditivity for** $p \neq 2$.
As mentioned in Remark 6.1.19, proving the strong subadditivity property for the case $p \neq 2$ remains a formidable open problem, despite the fact that it can be proven with a minor result for other classical capacities. Fourier space does not allow for this key result (at least not with current tools). Of course, it would be interesting to find a counterexample to strong subadditivity, as it could lead to useful constructions in the study of these new objects.

b). **Examples**.
In future research, it could be interesting to create constructions of sets that have positive Fourier capacity, and even classify them using some criterion. Of course, imposing conditions under which the Fourier capacity of a set is 0 would also be interesting, as it would allow us to differentiate between sets for which Theorem 6.3.10 does not provide relevant information and those for which it can generate interesting conclusions (about their dimension, measure, etc.).

c). **The relation between Fourier capacities and classical capacities**.
Since we now have a few properties of these new capacities, it would be interesting to investigate which tools might be useful for relating these Fourier capacities to more classical capacities, even though they already have a clear relationship with the capacities that use weights considered in this work. Formalizing any relationship that this capacity may have with any other could generate even more useful results within Potential theory, as this could yield results similar to the other capacities, with respect to sets with capacity 0 (removable sets).

d). **Deepening the Probabilistic Connections**.
Perhaps a promising avenue for future study would be to interpret the Fourier (p, k)-capacity in a probabilistic sense. The importance of negative definite functions

and their relationship with Lévy processes and Markov processes has already been mentioned throughout this work, so the application of these new weights controlled by negative definite functions may lead to interesting results in probability theory, possibly characterizing sets or processes using purely the Fourier (p, k)-capacity.

In conclusion, there is still much to study regarding these Fourier capacities, as the theory has only been covered in this work. It is clear that frequency space currently has its limitations, but we firmly believe that more interesting results can be generated around these objects. In particular, a future area of interest is to investigate further connections between geometric measure theory, Fourier analysis, and potential theory, perhaps applying these capacities to self-similar sets, fractal measures, and other geometric objects that are currently being studied.

Notation Index

This index summarizes the main notations and symbols used throughout this thesis.

Spaces and Sets

$\mathbb{R}^d$ d-dimensional Euclidean space.
$\mathcal{K}$ Collection of all compact subsets of $\mathbb{R}^d$.
$\mathcal{O}$ Collection of all open subsets of $\mathbb{R}^d$.
$C_0^\infty(\mathbb{R}^d)$ Space of infinitely differentiable functions with compact support (test functions, also denoted by $\mathcal{D}(\mathbb{R}^d)$).
$\mathcal{S}(\mathbb{R}^d)$ Schwartz space of rapidly decreasing smooth functions.
$\mathcal{D}'(\mathbb{R}^d)$ Space of distributions (continuous linear functionals on $C_0^\infty(\mathbb{R}^d)$).
$\mathcal{S}'(\mathbb{R}^d)$ Space of tempered distributions.
$L^p(\mathbb{R}^d)$ Space of Lebesgue measurable functions f such that $|f|^p$ is integrable.
$B_{p,k}$ Hörmander space with integrability exponent p, and weight k.
$N(\mathbb{R}^d)$ Space of negative definite functions.
$CN(\mathbb{R}^d)$ Space of continuous negative definite functions.

Measures and Dimension

$|x|$ Euclidean norm of $x \in \mathbb{R}^d$.
$m(E)$ **or** $m_d(E)$ d-dimensional Hausdorff measure of the set E.
$\mathcal{H}^s(E)$ s-dimensional Hausdorff measure of the set E.
$\dim_H(E)$ Hausdorff dimension of the set E.
$\dim_B(E)$ Box-counting (Minkowski) dimension of the set E.

Harmonic analysis and capacities

$\|f\|_p$ **or** $\|f\|_{L^p}$ The L^p norm of the function f.
$\hat{f}$ **or** $\mathcal{F}(f)$ Fourier transform of the function (or distribution) f.
$f * g$ Convolution of functions (or distributions) f and g.
$c_{p,\mathcal{F}}(K)$ Fourier p-capacity of the set K.
$c_{p,k}(K)$ Fourier (p,k)-capacity of the set K.
$|f|_{p,k}$ Weighted Fourier norm used in the definition of (p,k)-capacity.
k_A Corresponds to the weight function $k_A(\xi) = k(A\xi)$.

Inequalities and Equivalences

$f \lesssim g$ Indicates that $f(x) \leqslant Cg(x)$ for some constant $C > 0$.
$f \asymp g$ Indicates that $Dg(x) \leqslant f(x) \leqslant Cg(x)$ for some constants $D, C > 0$.
$f \asymp_c g$ Indicates that $c^{-1}g(x) \leqslant f(x) \leqslant cg(x)$ for some constant $c > 0$.

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