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Including binaries as energy sources in globular clusters: bypassing computational limitations

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A mi familia, mi Maca y mis amigos. Los amo.

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Resumen

La evolución de la dinámica de sistemas estelares densos, como los cúmulos globulares, es determinada por un conjunto de interacciones gravitatorias complejas. Si bien el comportamiento de estos sistemas tiende al colapso de su núcleo, las interacciones de tres cuerpos actúan como una fuente energética que revierte este proceso. Sin embargo, dado que el problema general de N cuerpos es caótico e analíticamente irresoluble, realizar un seguimiento de estas interacciones locales entre binarias y estrellas solitarias a lo largo de los mil millones de años de vida de un cúmulo resulta computacionalmente inviable.

Con el fin de evitar esta limitante, desarrollamos un modelo de ecuaciones Fokker-Planck acopladas, para modelar la evolución e impacto que los encuentros entre estrellas binaries y singulares tiene en en el cúmulo. Investigamos el impacto de las poblaciones binarias comparando tres modelos evolutivos distintos: un colapso puro donde el cúmulo esta compuesto netamente por estrellas singulares, un modelo que incorpora el calentamiento binario instantáneo de tres cuerpos y un modelo con una poblacion de binarias primordiales que sigue de forma autoconsistente la difusión, el compactamiento y agotamiento de dicha población.

Nuestra resultados indican que la presencia de una fracción binaria primordial (f_B) altera la cronología, profundidad y termodinámica del colapso. Establecemos que las poblaciones primordiales retrasan el inicio de las oscilaciones en el tamaño del núcleo, y que dicho retraso varía directamente con la cantidad inicial de binaries. Cabe destacar que los sistemas con poblaciones primordiales bajas ($f_B \leq 0,01$) se ven obligados a alcanzar densidades centrales máximas casi un orden de magnitud superiores a las provocadas por del modelo instantáneo. Esto se debe a que el calentamiento difusivo escala pobremente con la densidad, lo que proporciona una fuente energética suave que requiere una contracción más profunda para detener el colapso. Por último, demostramos que, en nuestro modelo, independientemente de la fracción binaria inicial, la fuerte dependencia de la densidad de la formación de sistemas de tres cuerpos garantiza que el rebote del núcleo se vea interrumpido por el endurecimiento y la formación de binarias.

Keywords – Binarias, cúmulos, dinámica estelar

Abstract

The dynamical evolution of dense stellar systems, such as globular clusters, is fundamentally driven by complex gravitational interactions. While the general behavior of these systems inevitably leads to gravothermal core collapse, multibody interactions, such binary-single scatterings, act as a critical heat source capable of arresting and reversing this collapse. However, because the general N-body problem is chaotic and analytically intractable, tracking these local binary-single interactions across the billion year lifespan of a cluster is computationally prohibitive.

To bypass the extreme computational cost of direct N-body integration, we develop a coupled, semi-analytical Fokker-Planck solver to model the evolution of star clusters. We investigate the thermodynamic impact of binary populations by comparing three distinct evolutionary models: a pure single-star gravothermal collapse, a model incorporating instantaneous three-body binary heating, and a comprehensive population model that self-consistently tracks the diffusive heating, hardening, and depletion of a primordial binary population.

Our simulations demonstrate that the presence of a primordial binary fraction (f_B) fundamentally alters the timing, depth, and thermodynamics of core collapse. We establish that primordial populations delay the onset of gravothermal oscillations, with the delay scaling directly with the initial fuel reservoir. Notably, systems with trace primordial populations ($f_B \leq 0.01$) are forced to reach peak central densities nearly an order of magnitude higher than those driven by pure three-body heating. This occurs due to diffusive heating scaling weakly with density, providing a softer thermodynamic brake that requires a deeper contraction to halt the collapse. Finally, we show that in our model regardless of the initial binary fraction, the steep density dependence of three-body formation ensures that core bounce is stopped by combined binary hardening and formation.

Keywords – Binary stars, star clusters, stellar dynamics

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Chapter 1

Introduction

All stars are formed in star cluster, and most stars are in binaries. This very fact should spark some curiosity in understanding how binaries affect their environments. And it does spark it! After all, one of the most studied subjects in astronomy is stellar dynamics. One of the earliest and most notable examples of this is the two-body problem, which is, as the name indicates, the formulation of the motions of two bodies undergoing a gravitational interaction. Newton solved this in 1687, a few years after he invented calculus (what a career!). After one studies and understand two bodies, the logical question is: *what about three bodies?* This is a problem that Newton could not solve.

The story is curious because in 1877 King Oscar II of Sweden and Norway set up a contest for his 60th birthday. He established a prize (2,500 Swedish kronor, if we do a rough estimate, this is about $90.000 in USD) for whoever solved the N-body problem (note that not the three-body but the N-body). The prize was claimed in 1889 by Poincaré who actually proved that for $N > 2$ bodies, the solution is not closed. This discovery is one of the earliest examples of published chaos theory. Just as a curiosity, while Poincaré won the prize in 1889, he noticed some errors in his approach, in withdrawing and fixing those errors, he ended up spending more money than he was awarded. The lesson here is clear: more money should be awarded to scientists (although, cash prizes for specific discoveries still exists, but the point still stands).

What is the history lesson for? Well, to start, I think it is a fun story, but beyond that, the takeaway is that gravitational interactions for system with more than two bodies (which is virtually, all systems in the universe), the dynamics are analytically intractable. This is very relevant for collisional systems like star clusters. Dense star clusters are fantastic laboratories for stellar dynamics and stellar evolution. However, they also present a massive computational hurdle: if the three-body problem is chaotic, tracking the individual gravitational interactions of 10^5 to 10^6 stars over billions of years is a monumentally expensive task. To solve these motions, numerical integration of Newton's equations is needed, this scales prohibitively even with the latest state-of-the-art algorithms.

In the advent of the discovery of high density clusters in the early universe thanks to the James Webb Space Telescope, the need for models than can help us understand these system accurately with lower computational expenses is vital. To bypass this, we can shift our perspective from the microscopic to the macroscopic. Instead of calculating exact orbital trajectories, we can treat the cluster statistically. One way to do this is by use of the Fokker-Planck equation, which ironically was the first method used to study the dynamics of cluster with high stellar counts, at the time when the available computational power required for N-body was not yet achieved.

But if Fokker-Planck models do exist, and have been for a few decades, why are we doing them now? While this pioneering models were very useful to understand the nature of core collapse and the evolution of star clusters, they were not able to capture chaotic dynamics, such as interaction between binary and single stars, which are a vital heat source for star clusters. Today, however, we can leverage recent analytical advances in the statistical solution of the three-body problem. By coupling these accurate phase space distributions with the classical Fokker-Planck formalism, we can construct a self-consistent statistical model that accurately quantifies the profound effects that binary populations have on global cluster dynamics, at a fraction of the computational cost of direct N-body integration.

Chapter 2

Theoretical framework

2.1. How to simulate a cluster

Clusters are systems with stellar counts ranging from 10^2 to 10^6. They are classified as *collisional systems*. Despite the nomenclature , this does not refer to actual physical collisions, though. It rather has to do with the relevance of gravitational interactions between the stars. While we refer broadly to 'clusters' throughout this work, the primary focus is on globular clusters.

The motion of the stars of a cluster can be studied via a rather simple set of equations,

$$\ddot{\mathbf{r}}_i = G \sum_{j=1, j\neq i}^{j=N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \tag{2.1.1}$$

where $\mathbf{r}_j$ is the position of the jth star at a time t, m_j its mass, and G is Newton's constant. Put very succintly: stellar dynamics is the study of the consequences of equation (2.1.1) (Heggie and Hut, 2003).

Although this formulation appears simple, Poincaré (Poincaré, 1885) demonstrated that for a self-gravitating system with N bodies, if $N > 2$ there is no analytical closed form solution for equation (2.1.1) . Therefore, in order to follow the motion of stellar systems, it is mandatory to use numerical methods.

Of the available approaches (see Amaro-Seoane (2018) and Spurzem and Kamlah (2023) for comprehensive reviews), N-body codes represent the most direct strategy.

By explicitly computing the exact gravitational interactions between all stars, they avoid the need for strong physical approximations beyond Newtonian gravity (although, some codes do include post-Newtonian terms, e.g. Mikkola and Merritt (2008); Trani and Spera (2020)). The primary limitation, however, is the immense computational cost; state-of-the-art simulations for $N = 10^6$ can require on the order of a year of computation time to complete (Arca Sedda et al., 2024).

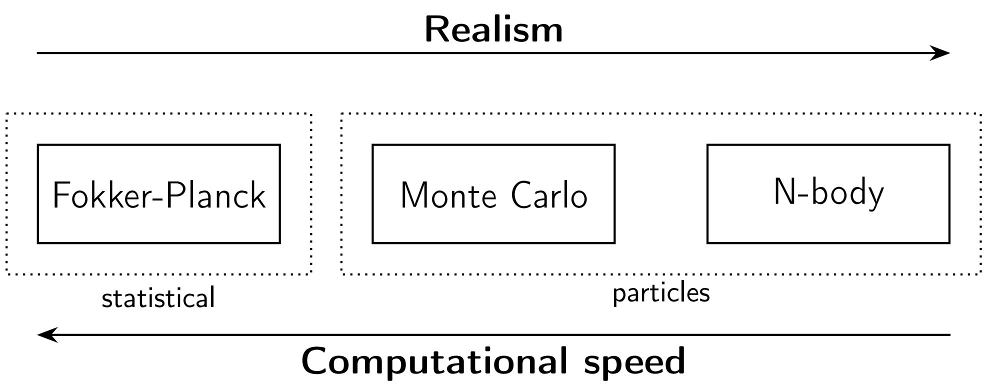

Figura 2.1.1: As the arrows indicate, there is a direct trade-off between physical fidelity and computational speed (adapted from Amaro-Seoane (2018)).

Figure 2.1.1 summarizes the widely available methods. Here we find a situation ubiquitous in all of computational astrophysics: the more comprehensive the physical model, the steeper the computational cost.

On the other side of the spectrum we find Fokker-Planck (FP) models. These are based on the homonymous equation. In this approximation, the effect of two-body relaxation processes is treated diffusively. The evolution of the distribution function of orbital energies for the stars is then evolved using the aforementioned FP equation. We discuss this formalism in more detain in Sec. 2.2.

In the middle of the road between realism and computational efficiency, we find the Monte Carlo (MC) approach. This technique assumes spherical symmetry (but see Vasiliev (2014) for an MC model that does not follow this assumption) but unlike FP, it actually has particles. The number of particles, however, can be less than that of the stars. While MC codes are particle based, relaxation is also treated as a diffusion process.

2.2. The Fokker-Planck equation

As we mentioned, the Fokker-Planck formalism is formulated under the assumption that two-body relaxation can be described by weak and distant encounters. In this framework, the FP equation, which has advection and diffusion operators, evolves the distribution function of orbital energies for the cluster members. The idea is to describe the phase-space distribution function (DF) $f(\vec{x}, \vec{v}, t)$, with $\vec{x}$ the positions, and $\vec{v}$ the velocities.

Now, cartesian coordinates are not the best suited for our and previous models' purposes. Also, from Jeans' theorem, for a spherical system the DF can depend on coordinates $(\vec{x}, \vec{v})$ only through the orbital energy E (Jeans, 1915). Thus, we follow the derivation outlined by Spitzer (1987) to obtain the FP equation in E-space. Let's note that orbital energy is given by,

$$E = \phi(r) - \frac{1}{2}v^2 \tag{2.2.1}$$

where v is the velocity and ϕ the potential. Now, we consider a spherical cluster, $N(E)dE$ is the number of stars with an energy in the interval dE, V denotes the volume accessible for such stars, from here it follows that $dV = 4\pi r^2 dr$. As such, we have

$$N(E)dE = \int dV \int f(E) 4\pi v^2 dv \tag{2.2.2}$$

Equation (2.2.1) yields $dE = -vdv$. Since we are mapping positive intervals in phase space, we use $|dv| = dE/v$, thus,

$$N(E) = 4\pi f(E) \int v dV = 16\pi^2 f(E) \int v r^2 dr \tag{2.2.3}$$

Since we want to compute the collision operator in $E-$space, we define $\Psi(E, \Delta E) d\Delta E$ as the probability that a during a time interval Δt a star with energy E will experience a change in energy ΔE. Conservation of the number of stars then leads to,

$$N(E, t + \Delta t) = \int N(E - \Delta E, t)\Psi(E - \Delta E, \Delta E) d\Delta E \tag{2.2.4}$$

We now Taylor-expand this up to second order given that the relevant collisions are perturbative in nature (Chandrasekhar, 1960; Leigh et al., 2022), obtaining,

$$\begin{aligned} N(E, t) + \Delta t \frac{\partial N}{\partial t} \approx \int \Big[& N(E)\Psi(E, \Delta E) \\ & - \frac{\partial}{\partial E}\Big(N(E)\Psi(E, \Delta E)\Big)\Delta E \\ & + \frac{1}{2}\frac{\partial^2}{\partial E^2}\Big(N(E)\Psi(E, \Delta E)\Big)\Delta E^2\Big] d\Delta E \end{aligned} \tag{2.2.5}$$

From here it follows that,

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial E}(N\langle \Delta E\rangle_V) + \frac{1}{2}\frac{\partial^2}{\partial E^2}(N\langle \Delta E^2\rangle_V) \tag{2.2.6}$$

here $\langle \Delta E\rangle_V$ and $\langle \Delta E^2\rangle_V$ are the first and second order diffusion coefficients, averaged over all stars of energy E within volume V.

A different form of equation (2.2.6) is one where ϕ varies with r. Essentially, for any E the accessible phase space volume goes from zero up to r_{max}, where $\phi(r_{max}) = E$. We can actually take this idea and define $p(E)$, which is the accessible phase space volume per unit interval of E, such that

$$p(E) = 4\int_0^{r_{max}} drvr^2 \tag{2.2.7}$$

with this we can rewrite equation (2.2.3) as,

$$N(E) = 4\pi^2 p(E) f(E) \tag{2.2.8}$$

Replace this into equation (2.2.6) we find the infamous equation,

$$4\pi^2 p(E)\frac{\partial f(E)}{\partial t} = -\frac{\partial}{\partial E}\left[-D_{EE}\frac{\partial f}{\partial E} - D_E f\right] \tag{2.2.9}$$

Here the flux coefficients are given by the expressions (Cohn, 1980),

$$
\begin{aligned}
D_{EE} &= 16\pi^3 \Gamma q(E) \int_0^E dE' f(E') \\
&\quad + 16\pi^3 \Gamma \int_E^{\phi(0)} dE' q(E') f(E')
\end{aligned} \tag{2.2.10}
$$

$$
D_E = -16\pi^3 \Gamma \int_E^{\phi(0)} dE' p(E') f(E') \tag{2.2.11}
$$

with $\Gamma = 4\pi^2 G^2 m^2 \ln \Lambda$, where G is the gravitational constant, m is the mass of a single star, and $\ln \Lambda$ is the Coulomb logarithm.

Additionally, $q(E)$ is defined as,

$$
q(E) = \frac{4}{3} \int_0^{\phi^{-1}(E)} dr r^2 v^3 \tag{2.2.12}
$$

and it is also related to $p(E)$ via

$$
q(E) = \int_E^{\phi(0)} dE' p(E') \tag{2.2.13}
$$

The most common numerical method to solve this equation was pioneered by Cohn (1979). The method was later revised and improved in Cohn (1980). It is also important to note that in the former, equation (2.2.9) was solved in (E, J)-space, while it was solved solely in E-space in the latter. It is also in this version were the Chang-Cooper scheme (Chang and Cooper, 1970) was introduced into the framework.

In order to solve this, we follow the classical algorithm, this means that we divide each integration step into two sub-steps:

1. **Fokker-Planck step:** the distribution $f(E)$ is advanced in time, with the potential held fixed. The diffusion coefficients are computed with the distribution of the previous step, which makes the equations linear.

2. **Adiabatic step:** after the distribution evolves, the new potential is estimated using the Poisson equation. Now, during this stage, the distribution function expressed in terms of the actions (phase space weights $p(E)$ and $q(E)$) are invariant (Lynden-Bell, 1973; Binney and Tremaine, 2008), allowing this to be done implicitly. In computational terms, this can be done with a Picard iteration.

This is algorithm is analogous to performing a smooth trajectory on (f, ϕ)-space (Cohn, 1985).

As mentioned, the adiabatic step requires to solve the Poisson equation, which can be expressed as,

$$\begin{aligned}\phi(r) = \frac{4\pi G}{r} \int_0^r dr' r'^2 \rho(r') \\ + 4\pi G \int_r^\infty dr' r' \rho(r')\end{aligned} \tag{2.2.14}$$

here $\rho(r)$ is the mass density profile given by,

$$\rho(r) = 2^{5/2}\pi m \int_0^{\phi(r)} dE[\phi(r) - E]^{1/2} f(E) \tag{2.2.15}$$

From here, we can self-consistently evolve the cluster. This also allows us to track different cluster properties, in particular we are interested in,

$$M(r) = 4\pi \int_0^r dr' r'^2 \rho(r') \tag{2.2.16}$$

$$\sigma^2(r) = \frac{3G}{\rho(r)} \int_r^\infty dr' \frac{M(r')\rho(r')}{r'^2} \tag{2.2.17}$$

Where $M(r)$ is the enclosed mass at radius r, and $\sigma^2(r)$ is the velocity dispersion of the cluster.

Throughout this work, we will frequently refer to the core radius of the system. It is important to note that this quantity can be defined in a few distinct ways depending on the theoretical or observational context. Observationally, the core radius is typically defined as the projected radius at which the surface brightness or surface mass density drops by a factor 2. As such, we use the classical theoretical King core radius (King, 1962),

$$r_c = \sqrt{\frac{3\sigma^2(0)}{4\pi G\rho(0)}} \tag{2.2.18}$$

we will also refer to two fundamental dynamical timescales. First, the central relaxation timescale t_{rc},

$$t_{rc} = \frac{6.5 \times 10^{-2}\sigma_c^3}{G^2 m\rho(0)\ln\Lambda} \tag{2.2.19}$$

where m is the mean stellar mass and $\ln\Lambda$ is the Coulomb logarithm. Second, we also use the half-mass relaxation timescale t_{rh}

$$t_{rh} = \frac{0.138\, M^{1/2} r_h^{3/2}}{G^{1/2} m\ln(0.4N)} \tag{2.2.20}$$

where M is the total cluster mass, N is the total number of stars, and r_h is the half-mass radius.

Physically, these two timescales isolate different regimes of the cluster's evolution. While t_{rc} dictates the rapid, localized interactions at the center that are directly responsible for accelerating the gravothermal core collapse, t_{rh} provides a macroscopic measure of the global cluster, tracking the longer interval required for energy redistribution across the bulk of the stellar distribution (Binney and Tremaine, 2008).

2.3. Binaries in star clusters

Star clusters are self-gravitating systems, as such, they are characterized by negative heat capacity (Lynden-Bell and Wood, 1968). This can be easily proven following the derivation presented in Lynden-Bell (1999). Let's imagine a self-gravitating system in virial equilibrium, then the kinetic energy T and potential energy W are related by,

$$2T + W = 0 \tag{2.3.1}$$

The total energy E is simply the sum of kinetic and potential energies. Then, from equation (2.3.1) we have that $W = -2T$, thus $E = -T$. Now, for a system of N particles, the kinetic energy T relates to temperature Θ via,

$$T = \frac{3}{2} N k_B \Theta \tag{2.3.2}$$

heat capacity C is defined as the change in internal energy with respect to temperature. And, replacing equation (2.3.2) in $E = -T$, we find that,

$$C = \frac{dE}{d\Theta} = -\frac{3}{2} N k_B \tag{2.3.3}$$

From here, it follows that $C < 0$ given that N and k_B are positive constants.

In the context of a star cluster, two-body relaxation will lead to the evaporation of the fastest stars in its central region. This will lead to a decrease in kinetic and potential energy, but a much larger kinetic energy loss. To balance this, the core will contract, increasing its density and also increasing the efficiency of the evaporation process. This will trigger additional contraction and the process will repeat itself. This feedback loop is what is known as core collapse.

As Mapelli (2018) points out, since we do not observe clusters with pathologically high central densities, some mechanism must break the aforementioned loop. This is where binaries come in.

Three-body interactions between hard binaries and single stars will inject kinetic energy into the cluster. Hard binaries are binaries whose binding energy is larger than the mean kinetic energy of the cluster members. To be more precise, the hard-soft boundary energy is defined as,

$$E_{HS} = \frac{1}{2} \bar{m} \sigma_c^2 \tag{2.3.4}$$

With $\bar{m}$ the mean stellar mass of a cluster member, and σ_c^2 the 3D central velocity dispersion. Hard binaries will, on average, become harder (or more compact) after an interaction with a single star. This is known as the Heggie-Hills law (Heggie, 1975a). Analogously, soft binaries will soften after an interaction with a single star.

The kinetic energy injected by hard binaries causes the central region to expand. Because of the cluster's negative heat capacity, this expansion allows the core to cool down, halting the core collapse process. Mapelli (2018) puts it very succinctly: binaries alter the thermodynamical state of the system without inputting extra energy into the system, just like Maxwell's demon.

2.4. A binary FP model for star clusters

We have established that binary-single interactions are very important for the life cycle of star clusters. But if three-body interactions are analytically intractable, how can we further quantify and understand their effects over cluster evolution? We can of course resort to the aforementioned N-body simulations, but what happens when we enter a mass regime in which this is not possible?

This computational bottleneck can be addressed by improving statistical models that can add the effects of three-body scattering outcomes to cluster evolution within a macroscopic FP framework. A significant step toward this goal was taken by Leigh et al. (2022), who implemented a FP model for binaries based on the statistical solution to the three-body problem presented in Stone and Leigh (2019). Their framework demonstrated good agreement with direct N-body experiments, providing a good foundation for dynamically evolving binary populations in dense stellar environments.

In this work, we build upon the formalism established by Leigh et al. (2022) to extend the applicability and precision of their model. Specifically, we present a refined implementation of the backreaction physics (see Sec. 2.5 in Leigh et al. (2022)), allowing for a more robust treatment of the energy exchange between the binary population and the single-star background. This refinement is particularly timely: Claeyssens et al. (2026) recently presented a sample of 222 high-redshift ($z > 0.5$) clusters with stellar densities comparable to those observed in nuclear star clusters. These environments are theorized to be primary nurseries for intermediate-mass black holes (IMBHs). Given the extreme computational cost associated with modeling such extreme density systems, our refined statistical approach provides a necessary and efficient candidate for investigating their dynamical life cycles.

Chapter 3

Methodology

3.1. Fokker-Planck for binary stars

We now introduce the FP formalism that will be utilized for binaries. Let $N(E_B, t)$ be the distribution of binary binding energies E_B at time t. Thus, its time evolution can be written as,

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial E_B}(D_B N) + \frac{1}{2}\frac{\partial^2}{\partial E_B^2}(D_{BB} N) \tag{3.1.1}$$

Where D_B and D_{BB} are the first and second order diffusion coefficients. These can be constructed following Spitzer (1987), where,

$$D_B = \int \Gamma(E_B)\mathcal{F}(E_B, E_f)\Delta E_B d\Delta E_B \tag{3.1.2}$$

$$D_{BB} = \int \Gamma(E_B)\mathcal{F}(E_B, E_f)\Delta E_B^2 d\Delta E_B \tag{3.1.3}$$

Here Γ is the gravitationally focused scattering rate given by,

$$\Gamma = \frac{3\pi n_s G^2(m_B + m_s)m_B}{\sigma E_B} \tag{3.1.4}$$

Where n_s is the single star number density, m_B and m_s are the binary and single star masses, and σ is the cluster velocity dispersion. Since we are considering a single mass stellar cluster, we have that $m_s = 1\,M_\odot$ and $m_B = 2\,M_\odot$. It follows that,

$$n_s = \rho\frac{1-f_B}{1+f_B} \qquad \frac{(m_B+m_s)m_B}{M_\odot} = 6 \tag{3.1.5}$$

with f_B the binary fraction of the cluster. This yields,

$$\Gamma(E_B) = \left(\frac{1-f_B}{1+f_B}\right)\left(\frac{18\pi\rho G^2}{\sigma E_B}\right) \tag{3.1.6}$$

Now, the distribution of final binary orbital energy E_f for hard binaries after single-binary encounters can be approximated by a power law (Valtonen and Karttunen, 2006),

$$\mathcal{F}_{VK}(|E_f|)d|E_f| = (n-1)|E_B|^{n-1}|E_f|^{-n}d|E_f| \tag{3.1.7}$$

The parameter n is an encapsulation of the total interaction angular momentum L given by,

$$n = 3 + 18\bar{L}^2 \tag{3.1.8}$$

where $\bar{L}$ is total angular momentum normalized by its maximum possible value.

Furthermore, we consider a population of hard binaries, so our energy ranges from the hard-soft boundary energy E_{HS} up to the contact criterion energy E_{coll}, these are given by,

$$E_{HS} = \frac{1}{2}\bar{m}\sigma_c^2 \tag{3.1.9}$$

$$E_{coll} = \frac{G\bar{m}^2}{R^2} \tag{3.1.10}$$

Where $\bar{m}$ is the mean mass of a cluster member, σ_c^2 is the central velocity dispersion, and R the radius of the test star species. For the single mass case $\bar{m} = 1M_\odot$, we also set $R = 1R_\odot$. Let's also note that E_{HS} will evolve over time.

Let us define the auxiliary function $\eta(j)$ to evaluate the energy boundaries:

$$\eta(j) = \frac{E_{\text{coll}}^{j-n} - E_B^{j-n}}{j-n} \tag{3.1.11}$$

The first- and second-order diffusion coefficients can then be compactly written as:

$$D_B(E_B) = \Gamma(E_B)(n-1)E_B^{n-1}\Big[\eta(2) - E_B\eta(1)\Big]$$

$$D_{BB}(E_B) = \Gamma(E_B)\begin{cases} (n-1)E_B^{n-1}\Big[\eta(3) - 2E_B\eta(2) + E_B^2\eta(1)\Big], & n > 3 \\ 2E_B^2\left[\ln\left(\frac{E_{\text{coll}}}{E_B}\right) + 2\frac{E_B}{E_{\text{coll}}} - \frac{1}{2}\frac{E_B^2}{E_{\text{coll}}^2} - \frac{3}{2}\right], & n = 3 \end{cases}$$

Let's now note that the only time-evolving component of the coefficients is the encounter rate. The encounter rate also evolves due to the density and the velocity dispersion, which means that our diffusion coefficients will evolve proportionally to the density over velocity dispersion ratio. This justifies the need for a two-zone model given that binary evolution will depend on the local properties of the environment.

3.1.1. A two-zone model for binaries

In order to better capture the evolution of the binary distribution, we follow a similar formalism to what was presented in Leigh et al. (2022). We divide the cluster into two zones: core and halo. Core is defined as the zone where $r < r_c$, and halo as the zone where $r_c < r < r_h$, with r_h the half-mass radius. Let $\mathcal{C}(E_B, t)$ and $\mathcal{H}(E_B, t)$ be the core and halo binary binding energy distributions, respectively. We can then write,

$$\frac{\partial \mathcal{C}}{\partial t} = -\frac{\partial}{\partial E_{\text{B}}}(D_{\text{B}}^c\mathcal{C}) + \frac{1}{2}\frac{\partial^2}{\partial E_B^2}(D_{BB}^c\mathcal{C}) - S_{c\to h} + S_{h\to c} \tag{3.1.12}$$

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial}{\partial E_B}(D_B^h\mathcal{H}) + \frac{1}{2}\frac{\partial^2}{\partial E_B^2}(D_{BB}^h\mathcal{H}) + S_{c\to h} - S_{h\to c} \tag{3.1.13}$$

Let's note the superscripts, indicating core and halo diffusion coefficients. $S_{c\to h}$ and $S_{h\to c}$ are flow terms, they represent binaries being ejected from core to halo

($c \rightarrow h$), and mass segregation from halo binaries to the core ($h \rightarrow c$). Leigh et al. (2022) presented these terms but in a simplified manner, but we now present our derivation of them.

3.1.1.1. Flow terms

Since we are studying a hard binary population, after each binary-single interaction, the binary binding energies will grow (Heggie, 1975b). At the same time, a recoil velocity will be imparted onto the binary. Eventually, the kick will be enough for the binary to leave the core but to stay at the halo.

We now calculate the probability of a three-body encounter producing such recoil velocities on our binary population. The rate at which a binary undergoes a scattering event that produces a relative change in binding energy Δ is given by the numerical fit from Heggie and Hut (1993).

$$\mathcal{R} \propto \Delta^{-\alpha}(1+\Delta)^{-4.5+\alpha}$$

Where α is a numerical fit value given by,

$$\alpha = \begin{cases} 1.049 - 0.912z + 0.530z^2 - 0.099z^3, & 0 < z < 1.5, \\ 0.813 - 0.474z + 0.261z^2 - 0.044z^3, & 1.5 < z < 3, \\ 0.54, & z > 3 \end{cases}$$

With $z = \log_{10}(E_B/E_{HS})$. Now, by energy conservation and for the equal mass m case we can show that,

$$\Delta = \left(\frac{V}{\sigma}\right)^2 \frac{3}{E_B/E_{HS}}$$

Where V is the final velocity of the binary after an encounter with a single star. From here, we can define,

$$\Delta_{ej} = \left(\frac{V_{ej}}{\sigma}\right)^2 \frac{3}{E_B/E_{HS}}, \qquad \Delta_{evap} = \left(\frac{V_{evap}}{\sigma}\right)^2 \frac{3}{E_B/E_{HS}}$$

Here V_{ej} is the velocity needed for a binary to migrate from the core of the cluster