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A Banach space mixed formulation for the unsteady Brinkman problem with spatially varying porosity

POR

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A mis padres Evelyn y Juan Carlos

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"El tiempo se bifurca perpetuamente
hacia innumerables futuros."
— Jorge Luis Borges

Dentro de todas las curvas que se manifiestan en el tiempo y en el espacio, uniendo un punto con otro, existen infinitas que comparten la misma longitud. Habitualmente procuramos encontrar aquella de longitud mínima y, para desconsuelo de muchos, ese empeño suele resultar ineficiente o incluso imposible. Si algo he aprendido en estos años, es que esa búsqueda es, en verdad, inútil para la escala humana. En su lugar, es preferible elegir el camino en el que cada tropiezo pueda recibirse con el menor dolor posible y con el máximo disfrute que la vida permita. A lo largo de esas curvas, en sus inevitables bifurcaciones, uno encuentra a otras personas que recorren trayectorias radicalmente distintas y que, al observarnos con cierta ingenuidad, revelan cómo la fortuna terminó enlazando caminos improbables. A todas ellas van mis agradecimientos.

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Abstract

In this thesis, we propose and analyze a new mixed formulation for the Brinkman equations with spatially varying porosity, modeling the time-dependent flow of an incompressible fluid through heterogeneous porous media. The formulation is developed within a Banach space framework and introduces the stress and vorticity tensors as additional unknowns. This approach eliminates the pressure, which can be recovered via post-processing, yielding a stress-velocity-vorticity system. The well-posedness of the continuous problem is proved under an appropriate small-porosity assumption, by employing monotone operator techniques together with recent advances on the solvability of perturbed saddle-point problems in Banach spaces. At the discrete level, we first introduce a semidiscrete continuous-in-time scheme employing finite element spaces stable for elasticity, such as the PEERS and Arnold–Falk–Winther elements. We prove the well-posedness of this scheme and derive the corresponding *a priori* error estimates. Subsequently, a fully discrete method is obtained by applying the backward Euler scheme for the time discretization, for which we also establish well-posedness and derive optimal convergence rates with respect to the spatial and temporal discretization parameters. Under this setting, momentum is conserved provided that the porosity and the permeability tensor are piecewise constant, and that the external force is a piecewise polynomial function. Finally, several two- and three-dimensional numerical experiments, involving both manufactured and non-manufactured solutions, are presented, which confirm the theoretical convergence rates and highlight the capability of the proposed method to handle challenging geometries featuring strong contrasts in physical parameters such as permeability and porosity.

Resumen

En esta tesis, proponemos y analizamos una nueva formulación mixta para las ecuaciones de Brinkman considerando porosidad variable en el espacio, modelando el flujo evolutivo de un fluido incompresible a través de medios porosos heterogéneos. La formulación se desarrolla en el marco de espacios de Banach e introduce los tensores de esfuerzo y vorticidad como incógnitas adicionales. Este enfoque permite eliminar la presión, la cual puede ser recuperada mediante post-proceso, conduciendo así a un sistema esfuerzo-velocidad-vorticidad. Se prueba que el problema continuo está bien puesto bajo una suposición apropiada de porosidad pequeña, empleando técnicas de operadores monótonos, junto con avances recientes en la solubilidad de problemas de punto de silla perturbados en espacios de Banach. A nivel discreto, primero introducimos un esquema semidiscreto continuo en tiempo empleando elementos finitos estables para elasticidad, tales como los elementos PEERS y Arnold–Falk–Winther. Probamos que este esquema está bien puesto y deducimos las correspondientes estimaciones de error *a priori*. Luego se obtiene un método totalmente discreto aplicando el esquema de Euler regresivo para la discretización en tiempo, para lo cual también establecemos que está bien puesto y deducimos tasas de convergencia óptimas con respecto a los parámetros de discretización tanto espacial como temporal. En este esquema, el momentum se conserva si es que la porosidad y el tensor de permeabilidad son constantes por trozos y la fuerza externa es una función polinomial a trozos. Finalmente, se presentan varios ensayos numéricos en dos y tres dimensiones, involucrando soluciones manufacturadas y no manufacturadas, los cuales confirman las tasas de convergencia teóricas y destacan la capacidad del método propuesto para manejar geometrías desafiantes en conjunto con fuertes contrastes entre parámetros físicos como la permeabilidad y la porosidad.

CHAPTER 1

Introduction

1.1 Flows in porous media

Flows in porous media play a central role in many branches of applied sciences, ranging from geophysical and biological systems to diverse engineering processes. Examples include subsurface flow problems, heat and mass transfer in pipes, liquid composite molding, the behavior and influence of osteonal structures, and computational fuel cell dynamics. The mathematical modeling of such flows requires a careful balance between macroscopic effective laws and microscopic fluid dynamics, depending on the characteristics of the medium and the regime of the flow. While simplified descriptions are often sufficient in the limiting cases of very low velocities, characteristic of Darcy-type flows, or in highly permeable media, where the behavior approaches that of Stokes flows, more refined models become necessary when viscous effects within the pore structure or interactions with adjacent free-flow regions cannot be neglected. This naturally leads to the need for models that bridge these distinct descriptions.

A natural starting point is the Stokes system, one of the fundamental models for fluid behavior. In fact, it provides the mathematical description of slow, viscous, incompressible flows, where inertial effects are negligible in comparison with the viscous forces. In this regime, the motion of the fluid is essentially dominated by the balance between the viscous stress and the pressure gradients. Typical examples include flows taking place at sufficiently small length scales, where the Reynolds number is so low that inertial effects can be neglected. In general, the Stokes model constitutes the basic linear setting which serves as the starting point for the analysis of more complex fluid models. In a domain $\Omega \subset \mathrm{R}^d$, with $d \in \{2,3\}$, the unsteady Stokes equations (see, for instance, [28, Chapter 72], or [33, Chapter 5]) read as follows

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{div}(2\,\mu\,\mathbf{e}(\mathbf{u})) + \nabla p \,=\, \mathbf{f}\,, \quad \mathrm{div}(\mathbf{u}) \,=\, 0 \quad \text{in} \quad \Omega \times (0,T]\,, \tag{1.1}$$

where $\mathbf{f} : \Omega \times [0,T] \to \mathrm{R}^d$ is a body force, $\mu > 0$ is the viscosity of the fluid, $\mathbf{u}$ is the fluid velocity, p is the pressure and $\mathbf{e}(\mathbf{u})$ is the symmetric part of the gradient. This model is complemented with suitable boundary conditions and an initial condition on the velocity. The first equation in (1.1) comes from momentum balance, whereas the second equation is derived from the mass balance and is often referred to as the incompressibility condition.

On the other hand, when the fluid occupies a porous medium, the microscopic structure of the solid matrix imposes strong restrictions on its motion. In this case, the velocity field does not evolve freely, but is forced to pass through a complex network of pores whose geometry cannot be resolved at the macroscopic scale. Under suitable averaging assumptions, typically involving a homogenization limit, the resulting macroscopic model is no longer given by the Stokes system, but by Darcy's law. The Darcy model describes slow, viscous flow through a rigid porous medium, incorporating possible storage effects arising from the compressibility of the fluid or the pore structure. At the macroscopic level, the momentum balance is reduced to an algebraic relation between the velocity and the pressure gradient. More precisely, for a given permeability tensor $\mathbf{K} : \Omega \to \mathrm{R}^{d\times d}$, Darcy's model (see, for instance, [42, Appendix A.2] or [27, Chapter 51]) reads

$$\mu\,\mathbf{K}^{-1}\,\mathbf{u} + \nabla p \,=\, \mathbf{f}\,, \quad s_0\,\frac{dp}{dt} + \mathrm{div}(\mathbf{u}) = g \quad \text{in} \quad \Omega \times (0,T]\,, \tag{1.2}$$

where $s_0 > 0$ is the storage coefficient and g is a prescribed source term. Here, $\mathbf{K}$ encodes the properties of the porous matrix, including pore size, anisotropy, connectivity, and related microscopic features. The first equation in (1.2) expresses the macroscopic momentum balance in a porous medium, stating that the velocity is no longer determined by a differential operator but is instead given by an algebraic constitutive relation in which the permeability and the viscosity quantify the resistance exerted by the solid matrix. The second equation represents the mass balance, where the first term accounts for fluid storage effects, arising either from slight compressibility of the fluid or the pore structure, whereas the second term measures the net volumetric flux. Their combination must match the source term g, which models fluid injection, extraction, or internal production.

Although Darcy's law provides a macroscopic description of flow through rigid porous matrices, it becomes insufficient when viscous shear effects within the pores are not negligible. In this intermediate regime, the microscopic structure still constrains the motion of the fluid, but the averaged velocity field behaves more like a viscous continuum than in the purely Darcy case. To account for this, Brinkman [11] introduced an additional viscous diffusion term into Darcy's momentum balance, leading to the so-called Brinkman equations

$$\rho \frac{\partial \mathbf{u}}{\partial t} - \mathbf{div}(2\,\mu\,\rho\,\mathbf{e}(\mathbf{u})) + \mu\,\mathbf{K}^{-1}\,\mathbf{u} + \nabla p \,=\, \mathbf{f}\,, \quad \operatorname{div}(\rho\,\mathbf{u}) \,=\, 0 \quad \text{in} \quad \Omega \times (0,T]\,, \tag{1.3}$$

where $\rho : \Omega \to \mathrm{R}$ is the porosity distribution. The third term corresponds to the Darcy resistance, whereas the second term is the viscous term that also appears in the Stokes model. Therefore, Brinkman's formulation can be interpreted as an interpolation between Darcy and Stokes, retaining shear stresses while still incorporating the resistive effects of the solid matrix. For this reason, the Brinkman model is particularly useful in coupling problems involving fluid-porous interfaces and, more generally, in flows through heterogeneous porous structures where the fully homogenized Darcy description is insufficient but a detailed pore-scale Stokes model remains impractical. In the model (1.3), the scalar field ρ represents the local porosity of the medium, that is, the fraction of the bulk volume that is effectively available for the fluid. More precisely, if $V_{\text{pore}}(\mathbf{x})$ denotes the pore volume and $V_{\text{bulk}}(\mathbf{x})$ the total representative elementary volume around $\mathbf{x} \in \Omega$, then

$$\rho(\mathbf{x}) = \frac{V_{\text{pore}}(\mathbf{x})}{V_{\text{bulk}}(\mathbf{x})}\,.$$

Hence, regions where $\rho(\mathbf{x})$ is close to 1 correspond to highly permeable zones with a large void fraction, whereas values $\rho(\mathbf{x}) \ll 1$ indicate an almost solid matrix with very limited pore space. The dependence on ρ in the viscous term and in the divergence constraint reflects the fact that only the pore volume contributes to transport and storage of the fluid, and therefore the effective transport properties depend on the spatial distribution of porosity.

In many practical situations, the porous matrix can be regarded as homogeneous at the macroscopic scale, so that the porosity may be assumed constant in Ω. This is the case, for instance, when the pore structure is statistically uniform in space or when the characteristic variations of the material are much smaller than the resolution scale of the model. Under this assumption, the parameter ρ may be factored out of the differential operators and the resulting equations simplify considerably. This is a common choice in classical porous media models. However, several relevant applications require accounting for spatially varying porosity. Heterogeneous geological formations, functionally graded biomaterials, layered filters and composite structures are typical examples where ρ exhibits nontrivial spatial changes. In such cases, keeping $\rho = \rho(\mathbf{x})$ allows the model to capture permeability contrast, variations in storage capacity, and anisotropic flow patterns induced by the geometry of the pores. Moreover, variable porosity is essential in coupling problems with free-flow regions, since it provides a smooth transition from almost free fluid to almost solid, without the need of explicit interface tracking. Consequently, retaining a variable porosity enhances the physical fidelity of the model and increases its flexibility to represent realistic media.

The three images[1] in Fig. 1.1 illustrate different types of porous structures that commonly appear in natural and engineered materials. The left image is an AI-generated idealization of a two-dimensional granular medium, visually resembling a consolidated sand or a packed-grain structure with irregular pore connectivity. The central image corresponds to a realistic porous microstructure reproduced from [1], representing an open-cell solid with complex pore

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[1] The left and right images in Fig. 1.1 were generated by the author using the Gemini AI image model. These illustrations are used only for conceptual purposes and do not represent experimental data.

geometry typically observed in rocks, foams, and filtration media. The right image is an AI-generated three-dimensional porous solid whose morphology resembles an open-cell foam or sponge-like material, a configuration frequently found in synthetic foams, catalytic supports, and lightweight structural materials. Together, these examples illustrate the geometric richness and variability of porous media, motivating the mathematical models studied in this thesis.

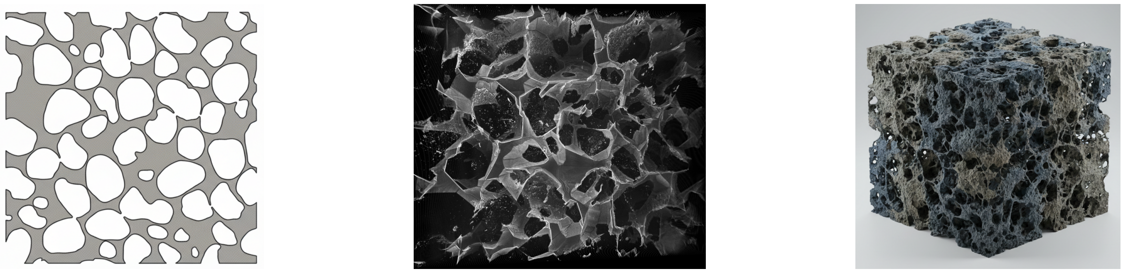


Figure 1.1: Examples of porous media: (left) AI-generated 2D granular structure, (center) realistic porous microstructure reproduced from [1], (right) AI-generated 3D open-cell foam-like structure.

1.2 Discretizations of the Brinkman equations

The mathematical analysis and numerical discretization of the Brinkman problem inherit the well-known difficulties associated with both the Darcy and the Stokes equations. A key distinction between these two models lies in the functional setting of the velocity. Namely, in the Stokes problem, velocities belong to $\mathbf{H}^1$, whereas in the Darcy case they are only in $\mathbf{H}(\mathrm{div})$. In the classical velocity-pressure formulation, this difference in regularity requires either the use of Stokes elements enriched with stabilization or penalty terms to enforce normal continuity, or the use of $\mathbf{H}(\mathrm{div})$-conforming elements with additional degrees of freedom to impose tangential continuity. These strategies allow, on the one hand, Stokes elements to capture the Darcy regime [12], and on the other hand, $\mathbf{H}(\mathrm{div})$-conforming finite elements to be extended consistently to the Stokes regime [34]. Another approach, also explored in the literature, is to employ divergence-preserving velocity reconstruction operators that map Stokes elements into an $\mathbf{H}(\mathrm{div})$-conforming space, leading naturally to weak Galerkin finite element formulations [38]. Beyond velocity-pressure formulations, alternative mixed strategies introduce additional unknowns, giving rise to pseudostress-based methods [30, 35] and vorticity-velocity-pressure schemes in both augmented and non-augmented forms [3, 6]. Some of the aforementioned approaches have also been investigated in the time-dependent setting. In particular, the companion model given by the unsteady Brinkman–Forchheimer equations has received considerable attention. This model extends the Brinkman formulation by including a nonlinear term in the velocity, which accounts for inertial effects that become relevant when the flow through the porous medium attains intermediate velocities. In [24], the authors analyze a pseudostress-velocity formulation of this problem and establish existence and uniqueness of solutions for the continuous, semidiscrete continuous-in-time, and fully discrete settings. Similar results are obtained in a velocity-vorticity-pressure formulation [5], as well as in a three-field method involving the velocity, its gradient, and the pseudostress tensor [23].

1.3 Thesis objectives

The aim of this work is to analyze the time-dependent Brinkman model under the assumption that the porosity may vary in space. Spatially varying porosity allows the Brinkman equations to capture local differences in fluid storage and resistance, variations in permeability, and a modified mass conservation law reflecting the pore volume, thereby representing the heterogeneous structure of real porous media. From a mathematical perspective, this variability introduces challenges similar to those encountered in the Stokes problem with variable density [22] or in the convective Brinkman–Forchheimer problem with variable porosity [20]. In turn, the analysis to be developed employs techniques similar to those used for the unsteady Brinkman–Forchheimer equations and related models [23, 24, 43]. To derive a mixed formulation, the stress and vorticity tensors are introduced, allowing the elimination of the pressure, which can be recovered by post-processing, and leading to a stress-velocity-vorticity formulation. This formulation is based on a Banach-space framework, providing natural flexibility to adapt the scheme to multiphysics problems. Such adaptability is particularly important and has motivated several studies on coupled problems where the Banach-space setting is essential [13, 14, 16, 19, 23, 24].

Under this framework, the techniques employed in [5, 23, 24] to establish well-posedness are no longer applicable. Indeed, while those approaches crucially rely on the monotonicity properties of the underlying operators, such monotonicity is lost in our formulation due to the introduction of the stress and vorticity variables with spatially varying porosity. To overcome this difficulty, we introduce an auxiliary problem, equivalent to the original one, but endowed with a monotone operator. This reformulation allows us, under a smallness assumption on the porosity, to prove existence and uniqueness of the continuous solution by combining classical results on monotone operators, recent advances on perturbed saddle-point problems [26], and a fixed-point strategy. To the best of the authors' knowledge, the strategy of recovering monotonicity through an auxiliary problem and establishing well-posedness via a fixed-point argument in Bochner spaces is novel, and appears to be applicable to other problems as well.

Once the solvability of the continuous problem has been established, similar arguments can be employed to derive the semidiscrete continuous-in-time and fully discrete schemes. For the spatial discretization, classical PEERS and Arnold–Falk–Winther elements are considered, while the backward Euler method is used for time stepping. Possible generalizations to other schemes are also feasible. In this setting, well-posedness of the discrete problem is established in a manner analogous to the continuous case, and an error analysis is carried out. Combined with the approximation properties of the finite element subspaces, this provides the theoretical rates of convergence in both space and time. With these choices of spatial and temporal discretizations, the fully discrete scheme inherits momentum conservation, a key feature for developing reliable numerical methods since it reflects the physical balance encoded in the continuous model.

The rest of this thesis is organized as follows. In the remainder of this chapter, we introduce the standard notation and functional spaces. In Chapter 2, we describe the model problem of interest and we focus on the derivation of the stress-velocity-vorticity mixed formulation. In Chapter 3, we establish the well-posedness of the weak mixed formulation through an auxiliary problem that is equivalent to the original one. In Chapter 4, we present a semidiscrete continuous-in-time scheme, provide particular families of stable finite element spaces, and derive error estimates for the proposed methods. Chapter 5 is devoted to the analysis of the fully discrete approximation. In Chapter 6, we present numerical examples in 2D and 3D that illustrate the theoretical results and highlight potential applications in challenging physical settings. Finally, in Chapter 7 we conclude by summarizing the strategies employed in the analysis and outlining possible directions for future work.

1.4 Preliminary notations

Let $\Omega \subset \mathrm{R}^d$, $d \in \{2,3\}$, denote a bounded domain with Lipschitz boundary Γ and let $\mathbf{n}$ be the outward unit normal vector on Γ. For $\mathrm{s} \geq 0$ and $\mathrm{p} \in [1,+\infty]$, we denote by $\mathrm{L}^\mathrm{p}(\Omega)$ and $\mathrm{W}^{\mathrm{s},\mathrm{p}}(\Omega)$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^\mathrm{p}(\Omega)}$ and $\|\cdot\|_{\mathrm{W}^{\mathrm{s},\mathrm{p}}(\Omega)}$, respectively. Note that $\mathrm{W}^{0,\mathrm{p}}(\Omega) = \mathrm{L}^\mathrm{p}(\Omega)$. If $\mathrm{p} = 2$, we write $\mathrm{H}^\mathrm{s}(\Omega)$ in place of $\mathrm{W}^{\mathrm{s},2}(\Omega)$, and denote the corresponding norm by $\|\cdot\|_{\mathrm{H}^\mathrm{s}(\Omega)}$. By $\mathbf{H}$ and $\mathbb{H}$ we will denote the corresponding vectorial and tensorial counterparts of a generic scalar functional space H. The $\mathrm{L}^2(\Omega)$ inner product for scalar, vector, or tensor valued functions is denoted by $(\cdot,\cdot)_\Omega$. The $\mathrm{L}^2(\Gamma)$ inner product or duality pairing is denoted by $\langle\cdot,\cdot\rangle_\Gamma$. Moreover, given a separable Banach space V endowed with the norm $\|\cdot\|_\mathrm{V}$, we let $\mathrm{L}^\mathrm{p}(0,T;\mathrm{V})$ be the space of classes of functions $f:(0,T)\to\mathrm{V}$ that are Bochner measurable and such that $\|f\|_{\mathrm{L}^\mathrm{p}(0,T;\mathrm{V})} < \infty$, with

$$\|f\|^\mathrm{p}_{\mathrm{L}^\mathrm{p}(0,T;\mathrm{V})} := \int_0^T \|f(t)\|^\mathrm{p}_\mathrm{V}\,dt\,, \quad \|f\|_{\mathrm{L}^\infty(0,T;\mathrm{V})} := \operatorname*{ess\,sup}_{t\in[0,T]} \|f(t)\|_\mathrm{V}\,.$$

In turn, for any vector field $\mathbf{v} := (v_i)_{i=1}^d$, we define the gradient, the symmetric part of the gradient, and the divergence operators as follows:

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,d}\,, \quad \mathbf{e}(\mathbf{v}) := \frac{1}{2}\Big(\nabla\mathbf{v} + (\nabla\mathbf{v})^\mathrm{t}\Big)\,, \quad \text{and} \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^d \frac{\partial v_j}{\partial x_j}\,.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1}^d$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$ and define the transpose, the trace, the tensor inner product and the deviatoric operator, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1}^{d}\,, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{d} \tau_{ii}\,, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{d} \tau_{ij}\,\zeta_{ij} \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{d}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}\,,$$

where $\mathbb{I}$ stands for the identity tensor. Furthermore, in the sequel, we will make use of the well-known Hölder inequality, given by

$$\int_\Omega |f\,g| \le \|f\|_{\mathrm{L}^{\mathrm{p}}(\Omega)}\,\|g\|_{\mathrm{L}^{\mathrm{q}}(\Omega)} \quad \forall\, f \in \mathrm{L}^{\mathrm{p}}(\Omega),\, \forall\, g \in \mathrm{L}^{\mathrm{q}}(\Omega), \quad \text{with} \quad \frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1\,,$$

and the Young inequality, which for all $a, b \ge 0$, $1/\mathrm{p} + 1/\mathrm{q} = 1$, and $\delta > 0$, establishes that

$$ab \,\le\, \frac{\delta^{\mathrm{p}/2}}{\mathrm{p}}\,a^{\mathrm{p}} + \frac{1}{\mathrm{q}\,\delta^{\mathrm{q}/2}}\,b^{\mathrm{q}}\,. \tag{1.4}$$

Next, for each $r \in [1, +\infty]$, we introduce the Banach space

$$\mathbb{H}(\mathbf{div}_r;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \;:\; \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^r(\Omega) \Big\}\,,$$

endowed with the natural norm

$$\|\boldsymbol{\tau}\|_{\mathbb{H}(\mathbf{div}_r;\Omega)} := \|\boldsymbol{\tau}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\mathbf{L}^r(\Omega)} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_r;\Omega)\,.$$

Additionally, we recall that, proceeding as in [29, eq. (1.43), Section 1.3.4], one can prove that for all $r \in \begin{cases} (1, +\infty] \text{ in } \mathrm{R}^2\,, \\ \left[\frac{6}{5}, +\infty\right] \text{ in } \mathrm{R}^3\,, \end{cases}$ there holds

$$\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{v} \rangle = \int_\Omega \Big\{ \boldsymbol{\tau} : \nabla\mathbf{v} + \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \Big\} \quad \forall\,(\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_r;\Omega) \times \mathbf{H}^1(\Omega)\,. \tag{1.5}$$

In addition, for all $\mathrm{p} \ge \mathrm{q}$, let $i_{\mathrm{p,q}} : \mathrm{L}^{\mathrm{p}}(\Omega) \to \mathrm{L}^{\mathrm{q}}(\Omega)$ denote the continuous inclusion, which satisfies

$$\|i_{\mathrm{p,q}}\| = |\Omega|^{(\mathrm{p}-\mathrm{q})/(\mathrm{pq})}\,, \tag{1.6}$$

and we also denote by $\mathbf{i}_{\mathrm{p,q}}$ its vector-valued counterpart, which also satisfies (1.6) if we replace

$i_{\mathrm{p},\mathrm{q}}$ by $\mathbf{i}_{\mathrm{p},\mathrm{q}}$. Finally, we recall that $\mathrm{H}^1(\Omega)$ is continuously embedded into $\mathrm{L}^{\mathrm{p}}(\Omega)$ for $\mathrm{p} \geq 1$ if $d = 2$, or $\mathrm{p} \in [1, 6]$ if $d = 3$. More precisely, we have the following inequality

$$\|w\|_{\mathrm{L}^{\mathrm{p}}(\Omega)} \leq \|i_{\mathrm{p}}\| \, \|w\|_{\mathrm{H}^1(\Omega)} \quad \forall\, w \in \mathrm{H}^1(\Omega)\,, \tag{1.7}$$

with $\|i_{\mathrm{p}}\| > 0$ depending only on $|\Omega|$ and p (see [40, Theorem 1.3.4]).

CHAPTER 2

The model problem and its mixed formulation

In this chapter, we present the model of interest and develop its mixed formulation based on the stress tensor, the velocity, and the vorticity tensor.

2.1 The model problem

Our model of interest is given by the unsteady Brinkman equations with spatially varying porosity (see, for instance, [11, 12, 20, 30]), which, as explained in Chapter 1, describes the transient flow of an incompressible fluid through a porous medium, combining viscous diffusion with a Darcy-type resistance term. The spatial variability of the porosity modifies the mass conservation law by weighting the storage of the fluid with the local pore volume, and leads to heterogeneous permeability effects. More precisely, given a porosity distribution $\rho : \Omega \to \mathrm{R}$, a body force $\mathbf{f} : \Omega \times [0, T] \to \mathrm{R}^d$, and a suitable initial datum $\mathbf{u}_0 : \Omega \to \mathrm{R}^d$, the system takes the form

$$
\begin{gathered}
\rho \frac{\partial \mathbf{u}}{\partial t} - \mathbf{div}(2\,\mu\,\rho\,\mathbf{e}(\mathbf{u})) + \mu\,\mathbf{K}^{-1}\,\mathbf{u} + \nabla p \,=\, \mathbf{f}\,, \quad \operatorname{div}(\rho\,\mathbf{u}) \,=\, 0 \quad \text{in} \quad \Omega \times (0,T]\,, \\
\mathbf{u} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma \times (0,T]\,, \quad \mathbf{u}(0) \,=\, \mathbf{u}_0 \quad \text{in} \quad \Omega\,, \quad (p,1)_\Omega \,=\, 0 \quad \text{in} \quad (0,T]\,,
\end{gathered}
\tag{2.1}
$$

where the unknowns are the velocity field $\mathbf{u}$ and the scalar pressure p. The constant $\mu > 0$ represents the viscosity, and $\mathbf{K}$ denotes a symmetric permeability tensor whose inverse belongs to $\mathbb{L}^\infty(\Omega)$. The equations are supplemented with a homogeneous Dirichlet condition, and further insights into the non-homogeneous case are provided later in Remark 3.3. The last equation in (2.1) serves to eliminate the indeterminacy in the pressure, which is commonly imposed to ensure the uniqueness of the pressure field.

Regarding the permeability tensor, we assume that $\mathbf{K}^{-1}$ is uniformly coercive. Namely, there exists a constant $C_{\mathbf{K}} > 0$ such that, for all $\mathbf{v} \in \mathrm{R}^d$,

$$
\mathbf{v} \cdot \mathbf{K}^{-1}\mathbf{v} \,\geq\, C_{\mathbf{K}}\,|\mathbf{v}|^2 \quad \text{in} \quad \Omega\,. \tag{2.2}
$$

In turn, we suppose that the porosity is positive and bounded, meaning that there exist constants ρ_0 and ρ_1 such that

$$
0 < \rho_0 \leq \rho(\mathbf{x}) \leq \rho_1 \qquad \text{a.e.} \quad \text{in} \quad \Omega\,. \tag{2.3}
$$

Let us now introduce a new stress-velocity-vorticity formulation for (2.1). To this end, we first rewrite the mass conservation equation in (2.1) as $\rho\,\operatorname{div}(\mathbf{u}) + \nabla\rho\cdot\mathbf{u} = 0$, which immediately gives

$$
\operatorname{div}(\mathbf{u}) = -\left(\frac{\nabla\rho}{\rho}\cdot\mathbf{u}\right) \quad \text{in} \quad \Omega \times (0,T]\,. \tag{2.4}
$$

Moreover, by integrating (2.4) over Ω and using the homogeneous Dirichlet condition from (2.1), we obtain the compatibility relation

$$
\left(\frac{\nabla\rho}{\rho}\cdot\mathbf{u}\,,\,1\right)_\Omega = 0 \quad \text{in} \quad (0,T]. \tag{2.5}
$$

We now define the Cauchy stress tensor $\widetilde{\boldsymbol{\sigma}}$ and the vorticity $\boldsymbol{\gamma}$ by

$$\widetilde{\boldsymbol{\sigma}} := 2\,\mu\,\rho\,\mathbf{e}(\mathbf{u}) - p\,\mathbb{I} \quad \text{and} \quad \boldsymbol{\gamma} := \frac{1}{2}\Big(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{t}}\Big)\,. \tag{2.6}$$

Taking the divergence of $\widetilde{\boldsymbol{\sigma}}$ and substituting it into the first equation of (2.1), yields

$$\rho\,\frac{\partial\mathbf{u}}{\partial t} + \mu\,\mathbf{K}^{-1}\,\mathbf{u} - \mathbf{div}(\widetilde{\boldsymbol{\sigma}}) = \mathbf{f} \quad \text{in} \quad \Omega\times(0,T]\,. \tag{2.7}$$

In turn, taking matrix trace to the stress tensor in (2.6) and using (2.4), we get

$$p = -\frac{1}{d}\Big(2\,\mu\,(\nabla\rho\cdot\mathbf{u}) + \mathrm{tr}(\widetilde{\boldsymbol{\sigma}})\Big) \quad \text{in} \quad \Omega\times(0,T]\,, \tag{2.8}$$

so replacing this into the constitutive equation of $\widetilde{\boldsymbol{\sigma}}$ (cf. (2.6)), dividing by $2\mu\rho$, and writing $\mathbf{e}(\mathbf{u}) = \nabla\mathbf{u} - \boldsymbol{\gamma}$ according to the definition of the vorticity, we obtain

$$\frac{1}{2\mu\rho}\,\widetilde{\boldsymbol{\sigma}}^{\mathrm{d}} = \nabla\mathbf{u} - \boldsymbol{\gamma} + \frac{1}{d}\left(\frac{\nabla\rho}{\rho}\cdot\mathbf{u}\right)\mathbb{I} \quad \text{in} \quad \Omega\times(0,T]\,. \tag{2.9}$$

Thus, from (2.7), (2.9), and using (2.8), we deduce that (2.1) can be equivalently rewritten as follows: Find $\widetilde{\boldsymbol{\sigma}}$, $\mathbf{u}$, and $\boldsymbol{\gamma}$, with $\widetilde{\boldsymbol{\sigma}}$ symmetric and $\boldsymbol{\gamma}$ skew-symmetric, in suitable spaces to be indicated below such that

$$\begin{aligned}
\frac{1}{2\mu\rho}\,\widetilde{\boldsymbol{\sigma}}^{\mathrm{d}} &= \nabla\mathbf{u} - \boldsymbol{\gamma} + \frac{1}{d}\left(\frac{\nabla\rho}{\rho}\cdot\mathbf{u}\right)\mathbb{I} \quad \text{in} \quad \Omega\times(0,T]\,,\\
\rho\frac{\partial\,\mathbf{u}}{\partial\,t} + \mu\,\mathbf{K}^{-1}\,\mathbf{u} - \mathbf{div}(\widetilde{\boldsymbol{\sigma}}) &= \mathbf{f} \quad \text{in} \quad \Omega\times(0,T]\,,\\
\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma\times(0,T]\,, &\quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in} \quad \Omega\,,\\
\Big(2\,\mu\,(\nabla\rho\cdot\mathbf{u}) + \mathrm{tr}(\widetilde{\boldsymbol{\sigma}})\,,\,1\Big)_{\Omega} &= 0 \quad \text{in} \quad (0,T]\,.
\end{aligned} \tag{2.10}$$

We note that the pressure has been completely eliminated from our system, and it can be recovered from ρ, $\mathbf{u}$, and $\widetilde{\boldsymbol{\sigma}}$ according to (2.8). In this context, the last equation of (2.10) is equivalent to the pressure uniqueness condition $(p,1)_\Omega = 0$ (cf. (2.1)). Moreover, it is worth noting that the first equation in (2.10) allows the gradient $\nabla\mathbf{u}$ to be recovered via post-processing from ρ, $\widetilde{\boldsymbol{\sigma}}$, $\boldsymbol{\gamma}$, and $\mathbf{u}$. Finally, enforcing the symmetry of $\widetilde{\boldsymbol{\sigma}}$ and the skew-symmetry of $\boldsymbol{\gamma}$ in (2.10) enables the vorticity to be obtained as the skew-symmetric part of $\nabla\mathbf{u}$, as defined in (2.6), which shows that, in fact, (2.10) is equivalent to (2.1).

2.2 The stress-velocity-vorticity weak formulation

In order to derive a weak formulation of (2.10) we initially consider $\mathbf{u}$ in $\mathbf{H}^1(\Omega)$, test the constitutive equation against a tensor field $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_\ell;\Omega)$, where $\ell \in (1,+\infty)$ if $d = 2$, or $\ell \in [6/5,+\infty)$ if $d = 3$, so that we are able to apply integration by parts according to (1.5) with the homogeneous Dirichlet condition, arriving at

$$\frac{1}{2\,\mu}\left(\frac{1}{\rho}\,\widetilde{\boldsymbol{\sigma}}^{\mathrm{d}},\boldsymbol{\tau}^{\mathrm{d}}\right)_\Omega + (\mathbf{u},\mathbf{div}(\boldsymbol{\tau}))_\Omega + (\boldsymbol{\gamma},\boldsymbol{\tau})_\Omega - \frac{1}{d}\left(\frac{\nabla\rho}{\rho}\cdot\mathbf{u},\mathrm{tr}(\boldsymbol{\tau})\right)_\Omega = 0\,. \tag{2.11}$$

Since the gradient of the velocity was eliminated, the above equation remains meaningful even when $\mathbf{u}$ is sought in a space larger than $\mathbf{H}^1(\Omega)$. Specifically, the second term suggests that $\mathbf{u}$ must be in $\mathbf{L}^s(\Omega)$, where s is the Hölder conjugate of ℓ, i.e. $1/s + 1/\ell = 1$. Moreover, the fourth term in (2.11) is estimated by using triple Hölder inequality and the fact that $\|\mathrm{tr}(\boldsymbol{\tau})\|_{\mathrm{L}^2(\Omega)} \le \sqrt{d}\,\|\boldsymbol{\tau}\|_{\mathbb{L}^2(\Omega)} \le \sqrt{d}\,\|\boldsymbol{\tau}\|_{\mathbb{H}(\mathbf{div}_\ell;\Omega)}$, thus obtaining

$$\left(\frac{\nabla\rho}{\rho}\cdot\mathbf{u},\mathrm{tr}(\boldsymbol{\tau})\right)_\Omega \le \sqrt{d}\,\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^r(\Omega)}\|\mathbf{u}\|_{\mathbf{L}^s(\Omega)}\,\|\boldsymbol{\tau}\|_{\mathbb{H}(\mathbf{div}_\ell;\Omega)}\,, \tag{2.12}$$

where $r := 2s/(s-2) \in (2,+\infty]$, assuming $s \ge 2$, with the convention that $r = +\infty$ if $s = 2$. Although considering $\nabla\rho/\rho \in \mathbf{L}^r(\Omega)$ with $r < +\infty$ is feasible in view of (2.12), we emphasize that in the subsequent analysis we shall repeatedly require the specific assumption $r = +\infty$. Nevertheless, we aim to preserve the generality of the Banach setting, that is, we keep $\mathbf{u} \in \mathbf{L}^s(\Omega)$ with s not necessarily equal to 2, while assuming $\nabla\rho/\rho \in \mathbf{L}^\infty(\Omega)$. At the end of this section, we provide additional comments on this assumption. Under this setting, we slightly simplify (2.12) by applying Cauchy–Schwarz and the continuous inclusion $\mathbf{i}_{s,2}$ (cf. (1.6)), thus obtaining

$$\left(\frac{\nabla\rho}{\rho}\cdot\mathbf{u},\mathrm{tr}(\boldsymbol{\tau})\right)_\Omega \le \sqrt{d}\,\|\mathbf{i}_{s,2}\|\,\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{u}\|_{\mathbf{L}^s(\Omega)}\,\|\boldsymbol{\tau}\|_{\mathbb{H}(\mathbf{div}_\ell;\Omega)}\,.$$

As a consequence of the previous discussion, the admissible ranges of s and ℓ are given by

$$s \in \begin{cases} [2,+\infty) & \text{if } d=2\,, \\ [2,6] & \text{if } d=3\,, \end{cases} \qquad \text{and} \qquad \ell = \frac{s}{s-1} \in \begin{cases} (1,2] & \text{if } d=2\,, \\ \big[6/5,2\big] & \text{if } d=3\,. \end{cases} \tag{2.13}$$

Now, returning to (2.11), we observe from the third term that it is enough to look for $\boldsymbol{\gamma}$ in $\mathbb{L}^2(\Omega)$ as $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_\ell;\Omega)$. Moreover, in order to enforce the required skew-symmetry, we further restrict $\boldsymbol{\gamma} \in \mathbb{L}^2_{\text{skew}}(\Omega)$, where

$$\mathbb{L}^2_{\text{skew}}(\Omega) := \Big\{ \boldsymbol{\eta} \in \mathbb{L}^2(\Omega) \ : \quad \boldsymbol{\eta}^{\mathrm{t}} = -\boldsymbol{\eta} \Big\}\,.$$

In turn, the symmetry of the stress tensor $\widetilde{\boldsymbol{\sigma}} \in \mathbb{L}^2(\Omega)$ is weakly enforced by

$$(\widetilde{\boldsymbol{\sigma}}, \boldsymbol{\eta})_\Omega = 0 \qquad \forall\, \boldsymbol{\eta} \in \mathbb{L}^2_{\text{skew}}(\Omega)\,. \tag{2.14}$$

Next, we test the momentum equation in (2.10) against $\mathbf{v} \in \mathbf{L}^s(\Omega)$, thereby obtaining

$$(\rho\,\partial_t\,\mathbf{u}, \mathbf{v})_\Omega + \mu\,(\mathbf{K}^{-1}\mathbf{u}, \mathbf{v})_\Omega - (\mathbf{div}(\widetilde{\boldsymbol{\sigma}}), \mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega \qquad \forall\, \mathbf{v} \in \mathbf{L}^s(\Omega)\,. \tag{2.15}$$

Regarding the first term, using (2.3) and applying Cauchy–Schwarz inequality we find that

$$(\rho\,\partial_t\,\mathbf{u}, \mathbf{v})_\Omega \le \rho_1\, \|\partial_t\,\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\, \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}\,,$$

which is finite due to the fact that $\mathbf{L}^s(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$ for all $s \ge 2$. Similarly, recalling that $\mathbf{K}^{-1} \in \mathbb{L}^\infty(\Omega)$, the second term in (2.15) is also well-defined since $\mathbf{u}, \mathbf{v} \in \mathbf{L}^s(\Omega)$ with $s \ge 2$. The third term in (2.15) forces the stress tensor $\widetilde{\boldsymbol{\sigma}}$ to belong to $\mathbb{H}(\mathbf{div}_\ell;\Omega)$. Despite the fact that the right-hand side of (2.15) is well defined under the sole assumption $\mathbf{f} \in \mathbf{L}^\ell(\Omega)$, we restrict ourselves to the smaller space $\mathbf{L}^2(\Omega)$, since this will be required in our analysis of the well-posedness of the weak formulation (cf. Theorems 3.8 and 3.9).

We now recall the decomposition $\mathbb{H}(\mathbf{div}_\ell;\Omega) = \mathbb{H}_0(\mathbf{div}_\ell;\Omega) \oplus \mathrm{R}\,\mathbb{I}$, where

$$\mathbb{H}_0(\mathbf{div}_\ell;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_\ell;\Omega) \quad : \quad (\mathrm{tr}(\boldsymbol{\tau}), 1)_\Omega = 0 \Big\}\,, \tag{2.16}$$

which means that, for all $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_\ell;\Omega)$, there exist unique components $\boldsymbol{\tau}_0 \in \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$ and $\lambda_{\boldsymbol{\tau}} \in \mathrm{R}$ such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \lambda_{\boldsymbol{\tau}}\,\mathbb{I}$. Moreover, it is easy to verify that

$$\lambda_{\boldsymbol{\tau}} = \frac{1}{d\,|\Omega|}\,(\mathrm{tr}(\boldsymbol{\tau}), 1)_\Omega\,. \tag{2.17}$$

Thus, applying this decomposition to the stress tensor, and using the last equation in (2.10) to simplify the scalar expression in (2.17), we deduce the existence of unique components $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$ and $\lambda_{\boldsymbol{\sigma}} \in \mathrm{R}$ such that

$$\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \lambda_{\boldsymbol{\sigma}}\,\mathbb{I} \quad \text{with} \quad \lambda_{\boldsymbol{\sigma}} = -\frac{2\mu}{d\,|\Omega|}\,(\nabla\rho, \mathbf{u})_\Omega\,. \tag{2.18}$$

In this regard, we notice that (2.11), (2.14) and (2.15) remain unaltered if $\widetilde{\boldsymbol{\sigma}}$ is replaced by $\boldsymbol{\sigma}$, and, hence, from now on we seek $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$ instead of $\widetilde{\boldsymbol{\sigma}}$. The original stress tensor can be recovered through post-processing via (2.18). Furthermore, using (2.5) along with the fact that $\boldsymbol{\gamma} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega)$, we notice that (2.11) trivially holds when $\boldsymbol{\tau}$ is any multiple of the identity tensor. Therefore, we may restrict ourselves to test in $\mathbb{H}_0(\mathbf{div}_\ell;\Omega)$ instead of the whole space.

In order to rewrite our system in a more suitable way for the analysis to be developed in the following chapters, we define the spaces

$$\mathbb{X} := \mathbb{H}_0(\mathbf{div}_\ell;\Omega) \quad \text{and} \quad \mathbf{Y} := \mathbf{L}^s(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)\,,$$

and set the notation

$$\underline{\mathbf{u}} := (\mathbf{u}, \boldsymbol{\gamma})\ ,\ \underline{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\eta}) \in \mathbf{Y}\,.$$

Under these definitions, it is natural to endow $\mathbf{Y}$ with the product space norm:

$$\|\underline{\mathbf{v}}\|_{\mathbf{Y}} := \|\mathbf{v}\|_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\eta}\|_{\mathbb{L}^2(\Omega)} \qquad \forall\,\underline{\mathbf{v}} \in \mathbf{Y}\,.$$

Hence, according to (2.11), (2.14) and (2.15), the weak formulation associated with (2.10) reads: Given $\mathbf{f} : [0,T] \to \mathbf{L}^2(\Omega)$ and $\mathbf{u}_0 \in \mathbf{L}^s(\Omega)$, find $(\boldsymbol{\sigma}, \underline{\mathbf{u}}) : [0,T] \to \mathbb{X} \times \mathbf{Y}$ such that $\mathbf{u}(0) = \mathbf{u}_0$ and, for a.e. $t \in (0,T)$,

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}(t)), \boldsymbol{\tau}] + [\mathbf{B}'(\underline{\mathbf{u}}(t)), \boldsymbol{\tau}] + [\mathbf{D}'_\rho(\underline{\mathbf{u}}(t)), \boldsymbol{\tau}] &= 0 & \forall\, \boldsymbol{\tau} \in \mathbb{X}\,, \\ \frac{\partial}{\partial t}[\mathbf{E}(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] - [\mathbf{B}(\boldsymbol{\sigma}(t)), \underline{\mathbf{v}}] + [\mathbf{C}(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] &= [\mathbf{F}(t), \underline{\mathbf{v}}] & \forall\, \underline{\mathbf{v}} \in \mathbf{Y}\,, \end{aligned} \tag{2.19}$$

where the operators $\mathbf{A} : \mathbb{X} \to \mathbb{X}'$, $\mathbf{B}, \mathbf{D}_\rho : \mathbb{X} \to \mathbf{Y}'$, $\mathbf{C}, \mathbf{E} : \mathbf{Y} \to \mathbf{Y}'$ are defined, respectively, as

$$[\mathbf{A}(\boldsymbol{\sigma}), \boldsymbol{\tau}] := \frac{1}{2\mu}\left(\frac{1}{\rho}\boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\tau}^{\mathrm{d}}\right)_\Omega\,, \tag{2.20a}$$

$$[\mathbf{B}(\boldsymbol{\tau}), \underline{\mathbf{v}}] := (\mathbf{v}, \mathbf{div}(\boldsymbol{\tau}))_\Omega + (\boldsymbol{\eta}, \boldsymbol{\tau})_\Omega\,, \tag{2.20b}$$

$$[\mathbf{D}_\rho(\boldsymbol{\tau}), \underline{\mathbf{v}}] := -\frac{1}{d}\left(\frac{\nabla\rho}{\rho}\cdot\mathbf{v}, \mathrm{tr}(\boldsymbol{\tau})\right)_\Omega\,, \tag{2.20c}$$

$$[\mathbf{C}(\underline{\mathbf{u}}), \underline{\mathbf{v}}] := \mu\,(\mathbf{K}^{-1}\,\mathbf{u}, \mathbf{v})_\Omega\,, \tag{2.20d}$$

$$[\mathbf{E}(\underline{\mathbf{u}}), \underline{\mathbf{v}}] := (\rho\,\mathbf{u}, \mathbf{v})_\Omega\,, \tag{2.20e}$$

and the right-hand side term $\mathbf{F} : [0,T] \to \mathbf{Y}'$ is given by

$$[\mathbf{F}(t), \underline{\mathbf{v}}] := (\mathbf{f}(t), \mathbf{v})_\Omega\,.$$

In all the terms above, $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators. Additionally, we let $\mathbf{B}' : \mathbf{Y} \to \mathbb{X}'$ be the operator defined by the relation $[\mathbf{B}'(\underline{\mathbf{v}}), \boldsymbol{\tau}] = [\mathbf{B}(\boldsymbol{\tau}), \underline{\mathbf{v}}]$ for all $(\boldsymbol{\tau}, \underline{\mathbf{v}}) \in \mathbb{X} \times \mathbf{Y}$. The operator $\mathbf{D}'_\rho : \mathbf{Y} \to \mathbb{X}'$ is defined analogously.

We conclude this chapter with additional comments on the assumptions considered herein. First, we note that the hypothesis $\nabla\rho/\rho \in \mathbf{L}^\infty(\Omega)$ is compatible both with the classical Hilbertian case $s = \ell = 2$ and with the Banach case $s, \ell \neq 2$ in (2.13). Although the more general assumption $\nabla\rho/\rho \in \mathbf{L}^r(\Omega)$ would be desirable, it cannot be accommodated within the techniques employed in this work. We refer to [22], where the authors study the stationary Stokes equations with variable density and impose a similar assumption, and to [20], which addresses the convective Brinkman–Forchheimer equations with variable porosity under a less restrictive setting, where $\nabla\rho/\rho$ is considered in $\mathbf{L}^r(\Omega)$ with $r < +\infty$. The techniques developed in the latter work, however, are not fully applicable here due to the unsteady nature of the model under consideration. In particular, the assumption $\nabla\rho/\rho \in \mathbf{L}^\infty(\Omega)$ is crucial in the proofs of Lemma 3.3 and Theorem 3.4.

On the other hand, it is also important to highlight that the use of Banach spaces rather than Hilbert spaces is motivated by the potential applicability of this work to the analysis of coupled models. For instance, the Brinkman model can be coupled with transport or heat equations. In fact, a similar stationary model is analyzed in [14], where the convective Brinkman–Forchheimer system is coupled with a nonlinear transport equation. In that setting, both the fluid velocity and the concentration of a chemical species transported by the flow are required to belong to a Lebesgue space L^p, with p necessarily greater than 2. In general, such couplings demand higher regularity of the shared unknowns, particularly when nonlinear interactions are involved. Further examples of couplings formulated in Banach space frameworks can be found in [17, 18, 31].

CHAPTER 3

Well-posedness of the model

In this chapter, we establish the solvability of (2.19). To this aim, we first collect some preliminary results that will be used in the forthcoming analysis.

3.1 Preliminary results

In what follows, a linear operator $\mathcal{A}$ from a real vector space E to its algebraic dual $E^\star$ is symmetric and monotone if, respectively,

$$[\mathcal{A}(x), y] = [\mathcal{A}(y), x] \quad \forall\, x, y \in E\,, \quad \text{and} \quad [\mathcal{A}(x), x] \geq 0 \quad \forall\, x \in E\,.$$

In addition, let us denote by $R(\mathcal{A})$ the range of $\mathcal{A}$. We also recall that the dual of a seminormed space is the space of all linear functionals that are continuous with respect to the seminorm.

The following result is a slight simplification of [42, Theorem IV.6.1(b)], whose proof is presented in Appendix A. It will play a key role in establishing the existence of a solution to (2.19).

Theorem 3.1. *Let the linear, symmetric and monotone operator $\mathcal{N}$ be given from the real vector space E to its algebraic dual $E^\star$, and let E'_* be the Hilbert space which is the topological dual of the seminormed space $(E, |\cdot|_*)$, where*

$$|x|_* = [\mathcal{N}(x), x]^{1/2} \qquad \forall\, x \in E\,. \tag{3.1}$$

Let $\mathcal{M} : E \to E'_*$ be an operator with domain $\mathcal{D} = \Big\{ x \in E \,:\, \mathcal{M}(x) \in E'_* \Big\}$. Assume that $\mathcal{M}$ is monotone and $R(\mathcal{N} + \mathcal{M}) = E'_*$. Then, for each $f \in \mathrm{W}^{1,1}(0,T; E'_*)$ and for each $u_0 \in \mathcal{D}$, there is a solution $u : [0,T] \to E$ of

$$\frac{\partial}{\partial t}\Big(\mathcal{N}(u(t))\Big) + \mathcal{M}\Big(u(t)\Big) \;=\; f(t) \quad \text{for a.e.} \quad 0 < t < T\,, \tag{3.2}$$

with

$$\mathcal{N}(u) \in \mathrm{W}^{1,\infty}(0,T; E'_*)\,, \quad u(t) \in \mathcal{D} \quad \text{for all } 0 \le t \le T\,, \quad \text{and} \quad \mathcal{N}(u(0)) = \mathcal{N}(u_0)\,.$$

One would like to write (2.19) in the form given by (3.2) and use this result to prove its well-posedness. However, it turns out that this is not possible, since the operator arising from the terms without time derivatives in (2.19) is not monotone. For this reason, we introduce an auxiliary formulation equivalent to (2.19) by defining the linear operator $\widetilde{\mathbf{B}} : \mathbb{X} \to \mathbf{Y}'$ as

$$[\widetilde{\mathbf{B}}(\boldsymbol{\tau}), \underline{\mathbf{v}}] \;:=\; [\mathbf{B}(\boldsymbol{\tau}), \underline{\mathbf{v}}] + [\mathbf{D}_\rho(\boldsymbol{\tau}), \underline{\mathbf{v}}] \qquad \forall\, \underline{\mathbf{v}} \in \mathbf{Y}\,,$$

and, for each $\boldsymbol{\zeta} \in \mathbb{X}$, we let $\widetilde{\mathbf{F}}_{\boldsymbol{\zeta}} : [0,T] \to \mathbf{Y}'$ be defined, for all $t \in [0,T]$, by

$$[\widetilde{\mathbf{F}}_{\boldsymbol{\zeta}}(t), \underline{\mathbf{v}}] \;:=\; [\mathbf{F}(t), \underline{\mathbf{v}}] - [\mathbf{D}_\rho(\boldsymbol{\zeta}), \underline{\mathbf{v}}] \qquad \forall\, \underline{\mathbf{v}} \in \mathbf{Y}\,.$$

The following result states the auxiliary problem and establishes its equivalence with (2.19). The proof is straightforward and is therefore omitted.

Lemma 3.2. *Let $\mathbf{f} : [0,T] \to \mathbf{L}^2(\Omega)$ and $\mathbf{u}_0 \in \mathbf{L}^s(\Omega)$. Then, $(\boldsymbol{\sigma}, \underline{\mathbf{u}}) : [0,T] \to \mathbb{X} \times \mathbf{Y}$ is solution* *to* (2.19) *if and only if* $\mathbf{u}(0) = \mathbf{u}_0$ *and, for a.e.* $t \in (0,T)$*,*

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}(t)), \boldsymbol{\tau}] + [\widetilde{\mathbf{B}}'(\underline{\mathbf{u}}(t)), \boldsymbol{\tau}] &= 0 & \forall\, \boldsymbol{\tau} \in \mathbb{X}\,, \\ \frac{\partial}{\partial t}\, [\mathbf{E}(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] - [\widetilde{\mathbf{B}}(\boldsymbol{\sigma}(t)), \underline{\mathbf{v}}] + [\mathbf{C}(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] &= [\widetilde{\mathbf{F}}_{\boldsymbol{\sigma}}(t), \underline{\mathbf{v}}] & \forall\, \underline{\mathbf{v}} \in \mathbf{Y}\,. \end{aligned} \tag{3.3}$$

Next, we establish stability properties of the operators involved in (2.19). In fact, employing Cauchy–Schwarz and Hölder inequalities, and the continuous inclusion $\mathbf{i}_{s,2}$ (cf. (1.6)), we find that

$$\Big|[\mathbf{A}(\boldsymbol{\sigma}), \boldsymbol{\tau}]\Big| \leq \frac{1}{2\mu\rho_0}\, \|\boldsymbol{\sigma}\|_{\mathbb{X}}\, \|\boldsymbol{\tau}\|_{\mathbb{X}}\,, \quad \Big|[\mathbf{B}(\boldsymbol{\tau}), \underline{\mathbf{v}}]\Big| \leq \|\boldsymbol{\tau}\|_{\mathbb{X}}\, \|\underline{\mathbf{v}}\|_{\mathbf{Y}}\,, \tag{3.4a}$$

$$\Big|[\mathbf{D}_\rho(\boldsymbol{\tau}), \underline{\mathbf{v}}]\Big| \leq \frac{\|\mathbf{i}_{s,2}\|}{\sqrt{d}}\, \left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)}\, \|\boldsymbol{\tau}\|_{\mathbb{X}}\, \|\underline{\mathbf{v}}\|_{\mathbf{Y}}\,, \tag{3.4b}$$

$$\Big|[\mathbf{E}(\underline{\mathbf{u}}), \underline{\mathbf{v}}]\Big| \leq \rho_1\, \|\mathbf{i}_{s,2}\|\, \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\, \|\mathbf{v}\|_{\mathbf{L}^s(\Omega)} \leq \rho_1\, \|\mathbf{i}_{s,2}\|^2\, \|\underline{\mathbf{u}}\|_{\mathbf{Y}}\, \|\underline{\mathbf{v}}\|_{\mathbf{Y}}\,, \tag{3.4c}$$

$$\Big|[\mathbf{C}(\underline{\mathbf{u}}), \underline{\mathbf{v}}]\Big| \leq \mu\, \|\mathbf{i}_{s,2}\|^2\, \|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}\, \|\underline{\mathbf{u}}\|_{\mathbf{Y}}\, \|\underline{\mathbf{v}}\|_{\mathbf{Y}}\,, \tag{3.4d}$$

$$\text{and} \quad \Big|[\mathbf{F}(t), \underline{\mathbf{v}}]\Big| \leq \|\mathbf{i}_{s,2}\|\, \|\mathbf{f}(t)\|_{\mathbf{L}^2(\Omega)}\, \|\underline{\mathbf{v}}\|_{\mathbf{Y}}\,. \tag{3.4e}$$

On the other hand, from (2.2) and (2.3), it follows that $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{E}$ are monotone. Indeed,

$$\Big|[\mathbf{A}(\boldsymbol{\tau}), \boldsymbol{\tau}]\Big| \geq \frac{1}{2\mu\rho_1}\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega)}\,, \quad \Big|[\mathbf{C}(\underline{\mathbf{v}}), \underline{\mathbf{v}}]\Big| \geq \mu\, C_{\mathbf{K}}\, \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}\,, \tag{3.5a}$$

$$\text{and} \quad \Big|[\mathbf{E}(\underline{\mathbf{v}}), \underline{\mathbf{v}}]\Big| \geq \rho_0\, \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}\,. \tag{3.5b}$$

We continue by establishing some inf-sup conditions needed for the subsequent analysis. We begin with the following condition for $\mathbf{B}$: there exists a positive constant β such that

$$\sup_{0 \neq \boldsymbol{\tau} \in \mathbb{X}} \frac{[\mathbf{B}(\boldsymbol{\tau}), \underline{\mathbf{v}}]}{\|\boldsymbol{\tau}\|_{\mathbb{X}}} \geq \beta\, \|\underline{\mathbf{v}}\|_{\mathbf{Y}} \qquad \forall\, \underline{\mathbf{v}} \in \mathbf{Y}\,. \tag{3.6}$$

The proof follows from a straightforward generalization of [32, Lemma 3.5] (see also [31, Lemma 3.4]), where the case $s = 4$ and $\ell = 4/3$ was analyzed, and is therefore omitted here. We remark that, to apply the same arguments as in these references, the continuous embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^s(\Omega)$ (cf. (1.7)) is required. Since $s \in [2,+\infty)$ for $d = 2$ and $s \in [2,6]$ for $d = 3$ (cf. (2.13)), this embedding holds in our setting.

Now, we let $\mathbb{V}$ denote the kernel of $\mathbf{B}$ (cf. (2.20b)), which is characterized by

$$\mathbb{V} := \Big\{ \boldsymbol{\tau} \in \mathbb{X} \ : \quad \mathbf{div}(\boldsymbol{\tau}) = 0 \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{t}} = \boldsymbol{\tau} \Big\}. \tag{3.7}$$

In turn, from a slight modification of [29, Lemma 2.3] (see also [16, Lemma 3.1]), there exists a positive constant c_ℓ such that

$$\|\boldsymbol{\tau}^{\mathrm{d}}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\mathbf{L}^\ell(\Omega)} \,\geq\, c_\ell \,\|\boldsymbol{\tau}\|_{\mathbb{L}^2(\Omega)} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}\,. \tag{3.8}$$

Then, for each $\boldsymbol{\tau} \in \mathbb{V}$, we have that $\|\boldsymbol{\tau}^{\mathrm{d}}\|_{\mathbb{L}^2(\Omega)} \geq c_\ell \,\|\boldsymbol{\tau}\|_{\mathbb{L}^2(\Omega)} = c_\ell \,\|\boldsymbol{\tau}\|_{\mathbb{X}}$. Consequently, the monotonicity property of $\mathbf{A}$ (cf. (3.5a)) translates into a coercivity property in $\mathbb{V}$, meaning that

$$\Big|[\mathbf{A}(\boldsymbol{\tau}), \boldsymbol{\tau}]\Big| \geq \frac{c_\ell^2}{2\mu\rho_1}\,\|\boldsymbol{\tau}\|_{\mathbb{X}}^2 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{V}\,. \tag{3.9}$$

Thus, noting that $\mathbf{A}$ and $\mathbf{C}$ are symmetric, having established (3.6) and (3.9), and bearing in mind that $\mathbf{C}$ is monotone (cf. (3.5a)), we can invoke [26, Theorem 3.4] to deduce that the following problem is well-posed: Given $(\mathcal{F},\mathcal{G}) \in \mathbb{X}' \times \mathbf{Y}'$, find $(\boldsymbol{\sigma}, \underline{\mathbf{u}}) \in \mathbb{X} \times \mathbf{Y}$ such that

$$\begin{aligned}
[\mathbf{A}(\boldsymbol{\sigma}), \boldsymbol{\tau}] + [\mathbf{B}'(\underline{\mathbf{u}}), \boldsymbol{\tau}] \;&=\; \mathcal{F}(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathbb{X}\,,\\
[\mathbf{B}(\boldsymbol{\sigma}), \underline{\mathbf{v}}] - [\mathbf{C}(\underline{\mathbf{u}}), \underline{\mathbf{v}}] \;&=\; \mathcal{G}(\underline{\mathbf{v}}) \quad \forall\, \underline{\mathbf{v}} \in \mathbf{Y}\,.
\end{aligned}$$

This means that there exists a positive constant Λ, depending only on β, c_ℓ, μ, ρ_0, ρ_1, $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$ and $|\Omega|$, such that, for all $(\boldsymbol{\zeta}, \underline{\mathbf{w}}) \in \mathbb{X} \times \mathbf{Y}$, there holds

$$\Lambda\,\|(\boldsymbol{\zeta}, \underline{\mathbf{w}})\|_{\mathbb{X}\times\mathbf{Y}} \leq \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\underline{\mathbf{v}})\in\mathbb{X}\times\mathbf{Y}} \frac{[\mathbf{A}(\boldsymbol{\zeta}), \boldsymbol{\tau}] + [\mathbf{B}'(\underline{\mathbf{w}}), \boldsymbol{\tau}] + [\mathbf{B}(\boldsymbol{\zeta}), \underline{\mathbf{v}}] - [\mathbf{C}(\underline{\mathbf{w}}), \underline{\mathbf{v}}]}{\|(\boldsymbol{\tau}, \underline{\mathbf{v}})\|_{\mathbb{X}\times\mathbf{Y}}}\,. \tag{3.10}$$

3.2 Construction of compatible initial data and stability

In this section, we begin by constructing initial data $\boldsymbol{\gamma}_0$ and $\boldsymbol{\sigma}_0$ compatible with $\mathbf{u}_0$, a necessary step to apply Theorem 3.1 in the context of (3.3). We subsequently derive a stability result for problem (2.19).

Lemma 3.3. *Assume that the initial condition $\mathbf{u}_0$ belongs to $\mathbf{L}^s(\Omega) \cap \mathbf{H}$, where*

$$\mathbf{H} := \Big\{ \mathbf{v} \in \mathbf{H}^1_0(\Omega) \quad : \quad \mathbf{div}(\rho\, \mathbf{e}(\mathbf{v})) \in \mathbf{L}^2(\Omega) \quad \text{and} \quad \operatorname{div}(\rho\, \mathbf{v}) = 0 \quad \text{in} \quad \Omega \Big\}. \tag{3.11}$$

Then there exist $\boldsymbol{\gamma}_0 \in \mathbb{L}^2_{\text{skew}}(\Omega)$ and $\boldsymbol{\sigma}_0 \in \mathbb{X}$ such that, if we set $\underline{\mathbf{u}}_0 := (\mathbf{u}_0, \boldsymbol{\gamma}_0) \in \mathbf{Y}$, there holds

$$\begin{pmatrix} \mathbf{A} & \widetilde{\mathbf{B}}' \\ -\widetilde{\mathbf{B}} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_0 \\ \underline{\mathbf{u}}_0 \end{pmatrix} \in \{\mathbf{0}\} \times \Big(\mathbf{L}^2(\Omega) \times \{\mathbf{0}\} \Big) .$$

Proof. Given $\mathbf{u}_0 \in \mathbf{L}^s(\Omega) \cap \mathbf{H}$, we define

$$\boldsymbol{\sigma}_0 := 2\mu\, \rho\, \mathbf{e}(\mathbf{u}_0) + \kappa\, \mathbb{I} \quad \text{and} \quad \boldsymbol{\gamma}_0 := \nabla \mathbf{u}_0 - \mathbf{e}(\mathbf{u}_0) \quad \text{in} \quad \Omega\,, \tag{3.12}$$

with $\kappa \in \mathrm{R}$ chosen so that $(\operatorname{tr}(\boldsymbol{\sigma}_0), 1)_\Omega = 0$. Since $\mathbf{u}_0 \in \mathbf{H}$, we have $\operatorname{div}(\rho\, \mathbf{u}_0) = 0$ in Ω, which, as in (2.4), implies

$$\operatorname{div}(\mathbf{u}_0) = -\left(\frac{\nabla \rho}{\rho} \cdot \mathbf{u}_0 \right) \quad \text{in} \quad \Omega\,. \tag{3.13}$$

Then, noting that $\operatorname{tr}(\boldsymbol{\sigma}_0) = 2\mu\, \rho \operatorname{div}(\mathbf{u}_0) + d\, \kappa$ and using (3.13), we find that κ is certainly given by

$$\kappa = \frac{2\mu}{d\,|\Omega|} \Big(\nabla \rho, \mathbf{u}_0 \Big)_\Omega .$$

Moreover, we observe that

$$\mathbf{div}(\boldsymbol{\sigma}_0) = 2\mu\, \mathbf{div}(\rho\, \mathbf{e}(\mathbf{u}_0)) \in \mathbf{L}^2(\Omega)\,,$$

so, consequently, $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}; \Omega) \subset \mathbb{X}$, where $\mathbb{H}_0(\mathbf{div}; \Omega) := \mathbb{H}_0(\mathbf{div}_2; \Omega)$. In turn, using once more the fact that $\mathbf{u}_0 \in \mathbf{H}$, we deduce $\boldsymbol{\gamma}_0 \in \mathbb{L}^2_{\text{skew}}(\Omega)$, with the skew-symmetry following directly from the definition of $\mathbf{e}(\mathbf{u}_0)$. In addition, from (3.12) we have $\frac{1}{2\mu\rho}\boldsymbol{\sigma}_0^{\mathrm{d}} = \mathbf{e}(\mathbf{u}_0)^{\mathrm{d}}$. Using this, the identity (3.13) and integrating by parts, which is valid since $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)$, we then perform straightforward algebraic manipulations to readily obtain

$$[\mathbf{A}(\boldsymbol{\sigma}_0), \boldsymbol{\tau}] + [\widetilde{\mathbf{B}}'(\underline{\mathbf{u}}_0), \boldsymbol{\tau}] = 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}\,. \tag{3.14}$$

In turn, one checks that

$$-[\widetilde{\mathbf{B}}(\boldsymbol{\sigma}_0), \underline{\mathbf{v}}] + [\mathbf{C}(\underline{\mathbf{u}}_0), \underline{\mathbf{v}}] = [\widetilde{\mathbf{G}}_0, \underline{\mathbf{v}}] \qquad \forall\, \underline{\mathbf{v}} \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)\,, \tag{3.15}$$

where $\widetilde{\mathbf{G}}_0 = (\widetilde{\mathbf{g}}_0, \mathbf{0})$, with

$$[\widetilde{\mathbf{g}}_0, \mathbf{v}] = -2\mu\Big(\mathbf{div}(\rho\,\mathbf{e}(\mathbf{u}_0)), \mathbf{v}\Big)_\Omega + \frac{2\mu}{d}\Big(\nabla\rho\cdot\mathbf{v}, \mathrm{div}(\mathbf{u}_0)\Big)_\Omega + \kappa\left(\frac{\nabla\rho}{\rho}, \mathbf{v}\right)_\Omega + \mu(\mathbf{K}^{-1}\mathbf{u}_0, \mathbf{v})_\Omega\,.$$

Thus, according to (3.14) and (3.15), we have arrived at

$$\begin{pmatrix} \mathbf{A} & \widetilde{\mathbf{B}}' \\ -\widetilde{\mathbf{B}} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_0 \\ \underline{\mathbf{u}}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \widetilde{\mathbf{G}}_0 \end{pmatrix}. \tag{3.16}$$

It remains to verify that $\widetilde{\mathbf{G}}_0$ belongs to $\mathbf{L}^2(\Omega) \times \{\mathbf{0}\}$. To this end, we apply the Cauchy–Schwarz inequality, use that $\nabla\rho/\rho \in \mathbf{L}^\infty(\Omega)$, exploit the identity (3.13), and then perform algebraic manipulations to obtain

$$\Big|[\widetilde{\mathbf{g}}_0, \mathbf{v}]\Big| \le \widetilde{C}_0 \left\{ \|\mathbf{div}(\rho\,\mathbf{e}(\mathbf{u}_0))\|_{\mathbf{L}^2(\Omega)} + \left(\left\|\frac{\nabla\rho}{\rho}\right\|^2_{\mathbf{L}^\infty(\Omega)} + \|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)} \right) \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} \right\} \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}\,, \tag{3.17}$$

with $\widetilde{C}_0 := \mu\, \max\Big\{2, 4\rho_1\, d^{-1} + 1\Big\}$. This shows that $\widetilde{\mathbf{g}}_0$ is a linear and bounded functional on $\mathbf{L}^2(\Omega)$, and, therefore, $\widetilde{\mathbf{G}}_0 \in \mathbf{L}^2(\Omega) \times \{\mathbf{0}\}$, as desired. □

Remark 3.1. *By a slight modification of the proof of Lemma 3.3, we also obtain compatible initial data for the original problem (2.19), constructed in the same way. More precisely, given*

$\mathbf{u}_0 \in \mathbf{L}^s(\Omega) \cap \mathbf{H}$, *and taking* $\boldsymbol{\sigma}_0$ *and* $\boldsymbol{\gamma}_0$ *constructed as in* (3.12), *we have*

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}' + \mathbf{D}'_\rho \\ -\mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_0 \\ \underline{\mathbf{u}}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{G}_0 \end{pmatrix}, \tag{3.18}$$

with $\mathbf{G}_0 := (\mathbf{g}_0, \mathbf{0})$, *where*

$$[\mathbf{g}_0, \mathbf{v}] := -2\mu \left(\mathbf{div}(\rho\, \mathbf{e}(\mathbf{u}_0)), \mathbf{v}\right)_\Omega + \mu \left(\mathbf{K}^{-1}\, \mathbf{u}_0, \mathbf{v}\right)_\Omega .$$

We next derive a stability result for the formulation (2.19), employing arguments that are similar in spirit to those in [24, Theorem 3.8] (see also [5, Theorem 3.9]).

Theorem 3.4. *Let* $(\boldsymbol{\sigma}, \underline{\mathbf{u}}) : [0,T] \to \mathbb{X} \times \mathbf{Y}$ *be a solution to* (2.19), *with* $\mathbf{u}(0) = \mathbf{u}_0 \in \mathbf{L}^s(\Omega) \cap \mathbf{H}$ *(cf. Lemma 3.3) and* $\mathbf{f} \in \mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))$. *Suppose that the porosity satisfies*

$$\left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \leq c_\rho\,, \quad \textit{with} \quad c_\rho := \frac{\sqrt{d}\,\Lambda}{2\,\|\mathbf{i}_{s,2}\|} \min\left\{1, \frac{\sqrt{\rho_0}}{4\rho_1}\right\}. \tag{3.19}$$

Then, $\boldsymbol{\sigma} \in \mathrm{L}^2(0,T;\mathbb{X})$, $\mathbf{u} \in \mathrm{H}^1(0,T;\mathbf{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathbf{L}^s(\Omega))$ *and* $\boldsymbol{\gamma} \in \mathrm{L}^2(0,T;\mathbb{L}^2_{\text{skew}}(\Omega))$. *In addition,* $\boldsymbol{\sigma}^{\mathrm{d}}(0) = \boldsymbol{\sigma}^{\mathrm{d}}_0$ *and* $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}_0$, *where* $\boldsymbol{\sigma}_0$ *and* $\boldsymbol{\gamma}_0$ *are given in* (3.12). *Moreover, there exists a positive constant* C_{B}, *depending only on* μ, ρ_0, ρ_1, $C_{\mathbf{K}}$, $\|\mathbf{K}^{-1}\|_{\mathbf{L}^\infty(\Omega)}$, Λ *and* $|\Omega|$, *such that*

$$\begin{aligned} &\|\boldsymbol{\sigma}\|^2_{\mathrm{L}^2(0,T;\mathbb{X})} + \|\mathbf{u}\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^s(\Omega))} + \|\mathbf{u}\|^2_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\gamma}\|^2_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} \\ &\qquad \leq C_{\mathrm{B}} \left\{ \|\mathbf{f}\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}(\mathbf{u}_0)\|^2_{\mathbb{L}^2(\Omega)} \right\}. \end{aligned} \tag{3.20}$$

Proof. Let $(\boldsymbol{\sigma}, \underline{\mathbf{u}}) : [0,T] \to \mathbb{X} \times \mathbf{Y}$ be a solution to (2.19). Then, we have the identity

$$[\mathbf{A}(\boldsymbol{\sigma}), \boldsymbol{\tau}] + [\mathbf{B}'(\underline{\mathbf{u}}), \boldsymbol{\tau}] + [\mathbf{B}(\boldsymbol{\sigma}), \underline{\mathbf{v}}] - [\mathbf{C}(\underline{\mathbf{u}}), \underline{\mathbf{v}}] = -[\mathbf{D}'_\rho(\underline{\mathbf{u}}), \boldsymbol{\tau}] + [\partial_t \mathbf{E}(\underline{\mathbf{u}}), \underline{\mathbf{v}}] - [\mathbf{F}, \underline{\mathbf{v}}]\,,$$

for all $(\boldsymbol{\tau}, \underline{\mathbf{v}}) \in \mathbb{X} \times \mathbf{Y}$. Using this into (3.10), and then applying the estimates (3.4b), (3.4c), (3.4e), and (3.19), we arrive at

$$\Lambda\, \|(\boldsymbol{\sigma}, \underline{\mathbf{u}})\|_{\mathbb{X} \times \mathbf{Y}} \leq \sup_{\mathbf{0} \neq (\boldsymbol{\tau}, \underline{\mathbf{v}}) \in \mathbb{X} \times \mathbf{Y}} \frac{[\partial_t \mathbf{E}(\underline{\mathbf{u}}), \underline{\mathbf{v}}] - [\mathbf{D}'_\rho(\underline{\mathbf{u}}), \boldsymbol{\tau}] - [\mathbf{F}, \underline{\mathbf{v}}]}{\|(\boldsymbol{\tau}, \underline{\mathbf{v}})\|_{\mathbb{X} \times \mathbf{Y}}}$$

$$\leq \rho_1 \|\mathbf{i}_{s,2}\| \, \|\partial_t \mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \frac{c_\rho}{\sqrt{d}} \|\mathbf{i}_{s,2}\| \, \|\mathbf{u}\|_{\mathbf{L}^s(\Omega)} + \|\mathbf{i}_{s,2}\| \, \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \, .$$

Now, using the fact that $c_\rho \leq \sqrt{d}\,\Lambda/(2\,\|\mathbf{i}_{s,2}\|)$, squaring and integrating over $[0, T]$, we obtain

$$\begin{aligned} C_1 \, \Big(\|\boldsymbol{\sigma}\|^2_{\mathrm{L}^2(0,T;\mathbb{X})} + \|\mathbf{u}\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^s(\Omega))} + \|\boldsymbol{\gamma}\|^2_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} \Big) \\ \leq \int_0^T \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \, dt + \rho_1^{-2} \, \|\mathbf{f}\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} \, , \end{aligned} \tag{3.21}$$

where $C_1 := \Lambda^2/(8\,\|\mathbf{i}_{s,2}\|^2 \, \rho_1^2)$.

In order to bound the integral on the right-hand side of (3.21), we differentiate in time the first equation in (2.19), test the system against $(\boldsymbol{\sigma}, \partial_t \underline{\mathbf{u}})$, and, then, after summing both equations, applying the monotonicity properties (cf. (3.5a) and (3.5b)), and using Cauchy–Schwarz and Young's inequalities (cf. (1.4)), we get

$$\begin{aligned} \partial_t \left(\frac{1}{4\mu} \left(\frac{1}{\rho} \boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\sigma}^{\mathrm{d}} \right)_\Omega + \frac{\mu}{2} \, (\mathbf{K}^{-1}\mathbf{u}, \mathbf{u})_\Omega \right) + \rho_0 \, \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} = (\mathbf{f}, \partial_t \mathbf{u})_\Omega + \frac{1}{d} \left(\frac{\nabla \rho}{\rho} \cdot \partial_t \mathbf{u}, \mathrm{tr}(\boldsymbol{\sigma}) \right)_\Omega \\ \leq \frac{1}{2\delta} \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\delta}{2} \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\tilde{\delta}}{2\sqrt{d}} \, c_\rho \, \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{c_\rho}{2\tilde{\delta}\sqrt{d}} \|\boldsymbol{\sigma}\|^2_{\mathbb{L}^2(\Omega)} \, , \end{aligned}$$

whence, by choosing $\delta = \rho_0$ and $\tilde{\delta} = \sqrt{d}\,\rho_0/(2\,c_\rho)$, it follows that

$$\partial_t \left(\frac{1}{2\mu} \left(\frac{1}{\rho} \boldsymbol{\sigma}^{\mathrm{d}}, \boldsymbol{\sigma}^{\mathrm{d}} \right)_\Omega + \mu \, (\mathbf{K}^{-1}\mathbf{u}, \mathbf{u})_\Omega \right) + \frac{\rho_0}{2} \, \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \leq \frac{1}{\rho_0} \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,c_\rho^2}{d\,\rho_0} \|\boldsymbol{\sigma}\|^2_{\mathbb{L}^2(\Omega)} \, . \tag{3.22}$$

Now, integrating from 0 to $t \in (0, T]$, and using (2.2), (2.3), together with the fact that $\mathbf{K}^{-1} \in \mathbb{L}^\infty(\Omega)$, we perform some algebraic manipulations so that the previous estimate becomes

$$\begin{aligned} \frac{1}{\mu\rho_0\rho_1} \|\boldsymbol{\sigma}^{\mathrm{d}}(t)\|^2_{\mathbb{L}^2(\Omega)} + \frac{2\mu C_{\mathbf{K}}}{\rho_0} \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|\partial_t \mathbf{u}(\mathrm{s})\|^2_{\mathbf{L}^2(\Omega)} \, d\mathrm{s} \\ \leq \frac{2}{\rho_0^2} \int_0^t \|\mathbf{f}(\mathrm{s})\|^2_{\mathbf{L}^2(\Omega)} \, d\mathrm{s} + \frac{4c_\rho^2}{d\rho_0^2} \int_0^t \|\boldsymbol{\sigma}(\mathrm{s})\|^2_{\mathbb{L}^2(\Omega)} \, d\mathrm{s} + \frac{1}{\mu\rho_0^2} \|\boldsymbol{\sigma}^{\mathrm{d}}(0)\|^2_{\mathbb{L}^2(\Omega)} \\ + \frac{2\mu}{\rho_0} \|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)} \, \|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} \, , \end{aligned} \tag{3.23}$$

for all $t \in (0, T]$. We now use that $4c_\rho^2/(d\rho_0^2) \leq C_1/2$ (cf. (3.19)), and, by a straightforward application of the definition of the Bochner norms, (3.23) implies that

$$\begin{aligned}\int_0^T \|\partial_t \mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)}\,dt \le \frac{2}{\rho_0^2}\,\|\mathbf{f}\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \frac{C_1}{2}\,\|\boldsymbol{\sigma}\|^2_{\mathrm{L}^2(0,T,\mathbb{L}^2(\Omega))} + \frac{1}{\mu\rho_0^2}\,\|\boldsymbol{\sigma}^{\mathrm{d}}(0)\|^2_{\mathbb{L}^2(\Omega)} \\ + \frac{2\,\mu}{\rho_0}\,\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}\,\|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} - \frac{2\,\mu\,C_{\mathbf{K}}}{\rho_0}\,\|\mathbf{u}\|^2_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))}\,,\end{aligned} \tag{3.24}$$

which proves that $\partial_t \mathbf{u} \in \mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))$. We have neglected the first term in (3.23) in order to simplify the estimate.

Next, to bound the term $\|\boldsymbol{\sigma}^{\mathrm{d}}(0)\|^2_{\mathbb{L}^2(\Omega)}$ in (3.24), we first notice that the first equation in (2.19) implies that the left-hand side, as a function from $[0,T]$ to R, belongs to the same $\mathrm{L}^\infty(0,T)$-class as the null function. Consequently, the left-hand side can be viewed as a continuous function in time and we can let $t \to 0^+$ in the first equation of (2.19), use the fact that $\mathbf{u}(0) = \mathbf{u}_0$, and subtract it from the first row of (3.18) with $\boldsymbol{\sigma}_0$ and $\boldsymbol{\gamma}_0$ constructed as in (3.12), thus obtaining

$$[\mathbf{A}(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}(0)), \boldsymbol{\tau}] + (\boldsymbol{\gamma}_0 - \boldsymbol{\gamma}(0), \boldsymbol{\tau})_\Omega = 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}\,. \tag{3.25}$$

In turn, by testing the second equation in (2.19) with $\underline{\mathbf{v}} = (\mathbf{0}, \boldsymbol{\eta}(t))$ and letting $t \to 0^+$, which is valid by the same reasoning mentioned above, we obtain that $\boldsymbol{\sigma}(0)$ is weakly symmetric. Since $\boldsymbol{\sigma}_0$ is also symmetric (cf. (3.12)), it follows that $\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}(0)$ is weakly symmetric. Then, testing (3.25) with $\boldsymbol{\tau} = \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}(0)$, and observing that the second term vanishes by weak symmetry, we get

$$[\mathbf{A}(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}(0)), \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}(0)] = 0\,.$$

In view of the monotonicity of $\mathbf{A}$ (cf. (3.5a)), this implies $\boldsymbol{\sigma}^{\mathrm{d}}(0) = \boldsymbol{\sigma}_0^{\mathrm{d}}$. As a consequence, using the inf-sup condition (3.6) together with (3.25), we obtain directly that $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}_0$. Moreover, from (3.12) and (2.3), we also deduce that

$$\|\boldsymbol{\sigma}^{\mathrm{d}}(0)\|_{\mathbb{L}^2(\Omega)} = \|\boldsymbol{\sigma}_0^{\mathrm{d}}\|_{\mathbb{L}^2(\Omega)} \le 2\mu\,\rho_1\,\|\mathbf{e}(\mathbf{u}_0)\|_{\mathbb{L}^2(\Omega)}\,.$$

Finally, replacing this into (3.24), and combining it with (3.21), gives (3.20), with constant

$$C_{\mathrm{B}} := \frac{2\,\max\left\{\rho_0^2 + 2\rho_1^2\,,\, 4\mu\,\rho_0^2\,\rho_1^2\,,\, 2\mu\,\rho_0\,\rho_1^2\,\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}\right\}}{\min\left\{C_1\,\rho_0^2\,\rho_1^2\,,\, 4\mu\,C_{\mathbf{K}}\,\rho_0\,\rho_1^2\right\}}\,.$$

□

3.3 A fixed-point strategy

In order to establish the well-posedness of (2.19), we shall prove that, under certain conditions on the porosity, the problem (3.3) has a unique solution. To that end, we propose a fixed-point strategy. Let us introduce the operator $\mathcal{J} : \mathrm{L}^2(0,T;\mathbb{X}) \to \mathrm{L}^2(0,T;\mathbb{X})$ as

$$\mathcal{J}(\boldsymbol{\zeta}) := \boldsymbol{\sigma} \qquad \forall\,\boldsymbol{\zeta} \in \mathrm{L}^2(0,T;\mathbb{X})\,,$$

where $(\boldsymbol{\sigma}, \underline{\mathbf{u}})$ is the unique solution (to be confirmed below) to

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}(t)),\boldsymbol{\tau}] + [\widetilde{\mathbf{B}}'(\underline{\mathbf{u}}(t)),\boldsymbol{\tau}] &= 0 && \forall\,\boldsymbol{\tau} \in \mathbb{X}\,, \\ \frac{\partial}{\partial t}\,[\mathbf{E}(\underline{\mathbf{u}}(t)),\underline{\mathbf{v}}] - [\widetilde{\mathbf{B}}(\boldsymbol{\sigma}(t)),\underline{\mathbf{v}}] + [\mathbf{C}(\underline{\mathbf{u}}(t)),\underline{\mathbf{v}}] &= [\widetilde{\mathbf{F}}_{\boldsymbol{\zeta}}(t),\underline{\mathbf{v}}] && \forall\,\underline{\mathbf{v}} \in \mathbf{Y}\,, \end{aligned} \tag{3.26}$$

for a.e. $t \in (0,T)$ and $\mathbf{u}(0) = \mathbf{u}_0$. We stress here that the operator $\mathcal{J}$ is naturally defined on $\mathrm{L}^2(0,T;\mathbb{X})$, as suggested by the stability result (cf. Theorem 3.4). Notice also that solving (3.3) is equivalent to finding a solution to the fixed-point equation

$$\mathcal{J}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}\,. \tag{3.27}$$

Now, to show that $\mathcal{J}$ is well-defined, we shall prove that (3.26) admits a unique solution by employing Theorem 3.1. For this purpose, we observe that (3.26) can be written in the form of (3.2) with

$$E = \mathbb{X} \times \mathbf{Y}\,, \quad u = (\boldsymbol{\sigma}, \underline{\mathbf{u}})\,, \quad \mathcal{N} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{pmatrix} \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} \mathbf{A} & \widetilde{\mathbf{B}}' \\ -\widetilde{\mathbf{B}} & \mathbf{C} \end{pmatrix}. \tag{3.28}$$

Let E'_* denote the Hilbert space defined as the dual of $(\mathbb{X} \times \mathbf{Y}, |\cdot|_*)$, where $|\cdot|_*$ is the seminorm induced by $\mathbf{E}$ (cf. (3.1)), and is given by

$$|(\boldsymbol{\tau}, \underline{\mathbf{v}})|_* = (\rho\, \mathbf{v}, \mathbf{v})_\Omega^{1/2} \qquad \forall\, (\boldsymbol{\tau}, \underline{\mathbf{v}}) \in \mathbb{X} \times \mathbf{Y}\,.$$

Since ρ is positive and bounded (cf. (2.3)), it is straightforward to show that E'_* is isomorphic to $\{\mathbf{0}\} \times \Big(\mathbf{L}^2(\Omega) \times \{\mathbf{0}\}\Big)$. Accordingly, we are able to define the spaces

$$E'_* := \{\mathbf{0}\} \times \Big(\mathbf{L}^2(\Omega) \times \{\mathbf{0}\}\Big) \quad \text{and} \quad \mathcal{D} := \Big\{(\boldsymbol{\tau}, \underline{\mathbf{v}}) \in \mathbb{X} \times \mathbf{Y} :\ \mathcal{M}(\boldsymbol{\tau}, \underline{\mathbf{v}}) \in E'_*\Big\}\,.$$

Notice that the range condition in Theorem 3.1 is equivalent to prove the existence of a solution to the following resolvent system: Find $(\boldsymbol{\sigma}, \underline{\mathbf{u}}) \in \mathbb{X} \times \mathbf{Y}$ such that

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}), \boldsymbol{\tau}] + [\widetilde{\mathbf{B}}'(\underline{\mathbf{u}}), \boldsymbol{\tau}] &= 0 && \forall\, \boldsymbol{\tau} \in \mathbb{X}\,, \\ [\widetilde{\mathbf{B}}(\boldsymbol{\sigma}), \underline{\mathbf{v}}] - [(\mathbf{E} + \mathbf{C})(\underline{\mathbf{u}}), \underline{\mathbf{v}}] &= -(\widehat{\mathbf{f}}, \mathbf{v})_\Omega && \forall\, \underline{\mathbf{v}} \in \mathbf{Y}\,, \end{aligned} \tag{3.29}$$

where $\widehat{\mathbf{f}} \in \mathbf{L}^2(\Omega)$ so that $(\mathbf{0}, (\widehat{\mathbf{f}}, \mathbf{0}))$ represents an arbitrary element of E'_*. In a similar way to how we proved the global inf-sup condition (3.10), we shall invoke [26, Theorem 3.4] to establish the well-posedness of (3.29). In this way, we now focus on verifying the hypotheses of this theorem, starting with the inf-sup condition of the operator $\widetilde{\mathbf{B}}$. Notice that, in order to relax the assumption on the datum ρ, in the following two intermediate lemmas we can suppose $\nabla\rho/\rho \in \mathbf{L}^r(\Omega)$ with $r = 2s/(s-2)$ as in (2.12).

Lemma 3.5. *Assume that the porosity satisfies*

$$\left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^r(\Omega)} \leq \frac{\sqrt{d}\,\beta}{2}\,, \tag{3.30}$$

with β as given in (3.6). Then, the following inf-sup condition holds:

$$\sup_{0\neq\boldsymbol{\tau}\in\mathbb{X}} \frac{[\widetilde{\mathbf{B}}(\boldsymbol{\tau}),\underline{\mathbf{v}}]}{\|\boldsymbol{\tau}\|_{\mathbb{X}}} \geq \frac{\beta}{2}\,\|\underline{\mathbf{v}}\|_{\mathbf{Y}} \qquad \forall\,\underline{\mathbf{v}}\in\mathbf{Y}\,.$$

Proof. Since $\mathbf{L}^\infty(\Omega)\hookrightarrow\mathbf{L}^r(\Omega)$, we can use (2.12) to bound $\mathbf{D}_\rho$, and then employ the assumption (3.30) along with the inf-sup condition of $\mathbf{B}$ (cf. (3.6)), to obtain

$$\sup_{0\neq\boldsymbol{\tau}\in\mathbb{X}} \frac{[\widetilde{\mathbf{B}}(\boldsymbol{\tau}),\underline{\mathbf{v}}]}{\|\boldsymbol{\tau}\|_{\mathbb{X}}} \geq \sup_{0\neq\boldsymbol{\tau}\in\mathbb{X}} \frac{[\mathbf{B}(\boldsymbol{\tau}),\underline{\mathbf{v}}]}{\|\boldsymbol{\tau}\|_{\mathbb{X}}} - \frac{1}{\sqrt{d}}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^r(\Omega)}\|\underline{\mathbf{v}}\|_{\mathbf{Y}} \geq \frac{\beta}{2}\,\|\underline{\mathbf{v}}\|_{\mathbf{Y}}\,.$$

□

Let $\widetilde{\mathbb{V}}$ be the kernel of the operator $\widetilde{\mathbf{B}}$, which, by standard duality and orthogonality arguments, can be characterized as

$$\widetilde{\mathbb{V}} = \left\{\boldsymbol{\tau}\in\mathbb{X} \;:\; \mathbf{div}(\boldsymbol{\tau}) = \frac{1}{d}\,\frac{\nabla\rho}{\rho}\,\mathrm{tr}(\boldsymbol{\tau}) \quad\text{and}\quad \boldsymbol{\tau}^{\mathrm{t}}=\boldsymbol{\tau}\right\}. \tag{3.31}$$

Employing this subspace, we now establish an inf-sup condition for $\mathbf{A}$.

Lemma 3.6. *Let c_ℓ be as in (3.8), and assume that the porosity satisfies*

$$\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^r(\Omega)} \leq \frac{\sqrt{d}\,c_\ell}{2}\,. \tag{3.32}$$

Then, there exists a positive constant α, depending only on μ, ρ_1 and c_ℓ, such that

$$\sup_{0\neq\boldsymbol{\tau}\in\widetilde{\mathbb{V}}} \frac{[\mathbf{A}(\boldsymbol{\sigma}),\boldsymbol{\tau}]}{\|\boldsymbol{\tau}\|_{\mathbb{X}}} \geq \alpha\,\|\boldsymbol{\sigma}\|_{\mathbb{X}} \qquad \forall\,\boldsymbol{\sigma}\in\widetilde{\mathbb{V}}\,.$$

Proof. Given $\boldsymbol{\sigma}\in\widetilde{\mathbb{V}}$, we have $\mathbf{div}(\boldsymbol{\sigma}) = \frac{1}{d}(\nabla\rho/\rho)\,\mathrm{tr}(\boldsymbol{\sigma})$ (cf. (3.31)). From this identity, we apply Hölder's inequality, use (3.32) and the fact that $\|\mathrm{tr}(\boldsymbol{\sigma})\|_{\mathrm{L}^2(\Omega)} \leq \sqrt{d}\,\|\boldsymbol{\sigma}\|_{\mathbb{L}^2(\Omega)}$, to get

$$\|\mathbf{div}(\boldsymbol{\sigma})\|_{\mathbf{L}^\ell(\Omega)} \leq \frac{1}{d}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^r(\Omega)}\|\mathrm{tr}(\boldsymbol{\sigma})\|_{\mathrm{L}^2(\Omega)} \leq \frac{c_\ell}{2}\,\|\boldsymbol{\sigma}\|_{\mathbb{L}^2(\Omega)}\,. \tag{3.33}$$

Combining this estimate with (3.8) gives $\frac{c_\ell}{2}\,\|\boldsymbol{\sigma}\|_{\mathbb{L}^2(\Omega)} \leq \|\boldsymbol{\sigma}^{\mathrm{d}}\|_{\mathbb{L}^2(\Omega)}$. Adding to both sides

$\frac{1}{2}\,\|\mathbf{div}(\boldsymbol{\sigma})\|_{\mathbf{L}^{\ell}(\Omega)}$ and using again (3.33), we obtain

$$\frac{c_\ell}{2}\,\|\boldsymbol{\sigma}\|_{\mathbb{L}^2(\Omega)}+\frac{1}{2}\,\|\mathbf{div}(\boldsymbol{\sigma})\|_{\mathbf{L}^{\ell}(\Omega)}\leq\|\boldsymbol{\sigma}^{\mathrm{d}}\|_{\mathbb{L}^2(\Omega)}+\frac{c_\ell}{4}\,\|\boldsymbol{\sigma}\|_{\mathbb{L}^2(\Omega)}\,.$$

Consequently,

$$\frac{1}{4}\,\min\{c_\ell,2\}\,\|\boldsymbol{\sigma}\|_{\mathbb{X}}\leq\|\boldsymbol{\sigma}^{\mathrm{d}}\|_{\mathbb{L}^2(\Omega)}\,.$$

Then, by using the boundedness of ρ (cf. (2.3)) together with the above estimate, we arrive at

$$\sup_{0\neq\boldsymbol{\tau}\in\widetilde{\mathbb{V}}}\frac{[\mathbf{A}(\boldsymbol{\sigma}),\boldsymbol{\tau}]}{\|\boldsymbol{\tau}\|_{\mathbb{X}}}\geq\frac{\|\boldsymbol{\sigma}^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega)}}{2\mu\rho_1\,\|\boldsymbol{\sigma}\|_{\mathbb{X}}}\geq\frac{\min\{c_\ell^2,4\}}{32\mu\rho_1}\,\|\boldsymbol{\sigma}\|_{\mathbb{X}}\,,$$

which completes the proof with $\alpha=\min\{c_\ell^2,4\}/(32\mu\rho_1)$. □

Lemma 3.7. *Suppose that the porosity satisfies* (3.30) *and* (3.32). *Then, for all* $\widehat{\mathbf{f}}\in\mathbf{L}^2(\Omega)$, *there exists a unique solution to* (3.29).

Proof. It is clear from the definition of $\mathbf{A}$ and $\mathbf{E}+\mathbf{C}$ (cf. (2.20a), (2.20d) and (2.20e)) that they induce symmetric bilinear forms. Furthermore, from the monotonicity of $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{C}$ (cf. (3.5a) and (3.5b)), we have

$$[\mathbf{A}(\boldsymbol{\tau}),\boldsymbol{\tau}]\geq\frac{1}{2\mu\rho_1}\,\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega)}\geq 0\quad\text{and}\quad[(\mathbf{E}+\mathbf{C})(\underline{\mathbf{v}}),\underline{\mathbf{v}}]\geq(\rho_0+\mu\,C_{\mathbf{K}})\,\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}\geq 0\,,$$

for all $\boldsymbol{\tau}\in\mathbb{X}$ and $\underline{\mathbf{v}}\in\mathbf{Y}$, which means that $\mathbf{A}$ and $\mathbf{E}+\mathbf{C}$ induce positive semi-definite bilinear forms. On the other hand, by Lemmas 3.5 and 3.6, we also have the inf-sup conditions required by [26, Theorem 3.4]. Thus, applying this result in our context, we conclude that, for each $\widehat{\mathbf{f}}\in\mathbf{L}^2(\Omega)$, (3.29) is well-posed. □

With this result at hand, we are in a position to prove the well-posedness of (3.26), and hence that $\mathscr{J}$ is well-defined.

Theorem 3.8. *Suppose that the porosity* ρ *satisfies* (3.19), (3.30) *and* (3.32). *Furthermore, let* $\mathbf{f}\in\mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega))\cap\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0\in\mathbf{L}^s(\Omega)\cap\mathbf{H}$ *(cf.* (3.11)*) be given. Then, for each* $\boldsymbol{\zeta}\in\mathrm{L}^2(0,T;\mathbb{X})$, *satisfying in addition that* $\boldsymbol{\zeta}\in\mathrm{W}^{1,1}(0,T;\mathbb{L}^2(\Omega))$, $\mathscr{J}(\boldsymbol{\zeta})$ *is well-defined.*

More precisely, there exists a unique solution $(\boldsymbol{\sigma}, \underline{\mathbf{u}})$ *to* (3.26) *such that* $\boldsymbol{\sigma} \in \mathrm{L}^2(0,T;\mathbb{X})$, $\mathbf{u} \in \mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathbf{L}^s(\Omega))$, $\boldsymbol{\gamma} \in \mathrm{L}^2(0,T;\mathbb{L}^2_{\mathrm{skew}}(\Omega))$, $\mathbf{u}(0) = \mathbf{u}_0$, $\boldsymbol{\sigma}^{\mathrm{d}}(0) = \boldsymbol{\sigma}_0^{\mathrm{d}}$, $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}_0$ *(cf.* (3.12)*), and* $\mathcal{J}(\boldsymbol{\zeta}) = \boldsymbol{\sigma}$*. Moreover, there exist positive constants* $\widetilde{C}_{\mathrm{B}}$ *and* $C_{\mathcal{J}}$*, with* $\widetilde{C}_{\mathrm{B}}$ *depending only on* μ, ρ_0, ρ_1, $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$, Λ, d *and* $|\Omega|$*, and* $C_{\mathcal{J}}$ *depending only on* ρ_0, ρ_1, Λ, d *and* $|\Omega|$*, such that*

$$\begin{aligned} \|\mathcal{J}(\boldsymbol{\zeta})\|_{\mathrm{L}^2(0,T;\mathbb{X})} \leq \widetilde{C}_{\mathrm{B}} \Big\{ \|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}(\mathbf{u}_0)\|_{\mathbb{L}^2(\Omega)} \Big\} \\ + C_{\mathcal{J}} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\boldsymbol{\zeta}\|_{\mathrm{L}^2(0,T;\mathbb{X})} . \end{aligned} \tag{3.34}$$

Proof. Recalling the notation introduced in (3.28), we note that $\mathcal{N}$ is linear, symmetric, and monotone, which follow directly from the properties of $\mathbf{E}$. Similarly, $\mathcal{M}$ is monotone since both $\mathbf{A}$ and $\mathbf{C}$ are monotone. Moreover, by Lemma 3.7, for all $(\mathbf{0}, (\widehat{\mathbf{f}}, \mathbf{0})) \in E'_*$, there exists a unique $(\boldsymbol{\sigma}, \underline{\mathbf{u}}) \in \mathbb{X} \times \mathbf{Y}$ such that $(\mathcal{N} + \mathcal{M})(\boldsymbol{\sigma}, \underline{\mathbf{u}}) = (\mathbf{0}, (\widehat{\mathbf{f}}, \mathbf{0}))$. This implies that $E'_* = R(\mathcal{N} + \mathcal{M})$. Furthermore, owing to the fact that $\mathbf{u}_0 \in \mathbf{L}^s(\Omega) \cap \mathbf{H}$, Lemma 3.3 ensures the existence of compatible initial data such that $(\boldsymbol{\sigma}_0, \underline{\mathbf{u}}_0) \in \mathcal{D}$. In turn, since $\mathbf{f} \in \mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega))$ and $\boldsymbol{\zeta} \in \mathrm{W}^{1,1}(0,T;\mathbb{L}^2(\Omega))$, we have that $\widetilde{\mathbf{F}}_{\boldsymbol{\zeta}} \in \mathrm{W}^{1,1}(0,T;E'_*)$. Thus, by Theorem 3.1, we deduce the existence of a solution $(\boldsymbol{\sigma}, \underline{\mathbf{u}})$ to (3.26), where $\mathbf{u} \in \mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega))$, $\mathcal{M}(\boldsymbol{\sigma}(t), \underline{\mathbf{u}}(t)) \in E'_*$, and $\mathbf{u}(0) = \mathbf{u}_0$. Moreover, by an argument similar to that in the proof of Theorem 3.4, we obtain $\boldsymbol{\sigma}^{\mathrm{d}}(0) = \boldsymbol{\sigma}_0^{\mathrm{d}}$, $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}_0$, and the desired regularity of the solution. In particular, proceeding as in (3.20), we get the following estimate for $\boldsymbol{\sigma}$:

$$\|\boldsymbol{\sigma}\|^2_{\mathrm{L}^2(0,T;\mathbb{X})} \leq C_1 \Big\{ \|\mathbf{f}\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}(\mathbf{u}_0)\|^2_{\mathbb{L}^2(\Omega)} \Big\} + C_2 \left\| \frac{\nabla \rho}{\rho} \right\|^2_{\mathbf{L}^\infty(\Omega)} \|\boldsymbol{\zeta}\|^2_{\mathrm{L}^2(0,T;\mathbb{X})} , \tag{3.35}$$

where

$$C_1 := \frac{16 \max \Big\{ \rho_0^2 + 2\rho_1^2 \,,\, 4\mu\, \rho_0^2\, \rho_1^2 \,,\, 2\mu\, \rho_0\, \rho_1^2\, \|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)} \Big\}}{\Lambda^2\, \rho_0^2\, \|\mathbf{i}_{s,2}\|^{-2}} \quad \text{and} \quad C_2 := \frac{16\, (\rho_0^2 + 4\rho_1^2)\, d^{-1}}{\Lambda^2\, \rho_0^2\, \|\mathbf{i}_{s,2}\|^{-2}} .$$

Then, by taking the square root in (3.35), some algebraic manipulations yield (3.34). To prove uniqueness of the solution, by linearity of the problem, it suffices to show that (3.26) admits only the trivial solution with the data $\mathbf{f} = \mathbf{0}$, $\mathbf{u}_0 = \mathbf{0}$, and $\boldsymbol{\zeta} = \mathbf{0}$. Certainly, one can establish estimates for $\mathbf{u}$ and $\boldsymbol{\gamma}$ similar to (3.35), once again relying on the arguments employed in the proof of Theorem 3.4. In this way, from (3.35), it turns out that if the data vanish, then the solution must be trivial. Therefore, (3.26) admits a unique solution, which implies that $\mathcal{J}$ is well-defined. This completes the proof. □

Remark 3.2. *Although the construction of the operator $\mathcal{J}$ requires the additional assumption $\boldsymbol{\zeta}$ in $\mathrm{W}^{1,1}(0,T;\mathbb{L}^2(\Omega))$, the fixed-point argument is carried out in the Banach space $\mathrm{L}^2(0,T;\mathbb{X})$. Once the fixed point is obtained, the time regularity of the solution follows by differentiating the system (3.26) with respect to time and proceeding exactly as in the proof of the stability result (cf. Theorem 3.4), thus obtaining $\partial_t\boldsymbol{\sigma} \in \mathrm{L}^2(0,T;\mathbb{X})$, which gives $\boldsymbol{\sigma} \in \mathrm{H}^1(0,T;\mathbb{X})$. In particular, $\boldsymbol{\sigma} \in \mathrm{W}^{1,1}(0,T;\mathbb{L}^2(\Omega))$ and the previous result can therefore be applied once again. This observation is fully consistent with the time regularity imposed on $\boldsymbol{\zeta}$ in Theorem 3.8. For the sake of brevity, we omit further details. We only note that, analogously to (3.23), the resulting estimates involve the initial conditions for the time derivatives of $\boldsymbol{\sigma}^{\mathrm{d}}$ and $\mathbf{u}$. Henceforth, we assume that $\|\partial_t\boldsymbol{\sigma}^{\mathrm{d}}(0)\|_{\mathbb{L}^2(\Omega)}$ and $\|\partial_t\mathbf{u}(0)\|_{\mathbf{L}^2(\Omega)}$ are finite.*

We finally are able to prove that the fixed-point equation (3.27) has a unique solution under certain assumptions on the porosity.

Theorem 3.9. *Suppose that the porosity ρ satisfies (3.19), (3.30) and (3.32). Furthermore, assume that*

$$C_{\mathcal{J}} \left\| \frac{\nabla\rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} < 1\,. \tag{3.36}$$

Then, given $\mathbf{f} \in \mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{u}_0 \in \mathbf{L}^s(\Omega) \cap \mathbf{H}$ (cf. (3.11)), there exists a unique $(\boldsymbol{\sigma}, \underline{\mathbf{u}})$ solution to (2.19) with $\mathbf{u}(0) = \mathbf{u}_0$, $\boldsymbol{\sigma}^{\mathrm{d}}(0) = \boldsymbol{\sigma}_0^{\mathrm{d}}$, $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}_0$ (cf. (3.12)), and $\mathcal{J}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}$. Moreover, $\boldsymbol{\sigma} \in \mathrm{L}^2(0,T;\mathbb{X})$, $\mathbf{u} \in \mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathbf{L}^s(\Omega))$ and $\boldsymbol{\gamma} \in \mathrm{L}^2(0,T;\mathbb{L}^2_{\mathrm{skew}}(\Omega))$, all of them satisfying (3.20) in Theorem 3.4.

Proof. We first notice that it is enough to prove that the fixed-point equation (3.27) has a unique solution. Once this is established, Lemma 3.2 ensures that (2.19) admits a unique solution, whose regularity then follows from Theorem 3.4. We therefore focus on the former. In this regard, by Theorem 3.8 the operator $\mathcal{J}$ is well-defined. We now show that $\mathcal{J}$ is Lipschitz continuous. Given $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \in \mathrm{L}^2(0,T;\mathbb{X})$, since (3.26) is linear, we have that $\mathcal{J}(\boldsymbol{\zeta}_1) - \mathcal{J}(\boldsymbol{\zeta}_2)$ corresponds to the unique solution to (3.26) with $\mathbf{f} = \mathbf{0}$, $\mathbf{u}_0 = \mathbf{0}$ and $\boldsymbol{\zeta} = \boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2$. Thus, applying the estimate (3.34), we obtain

$$\|\mathcal{J}(\boldsymbol{\zeta}_1) - \mathcal{J}(\boldsymbol{\zeta}_2)\|_{\mathrm{L}^2(0,T;\mathbb{X})} \leq C_{\mathcal{J}} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2\|_{\mathrm{L}^2(0,T;\mathbb{X})} ,$$

which implies that $\mathcal{J}$ is Lipschitz continuous. Moreover, using (3.36), $\mathcal{J}$ is a contractive operator on the Banach space $\mathrm{L}^2(0,T;\mathbb{X})$, so that the Banach fixed-point theorem ensures that $\mathcal{J}$ admits a unique fixed-point. Hence, (3.27) has a unique solution, as desired. $\square$

Remark 3.3. *It is worth noting that our analysis can be readily extended to the case of a non-homogeneous Dirichlet boundary condition in* (2.10)*, in a manner similar to [43, Section 2] (see also [21, Theorem 4.10]). Specifically, if we prescribe* $\mathbf{u} = \mathbf{u}_{\mathrm{D}}$ *on* $\Gamma \times (0,T]$ *for some time-dependent Dirichlet datum* $\mathbf{u}_{\mathrm{D}} : [0,T] \to \mathbf{H}^{1/2}(\Gamma)$*, a solution can be constructed as follows. Let* $(\boldsymbol{\sigma}_\Gamma, \underline{\mathbf{u}}_\Gamma) \in \mathbb{X} \times \mathbf{Y}$ *solve the problem*

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}_\Gamma(t)), \boldsymbol{\tau}] + [\mathbf{B}'(\underline{\mathbf{u}}_\Gamma(t)), \boldsymbol{\tau}] + [\mathbf{D}'_\rho(\underline{\mathbf{u}}_\Gamma(t)), \boldsymbol{\tau}] &= \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_{\mathrm{D}}(t)\rangle_\Gamma \quad \forall\, \boldsymbol{\tau} \in \mathbb{X} , \\ [\mathbf{E}(\underline{\mathbf{u}}_\Gamma(t)), \underline{\mathbf{v}}] - [\mathbf{B}(\boldsymbol{\sigma}_\Gamma(t)), \underline{\mathbf{v}}] + [\mathbf{C}(\underline{\mathbf{u}}_\Gamma(t)), \underline{\mathbf{v}}] &= [\mathbf{F}(t), \underline{\mathbf{v}}] \qquad\quad \forall\, \underline{\mathbf{v}} \in \mathbf{Y} , \end{aligned} \tag{3.37}$$

for all $t \in (0,T]$*. Notice that for each fixed* $t \in (0,T]$*, the problem* (3.37) *is indeed well-posed owing to a slight modification of the structure studied in Lemma 3.7 (see also* (3.29)*). Then, having* $\underline{\mathbf{u}}_\Gamma$ *as data, we may define* $(\boldsymbol{\sigma}_{\mathrm{H}}, \underline{\mathbf{u}}_{\mathrm{H}}) : [0,T] \to \mathbb{X} \times \mathbf{Y}$ *as the solution of the problem*

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}_{\mathrm{H}}(t)), \boldsymbol{\tau}] + [\mathbf{B}'(\underline{\mathbf{u}}_{\mathrm{H}}(t)), \boldsymbol{\tau}] + [\mathbf{D}'_\rho(\underline{\mathbf{u}}_{\mathrm{H}}(t)), \boldsymbol{\tau}] &= 0 \qquad\qquad\qquad\qquad\qquad \forall\, \boldsymbol{\tau} \in \mathbb{X} , \\ \partial_t\, [\mathbf{E}(\underline{\mathbf{u}}_{\mathrm{H}}(t)), \underline{\mathbf{v}}] - [\mathbf{B}(\boldsymbol{\sigma}_{\mathrm{H}}(t)), \underline{\mathbf{v}}] + [\mathbf{C}(\underline{\mathbf{u}}_{\mathrm{H}}(t)), \underline{\mathbf{v}}] &= [\mathbf{E}(\underline{\mathbf{u}}_\Gamma(t)) - \partial_t \mathbf{E}(\underline{\mathbf{u}}_\Gamma(t)), \underline{\mathbf{v}}] \quad \forall\, \underline{\mathbf{v}} \in \mathbf{Y} , \end{aligned} \tag{3.38}$$

which is also well-posed, as follows from the analysis developed in this section, by replacing the corresponding right-hand side in (2.19)*. Consequently, taking into account the linearity of both* (3.37) *and* (3.38)*, it is straightforward to verify that* $(\boldsymbol{\sigma}, \underline{\mathbf{u}}) = (\boldsymbol{\sigma}_\Gamma, \underline{\mathbf{u}}_\Gamma) + (\boldsymbol{\sigma}_{\mathrm{H}}, \underline{\mathbf{u}}_{\mathrm{H}})$ *is indeed a solution to the weak formulation of* (2.10) *with non-homogeneous Dirichlet boundary conditions.*

CHAPTER 4

Semidiscrete continuous-in-time approximation

In this chapter, we introduce and analyze the semidiscrete continuous-in-time approximation of (2.19). The solvability is established by adapting the arguments introduced in Chapter 3. Subsequently, we derive error estimates and identify the corresponding convergence rates.

4.1 Preliminaries

Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω made up of triangles K (when $d=2$) or tetrahedra K (when $d=3$) of diameter h_K, and define the mesh-size $h := \max\left\{h_K \ : \ K \in \mathcal{T}_h\right\}$. For a given integer $k \geq 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_k(K)$ be the space of polynomials of total degree at most k defined on K. Its vector and tensorial counterparts are denoted by $\mathbf{P}_k(K) := [\mathrm{P}_k(K)]^d$ and $\mathbb{P}_k(K) := [\mathrm{P}_k(K)]^{d\times d}$, respectively. In addition, we let $\mathbf{RT}_k(K) := \mathbf{P}_k(K) + \mathrm{P}_k(K)\,\mathbf{x}$ be the local Raviart–Thomas space of order k defined on K, where $\mathbf{x}$ stands for a generic vector in R^d. We denote by $\mathbb{RT}_k(K)$ the tensor space of functions whose rows lie in $\mathbf{RT}_k(K)$. Furthermore, we let b_K be the bubble function on K, which is given by the product of its $d+1$ barycentric coordinates. The local bubble space of order k is then set as

$$\mathbf{B}_k(K) := \begin{cases} \mathbf{curl}(b_K\,\mathrm{P}_k(K)) & \text{if } d = 2\,, \\ \mathbf{curl}(b_K\,\mathbf{P}_k(K)) & \text{if } d = 3\,, \end{cases}$$

where the curl operators are defined as $\mathbf{curl}(v) := \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right)$ for $v : K \to \mathrm{R}$ (if $d = 2$), and $\mathbf{curl}(\mathbf{v}) := \nabla \times \mathbf{v}$ for $\mathbf{v} : K \to \mathrm{R}^3$ (if $d = 3$). Finally, $\mathbb{B}_k(K)$ denotes the space of tensor functions in which each row belongs to $\mathbf{B}_k(K)$. With these notations at hand, we introduce the following global finite element spaces:

$$\begin{aligned} \mathbf{P}_k(\Omega) &:= \Big\{\mathbf{v}_h \in \mathbf{L}^2(\Omega) \ : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,, \\ \mathbb{P}_k(\Omega) &:= \Big\{\boldsymbol{\eta}_h \in \mathbb{L}^2(\Omega) \ : \quad \boldsymbol{\eta}_h|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,, \\ \mathbb{RT}_k(\Omega) &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) \ : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,, \\ \mathbb{B}_k(\Omega) &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) \ : \quad \boldsymbol{\tau}_h|_K \in \mathbb{B}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,. \end{aligned}$$

Let $\mathbb{X}_h \subset \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$, $\mathbf{H}_h^{\mathbf{u}} \subset \mathbf{L}^s(\Omega)$ and $\mathbb{H}_h^{\boldsymbol{\gamma}} \subset \mathbb{L}^2_{\text{skew}}(\Omega)$ be finite-dimensional subspaces forming a stable finite element triplet for the Banach spaces-based mixed elasticity with weakly imposed stress symmetry. This means that there exists a positive constant β_{d}, independent of h, such that

$$\sup_{0\neq\boldsymbol{\tau}_h\in\mathbb{X}_h} \frac{[\mathbf{B}(\boldsymbol{\tau}_h), \underline{\mathbf{v}}_h]}{\|\boldsymbol{\tau}_h\|_{\mathbb{X}}} \geq \beta_{\mathrm{d}}\, \|\underline{\mathbf{v}}_h\|_{\mathbf{Y}} \qquad \forall\, \underline{\mathbf{v}}_h := (\mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{Y}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\boldsymbol{\gamma}}\,. \tag{4.1}$$

We immediately stress that (4.1) is the discrete counterpart of the inf-sup condition (3.6). Furthermore, we point out that there exist several stable triples satisfying (4.1) with $s = \ell = 2$, which corresponds to the classical Hilbertian framework. Examples include the Amara–Thomas element [4], PEERS [8, 39], Stenberg [44], Arnold–Falk–Winther [9], and Cockburn–Gopalakrishnan–Guzmán [25] families. As established in [32, Lemma 4.8], if a triplet of finite element subspaces of $\mathbb{H}(\mathbf{div};\Omega)$, $\mathbf{L}^2(\Omega)$, and $\mathbb{L}^2_{\text{skew}}(\Omega)$ forms a stable triplet for linear elasticity in the Hilbertian setting, then, under certain assumptions detailed therein, these spaces also satisfy (4.1). That is, stability extends to the Banach setting. In particular, as shown in [32, Section 4.3.3], the Arnold–Falk–Winther and PEERS elements fulfill these assumptions. Therefore, we shall focus on these two families. To be precise, the Arnold–Falk–Winther element of order k, denoted AFW_k, consists of the following subspaces:

$$\widetilde{\mathbb{X}}_h := \mathbb{P}_{k+1}(\Omega) \cap \mathbb{H}(\mathbf{div};\Omega)\,, \quad \mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_k(\Omega)\,, \quad \text{and} \quad \mathbb{H}_h^{\gamma} := \mathbb{L}^2_{\mathrm{skew}}(\Omega) \cap \mathbb{P}_k(\Omega)\,. \tag{4.2}$$

In turn, the plane elasticity element with reduced symmetry of order k, denoted by PEERS_k, is defined by

$$\begin{aligned} \widetilde{\mathbb{X}}_h := \mathbb{RT}_k(\Omega) \oplus \mathbb{B}_k(\Omega)\,, \quad \mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_k(\Omega)\,, \\ \text{and} \quad \mathbb{H}_h^{\gamma} := \Big[C(\overline{\Omega})\Big]^{d\times d} \cap \mathbb{L}^2_{\mathrm{skew}}(\Omega) \cap \mathbb{P}_{k+1}(\Omega)\,. \end{aligned} \tag{4.3}$$

We notice that, even in the general framework where s and ℓ are not necessarily equal to 2, by setting $\mathbb{X}_h := \widetilde{\mathbb{X}}_h \cap \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$ both for AFW_k and for PEERS_k, the triplet $(\mathbb{X}_h, \mathbf{H}_h^{\mathbf{u}}, \mathbb{H}_h^{\gamma})$ is conforming with the continuous setting. Next, by letting $\underline{\mathbf{u}}_h := (\mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{Y}_h$, the semidiscrete continuous-in-time problem associated with (2.19) reads: Find $(\boldsymbol{\sigma}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbb{X}_h \times \mathbf{Y}_h$ such that, for a.e. $t \in (0,T)$,

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}_h(t)), \boldsymbol{\tau}_h] + [\mathbf{B}'(\underline{\mathbf{u}}_h(t)), \boldsymbol{\tau}_h] + [\mathbf{D}'_\rho(\underline{\mathbf{u}}_h(t)), \boldsymbol{\tau}_h] &= 0 && \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h\,, \\ \frac{\partial}{\partial t}[\mathbf{E}(\underline{\mathbf{u}}_h(t)), \underline{\mathbf{v}}_h] - [\mathbf{B}(\boldsymbol{\sigma}_h(t)), \underline{\mathbf{v}}_h] + [\mathbf{C}(\underline{\mathbf{u}}_h(t)), \underline{\mathbf{v}}_h] &= [\mathbf{F}(t), \underline{\mathbf{v}}_h] && \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h\,, \end{aligned} \tag{4.4}$$

and $\mathbf{u}_h(0) = \mathbf{u}_{h,0}$, where $(\boldsymbol{\sigma}_{h,0}, \underline{\mathbf{u}}_{h,0}) = (\boldsymbol{\sigma}_{h,0}, (\mathbf{u}_{h,0}, \boldsymbol{\gamma}_{h,0}))$ is a suitable approximation of $(\boldsymbol{\sigma}_0, \underline{\mathbf{u}}_0)$, which is the solution to (3.18). Namely, we choose $(\boldsymbol{\sigma}_{h,0}, \underline{\mathbf{u}}_{h,0}) \in \mathbb{X}_h \times \mathbf{Y}_h$ solving

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}_{h,0}), \boldsymbol{\tau}_h] + [\mathbf{B}'(\underline{\mathbf{u}}_{h,0}), \boldsymbol{\tau}_h] + [\mathbf{D}'_\rho(\underline{\mathbf{u}}_{h,0}), \boldsymbol{\tau}_h] &= 0 && \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h\,, \\ -[\mathbf{B}(\boldsymbol{\sigma}_{h,0}), \underline{\mathbf{v}}_h] + [\mathbf{C}(\underline{\mathbf{u}}_{h,0}), \underline{\mathbf{v}}_h] &= [\mathbf{G}_0, \underline{\mathbf{v}}_h] && \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h\,, \end{aligned} \tag{4.5}$$

where $\mathbf{G}_0 = (\mathbf{g}_0, \mathbf{0})$ is the linear functional defined as the right-hand side of (3.18). Notice that $\mathbf{G}_0 \in \mathbf{Y}'$. In fact, by applying Cauchy–Schwarz inequality and the continuous embedding $\mathbf{i}_{s,2}$

(cf. (1.6)), we have

$$\left|[\mathbf{G}_0, \underline{\mathbf{v}}]\right| \le C_0 \left\{\|\mathbf{div}(\rho\, \mathbf{e}(\mathbf{u}_0))\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}\right\} \|\underline{\mathbf{v}}\|_{\mathbf{Y}} \qquad \forall\, \underline{\mathbf{v}} \in \mathbf{Y}\,, \tag{4.6}$$

where $C_0 := 2\mu\, \|\mathbf{i}_{s,2}\|\, \max\{1, \|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}\}$. We stress here that we are assuming the hypothesis of Lemma 3.3, that is, $\mathbf{u}_0 \in \mathbf{L}^s(\Omega) \cap \mathbf{H}$. The well-posedness of (4.5) is established below, in Lemma 4.1.

4.2 Existence and uniqueness of a solution

Following the approach of the continuous formulation (cf. Chapter 3), we aim to prove the solvability of (4.4) by introducing a fixed-point strategy. To that end, we first let $\mathbb{V}_h$ be the discrete kernel of $\mathbf{B}$, which is given by

$$\mathbb{V}_h := \left\{\boldsymbol{\tau}_h \in \mathbb{X}_h \;:\quad [\mathbf{B}(\boldsymbol{\tau}_h), \underline{\mathbf{v}}_h] = 0 \qquad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h\right\}.$$

In turn, from (4.2) and (4.3), we notice that $\mathbf{div}(\mathbb{X}_h) \subset \mathbf{H}_h^{\mathbf{u}}$. Thus, $\mathbb{V}_h$ can be characterized as

$$\mathbb{V}_h = \left\{\boldsymbol{\tau}_h \in \mathbb{X}_h \;:\quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad (\boldsymbol{\eta}_h, \boldsymbol{\tau}_h)_\Omega = 0 \quad \forall\, \boldsymbol{\eta}_h \in \mathbb{H}_h^{\gamma}\right\}.$$

Notice that although $\mathbb{V}_h$ is not a subspace of $\mathbb{V}$ (cf. (3.7)), the coercivity of $\mathbf{A}$ also holds in the discrete kernel $\mathbb{V}_h$ (cf. (3.9)), since the divergence-free condition was the only property required in the argument. Consequently, bearing in mind (4.1) and following the same reasoning did it to prove (3.10), this time applying [26, Theorem 3.5], we obtain the existence of a positive constant Λ_{d}, depending only on β_{d}, c_ℓ, μ, ρ_0, ρ_1, $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$ and $|\Omega|$, such that, for all $(\boldsymbol{\zeta}_h, \underline{\mathbf{w}}_h) \in \mathbb{X}_h \times \mathbf{Y}_h$, there holds

$$\Lambda_{\mathrm{d}}\, \|(\boldsymbol{\zeta}_h, \underline{\mathbf{w}}_h)\|_{\mathbb{X}\times\mathbf{Y}} \le \sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h, \underline{\mathbf{v}}_h)\in\mathbb{X}_h\times\mathbf{Y}_h} \frac{[\mathbf{A}(\boldsymbol{\zeta}_h), \boldsymbol{\tau}_h] + [\mathbf{B}'(\underline{\mathbf{w}}_h), \boldsymbol{\tau}_h] + [\mathbf{B}(\boldsymbol{\zeta}_h), \underline{\mathbf{v}}_h] - [\mathbf{C}(\underline{\mathbf{w}}_h), \underline{\mathbf{v}}_h]}{\|(\boldsymbol{\tau}_h, \underline{\mathbf{v}}_h)\|_{\mathbb{X}\times\mathbf{Y}}}\,. \tag{4.7}$$

Having established this inf-sup condition, we are in a position to prove the well-posedness of (4.5).

Lemma 4.1. *Assume that* $\mathbf{u}_0 \in \mathbf{L}^s(\Omega) \cap \mathbf{H}$ *(cf. (3.11)) and the porosity satisfies*

$$\left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^r(\Omega)} \le \frac{\sqrt{d}\,\Lambda_{\mathrm{d}}}{2} \,. \tag{4.8}$$

Then, there exists a unique solution $(\boldsymbol{\sigma}_{h,0}, \underline{\mathbf{u}}_{h,0}) \in \mathbb{X}_h \times \mathbf{Y}_h$ *to (4.5). Moreover, there exists a positive constant* $C_{0,\mathrm{d}}$*, depending only on* Λ_{d} *and* C_0 *(cf. (4.6)), such that*

$$\|(\boldsymbol{\sigma}_{h,0}, \underline{\mathbf{u}}_{h,0})\|_{\mathbb{X}\times\mathbf{Y}} \le C_{0,\mathrm{d}} \left\{ \|\mathbf{div}(\rho\, \mathbf{e}(\mathbf{u}_0))\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} \right\} . \tag{4.9}$$

Proof. Similarly to [13, eq. (4.17)–(4.18)], we employ the inf-sup condition (4.7), the stability property of $\mathbf{D}_\rho$ (cf. (3.4b)), and the assumption (4.8), to establish a global inf-sup condition analogous to (4.7), but incorporating $\mathbf{D}_\rho$. Similarly, it can be verified that this condition also holds when taking the supremum over the other component. Therefore, by invoking the Banach–Nečas–Babuška Theorem (see, e.g. [27, Theorem 25.15]), we conclude that (4.5) is well-posed, together with the corresponding *a priori* estimate. For the sake of brevity, we omit further details and refer the reader to [13, Lemma 4.3] for a similar analysis. □

Observe that (4.7) enables the stability of (4.4) to be established by following the same arguments as in the continuous case (cf. Theorem 3.4). Although the precise statement is deferred to Theorem 4.7, this observation motivates the introduction of a fixed-point operator $\mathcal{J}_{\mathrm{d}} : \mathrm{L}^2(0,T;\mathbb{X}_h) \to \mathrm{L}^2(0,T;\mathbb{X}_h)$ on the space $\mathrm{L}^2(0,T;\mathbb{X}_h)$, in a similar fashion as in (3.26). The operator $\mathcal{J}_{\mathrm{d}}$ is then defined by

$$\mathcal{J}_{\mathrm{d}}(\boldsymbol{\zeta}_h) := \boldsymbol{\sigma}_h \qquad \forall\, \boldsymbol{\zeta}_h \in \mathrm{L}^2(0,T;\mathbb{X}_h)\,,$$

where $(\boldsymbol{\sigma}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbb{X}_h \times \mathbf{Y}_h$ is the unique solution (to be confirmed below) to

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}_h(t)), \boldsymbol{\tau}_h] + [\widetilde{\mathbf{B}}'(\underline{\mathbf{u}}_h(t)), \boldsymbol{\tau}_h] &= 0 && \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h\,, \\ \frac{\partial}{\partial t}\,[\mathbf{E}(\underline{\mathbf{u}}_h(t)), \underline{\mathbf{v}}_h] - [\widetilde{\mathbf{B}}(\boldsymbol{\sigma}_h(t)), \underline{\mathbf{v}}_h] + [\mathbf{C}(\underline{\mathbf{u}}_h(t)), \underline{\mathbf{v}}_h] &= [\widetilde{\mathbf{F}}_{\boldsymbol{\zeta}_h}(t), \underline{\mathbf{v}}_h] && \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h\,, \end{aligned} \tag{4.10}$$

for a.e. $t \in (0,T)$ with $\mathbf{u}_h(0) = \mathbf{u}_{h,0}$, where $(\boldsymbol{\sigma}_{h,0}, \underline{\mathbf{u}}_{h,0}) \in \mathbb{X}_h \times \mathbf{Y}_h$ is the unique solution (to be confirmed in Lemma 4.5) to

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}_{h,0}), \boldsymbol{\tau}_h] + [\widetilde{\mathbf{B}}'(\underline{\mathbf{u}}_{h,0}), \boldsymbol{\tau}_h] &= 0 && \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h\,, \\ -[\widetilde{\mathbf{B}}(\boldsymbol{\sigma}_{h,0}), \underline{\mathbf{v}}_h] + [\mathbf{C}(\underline{\mathbf{u}}_{h,0}), \underline{\mathbf{v}}_h] &= [\widetilde{\mathbf{G}}_0, \underline{\mathbf{v}}_h] && \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h\,, \end{aligned} \tag{4.11}$$

with $\widetilde{\mathbf{G}}_0 = (\widetilde{\mathbf{g}}_0, \mathbf{0})$ the linear functional on the right-hand side of (3.16). It follows from (3.17) and the continuous embedding $\mathbf{i}_{s,2}$ (cf. (1.6)) that $\widetilde{\mathbf{G}}_0 \in \mathbf{Y}'$. To prove the unique solvability of (4.10), we first establish the discrete counterpart of Lemma 3.5. The proof is analogous to the continuous case, employing the discrete inf-sup condition (4.1) in place of (3.6).

Lemma 4.2. *Assume that the porosity satisfies*

$$\left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^r(\Omega)} \le \frac{\sqrt{d}\,\beta_{\mathrm{d}}}{2}\,, \tag{4.12}$$

with β_{d} as given in (4.1). Then, the following inf-sup condition holds:

$$\sup_{0 \neq \boldsymbol{\tau}_h \in \mathbb{X}_h} \frac{[\widetilde{\mathbf{B}}(\boldsymbol{\tau}_h), \underline{\mathbf{v}}_h]}{\|\boldsymbol{\tau}_h\|_{\mathbb{X}}} \ge \frac{\beta_{\mathrm{d}}}{2}\, \|\underline{\mathbf{v}}_h\|_{\mathbf{Y}} \qquad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h\,. \tag{4.13}$$

Next, we define $\widetilde{\mathbb{V}}_h$ as the discrete kernel of the operator $\widetilde{\mathbf{B}}$, namely

$$\widetilde{\mathbb{V}}_h := \Big\{ \boldsymbol{\tau}_h \in \mathbb{X}_h \;:\; [\widetilde{\mathbf{B}}(\boldsymbol{\tau}_h), \underline{\mathbf{v}}_h] = 0 \qquad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h \Big\}. \tag{4.14}$$

Then, we have the following result, which serves as the discrete counterpart of Lemma 3.6.

Lemma 4.3. *Let c_ℓ be as in (3.8) and assume that the porosity satisfies*

$$\left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \leq \frac{\sqrt{d}\, c_\ell}{2\, \|\mathbf{i}_{s,2}\|}\,. \tag{4.15}$$

Then, with the same constant α as in Lemma 3.6, which is independent of h, it holds

$$\sup_{0 \neq \boldsymbol{\tau}_h \in \widetilde{\mathbb{V}}_h} \frac{[\mathbf{A}(\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h]}{\|\boldsymbol{\tau}_h\|_{\mathbb{X}}} \geq \alpha\, \|\boldsymbol{\sigma}_h\|_{\mathbb{X}} \qquad \forall\, \boldsymbol{\sigma}_h \in \widetilde{\mathbb{V}}_h\,. \tag{4.16}$$

Proof. Let $\boldsymbol{\sigma}_h \in \widetilde{\mathbb{V}}_h$. Since $\mathbf{div}(\boldsymbol{\sigma}_h) \in \mathbf{H}_h^{\mathbf{u}}$, we can use (4.14) with $\underline{\mathbf{v}}_h = (\mathbf{div}(\boldsymbol{\sigma}_h), \mathbf{0}) \in \mathbf{Y}_h$. Then, we apply Cauchy–Schwarz inequality along with (4.15), thus obtaining

$$\|\mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{\mathbf{L}^2(\Omega)} = \frac{1}{d} \left(\frac{\nabla \rho}{\rho} \cdot \mathbf{div}(\boldsymbol{\sigma}_h), \mathrm{tr}(\boldsymbol{\sigma}_h) \right)_\Omega \leq \frac{c_\ell}{2\, \|\mathbf{i}_{s,2}\|} \, \|\mathbf{div}(\boldsymbol{\sigma}_h)\|_{\mathbf{L}^2(\Omega)} \, \|\boldsymbol{\sigma}_h\|_{\mathbb{L}^2(\Omega)}\,.$$

Now, using this estimate and the continuous embedding $\mathbf{i}_{2,\ell}$, which satisfies $\|\mathbf{i}_{2,\ell}\| = |\Omega|^{(2-\ell)/(2\ell)} = |\Omega|^{(s-2)/(2s)} = \|\mathbf{i}_{s,2}\|$ (cf. (1.6)), we find that

$$\|\mathbf{div}(\boldsymbol{\sigma}_h)\|_{\mathbf{L}^\ell(\Omega)} \leq \|\mathbf{i}_{s,2}\| \, \|\mathbf{div}(\boldsymbol{\sigma}_h)\|_{\mathbf{L}^2(\Omega)} \leq \frac{c_\ell}{2} \, \|\boldsymbol{\sigma}_h\|_{\mathbb{L}^2(\Omega)}\,.$$

Having this established, the rest of the argument proceeds exactly as in the proof of Lemma 3.6, thereby showing that (4.16) holds with the same constant. We omit further details. □

Lemma 4.4. *Suppose that the porosity ρ satisfies (4.12) and (4.15). Then, for all $\widehat{\mathbf{f}} \in \mathbf{L}^2(\Omega)$, there exists a unique solution $(\boldsymbol{\sigma}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_h \times \mathbf{Y}_h$ to the problem*

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h] + [\widetilde{\mathbf{B}}'(\underline{\mathbf{u}}_h), \boldsymbol{\tau}_h] &= 0 && \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h\,, \\ [\widetilde{\mathbf{B}}(\boldsymbol{\sigma}_h), \underline{\mathbf{v}}_h] - [(\mathbf{E} + \mathbf{C})(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}_h] &= -(\widehat{\mathbf{f}}, \mathbf{v}_h)_\Omega && \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h\,, \end{aligned} \tag{4.17}$$

Proof. Bearing in mind Lemmas 4.2 and 4.3, and recalling that $\mathbf{A}$ and $\mathbf{E}+\mathbf{C}$ induce symmetric bilinear forms, we apply [26, Theorem 3.5] to conclude. □

Certainly, Lemma 4.4 proves the range condition of Theorem 3.1 in our discrete setting, thereby establishing the discrete counterpart of Lemma 3.7. Moreover, by neglecting the operator $\mathbf{E}$ in (4.17), we also prove that, under the same assumptions as in Lemma 4.4, the problem associated with the initial conditions (4.11) has a unique solution. We state this in the following result.

Lemma 4.5. *Assume that* $\mathbf{u}_0 \in \mathbf{L}^s(\Omega) \cap \mathbf{H}$ *(cf. Lemma 3.3) and that the porosity* ρ *satisfies* (4.12) *and* (4.15). *Then, the problem* (4.11) *has a unique solution* $(\boldsymbol{\sigma}_{h,0}, \underline{\mathbf{u}}_{h,0}) \in \mathbb{X}_h \times \mathbf{Y}_h$. *Moreover, there exists a positive constant* $\widetilde{C}_{0,\mathrm{d}}$, *depending only on* α, β_{d}, $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$, ρ_0, *and* $\widetilde{C}_0$ *(cf.* (3.17)*), and thus independent of* h, *such that*

$$\|\boldsymbol{\sigma}_{h,0}\|_{\mathbb{X}} + \|\underline{\mathbf{u}}_{h,0}\|_{\mathbf{Y}} \leq \widetilde{C}_{0,\mathrm{d}} \left\{ \|\mathbf{div}(\rho\,\mathbf{e}(\mathbf{u}_0))\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} \right\}. \tag{4.18}$$

Proof. Similarly to the proof of Lemma 4.4, we invoke [26, Theorem 3.5] to ensure existence and uniqueness of the solution. Moreover, we use the *a priori* estimate provided by the same result, together with the continuity of $\widetilde{\mathbf{G}}_0$ established in (3.17), to obtain (4.18). We omit further details. □

The following result shows that (4.10) has a unique solution, which means that $\mathcal{J}_{\mathrm{d}}$ is well-defined.

Theorem 4.6. *Suppose that the porosity* ρ *satisfies* (4.12) *and* (4.15). *In addition, let* $\mathbf{f} \in \mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{L}^s(\Omega) \cap \mathbf{H}$ *be given, and let* $(\boldsymbol{\sigma}_{h,0}, \underline{\mathbf{u}}_{h,0})$ *be solution to* (4.5). *Then, for each* $\boldsymbol{\zeta}_h \in \mathrm{L}^2(0,T;\mathbb{X}_h)$ *satisfying in addition that* $\boldsymbol{\zeta}_h \in \mathrm{W}^{1,1}(0,T;\mathbb{L}^2(\Omega))$, $\mathcal{J}_{\mathrm{d}}(\boldsymbol{\zeta}_h)$ *is well-defined. More precisely, there exists a unique solution* $(\boldsymbol{\sigma}_h, \underline{\mathbf{u}}_h)$ *to* (4.10) *such that* $\boldsymbol{\sigma}_h \in \mathrm{L}^2(0,T;\mathbb{X}_h)$, $\mathbf{u}_h \in \mathrm{W}^{1,\infty}(0,T;\mathbf{H}_h^{\mathbf{u}})$, $\boldsymbol{\gamma}_h \in \mathrm{L}^2(0,T;\mathbb{H}_h^{\boldsymbol{\gamma}})$, $\mathbf{u}_h(0) = \mathbf{u}_{h,0}$, $\boldsymbol{\sigma}_h^{\mathrm{d}}(0) = \boldsymbol{\sigma}_{h,0}^{\mathrm{d}}$, $\boldsymbol{\gamma}_h(0) = \boldsymbol{\gamma}_{h,0}$, *and* $\mathcal{J}_{\mathrm{d}}(\boldsymbol{\zeta}_h) = \boldsymbol{\sigma}_h$. *Moreover, there exist positive constants* $\widetilde{C}_{\mathrm{B,d}}$ *and* $C_{\mathcal{J}_{\mathrm{d}}}$, *with* $\widetilde{C}_{\mathrm{B,d}}$ *depending only on* μ, ρ_0, ρ_1, $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$, Λ_{d}, $\widetilde{C}_{0,\mathrm{d}}$, d *and* $|\Omega|$, *and* $C_{\mathcal{J}_{\mathrm{d}}}$ *depending only on* ρ_0, ρ_1, Λ_{d}, d *and* $|\Omega|$, *such that*

$$\begin{aligned} \|\mathcal{J}_{\mathrm{d}}(\boldsymbol{\zeta}_h)\|_{\mathrm{L}^2(0,T;\mathbb{X})} \leq \widetilde{C}_{\mathrm{B,d}} & \left\{ \|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{div}(\rho\,\mathbf{e}(\mathbf{u}_0))\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} \right\} \\ & + C_{\mathcal{J}_{\mathrm{d}}} \left\| \frac{\nabla\rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\boldsymbol{\zeta}_h\|_{\mathrm{L}^2(0,T;\mathbb{X})}. \end{aligned} \tag{4.19}$$

Proof. Using the fact that $\mathbb{X}_h \subset \mathbb{X}$ and $\mathbf{Y}_h \subset \mathbf{Y}$, the proof is identical to the proof of

Theorem 3.8, this time relying on the discrete inf-sup conditions established in this chapter (cf. (4.1), (4.13), and (4.16)), the discrete initial conditions (cf. (4.11)), and the estimate (4.18) given in Lemma 4.5. □

We finally obtain the main result of this section, which is the existence and uniqueness of a solution to (4.4) along with the stability of the discrete problem. We note in advance that the time regularity of $\boldsymbol{\sigma}_h$ can be improved to $\mathrm{H}^1(0,T;\mathbb{X}_h)$ by arguments analogous to those presented in Remark 3.2.

Theorem 4.7. *Suppose that the porosity ρ satisfies (4.12), (4.15),*

$$\left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \le c_{\rho,\mathrm{d}} \quad \textit{with} \quad c_{\rho,\mathrm{d}} := \frac{\sqrt{d}\,\Lambda_{\mathrm{d}}}{2\,\|\mathbf{i}_{s,2}\|} \, \min\left\{ 1, \frac{\sqrt{\rho_0}}{4\rho_1} \right\}, \tag{4.20}$$

and

$$C_{\mathcal{J}_{\mathrm{d}}} \left\| \frac{\nabla \rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} < 1\,. \tag{4.21}$$

Then, given $\mathbf{f} \in \mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))$ and $\mathbf{u}_0 \in \mathbf{L}^s(\Omega) \cap \mathbf{H}$, and denoting by $(\boldsymbol{\sigma}_{h,0}, \underline{\mathbf{u}}_{h,0})$ the unique solution to (4.5) (cf. Lemma 4.5), there exists a unique solution to (4.4) with $\boldsymbol{\sigma}_h \in \mathrm{L}^2(0,T;\mathbb{X}_h)$, $\mathbf{u}_h \in \mathrm{W}^{1,\infty}(0,T;\mathbf{H}^{\mathbf{u}}_h)$, $\boldsymbol{\gamma}_h \in \mathrm{L}^2(0,T;\mathbb{H}^{\boldsymbol{\gamma}}_h)$, $\mathbf{u}_h(0) = \mathbf{u}_{h,0}$, $\boldsymbol{\sigma}^{\mathrm{d}}_h(0) = \boldsymbol{\sigma}^{\mathrm{d}}_{h,0}$, $\boldsymbol{\gamma}_h(0) = \boldsymbol{\gamma}_{h,0}$, and $\mathcal{J}_{\mathrm{d}}(\boldsymbol{\sigma}_h) = \boldsymbol{\sigma}_h$. Moreover, there exists a positive constant $C_{\mathrm{B,d}}$, depending only on μ, ρ_0, ρ_1, $C_{\mathbf{K}}$, $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$, Λ_{d}, $C_{0,\mathrm{d}}$, d and $|\Omega|$, such that

$$\begin{aligned} &\|\boldsymbol{\sigma}_h\|^2_{\mathrm{L}^2(0,T;\mathbb{X})} + \|\mathbf{u}_h\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^s(\Omega))} + \|\mathbf{u}_h\|^2_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\gamma}_h\|^2_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} \\ &\qquad \le C_{\mathrm{B,d}} \left\{ \|\mathbf{f}\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{div}(\rho\,\mathbf{e}(\mathbf{u}_0))\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} \right\}. \end{aligned} \tag{4.22}$$

Proof. Employing the same arguments as in the proof of Theorem 3.9, from (4.19) one verifies that $\mathcal{J}_{\mathrm{d}}$ is Lipschitz continuous with constant $C_{\mathcal{J}_{\mathrm{d}}}\,\|\nabla\rho/\rho\|_{\mathbf{L}^\infty(\Omega)}$. Under the assumption (4.21), the fixed-point operator is a contraction, and the Banach fixed-point theorem yields the solvability of (4.4). The stability estimate (4.22) then follows by the same reasoning as in Theorem 3.4, this time relying on the discrete global inf-sup condition (4.7) and applying the assumption (4.20). Finally, in analogy with Lemma 3.2, it is straightforward to show that (4.4) is equivalent to (4.10) with $\boldsymbol{\zeta}_h = \boldsymbol{\sigma}_h$, and that the initial conditions coincide. We omit further details. $\square$

Remark 4.1. *The analysis developed in this section remains valid for any other triplet of finite element spaces satisfying* (4.1)*, provided that the sole requirement* $\mathbf{div}(\mathbb{X}_h) \subset \mathbf{H}_h^{\mathbf{u}}$ *is fulfilled.*

4.3 Error analysis

We now proceed to establish the rates of convergence. For the sake of clarity, we restrict the analysis to the PEERS element (cf. (4.3)), and indicate at the end of this section, in Remark 4.2, the minor adjustments required for the AFW element (cf. (4.2)) or any other choice of triples. In this way, let $\boldsymbol{\mathcal{P}}_h^k : \mathbf{L}^s(\Omega) \to \mathbf{H}_h^{\mathbf{u}}$ and $\mathcal{P}_h^{k+1} : \mathbb{L}^2_{\mathrm{skew}}(\Omega) \to \mathbb{H}_h^{\gamma}$ be the L^2-projection operators, satisfying

$$\begin{aligned} \Big(\mathbf{u} - \boldsymbol{\mathcal{P}}_h^k(\mathbf{u}), \mathbf{v}_h\Big)_\Omega &= 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\,, \\ \Big(\boldsymbol{\gamma} - \mathcal{P}_h^{k+1}(\boldsymbol{\gamma}), \boldsymbol{\eta}_h\Big)_\Omega &= 0 \qquad \forall\, \boldsymbol{\eta}_h \in \mathbb{H}_h^{\gamma}\,, \end{aligned} \tag{4.23}$$

and, given $\mathrm{p} \geq 2\,d/(d+2)$, we consider the space

$$\mathbb{H}_\mathrm{p} := \Big\{ \boldsymbol{\tau} \in \mathbb{X} \;:\; \boldsymbol{\tau}|_K \in \mathbb{W}^{1,\mathrm{p}}(K) \qquad \forall\, K \in \mathcal{T}_h \Big\}\,,$$

so that we may define $\boldsymbol{\Pi}_h^k : \mathbb{H}_\mathrm{p} \to \mathbb{RT}_k(\Omega)$ as the tensorial counterpart of the Raviart–Thomas interpolation operator, which satisfies the well-known commuting diagram property (cf. [10, Section 2.5.2] or [29, Section 3.4.1])

$$\mathbf{div}(\boldsymbol{\Pi}_h^k(\boldsymbol{\tau})) = \boldsymbol{\mathcal{P}}_h^k(\mathbf{div}(\boldsymbol{\tau})) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_\mathrm{p}\,. \tag{4.24}$$

Furthermore, recalling the decomposition (2.16), let us define $\boldsymbol{\Pi}_{h,0}^k : \mathbb{H}_\mathrm{p} \to \mathbb{RT}_k(\Omega) \cap \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$ such that, for each $\boldsymbol{\tau} \in \mathbb{H}_\mathrm{p}$, $\boldsymbol{\Pi}_{h,0}^k(\boldsymbol{\tau})$ is the $\mathbb{H}_0(\mathbf{div}_\ell;\Omega)$-component of $\boldsymbol{\Pi}_h^k(\boldsymbol{\tau})$. Then, we notice that the range of $\boldsymbol{\Pi}_{h,0}$ is contained in $\mathbb{X}_h$, as $\mathbb{RT}_k(\Omega) \cap \mathbb{H}_0(\mathbf{div}_\ell;\Omega) \subset \mathbb{X}_h$. Moreover, one readily checks that the property (4.24) also holds when $\boldsymbol{\Pi}_h^k$ is replaced by $\boldsymbol{\Pi}_{h,0}^k$.

Now, let us define the errors $\mathbf{e}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ and $\mathbf{e}_{\underline{\mathbf{u}}} := (\mathbf{e}_{\mathbf{u}}, \mathbf{e}_{\boldsymbol{\gamma}}) = (\mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)$, and consider the decompositions

$$\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\delta}_{\boldsymbol{\sigma}} + \boldsymbol{\theta}_{\boldsymbol{\sigma}} \quad \text{and} \quad \mathbf{e}_{\underline{\mathbf{u}}} = \boldsymbol{\delta}_{\underline{\mathbf{u}}} + \boldsymbol{\theta}_{\underline{\mathbf{u}}} = (\boldsymbol{\delta}_{\mathbf{u}} + \boldsymbol{\theta}_{\mathbf{u}}, \boldsymbol{\delta}_{\boldsymbol{\gamma}} + \boldsymbol{\theta}_{\boldsymbol{\gamma}}), \tag{4.25}$$

where

$$\boldsymbol{\delta}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \mathbf{\Pi}_{h,0}^{k}(\boldsymbol{\sigma}), \quad \boldsymbol{\theta}_{\boldsymbol{\sigma}} := \mathbf{\Pi}_{h,0}^{k}(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_h, \quad \boldsymbol{\delta}_{\mathbf{u}} := \mathbf{u} - \boldsymbol{\mathcal{P}}_h^{k}(\mathbf{u}), \quad \boldsymbol{\theta}_{\mathbf{u}} := \boldsymbol{\mathcal{P}}_h^{k}(\mathbf{u}) - \mathbf{u}_h,$$

$$\boldsymbol{\delta}_{\boldsymbol{\gamma}} := \boldsymbol{\gamma} - \boldsymbol{\mathcal{P}}_h^{k+1}(\boldsymbol{\gamma}), \quad \text{and} \quad \boldsymbol{\theta}_{\boldsymbol{\gamma}} := \boldsymbol{\mathcal{P}}_h^{k+1}(\boldsymbol{\gamma}) - \boldsymbol{\gamma}_h.$$

Subtracting the discrete problem (4.4) from the continuous one (2.19), we obtain the following error system:

$$\begin{aligned} [\mathbf{A}(\mathbf{e}_{\boldsymbol{\sigma}}), \boldsymbol{\tau}_h] + [\mathbf{B}'(\mathbf{e}_{\underline{\mathbf{u}}}), \boldsymbol{\tau}_h] + [\mathbf{D}'_{\rho}(\mathbf{e}_{\underline{\mathbf{u}}}), \boldsymbol{\tau}_h] &= 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h, \\ \frac{\partial}{\partial t}[\mathbf{E}(\mathbf{e}_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] - [\mathbf{B}(\mathbf{e}_{\boldsymbol{\sigma}}), \underline{\mathbf{v}}_h] + [\mathbf{C}(\mathbf{e}_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] &= 0 \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h. \end{aligned} \tag{4.26}$$

In turn, using the projection properties (4.23) and (4.24), we find that

$$[\mathbf{B}'(\boldsymbol{\delta}_{\underline{\mathbf{u}}}), \boldsymbol{\tau}_h] = (\boldsymbol{\tau}_h, \boldsymbol{\delta}_{\boldsymbol{\gamma}})_{\Omega} \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h, \quad \text{and} \quad [\mathbf{B}(\boldsymbol{\delta}_{\boldsymbol{\sigma}}), \underline{\mathbf{v}}_h] = (\boldsymbol{\delta}_{\boldsymbol{\sigma}}, \boldsymbol{\eta}_h)_{\Omega} \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h.$$

Hence, the error system (4.26) can be rewritten as

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\theta}_{\boldsymbol{\sigma}}), \boldsymbol{\tau}_h] + [\mathbf{B}'(\boldsymbol{\theta}_{\underline{\mathbf{u}}}), \boldsymbol{\tau}_h] + [\mathbf{D}'_{\rho}(\boldsymbol{\theta}_{\underline{\mathbf{u}}}), \boldsymbol{\tau}_h] &= -[\mathbf{A}(\boldsymbol{\delta}_{\boldsymbol{\sigma}}), \boldsymbol{\tau}_h] - (\boldsymbol{\tau}_h, \boldsymbol{\delta}_{\boldsymbol{\gamma}})_{\Omega} - [\mathbf{D}'_{\rho}(\boldsymbol{\delta}_{\underline{\mathbf{u}}}), \boldsymbol{\tau}_h], \\ \frac{\partial}{\partial t}[\mathbf{E}(\boldsymbol{\theta}_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] - [\mathbf{B}(\boldsymbol{\theta}_{\boldsymbol{\sigma}}), \underline{\mathbf{v}}_h] + [\mathbf{C}(\boldsymbol{\theta}_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] &= -\frac{\partial}{\partial t}[\mathbf{E}(\boldsymbol{\delta}_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] + (\boldsymbol{\delta}_{\boldsymbol{\sigma}}, \boldsymbol{\eta}_h)_{\Omega} - [\mathbf{C}(\boldsymbol{\delta}_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h], \end{aligned} \tag{4.27}$$

for all $(\boldsymbol{\tau}_h, \underline{\mathbf{v}}_h) \in \mathbb{X}_h \times \mathbf{Y}_h$.

On the other hand, similar notation is introduced for the discrete initial conditions system (4.5). Let us consider $\mathbf{e}_{\boldsymbol{\sigma}_0} := \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}$ and $\mathbf{e}_{\underline{\mathbf{u}}_0} := (\mathbf{e}_{\mathbf{u}_0}, \mathbf{e}_{\boldsymbol{\gamma}_0}) = (\mathbf{u}_0 - \mathbf{u}_{h,0}, \boldsymbol{\gamma}_0 - \boldsymbol{\gamma}_{h,0})$, with the corresponding decompositions

$$\mathbf{e}_{\boldsymbol{\sigma}_0} = \boldsymbol{\delta}_{\boldsymbol{\sigma}_0} + \boldsymbol{\theta}_{\boldsymbol{\sigma}_0} \quad \text{and} \quad \mathbf{e}_{\underline{\mathbf{u}}_0} = \boldsymbol{\delta}_{\underline{\mathbf{u}}_0} + \boldsymbol{\theta}_{\underline{\mathbf{u}}_0} = (\boldsymbol{\delta}_{\mathbf{u}_0} + \boldsymbol{\theta}_{\mathbf{u}_0}, \boldsymbol{\delta}_{\boldsymbol{\gamma}_0} + \boldsymbol{\theta}_{\boldsymbol{\gamma}_0}), \tag{4.28}$$

where

$$\boldsymbol{\delta}_{\boldsymbol{\sigma}_0} := \boldsymbol{\sigma}_0 - \boldsymbol{\Pi}_{h,0}^k(\boldsymbol{\sigma}_0)\,, \quad \boldsymbol{\theta}_{\boldsymbol{\sigma}_0} := \boldsymbol{\Pi}_{h,0}^k(\boldsymbol{\sigma}_0) - \boldsymbol{\sigma}_{h,0}\,, \quad \boldsymbol{\delta}_{\mathbf{u}_0} := \mathbf{u}_0 - \boldsymbol{\mathcal{P}}_h^k(\mathbf{u}_0)\,,$$

$$\boldsymbol{\theta}_{\mathbf{u}_0} := \boldsymbol{\mathcal{P}}_h^k(\mathbf{u}_0) - \mathbf{u}_{h,0}\,, \quad \boldsymbol{\delta}_{\boldsymbol{\gamma}_0} := \boldsymbol{\gamma}_0 - \mathbb{P}_h^{k+1}(\boldsymbol{\gamma}_0)\,, \quad \text{and} \quad \boldsymbol{\theta}_{\boldsymbol{\gamma}_0} := \mathbb{P}_h^{k+1}(\boldsymbol{\gamma}_0) - \boldsymbol{\gamma}_{h,0}\,.$$

Thus, by subtracting the discrete initial conditions system (4.5) from the continuous one (3.18), we obtain the error system

$$\begin{aligned} [\mathbf{A}(\mathbf{e}_{\boldsymbol{\sigma}_0}),\boldsymbol{\tau}_h] + [\mathbf{B}'(\mathbf{e}_{\underline{\mathbf{u}}_0}),\boldsymbol{\tau}_h] + [\mathbf{D}'_\rho(\mathbf{e}_{\underline{\mathbf{u}}_0}),\boldsymbol{\tau}_h] &= 0 \quad \forall\,\boldsymbol{\tau}_h \in \mathbb{X}_h\,, \\ -[\mathbf{B}(\mathbf{e}_{\boldsymbol{\sigma}_0}),\underline{\mathbf{v}}_h] + [\mathbf{C}(\mathbf{e}_{\underline{\mathbf{u}}_0}),\underline{\mathbf{v}}_h] &= 0 \quad \forall\,\underline{\mathbf{v}}_h \in \mathbf{Y}_h\,. \end{aligned} \tag{4.29}$$

We now establish the main result of this section.

Theorem 4.8. *Assume that the hypotheses of Theorems 3.9 and 4.7 hold. Furthermore, suppose that the porosity satisfies*

$$\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^\infty(\Omega)} \leq \frac{\sqrt{d}\,\rho_0\,\Lambda_{\mathrm{d}}}{8\sqrt{2}\,\rho_1\,\|\mathbf{i}_{s,2}\|}\,. \tag{4.30}$$

Let $(\boldsymbol{\sigma},\underline{\mathbf{u}})$ *and* $(\boldsymbol{\sigma}_h,\underline{\mathbf{u}}_h)$ *be the unique solutions of the continuous and semidiscrete problems (2.19) and (4.4), respectively, with the regularity specified in Theorems 3.9 and 4.7. Assume further that* $\boldsymbol{\sigma} \in \mathbb{H}_{\mathrm{p}}$ *for some* $\mathrm{p} \geq 2d/(d+2)$*. Then, there exists a positive constant* $\mathcal{C}$*, independent of* h*, such that*

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|^2_{\mathrm{L}^2(0,T;\mathbb{X})} + \|\mathbf{e}_{\mathbf{u}}\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^s(\Omega))} + \|\mathbf{e}_{\mathbf{u}}\|^2_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\boldsymbol{\gamma}}\|^2_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} \leq \mathcal{C}\,\mathrm{E}_{\boldsymbol{\delta}}\,, \tag{4.31}$$

where

$$\begin{aligned} \mathrm{E}_{\boldsymbol{\delta}} :=\ & \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathrm{L}^2(0,T;\mathbb{X})} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^s(\Omega))} + \|\boldsymbol{\delta}_{\boldsymbol{\gamma}}\|^2_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} + \|\partial_t\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} \\ & + \|\partial_t\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\partial_t\boldsymbol{\delta}_{\boldsymbol{\gamma}}\|^2_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathrm{L}^\infty(0,T;\mathbb{X})} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathrm{L}^\infty(0,T;\mathbf{L}^s(\Omega))} \\ & + \|\boldsymbol{\delta}_{\boldsymbol{\gamma}}\|^2_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega))} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(0)\|^2_{\mathbb{X}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}_0}\|^2_{\mathbb{X}} + \|\boldsymbol{\delta}_{\mathbf{u}_0}\|^2_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\gamma}_0}\|^2_{\mathbb{L}^2(\Omega)}\,. \end{aligned}$$

Proof. First, notice that since $\boldsymbol{\sigma} \in \mathbb{H}_{\mathrm{p}}$, $\boldsymbol{\Pi}_{h,0}^k(\boldsymbol{\sigma})$ is well-defined. Then, proceeding similarly as in the proof of Theorem 3.4, we use the discrete inf-sup condition (4.7), the error system (4.27), and the assumption $\|\nabla\rho/\rho\|_{\mathbf{L}^\infty(\Omega)} \le \sqrt{d}\,\Lambda_{\rm d}/(2\,\|\mathbf{i}_{s,2}\|)$ (cf. (4.20)), so that we obtain

$$\begin{aligned} \frac{\Lambda_{\rm d}}{2}\,\|(\boldsymbol{\theta}_{\boldsymbol{\sigma}},\boldsymbol{\theta}_{\underline{\mathbf{u}}})\|_{\mathbb{X}\times\mathbf{Y}} \le C_1\,\Big(\|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} + \|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)} + \|\partial_t\,\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}\Big) \\ + \rho_1\,\|\mathbf{i}_{s,2}\|\,\|\partial_t\boldsymbol{\theta}_{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}\,, \end{aligned} \tag{4.32}$$

where C_1 is a positive constant depending only on μ, ρ_0, ρ_1, $|\Omega|$, $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$, and $\|\nabla\rho/\rho\|_{\mathbf{L}^r(\Omega)}$. By squaring (4.32), integrating from 0 to $t \in (0,T]$, and performing some algebraic manipulations, we arrive at

$$\begin{aligned} &\frac{\Lambda_{\rm d}^2}{8\,\rho_1^2\,\|\mathbf{i}_{s,2}\|^2}\int_0^t \Big\{\|\boldsymbol{\theta}_{\boldsymbol{\sigma}}(\mathrm{s})\|_{\mathbb{X}}^2 + \|\boldsymbol{\theta}_{\mathbf{u}}(\mathrm{s})\|_{\mathbf{L}^s(\Omega)}^2 + \|\boldsymbol{\theta}_{\boldsymbol{\gamma}}(\mathrm{s})\|_{\mathbb{L}^2(\Omega)}^2\Big\}\,d\mathrm{s} \\ &\quad \le \widetilde{C}_1\int_0^t \Big\{\|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(\mathrm{s})\|_{\mathbb{X}}^2 + \|\boldsymbol{\delta}_{\mathbf{u}}(\mathrm{s})\|_{\mathbf{L}^s(\Omega)}^2 + \|\boldsymbol{\delta}_{\boldsymbol{\gamma}}(\mathrm{s})\|_{\mathbb{L}^2(\Omega)}^2 + \|\partial_t\,\boldsymbol{\delta}_{\mathbf{u}}(\mathrm{s})\|_{\mathbf{L}^2(\Omega)}^2\Big\}\,d\mathrm{s} \\ &\qquad + \int_0^t \|\partial_t\,\boldsymbol{\theta}_{\mathbf{u}}(\mathrm{s})\|_{\mathbf{L}^2(\Omega)}^2\,d\mathrm{s}\,, \end{aligned} \tag{4.33}$$

with $\widetilde{C}_1$ having the same dependence on the data and parameters as C_1.

In order to bound the last term in (4.33), we differentiate in time the first row of (4.27), test with $(\boldsymbol{\tau}_h,\underline{\mathbf{v}}_h) = (\boldsymbol{\theta}_{\boldsymbol{\sigma}},\partial_t\,\boldsymbol{\theta}_{\underline{\mathbf{u}}})$, and use the identity $(\boldsymbol{\delta}_{\boldsymbol{\sigma}},\partial_t\,\boldsymbol{\theta}_{\boldsymbol{\gamma}})_\Omega = \partial_t\,(\boldsymbol{\delta}_{\boldsymbol{\sigma}},\boldsymbol{\theta}_{\boldsymbol{\gamma}})_\Omega - (\partial_t\,\boldsymbol{\delta}_{\boldsymbol{\sigma}},\boldsymbol{\theta}_{\boldsymbol{\gamma}})_\Omega$, thus obtaining

$$\begin{aligned} &\frac{1}{2\,\mu}\left(\frac{1}{\rho}\,\partial_t\,\boldsymbol{\theta}_{\boldsymbol{\sigma}}^{\rm d},\boldsymbol{\theta}_{\boldsymbol{\sigma}}^{\rm d}\right)_\Omega + \Big(\rho\,\partial_t\,\boldsymbol{\theta}_{\mathbf{u}},\partial_t\,\boldsymbol{\theta}_{\mathbf{u}}\Big)_\Omega + \mu\,\Big(\mathbf{K}^{-1}\,\boldsymbol{\theta}_{\mathbf{u}},\partial_t\,\boldsymbol{\theta}_{\mathbf{u}}\Big)_\Omega \\ &\quad = -\frac{1}{2\mu}\left(\frac{1}{\rho}\,\partial_t\,\boldsymbol{\delta}_{\boldsymbol{\sigma}}^{\rm d},\boldsymbol{\theta}_{\boldsymbol{\sigma}}^{\rm d}\right)_\Omega - \Big(\rho\,\partial_t\,\boldsymbol{\delta}_{\mathbf{u}},\partial_t\,\boldsymbol{\theta}_{\mathbf{u}}\Big)_\Omega - (\boldsymbol{\theta}_{\boldsymbol{\sigma}},\partial_t\,\boldsymbol{\delta}_{\boldsymbol{\gamma}})_\Omega + \partial_t\,(\boldsymbol{\delta}_{\boldsymbol{\sigma}},\boldsymbol{\theta}_{\boldsymbol{\gamma}})_\Omega \\ &\qquad - (\partial_t\,\boldsymbol{\delta}_{\boldsymbol{\sigma}},\boldsymbol{\theta}_{\boldsymbol{\gamma}})_\Omega + \frac{1}{d}\left(\frac{\nabla\rho}{\rho}\cdot\partial_t\,\mathbf{e}_{\mathbf{u}},\mathrm{tr}(\boldsymbol{\theta}_{\boldsymbol{\sigma}})\right)_\Omega - \mu\,\Big(\mathbf{K}^{-1}\,\boldsymbol{\delta}_{\mathbf{u}},\partial_t\,\boldsymbol{\theta}_{\mathbf{u}}\Big)_\Omega\,. \end{aligned}$$

Now we use the monotonicity properties (cf. (3.5a) and (3.5b)), Cauchy–Schwarz and Young's inequalities (cf. (1.4)), so that, after some algebraic manipulations, the previous estimate becomes

$$\begin{aligned}\frac{1}{4\,\mu}\,\partial_t\left(\frac{1}{\rho}\,\boldsymbol{\theta}_{\boldsymbol{\sigma}}^{\mathrm{d}},\boldsymbol{\theta}_{\boldsymbol{\sigma}}^{\mathrm{d}}\right)_{\Omega} &+ \frac{\rho_0}{8}\,\|\partial_t\,\boldsymbol{\theta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\mu}{2}\,\partial_t\left(\mathbf{K}^{-1}\,\boldsymbol{\theta}_{\mathbf{u}},\boldsymbol{\theta}_{\mathbf{u}}\right)_{\Omega} \\ &\leq C_2\left(\|\partial_t\,\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t\,\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{L}^s(\Omega)} + \|\partial_t\,\boldsymbol{\delta}_{\boldsymbol{\gamma}}\|^2_{\mathbb{L}^2(\Omega)}\right) \\ &\quad + \delta_1\,\|\boldsymbol{\theta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}} + \delta_2\,\|\boldsymbol{\theta}_{\boldsymbol{\gamma}}\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{d\,\rho_0}\left\|\frac{\nabla\rho}{\rho}\right\|^2_{\mathbf{L}^\infty(\Omega)}\|\boldsymbol{\theta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}} + \partial_t\,(\boldsymbol{\delta}_{\boldsymbol{\sigma}},\boldsymbol{\theta}_{\boldsymbol{\gamma}})_{\Omega}\,,\end{aligned} \tag{4.34}$$

with C_2 depending only on μ, ρ_0, ρ_1, $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$, and δ_1, δ_2 arbitrary positive numbers to be chosen later. Then, integrating (4.34) from 0 to $t \in (0,T]$, and using the assumption (4.30), we get

$$\begin{aligned}\frac{1}{4\,\mu\,\rho_1}\,\|\boldsymbol{\theta}_{\boldsymbol{\sigma}}^{\mathrm{d}}(t)\|^2_{\mathbb{L}^2(\Omega)} &+ \frac{\mu\,C_{\mathbf{K}}}{2}\,\|\boldsymbol{\theta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\rho_0}{8}\int_0^t\|\partial_t\,\boldsymbol{\theta}_{\mathbf{u}}(\mathrm{s})\|^2_{\mathbf{L}^2(\Omega)}\,d\mathrm{s} \\ &\leq C_2\int_0^t\left(\|\partial_t\,\boldsymbol{\delta}_{\boldsymbol{\sigma}}(\mathrm{s})\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t\,\boldsymbol{\delta}_{\mathbf{u}}(\mathrm{s})\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\delta}_{\mathbf{u}}(\mathrm{s})\|^2_{\mathbf{L}^s(\Omega)} + \|\partial_t\,\boldsymbol{\delta}_{\boldsymbol{\gamma}}(\mathrm{s})\|^2_{\mathbb{L}^2(\Omega)}\right)d\mathrm{s} \\ &\quad + \int_0^t\left(\delta_1\,\|\boldsymbol{\theta}_{\boldsymbol{\sigma}}(\mathrm{s})\|^2_{\mathbb{X}} + \delta_2\,\|\boldsymbol{\theta}_{\boldsymbol{\gamma}}(\mathrm{s})\|^2_{\mathbb{L}^2(\Omega)}\right)d\mathrm{s} + \frac{\rho_0\,\Lambda_{\mathrm{d}}^2}{128\,\rho_1^2\,\|\mathbf{i}_{s,2}\|^2}\int_0^t\|\boldsymbol{\theta}_{\boldsymbol{\sigma}}(\mathrm{s})\|^2_{\mathbb{X}}\,d\mathrm{s} \\ &\quad + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(t)\|_{\mathbb{L}^2(\Omega)}\,\|\boldsymbol{\theta}_{\boldsymbol{\gamma}}(t)\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(0)\|_{\mathbb{L}^2(\Omega)}\,\|\boldsymbol{\theta}_{\boldsymbol{\gamma}}(0)\|_{\mathbb{L}^2(\Omega)} + \frac{1}{4\mu\,\rho_0}\,\|\boldsymbol{\theta}_{\boldsymbol{\sigma}}^{\mathrm{d}}(0)\|^2_{\mathbb{L}^2(\Omega)} \\ &\quad + \frac{\mu}{2}\,\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}\,\|\boldsymbol{\theta}_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)}\,.\end{aligned} \tag{4.35}$$

To bound the term $\|\boldsymbol{\theta}_{\boldsymbol{\gamma}}(t)\|_{\mathbb{L}^2(\Omega)}$, we observe from the first equation in (4.27) that

$$[\widetilde{\mathbf{B}}'(\boldsymbol{\theta}_{\underline{\mathbf{u}}}),\boldsymbol{\tau}_h] = -[\mathbf{A}(\mathbf{e}_{\boldsymbol{\sigma}}),\boldsymbol{\tau}_h] - (\boldsymbol{\delta}_{\boldsymbol{\gamma}},\boldsymbol{\tau}_h)_{\Omega} - [\mathbf{D}_\rho'(\boldsymbol{\delta}_{\underline{\mathbf{u}}}),\boldsymbol{\tau}_h] \qquad \forall\,\boldsymbol{\tau}_h \in \mathbb{X}_h\,.$$

Then, applying the discrete inf-sup condition of $\widetilde{\mathbf{B}}$ (cf. (4.13)), together with the Cauchy–Schwarz inequality, yields

$$\begin{aligned}\|\boldsymbol{\theta}_{\boldsymbol{\gamma}}(t)\|_{\mathbb{L}^2(\Omega)} &\leq \frac{1}{\beta_{\mathrm{d}}\,\mu\,\rho_0}\,\|\boldsymbol{\theta}_{\boldsymbol{\sigma}}^{\mathrm{d}}(t)\|_{\mathbb{L}^2(\Omega)} + \frac{1}{\beta_{\mathrm{d}}\,\mu\,\rho_0}\,\|\boldsymbol{\delta}_{\boldsymbol{\sigma}}^{\mathrm{d}}(t)\|_{\mathbb{L}^2(\Omega)} \\ &\quad + \frac{2}{\beta_{\mathrm{d}}}\,\|\boldsymbol{\delta}_{\boldsymbol{\gamma}}(t)\|_{\mathbb{L}^2(\Omega)} + \frac{2}{\sqrt{d}\,\beta_{\mathrm{d}}}\left\|\frac{\nabla\rho}{\rho}\right\|_{\mathbf{L}^r(\Omega)}\|\boldsymbol{\delta}_{\mathbf{u}}(t)\|_{\mathbf{L}^s(\Omega)}\,.\end{aligned}$$

Consequently, by suitably applying Young's inequality (cf. (1.4)), we obtain

$$\|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(t)\|_{\mathbb{L}^2(\Omega)}\,\|\boldsymbol{\theta}_{\boldsymbol{\gamma}}(t)\|_{\mathbb{L}^2(\Omega)} \leq \frac{1}{4\mu\,\rho_1}\,\|\boldsymbol{\theta}^{\mathrm{d}}_{\boldsymbol{\sigma}}(t)\|^2_{\mathbb{L}^2(\Omega)} + C_3\left(\|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(t)\|^2_{\mathbb{X}} + \|\boldsymbol{\delta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\gamma}}(t)\|^2_{\mathbb{L}^2(\Omega)}\right), \tag{4.36}$$

with $C_3 > 0$ depending only on β_{d}, μ, ρ_0, ρ_1 and $\|\nabla\rho/\rho\|_{\mathbf{L}^r(\Omega)}$.

On the other hand, to bound the last three terms in (4.35), we first observe that, similarly as in (4.32), by using the discrete global inf-sup condition (4.7), together with Cauchy–Schwarz and Young's inequalities, and assumption (4.20), it follows from (4.29) that

$$\|\boldsymbol{\theta}_{\boldsymbol{\sigma}_0}\|^2_{\mathbb{X}} + \|\boldsymbol{\theta}_{\mathbf{u}_0}\|^2_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\theta}_{\boldsymbol{\gamma}_0}\|^2_{\mathbb{L}^2(\Omega)} \leq C_4\left(\|\boldsymbol{\delta}_{\boldsymbol{\sigma}_0}\|^2_{\mathbb{X}} + \|\boldsymbol{\delta}_{\mathbf{u}_0}\|^2_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\gamma}_0}\|^2_{\mathbb{L}^2(\Omega)}\right), \tag{4.37}$$

with $C_4 > 0$ depending only on Λ_{d}, μ, $|\Omega|$, $\|\nabla\rho/\rho\|_{\mathbf{L}^\infty(\Omega)}$, $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$ and ρ_0. In turn, we recall from Theorems 3.9 and 4.7 that $\boldsymbol{\sigma}^{\mathrm{d}}(0) = \boldsymbol{\sigma}^{\mathrm{d}}_0$ and $\boldsymbol{\sigma}^{\mathrm{d}}_h(0) = \boldsymbol{\sigma}^{\mathrm{d}}_{h,0}$, which allows us to estimate

$$\begin{aligned}\|\boldsymbol{\theta}^{\mathrm{d}}_{\boldsymbol{\sigma}}(0)\|_{\mathbb{L}^2(\Omega)} &= \left\|\boldsymbol{\Pi}^k_{h,0}(\boldsymbol{\sigma}(0))^{\mathrm{d}} - \boldsymbol{\sigma}^{\mathrm{d}}(0) + \boldsymbol{\sigma}^{\mathrm{d}}_0 - \boldsymbol{\Pi}^k_{h,0}(\boldsymbol{\sigma}_0)^{\mathrm{d}} + \boldsymbol{\Pi}^k_{h,0}(\boldsymbol{\sigma}_0)^{\mathrm{d}} - \boldsymbol{\sigma}^{\mathrm{d}}_{h,0}\right\|_{\mathbb{L}^2(\Omega)} \\ &\leq \|\boldsymbol{\delta}^{\mathrm{d}}_{\boldsymbol{\sigma}}(0)\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{\delta}^{\mathrm{d}}_{\boldsymbol{\sigma}_0}\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{\theta}^{\mathrm{d}}_{\boldsymbol{\sigma}_0}\|_{\mathbb{L}^2(\Omega)} \leq \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(0)\|_{\mathbb{X}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}_0}\|_{\mathbb{X}} + \|\boldsymbol{\theta}_{\boldsymbol{\sigma}_0}\|_{\mathbb{X}}\,.\end{aligned} \tag{4.38}$$

Moreover, by the same results, $\mathbf{u}(0) = \mathbf{u}_0$, $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}_0$, $\mathbf{u}_h(0) = \mathbf{u}_{h,0}$, and $\boldsymbol{\gamma}_h(0) = \boldsymbol{\gamma}_{h,0}$, which implies that $\boldsymbol{\theta}_{\mathbf{u}_0} = \boldsymbol{\theta}_{\mathbf{u}}(0)$ and $\boldsymbol{\theta}_{\boldsymbol{\gamma}_0} = \boldsymbol{\theta}_{\boldsymbol{\gamma}}(0)$. Hence, substituting these facts into (4.37), and then combining it with (4.38), we obtain that

$$\|\boldsymbol{\theta}^{\mathrm{d}}_{\boldsymbol{\sigma}}(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{\theta}_{\mathbf{u}}(0)\|^2_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\theta}_{\boldsymbol{\gamma}}(0)\|^2_{\mathbb{L}^2(\Omega)} \leq C_5\,\mathrm{E}_{\boldsymbol{\delta},0}\,, \tag{4.39}$$

where $C_5 > 0$ depends only on C_4, and $\mathrm{E}_{\boldsymbol{\delta},0}$ is defined as

$$\mathrm{E}_{\boldsymbol{\delta},0} := \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(0)\|^2_{\mathbb{X}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}_0}\|^2_{\mathbb{X}} + \|\boldsymbol{\delta}_{\mathbf{u}_0}\|^2_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\gamma}_0}\|^2_{\mathbb{L}^2(\Omega)}\,.$$

Thus, by replacing (4.36) and (4.39) into (4.35), and then applying Young's inequality (cf. (1.4)) together with some algebraic manipulations, we deduce that

$$
\begin{aligned}
&\frac{\mu\, C_{\mathbf{K}}}{2}\, \|\boldsymbol{\theta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\rho_0}{8} \int_0^t \|\partial_t \boldsymbol{\theta}_{\mathbf{u}}(\mathrm{s})\|^2_{\mathbf{L}^2(\Omega)}\, ds \\
&\quad \leq C_2 \int_0^t \Big(\|\partial_t\, \boldsymbol{\delta}_{\boldsymbol{\sigma}}(\mathrm{s})\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t\, \boldsymbol{\delta}_{\mathbf{u}}(\mathrm{s})\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\delta}_{\mathbf{u}}(\mathrm{s})\|^2_{\mathbf{L}^s(\Omega)} + \|\partial_t\, \boldsymbol{\delta}_{\boldsymbol{\gamma}}(\mathrm{s})\|^2_{\mathbb{L}^2(\Omega)} \Big)\, ds \\
&\quad + C_6 \Big(\|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(t)\|^2_{\mathbb{X}} + \|\boldsymbol{\delta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\gamma}}(t)\|^2_{\mathbb{L}^2(\Omega)} + \mathrm{E}_{\boldsymbol{\delta},0} \Big) \\
&\quad + \int_0^t \Big(\delta_1\, \|\boldsymbol{\theta}_{\boldsymbol{\sigma}}(\mathrm{s})\|^2_{\mathbb{X}} + \delta_2\, \|\boldsymbol{\theta}_{\boldsymbol{\gamma}}(\mathrm{s})\|^2_{\mathbb{L}^2(\Omega)} \Big)\, ds + \frac{\rho_0\, \Lambda_{\mathrm{d}}^2}{128\, \rho_1^2\, \|\mathbf{i}_{s,2}\|^2} \int_0^t \|\boldsymbol{\theta}_{\boldsymbol{\sigma}}(\mathrm{s})\|^2_{\mathbb{X}}\, ds\,,
\end{aligned}
\tag{4.40}
$$

with $C_6 > 0$ depending only on C_3 and C_5. Then, we substitute (4.40) into (4.33), and choose δ_1 and δ_2 sufficiently small, thereby yielding

$$
\begin{aligned}
&\int_0^t \Big(\|\boldsymbol{\theta}_{\boldsymbol{\sigma}}(\mathrm{s})\|^2_{\mathbb{X}} + \|\boldsymbol{\theta}_{\mathbf{u}}(\mathrm{s})\|^2_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\theta}_{\boldsymbol{\gamma}}(\mathrm{s})\|^2_{\mathbb{L}^2(\Omega)} \Big) ds + \|\boldsymbol{\theta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \leq \widehat{C}_1 \Bigg\{ \int_0^t \Big(\|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(\mathrm{s})\|^2_{\mathbb{X}} + \|\boldsymbol{\delta}_{\mathbf{u}}(\mathrm{s})\|^2_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\gamma}}(\mathrm{s})\|^2_{\mathbb{L}^2(\Omega)} \Big)\, ds \\
&\qquad + \int_0^t \Big(\|\partial_t \boldsymbol{\delta}_{\mathbf{u}}(\mathrm{s})\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t \boldsymbol{\delta}_{\boldsymbol{\sigma}}(\mathrm{s})\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \boldsymbol{\delta}_{\boldsymbol{\gamma}}(\mathrm{s})\|^2_{\mathbb{L}^2(\Omega)} \Big)\, ds \Bigg\} \\
&\quad + \widehat{C}_2 \Big(\|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(t)\|^2_{\mathbb{X}} + \|\boldsymbol{\delta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^s(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\gamma}}(t)\|^2_{\mathbb{L}^2(\Omega)} + \mathrm{E}_{\boldsymbol{\delta},0} \Big)\,,
\end{aligned}
\tag{4.41}
$$

with $\widehat{C}_1$ and $\widehat{C}_2$ depending on the previous constants, physical parameters and data. Finally, by using the error decompositions (4.25) together with (4.41), we obtain (4.31), as desired. $\square$

Next, in order to obtain the theoretical rates of convergence for the semidiscrete scheme (4.4), we recall the approximation properties associated with the finite element spaces (cf. (4.2) and (4.3)), and the operators $\boldsymbol{\mathcal{P}}_h^k$, $\mathbb{P}_h^{k+1}$, and $\boldsymbol{\Pi}_h^k$. These properties basically follow from classical interpolation estimates of Sobolev spaces and the commuting diagram property (cf. (4.24)). For details we refer to [29, Section 3.4.4], [10, Section 2.5.5], or [16, Section 4.2.1].

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ There exists a positive constant C, independent of h, such that for each $\vartheta \in (0, k+1]$ and for each $\boldsymbol{\tau} \in \mathbb{H}^{\vartheta}(\Omega) \cap \mathbb{X}$, with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{\vartheta,\ell}(\Omega)$, there holds

$$
\|\boldsymbol{\tau} - \boldsymbol{\Pi}_h^k(\boldsymbol{\tau})\|_{\mathbb{X}} \leq C\, h^{\vartheta} \Big\{ \|\boldsymbol{\tau}\|_{\mathbb{H}^{\vartheta}(\Omega)} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\mathbf{W}^{\vartheta,\ell}(\Omega)} \Big\}\,.
$$

$(\mathbf{AP}_h^{\mathbf{u}})$ There exists a positive constant C, independent of h, such that for each $\vartheta \in [0, k+1]$ and for each $\mathbf{v} \in \mathbf{W}^{\vartheta,s}(\Omega)$, there holds

$$\|\mathbf{v} - \boldsymbol{\mathcal{P}}_h^k(\mathbf{v})\|_{\mathbf{L}^s(\Omega)} \leq C\, h^{\vartheta}\, \|\mathbf{v}\|_{\mathbf{W}^{\vartheta,s}(\Omega)} \,.$$

$(\mathbf{AP}_h^{\boldsymbol{\gamma}})$ There exists a positive constant C, independent of h, such that for each $\vartheta \in [0, k+1]$ and for each $\boldsymbol{\eta} \in \mathbb{H}^{\vartheta}(\Omega) \cap \mathbb{L}^2_{\mathrm{skew}}(\Omega)$, there holds

$$\|\boldsymbol{\eta} - \boldsymbol{\mathscr{P}}_h^{k+1}(\boldsymbol{\eta})\|_{\mathbb{L}^2(\Omega)} \leq C\, h^{\vartheta}\, \|\boldsymbol{\eta}\|_{\mathbb{H}^{\vartheta}(\Omega)} \,.$$

It is worth noting that $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ is stated in terms of $\boldsymbol{\Pi}_h^k$ instead of $\boldsymbol{\Pi}_{h,0}^k$. However, it is not difficult to prove that, for all $\boldsymbol{\tau} \in \mathbb{H}^{\vartheta}(\Omega) \cap \mathbb{X}$,

$$\|\boldsymbol{\tau} - \boldsymbol{\Pi}_{h,0}^k(\boldsymbol{\tau})\|_{\mathbb{X}} \leq (1 + d^2\, |\Omega|^2)^{1/2}\, \|\boldsymbol{\tau} - \boldsymbol{\Pi}_h^k(\boldsymbol{\tau})\|_{\mathbb{X}} \,,$$

so the approximation property also holds for this operator, up to a multiplicative constant independent of h. In this way, it follows that, under an extra regularity assumption on the exact solution, there exist positive constants $C(\boldsymbol{\sigma})$, $C(\mathbf{u})$, $C(\boldsymbol{\gamma})$, $C(\partial_t\boldsymbol{\sigma})$, $C(\partial_t\mathbf{u})$, and $C(\partial_t\boldsymbol{\gamma})$, whose explicit expression are obtained from the right-hand side of the foregoing approximation properties, such that

$$\begin{gathered}
\|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} \leq C(\boldsymbol{\sigma})\, h^{\vartheta}\,, \quad \|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{L}^s(\Omega)} \leq C(\mathbf{u})\, h^{\vartheta}\,, \quad \|\boldsymbol{\delta}_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)} \leq C(\boldsymbol{\gamma})\, h^{\vartheta}\,, \\
\|\partial_t\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{L}^2(\Omega)} \leq C(\partial_t\boldsymbol{\sigma})\, h^{\vartheta}\,, \quad \|\partial_t\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} \leq C(\partial_t\mathbf{u})\, h^{\vartheta}\,, \quad \|\partial_t\boldsymbol{\delta}_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)} \leq C(\partial_t\boldsymbol{\gamma})\, h^{\vartheta}\,.
\end{gathered} \tag{4.42}$$

The following result establishes the theoretical rates of convergence of the semidiscrete continuous-in-time scheme (4.4).

Theorem 4.9. *Assume the same hypotheses as in Theorem 4.8. Furthermore, suppose that there exists* $\vartheta \in (0, k+1]$ *such that* $\boldsymbol{\sigma} \in \mathbb{H}^{\vartheta}(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{\vartheta,\ell}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{\vartheta,s}(\Omega)$ *and* $\boldsymbol{\gamma} \in \mathbb{H}^{\vartheta}(\Omega)$. *Then, there exists a positive constant* $\mathcal{C}(\boldsymbol{\sigma}, \underline{\mathbf{u}})$, *depending only on* $\mathcal{C}$ *(cf. Theorem 4.8)* *and the constants defined in* (4.42)*, such that*

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathrm{L}^2(0,T;\mathbb{X})} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{L}^s(\Omega))} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))} \leq \mathcal{C}(\boldsymbol{\sigma}, \underline{\mathbf{u}})\, h^{\vartheta}\,. \qquad (4.43)$$

Proof. It follows from using (4.42) into (4.31), and performing some algebraic manipulations. We omit further details. □

Remark 4.2. *The error analysis remains valid for any triplet of finite element spaces that are stable for the semidiscrete continuous-in-time scheme (cf.* (4.4)*), provided that the orthogonal projectors* $\boldsymbol{\mathcal{P}}_h^k$ *and* $\mathbb{P}_h^{k+1}$ *are available, and that a mixed interpolation operator satisfying the commuting diagram property* (4.24) *exists. In particular, for the choice of AFW elements (cf.* (4.2)*), we can employ the BDM interpolation operator (cf. [10, Section 2.5.1]), which also satisfies* (4.24)*. Furthermore, since property* ($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) *also holds for this operator (cf. [10, Proposition 2.5.4]), Theorem 4.9 remains unchanged.*

CHAPTER 5

Fully discrete approximation

In this chapter, we introduce and analize a fully discrete approximation of (2.19). For this purpose, we employ the backward Euler method for the time discretization. Let Δt be the time step, $T = N\Delta t$, and let $t_n = n\Delta t$, for each $n \in \{0, \ldots, N\}$. Let $d_t u^n = (\Delta t)^{-1}\,(u^n - u^{n-1})$ be the first order (backward) discrete time derivative, where $u^n := u(t_n)$. Then, the fully discrete method reads: Given $\mathbf{f}^n \in \mathbf{L}^2(\Omega)$ and $(\boldsymbol{\sigma}_h^0, \underline{\mathbf{u}}_h^0) = (\boldsymbol{\sigma}_{h,0}, (\mathbf{u}_{h,0}, \boldsymbol{\gamma}_{h,0}))$ satisfying (4.5), find $(\boldsymbol{\sigma}_h^n, \underline{\mathbf{u}}_h^n) := (\boldsymbol{\sigma}_h^n, (\mathbf{u}_h^n, \boldsymbol{\gamma}_h^n)) \in \mathbb{X}_h \times \mathbf{Y}_h$, for $n \in \{1, \ldots, N\}$, such that

$$\begin{aligned}
[\mathbf{A}(\boldsymbol{\sigma}_h^n), \boldsymbol{\tau}_h] + [\mathbf{B}'(\underline{\mathbf{u}}_h^n), \boldsymbol{\tau}_h] + [\mathbf{D}'_\rho(\underline{\mathbf{u}}_h^n), \boldsymbol{\tau}_h] &= 0 && \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h\,, \\
d_t\,[\mathbf{E}(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] - [\mathbf{B}(\boldsymbol{\sigma}_h^n), \underline{\mathbf{v}}_h] + [\mathbf{C}(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] &= [\mathbf{F}^n, \underline{\mathbf{v}}_h] && \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h\,,
\end{aligned} \tag{5.1}$$

where $[\mathbf{F}^n, \underline{\mathbf{v}}_h] := (\mathbf{f}^n, \mathbf{v}_h)_\Omega$.

In what follows, for a separable Banach space V equipped with the norm $\|\cdot\|_V$, we define the following discrete-in-time norms:

$$\|u\|^2_{\ell^2(0,T;V)} := \Delta t \sum_{n=1}^{N} \|u^n\|^2_V \quad \text{and} \quad \|u\|_{\ell^\infty(0,T;V)} := \max_{1\le n\le N} \|u^n\|_V \,.$$

Endowed with these norms, we define the Banach spaces $\ell^2(0,T;V)$ and $\ell^\infty(0,T;V)$ respectively as

$$\begin{aligned} \ell^2(0,T;V) &:= \Big\{ u = (u^1,\dots,u^N) \in V^N \ : \ \|u\|_{\ell^2(0,T;V)} < +\infty \Big\}, \quad \text{and} \\ \ell^\infty(0,T;V) &:= \Big\{ u = (u^1,\dots,u^N) \in V^N \ : \ \|u\|_{\ell^\infty(0,T;V)} < +\infty \Big\}. \end{aligned}$$

We begin our analysis of the fully discrete scheme (5.1) by establishing a stability result.

Theorem 5.1. *Suppose that the hypotheses of Theorem 3.9 and Lemma 4.1 hold. Assume further that the porosity satisfies*

$$\left\| \frac{\nabla\rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \le \frac{\sqrt{d}\,\Lambda_{\mathrm{d}}\,\rho_0}{8\,\rho_1\,\|\mathbf{i}_{s,2}\|} \,. \tag{5.2}$$

Let $(\boldsymbol{\sigma}_h^n, \underline{\mathbf{u}}_h^n) = (\boldsymbol{\sigma}_h^n, (\mathbf{u}_h^n, \boldsymbol{\gamma}_h^n)) \in \mathbb{X}_h \times \mathbf{Y}_h$ *be a solution to* (5.1) *with* $(\boldsymbol{\sigma}_h^0, \underline{\mathbf{u}}_h^0) = (\boldsymbol{\sigma}_{h,0}, (\mathbf{u}_{h,0}, \boldsymbol{\gamma}_{h,0}))$ *satisfying* (4.5), *and* $\mathbf{f}^n \in \mathbf{L}^2(\Omega)$ *with* $n \in \{1,\dots,N\}$. *Then, there exists a positive constant* $\widehat{C}_{\mathrm{B}}$, *depending only on* μ, ρ_0, ρ_1, $C_{\mathbf{K}}$, Λ_{d}, $C_{0,\mathrm{d}}$ *(cf.* (4.9)*)*, $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$ *and* $|\Omega|$, *such that*

$$\begin{aligned} &\|\boldsymbol{\sigma}_h\|^2_{\ell^2(0,T;\mathbb{X})} + \|\mathbf{u}_h\|^2_{\ell^2(0,T;\mathbf{L}^s(\Omega))} + \|\mathbf{u}_h\|^2_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\gamma}_h\|^2_{\ell^2(0,T;\mathbb{L}^2(\Omega))} \\ &\quad \le \widehat{C}_{\mathrm{B}} \Big\{ \|\mathbf{f}\|^2_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{div}(\rho\,\mathbf{e}(\mathbf{u}_0))\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} \Big\} . \end{aligned} \tag{5.3}$$

Proof. Following a similar approach to that in Theorem 3.4, we first apply the discrete global inf-sup condition (4.7), use the system (5.1), and recall that $\|\nabla\rho/\rho\|_{\mathbf{L}^r(\Omega)} \le \sqrt{d}\,\Lambda_{\mathrm{d}}/2$ (cf. (4.8)), to obtain

$$\frac{\Lambda_{\mathrm{d}}}{2}\,\|(\boldsymbol{\sigma}_h^n, \underline{\mathbf{u}}_h^n)\|_{\mathbb{X}\times\mathbf{Y}} \le \rho_1\,\|\mathbf{i}_{s,2}\|\,\|d_t\mathbf{u}_h^n\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{i}_{s,2}\|\,\|\mathbf{f}^n\|_{\mathbf{L}^2(\Omega)} \,,$$

which, upon squaring, summing over the time steps $n \in \{1,\dots,N\}$, and multiplying by Δt, becomes the discrete counterpart of (3.21),

$$
\begin{aligned}
&\widehat{C}_1 \Big\{ \|\boldsymbol{\sigma}_h\|^2_{\ell^2(0,T;\mathbb{X})} + \|\mathbf{u}_h\|^2_{\ell^2(0,T;\mathbf{L}^s(\Omega))} + \|\boldsymbol{\gamma}_h\|^2_{\ell^2(0,T;\mathbb{L}^2(\Omega))} \Big\} \\
&\qquad \leq \|d_t \mathbf{u}_h\|^2_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \rho_1^{-2}\,\|\mathbf{f}\|^2_{\ell^2(0,T;\mathbf{L}^2(\Omega))}\,,
\end{aligned}
\tag{5.4}
$$

where $\widehat{C}_1 := \Lambda_{\mathrm{d}}^2/(8\,\|\mathbf{i}_{s,2}\|^2\,\rho_1^2)$.

In order to bound the first term on the right-hand side of (5.4), we note that a discrete time differentiation of the first equation in (5.1) can be obtained merely through algebraic manipulations, which yield

$$
[\mathbf{A}(d_t\boldsymbol{\sigma}_h^n), \boldsymbol{\tau}_h] + [\mathbf{B}'(d_t\underline{\mathbf{u}}_h^n), \boldsymbol{\tau}_h] + [\mathbf{D}'_\rho(d_t\underline{\mathbf{u}}_h^n), \boldsymbol{\tau}_h] = 0 \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h\,. \tag{5.5}
$$

In particular, testing with $\boldsymbol{\tau}_h = \boldsymbol{\sigma}_h^n$ and using the second row of (5.1) with $\underline{\mathbf{v}}_h = d_t\underline{\mathbf{u}}_h^n$ to handle the second term of (5.5), we arrive at

$$
[\mathbf{A}(d_t\boldsymbol{\sigma}_h^n), \boldsymbol{\sigma}_h^n] + [\mathbf{E}(d_t\underline{\mathbf{u}}_h^n), d_t\underline{\mathbf{u}}_h^n] + [\mathbf{C}(\underline{\mathbf{u}}_h^n), d_t\underline{\mathbf{u}}_h^n] = [\mathbf{F}^n, d_t\underline{\mathbf{u}}_h^n] - [\mathbf{D}'_\rho(d_t\underline{\mathbf{u}}_h^n), \boldsymbol{\sigma}_h^n]\,. \tag{5.6}
$$

In turn, owing to the linearity of $\mathbf{A}$ and $\mathbf{C}$, simple algebraic manipulations show that

$$
\begin{aligned}
[\mathbf{A}(d_t\boldsymbol{\sigma}_h^n), \boldsymbol{\sigma}_h^n] &= \frac{1}{2}\, d_t[\mathbf{A}(\boldsymbol{\sigma}_h^n), \boldsymbol{\sigma}_h^n] + \frac{\Delta t}{2}\,[\mathbf{A}(d_t\boldsymbol{\sigma}_h^n), d_t\boldsymbol{\sigma}_h^n]\,,\\
[\mathbf{C}(\underline{\mathbf{u}}_h^n), d_t\underline{\mathbf{u}}_h^n] &= \frac{1}{2}\, d_t[\mathbf{C}(\underline{\mathbf{u}}_h^n), \underline{\mathbf{u}}_h^n] + \frac{\Delta t}{2}\,[\mathbf{C}(d_t\underline{\mathbf{u}}_h^n), d_t\underline{\mathbf{u}}_h^n]\,,
\end{aligned}
\tag{5.7}
$$

so, substituting (5.7) into (5.6) and using the monotonicity properties (cf. (3.5a) and (3.5b)), together with Cauchy–Schwarz and Young's inequalities, leads to the discrete version of (3.22),

$$
\begin{aligned}
&\left(\frac{\rho_0}{4} + \frac{\mu\, C_{\mathbf{K}}\,\Delta t}{2}\right) \|d_t\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \frac{\Delta t}{4\mu\,\rho_1}\,\|(d_t\boldsymbol{\sigma}_h^n)^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2}\, d_t\Big\{[\mathbf{A}(\boldsymbol{\sigma}_h^n), \boldsymbol{\sigma}_h^n] + [\mathbf{C}(\underline{\mathbf{u}}_h^n), \underline{\mathbf{u}}_h^n]\Big\} \\
&\qquad \leq \frac{1}{d\rho_0}\,\left\|\frac{\nabla\rho}{\rho}\right\|^2_{\mathbf{L}^\infty(\Omega)} \|\boldsymbol{\sigma}_h^n\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2\,\rho_0}\,\|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)}\,.
\end{aligned}
\tag{5.8}
$$

Next, using (5.2) to bound $\|\nabla\rho/\rho\|^2_{\mathbf{L}^\infty(\Omega)}/(d\rho_0) \leq (\rho_0\widehat{C}_1)/8$, summing over the time steps $n \in$

$\{1,\dots,m\}$, with $m \in \{1,\dots,N\}$, multiplying by Δt, and invoking once more the monotonicity of $\mathbf{A}$ and $\mathbf{C}$ (cf. (3.5a)) together with the stability properties (cf. (3.4a) and (3.4d)), we obtain

$$\begin{aligned}\left(\frac{\rho_0}{4} + \frac{\mu\, C_{\mathbf{K}}\, \Delta t}{2}\right) \Delta t \sum_{n=1}^{m} \|d_t \mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{4\mu\rho_1}\, \|(\boldsymbol{\sigma}_h^m)^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega)} + \frac{\mu\, C_{\mathbf{K}}}{2}\, \|\mathbf{u}_h^m\|^2_{\mathbf{L}^2(\Omega)} \\ \leq \frac{\rho_0 \widehat{C}_1\, \Delta t}{8} \sum_{n=1}^{m} \|\boldsymbol{\sigma}_h^n\|^2_{\mathbb{X}} + \frac{\Delta t}{2\rho_0} \sum_{n=1}^{m} \|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{4\mu\rho_0}\, \|\boldsymbol{\sigma}_h^0\|^2_{\mathbb{X}} \\ + \frac{\mu}{2\,\|\mathbf{i}_{s,2}\|^2}\, \|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}\, \|\mathbf{u}_h^0\|^2_{\mathbf{L}^s(\Omega)}\,.\end{aligned}$$

Notice that we have neglected the second term in (5.8). We now use the fact that $\boldsymbol{\sigma}_h^0 = \boldsymbol{\sigma}_{h,0}$ and $\mathbf{u}_h^0 = \mathbf{u}_{h,0}$, together with the estimate (4.18), and after some algebraic manipulations, and the omission of some terms for clarity, we obtain the discrete counterpart of (3.23),

$$\begin{aligned}\Delta t \sum_{n=1}^{m} \|d_t \mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\mu\, C_{\mathbf{K}}}{\rho_0}\, \|\mathbf{u}_h^m\|^2_{\mathbf{L}^2(\Omega)} \leq \frac{2\Delta t}{\rho_0^2} \sum_{n=1}^{m} \|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)} + \frac{\widehat{C}_1\, \Delta t}{2} \sum_{n=1}^{m} \|\boldsymbol{\sigma}_h^n\|^2_{\mathbb{X}} \\ + \widehat{C}_2 \left\{ \|\mathbf{div}(\rho\, \mathbf{e}(\mathbf{u}_0))\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} \right\},\end{aligned} \tag{5.9}$$

with $\widehat{C}_2$ depending only on $C_{0,\mathrm{d}}$ (cf. (4.18)), μ, ρ_0, $|\Omega|$ and $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$. Since $m \in \{1,\dots,N\}$ is arbitrary, (5.9) implies

$$\begin{aligned}\|d_t \mathbf{u}_h\|^2_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \frac{2\mu\, C_{\mathbf{K}}}{\rho_0}\, \|\mathbf{u}_h\|^2_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))} \leq \frac{2}{\rho_0^2}\, \|\mathbf{f}\|^2_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \frac{\widehat{C}_1}{2}\, \|\boldsymbol{\sigma}_h\|^2_{\ell^2(0,T;\mathbb{X})} \\ + \widehat{C}_2 \left\{ \|\mathbf{div}(\rho\, \mathbf{e}(\mathbf{u}_0))\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} \right\}.\end{aligned} \tag{5.10}$$

Thus, substituting (5.10) into (5.4) yields (5.3) with constant

$$\widehat{C}_{\mathrm{B}} := \frac{\max\left\{\rho_1^{-2} + 2\,\rho_0^{-2}, \widehat{C}_2\right\}}{\min\{\widehat{C}_1/2, 2\mu C_{\mathbf{K}}\, \rho_0^{-1}\}}\,.$$

□

Certainly, one possible way to prove the well-posedness of (5.1) is to use an induction argument to handle the discrete time derivative and to follow a similar approach to that in Lemma 4.5. This consists in establishing the well-posedness of (5.1) while neglecting the operator $\mathbf{D}'_\rho$, deducing a global inf-sup condition, and then assuming that $\|\nabla\rho/\rho\|_{\mathbf{L}^\infty(\Omega)}$ is sufficiently small, depending on the global inf-sup constant, to incorporate the term associated with $\mathbf{D}'_\rho$. However, this constant depends on the time step Δt, which in turn implies that $\|\nabla\rho/\rho\|_{\mathbf{L}^\infty(\Omega)}$ must be smaller than a constant depending on the time step. To overcome this difficulty, we proceed similarly to Chapters 3 and 4, introducing an auxiliary problem and a fixed-point strategy to establish the well-posedness of (5.1). In fact, let $\widehat{\mathcal{J}}_{\mathrm{d}} : \ell^2(0,T;\mathbb{X}_h) \to \ell^2(0,T;\mathbb{X}_h)$ be the operator defined by

$$\widehat{\mathcal{J}}_{\mathrm{d}}(\boldsymbol{\zeta}_h) := \boldsymbol{\sigma}_h\,,$$

where $(\boldsymbol{\sigma}_h, \underline{\mathbf{u}}_h)$ is the unique solution (to be confirmed below) to

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}_h^n), \boldsymbol{\tau}_h] + [\widetilde{\mathbf{B}}'(\underline{\mathbf{u}}_h^n), \boldsymbol{\tau}_h] &= 0 && \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_h\,, \\ d_t\,[\mathbf{E}(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] - [\widetilde{\mathbf{B}}(\boldsymbol{\sigma}_h^n), \underline{\mathbf{v}}_h] + [\mathbf{C}(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] &= [\widetilde{\mathbf{F}}^n_{\boldsymbol{\zeta}_h}, \underline{\mathbf{v}}_h] && \forall\, \underline{\mathbf{v}}_h \in \mathbf{Y}_h\,, \end{aligned} \tag{5.11}$$

for all $n \in \{1, \ldots, N\}$ with $(\boldsymbol{\sigma}_h^0, \underline{\mathbf{u}}_h^0)$ given by (4.5). Here, $[\widetilde{\mathbf{F}}^n_{\boldsymbol{\zeta}_h}, \underline{\mathbf{v}}_h] := (\mathbf{f}^n, \mathbf{v}_h)_\Omega - [\mathbf{D}(\boldsymbol{\zeta}_h^n), \underline{\mathbf{v}}_h]$.

Now, we notice that establishing the well-posedness of (5.1) is equivalent to prove that there exists a unique solution to the fixed-point equation

$$\widehat{\mathcal{J}}_{\mathrm{d}}(\boldsymbol{\sigma}_h) = \boldsymbol{\sigma}_h\,. \tag{5.12}$$

The following result asserts that (5.11) is well-posed and a stability result for the fixed-point system. Notice that, owing to the fully discrete nature of the problem, Theorem 3.1 need not be invoked, and hence no additional time regularity for $\boldsymbol{\sigma}_h$ and $\boldsymbol{\zeta}_h$, nor any assumptions on the initial conditions of their time derivatives, are required.

Theorem 5.2. *Suppose that the hypotheses of Theorem 3.9 and Lemma 4.5 hold. Then, $\widehat{\mathcal{J}}_{\mathrm{d}}$ is well-defined. More precisely, given $\boldsymbol{\zeta}_h \in \ell^2(0,T;\mathbb{X}_h)$, there exists a unique solution $(\boldsymbol{\sigma}_h, \underline{\mathbf{u}}_h)$ to (5.11), with $\boldsymbol{\sigma}_h \in \ell^2(0,T;\mathbb{X}_h)$, $\mathbf{u}_h \in \ell^\infty(0,T;\mathbf{H}_h^{\mathbf{u}})$, and $\boldsymbol{\gamma}_h \in \ell^2(0,T;\mathbb{H}_h^{\boldsymbol{\gamma}})$. Moreover, there exist positive constants $\widehat{C}_{\mathrm{B,d}}$ and $C_{\widehat{\mathcal{J}}_{\mathrm{d}}}$, with $\widehat{C}_{\mathrm{B,d}}$ depending only on μ, ρ_0, ρ_1, $\|\mathbf{K}^{-1}\|_{\mathbb{L}^\infty(\Omega)}$, Λ_{d}, $\widetilde{C}_{0,\mathrm{d}}$ (cf. (4.18)), d and $|\Omega|$, and $C_{\widehat{\mathcal{J}}_{\mathrm{d}}}$ depending only on ρ_0, ρ_1, Λ_{d}, d and $|\Omega|$, such that*

$$
\begin{aligned}
\|\widehat{\mathcal{J}}_{\mathrm{d}}(\boldsymbol{\zeta}_h)\|_{\ell^2(0,T;\mathbb{X})} \leq \widehat{C}_{\mathrm{B,d}} \Big\{ \|\mathbf{f}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{div}(\rho\, \mathbf{e}(\mathbf{u}_0))\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} \Big\} \\
+ C_{\widehat{\mathcal{J}}_{\mathrm{d}}} \left\| \frac{\nabla\rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} \|\boldsymbol{\zeta}_h\|_{\ell^2(0,T;\mathbb{X}_h)} \,.
\end{aligned} \tag{5.13}
$$

Proof. Let $\boldsymbol{\zeta}_h \in \ell^2(0,T;\mathbb{X}_h)$ be given and recall from Lemma 4.5 that we have the discrete initial conditions $(\boldsymbol{\sigma}_h^0, \underline{\mathbf{u}}_h^0)$ satisfying (4.18). We then proceed to establish the well-posedness of (5.11) at each time step by induction. In fact, assuming that $\mathbf{u}_h^{n-1}$ is known, we prove the existence and uniqueness of the problem by following the same arguments as in Lemma 4.4. Consequently, we obtain the existence and uniqueness of $\boldsymbol{\sigma}_h \in (\mathbb{X}_h)^N$ and $\underline{\mathbf{u}}_h \in (\mathbf{Y}_h)^N$ satisfying (5.11). In turn, to prove (5.13), one proceeds as in the proof of Theorem 5.1, arriving at (5.13) with a constant independent of h and Δt. Further details are omitted. □

Theorem 5.3. *Suppose that the hypotheses of Theorems 5.1 and 5.2 hold. Assume further that the porosity satisfies*

$$
C_{\widehat{\mathcal{J}}_{\mathrm{d}}} \left\| \frac{\nabla\rho}{\rho} \right\|_{\mathbf{L}^\infty(\Omega)} < 1 \,. \tag{5.14}
$$

Then, given $(\boldsymbol{\sigma}_h^0, \underline{\mathbf{u}}_h^0) = (\boldsymbol{\sigma}_{h,0}, (\mathbf{u}_{h,0}, \boldsymbol{\gamma}_{h,0}))$ *satisfying* (4.5) *and* $\mathbf{f}^n \in \mathbf{L}^2(\Omega)$ *with* $n \in \{1,\dots,N\}$, *there exists a unique solution* $(\boldsymbol{\sigma}_h^n, \underline{\mathbf{u}}_h^n)$ *to the fully discrete scheme* (5.1)*. Moreover, the solution satisfy the stability estimate* (5.3)*.*

Proof. The existence and uniqueness is achieved by arguments similar to those of the proof of Theorems 3.9 and 4.7. In fact, it is straightforward to verify that $\widehat{\mathcal{J}}_{\mathrm{d}}$ is Lipschitz continuous with constant $C_{\widehat{\mathcal{J}}_{\mathrm{d}}} \|\nabla\rho/\rho\|_{\mathbf{L}^\infty(\Omega)}$. Then, by (5.14), it follows that $\widehat{\mathcal{J}}_{\mathrm{d}}$ is a contractive operator in the Banach space $\ell^2(0,T;\mathbb{X}_h)$. Thus, by the Banach fixed-point theorem, there exists a unique solution to (5.12), which is equivalent to the existence and uniqueness of solution to (5.1). The stability follows from Theorem 5.1. This completes the proof. □

Remark 5.1. *We emphasize that the fully discrete scheme* (5.1) *yields exact conservation of momentum when* ρ *and* $\mathbf{K}$ *are piecewise constant, and* $\mathbf{f}^n \in \mathbf{H}_h^{\mathbf{u}}$, *for each* $n \in \{1,\dots,N\}$. *In this case,* $\rho\, d_t \mathbf{u}_h^n$, $\mathbf{K}^{-1}\mathbf{u}_n^n$, *and* $\mathbf{f}^n$ *all belong to* $\mathbf{H}_h^{\mathbf{u}}$. *This fact, together with the inclusion*

$\mathbf{div}(\mathbb{X}_h) \subset \mathbf{H}_h^{\mathbf{u}}$, *implies from the second equation in* (5.1) *that, for every* $n \in \{1, \ldots, N\}$,

$$\rho\, d_t \mathbf{u}_h^n + \mu\, \mathbf{K}^{-1}\, \mathbf{u}_h^n - \mathbf{div}(\boldsymbol{\sigma}_h^n) = \mathbf{f}^n \quad in \quad \Omega\,. \tag{5.15}$$

Furthermore, if the data is not piecewise constant or $\mathbf{f}^n \notin \mathbf{H}_h^{\mathbf{u}}$, (5.15) *can only be obtained in an approximate sense, by replacing* $\rho\, d_t \mathbf{u}_h^n$, $\mathbf{K}^{-1}\, \mathbf{u}_h^n$ *and* $\mathbf{f}^n$ *with* $\boldsymbol{\mathcal{P}}_h^k(\rho\, d_t \mathbf{u}_h^n)$, $\boldsymbol{\mathcal{P}}_h^k(\mathbf{K}^{-1}\, \mathbf{u}_h^n)$ *and* $\boldsymbol{\mathcal{P}}_h^k(\mathbf{f}^n)$, *where* $\boldsymbol{\mathcal{P}}_h^k$ *is defined as in* (4.23). *The numerical verification of this property is illustrated in Chapter 6.*

In what follows, we establish the rates of convergence associated with the fully discrete scheme (5.1). To this end, we subtract the fully discrete system (5.1) from its continuous counterpart (2.19) at each time step $n \in \{1, \ldots, N\}$, yielding the following error system:

$$\begin{aligned}
[\mathbf{A}(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n), \boldsymbol{\tau}_h] + [\mathbf{B}'(\underline{\mathbf{u}}^n - \underline{\mathbf{u}}_h^n), \boldsymbol{\tau}_h] + [\mathbf{D}'_\rho(\underline{\mathbf{u}}^n - \underline{\mathbf{u}}_h^n), \boldsymbol{\tau}_h] &= 0 \\
(\rho\, d_t(\mathbf{u}^n - \mathbf{u}_h^n), \mathbf{v}_h)_\Omega - [\mathbf{B}(\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n), \underline{\mathbf{v}}_h] + [\mathbf{C}(\underline{\mathbf{u}}^n - \underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] &= (\rho\, \mathbf{r}_n(\mathbf{u}), \mathbf{v}_h)_\Omega
\end{aligned}$$

for all $(\boldsymbol{\tau}_h, \underline{\mathbf{v}}_h) \in \mathbb{X}_h \times \mathbf{Y}_h$, where $\mathbf{r}_n$ is the difference between the continuous and discrete time derivatives, that is,

$$\mathbf{r}_n(\mathbf{u}) := d_t \mathbf{u}^n - \partial_t \mathbf{u}(t_n)\,.$$

Additionally, we recall from [15, Lemma 4] that, if $\mathbf{u} \in \mathrm{H}^2(0, T; \mathbf{L}^2(\Omega))$, there holds

$$\Delta t \sum_{n=1}^{N} \|\mathbf{r}_n(\mathbf{u})\|_{\mathbf{L}^2(\Omega)}^2 \le C(\partial_{tt}\mathbf{u})\, (\Delta t)^2\,, \quad \text{with} \quad C(\partial_{tt}\mathbf{u}) := C\, \|\partial_{tt}\mathbf{u}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))}^2\,,$$

for some positive constant C, independent of Δt. Thus, we now state the theoretical rates of convergence associated with the fully discrete scheme (5.1). The proof follows the same structure as that of Theorem 4.8, relying on the approximation properties detailed in Chapter 4, and more precisely in (4.42), thereby yielding a result analogous to Theorem 4.9. Naturally, all arguments must be adapted to the discrete-in-time setting, in a way similar to the proof of Theorem 5.1. For the sake of brevity, we omit further details and restrict ourselves to stating the result.

Theorem 5.4. *Suppose that there exists* $\vartheta \in (0, k+1]$ *such that the assumptions of Theorem 4.9 hold. Assume further that the hypotheses of Theorem 5.3 hold and that* $\mathbf{u} \in \mathrm{H}^2(0,T;\mathbf{L}^2(\Omega))$. *Then, for the solution of the fully discrete scheme* (5.1)*, there exists a positive constant* $\widehat{\mathcal{C}}(\boldsymbol{\sigma}, \underline{\mathbf{u}})$*, independent of* h *and* Δt*, but depending on the exact solutions and* $\widehat{C}_{\mathrm{B}}$ *(cf.* (5.3)*), such that*

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\ell^2(0,T;\mathbb{X})} + \|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{L}^s(\Omega))} + \|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\ell^2(0,T;\mathbb{L}^2(\Omega))} \leq \widehat{\mathcal{C}}(\boldsymbol{\sigma}, \underline{\mathbf{u}})\,(h^\vartheta + \Delta t)\,.$$

Finally, inspired by the first equation in (2.10), (2.8), and (2.18), we observe that the gradient of the velocity $\nabla\mathbf{u}$, the pressure p, and the original stress tensor $\widetilde{\boldsymbol{\sigma}}$, can be approximated through a post-processing procedure as

$$\begin{aligned}
[\nabla\mathbf{u}]_h^n &:= \frac{1}{2\mu\rho}\,(\boldsymbol{\sigma}_h^n)^{\mathrm{d}} + \boldsymbol{\gamma}_h^n - \frac{1}{d}\left(\frac{\nabla\rho}{\rho}\cdot\mathbf{u}_h^n\right)\mathbb{I}\,,\\
p_h^n &:= -\frac{1}{d}\left\{2\mu\,(\nabla\rho\cdot\mathbf{u}_h^n) + \mathrm{tr}(\boldsymbol{\sigma}_h^n)\right\} - \lambda_{\boldsymbol{\sigma}_h^n}\,,\\
\text{and}\quad \widetilde{\boldsymbol{\sigma}}_h^n &:= \boldsymbol{\sigma}_h^n + \lambda_{\boldsymbol{\sigma}_h^n}\,\mathbb{I}\,,\quad \text{with}\quad \lambda_{\boldsymbol{\sigma}_h^n} := -\frac{2\mu}{d\,|\Omega|}\,(\nabla\rho, \mathbf{u}_h^n)_\Omega\,,
\end{aligned} \tag{5.16}$$

for all $n \in \{1, \ldots, N\}$, where $[\nabla\mathbf{u}]_h$, p_h and $\widetilde{\boldsymbol{\sigma}}_h$ denote the respective approximations of the variables of interest. Consequently, from the rates of convergence of $\boldsymbol{\sigma}$, $\mathbf{u}$, and $\boldsymbol{\gamma}$ established in the previous theorem, it follows directly that the same rates are inherited by $[\nabla\mathbf{u}]_h$, p_h and $\widetilde{\boldsymbol{\sigma}}_h$.

Lemma 5.5. *Suppose the same assumptions as in Theorem 5.4. Then, there exists a positive constant* $\widetilde{\mathcal{C}}(\boldsymbol{\sigma}, \underline{\mathbf{u}})$*, independent of* h *and* Δt*, but depending on the exact solutions and* $\widehat{C}_{\mathrm{B}}$ *(cf.* (5.3)*), such that*

$$\|\mathbf{e}_{\nabla\mathbf{u}}\|_{\ell^2(0,T;\mathbb{L}^2(\Omega))} + \|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))} + \|\mathbf{e}_{\widetilde{\boldsymbol{\sigma}}}\|_{\ell^2(0,T;\mathbb{X})} \leq \widetilde{\mathcal{C}}(\boldsymbol{\sigma}, \underline{\mathbf{u}})\,(h^\vartheta + \Delta t)\,,$$

where $\mathbf{e}_{\nabla\mathbf{u}} := \nabla\mathbf{u} - [\nabla\mathbf{u}]_h$, $\mathbf{e}_p := p - p_h$, *and* $\mathbf{e}_{\widetilde{\boldsymbol{\sigma}}} := \widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h$.

Remark 5.2. *In the fully discrete scheme* (5.1)*, we restrict ourselves to the backward Euler method merely for simplicity. Nevertheless, the analysis in Chapter 5 can be readily extended to other time discretizations, including BDF schemes and the Crank–Nicholson method.*

CHAPTER 6

Numerical results

In this chapter, we present three numerical experiments that illustrate the performance of the fully discrete method (5.1). The implementation was carried out using the open-source finite element library `FEniCS` [2]. We consider quasi-uniform triangulations and the finite element subspaces associated with PEERS_k and AFW_k, as described in Chapter 4. Examples 1 and 2 aim to verify the expected rates of convergence in two- and three-dimensional domains, respectively, and to corroborate numerically that the conservation of momentum (5.15) holds. In these cases, the total simulation time is set to $T = 10^{-2}$ with a time step of $\Delta t = 10^{-3}$, which is sufficiently small to ensure that the temporal discretization error does not influence the observed convergence rates. Finally, Example 3 examines the flow of a free fluid around a porous obstacle under various operating conditions, highlighting the applicability of the proposed method to complex geometries and diverse physical scenarios.

For the first two examples, in addition to the errors in the velocity and vorticity, we also compute the errors associated with the original Cauchy stress tensor and the pressure obtained from (5.16) and Lemma 5.5, while omitting the computation of the velocity gradient for simplicity. In addition, we compute the error associated with the conservation of momentum (5.15) as

$$\mathbf{e}_{\mathtt{m},k} := \boldsymbol{\mathcal{P}}_h^k(\rho\, d_t \mathbf{u}_h) + \mu\, \boldsymbol{\mathcal{P}}_h^k(\mathbf{K}^{-1}\, \mathbf{u}_h) - \mathbf{div}(\boldsymbol{\sigma}_h) - \boldsymbol{\mathcal{P}}_h^k(\mathbf{f})\,.$$

We recall that the experimental rates of convergence are defined as

$$r_\diamond := \frac{\log(\mathbf{e}_\diamond/\mathbf{e}'_\diamond)}{\log(h/h')} \qquad \text{for } \diamond \in \{\widetilde{\boldsymbol{\sigma}}, \mathbf{u}, \boldsymbol{\gamma}, p\}\,,$$

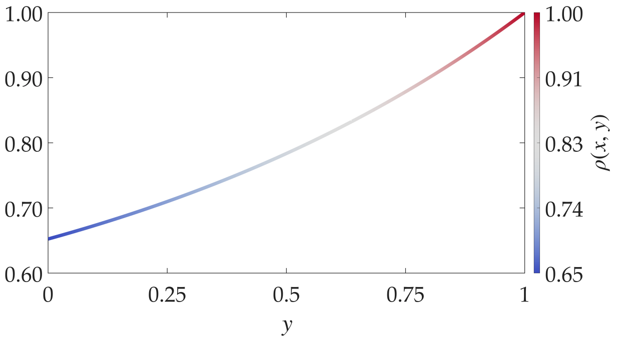
where h and h' denote two consecutive mesh sizes with errors $\mathbf{e}_\diamond$ and $\mathbf{e}'_\diamond$.

Finally, we remark that the zero-mean constraint on $\operatorname{tr}(\boldsymbol{\sigma}_h)$ over Ω is imposed via a scalar Lagrange multiplier, which amounts to adding one row and one column to the matrix system corresponding to (5.1).

Example 1: Convergence against smooth exact solutions in a 2D domain

In this test, we analyze the convergence with respect to the spatial discretization using a manufactured solution. The computational domain is the square $\Omega := (0,1)^2$, and we set $s = 4$, which yields $\ell = 4/3$ (cf. (2.13)). The viscosity is fixed at $\mu = 1$, and the permeability tensor is given by $\mathbf{K} := 10^{-2}\,\mathbb{I}$. Following [20], the porosity function is defined through an exponential profile, while the source term $\mathbf{f}$ is adjusted so that the manufactured solution coincides with the prescribed analytical functions (cf. (2.1)). These functions are depicted in Figure 6.1. The model problem is complemented with the corresponding Dirichlet boundary condition and suitable initial data.

Tables 6.1 and 6.2 report the convergence history for a sequence of quasi-uniform mesh refinements using both PEERS$_k$ and AFW$_k$ elements, for $k \in \{0,1\}$. The results confirm that the optimal spatial convergence rates $\mathcal{O}(h^{k+1})$ predicted by Theorem 5.4 and Lemma 5.5 are achieved. Table 6.3 shows that, although the data is not piecewise constant, the error associated with the conservation of momentum (5.15) is close to zero. In Figure 6.2, we display some solutions at the final time obtained with the AFW$_1$ discretization with meshsize $h = 0.014$ and



$$\rho(x,y) := 0.45 + 0.55\,\exp(-(1-y))\,,$$

$$\mathbf{u} = \exp(t)\,\rho(x,y)^{-1}\begin{pmatrix}\sin(\pi x)\,\cos(\pi y)\\ -\cos(\pi x)\,\sin(\pi y)\end{pmatrix},$$

$$p = \exp(t)\,\cos(\pi x)\,\exp(y)\,.$$

Figure 6.1: [Example 1] Graph of the porosity function (left) and analytical expressions of the porosity and manufactured solutions (right).

20,000 triangle elements, representing 481,201 `DOF`.

PEERS$_0$ discretization											
		$\|\mathbf{e}_{\widetilde{\boldsymbol{\sigma}}}\|_{\ell^2(0,T;\mathbb{X})}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{L}^s(\Omega))}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\ell^2(0,T;\mathbb{L}^2(\Omega))}$		$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$	
`DOF`	h	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`
266	0.354	6.02E-01	–	3.33E-02	–	2.68E-01	–	3.04E-02	–	8.94E-02	–
1010	0.177	2.92E-01	1.047	1.70E-02	0.968	1.36E-01	0.980	7.64E-03	1.995	4.44E-02	1.009
3938	0.088	1.36E-01	1.101	8.57E-03	0.992	6.80E-02	0.996	2.50E-03	1.610	1.93E-02	1.200
15554	0.044	6.45E-02	1.076	4.29E-03	0.998	3.40E-02	0.999	9.58E-04	1.386	8.56E-03	1.175
54362	0.024	3.37E-02	1.032	2.29E-03	0.999	1.82E-02	1.000	3.85E-04	1.449	4.34E-03	1.081
150602	0.014	2.01E-02	1.012	1.37E-03	1.000	1.09E-02	1.000	1.80E-04	1.488	2.56E-03	1.031

AFW$_0$ discretization											
		$\|\mathbf{e}_{\widetilde{\boldsymbol{\sigma}}}\|_{\ell^2(0,T;\mathbb{X})}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{L}^s(\Omega))}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\ell^2(0,T;\mathbb{L}^2(\Omega))}$		$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$	
`DOF`	h	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`
321	0.354	5.39E-01	–	3.33E-02	–	2.68E-01	–	8.60E-02	–	2.89E-02	–
1217	0.177	2.36E-01	1.189	1.70E-02	0.966	1.36E-01	0.981	4.33E-02	0.991	1.31E-02	1.143
4737	0.088	1.13E-01	1.072	8.57E-03	0.991	6.80E-02	0.995	2.16E-02	0.999	6.27E-03	1.060
18689	0.044	5.54E-02	1.022	4.29E-03	0.998	3.40E-02	0.999	1.08E-02	1.000	3.10E-03	1.017
65281	0.024	2.94E-02	1.007	2.29E-03	0.999	1.82E-02	1.000	5.77E-03	1.000	1.65E-03	1.005
180801	0.014	1.76E-02	1.002	1.37E-03	1.000	1.09E-02	1.000	3.46E-03	1.000	9.88E-04	1.002

Table 6.1: [Example 1, $k = 0$] Number of degrees of freedom, meshsizes, errors, and rates of convergence.

PEERS$_1$ discretization											
		$\|\mathbf{e}_{\widetilde{\boldsymbol{\sigma}}}\|_{\ell^2(0,T;\mathbb{X})}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{L}^s(\Omega))}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\ell^2(0,T;\mathbb{L}^2(\Omega))}$		$\|\mathbf{e}_{p}\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$	
`DOF`	h	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`
818	0.354	7.84E-02	–	5.43E-03	–	3.62E-02	–	8.99E-03	–	6.79E-03	–
3170	0.177	1.87E-02	2.067	1.39E-03	1.968	9.18E-03	1.977	2.62E-03	1.779	2.00E-03	1.762
12482	0.088	4.59E-03	2.029	3.49E-04	1.992	2.31E-03	1.994	7.54E-04	1.797	5.50E-04	1.863
49538	0.044	1.14E-03	2.011	8.75E-05	1.997	5.77E-04	1.999	2.02E-04	1.900	1.44E-04	1.937
173522	0.024	3.23E-04	2.005	2.49E-05	1.998	1.64E-04	2.000	5.87E-05	1.965	4.16E-05	1.973
481202	0.014	1.16E-04	1.995	9.00E-06	1.995	5.91E-05	1.999	2.13E-05	1.986	1.51E-05	1.987

AFW$_1$ discretization											
		$\|\mathbf{e}_{\widetilde{\boldsymbol{\sigma}}}\|_{\ell^2(0,T;\mathbb{X})}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{L}^s(\Omega))}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\ell^2(0,T;\mathbb{L}^2(\Omega))}$		$\|\mathbf{e}_{p}\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$	
`DOF`	h	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`
817	0.354	7.86E-02	–	5.40E-03	–	3.62E-02	–	1.16E-02	–	3.20E-03	–
3169	0.177	1.73E-02	2.181	1.39E-03	1.960	9.19E-03	1.978	2.97E-03	1.967	7.38E-04	2.115
12481	0.088	4.05E-03	2.098	3.49E-04	1.990	2.31E-03	1.994	7.50E-04	1.986	1.88E-04	1.976
49537	0.044	9.86E-04	2.037	8.75E-05	1.997	5.77E-04	1.999	1.88E-04	1.994	4.78E-05	1.973
173521	0.024	2.78E-04	2.014	2.49E-05	1.998	1.64E-04	2.000	5.37E-05	1.997	1.37E-05	1.984
481201	0.014	1.00E-04	1.997	9.00E-06	1.995	5.91E-05	1.999	1.93E-05	1.997	4.99E-06	1.983

Table 6.2: [Example 1, $k = 1$] Number of degrees of freedom, meshsizes, errors, and rates of convergence.

PEERS$_k$ discretization						
h	0.354	0.177	0.088	0.044	0.024	0.014
$\|\mathbf{e}_{\mathtt{m},0}\|_{\ell^\infty(0,T;\ell^\infty(\Omega))}$	1.25E-12	1.71E-12	3.66E-12	1.34E-11	3.21E-11	9.89E-11
$\|\mathbf{e}_{\mathtt{m},1}\|_{\ell^\infty(0,T;\ell^\infty(\Omega))}$	4.97E-12	2.32E-10	4.74E-09	1.58E-09	2.72E-09	3.38E-09

AFW$_k$ discretization						
h	0.354	0.177	0.088	0.044	0.024	0.014
$\|\mathbf{e}_{\mathtt{m},0}\|_{\ell^\infty(0,T;\ell^\infty(\Omega))}$	2.75E-12	3.64E-12	7.18E-12	1.98E-11	5.81E-11	2.04E-10
$\|\mathbf{e}_{\mathtt{m},1}\|_{\ell^\infty(0,T;\ell^\infty(\Omega))}$	2.94E-12	5.57E-12	1.34E-11	5.08E-11	1.66E-10	4.89E-10

Table 6.3: [Example 1, $k = 0, 1$] Conservation of momentum for the fully discrete scheme.

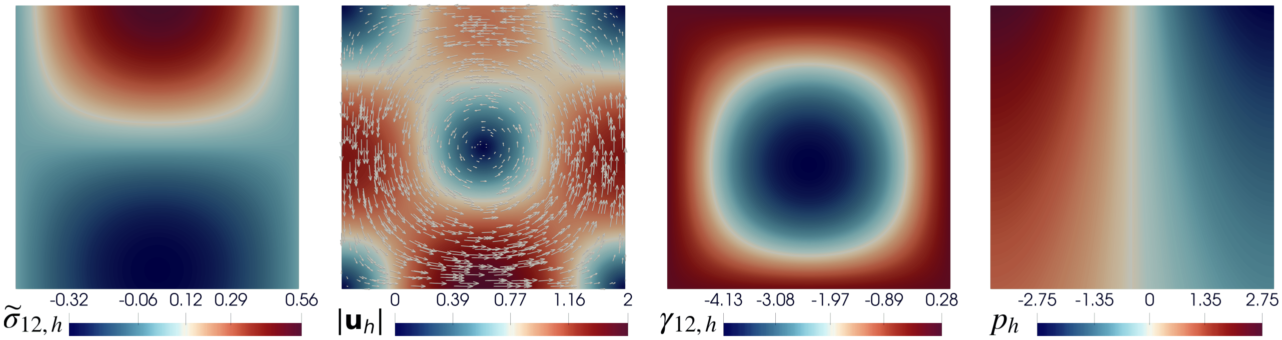
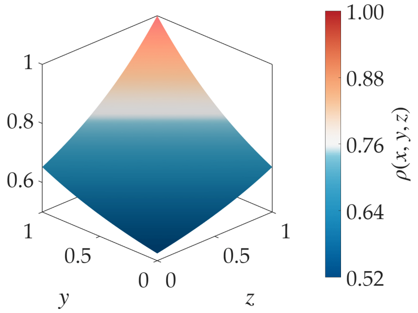


Figure 6.2: [Example 1] Computed stress component, magnitude of the velocity, vorticity component, and pressure field.

Example 2: Convergence against smooth exact solutions in a 3D domain

In the second numerical test, we study the convergence with respect to the spatial discretization using a manufactured solution in the unit cube $\Omega := (0,1)^3$. We set $s = 3$, which yields $\ell = 3/2$ (cf. (2.13)). Similarly to the first example, the viscosity is $\mu = 1$, the permeability tensor is given by $\mathbf{K} := 10^{-2}\,\mathbb{I}$, and the porosity function along with the manufactured solutions are given in Figure 6.3. The datum $\mathbf{f}$ is computed according to this (cf. (2.1)).



$$\rho(x,y,z) := 0.45 + 0.55\,\exp(-(2-y-z))\,,$$

$$\mathbf{u} = \exp(t)\,\rho(x,y,z)^{-1}\begin{pmatrix} \sin(\pi x)\,\cos(\pi y)\,\cos(\pi z) \\ -2\cos(\pi x)\,\sin(\pi y)\,\cos(\pi z) \\ \cos(\pi x)\,\cos(\pi y)\,\sin(\pi z) \end{pmatrix},$$

$$p = \exp(t)\,\cos(\pi x)\,\exp(y+z)\,.$$

Figure 6.3: [Example 2] Graph of the porosity function (left) and analytical expressions of the porosity and manufactured solutions (right).

Table 6.4 shows the convergence history for a sequence of quasi-uniform mesh refinements using both PEERS_0 and AFW_0 elements. Once again, the optimal spatial convergence rates $\mathcal{O}(h^{k+1})$ predicted by Theorem 5.4 and Lemma 5.5 are confirmed. Regarding the conservation of momentum, Table 6.5 shows a behavior similar to that observed in the first example. Figure 6.4 displays the approximated solutions at the final time obtained with the PEERS_0 discretization on a mesh with size $h = 0.0962$ and 34,992 tetrahedral elements, corresponding to 656,266 degrees of freedom.

Example 3: Free fluid flow around a porous obstacle

Our final test aims to evaluate the performance of the proposed method in a more complex physical configuration, where a fluid interacts with a porous obstacle acting as a filter. This setting allows us to examine how the formulation captures the coupling between the free flow

PEERS$_0$ discretization

		$\|\mathbf{e}_{\widetilde{\boldsymbol{\sigma}}}\|_{\ell^2(0,T;\mathbb{X})}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{L}^s(\Omega))}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\ell^2(0,T;\mathbb{L}^2(\Omega))}$		$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$	
`DOF`	h	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`
7576	0.433	1.08E+00	–	4.19E-02	–	3.65E-01	–	5.04E-02	–	1.30E-01	–
25006	0.289	7.16E-01	1.001	2.84E-02	0.957	2.47E-01	0.968	2.19E-02	2.054	8.78E-02	0.960
58636	0.216	5.29E-01	1.058	2.14E-02	0.979	1.86E-01	0.985	1.30E-02	1.805	6.37E-02	1.114
195808	0.144	3.40E-01	1.087	1.43E-02	0.989	1.24E-01	0.993	6.85E-03	1.588	3.89E-02	1.219
656266	0.096	2.19E-01	1.089	9.58E-03	0.995	8.29E-02	0.997	3.78E-03	1.462	2.33E-02	1.268

AFW$_0$ discretization

		$\|\mathbf{e}_{\widetilde{\boldsymbol{\sigma}}}\|_{\ell^2(0,T;\mathbb{X})}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{L}^s(\Omega))}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\ell^2(0,T;\mathbb{L}^2(\Omega))}$		$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$	
`DOF`	h	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`	`error`	`rate`
10081	0.433	9.49E-01	–	4.18E-02	–	3.65E-01	–	1.03E-01	–	5.88E-02	–
33049	0.289	6.01E-01	1.127	2.84E-02	0.957	2.47E-01	0.969	6.95E-02	0.964	3.41E-02	1.345
77185	0.216	4.38E-01	1.097	2.14E-02	0.978	1.86E-01	0.985	5.23E-02	0.984	2.37E-02	1.261
256609	0.144	2.85E-01	1.062	1.43E-02	0.989	1.24E-01	0.992	3.50E-02	0.993	1.48E-02	1.165
857305	0.096	1.87E-01	1.032	9.58E-03	0.995	8.29E-02	0.997	2.34E-02	0.997	9.52E-03	1.085

Table 6.4: [Example 2, $k = 0$] Number of degrees of freedom, meshsizes, errors, and rates of convergence.

PEERS$_0$ discretization

h	0.433	0.289	0.216	0.144	0.096
$\|\mathbf{e}_{\mathrm{m},0}\|_{\ell^\infty(0,T;\ell^\infty(\Omega))}$	7.96E-12	7.28E-12	7.30E-12	8.41E-12	9.32E-12

AFW$_0$ discretization

h	0.433	0.289	0.216	0.144	0.096
$\|\mathbf{e}_{\mathrm{m},0}\|_{\ell^\infty(0,T;\ell^\infty(\Omega))}$	2.96E-12	3.87E-12	3.87E-12	4.56E-12	4.55E-12

Table 6.5: [Example 2, $k = 0$] Conservation of momentum for the fully discrete scheme.

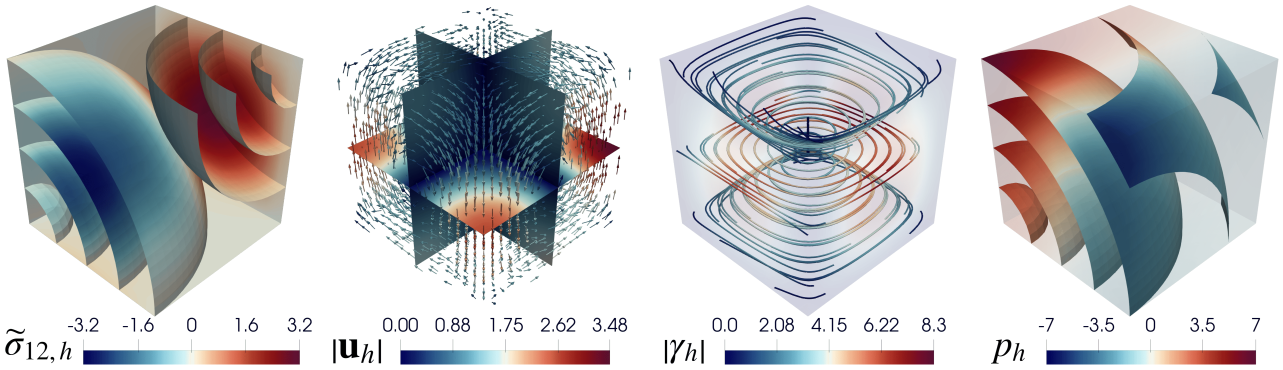


Figure 6.4: [Example 2] Computed stress component, magnitude of the velocity, vorticity streamlines, and pressure field.

and the porous medium, as well as the influence of anisotropy and permeability contrasts on the global behavior of the flow. This benchmark problem was first introduced in [36] and later examined in [7,41]. We consider a channel Ω with dimensions 0.75 [m] $\times$ 0.25 [m], through which a fluid of viscosity $\mu = 1.5{\cdot}10^{-5}$ [m^2/s] flows due to a left-to-right pressure drop of $1{\cdot}10^{-6}$ [m^2/s^2]. The channel consists of two regions. The first one, Ω_{free}, is a free-fluid domain, whereas the second region represents a heterogeneous porous filter composed of an outer isotropic subregion Ω_{iso} and an inner anisotropic subregion Ω_{an}. More precisely, the domain under consideration is given by $\Omega = \Omega_{\text{free}} \cup \Omega_{\text{iso}} \cup \Omega_{\text{an}}$, where

$$\Omega_{\text{an}} := (0.32, 0.385) \times (0.05, 0.12)\,, \quad \Omega_{\text{iso}} := (0.25, 0.5) \times (0.2) \setminus \Omega_{\text{an}}\,,$$

$$\text{and} \quad \Omega_{\text{free}} := (0, 0.75) \times (0, 0.25) \setminus (\Omega_{\text{an}} \cup \Omega_{\text{iso}})\,.$$

We denote by $\Gamma = \Gamma_{\text{top}} \cup \Gamma_{\text{bottom}} \cup \Gamma_{\text{right}} \cup \Gamma_{\text{left}}$ the boundary of Ω, naturally partitioned into its corresponding sides. We illustrate the geometrical setting in Figure 6.5. There is no transition region between the free fluid and porous regions.

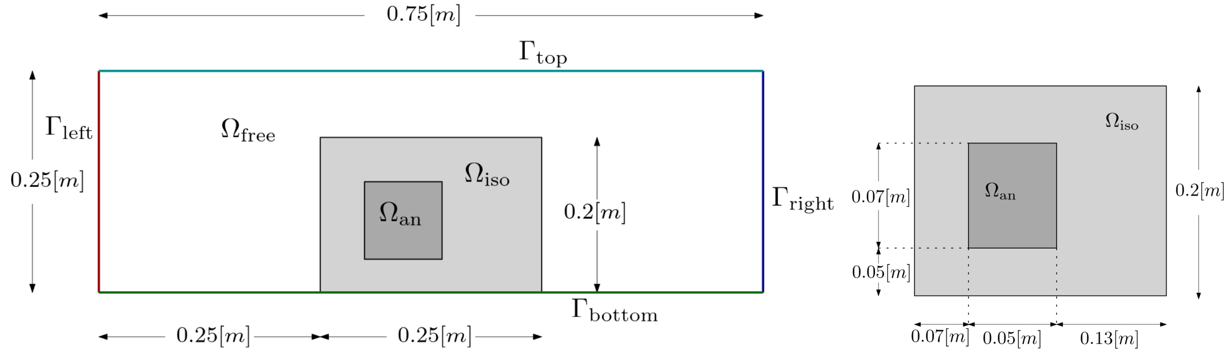


Figure 6.5: [Example 3] Geometrical configuration of the numerical experiment. The left panel shows the channel Ω, while the right panel depicts a detailed view of the porous filter consisting of isotropic and anisotropic regions.

In the porous filter, the permeability tensor $\mathbf{K}_*$, with $* \in \{\text{iso}, \text{an}\}$, is given by

$$\mathbf{K}_* := \mathbf{M}_*\,\mathbf{C}_*\,\mathbf{M}_*^{-1}\,, \quad \mathbf{M}_* := \begin{pmatrix} \cos(\alpha_*) & -\sin(\alpha_*) \\ \sin(\alpha_*) & \cos(\alpha_*) \end{pmatrix}, \quad \mathbf{C}_* := \begin{pmatrix} k_*/\beta_* & 0 \\ 0 & k_* \end{pmatrix}, \tag{6.1}$$

where k_* and β_* are positive parameters and α_* is the anisotropy angle. Clearly, $\alpha_{\text{iso}} = 0$ owing to the isotropy. We consider the parameters of the model as

$$\mathbf{K} := \begin{cases} +\infty & \text{in} \quad \Omega_{\text{free}}\,, \\ \mathbf{K}_{\text{iso}} & \text{in} \quad \Omega_{\text{iso}}\,, \\ \mathbf{K}_{\text{an}} & \text{in} \quad \Omega_{\text{an}}\,, \end{cases} \quad \text{and} \quad \rho := \begin{cases} 1 & \text{in} \quad \Omega_{\text{free}}\,, \\ \rho_{\text{iso}} & \text{in} \quad \Omega_{\text{iso}}\,, \\ \rho_{\text{an}} & \text{in} \quad \Omega_{\text{an}}\,. \end{cases}$$

In the forthcoming presentation, we shall consider different porosity functions and parameters for the permeability tensor in the porous region. We notice also that the fact that the permeability tensor is identically $+\infty$ in the free-fluid region means that $\mathbf{K}^{-1} \equiv \mathbf{0}$ in Ω_{free}.

No-slip boundary conditions are prescribed for the fluid velocity along the top and bottom walls of the channel, whereas the pressure drop is induced by a traction difference between the left and right boundaries. Namely,

$$\begin{gathered} \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_{\text{top}} \cup \Gamma_{\text{bottom}} \times (0,T]\,, \quad \widetilde{\boldsymbol{\sigma}}\mathbf{n} = -p_{\text{in}}\,\mathbf{n} \quad \text{on} \quad \Gamma_{\text{left}} \times (0,T]\,, \\ \text{and} \quad \widetilde{\boldsymbol{\sigma}}\mathbf{n} = -p_{\text{out}}\,\mathbf{n} \quad \text{on} \quad \Gamma_{\text{right}} \times (0,T]\,, \end{gathered} \tag{6.2}$$

where $p_{\text{in}} = p_{\text{ref}} + 1 \cdot 10^{-6}\,[\text{m}^2/\text{s}^2]$, $p_{\text{out}} = p_{\text{ref}}$ and $p_{\text{ref}} := 1 \cdot 10^{-6}\,[\text{m}^2/\text{s}^2]$. These conditions translate the Dirichlet boundary condition in (2.1) of our model into a mixed-type boundary condition. The analysis developed in the previous chapters can readily be adapted to handle this case. In particular, one may employ a lifting of the normal trace in (6.2) and introduce a change of variable for the stress tensor, so that the formulation derived in Section 2.2 is now posed on $\mathbb{H}_N(\mathbf{div}_\ell;\Omega)$ instead of $\mathbb{H}_0(\mathbf{div}_\ell;\Omega)$, where $\mathbb{H}_N(\mathbf{div}_\ell;\Omega)$ stands for tensors in $\mathbb{H}(\mathbf{div}_\ell;\Omega)$ with vanishing normal trace on $\Gamma_{\text{left}} \cup \Gamma_{\text{right}}$. Then, the analysis proceeds in a similar manner, provided that suitable geometric conditions on the domain hold, for instance assumptions on the maximal interior angle. In our case, the domain is rectangular and therefore convex, so no technical issues arise. We omit further details and refer the reader to [19, eqs. (3.25)–(3.30)] for a more complete discussion. Finally, we set as initial condition and source term $\mathbf{u}_0 = \mathbf{0}$ and $\mathbf{f} = \mathbf{0}$, respectively.

In all the following experiments, we consider a total simulation time of $T = 80\,[\text{s}]$ and the time step size is set as $\Delta t = 4\,[\text{s}]$. The mesh consists of 63,221 triangles and the element sizes goes from $4 \cdot 10^{-3}$ in the bulk fluid to $1 \cdot 10^{-4}$ close to the interfaces (see Figure 6.9). For the spatial discretization, we employ the AFW_0 finite element triple (cf. (4.2)), which results in a total of 569,995 degrees of freedom in the implementation.

In our first experiment, we consider a simple case when the porosity function is constant in each region, given by $\rho_{\mathrm{iso}} \equiv 0.5$ and $\rho_{\mathrm{an}} \equiv 0.25$. Regarding the permeability tensor, we take $\alpha_{\mathrm{iso}} = 0$, $\beta_{\mathrm{iso}} = 1$, $k_{\mathrm{iso}} = 1 \cdot 10^{-4}$, $\alpha_{\mathrm{an}} = -\pi/4$, $\beta_{\mathrm{an}} = 100$, and $k_{\mathrm{an}} = 1 \cdot 10^{-6}$. In the free-fluid region, the flow circulates freely and tends to avoid the porous region, concentrating on the upper area of the filter due to the porous and permeability effects. The isotropic region, with higher permeability and porosity, allows the fluid to flow more freely, while the anisotropic region acts as a barrier that restricts flow in certain directions due to both its lower permeability and the directional dependence encoded by the rotation angle α_{an}. In turn, if we repeat the experiment with $k_{\mathrm{an}} = 1 \cdot 10^{-4}$, the contrast between the isotropic and anisotropic regions decreases significantly. In this case, the anisotropic layer becomes more permeable, allowing the fluid to penetrate and traverse it with less resistance. Consequently, the flow field tends to distribute more uniformly across the porous domain, rather than being diverted around the anisotropic region as in the previous configuration. In both scenarios, we observe that the conservation of momentum error is close to zero. Indeed, bearing in mind that the porosity is constant and $\mathbf{f} \equiv \mathbf{0}$, in order to verify that (5.15) holds, we compute $\|\rho\, d_t(\mathbf{u}_h) + \mu\, \boldsymbol{\mathcal{P}}_h^k(\mathbf{K}^{-1}\, \mathbf{u}_h) - \mathbf{div}(\boldsymbol{\sigma}_h)\|_{\ell^\infty(0,T;\ell^\infty(\Omega))}$, obtaining $9.54 \cdot 10^{-7}$ in both cases. In Figure 6.6, we display these results at the final time T. We only show the velocity profile for the sake of brevity.

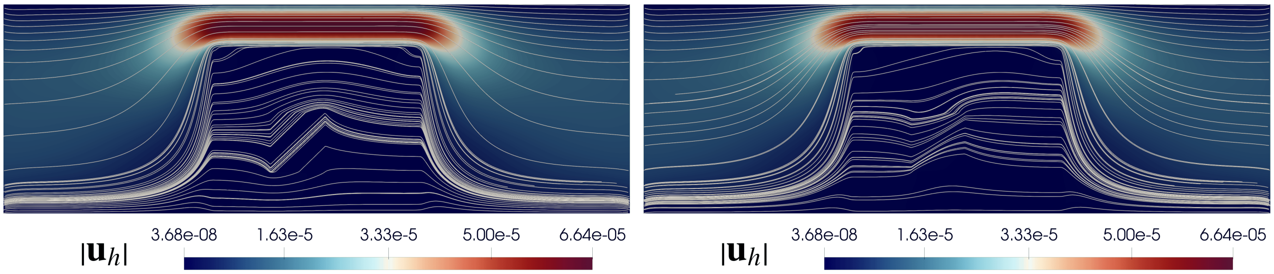
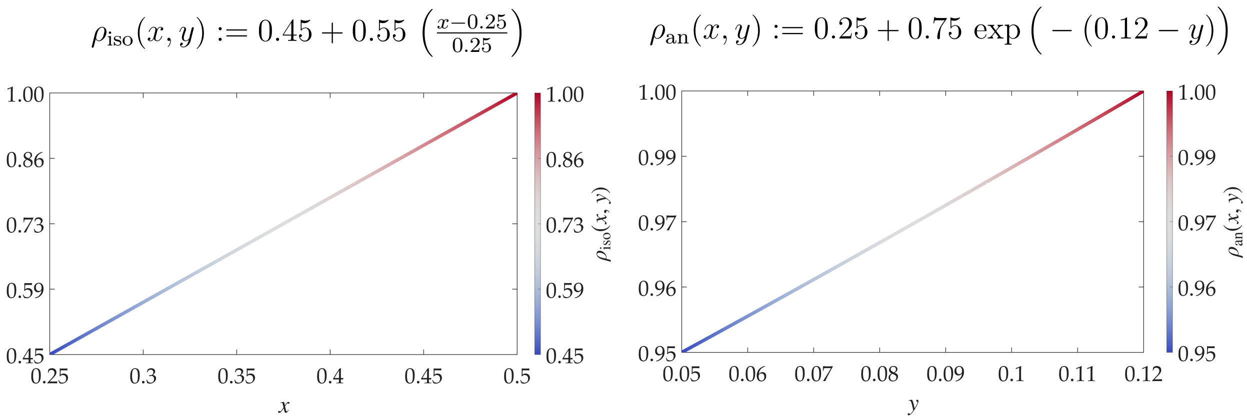


Figure 6.6: [Example 3] Computed streamlines of the velocity with piecewise constant porosity and $k_{\mathrm{an}} = 1 \cdot 10^{-6}$ (left) and $k_{\mathrm{an}} = 1 \cdot 10^{-4}$ (right).

As a second experiment, we consider the non-piecewise constant porosity functions given in

Figure 6.7. In the isotropic region, we prescribe a linear variation of the porosity, representing a gradual compaction or deposition of the porous material. Physically, this choice models a medium whose microstructure becomes progressively denser along the horizontal axis, such as might occur due to sedimentation or pressure-induced compaction in filtration processes. On the other hand, in the anisotropic region, we consider an exponential porosity profile, decreasing from the outer boundary toward the interior of the inclusion. This choice mimics a boundary-layer-type behavior, where the porous structure becomes gradually denser as one moves inward, reflecting processes such as clogging, compression, or material deposition within the anisotropic medium. Such a profile provides a smooth yet strongly contrasting variation that highlights how the anisotropic permeability interacts with spatially varying microstructural properties. Notice that this choice is similar to those considered in Examples 1 and 2.



$$\rho_{\mathrm{iso}}(x,y) := 0.45 + 0.55\left(\frac{x-0.25}{0.25}\right)$$

$$\rho_{\mathrm{an}}(x,y) := 0.25 + 0.75\,\exp\Big(-(0.12-y)\Big)$$

Figure 6.7: [Example 3] Porosity functions in the isotropic (left) and anisotropic (right) regions.

The permeability tensor is computed with parameters $\alpha_{\mathrm{iso}} = 0$, $\beta_{\mathrm{iso}} = 1$, $k_{\mathrm{iso}} = 1 \cdot 10^{-6}$, $\alpha_{\mathrm{an}} = \pi/4$, $\beta_{\mathrm{an}} = 100$, and $k_{\mathrm{an}} = 1 \cdot 10^{-6}$. The anisotropy angle α_{an}, unlike in the first test, takes a positive value, meaning that the principal permeability directions are rotated so that the fluid enters the anisotropic region from the upper side and exits through the lower one. Figure 6.8 shows the numerical results obtained with these parameters, which confirm the physical intuition regarding the direction of the flow induced by the anisotropy orientation. As in Figure 6.6, the fluid tends to move upward and bypass the porous filter, avoiding direct penetration through it. Similarly, inside the porous medium, the fluid still tends to circumvent the anisotropic region, although this effect is less pronounced because the permeability contrast is lower than in the previous test. In Figure 6.9, a zoomed view of the domain interfaces is presented, focusing on the upper-left corner of Ω_{iso} and the entire interaction between Ω_{an} and its surrounding region. We can observe that the fluid tends to concentrate near the corners, a behavior mainly driven by the anisotropy angle, which redirects the preferential flow paths along the principal directions of permeability. This effect highlights how the orientation of the anisotropic axes influences the local acceleration and deflection of the flow at the interface between both materials. Finally, we compute the error associated with the conservation of momentum (cf. (5.15)), now considering the non-piecewise constant porosity defined above and the datum $\mathbf{f} \equiv 0$ in Ω, thus obtaining

$$\|\boldsymbol{\mathcal{P}}_h^k(\rho\, d_t \mathbf{u}_h) + \mu \boldsymbol{\mathcal{P}}_h^k(\mathbf{K}^{-1}\, \mathbf{u}_h) - \mathbf{div}(\boldsymbol{\sigma}_h)\|_{\ell^\infty(0,T;\ell^\infty(\Omega))} = 9.76 \cdot 10^{-7}\,,$$

which confirms that momentum is conserved.

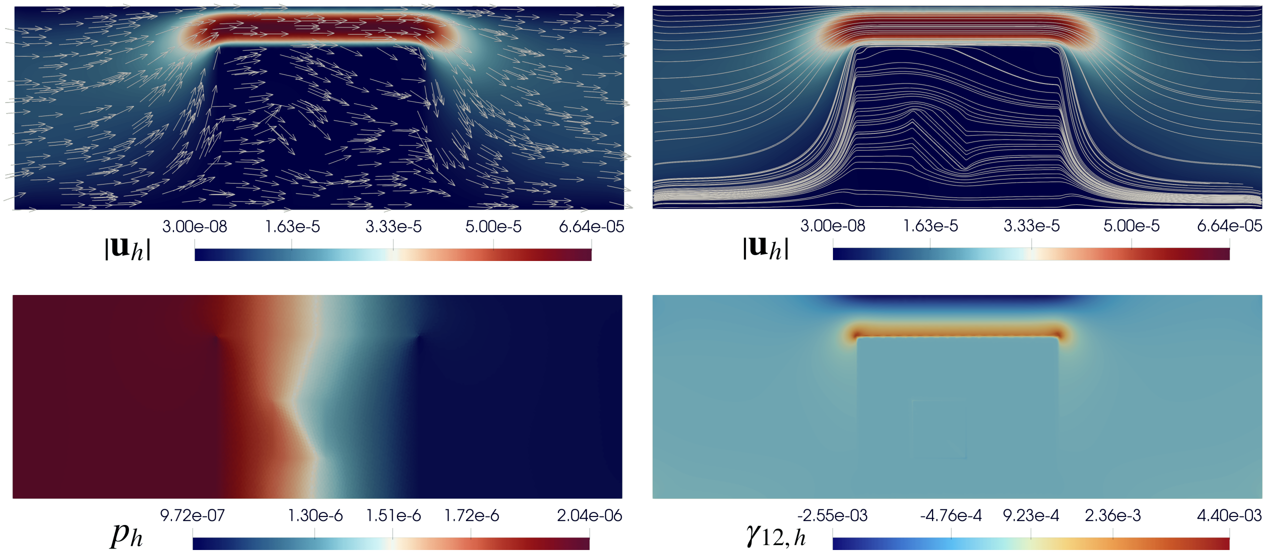


Figure 6.8: [Example 3] Computed velocity and its streamlines (top), and the pressure and component of the vorticity (bottom), with porosity function varying in space.

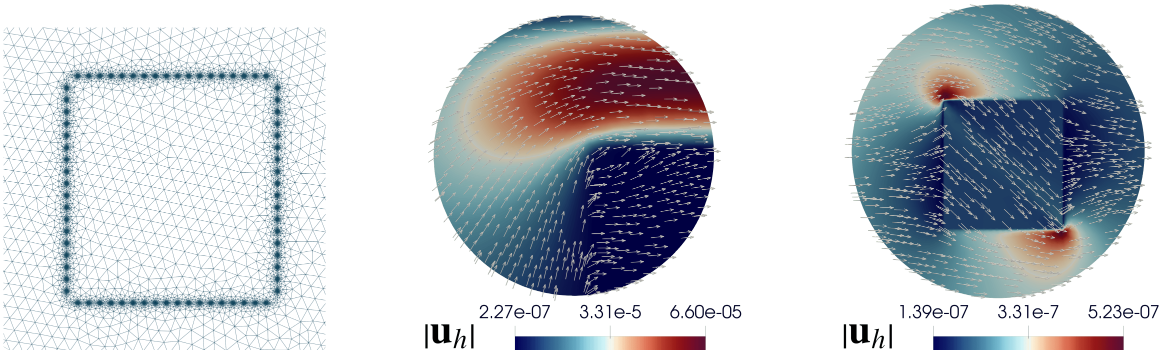


Figure 6.9: [Example 3] Zoom view of the computational mesh near the interface between Ω_{iso} and Ω_{an} (left). Zoom view of the computed velocity at the interfaces: free-fluid versus porous filter (center), and isotropic versus anisotropic regions (right).

CHAPTER 7

Conclusions and future works

In this thesis, we have introduced a new stress-velocity-vorticity formulation for the time-dependent Brinkman problem with spatially varying porosity, together with its mixed finite element approximation. This model is of significant interest for describing flows through porous media and serves as an intermediate regime between the Darcy and Stokes equations. The proposed formulation offers several advantages:

(i) It naturally incorporates the porosity gradients and recovers the classical constant-porosity formulation as a particular case, for which all assumptions involving $\|\nabla\rho/\rho\|_{\mathbf{L}^\infty(\Omega)}$ are no longer required.

(ii) It enables the direct computation of physically meaningful quantities such as the Cauchy stress and vorticity tensors, while the pressure, eliminated from the formulation, as well as the velocity gradient, can be easily reconstructed through a simple post-processing step.

In addition, the analysis developed here leads to the following conclusions:

(i) The theoretical analysis relies heavily on monotone operator techniques and is made possible by introducing a fixed-point strategy formulated in a suitable Bochner space.

(ii) We have rigorously established the well-posedness of both the semi-discrete continuous-in-time and the fully discrete schemes, and we have developed the corresponding error analysis. Moreover, the proposed method inherits from the continuous problem the important property of being momentum conservative.

(iii) From the numerical perspective, we have illustrated, through simulations implemented in `FEniCS`, the robustness and accuracy of the method, even in scenarios involving challenging physical parameters.

As a consequence of the work developed in this thesis, we currently have the following preprint under review:

> A. J. Bustos and S. Caucao, *A Banach space mixed formulation for the unsteady Brinkman problem with spatially varying porosity.* Universidad de Concepción, (2025). Available in the CI^{2}MA repository.

Finally, we remark that the results obtained in this thesis have motivated to several ongoing and future projects. Some of them are described below:

(i) **A posteriori error analysis**: The last numerical example highlights the importance of mesh refinement near material interfaces, as illustrated in Figure 6.9. This observation motivates future work on developing an *a posteriori* error analysis for the proposed method, which could then be employed to guide adaptive mesh refinement and achieve more accurate approximations in this class of problems.

(ii) **Multiphysics problems**: As discussed throughout this manuscript, the analysis carried out in Banach spaces, beyond its intrinsic mathematical interest, is motivated by the coupling of the Brinkman problem with other models. In particular, it is of significant interest to extend the present analysis to the interaction of this model with heat or transport equations.

Appendix A: On the solvability of an abstract evolution problem

This appendix is devoted to the proof of Theorem 3.1. The arguments follow the general ideas in [42, Chapter IV], and in particular those of [42, Theorem IV.6.1(b)]. Nevertheless, our aim is to provide a self-contained proof, including all the necessary preliminaries. We begin by recalling several definitions and results concerning accretive operators. Next, we establish a key result on evolution equations in Hilbert spaces. We then include an interlude describing the relationship between pre-Hilbert and semi-inner product spaces, which offers a convenient framework for passing from just a semi-inner product space to a Hilbert space by duality. Finally, we conclude by proving the main result, which relies on all the preceding developments.

A.1 Prelude: Accretive operators

Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and let A be a relation on H, that is, A is a subset of $H \times H$. We define the domain, range and inverse of A as

$$\mathcal{D}(A) := \Big\{ x \in H \ : \quad (x, y) \in A \quad \text{for some } y \in H \Big\},$$

$$\begin{aligned} R(A) &:= \Big\{ y \in H \ : \quad (x,y) \in A \quad \text{for some } x \in H \Big\} , \\ A^{-1} &:= \Big\{ (y,x) \in H \times H \ : \quad (x,y) \in A \Big\} , \end{aligned}$$

respectively. We may view A as a mapping that assigns to each $x \in H$ the subset $A(x) = \{y \in H \ : \ (x,y) \in A\}$. Then A is a function precisely when each set $A(x)$ consists of a single point. In this sense, we refer to A as a relation or, equivalently, as a multivalued operator. Furthermore, we may define the operations

$$\begin{aligned} &\lambda A := \Big\{ (x, \lambda y) \in H \times H \ : \quad (x,y) \in A \Big\} \qquad \forall \, \lambda \in \mathrm{R} \, , \quad \text{and} \\ &A + B := \Big\{ (x, y+z) \in H \times H \ : \quad (x,y) \in A \quad \text{and} \quad (x,z) \in B \Big\} . \end{aligned}$$

We notice that $\mathcal{D}(\lambda A) = \mathcal{D}(A)$ if $\lambda \neq 0$ and $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$. From now on, we denote by I the identity operator.

Definition A.1. *An operator A on H is said to be accretive if, for all $(x_j, w_j) \in A$, $j \in \{1,2\}$, then*

$$(w_1 - w_2, x_1 - x_2)_H \geq 0 \, .$$

Furthermore, A is called m-accretive if, in addition, $R(I + A) = H$.

The following two lemmas provide characterizations of accretive and m-accretive operators. Their proofs are mainly algebraic and are therefore omitted. For further details, see [42, Corollary IV.1.3 and Lemma IV.1.3].

Lemma A.1. *The following are equivalent:*

(i) A is accretive,

(ii) $(I + \alpha A)^{-1}$ is a contraction on $R(I + \alpha A)$ for all $\alpha > 0$.

Lemma A.2. *The following are equivalent:*

(i) A is accretive and $R(I + \alpha A) = H$ for some $\alpha > 0$,

(ii) *A is m-accretive,*

(iii) *A is accretive and $R(I+\alpha A)=H$ for all $\alpha>0$.*

Given an m-accretive operator A and $\alpha>0$, Lemma A.2 implies that $R(I+\alpha A)=H$, and, consequently, $\mathcal{D}((I+\alpha\, A)^{-1})=H$. Moreover, for any $y\in H$, the problem of finding $x\in H$ such that

$$(I+\alpha\, A)^{-1}(y)=x$$

admits a unique solution, since $(I+\alpha\, A)^{-1}$ is a contraction on $R(I+\alpha A)=H$ as stated in Lemma A.1. Therefore, we may define the single-valued operators

$$J_\alpha := (I+\alpha A)^{-1} \qquad \forall\, \alpha>0\,,$$

which are called the resolvents of A. Observe that $y=J_\alpha(x)$ if and only if $\frac{1}{\alpha}(x-y)\in A(y)$, whence

$$\frac{1}{\alpha}\,(x-J_\alpha(x))\in A(J_\alpha(x)) \qquad \forall\, x\in \mathrm{H}\,, \quad \forall\, \alpha>0\,. \tag{A.1}$$

Proposition A.1. *Let A be an m-accretive operator on H. Then, A is maximal accretive, i.e. if B is accretive and $A\subset B$, then $A=B$.*

Proof. Let $(x,y)\in B$ and define $z:=(I+A)^{-1}(y+x)$. Then, $z\in\mathcal{D}(A)\subset\mathcal{D}(B)$ and $y+x\in(I+A)(z)\subset(I+B)(z)$, so $y+x-z\in B(z)$. Using that B is accretive with the pair $(x,y),(z,y+x-z)\in B$, we deduce that

$$0\le (y-(y+x-z),x-z)_H=(-x+z,x-z)_H=-\|x-z\|_H^2\,,$$

which implies $x=z$. Therefore, $x\in\mathcal{D}(A)$, and, since $y+x\in(I+A)(z)$, it follows that $y\in A(x)$. Thus, $(x,y)\in A$, as desired. □

Hereafter, if S is a subset of a topological space X, we denote by $\overline{S}$ the closure of S in X.

Proposition A.2. *Let A be an m-accretive operator on H. Then, for all $x\in H$, the set $A(x)$ is closed and convex.*

Proof. Let us define the relation $\widetilde{A} \subset H \times H$ by

$$(x, y) \in \widetilde{A} \quad \text{if and only if} \quad x \in \mathcal{D}(A) \quad \text{and} \quad y \in \overline{A(x)}\,.$$

It is readily verified that $A \subset \widetilde{A}$ and that $\widetilde{A}$ is also accretive. By Proposition A.1, it follows that $A = \widetilde{A}$. Thus, $A(x) = \overline{A(x)}$ for all $x \in \mathcal{D}(A)$. This proves that $A(x)$ is closed.

On the other hand, to prove that $A(x)$ is convex, we define $A_c \subset H \times H$ by

$$(x, y) \in A_c \quad \text{if and only if} \quad x \in \mathcal{D}(A) \quad \text{and} \quad y \in \mathrm{Conv}(A(x))\,,$$

where $\mathrm{Conv}(A(x))$ denotes the convex hull of $A(x)$. As before, it is readily verified that $A \subset A_c$ and A_c is also accretive. Hence, $A_c = A$, which implies that for all $x \in \mathcal{D}(A)$, $A(x) = \mathrm{Conv}(A(x))$, that is, $A(x)$ is convex. This completes the proof. □

Proposition A.3. *Let A be an m-accretive operator on H and let $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset A$ be a sequence such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ in H. Then*

(i) If $\liminf\limits_{n\to\infty} (x_n, y_n)_H \le (x, y)_H$, then $(x, y) \in A$.

(ii) If $\limsup\limits_{n\to\infty} (x_n, y_n)_H \le (x, y)_H$, then $\lim\limits_{n\to\infty} (x_n, y_n)_H = (x, y)_H$.

Proof.

(i) Consider the relation $\widetilde{A} := A \cup \{(x, y)\}$. Clearly, $A \subset \widetilde{A}$. Let us prove that $\widetilde{A}$ is accretive. Since A is accretive, for all $(u, v) \in A$,

$$0 \le (v - y_n, u - x_n)_H = (v, u)_H - (v, x_n)_H - (y_n, u)_H + (y_n, x_n)_H\,.$$

Then, taking $\liminf$, using the assumption and the weak convergences,

$$0 \le (v, u)_H - (v, x)_H - (y, u)_H + (y, x)_H = (x - y, u - x)_H\,.$$

Thus, $\widetilde{A}$ is accretive. Therefore, by Proposition A.1, we conclude that $\widetilde{A} = A$, which means that $(x, y) \in A$.

(ii) If the lim sup assumption holds, the lim inf assumption of the previous item also holds, and, thus, $(x, y) \in A$. Then, using that A is accretive,

$$(y - y_n, x - x_n)_H \geq 0 \qquad \forall\, n \in \mathbb{N}\,.$$

Taking lim inf, we obtain that

$$\liminf_{n\to\infty} (x_n, y_n)_H \geq (x, y)_H\,,$$

which along with the lim sup assumption, gives $\lim\limits_{n\to\infty} (x_n, y_n)_H = (x, y)_H$, as desired.

□

Definition A.2. *Let A be an m-accretive operator on H. The Yosida approximations of A are defined by*

$$A_\alpha := \frac{1}{\alpha}\,(I - J_\alpha) \qquad \forall\, \alpha > 0\,.$$

Certainly, from (A.1) it follows that $A_\alpha(x) \in A(J_\alpha(x))$ for all $x \in H$ and $\alpha > 0$. Moreover, using once again the characterization of J_α (A.1), it is straightforward to verify that $y = A_\alpha(x)$ if and only if $y \in A(x - \alpha y)$.

Since A is m-accretive, by Proposition A.2, $A(x)$ is closed and convex for all $x \in H$. This allows us to define the minimal section operator $A^0 : H \to H$ by

$$A^0(x) = \mathrm{Proj}_{A(x)}(0) = \operatorname*{arg\,min}_{y\in A(x)}\ \|y\|_H\,.$$

Theorem A.3. *Let A be an m-accretive operator on H.*

(i) Each A_α is m-accretive and Lipschitz with constant $1/\alpha$.

(ii) For each $x \in \mathcal{D}(A)$, $\|A_\alpha(x)\|_H$ converges upward to $\|A^0(x)\|_H$, $\lim\limits_{\alpha\to 0} A_\alpha(x) = A^0(x)$ and

$$\|A_\alpha(x) - A^0(x)\|_H^2 \leq \|A^0(x)\|_H^2 - \|A_\alpha(x)\|_H^2 \qquad \forall\, \alpha > 0\,. \tag{A.2}$$

Proof. See [42, Theorem IV.1.1]. □

A.2 Transitions: The theorem of Kato

An important tool in the development of the theory of abstract evolution equations is the use of Yosida approximations. Later in this section, we shall prove Kato's theorem, which establishes the existence and uniqueness of a solution to the problem

$$\frac{du}{dt}(t) + A(u(t)) \ni \omega\, u(t) + f(t) \qquad \text{for a.e.} \quad t \in (0,T)\,,$$

where A is an m-accretive operator in the Hilbert space H, $\omega \geq 0$, $f : [0,T] \to H$ is absolutely continuous and $u(0) = u_0$. In general, if A is replaced by its Yosida approximation A_α, with $\alpha > 0$, the proof of existence and uniqueness becomes simpler. In this way, we can construct a sequence of solutions u_α corresponding to each approximated problem and then show that $\lim\limits_{\alpha\to 0} u_\alpha$ exists and solves the original problem. Therefore, our first step will be to prove the existence and uniqueness of the solution for each approximation.

Lemma A.4. *Let A be m-accretive in the Hilbert space H and $\omega \geq 0$. For each $u_0 \in \mathcal{D}(A)$, absolutely continuous $f : [0,T] \to H$, and $\alpha > 0$, there is a unique $u_\alpha \in C^1([0,T], H)$, such that $u_\alpha(0) = u_0$ and*

$$\frac{du_\alpha}{dt}(t) + A_\alpha(u_\alpha(t)) = \omega\, u_\alpha(t) + f(t) \qquad \textit{for a.e.} \quad t \in [0,T]\,. \tag{A.3}$$

Proof. If u_α is a solution of (A.3), then by integrating (A.3) from 0 to $t \in (0,T]$, we obtain

$$u_\alpha(t) = u_0 + \int_0^t \Big\{ - A_\alpha(u_\alpha(s)) + \omega\, u_\alpha(s) + f(s) \Big\}\, ds\,, \tag{A.4}$$

which motivates the definition of the operator $\Phi_\alpha : C([0,T], H) \to C([0,T], H)$ such that for each $v \in C([0,T], H)$,

$$\Phi_\alpha(v)(t) := u_0 + \int_0^t \Big\{ - A_\alpha(v(s)) + \omega\, v(s) + f(s) \Big\}\, ds \qquad \forall\, t \in [0,T]\,.$$

We shall consider the weighted norm $\| \cdot \|_\lambda : C([0,T], H) \to \mathrm{R}$, for some $\lambda > 0$ to be chosen later, defined by

$$\|v\|_\lambda := \sup_{0\leq t\leq T} e^{-\lambda t}\,\|v(t)\|_H \qquad \forall\, v \in C([0,T],H)\,,$$

which is equivalent to the usual norm in $C([0,T],H)$. We now proceed to prove that Φ_α is a contractive operator on $C([0,T],H)$ with this norm. Given $v_1, v_2 \in C([0,T],H)$, we have

$$\|\Phi_\alpha(v_1)(t) - \Phi_\alpha(v_2)(t)\|_H \leq \int_0^t \Big\{\|A_\alpha(v_1(s)) - A_\alpha(v_2(s))\|_H + \omega\,\|v_1(s) - v_2(s)\|_H\Big\}\,ds\,,$$

whence using that A_α is Lipschitz with constant $1/\alpha$ (cf. Theorem A.3) and denoting $L := 1/\alpha + \omega$, we obtain that

$$\begin{aligned}\|\Phi_\alpha(v_1)(t) - \Phi_\alpha(v_2)(t)\|_H &\leq L\int_0^t \|v_1(s) - v_2(s)\|_H\,ds\\ &\leq L\,\|v_1 - v_2\|_\lambda \int_0^t e^{\lambda s}\,ds = \frac{L}{\lambda}\,\|v_1 - v_2\|_\lambda\,(e^{\lambda t} - 1)\,.\end{aligned}$$

Thus,

$$\|\Phi_\alpha(v_1) - \Phi_\alpha(v_2)\|_\lambda \leq \frac{L}{\lambda}\,\|v_1 - v_2\|_\lambda \sup_{0\leq t\leq T}(1 - e^{-\lambda t}) \leq \frac{L}{\lambda}\,\|v_1 - v_2\|_\lambda\,.$$

Hence, by chosing $\lambda := 2L$, we conclude that Φ_α is a contractive operator in the Banach space $C([0,T],H)$ with the norm $\|\cdot\|_\lambda$. Applying the Banach fixed-point theorem, we deduce that there is a unique $u_\alpha \in C([0,T],H)$ such that $\Phi_\alpha(u_\alpha) = u_\alpha$, which means that (A.4) holds for all $t \in [0,T]$, thereby implying (A.3). Moreover, since every term in the integrand in (A.4) is continuous, it follows that the derivative of u_α is also continuous. Hence, $u_\alpha \in C^1([0,T],H)$. This completes the proof. $\square$

The following lemma, which is a version of the so-called Gronwall inequality, will be used repeatedly in the main result of this section.

Lemma A.5 (Gronwall inequality)**.** *Let* $a, b \in \mathrm{L}^1(0,T)$ *with* $b(t) \geq 0$ *a.e. in* $(0,T)$*, and let* $v : [0,T] \to \mathrm{R}^+$ *be an absolutely continuous function that satisfies, for some* $0 \leq \delta < 1$*,*

$$(1-\delta)\,v'(t) \leq a(t)\,v(t) + b(t)\,v^\delta(t) \qquad \textit{for a.e.} \quad t \in [0,T]\,.$$

Then,

$$v^{1-\delta}(t) \leq v^{1-\delta}(0)\,\eta(0) + \int_0^t \eta(s)\,b(s)\,ds \qquad \forall\, t \in [0,T]\,,$$

where

$$\eta(s) := \exp\left(\int_s^t a(\xi)\,d\xi\right).$$

Proof. See [42, Lemma IV.4.1]. □

We are now in a position to prove the existence and uniqueness of a solution to the abstract evolution problem introduced at the beginning of this section. This result is originally due to Kato [37].

Theorem A.6 (Kato)**.** *Let A be m-accretive in the Hilbert space H and $\omega \geq 0$. For each $u_0 \in \mathcal{D}(A)$ and absolutely continuous $f : [0,T] \to H$, there is a unique absolutely continuous $u : [0,T] \to H$, such that $u(0) = u_0$ and*

$$\frac{du}{dt}(t) + A(u(t)) \ni \omega\,u(t) + f(t) \qquad \text{for a.e.} \quad t \in (0,T)\,. \tag{A.5}$$

Furthermore, u is Lipschitz continuous and $u(t) \in \mathcal{D}(A)$ for every $t \geq 0$.

Proof. We first prove uniqueness. Let u_1 and u_2 be two solutions of (A.5). Since A is accretive, it follows that

$$\frac{1}{2}\,\frac{d}{dt}\,\|u_1(t) - u_2(t)\|_H^2 \leq \omega\,\|u_1(t) - u_2(t)\|_H^2 \qquad \forall\, t \geq 0\,,$$

which, by using Gronwall inequality with $\delta = 0$ (cf. Lemma A.5), yields

$$\|u_1(t) - u_2(t)\|_H \leq e^{\omega\,t}\,\|u_1(0) - u_2(0)\|_H \qquad \forall\, t \geq 0\,.$$

Thus, since $u_1(0) = u_2(0) = u_0$, any solution, if it exists, must be unique.

Now, in order to prove existence, for each $\alpha > 0$, we let $u_\alpha \in C^1([0,T],H)$ be the unique solution (cf. Lemma A.4) to

$$\frac{du_\alpha}{dt}(t) + A_\alpha(u_\alpha(t)) = \omega\,u_\alpha(t) + f(t) \qquad \forall\, t \in [0,T]\,, \tag{A.6}$$

with $u_\alpha(0) = u_0$. We observe that, if $h > 0$, then $u_\alpha(t+h)$ is a solution of (A.6) with $f(t)$ replaced by $f(t+h)$. Using the accretivity of A_α, a few algebraic manipulations yield

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\,\|u_\alpha(t+h) - u_\alpha(t)\|_H^2 &\leq \omega\,\|u_\alpha(t+h) - u_\alpha(t)\|_H^2 \\ &+ \|f(t+h) - f(t)\|_H\,\|u_\alpha(t+h) - u_\alpha(t)\|_H\,.\end{aligned}$$

By applying Gronwall inequality with $\delta = 1/2$ (cf. Lemma A.5) and letting $h \to 0$, we arrive at

$$\|u_\alpha'(t)\|_H \leq e^{\omega\,t}\,\Big\| - A_\alpha(u_0) + \omega\,u_0 + f(0)\Big\|_H + \int_0^t e^{\omega\,(t-s)}\,\|f'(s)\|_H\,ds\,,$$

where the prime indicates differentiation in time. This, together with the fact that $\|A_\alpha(u_0)\|_H \leq \|A^0(u_0)\|_H$ (cf. (A.2)), yields

$$\|u_\alpha'(t)\|_H \leq e^{\omega\,t}\,\|A^0(u_0)\|_H + e^{\omega\,t}\|\omega\,u_0 + f(0)\|_H + \int_0^t e^{\omega\,(t-s)}\,\|f'(s)\|_H\,ds\,. \tag{A.7}$$

This implies that $\{u_\alpha'\}$ is a bounded sequence in $C([0,T],H)$. Moreover, from this fact and employing (A.6), it follows that $\{u_\alpha\}$ and $\{A_\alpha(u_\alpha)\}$ are also bounded in $C([0,T],H)$.

We now proceed to prove that the sequence $\{u_\alpha\}$ is Cauchy in $C([0,T],H)$. To this end, let $\alpha, \beta > 0$, and use (A.6) to obtain

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\,\|u_\alpha(t) - u_\beta(t)\|_H^2 \\ &\quad = \omega\,\|u_\alpha(t) - u_\beta(t)\|_H^2 - \Big(A_\alpha(u_\alpha(t)) - A_\beta(u_\beta(t)), u_\alpha(t) - u_\beta(t)\Big)_H\,.\end{aligned} \tag{A.8}$$

Writing $u_* = \alpha\,A_*(u_*) + J_*(u_*)$ for $* \in \{\alpha,\beta\}$, and omitting the time dependence for clarity, the last term in (A.8) can be rewritten as

$$\begin{aligned}\Big(A_\alpha(u_\alpha) - A_\beta(u_\beta), u_\alpha - u_\beta\Big)_H &= \Big(A_\alpha(u_\alpha) - A_\beta(u_\beta), \alpha\,A_\alpha(u_\alpha) - \beta\,A_\beta(u_\beta)\Big)_H \\ &+ \Big(A_\alpha(u_\alpha) - A_\beta(u_\beta), J_\alpha(u_\alpha) - J_\beta(u_\beta)\Big)_H\,.\end{aligned} \tag{A.9}$$

As mentioned just below of Definition A.2, we have $A_*(u_*) \in A(J_*(u_*))$ for each $* \in \{\alpha,\beta\}$. This, along with the fact that A is accretive, implies that the last term in (A.9) is non-negative.

Thus,

$$\Big(A_\alpha(u_\alpha) - A_\beta(u_\beta), u_\alpha - u_\beta\Big)_H \geq \Big(A_\alpha(u_\alpha) - A_\beta(u_\beta), \alpha\, A_\alpha(u_\alpha) - \beta\, A_\beta(u_\beta)\Big)_H . \tag{A.10}$$

By performing some algebraic manipulations and using Cauchy–Schwarz and Young's inequalities, we obtain that

$$\begin{aligned}\Big(A_\alpha(u_\alpha) - A_\beta(u_\beta), u_\alpha - u_\beta\Big)_H &\geq \alpha\, \|A_\alpha(u_\alpha)\|_H^2 + \beta\, \|A_\beta(u_\beta)\|_H^2 - (\alpha+\beta)\,(A_\alpha(u_\alpha), A_\beta(u_\beta))_H \\ &\geq \alpha\, \|A_\alpha(u_\alpha)\|_H^2 + \beta\, \|A_\beta(u_\beta)\|_H^2 - \alpha \left(\|A_\alpha(u_\alpha)\|_H^2 + \frac{1}{4}\, \|A_\beta(u_\beta)\|_H^2\right) \\ &\quad - \beta\, \left(\frac{1}{4}\, \|A_\alpha(u_\alpha)\|_H^2 + \|A_\beta(u_\beta)\|_H^2\right) \\ &\geq -\frac{\alpha+\beta}{4}\, K^2 ,\end{aligned}$$

where $K := \sup\Big\{\|A_\alpha(u_\alpha(t))\|_H \;:\; 0 \leq t \leq T \quad \text{and} \quad \alpha > 0\Big\}$. Notice that this supremum is indeed finite, as the sequence $\{A_\alpha(u_\alpha)\}$ is bounded in $C([0,T],H)$. Hence, upon writing the time dependence explicitly again,

$$-\Big(A_\alpha(u_\alpha(t)) - A_\beta(u_\beta(t)), u_\alpha(t) - u_\beta(t)\Big)_H \leq \frac{\alpha+\beta}{4}\, K^2 .$$

Putting this estimate into (A.8), we find that

$$\frac{d}{dt}\, \|u_\alpha(t) - u_\beta(t)\|_H^2 \leq 2\omega\, \|u_\alpha(t) - u_\beta(t)\|_H^2 + \frac{\alpha+\beta}{2}\, K^2 ,$$

and using Gronwall inequality with $\delta = 0$ (cf. Lemma A.5), we discover

$$\|u_\alpha(t) - u_\beta(t)\|_H^2 \leq \frac{\alpha+\beta}{4\,\omega}\, K^2\,(e^{2\omega\, t} - 1) \qquad \forall\, t \in [0,T] .$$

Therefore, $\{u_\alpha\}$ is uniformly Cauchy and, consequently, there exists a function $u \in C([0,T],H)$ satisfying

$$\|u_\alpha(t) - u(t)\|_H^2 \leq \frac{\alpha}{4\,\omega}\, K^2\,(e^{2\omega\, t} - 1) \qquad \forall\, t \in [0,T] , \quad \forall\, \alpha > 0 ,$$

which, in particular, gives us that $u_\alpha \to u$ in $\mathrm{L}^2(0,T;H)$. Since $\{A_\alpha(u_\alpha)\}_\alpha$ is bounded in

$C([0,T];H)$, we also have that is bounded in $\mathrm{L}^2(0,T;H)$, and, consequently, there is a subsequence (not relabeled) such that $A_\alpha(u_\alpha) \rightharpoonup \xi$ in $\mathrm{L}^2(0,T;H)$, for some $\xi \in \mathrm{L}^2(0,T;H)$. Analogously for $\{u'_\alpha\}_\alpha$, it follows that there exists a subsequence weakly convergent. Moreover, by taking $\alpha \to 0$ in the equation

$$u_\alpha(t) = u_0 + \int_0^t u'_\alpha(s)\,ds\,,$$

we discover that $u'_\alpha \rightharpoonup u'$. On the other hand, we recall from Definition A.2 that $u_\alpha - J_\alpha(u_\alpha) = \alpha\,A_\alpha(u_\alpha)$, which along with the boundedness of $\{A_\alpha(u_\alpha)\}_\alpha$ yields

$$\|u_\alpha - J_\alpha(u_\alpha)\|_{C([0,T],H)} \leq \alpha\,\|A_\alpha(u_\alpha)\|_{C([0,T],H)} \xrightarrow{\alpha\to 0} 0\,,$$

whence $J_\alpha(u_\alpha) \to u$ in $C([0,T],H)$ and, thus, we also obtain convergence in $\mathrm{L}^2(0,T;H)$. Therefore, we can take limit in (A.6) to find that

$$u' + \xi = \omega\,u + f \quad \text{in} \quad \mathrm{L}^2(0,T;H)\,. \tag{A.11}$$

It remains to prove that $\xi \in A(u)$ to obtain (A.5). To that end, we introduce the realization $\mathcal{A} \subset \mathrm{L}^2(0,T;H) \times \mathrm{L}^2(0,T;H)$ as a relation defined by

$$(z,w) \in \mathcal{A} \quad \text{if and only if} \quad w(t) \in A(z(t)) \quad \text{for a.e.} \quad t \in (0,T)\,. \tag{A.12}$$

Since A is m-accretive in H, it is not difficult to see that $\mathcal{A}$ is m-accretive in $\mathrm{L}^2(0,T;H)$. Then, putting $x_\alpha := J_\alpha(u_\alpha)$ and $y_\alpha := A_\alpha(u_\alpha)$, we have that $(x_\alpha, y_\alpha) \in \mathcal{A}$. We already know that $x_\alpha \to u$ in $\mathrm{L}^2(0,T;H)$ and $y_\alpha \rightharpoonup \xi$ in $\mathrm{L}^2(0,T;H)$. Consequently,

$$\int_0^T (x_\alpha(t), y_\alpha(t))_H\,dt \xrightarrow{\alpha\to 0} \int_0^T (u(t),\xi(t))_H\,dt\,. \tag{A.13}$$

By applying Lemma A.3 to the realization $\mathcal{A}$ (cf. (A.12)) on the Hilbert space $\mathrm{L}^2(0,T;H)$, together with the sequences $\{x_\alpha\}$ and $\{y_\alpha\}$ and the convergence in (A.13), we obtain that

$(u,\xi)\in\mathcal{A}$; that is, $\xi(t)\in A(u(t))$ for a.e. $t\in(0,T)$. Therefore, (A.11) becomes

$$u'(t)+A(u(t))\ni\omega\,u(t)+f(t)\qquad\text{for a.e.}\quad t\in(0,T)\,,$$

which proves (A.5). Furthermore, since $u_\alpha\to u$ in $C([0,T],H)$, we have that $u_\alpha(0)\to u(0)$, i.e. $u(0)=u_0$. We now prove that u is Lipschitz continuous. In fact, each u_α is Lipschitz, since for all $t\geq s$,

$$\|u_\alpha(t)-u_\alpha(s)\|_H\leq\int_s^t\|u'_\alpha(t)-u'_\alpha(s)\|_H\leq C\,(t-s)\,,$$

where C is independent of t and s, owing to the uniformly boundedness of u'_α (cf. (A.7)). Hence,

$$\|u(t)-u(s)\|_H=\lim_{\alpha\to0}\|u_\alpha(t)-u_\alpha(s)\|_H\leq C\,(t-s)\,,$$

so u is Lipschitz continuous. This completes the proof. □

A.3 Interlude: Semi-inner product spaces and duality

Let E be a vector space endowed with the inner product $(\cdot,\cdot)$. We assume that E is a pre-Hilbert space, i.e. E with the induced norm is not necessarily complete. Recall that the dual space E', endowed with the usual dual norm $\|\cdot\|_{E'}$ is a Banach space. Let $T:E\to E'$ be the map defined by

$$[T(u),v]:=(u,v)\qquad\forall\,u,v\in E\,,$$

where $[\cdot,\cdot]$ denotes the dual pairing between E' and E. It is easy to show that T is a linear isometry, so it is injective. Notice that if T were surjective, then it would coincide with the Riesz map and E would be complete, which, in general, is not the case here. Our purpose is to show that $R(T)$ is dense in E' and that E' is actually a Hilbert space. To that end, we first define the inner product $(\!(\cdot,\cdot)\!):R(T)\times R(T)\to\mathrm{R}$ by

$$(\!(f,g)\!):=(T^{-1}(f),T^{-1}(g))\qquad\forall\,f,g\in R(T)\,,$$

which is well defined due to the injectivity of T. We now extend this definition to $\overline{R(T)}$ by density. Let $f, g \in \overline{R(T)}$ and $\{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n\in\mathbb{N}} \subset R(T)$ such that $f_n \to f$ and $g_n \to g$ in E'. We define

$$(\!(f, g)\!) := \lim_{n\to\infty} (T^{-1}(f_n), T^{-1}(g_n)) \,. \tag{A.14}$$

This definition clearly extends the previous one. We proceed to prove that $(\!(\cdot,\cdot)\!)$ is properly defined, which means that the limit exists and the definition does not depend on the particular sequences. To prove the former, we use the triangle and Cauchy–Schwarz inequalities along with the fact that T is an isometry to obtain

$$\begin{aligned}
&\Big|(T^{-1}(f_n), T^{-1}(g_n)) - (T^{-1}(f_m), T^{-1}(g_m))\Big| \\
&\quad = \Big|(T^{-1}(f_n) - T^{-1}(f_m), T^{-1}(g_n)) + (T^{-1}(f_m), T^{-1}(g_n) - T^{-1}(g_m))\Big| \\
&\quad \le \|T^{-1}(f_n) - T^{-1}(f_m)\| \, \|T^{-1}(g_n)\| + \|T^{-1}(f_m)\| \, \|T^{-1}(g_n) - T^{-1}(g_m)\| \\
&\quad = \|f_n - f_m\|_{E'} \, \|g_n\|_{E'} + \|f_n\|_{E'} \, \|g_n - g_m\|_{E'} \xrightarrow{n,m\to\infty} 0 \,.
\end{aligned}$$

Thus, $\{(T^{-1}(f_n), T^{-1}(g_n))\}_{n\in\mathbb{N}}$ is Cauchy in R, and, consequently, it is convergent.

On the other hand, (A.14) does not depend on the particular choice of sequences. In fact, let $\{\widetilde{f}_n\}_{n\in\mathbb{N}}, \{\widetilde{g}_n\}_{n\in\mathbb{N}} \subset R(T)$ be another pair of sequences satisfying that $\widetilde{f}_n \to f$ and $\widetilde{g}_n \to g$ in E'. Then,

$$\begin{aligned}
&\Big|(T^{-1}(f_n), T^{-1}(g_n)) - (T^{-1}(\widetilde{f}_n), T^{-1}(\widetilde{g}_n))\Big| \\
&\quad = \Big|(T^{-1}(f_n) - T^{-1}(\widetilde{f}_n), T^{-1}(g_n)) + (T^{-1}(\widetilde{f}_n), T^{-1}(g_n) - T^{-1}(\widetilde{g}_n))\Big| \\
&\quad \le \|T^{-1}(f_n) - T^{-1}(\widetilde{f}_n)\| \, \|T^{-1}(g_n)\| + \|T^{-1}(\widetilde{f}_n)\| \, \|T^{-1}(g_n) - T^{-1}(\widetilde{g}_n)\| \\
&\quad = \|f_n - \widetilde{f}_n\|_{E'} \, \|g_n\|_{E'} + \|\widetilde{f}_n\|_{E'} \, \|g_n - \widetilde{g}_n\|_{E'} \xrightarrow{n\to\infty} 0 \,,
\end{aligned}$$

so the limits coincide, which proves that (A.14) is well defined.

Now, notice that $(\!(f, f)\!)^{1/2} = \|f\|_{E'}$ for all $f \in \overline{R(T)}$. Therefore, we have proven that $(\overline{R(T)}, (\!(\cdot,\cdot)\!))$ is a Hilbert space. In what follows, we prove that $\overline{R(T)} = E'$.

Given $f \in E'$, we may define $\phi_f : \overline{R(T)} \to \mathrm{R}$ by

$$[\phi_f, g] := \lim_{n\to\infty} [f, T^{-1}(g_n)] \qquad \forall\, g \in R(T)\,,$$

where $\{g_n\}_{n\in\mathbb{N}}$ denotes any sequence in $R(T)$ such that $g_n \to g$ in E'. Analogously as before, it can be proven that this definition does not depend on the choice of the sequence. Furthermore, it is straightforward to prove that $\phi_f \in \overline{R(T)}'$. Since $\overline{R(T)}$ is a Hilbert space, we may use the Riesz representation theorem to deduce that there exists a unique $\widehat{f} \in \overline{R(T)}$ such that

$$[\phi_f, g] = (\!(\widehat{f}, g)\!) \qquad \forall\, g \in \overline{R(T)}\,.$$

Moreover, given a sequence $\{\widehat{f}_n\}_{n\in\mathbb{N}} \subset R(T)$ converging to $\widehat{f}$, we have that, for all $g \in R(T)$,

$$(\!(\widehat{f}, g)\!) = \lim_{n\to\infty}(T^{-1}(\widehat{f}_n), T^{-1}(g)) = \lim_{n\to\infty}[T(T^{-1}(\widehat{f}_n)), T^{-1}(g)] = \lim_{n\to\infty}[\widehat{f}_n, T^{-1}(g)] = [\widehat{f}, T^{-1}(g)]\,.$$

Then,

$$\|f-\widehat{f}\|_{E'} = \sup_{0\neq u\in E} \frac{[f-\widehat{f}, u]}{\|u\|} = \sup_{0\neq g\in R(T)} \frac{[f-\widehat{f}, T^{-1}(g)]}{\|T^{-1}(g)\|} = \sup_{0\neq g\in R(T)} \frac{[\phi_f, g] - (\!(\widehat{f}, g)\!)}{\|g\|} = 0\,,$$

which means that $f = \widehat{f}$. Hence, $f \in \overline{R(T)}$ and, consequently, $\overline{R(T)} = E'$. In particular, E' is a Hilbert space.

Certainly, we have shown that E' is a Hilbert space containing a dense subspace that is isometrically isomorphic to E. The unique space satisfying these properties, up to isometric isomorphism, is the completion of E. Hence, E' can be identified with the completion of E. We summarize these results in the following theorem.

Theorem A.7. *Let E be a vector space endowed with the inner product $(\cdot,\cdot)$. Then, E' is a Hilbert space and it is identified with the completion of E.*

For the purposes of this manuscript, the situation in which the vector space E is endowed with a semi-inner product is even more relevant. In this case, we can also obtain a similar result. We shall consider then the following setting. Denote by $E^\star$ the algebraic dual of E

and let $\mathcal{N} : E \to E^\star$ be a linear, symmetric and monotone operator, and define the semi-inner product

$$\begin{aligned} (\cdot,\cdot)_* : E \times E &\to \mathrm{R} \\ (x,y) &\mapsto (x,y)_* := [\mathcal{N}(x), y] \,. \end{aligned}$$

Denote by $|\cdot|_*$ the induced seminorm and by E_* the corresponding seminormed space. Let $\mathcal{K}$ be the kernel of $\mathcal{N}$. The topological dual of E_* is defined as

$$E'_* := \left\{ f \in E^\star \;\middle|\; \exists\, C > 0 \;:\; \forall\, x \in E \;:\; |f(x)| \leq C\,|x|_* \right\},$$

and it is endowed with the norm

$$\|f\|_{E'_*} := \sup_{u \in E \setminus \mathcal{K}} \frac{|f(u)|}{|u|_*} \qquad \forall\, f \in E'_* \,.$$

Now, define the quotient space

$$\widehat{E}_* := E/\mathcal{K} = \left\{ [u] \;:\; u \in E \right\}, \quad \text{where} \quad [u] := \left\{ v \in E \;:\; \quad v - u \in \mathcal{K} \right\},$$

and the quotient map $\pi : E \to \widehat{E}_*$ by $\pi(u) := [u]$ for all $u \in E$. We endow the space $\widehat{E}_*$ with the inner product

$$(\widehat{u}, \widehat{v})_{\widehat{E}_*} := (u, v)_* \,, \tag{A.15}$$

for all $\widehat{u}, \widehat{v} \in \widehat{E}_*$, where $u \in \pi^{-1}(\widehat{u})$ and $v \in \pi^{-1}(\widehat{v})$. It is easy to verify that this definition does not depend on the particular choice of representative of each class. Denote by $\|\cdot\|_{\widehat{E}_*}$ the induced norm and by W the completion of $\widehat{E}_*$. Hence, by Theorem A.7, it follows that $\widehat{E}'_*$ is a Hilbert space and it is isometrically isomorphic to W.

Since W is the completion of $\widehat{E}_*$, there exists a linear and bounded operator $j : \widehat{E}_* \to W$ that is injective, isometric and has dense range. Let us define the linear and continuous operator $q := j \circ \pi : E_* \to W$. Notice that

$$\|q(u)\|_W = \|j(\pi(u))\|_W = \|\pi(u)\|_{\widehat{E}_*} = |u|_* \qquad \forall\, u \in E_* \,,$$

so $\ker(q) = \mathcal{K}$, which means that q is not injective. In addition, since π is surjective and j has dense range, it follows that q has dense range. Let us introduce the adjoint of q as the operator $q' : W' \to E'_*$ defined by

$$q'(f) := f \circ q \qquad \forall\, f \in W' , \tag{A.16}$$

which is a linear and bounded operator. It is straightforward to prove that q' is an isometry. Since W' and E'_* are normed vector spaces, this fact implies that q' is injective. We proceed to prove that it is also surjective. Let $g \in E'_*$ and define the map $\widehat{g} : \widehat{E}_* \to \mathrm{R}$ by

$$\widehat{g}(\widehat{u}) := g(u) \qquad \forall\, \widehat{u} \in \widehat{E}_* \quad \text{with} \quad u \in \pi^{-1}(\widehat{u}) \, .$$

Using that g is continuous in E_*, it can be proven that $\widehat{g}$ is well defined, meaning that it does not depend on the representative of the class. Moreover, $\widehat{g} \in \widehat{E}'_*$. Then, we may define $f : R(j) \to \mathrm{R}$ by

$$f(w) := \widehat{g}(j^{-1}(w)) \, ,$$

which is properly defined, since j is injective. One can verify that f is a bounded and linear functional on $R(j)$. By Hahn–Banach theorem, it follows that there is a continuous linear extension $F \in W'$. We now notice that, for each $u \in E_*$,

$$q'(F)(u) = F(q(u)) = F(j(\pi(u))) = f(j(\pi(u))) = \widehat{g}(\pi(u)) = g(u) \, ,$$

which means that $q'(F) = g$. Therefore, q' is surjective.

We have shown that q' is an isometric isomorphism between W' and E'_*. Since W is a Hilbert space, the Riesz representation theorem implies that W and W' are isometrically isomorphic. We thus obtain the following result.

Theorem A.8. *Let E be a real vector space and let $\mathcal{N} : E \to E^\star$ be a linear, symmetric and monotone operator, whose kernel is denoted by $\mathcal{K}$. Let $(\cdot,\cdot)_*$ be the semi-inner product induced by $\mathcal{N}$, and denote by $|\cdot|_*$ the corresponding seminorm. Set $E_* := (E, |\cdot|_*)$ and define the quotient space $\widehat{E}_* := E/\mathcal{K}$ endowed with the inner product (A.15). Furthermore, let W denote the completion of $\widehat{E}_*$ and let T be the isometric isomorphism from $\widehat{E}'_*$ onto W. Additionally,* *let* $\mathcal{R} : W \to W'$ *be the Riesz mapping and* $q' : W' \to E'_*$ *be defined as in* (A.16)*. Then, we have the following isometric isomorphisms:*

$$\widehat{E}'_* \xrightarrow{T} W \xrightarrow{\mathcal{R}} W' \xrightarrow{q'} E'_* \,.$$

A.4 Finale: Solvability of an evolution problem

In what follows, we adopt the same notations and definitions as in Theorem A.8 and, additionally, we consider a monotone operator $\mathcal{M} : \mathcal{D} \to E'_*$, where $\mathcal{D} \subset E$ is the domain of $\mathcal{M}$. In this section, our aim is to study the solvability of the problem

$$\frac{d}{dt}\Big(\mathcal{N}(u(t))\Big) + \mathcal{M}\Big(u(t)\Big) \;=\; f(t) \quad \text{for a.e.} \quad 0 < t < T\,,$$

where $f : [0, T] \to E'_*$ and $u(0) = u_0$ are given. To this end, we employ the tools developed in the previous section to reformulate this problem within the framework of Kato's theorem (cf. Theorem A.6).

Using the definitions of $(\cdot, \cdot)_*$ and $(\cdot, \cdot)_{\widehat{E}_*}$, together with the fact that j is an isometry and the definition of the mapping q (cf. (A.16)), we find that, for all $x, y \in E$,

$$[\mathcal{N}(x), y] = (x, y)_* = (\pi(x), \pi(y))_{\widehat{E}_*} = (j(\pi(x)), j(\pi(y)))_W = (q(x), q(y))_W \,. \tag{A.17}$$

Moreover, employing now the Riesz map $\mathcal{R} : W \to W'$, which satisfies

$$[\mathcal{R}(v), w] = (v, w)_W \qquad \forall\, v, w \in W\,,$$

and the definition of q' (cf. (A.16)), the identity (A.17) is rewritten as

$$[\mathcal{N}(x), y] = [\mathcal{R}(q(x)), q(y)] = \Big[(q' \circ \mathcal{R} \circ q)(x), y\Big] \qquad \forall\, x, y \in E\,,$$

which proves the following factorization of $\mathcal{N}$,

$$\mathcal{N} = q' \circ \mathcal{R} \circ q\,. \tag{A.18}$$

On the other hand, to obtain a similar factorization for $\mathcal{M}$, let us introduce the relation $\mathcal{M}_0 \subset W \times W'$ defined by

$$g \in \mathcal{M}_0(\widehat{x}) \quad \text{if and only if} \quad \exists\, x \in \mathcal{D} \;:\; q(x) = \widehat{x} \quad \text{and} \quad q'(g) = \mathcal{M}(x)\,. \tag{A.19}$$

In general, $\mathcal{M}_0$ is not a single-valued operator, since q is not injective. Observe also that $\mathcal{D}(\mathcal{M}_0) = \Big\{ q(x) \in W \;:\; x \in \mathcal{D} \Big\}$. Moreover,

$$\mathcal{M} = q' \circ \mathcal{M}_0 \circ q\,. \tag{A.20}$$

The following lemma presents two equivalences that relate all these operators. The proof follows directly from (A.18), (A.19), and (A.20), and is therefore omitted.

Lemma A.9. *There hold:*

- *(i) For each $\widehat{x} = q(x) \in W$, $q'(g) = \mathcal{M}(x)$ if and only if $g \in \mathcal{M}_0(\widehat{x})$, and, in this case, we have $[q'(g), x] = g(\widehat{x})$.*
- *(ii) The equality $q'(g) = (\mathcal{N} + \mathcal{M})(x)$ holds if and only if $g \in (\mathcal{R} + \mathcal{M}_0)(q(x))$.*

Using this lemma, it is straightforwardly obtained the following result.

Corollary A.10. *There hold:*

- *(i) $\mathcal{M} : \mathcal{D} \to E'_*$ is monotone if and only if $\mathcal{M}_0 : \mathcal{D}(\mathcal{M}_0) \to W'$ is monotone.*
- *(ii) q' is a bijection of $R(\mathcal{R} + \mathcal{M}_0)$ onto $R(\mathcal{N} + \mathcal{M})$.*

We are finally ready to establish Theorem 3.1. For the reader's convenience, we recall the statement of the theorem.

Theorem A.11. *Let the linear, symmetric and monotone operator* $\mathcal{N}$ *be given from the real vector space* E *to its algebraic dual* $E^\star$*, and let* E'_* *be the Hilbert space which is the dual of the seminormed space* $(E, |\cdot|_*)$*, where*

$$|x|_* = [\mathcal{N}(x), x]^{1/2} \qquad \forall\, x \in E\,.$$

Let $\mathcal{M} : \mathcal{D} \to E'_*$ *be a monotone operator, with* $\mathcal{D} \subset E$*, and suppose that* $R(\mathcal{N} + \mathcal{M}) = E'_*$*. Then, for each* $f \in \mathrm{W}^{1,1}(0,T;E'_*)$ *and for each* $u_0 \in \mathcal{D}$*, there is a solution* $u : [0,T] \to E$ *of*

$$\frac{d}{dt}\Big(\mathcal{N}(u(t))\Big) + \mathcal{M}\Big(u(t)\Big) \;=\; f(t) \quad \textit{for a.e.} \quad 0 < t < T\,, \tag{A.21}$$

with

$$\mathcal{N}(u) \in \mathrm{W}^{1,\infty}(0,T;E'_*)\,, \quad u(t) \in \mathcal{D} \quad \textit{for all } 0 \le t \le T\,, \quad \textit{and} \quad \mathcal{N}(u(0)) = \mathcal{N}(u_0)\,.$$

Proof. We begin by showing that the existence of a solution to (A.21) is equivalent to the existence of a solution to a related problem formulated within the framework of (A.5). Suppose that $u : [0,T] \to E$ is a solution to (A.21). Then, since $q' : W' \to E'_*$ is an isomorphism, by combining (A.18) and (A.20), we find that $\widehat{u} := \mathcal{R} \circ q \circ u : [0,T] \to W'$ satisfies

$$\frac{d\widehat{u}}{dt}(t) + (\mathcal{M}_0 \circ \mathcal{R}^{-1})(\widehat{u}(t)) \ni \widehat{f}(t) \quad \text{for a.e.} \quad 0 < t < T\,, \tag{A.22}$$

where $\widehat{f} := (q')^{-1} \circ f$. Observe that we have lost the equality, which has been replaced by an inclusion, since $\mathcal{M}_0$ is not a single-valued operator. We now suppose that $\widehat{u} : [0,T] \to W'$ solves (A.22). Then, $\mathcal{R}^{-1} \circ \widehat{u}(t) \in \mathcal{D}(\mathcal{M}_0)$ for a.e. $t \in (0,T)$, which implies that, for almost every t, there exists $u(t) \in \mathcal{D}$ such that $\mathcal{R}^{-1} \circ \widehat{u}(t) = q(u(t))$. Hence, using once again the isomorphism q' together with (A.18) and (A.20), we deduce that u satisfies the original equation (A.21).

Now, in order to use Theorem A.6, we must prove that $\mathcal{M}_0 \circ \mathcal{R}^{-1} \subset W' \times W'$ is an m-accretive operator. Since $\mathcal{M}$ is monotone, by Corollary A.10 it follows that $\mathcal{M}_0$ is monotone.

Then, for all $(f_i, g_i) \in W' \times W'$ such that $(\mathcal{M}_0 \circ \mathcal{R}^{-1})(f_i) = g_i$, $i \in \{1, 2\}$, there holds

$$\begin{aligned}(g_1 - g_2, f_1 - f_2)_{W'} = (\mathcal{R}^{-1}(g_1 - g_2), \mathcal{R}^{-1}(f_1 - f_2))_W = [g_1 - g_2, \mathcal{R}^{-1}(f_1 - f_2)]\\ = [\mathcal{M}_0(\mathcal{R}^{-1}(f_1)) - \mathcal{M}_0(\mathcal{R}^{-1}(f_2)), \mathcal{R}^{-1}(f_1) - \mathcal{R}^{-1}(f_2)] \geq 0\,,\end{aligned}$$

where we used the monotonicity of $\mathcal{M}_0$. This proves that $\mathcal{M}_0 \circ \mathcal{R}^{-1}$ is accretive. It remains to prove the range condition $R(I + \mathcal{M}_0 \circ \mathcal{R}^{-1}) = W'$. From the hypothesis on the range and Corollary A.10, we have that q' is a bijection of $R(\mathcal{R} + \mathcal{M}_0)$ onto E'_*. This means that $R(\mathcal{R} + \mathcal{M}_0) = W'$. Thus, for all $g \in W'$, there exists $w \in W$ such that

$$(\mathcal{R} + \mathcal{M}_0)(w) = g\,.$$

Since $\mathcal{R}$ is an isomorphism, the above equation is equivalent to say that for all $g \in W'$, there exists $f \in W'$ such that

$$(I + \mathcal{M}_0 \circ \mathcal{R}^{-1})(f) = g\,,$$

where $f = \mathcal{R}(w)$. Thus, $R(I + \mathcal{M}_0 \circ \mathcal{R}^{-1}) = W'$. Therefore, $\mathcal{M}_0 \circ \mathcal{R}^{-1}$ is m-accretive.

Finally, by applying Theorem A.6 in the context of (A.22), we conclude the desired result.

□

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