Universidad de Concepción
Dirección de Postgrado
Facultad de Ciencias Físicas y Matemáticas
Programa de Doctorado en Ciencias Aplicadas
con Mención en Ingeniería Matemática

PROBLEMAS NO CONVEXOS EN CONTROL ÓPTIMO Y OPTIMIZACIÓN

NON-CONVEX PROBLEMS IN OPTIMAL CONTROL AND OPTIMIZATION

Tesis presentada a la Facultad de Ciencias Físicas y Matemáticas de la Universidad de Concepción para optar al grado académico de Doctor en Ciencias Aplicadas con Mención en Ingeniería Matemática

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Concepción, Chile
2026

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Concepción, Marzo de 2026

Dedicado a la memoria de mi abuelo Osvaldo Thiele Buck.

Agradecimientos

Agradezco a todos quienes me brindaron su apoyo durante todo este tiempo. Agradezco, en particular, a mi director de tesis, el profesor Fabián Flores-Bazán, por su buena disposición y paciencia para compartir su conocimiento y guiarme durante nuestra investigación. Al profesor Dihn Hoang Nguyen, quien me introdujo a su área de experticia y siempre mantuvo una actitud positiva. De la misma manera, a Stephanie Caro, con quien tuve el agrado de trabajar durante varios meses, durante los cuales fue una fuente de conocimiento, tanto en lo académico, como fuera de lo académico. Agradezco también a todo el personal del Centro de Investigación en Ingeniería Matemática (CI^{2}MA), tanto profesores como funcionarios que me brindaron un lugar de trabajo con excelente ambiente laboral, que ha hecho de esta, una experiencia inolvidable. Agradezco a mi familia por su apoyo incondicional, que me ha guiado desde el comienzo. Gracias por todo. Asimismo, agradezco a mis amigos por los buenos tiempos que hemos pasado durante todo este proceso.

Agradezco también a la Agencia Nacional de Investigación y Desarrollo (ANID) por financiar mis estudios a través de ANID BECAS/DOCTORADO NACIONAL 21221518.

Parte del material de esta tesis corresponde al de los proyectos de investigación financiados parcialmente por ANID-Chile a través de FONDECYT 1252131 y BASAL FB210005.

Resumen

Esta tesis se centra en el estudio de problemas no convexos a través de dos enfoques diferentes. En primer lugar, centrándose en las condiciones de optimalidad para problemas con conjuntos de restricciones geométricas no convexas particulares, sin suponer la convexidad de las funciones; y, en segundo lugar, considerando una familia especial de funciones cuasiconvexas.

Desarrollamos descripciones algebraicas del cono normal límite para distintos tipos de conjuntos, incluida la unión de dos poliedros y una superficie cuadrática única o la unión de dos superficies cuadráticas. A continuación, estas descripciones se utilizan para estudiar las condiciones de optimalidad M-estacionarias, lo que da lugar a resultados sobre la unicidad local y las propiedades de estabilidad del conjunto de soluciones. Nuestros hallazgos sobre las condiciones de optimalidad son nuevos y pueden considerarse extensiones de resultados conocidos que emplean condiciones KKT en problemas sin restricciones geométricas. Además, presentamos un ejemplo de un problema de optimización, al que se aplican nuestros resultados, en el que las condiciones M-estacionarias identifican correctamente al minimizador, mientras que otros enfoques fallan

Posteriormente, ampliamos ciertos resultados conocidos del análisis convexo a las funciones cuasiconvexas, demostrando que, al igual que con las funciones convexas, algunos comportamientos globales pueden entenderse examinando el comportamiento en puntos individuales. Por último, nos centramos en una clase particular de funciones cuasiconvexas para demostrar la existencia de soluciones y la semicontinuidad inferior de la función de valor, complementando o generalizando así los resultados conocidos en el caso convexo.

Abstract

This thesis is concerned with the study of non-convex problems through two different approaches: first, by focusing on optimality conditions for problems with particular non-convex geometric constraint sets, without assuming convexity of the functions; and secondly, by considering a special family of quasiconvex functions.

We develop algebraic descriptions of the limiting normal cone for distinct types of sets, including the union of two polyhedra and either a single quadric surface or the union of two quadric surfaces. These descriptions are then used to study M-stationary optimality conditions, yielding results on local uniqueness as well as stability properties of the solution set. Our findings regarding optimality conditions are new and can be viewed as extensions of known results that employ KKT conditions in problems without geometric constraints. In addition, we present an example of an optimization problem where M-stationary conditions correctly identify the minimizer, to which our results apply, whereas other approaches fail.

Subsequently, we extend certain known results from convex analysis to quasiconvex functions, showing that, as with convex functions, some global behaviors can be understood by examining the behavior at individual points. Finally, we focus on a particular class of quasiconvex functions to prove existence of solutions and lower semicontinuity of the value function, thereby complementing or generalizing known results in the convex case.

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Chapter 1

Introducción

1.1 Versión en español

La noción de convexidad juega un papel crítico en la optimización, ya sea en el establecimiento de la existencia de soluciones, la derivación de condiciones de optimalidad;o incluso, en la convergencia de métodos numéricos utilizados en la discretización. No obstante, no todos los problemas involucran funciones convexas o restricciones convexas, por lo que a menudo es necesario trabajar fuera de este marco y desarrollar técnicas que no dependan de la convexidad para abordar ciertos problemas que aparecen en aplicaciones.

El objetivo principal de esta tesis será tratar problemas de la forma

$$\min\{f(x) :\ g(x) \leq 0,\ h(x) = 0,\ x \in C\}, \tag{1.1.1}$$

sin asumir convexidad. Existen resultados que abordan este problema sin requerir hipótesis adicionales; una condición de optimalidad bastante conocida establece que el gradiente de f en un minimizador pertenece al polar del cono contingente del conjunto factible en dicho minimizador (véase, por ejemplo, [56, Theorem 6.12]). Pese a que este resultado es válido sin ninguna hipotesis extra, salvo diferenciabilidad de la función objetivo, en la práctica suele ser inutilizable debido a la dificultad inherente de calcular el polar del cono tangente del conjunto factible.

Por esta razón, si asumimos que (1.1.1) no es (del todo) convexo, usualmente nos restringimos a una familia de problemas con una estructura prescrita. Por ejemplo, las funciones f, g, h pueden ser lineales o cuadráticas [50, 23]; alternativamente, el conjunto de restricciones geométricas C o el conjunto factible completo puede poseer una estructura o caracterización geométrica específica [9, 21]; o incluso, una o más de las funciones involucradas pueden ser explícitamente conocidas [59].

Existen dos formas principales de abordar (1.1.1). La primera consiste en construir resultados desde cero sin depender de la convexidad. Este enfoque es amplio y puede producir resultados muy generales o resultados adaptados a programas con tipos específicos de restricciones. En particular, nos centraremos en condiciones de optimalidad, donde la principal dificultad radica en manejar la restricción geométrica dada por el conjunto (no convexo) C. Las técnicas para el caso convexo no pueden generalizarse de manera directa, ya que dependen de nociones de direcciones normales que no se extienden naturalmente a conjuntos no convexos. Interpretaremos las direcciones normales a conjuntos no convexos en el sentido del cono normal limitante (o de Mordukhovich), introducido en [42], lo que conduce a la condición de optimalidad asociada conocida como condición M-estacionaria. Nuestro objetivo principal con este enfoque será estudiar dicha condición e imponer hipótesis apropiadas para establecer unicidad local y ciertos resultados de estabilidad.

El segundo enfoque se basa en considerar una clase más amplia de funciones. En particular, estudiaremos funciones cuasiconvexas, que generalizan las funciones convexas manteniendo algunas de sus propiedades útiles. Esto nos permite extender resultados conocidos en el caso convexo.

Aquí nos centraremos principalmente en problemas de programación semi-infinita [51], lo que significa que consideramos una cantidad finita de variables y un número infinito de restricciones. Nuestras principales herramientas para este análisis provienen del análisis asintótico [2]. En particular, estudiaremos funciones asintóticas como un medio para describir el comportamiento tanto de las funciones objetivo como del conjunto factible [19, 20, 29, 30].

La estructura de esta tesis es la siguiente. En el Capítulo 2, introducimos las definiciones y notaciones que utilizaremos en los capítulos posteriores.

El Capítulo 3 está estructurado de la siguiente manera. En la Sección 3.2, se desarrolla un estudio profundo de conos normales limitantes a la unión de conjuntos convexos cerrados y, en particular, cuando los conjuntos son conos convexos cerrados. El caso de poliedros convexos se aborda en la Sección 3.3, donde se establece una fórmula de representación para el cono normal limitante a la unión de dos poliedros convexos y algunas de sus consecuencias: Teorema 3.1, Corolario 3.2. La Sección 3.4 proporciona unicidad local para la M-estacionariedad, lo que sirve para establecer cotas de error en un problema de optimización no lineal bajo un subespacio afín y un conjunto geométrico no convexo, como se presenta en la Sección 3.5. Los resultados contenidos en este capítulo fueron publicados en el artículo:

- [24] Flores-Bazás, F., Nguyen, D.H., Thiele, F. Limiting normal cone to quadric surfaces, local uniqueness and error bound in M-stationarity. *SIAM, J. Optim.*, 35(4):2323–2342, 2025.

El Capítulo 4 está estructurado de la siguiente manera. En la Sección 4.2, establecemos la fórmula para el cono normal limitante a una superficie cuadrática y a la unión de dos superficies cuadráticas. La unicidad local de puntos M-estacionarios para el problema (4.1.4) bajo una sola cuadrática se obtiene en la Sección 4.3. En la Sección 4.4, se establece una cota de error local en puntos M-estacionarios para una clase de problemas de optimización no lineal. Esto se basa en tres condiciones: unicidad local de un punto M-estacionario, la propiedad de contención externa local (introducida en la Definición 4.1) para el cono normal limitante al conjunto de restricciones geométricas en el mismo punto y la Lipschitzianidad superior para la correspondencia multivaluada de puntos M-estacionarios. El contenido de este capítulo se reporta en el siguiente preprint:

- [25] Flores-Bazás, F., Nguyen, D.H., Thiele, F. Limiting normal cone to quadric surfaces, local uniqueness and error bound in M-stationarity. *Preprint 2026-01, Departamento de Ingeniería Matemática, Universidad de Concepción*, Chile, 2026.

El Capítulo 5 está estructurado de la siguiente manera. La Sección 5.2 establece resultados de existencia bajo restricciones geométricas poliedrales para el programa cuasiconvexo semi-infinito. La Sección 5.3 extiende estos resultados para considerar otros tipos de restricciones geométricas. Finalmente, la Sección 5.4 está dedicada al estudio de la semicontinuidad inferior de la función valor, diferenciando entre el caso semi-infinito y el caso finito. Los resultados contenidos en este capítulo fueron publicados en el artículo:

- [8] Caro, S., Thiele, F. Asymptotic analysis for a class of quasiconvex semi-infinite programming problems. *J. Optim. Theory Appl.* 207(58), 2025.

El problema fue sugerido por mi profesor guía, quien lo planteó a la Dra. Stephanie Caro y a mí. Durante las etapas iniciales del estudio, sostuvimos reuniones con él para analizar y definir los lineamientos del trabajo.

1.2 English version

The notion of convexity plays a critical role in optimization, whether in establishing the existence of solutions, deriving optimality conditions or even convergence of numerical methods used in discretization. Nonetheless, not all problems involve convex functions or convex constraints, so it is often necessary to work outside this framework and develop techniques that do not rely on convexity to deal with some problems found in applications.

The main focus of this thesis will be problems that take the form

$$\min\{f(x): \ g(x) \leq 0, \ h(x) = 0, \ x \in C\}, \tag{1.2.1}$$

without assuming convexity. There are results addressing this problem without requiring additional assumptions, one well-known optimality condition states that the gradient of f at a minimizer belongs to the polar of the contingent cone of the feasible set at the minimizer (see [56, Theorem 6.12], for example). While this result always holds true without any extra hypotheses, other than differentiability of the cost function, it is usually not usable in practice, due to the inherent difficulty of computing the polar to the tangent cone of the feasible set.

Because of this, if we are to assume that (1.2.1) is not (fully) convex, usually we limit ourselves to a family of problems with a prescribed structure. For instance, the functions f, g, h may be linear or quadratic [50, 23]; alternatively, the geometric constraint set C or the entire feasible set may possess a specific structure or geometric characterization [9, 21]; or even, one or more of the functions involved may be explicitly known [59].

There are two main ways we approach (1.2.1). The first consists in building results from the ground up without relying on convexity. This approach is broad and can yield either very general results or results tailored to programs with specific types of constraints. In particular, we will focus on optimality conditions, where the main difficulty lies in handling the geometric constraint given by the (non-convex) set C. Techniques from the convex case cannot be generalized straightforwardly, since they rely on notions of normal directions that do not extend naturally to non-convex sets. We will interpret normal directions to non-convex sets in the sense of the limiting (or Mordukhovich) normal cone, introduced in [42], which leads to the associated optimality condition known as the M-stationary condition. Our main goal with this approach will be to study this condition and impose appropriate hypotheses to establish local uniqueness and certain stability results.

The second approach relies on considering a broader class of functions. In particular, we will study quasiconvex functions, which generalize convex functions while retaining some of their useful properties. This allows us to extend known results for the convex case.

Here we mainly focus on a semi-infinite programming problems [51], meaning that we consider a finite amount of variables and an infinite number of constraints. Our main tools for this come from asymptotic analysis [2]. In particular, we will study asymptotic functions as a means to describe the behavior of both the objective functions and the feasible set [19, 20, 29, 30].

The structure of this thesis is as follows. In Chapter 2, we introduce the definitions and notations that we will use in the later chapters.

Chapter 3 is structured as follows. In Section 3.2, a deep study of limiting normal cones to the union of closed convex sets is developed and, particularly, when the sets are closed convex cones. The case of convex polyhedra is dealt with in Section 3.3, where a representation formula for the limiting normal cone to the union of two convex polyhedra and some of its consequences are established: Theorem 3.1, Corollary 3.2. Section 3.4 provides local uniqueness for M-stationarity, which serves to establish error estimates for a nonlinear optimization problem under an affine subspace and a nonconvex geometric constraint set, as presented in Section 3.5. The results contained in this chapter were published in the article:

- [24] Flores-Bazás, F., Nguyen, D.H., Thiele, F. Limiting normal cone to quadric surfaces, local uniqueness and error bound in M-stationarity. *SIAM, J. Optim.*, 35(4):2323–2342, 2025.

Chapter 4 is structured as follows. In Section 4.2, we establish the formula for the limiting normal cone to single quadric surface and to the union of two quadric surfaces. Local uniqueness of M-stationary points for problem (4.1.4) under a single quadric is obtained in Section 4.3. In Section 4.4, a local error-bound at M-stationary points for a class of nonlinear optimization problems is established. This is based on three conditions: local uniqueness of an M-stationary point, local outer contained property (as introduced in Definition 4.1) for the limiting normal cone to the geometric constraints set at the same point, and upper Lipschitzianity for the M-stationary points set-valued mapping. The contents of this chapter are reported in the following preprint:

- [25] Flores-Bazás, F., Nguyen, D.H., Thiele, F. Limiting normal cone to quadric surfaces, local uniqueness and error bound in M-stationarity. *Preprint 2026-01, Departamento de Ingeniería Matemática, Universidad de Concepción*, Chile, 2026.

Chapter 5 is structured as follows. Section 5.2 establishes existence results under polyhedral geometric constraints for the semi-infinite quasiconvex program. Section 5.3 extends these results to consider other types of geometric constraints. Finally, Section 5.4 is dedicated to studying the lower semicontinuity of the value function, differentiating between the semi-infinite and finite case. The results contained in this chapter were published in the article:

- [8] Caro, S., Thiele, F. Asymptotic analysis for a class of quasiconvex semi-infinite programming problems. *J. Optim. Theory Appl.* 207(58), 2025.

The problem was suggested by my advisor, who presented it to Dr. Stephanie Caro and me. During the initial stages of the study, we held meetings with him to analyze and define the guidelines for the work.

Chapter 2

Notations and Preliminaries

In this chapter, we present the basic notations and the terminology that will be used throughout this thesis.

2.1 Tangent and Normal Cones

Throughout this thesis, given a set $A \subseteq \mathbb{R}^n$, its closure is denoted by $\mathrm{cl}(A)$ or $\overline{A}$; its convex hull by $\mathrm{co}(A)$ which is the smallest convex set containing A; its topological interior by int A; ri A or ri (A) is the relative interior of A, $A^\perp$ the orthogonal complement of A and ι_A the indicator function of A. Moreover, for $B \subseteq \mathbb{R}^n$, $A + B$ is the Minkowski sum and given $\Lambda \subseteq \mathbb{R}$, ΛA denotes the product of A and $\Lambda \subseteq \mathbb{R}$, with the convention $\Lambda\emptyset = \emptyset A = \emptyset$. cone$(A)$ stands for the smallest cone containing A, that is,

$$\mathrm{cone}(A) = \bigcup_{t \geq 0} tA,$$

whereas $\overline{\mathrm{cone}}(A)$ denotes the smallest closed cone containing A. Obviously $\overline{\mathrm{cone}}(A) = \overline{\mathrm{cone}(A)}$. By $\langle \cdot, \cdot \rangle$ we denote the inner or scalar product in $\mathbb{R}^n$, whose elements are considered column vectors. Thus, $\langle a, b \rangle = a^\top b$ for all $a, b \in \mathbb{R}^n$.

Given a nonempty set $P \subseteq \mathbb{R}^n$, it is a cone if $tP \subseteq P$, for all $t \geq 0$. The polar cone of P, P^*, is defined as

$$P^* = \{\xi \in \mathbb{R}^n : \ \langle \xi, p\rangle \geq 0 \ \ \forall\, p \in P\}.$$

It is well known that, whenever P is a closed convex cone, we have (the bipolar theorem) $P = P^{**} \doteq (P^*)^*$, and in general we have $P^{**} = \overline{\text{co}}(\text{cone } P)$, where $\overline{\text{co}}\ A$ is the smallest closed and convex set containing A. We equip $\mathbb{R}^n$ with the Euclidean norm $\|x\| \doteq \sqrt{\langle x, x\rangle}$, so that the Projection operator of x onto a closed set C is denoted by $\text{Proj}(C; x)$ and given in terms of the Euclidean norm. For $a \in \mathbb{R}^n$, $a^\perp$ stands for the ($n-1$ dimensional) hyperplane orthogonal to a. Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a set-valued mapping; its graph and domain are the sets, respectively,

$$\text{gph } F \doteq \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m : \ z \in F(x)\}, \ \text{Dom } F \doteq \{x \in \mathbb{R}^n : \ F(x) \neq \emptyset\}.$$

The (sequential) upper limit, in the Painlevé-Kuratowsky sense, at a point $\bar{x} \in \mathbb{R}^n$, is defined as

$$\operatorname*{Limsup}_{x \to \bar{x}} F(x) \doteq \{y \in \mathbb{R}^m : \ \exists (x_k, y_k) \in \text{gph } F \ \forall\, k \in \mathbb{N}, x_k \to \bar{x}, \ y_k \to y\}.$$

Given a set $\Omega \subseteq \mathbb{R}^n$ we denote

$$\operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in \Omega}} F(x) \doteq \{y \in \mathbb{R}^m : \ \exists (x_k, y_k) \in \text{gph } F, \ x_k \in \Omega, x_k \to \bar{x}, \ y_k \to y\}.$$

Finally, $B(x; \varepsilon)$ stands for the open ball centered at $x \in \mathbb{R}^n$ and radius $\varepsilon > 0$.

It is straightforward to prove the following result.

Proposition 2.1. *Let $D_1, D_2 \subseteq \mathbb{R}^n$, $\bar{x} \in D_1 \cap D_2$ and a set-valued mapping $F : D_1 \cup D_2 \to 2^{\mathbb{R}^n}$. We have*

$$\operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in D_1 \cup D_2}} F(x) = \operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in D_1 \setminus D_2}} F(x) \cup \operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in D_2 \setminus D_1}} F(x) \cup \operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in D_1 \cap D_2}} F(x);$$

$$\operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in D_1 \cup D_2}} F(x) = \operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in D_1}} F(x) \cup \operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in D_2}} F(x).$$

In what follows, we recall the notions of contingent and limiting normal cones, well known in variational analysis. Some of their main properties may be found in the book [56].

Definition 2.1. *Let* $\emptyset \neq C \subseteq \mathbb{R}^n$ *and* $\bar{x} \in \overline{C}$*:*

(i) *the (Bouligand-Severi) contingent cone of* C *at* $\bar{x}$*, denoted by* $T(C;\bar{x})$*, is the closed cone*

$$T(C;\bar{x}) \doteq \{v \in \mathbb{R}^n : \ \exists\, t_k > 0, \exists\, x_k \in C, \ x_k \to \bar{x}, \ t_k(x_k - \bar{x}) \to v\}.$$

(ii) *If* $\bar{x} \in C$*, the limiting (also named Mordukhovich or basic) normal cone of* C *at* $\bar{x} \in C$*, is defined as*

$$N_M(C;\bar{x}) \doteq \operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in C}} \hat{N}(C;x),$$

where Limsup *is the upper limit mentioned before and* $\hat{N}(C;x)$ *is called the prenormal cone, or Fréchet (or regular) normal cone, of* C *at* x*, defined by*

$$\hat{N}(C;x) \doteq \Big\{ v \in \mathbb{R}^n : \limsup_{\substack{u \to x \\ u \in C}} \frac{\langle v, u - x\rangle}{\|u - x\|} \leq 0 \Big\}.$$

It should be noticed that, in case C is locally closed around $\bar{x} \in C$ (meaning that there is a neighborhood U of $\bar{x}$ for which $C \cap U$ is closed), then [44, Theorem 1.6] gives us a simpler representation:

$$\begin{aligned} N_M(C;\bar{x}) &= \operatorname*{Limsup}_{x \to \bar{x}} \left[\operatorname{cone}(x - \operatorname{Proj}(C;x))\right] \\ &= \{v \in \mathbb{R}^n : \ \exists\, x_k \to \bar{x}, \ t_k \geq 0, \ y_k \in \operatorname{Proj}(C;x_k), \ t_k(x_k - y_k) \to v\}. \end{aligned} \tag{2.1.1}$$

Remark 2.1. *The sequences appearing in* (2.1.1) *have the following properties:*

- $x_k - y_k \to 0$*: this follows since* $y_k \in \operatorname{Proj}(C;x_k)$ *and* $\bar{x} \in C$*, one obtains*

$\|x_k - y_k\| \le \|x_k - \bar{x}\|$.

- $y_k \to \bar{x}$: *it is a consequence of* $\|y_k - \bar{x}\| \le \|y_k - x_k\| + \|x_k - \bar{x}\|$.

Whenever we are dealing with a convex set C, we will use $N(C;\bar{x})$ to refer the clasical (outward) normal cone at $\bar{x} \in C$:

$$N(C;\bar{x}) \doteq \{\xi \in \mathbb{R}^n : \ \langle \xi, x - \bar{x} \rangle \le 0, \ \ \forall\ x \in C\};$$

$N(C;\bar{x}) = \emptyset$ if $\bar{x} \notin C$. It is not hard to prove that

$$T(C;\bar{x}) = \overline{\bigcup_{t \ge 0} t(C - \bar{x})}.$$

Thus, $(T(C;\bar{x}))^* = -N(C;\bar{x}) = N_M(C;\bar{x})$ provided $\bar{x} \in C$ and C is convex.

Let us go back to consider any set C locally closed around $\bar{x} \in C$. In what follows, given $\bar{x} \in C$, $N_C(C;\bar{x})$ stands for the Clarke normal cone defined as $N_C(C;\bar{x}) \doteq -[T_C(C;\bar{x})]^*$, where $T_C(C;\bar{x})$ is the Clarke tangent cone [10]. From [44, Theorem 3.57], it follows that $N_C(C;\bar{x}) = \overline{\text{co}}\ N_M(C;\bar{x})$.

In general, we have (see Corollary 1.11 in [44])

$$-(T(C;\bar{x}))^* = \hat{N}(C;\bar{x}) \subseteq N_M(C;\bar{x}) \subseteq N_C(C;\bar{x}). \tag{2.1.2}$$

Thus, by using (2.1.1), one infers

$$\text{gph } N_M(C;\cdot) = \overline{\text{gph}[-(T(C;\cdot))^*]} = \overline{\text{gph } \hat{N}(C;\cdot)}, \tag{2.1.3}$$

which means that $N_M(C;\cdot)$ has closed graph, provided C is closed. Actually, this closedness also follows from the robustness property (see page 11 in [44]):

$$N_M(C;\bar{x}) \doteq \operatorname*{Limsup}_{x \to \bar{x}} N_M(C;x).$$

Furthermore, we denote $\mathbb{R}_+ \doteq [0,+\infty[$, $\mathbb{R}_{++} \doteq]0,+\infty[$ and $\mathbb{R}^n_{++} = \text{int } \mathbb{R}^n_+$.

We recall that

$$C^* = (\text{co } C)^* = (\text{cone } C)^* = (\overline{C})^*.$$

The following proposition collects some interesting results linking (limiting) normal cones and contingent ones.

Proposition 2.2. *Let C be any nonempty set in $\mathbb{R}^n$. The following assertions hold.*

(*a*) *Assume that $0 \in C$.*

(*a*1) $N(\text{cone}(\text{co } C); 0) = N(\text{co } C; 0) = N(\overline{\text{cone}}(\text{co } C); 0) = N(\overline{\text{co}} \ C; 0)$.

(*a*2) $T(0; \text{cone } C) = \overline{\text{cone}} \ C$.

(*a*3) $[T(\text{co } C; 0)]^* = -N(\text{co } C; 0) = C^* = (\overline{\text{cone}} \ C)^* = [T(\text{cone } C; 0)]^*$.

(*b*) *Let C_1, C_2 be nonempty sets in $\mathbb{R}^n$, $C_1 \cup C_2$ be locally closed around $\bar{x}$.*

(*b*1) *If $\bar{x} \in C_1 \cap C_2$ and C_1, C_2 are convex sets, then*

$$N(\overline{\text{co}}(C_1 \cup C_2); \bar{x}) = N(C_1; \bar{x}) \cap N(C_2; \bar{x}) \subseteq N_M(C_1 \cup C_2; \bar{x}).$$

(*b*2) *If $\bar{x} \in C_1$, $\text{dist}(\bar{x}, C_2) > 0$ with C_1 being locally closed around $\bar{x}$, then*

$$N_M(C_1 \cup C_2; \bar{x}) = N_M(C_1; \bar{x}).$$

(*c*) *Let $\bar{x} \in C$ and C being locally closed around $\bar{x}$. Then*

$$(C - \bar{x})^* = N(\bar{x}; \text{co } C) = N(\overline{\text{co}} \ C; \bar{x}) \subseteq N_M(C; \bar{x}).$$

Proof. (*a*): The proof of these results are standard.
(*b*1): We prove only the inclusion, since the equality follows from (*a*). Let $v \in N(\overline{\text{co}}(C_1 \cup C_2); \bar{x})$. Then for all $t > 0$, $\text{Proj}(\overline{\text{co}}(C_1 \cup C_2); \bar{x} + tv) = \bar{x}$. By assumptions on $\bar{x}$, $\text{Proj}(C_i; \bar{x} + tv) = \bar{x}$ for $i = 1, 2$. Thus $\bar{x} \in \text{Proj}(C_1 \cup C_2; \bar{x} + tv)$. Now, by setting $x_k \doteq \bar{x} + k^{-1}v$, $y_k \doteq \bar{x}$, $t_k \doteq k$, one obtains $y_k \in \text{Proj}(C_1 \cup C_2; x_k)$, $x_k \to \bar{x}$, $t_k(x_k - y_k) = v$, meaning $v \in N_M(C_1 \cup C_2; \bar{x})$.
(*b*2): We first prove "$\subseteq$". Let $x_k \to \bar{x}$, $y_k \in \text{Proj}(C_1 \cup C_2; x_k)$, $t_k > 0$ satisfying $t_k(x_k - y_k) \to v$. Since $y_k \notin C_2$, $y_k \in C_1$ for all k sufficiently large and therefore $y_k \in P(x_k; C_1)$. Then $v \in N_M(\bar{x}; C_1)$.
"$\supseteq$": Set $\varepsilon \doteq \text{dist}(\bar{x}, C_2) > 0$. Take any $v \in N_M(C_1; \bar{x})$. Then there exist $x_k \to \bar{x}$,

$y_k \in \mathrm{Proj}(C_1; x_k)$, $t_k > 0$, $t_k(x_k - y_k) \to v$. By assuming $\max\{\|x_k - \bar{x}\|, \|y_k - \bar{x}\|\} < \varepsilon/3$ for all $k \in \mathbb{N}$, we get by triangle inequality, $\mathrm{dist}(x_k; C_2) > 2\varepsilon/3$. This implies $\mathrm{dist}(x_k; C_1) \leq 2\varepsilon/3 < \mathrm{dist}(x_k; C_2)$. Hence $\mathrm{dist}(x_k; C_1) = \mathrm{dist}(x_k; C_1 \cup C_2)$ and so $y_k \in \mathrm{Proj}(C_1 \cup C_2; x_k)$, proving that $v \in N_M(C_1 \cup C_2; \bar{x})$.
(c): It is similar to $(b1)$. □

2.2 Quasi-convex Functions and Asymptotic Analysis

Let I be a possibly infinite index set, we consider the locally convex product space $\mathbb{R}^I$, whose topological dual space, $\mathbb{R}^{(I)}$, is constructed via the mappings $\alpha : I \to \mathbb{R}$ with $|\operatorname{supp} \alpha| < +\infty$, where $\operatorname{supp} \alpha := \{i \in I : \alpha_i = \alpha(i) \neq 0\}$ is called the support of α and $|\cdot|$ stands for the cardinality of the set. We denote by $\mathbb{R}^{(I)}_+$ the positive cone of $\mathbb{R}^{(I)}$, that is, $\mathbb{R}^{(I)}_+ = \{\alpha : I \to \mathbb{R}_+ : |\operatorname{supp} \alpha| < +\infty\}$.

The recession and asymptotic cone of A are respectively defined by

$$\begin{aligned} A^{\mathrm{rec}} &= \{v \in \mathbb{R}^n : \ x + tv \in A \text{ for all } x \in A, \ t \geq 0\}, \\ A^{\infty} &= \{v \in \mathbb{R}^n : \ \exists t_k \searrow 0, \ x_k \in A \text{ with } t_k x_k \to v\}. \end{aligned}$$

These sets coincide when A is closed and convex.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a function, we define the effective domain and the epigraph of f as $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x, \lambda) \in \mathbb{R}^{n+1} : f(x) \leq \lambda\}$, respectively. Moreover, given a real number λ, $[f \leq \lambda]$ and $[f < \lambda]$ denote the sublevel set and the strict sublevel set, respectively. Furthermore, the function f is said proper if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$, it will be convex when its epigraph is convex, coercive when $f(x) \to +\infty$ as $\|x\| \to +\infty$ and quasiconvex when the sets $[f \leq \lambda]$ are convex for all $\lambda \in \mathbb{R}$. We refer to a function f as nonincreasing whenever $t_1 < t_2$ implies $f(t_2) \leq f(t_1)$ and nondecreasing whenever $t_1 < t_2$ implies $f(t_2) \geq f(t_1)$.

Definition 2.2. *A proper, lower semicontinuous (lsc, in brief) function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *belongs to* $\mathcal{F}$ *if, for any* $\alpha > 0$*, any convergent sequence* $\{\varepsilon^k\} \subset \mathbb{R}$*, any* *sequence* $\{x^k\} \subset \mathbb{R}^n$*, and for any* $d \in \mathbb{R}$ *satisfying*

$$x^k \in [f \leq \varepsilon^k], \quad \|x^k\| \to \infty, \quad x^k/\|x^k\| \to d, \quad \liminf_{\substack{h\to d \\ t\to\infty}} \frac{f(th)}{t} = 0$$

there exists $\bar{k}$ *such that* $x^k - \alpha d \in [f \leq \varepsilon^k]$*, for all* $k \geq \bar{k}$*.*

Several "asymptotic functions" associated with a function f appear in the literature. The standard asymptotic function is the function whose epigraph is equal to the asymptotic cone of the epigraph of f and is denoted by f^∞. Another is the q-asymptotic function, introduced in [18], which is defined for $v \in \mathbb{R}^n$ by

$$f_q^\infty(v) := \sup_{x\in \mathrm{dom}\ f} \sup_{t>0} \frac{f(x+tv) - f(x)}{t}.$$

The last one we consider is related to the q-asymptotic functions, the zero-scale asymptotic function $f_{0,q}^\infty$, introduced in [19], is defined by

$$f_{0,q}^\infty(v) := \sup_{x\in \mathrm{dom}\ f} \sup_{t>0} \left(f(x+tv) - f(x)\right).$$

Definition 2.3. *It is said that a function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ *belongs to* $\mathcal{C}$ *if for all* $x \in \mathrm{dom}\ f$ *and* $v \in (\mathrm{dom}\ f)^\infty$*, the mapping* $t \mapsto f(x+tv)$*,* $t > 0$*, is nonincreasing, or* $\lim_{t\to+\infty} f(x+tv) = +\infty$*.*

In the previous definition, if the function f is proper, lsc and quasiconvex, we can replace the limit by $\sup_{t>0} f(x+tv) = +\infty$ (see the proof of [30, Proposition 3.2]).

It is known that both convex and coercive functions belong to the class $\mathcal{C}$. Examples of functions in the class $\mathcal{C}$ can be found in [17, 26].

Proposition 2.3. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper function, and* $v \in \mathbb{R}^n$*,* $v \neq 0$*. The following are equivalent:*

(*a*) $f_q^\infty(v) \leq 0$;

(*b*) $v \in (\mathrm{dom}\ f)^{\mathrm{rec}}$*, and for all* $x \in \mathrm{dom}\ f$*, the function* $t \mapsto f(x+tv)$*,* $t \geq 0$*, is nonincreasing.*

Proof. The proof follows immediately from the definition of f_q^∞. $\square$

We recall a very nice description of quasiconvex and lsc functions defined on the real line. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function, then the mapping $t \mapsto f(x + tv)$, $t \geq 0$ is quasiconvex.

Proposition 2.4. *Given $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, i \in I$ a family of quasiconvex lsc and proper functions in class $\mathcal{C}$, I an arbitrary index set and for all $i, j \in I$,* $\mathrm{dom}\ f_i = \mathrm{dom}\ f_j$*. Then, the function $f := \sup_{i \in I} f_i$ also belongs to $\mathcal{C}$.*

Proof. If for some $i \in I$ one has $\sup_{t \geq 0} f_i(x + tv) = +\infty$, then it is evident that $\sup_{t \geq 0} f(x + tv) = +\infty$. Otherwise, all mappings $t \mapsto f_i(x + tv), t \geq 0$ for $i \in I$ are nonincreasing and clearly $t \mapsto f(x + tv), t \geq 0$, is also nonincreasing. $\square$

Proposition 2.5. *Given $f_1, \cdots, f_m : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ a family of quasiconvex lsc and proper functions belonging to class $\mathcal{C}$ and $f := \max_{1 \leq i \leq m} f_i$. Then, for all $v \in (\mathrm{dom} f)^\infty$ the zero-scale asymptotic function of f satisfies*

$$f_{0,q}^\infty(v) = \max_{1 \leq i \leq m} (f_i)_{0,q}^\infty(v). \tag{2.2.1}$$

Proof. Take any $v \in (\mathrm{dom} f)^\infty$. The inequality "$\leq$" follows directly from the definition of $f_{0,q}^\infty$ and the observation that for every $x \in \mathrm{dom}\, f$ and any $t \geq 0$, we have

$$\max_{1 \leq i \leq m} f_i(x + tv) - \max_{1 \leq i \leq m} f_i(x) \leq \max_{1 \leq i \leq m} (f_i(x + tv) - f_i(x)).$$

Moreover, if for some $i \in \{1, \cdots, m\}$ we have $(f_i)_{0,q}^\infty(v) = +\infty$, then $f_{0,q}^\infty(v) = +\infty$ (see Proposition 2.4), and equality (2.2.1) is achieved. Otherwise, if $(f_i)_{0,q}^\infty(v) = 0$ for all $i \in \{1, \cdots, m\}$, then $f_{0,q}^\infty(v) = 0$ (see also Proposition 2.4), and again the equality (2.2.1) holds. $\square$

Similar as how we can describe the asymptotic function f^∞ for a convex function along a direction v by studying its behavior along any half line with direction v, we are interested in describing the behavior of a quasiconvex function along parallel half lines, when f is not necessarily lsc.

Proposition 2.6. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex proper function and $v \in (\text{dom } f)^\infty$. Assume that $\sup_{t\geq 0} f(\bar{x}+tv) = +\infty$. The following assertions hold:*

(*a*) *If $\bar{x} \in \text{ri dom } f$, then $\sup_{t\geq 0} f(x+tv) = +\infty$ for all $x \in \text{dom } f$.*

(*b*) *If f is lsc and $\bar{x} \in \text{dom } f$, then $\sup_{t\geq 0} f(x+tv) = +\infty$ for all $x \in \text{dom } f$.*

Proof. (a): Take any $x \in \text{dom } f$. Then, since $\bar{x} \in \text{ri dom } f$, there exists $\lambda_1 > 1$ such that $x_1 := x + \lambda_1(\bar{x} - x) \in \text{dom } f$. Thus $\bar{x} \in [x, x_1]$. From the choice of x_1, we have for all $t \geq 0$

$$\bar{x} + tv = \lambda_1^{-1} x_1 + \left(1 - \lambda_1^{-1}\right)\left(x + \lambda_1(\lambda_1 - 1)^{-1} tv\right)$$

and so $\bar{x} + tv \in [x_1, x + \lambda_1(\lambda_1-1)^{-1}tv]$ for all $t \geq 0$. From the quasiconvexity of f, it follows that $f(\bar{x}+tv) \leq \max\{f(x_1), f(x + \lambda_1(\lambda_1-1)^{-1}tv)\}$, and since $x_1 \in \text{dom } f$ and $\sup_{t\geq 0} f(\bar{x}+tv) = +\infty$, necessarily

$$\sup_{t\geq 0} f(x+tv) = \sup_{t\geq 0} f(x + \lambda_1(\lambda_1-1)^{-1}tv) = +\infty.$$

(b): Assume that $\sup_{t\geq 0} f(x+tv) < +\infty$ for some $x \in \text{dom } f$. Let $\lambda > \max\{f(\bar{x}), \sup_{t\geq 0} f(x+tv)\}$. The set $S := [f \leq \lambda]$ is closed and convex and $\bar{x} \in S$. By [2, Proposition 2.1.5], $v \in S^\infty$ and also $\sup_{t\geq 0} f(\bar{x}+tv) < \lambda$, a contradiction. □

The following example shows that if $\bar{x}$ is not in the relative interior of dom f, the conclusion in (a) may fail.

Example 2.1. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f(x_1, x_2) = \begin{cases} 0, & \text{if } x_2 \in [0,1[, \\ x_1, & \text{if } x_2 = 1, \\ +\infty, & \text{elsewhere.} \end{cases}$$

Obviously, $\text{dom } f = \mathbb{R} \times [0,1]$. *The sublevel set $[f \leq \alpha]$ is given by $[f \leq \alpha] = \mathbb{R} \times [0,1[\cup]-\infty, \alpha] \times \{1\}$ if $\alpha \geq 0$, and $[f \leq \alpha] =]-\infty,\alpha] \times \{1\}$ if $\alpha < 0$. Thus,* $(\text{dom } f)^\infty = \mathbb{R} \times \{0\}$, *and f is quasiconvex. For $v = (1,0) \in (\text{dom } f)^\infty$, we obtain*

$$f((x_1,x_2) \pm t(1,0)) = 0, \ \forall \ (x_1,x_2) \in \mathbb{R} \times [0,1[, \forall \ t \geq 0,$$

but

$$\sup_{t\geq 0} f((x_1,1)+t(1,0)) = +\infty, \forall\, x_1 \in \mathbb{R}.$$

The q-asymptotic function, along with the class $\mathcal{C}$, allow us to compute the asymptotic cone of every sublevel set of any quasiconvex and lsc function. Part (a) of the next lemma provides a similar formula as in the convex case.

Lemma 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex proper function. The following assertions hold:*

(a) *assume, in addition, that f is lsc and belongs to $\mathcal{C}$. If $[f \leq \lambda] \neq \emptyset$ for some $\lambda \in \mathbb{R}$, then*

$$[f \leq \lambda]^\infty = [f_q^\infty \leq 0] = \{v \in \mathbb{R}^n : f_{0,q}^\infty(v) = 0\}.$$

(b) *Let $-v, v \in (\mathrm{dom}\ f)^\infty$ and $v \neq 0$. Then*

$$f_q^\infty(v) < 0 \implies f_q^\infty(-v) > 0;\ f_q^\infty(v) = 0 \implies f_q^\infty(-v) \geq 0.$$

Proof. (a): See the proof in [26, Lemma 2.1 (a)].

(b): According to Proposition 3.8 in [30], f_q^∞ is convex; thus, we obtain $0 = f_q^\infty(0) \leq \frac{1}{2}f_q^\infty(v) + \frac{1}{2}f_q^\infty(-v)$, from which the desired result follows. □

We now analyze the behavior along a direction $-v$, knowing that f is nonincreasing along v.

Proposition 2.7. *Assume that $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper quasiconvex function belonging to $\mathcal{C}$, $\bar{x} \in \mathrm{ri\ dom}\ f$ and $v, -v \in (\mathrm{dom}\ f)^\infty$. Additionally, suppose that f is bounded from above along a half-line $\bar{x} + \mathbb{R}_+ v$. Then either*

a) *$f(x+tv) = f(x)$ for all $t \in \mathbb{R}$ and all $x \in \mathrm{ri\ dom}\ f$; or*

b) *there exists $\bar{t} > 0$ such that $f(\bar{x}) < f(\bar{x} - \bar{t}v)$, in which case the mappings*

$t \mapsto f(x - tv)$ with $t \geq 0$ for all $x \in \operatorname{dom} f$ are nondecreasing with

$$\sup_{t \to +\infty} f(x - tv) = +\infty.$$

If f is lsc, the same is true if $\bar{x} \in \operatorname{dom} f$ and the conclusion in a) holds for all $x \in \operatorname{dom} f$.

Proof. From the definition of class $\mathcal{C}$, necessarily the mapping $t \mapsto f(\bar{x} + tv)$ in nonincreasing for $t \geq 0$. Then, part (a) in Proposition 2.6 implies that the same holds for every mapping $t \mapsto f(x + tv)$, with $x \in \operatorname{ri} \operatorname{dom} f$ and $t \geq 0$.

If $f(\bar{x}) < f(\bar{x} - \bar{t}v)$, then $t \mapsto f(\bar{x} - tv)$ is not nonincreasing and the definition of class $\mathcal{C}$ implies $\lim_{t \to +\infty} f(\bar{x} - tv) = +\infty$, then (a) in Proposition 2.6 implies that $\sup_{t \geq 0} f(x - tv) = +\infty$ for all $x \in \operatorname{dom} f$.

If no such $\bar{t} > 0$ exists, we conclude, from the definition of the class $\mathcal{C}$ and (a) in Proposition 2.6, that $t \mapsto f(x - tv)$ is nonincreasing for all $x \in \operatorname{ri} \operatorname{dom} f$. Then $f(x + tv) \leq f(x) = f(x + tv - tv) \leq f(x + tv)$ for every $t \geq 0$ and every $x \in \operatorname{ri} \operatorname{dom} f$, which proves that f is constant along every line with direction v with starting point in $\operatorname{ri} \operatorname{dom} f$.

If f is lsc, we conclude using (b) in Proposition 2.6. □

Below, we present an extension to [14, Proposition 4.1], which also happens to be an extension to [54, Theorem 8.6].

Proposition 2.8. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lsc quasiconvex function that belongs to class $\mathcal{C}$, $v \in \mathbb{R}^n \backslash \{0\}$. The following statements are equivalent:*

(a) the mapping $t \mapsto f(x + tv)$ is nonincreasing for all $x \in \operatorname{dom} f$,

(b) there exists $x \in \operatorname{dom} f$ such that $\liminf_{t \to +\infty} f(x + tv) < +\infty$.

Proof. This is a consequence of Proposition 2.6 together with the definition of the class $\mathcal{C}$. □

The following proposition shows an extension of [14, Proposition 4.2] by quasiconvex functions belonging to the class $\mathcal{C}$, here we use the concept of ba or ia-direction of recession introduced in [14].

Definition 2.4. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lsc proper function. A non-null vector $v \in \mathbb{R}^n$ is a ba-direction of recession of f if and only if there are two half lines of direction v along which f has different limits inferior. A non-null vector $v \in \mathbb{R}^n$ is an ia-direction of recession of f if and only if the limit inferior of f along any half-line of direction v amounts to $\inf_{\mathbb{R}^n} f$.*

Proposition 2.9. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper quasiconvex function that belongs to class $\mathcal{C}$ and $v \in \mathbb{R}^n \backslash \{0\}$. The following statements are equivalent*

(a) $f_q^\infty(v) \leq 0$,

(b) v *is a ba, or an ia-direction of recession of* f.

Proof. $(a) \Rightarrow (b)$: We follow the proof given by [14, Proposition 4.2] considering a direction of recession v satisfying $f_q^\infty(v) \leq 0$.

$(b) \Rightarrow (a)$: Let v be a ba-direction of recession of f, then there exist two half-lines with different values for $\liminf$, therefore one of them is strictly less than $+\infty$ and by Proposition 2.8, $f_q^\infty(v) \leq 0$.

If we consider w an ia-direction of recession of f, since f is a proper function, the infimum over $\mathbb{R}$ is not $+\infty$ and also

$$\liminf_{t \to +\infty} f(y + tw) < +\infty,$$

so by Proposition 2.8, we conclude $f_q^\infty(w) \leq 0$. □

Looking at [55, Lemma 1], we notice that the same result can be obtained for quasiconvex functions.

Lemma 2.2. *Let $P \neq \emptyset$ be a convex subset of $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ quasiconvex function. Then f is lsc in P if and only if f is lsc along each line segment $\{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\} \subseteq P$.*

Proof. For each $\mu \in \mathbb{R}$, the set $[f \leq \mu] \cap P$ is convex because f is a quasiconvex function. So, we can refer to the proof provided in [55, Lemma 1]. □

Chapter 3

Limiting normal cones to the union of two convex sets and M-stationarity: local uniquenessand stability in mathematical programming

3.1 Introduction

The limiting normal cone has proved to be very useful in developing necessary optimality conditions under the presence of equilibrium constraints that very often appear in structural optimization, where, for instance, those constraints are expressed as the union of convex sets, or more specifically, as a finite union of convex polyhedra. About the use of techniques from nonlinear programming to develop practical methods in optimal control, the book [4] is a good source of information.
As a motivation, let us briefly describe a nonlinear optimization problem arising after discretizing a control problem governed by the sweeping process:

$$\dot{x}(t) \in -N(C; x(t)) + g(x(t), u(t)), \tag{3.1.1}$$

where $x(0) = x_0$ and $u(t) \in U \subseteq \mathbb{R}^m$ (see for instance, [7, 11] for the case when C is time-dependant), $C \subseteq \mathbb{R}^n$ is a convex set, and $N(C; X)$ denotes the standard normal cone of C at $x \in C$. For simplicity, we choose $g \equiv 0$ and C to be the half-space $C = \{x \in \mathbb{R}^n : \langle a, x\rangle \ \leq \alpha\}$. By discretizing (3.1.1), we get

$$y_k \doteq \frac{x_{k+1} - x_k}{h} \in -N(C; x_k),$$

which is equivalent to $(x_k, y_k) \in \operatorname{gph}(-N(C; \cdot))$ and the right-hand side is the graph of the normal cone mapping that can be explicitly calculated as follows:

$$\operatorname{gph}(-N(C; \cdot)) = (C \times \{0\}) \cup (\{x \in \mathbb{R}^n : \langle a, x\rangle \ = \alpha\} \times \mathbb{R}_+\{a\}).$$

Thus, we arrive to an optimization problem having as a geometric constraint set being the union of two convex polyhedra P_1 and P_2 which has nonempty intersection.

We have to mention that a class of optimization problems, with convex objective function, under a finite union of convex sets with a different perspective, was analyzed in [9].

The limiting normal cone: some advantages and our main contributions

Let us consider the following class of optimization problems:

$$\min\{f(x) : \ g_i(x) \leq 0, \ i = 1, 2, \ldots, m, \ x \in C\}, \tag{3.1.2}$$

where f and each g_i are continuously differentiable in a neighborhood of $\bar{x} \in C$. Without any further assumption, it is known (see [43] for instance, where even the nonsmooth situation is treated) that if $\bar{x}$ is a local solution to (3.1.2), then there exist $\lambda_0 \geq 0$, $\lambda_i \geq 0$, $i = 1, \ldots, m$, not all zeros, such that

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) \in -N_M(C; \bar{x}), \tag{3.1.3}$$

where $I(\bar{x})$ is the active index set at $\bar{x}$ and $N_M(C; \bar{x})$ is the Mordukhovich normal cone of C at $\bar{x}$ as recalled in Definition 2.1 and introduced in [42]. Outrata named (3.1.3) for the first time M-stationarity in [47] (see also [15] for further developments), in contrast to the strong stationarity, which means KKT conditions ($\lambda_0 = 1$):

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) \in [T(C;\bar{x})]^*, \tag{3.1.4}$$

for suitable multipliers λ_i's, as above. Recall that P^* stands for the polar cone of P and $T(C;\bar{x})$ for the (Bouligand-Severi) contingent cone (also known as the tangent cone), as defined in Chapter 2. Obviously, to force $\lambda_0 = 1$, it requires a constraint qualification (CQ) condition. Interested readers may find some CQ conditions for (3.1.3) with $\lambda_0 = 1$, in [44], [45, Section 5.1] and [56, Corollary 6.15], as well as in [46, Proposition 6.4]. Again, under no additional assumptions, it is known by [22, Theorem 3.1] that the fulfillment of (3.1.4) is equivalent to the nonexistence of solution to the system:

$$\langle \nabla f(\bar{x}), v \rangle \;<\; 0, \quad v \in \overline{\text{co}}\, T(C;\bar{x}), \tag{3.1.5}$$

$$\langle \nabla g_i(\bar{x}), v \rangle \;<\; 0, \quad \forall\, i \in I(\bar{x}). \tag{3.1.6}$$

There are situations with $\nabla f(\bar{x}) \neq 0$ in which (3.1.5)-(3.1.6) has no solution and hence (3.1.4) holds: for instance see Example 3.3 in [22]. Take $m = 1$, $n = 2$, $f(x_1, x_2) = x_1$, $g_1(x_1, x_2) = x_2$ and $C = (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+)$ with $\bar{x} = (0,0)$. It is easy to show that the inclusion $[T(C;\bar{x})]^* \subseteq -N_M(C;\bar{x})$ is strict. This asserts that (3.1.4) provides much more information than (3.1.3) and, because of this fact, the term "strong stationarity" arises, see also (2.1.2). As outlined in [16], a class of optimization problems for which the system (3.1.5)-(3.1.6) has no solution comes out when $\nabla f(\bar{x}) \in \overline{\text{co}}\, T(C;\bar{x})$ and $\overline{\text{co}}\, T(C;\bar{x}) = [T(C;\bar{x})]^*$.
For the above example, one obtains that the graph of $-[T(C;\cdot)]^*$ is not closed. Indeed, it holds $(1,0) \in [T(C;(0,x_2))]^*$ for all $x_2 \geq 0$, therefore $((0,1/k),(1,0)) \in \text{gph}\ [T(C;\cdot)]^*$ for all $k \in \mathbb{N}$ but $((0,0),(1,0)) \notin \text{gph}\ [T(C;\cdot)]^*$. We actually know that, in general, the graph of $N_M(C;\cdot)$ is the closure of the graph of $-[T(C;\cdot)]^*$, this is (2.1.3). See also (2.1.3).

Let us describe another advantage of the limiting normal cone $N_M(P_1 \cup P_2; x)$ given by any convex polyhedra P_1 and P_2. First of all, our Theorem 3.1 establishes a representation for $N_M(P_1 \cup P_2; x)$, which is the basis for deriving another description for gph $N_M(P_1 \cup P_2; \cdot)$ (see Theorem 3.2) being equal to the union of finitely many convex polyhedra. Secondly, again based on Theorem 3.1, Corollary 3.2 shows a geometrical local behaviour of $N_M(P_1 \cup P_2; \bar{x})$ when $\bar{x} \in P_1 \cap P_2$ in contrast to its counterpart $[T(P_1 \cup P_2; \bar{x})]^*$. To be more precise, we establish the existence of $\varepsilon > 0$ such that $N_M(P_1 \cup P_2; x) \subseteq N_M(P_1 \cup P_2; \bar{x})$ for all $x \in B(\bar{x}; \varepsilon)$. An inclusion of this type is not valid for $[T(P_1 \cup P_2; \cdot)]^*$, in general. Actually, this inclusion and the representation formula for $N_M(P_1 \cup P_2; x)$ are extremely important in the study of stability or error estimates in nonlinear programming, as shown in Sections 3.4 and 3.5.

3.2 Limiting normal cones to the union of convex sets

This section establishes some formulae for the limiting normal cone to general closed sets, to cones, and finally to the union of closed convex sets. Some inner or outer estimates for such cones are also obtained. These formulae are based on the sequential representation of limiting normal cones as presented in [44] given for locally closed sets.

The following proposition expresses the limiting normal cone at the reference point in terms of same normal cones at points nearby. Part (a) actually is a consequence of the closedness of the graph of $N_M(C; \cdot)$, but we still want to provide a self-contained proof. More precise expressions can be formulated for closed cones. Notice that no further structural assumptions on the involved closed and convex cones are considered. In particular, Part (c) does not require the assumption $C_1 \cap C_2 = \{0\}$ as occurs in [32, Proposition 3.1].

Proposition 3.1. *Let $C \subseteq \mathbb{R}^n$ be a nonempty set. The following assertions hold:*

(a) if C is locally closed around $\bar{x} \in C$, then

$$N_M(C; \bar{x}) = \bigcap_{\epsilon > 0} \bigcup_{x \in B(\bar{x}; \epsilon) \cap C} N_M(C; x).$$

(*b*) *If* C *is a closed cone, then* $N_M(C;\bar{x}) = N_M(C;t\bar{x})$ *for all* $t > 0$, $\bar{x} \in C \setminus \{0\}$ *and*

$$N_M(C;0) = \left[\bigcup_{\substack{x\in C \\ \|x\|=1}} N_M(C;x) \right] \cup N(\text{co } C;0). \tag{3.2.1}$$

(*c*) *Assume that each* C_i, $i = 1, 2$, *is a closed convex cone. Then*

$$N_M(C_1 \cup C_2;0) = \tag{3.2.2}$$

$$\bigcup_{i=1}^{2} \bigcup_{\substack{x\in(\text{rbd}C_i)\setminus C_j \\ \|x\|=1 \\ j\neq i}} N(C_i;x) \cup (C_i)^{\perp} \cup \bigcup_{\substack{x\in C_1\cap C_2 \\ \|x\|=1}} N_M(C_1 \cup C_2;x) \cup N(\text{co}(C_1 \cup C_2);0).$$

Proof. (*a*): The inclusion "⊆" is obvious. Let us prove the opposite inclusion. Take any $v \in \bigcap_{\epsilon>0} \bigcup_{x\in B(\bar{x};\epsilon)\cap C} N_M(C;x)$. Then, for all $\epsilon > 0$, there exists $y_\epsilon \in B(\bar{x};\epsilon) \cap C$ such that $v \in N_M(C;y_\epsilon)$. By (2.1.1), we can find $z_\epsilon^k \to y_\epsilon$ as $k \to +\infty$, $u_\epsilon^k \in \text{Proj}(C;z_\epsilon^k)$ and $t_\epsilon^k > 0$ such that $t_\epsilon^k(z_\epsilon^k - u_\epsilon^k) \to v$ as $k \to +\infty$. Furthermore, for all $\epsilon > 0$, among those z_ϵ^k, t_ϵ^k and u_ϵ^k, there exist z_ϵ, t_ϵ and u_ϵ such that

$$u_\epsilon \in \text{Proj}(C;z_\epsilon),\ t_\epsilon > 0,\ \|t_\epsilon(z_\epsilon - u_\epsilon) - v\| < \epsilon \text{ and } \|z_\epsilon - y_\epsilon\| < \epsilon.$$

Since $y_\epsilon \to \bar{x}$ as $\epsilon \downarrow 0$, we have $z_\epsilon \to \bar{x}$ as $\epsilon \downarrow 0$ and $t_\epsilon(z_\epsilon - u_\epsilon) \to v$ as $\epsilon \downarrow 0$. Therefore $v \in N_M(C;\bar{x})$.

(*b*): This first part easily follows from (2.1). We now prove the inclusion "⊇" in (3.2.1). From the first part, it follows that, for any $\epsilon > 0$ we have

$$\bigcup_{\substack{x\in B(\bar{x};\epsilon)\cap C \\ x\neq\bar{x}}} N_M(C;x) = \bigcup_{\substack{x\in C \\ \|x\|=1}} N_M(C;x).$$

This together with the inclusion in (*c*) of Proposition 2.2 completes the proof of the inclusion "⊇".

We now prove the inclusion "⊆". Let $0 \neq v = \lim_{k\to+\infty} t_k(x_k - y_k)$, with $t_k > 0$, $x_k \to 0$ and $y_k \in \text{Proj}(C;x_k)$. It is known that $y_k \to 0$. We can assume that $\|v\| = 1$ and consider two cases: (1) $y_k \neq 0$ for infinitely many k, and (2) $y_k = 0$ except for finitely many k. In the first case, by the triangle inequality, we obtain

$$y_k \in \text{Proj}(C; y_k + l(x_k - y_k))$$

for all $l \in]0,1[$. Since C is a cone, we also have that $ty_k \in \text{Proj}(C; ty_k + tl(x_k - y_k))$ for all $t > 0$ and all $l \in]0,1[$. Hence, up to some subsequences, we have $\frac{y_k}{\|y_k\|} \to y \in C$, with $\|y\| = 1$ and also $\frac{y_k}{\|y_k\|} \in \text{Proj}\left(C; \frac{y_k}{\|y_k\|} + \frac{l}{\|y_k\|}(x_k - y_k)\right)$. In particular, we consider $l_k \searrow 0$ converging fast enough so that $\frac{y_k}{\|y_k\|} + \frac{l_k}{\|y_k\|}(x_k - y_k) \to y$ (it suffices to consider $l_k = \|y_k\|^2$). Then we have

$$\lim_{k\to+\infty} \frac{1}{\frac{l_k}{\|y_k\|}\|x_k - y_k\|}\left(\frac{y_k}{\|y_k\|} + \frac{l_k}{\|y_k\|}(x_k - y_k) - \frac{y_k}{\|y_k\|}\right) =$$

$$= \lim_{k\to+\infty} \frac{\|y_k\|}{l_k\|x_k - y_k\|}\left(\frac{l_k}{\|y_k\|}(x_k - y_k)\right) = \lim_{k\to+\infty} \frac{1}{\|x_k - y_k\|}(x_k - y_k) = v$$

with $\frac{y_k}{\|y_k\|} \in \text{Proj}\left(C; \frac{y_k}{\|y_k\|} + \frac{l_k}{\|y_k\|}(x_k - y_k)\right)$ and $\frac{y_k}{\|y_k\|} + \frac{l_k}{\|y_k\|}(x_k - y_k) \to y$. Thus $v \in N_M(C; y)$.

We now consider the second case, that is, when $y_k \neq 0$ for finitely many k. In this situation, $\text{Proj}(C; x_k) = \{0\}$. This yields $\|x_k\| \leq \|x_k - x\|$ for all $x \in C$, which, in turn, gives $2\langle x_k, x\rangle \leq \|x\|^2$ for all $x \in C$. Since C is a cone, the previous inequality implies $\langle x_k, x\rangle \leq 0$ for all $x \in C$ and so also for all $x \in \text{co } C$. This implies $x_k \in N(\text{co } C; 0)$. Finally, since the normal cone to a convex set is a closed cone, $v = \lim_k t_k(x_k - 0) \in N(\text{co } C; 0)$. This completes the proof of (3.2.1).

(c): Use (b) and the following equalities

$$C_1 \cup C_2 = (C_1 \setminus C_2) \cup (C_2 \setminus C_1) \cup (C_1 \cap C_2);\ \ C_i \setminus C_j = (\text{ri } C_i) \setminus C_j \cup (\text{rbd } C_i) \setminus C_j;$$

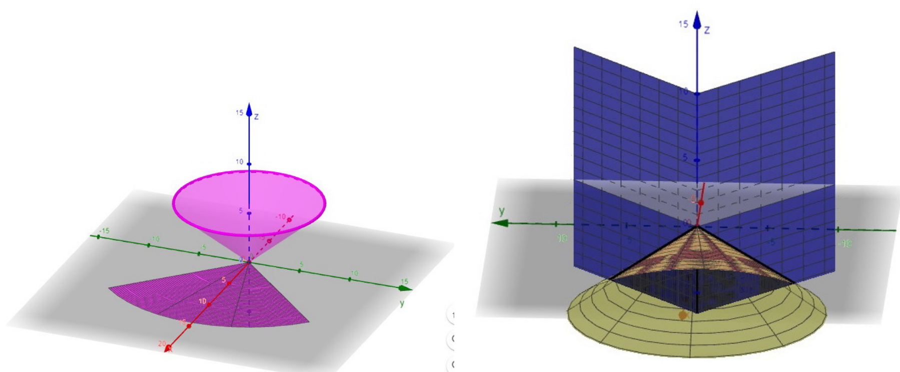
$$-N(C_i; x) = (C_i - x)^* = (C_i - x)^\perp = (C_i)^\perp = -(C_i)^\perp, \ \text{ provided } x \in \text{ri } C_i,$$

to conclude with the desired result. □

Next instance shows an application of Part (c) of Proposition 3.1.

Example 3.1. *Consider the two convex sets:* $C_1 = \{(x, y, z) : \ x^2 + y^2 \leq z\}$ *and*

$C_2 = \{(x, y, z) : \ x \geq |y|, z = 0\}$ *and the set* $C \doteq C_1 \cup C_2$*, whose picture is drawn in Figure 3.2.1* (*a*)*; whereas 3.2.1* (*b*) *shows its limiting normal cone at* $(0, 0, 0)$*. This cone is, according to* (3.2.2)*, the union of three sets: the blue set corresponds to the set* $\{(x, y, z) \in \mathbb{R}^3 : \ |y| = -x\}$*, which is obtained by taking points in relative boundary of* C_2 *with norm one; the light green set* $\{(x, y, z) \in \mathbb{R}^3 : \ z = -\sqrt{x^2 + y^2}\}$ *is obtained by taking points in the relative boundary of* C_1 *with norm one and the set* $\{(x, y, z) : \ z \leq -\sqrt{x^2 + y^2}, \ -x \geq |y|\} = N(\text{co } (C_1 \cup C_2); 0)$*. The second part in* (3.2.2) *is actually empty since* $(C_1 \cap C_2) \cap \{(x, y, z) : \ x^2 + y^2 + z^2 = 1\} = \emptyset$*.*



(a) Set $C = C_1 \cup C_2$

(b) Limiting normal cone to C at $(0, 0, 0)$, the "ice-cream cone" is not solid, except for the shaded part

Figure 3.2.1: Example 3.1

Under a further assumption on the origin, (3.2.1) can be refined.

Corollary 3.1. *If* $C \subseteq \mathbb{R}^n$ *is a closed cone and* $0 \in \text{ri}(\text{co } C)$*, then*

$$N_M(C; 0) = \bigcup_{\substack{x \in C \\ \|x\|=1}} N_M(C; x).$$

Proof. By assumption co C = cone(co C) is a subspace and so $N(\text{co } C; 0) = (\text{co } C)^{\perp} \subseteq N_M(C; x)$ for all $x \in C$. □

The next proposition establishes a very useful result that will be used in many places. It allows us to pass from projecting onto a large set, to projecting onto a smaller set.

Proposition 3.2. *Let C_1, C_2 be nonempty closed and convex sets in $\mathbb{R}^n$ and let $\bar{x} \in C_1 \cap C_2$. Assume that there exist $x_k \in C_1 \setminus C_2$, $x_k \to \bar{x}$, $v_k \to v$, $v_k \in N(C_1; x_k)$. Then there exists $t_k \downarrow 0$ such that* $\mathrm{dist}(x_k + t_k v_k; C_2) > \mathrm{dist}(x_k + t_k v_k; C_1)$. *Consequently,* $\mathrm{Proj}(C_1 \cup C_2; x_k + t_k v_k) = \mathrm{Proj}(C_1; x_k + t_k v_k) = x_k$. *Thus, $v \in N_M(C_1 \cup C_2; \bar{x})$.*

Proof. By assumption, we can take $t_k > 0$ such that $t_k \|v_k\| < \frac{1}{2}\mathrm{dist}(x_k, C_2)$. Then $t_k \downarrow 0$ because $\mathrm{dist}(x_k, C_2) \to 0$. Moreover,

$$\begin{aligned} \mathrm{dist}(x_k + t_k v_k; C_2) &\geq \mathrm{dist}(x_k; C_2) - \mathrm{dist}(x_k + t_k v_k; x_k) = \mathrm{dist}(x_k; C_2) - t_k \|v_k\| \\ &\geq \frac{1}{2}\mathrm{dist}(x_k; C_2). \end{aligned}$$

On the other hand, $v_k \in N(C_1; x_k)$ implies $\mathrm{Proj}(C_1; x_k + t_k v_k) = x_k$, which means that $\mathrm{dist}(x_k + t_k v_k; C_1) = t_k \|v_k\| < \frac{1}{2}\mathrm{dist}(x_k; C_2)$. Thus $\mathrm{Proj}(C_1 \cup C_2; x_k + t_k v_k) = \mathrm{Proj}(C_1; x_k + t_k v_k) = \{x_k\}$. Therefore by setting $\tilde{x}_k = x_k + t_k v_k$, $\tilde{y}_k = x_k$ and $\tilde{t}_k = t_k^{-1}$, one obtains $\tilde{x}_k \to \bar{x}$, $\tilde{t}_k(\tilde{x}_k - \tilde{y}_k) = v_k \to v$ and so $v \in N_M(C_1 \cup C_2; \bar{x})$. □

The following result provides some inner and outer estimates for the limiting normal cone to the union of two closed convex sets. Recall that $N(C; x) = \emptyset$ if $x \notin C$.

Lemma 3.1. *Let C_1, C_2 be convex sets in $\mathbb{R}^n$. The following hold:*

(*a*) *Assume that C_1 is locally closed around $\bar{x}$. Then*

$$\operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in C_1 \setminus C_2}} N(C_1; x) = \bigcap_{\epsilon > 0} \overline{\bigcup_{x \in B(\bar{x}, \epsilon) \cap (C_1 \setminus C_2)} N(C_1; x)};$$

(*b*) *Assume that $\bar{x} \in C_1 \cap C_2$, with $C_1 \cup C_2$ locally closed around $\bar{x}$. Then*

$$\operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in C_1 \setminus C_2}} N(C_1; x) \cup \operatorname*{Limsup}_{\substack{x \to \bar{x} \\ x \in C_2 \setminus C_1}} N(C_2; x) \cup N(\mathrm{co}(C_1 \cup C_2); \bar{x}) \subseteq N_M(C_1 \cup C_2; \bar{x}).$$

(*c*) *Asume that $\bar{x} \in C_1 \cup C_2$ and $C_1 \cup C_2$ be locally closed around $\bar{x}$. Then*

$$N_M(C_1 \cup C_2; \bar{x}). \subseteq N(C_1; \bar{x}) \cup N(C_2; \bar{x}).$$

Proof. (a): Let $v \in \operatorname*{Limsup}_{\substack{x\to\bar{x}\\ x\in C_1\setminus C_2}} N(C_1;x)$. Then, there exist $x_k \in C_1 \setminus C_2$ and $v_k \in N(C_1;x_k)$ with $x_k \to \bar{x}$ and $v_k \to v$. This clearly proves the inclusion "$\subseteq$". For the reverse inclusion, simply take $\epsilon = k^{-1}$ and the proof is completed.

(b): The inclusion $N(\text{co}(C_1 \cup C_2);\bar{x}) \subseteq N_M(C_1 \cup C_2;\bar{x})$ appears in Proposition 2.2. We now check the inclusion $\operatorname*{Limsup}_{\substack{x\to\bar{x}\\ x\in C_i\setminus C_j}} N(C_i;x) \subseteq N_M(C_1 \cup C_2;\bar{x})$ for $i \neq j$. We consider only the case $i = 1$, since for $i = 2$ is entirely similar.

Let $v \in \operatorname*{Limsup}_{\substack{x\to\bar{x}\\ x\in C_1\setminus C_2}} N(C_1;x)$. Then, there exist $x_k \in C_1 \setminus C_2$ and $v_k \in N(C_1;x_k)$ with $x_k \to \bar{x}$ and $v_k \to v$. By applying Proposition 3.2, we conclude that $v \in N_M(C_1 \cup C_2;\bar{x})$ and the proof is completed.

(c): If $v \in N_M(C_1 \cup C_2;\bar{x})$, then $v = \lim_k t_k(x_k - y_k)$ with $t_k > 0$, $x_k \to \bar{x}$ and $y_k \in \text{Proj}(C_1 \cup C_2;x_k)$. If there exists a subsequence $y_{k_l} \in C_1$, then we have $v = \lim_k t_{k_l}(x_{k_l} - y_{k_l})$. Therefore $v \in N_M(C_1;\bar{x}) = N(C_1;\bar{x})$. If no such a subsequence exists, then there must be a subsequence $y_{k_l} \in C_2$ and consequently $v \in N(C_2;\bar{x})$. □

3.3 Limiting normal cones to the union of two convex polyhedra: a representation formula with applications to graphs

In this section all the involved sets are convex polyhedra. So, due to their geometric structure, more precise expressions for the limiting normal cone will be established. In particular, the case of the union of two polyhedral sets is analyzed, ensuring its limiting normal cone is a finite union of polyhedra, see Theorem 3.1. An application to determine the polyhedral nature of graph of mappings that are sum of linear transformations and limiting normal cone to the union of two polyhedra, is given, see Proposition 3.2 and Theorem 3.3. We refer the readers to [32] for another kind of representation of the limiting normal cone to a finite union of closed, convex cones that their pairwise intersections reduce to the origin. The other situation considered in [32] is that when the cones are halfspaces.

Let us consider polyhedra of the form:

$$P_i \doteq \{x \in \mathbb{R}^n : \ \langle a^i_j, x\rangle \leq \alpha^i_j \ \text{ for } j = 1, \ldots, m_i\}, \quad i = 1, 2.$$

Set $J_i \doteq \{1, \ldots, m_i\}$. Given $x \in P_i$, its set of active indices is defined by

$$I_i(x) \doteq \{j \in J_i : \ \langle a^i_j, x\rangle = \alpha^i_j\}.$$

It is well-known that (see [56, Theorem 6.46], for instance) the cone $N(P_i; x)$ is a polyhedron and, more precisely, one obtains

$$N(P_i; x) = \text{co cone}\{a^i_j : \ j \in I_i(x)\} = \Big\{ \sum_{j\in I_i(x)} t_j a^i_j : \ t_j \geq 0, \ j \in I_i(x)\Big\}. \qquad (3.3.1)$$

As a consequence, $A(P_i)$ is also a polyhedron for any real matrix A.

What follows describes the local behaviour of the active indices set of points around a fixed $\bar{x} \in P_1 \cap P_2$ with respect to $I_i(\bar{x})$. We believe the next lemma is important by itself and, as far as these authors know, it is new.

Lemma 3.2. *Let P_i be a polyhedron for $i = 1, 2$, as above and $\bar{x} \in P_1 \cap P_2$. The following statements hold.*

(*a*) *The set $\mathcal{E}(\bar{x}) \doteq \{\epsilon > 0 : \ I_i(x) \subseteq I_i(\bar{x})$ for $i = 1, 2, \ \forall\, x \in B(\bar{x}; \epsilon)\}$ is nonempty. Consequently, if $\epsilon \in \mathcal{E}(\bar{x})$, then $N(P_i; x) \subseteq N(\bar{x}; P_i)$ for all $x \in B(\bar{x}; \epsilon)$ and $i = 1, 2$.*

(*b*) *Let $\epsilon \in \mathcal{E}(\bar{x})$ defined in (a). If $x \in B(\bar{x}; \epsilon) \cap P_1$, then $I_1(x) = I_1(z)$ for all $z \in\,]x, \bar{x}[$.*

(*c*) *Let $\epsilon \in \mathcal{E}(\bar{x})$. If $x \in B(\bar{x}; \epsilon) \cap (P_1 \setminus P_2)$, then $[x, \bar{x}] \cap P_2 = \{\bar{x}\}$.*

(*d*) *Let $\epsilon_i \in \mathcal{E}(\bar{x})$, $i = 1, 2$, $\epsilon_2 > \epsilon_1$. If $x \in B(\bar{x}; \epsilon_2) \cap (P_1 \setminus P_2)$, then there exists*

$x' \in B(\bar{x};\epsilon_1) \cap (P_1 \setminus P_2)$ such that $I_i(x) = I_i(x')$ for $i = 1, 2$. As a consequence,

$$\begin{aligned} \bigcup_{x\in B(\bar{x},\epsilon_1)\cap(P_1\setminus P_2)} N(P_1;x) &= \bigcup_{x\in B(\bar{x},\epsilon_2)\cap(P_1\setminus P_2)} N(P_1;x) \\ &= \bigcup_{R\in R(\bar{x})} \operatorname{co}\operatorname{cone}\{a_j^1 :\ j \in R\}, \qquad (3.3.2) \end{aligned}$$

is closed. Here, $R(\bar{x}) = \{I_1(x) \subseteq I_1(\bar{x}) :\ x \in B(\bar{x};\epsilon_1) \cap (P_1 \setminus P_2)\}$.

Proof. (a): Suppose to the contrary that there exist a sequence $x_k \to \bar{x}$, and $i_0 \in \{1,2\}$ such that $I_{i_0}(x_k) \not\subseteq I_{i_0}(\bar{x})$. Thus, there exists $j_k \in I_{i_0}(x_k) \setminus I_{i_0}(\bar{x})$. By finiteness, there is $j_0 \in I_{i_0}(x_k) \setminus I_{i_0}(\bar{x})$ for all k sufficiently large. This implies $j_0 \in I_{i_0}(\bar{x})$, a contradiction, proving the result.

(b): Take any $t \in\ ]0,1[$ and set $z = tx + (1-t)\bar{x}$. For all $j \in I_1(x) \subseteq I_1(\bar{x})$, we have $\langle a_j^1, x\rangle = \langle a_j^1, \bar{x}\rangle = \alpha_j^1$ and so $\langle a_j^1, z\rangle = \langle a_j^1, tx+(1-t)\bar{x}\rangle = \alpha_j^1$, which means $j \in I_1(z)$, showing $I_1(x) \subseteq I_1(z)$. Now, if $j \in I_1(z) \setminus I_1(x)$, then $\langle a_j^1, x\rangle < \alpha_j^1$ and $\langle a_j^1, \bar{x}\rangle = \alpha_j^1$. Then $\langle a_j^1, z\rangle = \langle a_j^1, tx + (1-t)\bar{x}\rangle < \alpha_j^1$, that is, $j \notin I_1(z)$, which is impossible. Thus, $I_1(z) \subseteq I_1(x)$ and so $I_1(z) = I_1(x)$.

(c): Take any $x_0 \in\ ]x, \bar{x}[\ \cap P_2$. Then $\langle a_j^2, x_0\rangle \leq \alpha_j^2 < \langle a_j^2, x\rangle$ for some $j \in m_2$ and so $\langle a_j^2, x_1\rangle = \alpha_j^2$ for some $x_1 \in\ ]x, x_0] \cap P_1$, which means $j \in I_2(x_1)$. On the other hand, by writing $\bar{x} = x_0 + \lambda(x_0 - x)$ for some $\lambda > 0$, one gets $\langle a_j^2, \bar{x}\rangle < \alpha_j^2$, that is, $j \notin I_2(\bar{x})$. Since $x_1 \in\ ]x, x_0] \subseteq [x, \bar{x}] \subseteq B(\bar{x};\epsilon)$, the choice of ϵ yields $I_2(x_1) \subseteq I_2(\bar{x})$, reaching a contradiction and the conclusion follows.

(d): Assume that $\|x - \bar{x}\| \geq \epsilon_1$ since otherwise there is nothing to do. Now, we simply take any $x' \in\ ]x, \bar{x}[$ with $\|x' - \bar{x}\| < \epsilon_1$. By (b) and (c), one gets $I_i(x) = I_i(x')$ for $i = 1, 2$. This along with the representation (3.3.1) yields the equalities in (3.3.2). The closedness follows from the fact that the set on the right-hand side of (3.3.2) is a finite union of polyhedra. □

Due to the special polyhedral structure, we are able to establish a formula for the limiting normal cone to the union of two convex polyhedra. Such a new representation formula also expresses that such a normal cone is a union of finitely many polyhedrons. In case P_1 and P_2 are polyhedral cones, it can be observed that the next representation might be also obtained as a consequence of (14) in [32]. However, we prefer to keep our proof since it is self-contained and follows our line of reasoning. A comparison between both results is given in Remark 3.1.

Theorem 3.1. *Let $\bar{x} \in P_1 \cap P_2$ and $\epsilon > 0$ belonging to $\mathcal{E}(\bar{x})$ introduced in Lemma 3.2. Then*

$$N_M(P_1 \cup P_2; \bar{x}) =$$

$$= \left[\bigcup_{x \in B(\bar{x},\epsilon) \cap (P_1 \setminus P_2)} N(P_1; x) \right] \bigcup \left[\bigcup_{x \in B(\bar{x},\epsilon) \cap (P_2 \setminus P_1)} N(P_2; x) \right] \bigcup N(\mathrm{co}(P_1 \cup P_2); \bar{x}),$$

which is closed since it is a finite union of polyhedra by Lemma 3.2(d).

Proof. We first prove the inclusion "$\supseteq$" by spliting it into three cases.
Case 1: Let $x \in B(\bar{x}, \epsilon) \cap (P_1 \setminus P_2)$ and $v \in N(P_1; x)$. Set $x_k = k^{-1}x + (1 - k^{-1})\bar{x}$, $k \in \mathbb{N}$. By (b) and (c) of Lemma 3.2, we obtain $d_k := \mathrm{dist}(x_k, P_2) > 0$ for all $k \in \mathbb{N}$ and $I_1(x) = I_1(x_k)$. This implies $N(P_1; x) = N(P_1; x_k)$ and so $v \in N(P_1; x_k)$ thanks to (3.3.1). We are in a position to apply Proposition 3.2 to conclude that $v \in N_M(P_1 \cup P_2; \bar{x})$.
Case 2: $x \in B(\bar{x}, \epsilon) \cap (P_2 \setminus P_1)$ and $v \in N(P_2; x)$ is similar to Case 1.
Case 3: From Proposition 2.2, it follows that $N(\mathrm{co}(P_1 \cup P_2); \bar{x}) \subseteq N_M(P_1 \cup P_2; \bar{x})$.
We now prove the inclusion "$\subseteq$". Let $x_k \to \bar{x}$, $y_k \in \mathrm{Proj}(P_1 \cup P_2; x_k)$ and $t_k > 0$ such that $t_k(x_k - y_k) \to v \in N_M(P_1 \cup P_2; \bar{x})$; we also have $y_k \to \bar{x}$. Up to subsequences, we have $y_k \in P_1 \setminus P_2$, $y_k \in P_2 \setminus P_1$ or $y_k \in P_1 \cap P_2$.
In case $y_k \in P_1 \setminus P_2$ for all k, we have $y_k = \mathrm{Proj}(P_1; x_k)$, which implies $t_k(x_k - y_k) \in N(P_1; y_k)$, with $y_k \in B(\bar{x}, \epsilon)$ for all k. Thus

$$t_k(x_k - y_k) \in \bigcup_{x \in B(\bar{x},\epsilon) \cap (P_1 \setminus P_2)} N(P_1; x) \quad \forall\, k \in \mathbb{N}. \tag{3.3.3}$$

Since the normal cone at every $x \in B(\bar{x}, \epsilon)$ is a finite union of polyhedra, the union in the right side of (3.3.3) is also closed (see Lemma 3.2(d)) and so

$$\lim_{k \to \infty} t_k(x_k - y_k) = v \in \bigcup_{x \in B(\bar{x},\epsilon) \cap (P_1 \setminus P_2)} N(P_1; x).$$

By symmetry, in case $y_k \in P_2 \setminus P_1$ for all k, we obtain similar to the previous case,

$$\lim_{k\to\infty} t_k(x_k - y_k) = v \in \bigcup_{x\in B(\bar{x},\epsilon)\cap(P_2\setminus P_1)} N(P_2; x).$$

Finally, if $y_k \in P_1 \cap P_2$ for all $k \in \mathbb{N}$, we obtain, as above, $t_k(x_k - y_k) \in N(P_1; y_k) \cap N(P_2; y_k) = N(\mathrm{co}(P_1 \cup P_2); y_k)$ by $(b1)$ of Proposition 2.2. Hence, $v \in N(\mathrm{co}(P_1 \cup P_2); \bar{x})$. This completes the proof. $\square$

The following remark makes a comparison with a similar result established in [32].

Remark 3.1. (i) *We must point out that when P_1 and P_2 are polyhedral cones and $\bar{x} = 0$, the formula in Theorem 3.1 reduces to the formula (14) in [32], once one notices that $\hat{N}(P_i; x) = N(P_i; x)$ and the fact that $N(\mathrm{co}(P_1 \cup P_2); 0) = -(P_1 \cup P_2)^* = -P_1^* \cap P_2^*$ in view of Proposition 2.2.*
(ii) *The authors in [32, Page 59] show that*

$$N_M(P_1 \cup P_2; \bar{x}) = N_M\Big(\bigcup_{i\in I(\bar{x})} T(P_i; \bar{x}); 0\Big), \quad I(\bar{x}) \doteq \{i : \ \bar{x} \in P_i\},$$

so, they conclude that it suffices to compute the limiting normal cone to the finite union of polyhedral cones. Such a formula is expressed in (14) of the cited work. But then, it is not easy to manipulate the term $\hat{N}(T(P_1; \bar{x}); x) = N(T(P_1; \bar{x}); x)$ for $x \in T(P_1; \bar{x}) \setminus T(P_2; \bar{x})$. We find it inconvenient for our purposes.

The next corollary provides a new geometrical local behaviour of the limiting normal cone near points in $P_1 \cap P_2$. This property fails for $[T(P_1 \cup P_2; \bar{x})]^*$ as mentioned in the introduction.

Corollary 3.2. *Let P_1 and P_2 be polyhedra as above and $\bar{x} \in P_1 \cap P_2$. Then, there exists $\varepsilon \in \mathcal{E}(\bar{x})$ (as defined in Lemma 3.2) such that*

$$N_M(P_1 \cup P_2; x) \subseteq N_M(P_1 \cup P_2; \bar{x}), \quad \forall\ x \in B(\bar{x}; \varepsilon).$$

Proof. Let $\varepsilon_1 \in \mathcal{E}(\bar{x})$ for which the representation formula for $N_M(P_1 \cup P_2; \bar{x})$ holds. If $x \in B(\bar{x}; \varepsilon_1) \cap (P_1 \setminus P_2)$ (the case where $x \in B(\bar{x}; \varepsilon_1) \cap (P_2 \setminus P_1)$ is similar), then from such a representation it follows that (see $(b2)$ of Proposition 2.2)

$$N_M(P_1 \cup P_2; x) = N(P_1; x) \subseteq N_M(P_1 \cup P_2; \bar{x}).$$

In case $x \in P_1 \cap P_2$, take any $\varepsilon_2 \in \mathcal{E}(x)$ such that $B(x; \varepsilon_2) \subseteq B(\bar{x}; \varepsilon_1)$. Then, again by the representation formula for $N_M(P_1 \cup P_2; x)$ for ε_2, we infer immediately the inclusion $N_M(P_1 \cup P_2; x) \subseteq N_M(P_1 \cup P_2; \bar{x})$, since we also have $N(\text{co}(P_1 \cup P_2); x) \subseteq N(\text{co}(P_1 \cup P_2); \bar{x})$, because of $I_i(x) \subseteq I_i(\bar{x})$ and $\text{co}(P_1 \cup P_2)$ is again a polyhedron. Finally, by taking $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$, we conclude the proof. □

We now consider mappings which are the sum of a linear transformation and limiting normal cones to a union of two convex polyhedra and aim to prove that the graph of such mappings is a finite union of polyhedra. We start by analyzing a geometrical representation for the graph of the mapping $\mathcal{G}(x) = N_M(P_1 \cup P_2; x)$. We immediately obtain the inclusion $\text{gph}(\mathcal{H}) \subseteq \text{gph}(\mathcal{G})$ restricted to $P_1 \cap P_2$, where $\mathcal{H}(x) \doteq N(\text{co}(P_1 \cup P_2); x)$.

In order to avoid the use of many indices, we use the following representation for the polyhedra P_1 and P_2.

$$P_1 = \{x \in \mathbb{R}^n : \ \langle a_j, x \rangle \leq \alpha_j, \ \forall j \in J_1\}, \ J_1 \doteq \{1, 2, \ldots, p_1\}.$$

$$P_2 = \{x \in \mathbb{R}^n : \ \langle b_j, x \rangle \leq \beta_j, \ \forall j \in J_1\}, \ J_2 \doteq \{1, 2, \ldots, p_2\}.$$

Given any $M \subseteq J_1$ and $N \subseteq J_2$, we define

$$D(M) \doteq \overline{\{x \in P_1 \setminus P_2 : \ \langle a_j, x \rangle = \alpha_j, \ j \in M\}},$$

$$E(N) \doteq \overline{\{x \in P_2 \setminus P_1 : \ \langle b_j, x \rangle = \beta_j, \ j \in N\}},$$

$$F(M) \doteq \text{co cone}\{a_j : \ j \in M\}, \quad G(N) \doteq \text{co cone}\{b_j : \ j \in N\}.$$

We obtain the following equalities:

$$\begin{aligned} D(M) &= \overline{\{x \in P_1 : \ \langle a_i, x\rangle = \alpha_i, \ i \in M; \ \langle b_j, x\rangle > \beta_j \text{ for some } j \in J_2\}} \\ &= \bigcup_{j\in N} \overline{\{x \in P_1 : \ \langle a_i, x\rangle = \alpha_i, \ i \in M; \ \langle b_j, x\rangle > \beta_j\}} \\ &= \bigcup_{j\in N} \{x \in P_1 : \ \langle x, a_i\rangle = \alpha_i, \ i \in M; \ \langle x, b_j\rangle \geq \beta_j\}, \end{aligned}$$

where the last equality holds provided each set involved in the union is nonempty.

The following result gives us the expected representation for the graph of $\mathcal{G}$, which will be used later.

Theorem 3.2. *The following representation for the graph of $\mathcal{G}$ holds*

$$\operatorname{gph}(\mathcal{G}) = \left[\bigcup_{M\subseteq\{1,\ldots,p_1\}} D(M)\times F(M)\right] \cup \left[\bigcup_{N\subseteq\{1,\ldots,p_2\}} E(N)\times G(N)\right] \cup \operatorname{gph}(\mathcal{H}_{|P_1\cap P_2})$$

and therefore it is the union of finitely many polyhedrons. Here, $\mathcal{H}_{|P_1\cap P_2}$ is the restriction of $\mathcal{H}$ to $P_1 \cap P_2$, which is defined as $\mathcal{H}_{|P_1\cap P_2}(x) = \mathcal{H}(x)$ if $x \in P_1 \cap P_2$, and $\mathcal{H}_{|P_1\cap P_2}(x) = \emptyset$ elsewhere.

Proof. First, we show the inclusion:

$$\left[\bigcup_{M\subseteq\{1,\ldots,p_1\}} D(M)\times F(M)\right] \cup \left[\bigcup_{N\subseteq\{1,\ldots,p_2\}} E(N)\times G(N)\right] \cup \operatorname{gph}(\mathcal{H}_{|P_1\cap P_2}) \subseteq \operatorname{gph}(\mathcal{G}).$$

To that end, let $(x, y) \in D(M) \times F(M)$ for some $M \subseteq J_1$ (the case $(x, y) \in E(N) \times G(N)$ for some $N \subseteq J_2$ is analyzed in a similar manner). Since P_1 is closed, $x \in P_1$. If $x \in P_1 \setminus P_2$, then by $(b2)$ of Proposition 2.2, $N_M(P_1 \cup P_2; x) = N(P_1; x)$. We also have M is a subset of the active indices set of x in P_1, which implies $F(M) \subseteq N(P_1; x)$. It turns out that $y \in F(M) \subseteq N(P_1; x) = N_M(P_1 \cup P_2; x)$, that is, $(x, y) \in \operatorname{gph}(\mathcal{G})$.
If $x \in P_1 \cap P_2$, then one can choose a sequence $x_k \in P_1$ satisfying $x_k \to x$ and $\langle a_i, x_k\rangle = \alpha_i$, for all $i \in M$. Thus $(x_k, y) \in \operatorname{gph}(\mathcal{G})$. From Corollary 3.2 it follows that $(x, y) \in \operatorname{gph}(\mathcal{G})$. The previous reasoning together with $\operatorname{gph}(\mathcal{H}_{|P_1\cap P_2}) \subseteq \operatorname{gph}(\mathcal{G})$

completes the proof of the desired inclusion.

We now prove the reverse inclusion. Let $(\bar{x}, \bar{y}) \in \text{gph}(\mathcal{G})$, we distinguish various cases. If $\bar{x} \in P_1 \setminus P_2$ (the case $\bar{x} \in P_2 \setminus P_1$ is entirely similar), then $\bar{y} \in \mathcal{G}(\bar{x}) = N(P_1; \bar{x}) = F(I_1(\bar{x}))$ and clearly $\bar{x} \in D(I_1(\bar{x}))$, where $I_1(\bar{x})$ is the set of active indices of $\bar{x}$ in P_1. This shows the required result.

We now assume that $\bar{x} \in P_1 \cap P_2$ and use the following representation:

$$\mathcal{G}(\bar{x}) = \left[\bigcup_{x \in B(\bar{x},\epsilon) \cap (P_1 \setminus P_2)} N(P_1; x) \right] \bigcup \left[\bigcup_{x \in B(\bar{x},\epsilon) \cap (P_2 \setminus P_1)} N(P_2; x) \right] \bigcup \mathcal{H}(\bar{x}),$$

for an appropriate $\epsilon > 0$.

In case $\bar{y} \in N(\text{co}(P_1 \cup P_2); \bar{x}) = \mathcal{H}_{|P_1 \cap P_2}(\bar{x})$, the desired results follows.

If $\bar{y} \in N(P_1; x)$ for some $x \in B(\bar{x}, \epsilon) \cap (P_1 \setminus P_2)$, satisfying $I_i(x) \subseteq I_i(\bar{x})$, $i = 1, 2$. Thus, $\bar{y} \in N(P_1; x) = \text{co cone}\{a_i : \ i \in I_1(x)\} = F(I_1(x)) \subseteq F(I_1(\bar{x}))$ and $\bar{x} \in D(I_1(\bar{x}))$. Hence $(\bar{x}, \bar{y}) \in D(I_1(\bar{x})) \times F(I_1(\bar{x}))$.

If $\bar{y} \in N(P_2; x)$ for some $x \in B(\bar{x}, \epsilon) \cap (P_2 \setminus P_1)$, we proceed in a similar manner. This concludes the proof of the reverse inclusion and the proof is complete. □

From the preceding proposition it follows our main result in this section.

Theorem 3.3. *Consider a mapping $G : \mathbb{R}^{nm} \rightrightarrows \mathbb{R}^{nm}$, with $G = (G_1, \ldots, G_m)$ given by:*

$$G_i(x_1, \ldots, x_m) = A_i(x_1, \ldots, x_m) + N_M(C_i; x_i),$$

where $x = (x_1, \ldots, x_m) \in \mathbb{R}^{nm}$, each $A_i \in \mathbb{R}^{n \times nm}$ is a real matrix and each C_i may be union of two polyhedrons. Then, the set

$$\text{gph } G = \{(x, y) : \ y \in G(x)\}$$

is a finite union of polyhedra, hence closed.

An application of the above theorem is given next.

Example 3.2. *The following mapping arises when writing the first-order optimality condition for a suitable quadratic optimization problem under a geometric constraints set which is the union of two polyhedra, P_1 and P_2, for instance. Whenever $\lambda \geq 0$,* *such a mapping is:*

$$\mathcal{F}(z,\lambda,\mu) = \begin{bmatrix} Qz + A^\top\lambda + B^\top\mu \\ Az \\ Bz \end{bmatrix} - \begin{bmatrix} -N_M(P_1 \cup P_2; z) \\ N(\mathbb{R}^n_+; \lambda) \\ 0 \end{bmatrix}.$$

We can apply Theorem 3.3 to $-\mathcal{F}$ *with* $m = 3$*,* $C_1 = P_1 \cup P_2$*,* $C_2 = \mathbb{R}^n_+$*,* $C_3 = \mathbb{R}^n$*, and*

$$A_1 = \begin{bmatrix} -Q & A^\top & B^\top \end{bmatrix}, \quad A_2 = \begin{bmatrix} A & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} B & 0 & 0 \end{bmatrix},$$

to conclude that gph $\mathcal{F}$ *is the union of a finitely many polyhedrons.*

3.4 M-stationarity: local uniqueness in nonlinear programming with geometric constraints

The purpose of this section is to analyze local uniqueness property and error estimates for a class of quadratic optimization problems under a geometric constraints set given by the union of two convex polyhedra, without constraint qualificiation. Our results to be established presently extend those given by Hager and Gowda in [31] where the case $P_1 = P_2$ to be a polyhedron is treated. More precisely, we consider the following problem

$$\mu_0 \doteq \min\{f(z) : \ Bz = s, \ z \in P_1 \cup P_2\}, \tag{3.4.1}$$

where, for $i = 1, 2$, $P_i \doteq \{z \in \mathbb{R}^n : \ H_i z \leq d_i\}$, $B \in \mathbb{R}^{k\times n}$, $H_i \in \mathbb{R}^{m_i \times n}$ are given matrices with real entries; the vectors d_i and s are also given. Problem (3.4.1) has the equivalent form:

$$\min\Big\{\min[f(z) : \ Bz = s, z \in P_1]; \min[f(z) : \ Bz = s, z \in P_2]\Big\}. \tag{3.4.2}$$

So, as expected, the study of problem (3.4.1) is linked to the following problems:

$$\min\{f(z) : \ Bz = s, z \in P_1\}; \tag{3.4.3}$$

$$\min\{f(z) : \ Bz = s, z \in P_2\}. \tag{3.4.4}$$

We will deal with M-*stationarity* since limiting normal cones are involved. Following [47, 15], given a feasible point z to problem (3.4.1), we say that $\bar{z}$ is called a M-stationary point if there exists $\mu \in \mathbb{R}^k$ such that

$$\nabla f(z) + B^\top \mu \in -N_M(P_1 \cup P_2; z), \; Bz = s. \tag{3.4.5}$$

Some constraint qualification conditions implying (3.4.5) can be found in [44], [45, Section 5.1] and [56, Corollary 6.15], as well as in [46, Proposition 6.4] and [48].

It is straightforward to check that (say, z is a KKT-point for the first minimization problem appearing inside the brackets in (3.4.2))

$$\nabla f(z) + B^\top \mu \in -N(P_1; z), \; Bz = s, \tag{3.4.6}$$

is equivalent to:

$$\exists \, \gamma_1 \in \mathbb{R}^{m_1}_+, \; \nabla f(z) + B^\top \mu + H_1^\top \gamma_1 = 0, \; Bz = s, \; H_1 z - d_1 \in N(\mathbb{R}^{m_1}_+; \gamma_1); \tag{3.4.7}$$

Similarly,

$$\nabla f(z) + B^\top \mu \in -N(P_2; z), \; Bz = s, \tag{3.4.8}$$

is equivalent to:

$$\exists \, \gamma_2 \in \mathbb{R}^{m_2}_+, \; \nabla f(z) + B^\top \mu + H_2^\top \gamma_2 = 0, \; Bz = s, \; H_2 z - d_2 \in N(\mathbb{R}^{m_2}_+; \gamma_2). \tag{3.4.9}$$

In what follows, the + and 0 subscripts are used to denote the subvectors associated with those indices i for which $(\bar{\gamma}_j)_i > 0$, $(\bar{\gamma}_j)_i = 0 = (H_j \bar{z} - d_j)_i$, $j = 1, 2$, respectively. Actually, (c) and (d) are interesting cases which deal with $\bar{z} \in P_1 \cap P_2$. Parts (a) and (b) are consequences of Proposition 1 in [31]. However, for (d) it requires the result of Corollary 3.2.

Theorem 3.4. *Let f be a function twice continuously differentiable in a neighborhood of $\bar{z}$. The following statements hold.*

(a) *If $\bar{z} \in P_1 \setminus P_2$, $(\bar{z}, \bar{\mu}, \bar{\gamma}_1)$ is a solution to (3.4.7) and there exists a scalar $\alpha > 0$*

such that

$$w^\top \nabla^2 f(\bar{z}) w \geq \alpha \|w\|^2,$$

for all $w \neq 0$ *satisfying* $Bw = 0$, $(H_1 w)_+ = 0$ *and* $(H_1 w)_0 \leq 0$, *then* $N_M(P_1 \cup P_2; \bar{z}) = N(P_1; \bar{z})$ *and there exists a neighborhood* $\mathcal{N}$ *of* $\bar{z}$ *with the property that if* $z \in \mathcal{N}$ *and* (z, μ) *is a solution to* (3.4.5), *then* $z = \bar{z}$.

(*b*) *If* $\bar{z} \in P_2 \setminus P_1$, $(\bar{z}, \bar{\mu}, \bar{\gamma}_2)$ *is a solution to* (3.4.9) *and there exists a scalar* $\alpha > 0$ *such that*

$$w^\top \nabla^2 f(\bar{z}) w \geq \alpha \|w\|^2,$$

for all $w \neq 0$ *satisfying* $Bw = 0$, $(H_2 w)_+ = 0$ *and* $(H_2 w)_0 \leq 0$, *then* $N_M(P_1 \cup P_2; \bar{z}) = N(P_2; \bar{z})$ *and there exists a neighborhood* $\mathcal{N}$ *of* $\bar{z}$ *with the property that if* $z \in \mathcal{N}$ *and* (z, μ) *is a solution to* (3.4.5), *then* $z = \bar{z}$.

(*c*) *If* $\bar{z} \in P_1 \cap P_2$, $(\bar{z}, \bar{\mu}, \bar{\gamma}_1)$ *and* $(\bar{z}, \bar{\mu}, \bar{\gamma}_2)$ *are solutions to* (3.4.7) *and* (3.4.9), *respectively, and there exists a scalar* $\alpha > 0$ *such that*

$$w^\top \nabla^2 f(\bar{z}) w \geq \alpha \|w\|^2,$$

for all $w \neq 0$ *satisfying* $Bw = 0$, *and, either* $(H_1 w)_+ = 0$ *and* $(H_1 w)_0 \leq 0$, *or* $(H_2 w)_+ = 0$ *and* $(H_2 w)_0 \leq 0$, *then there exists a neighborhood* $\mathcal{N}$ *of* $\bar{z}$ *with the property that if* $z \in \mathcal{N}$ *and* (z, μ) *is a solution to* (3.4.5), *then* $z = \bar{z}$.

(*d*) *If* $\bar{z} \in P_1 \cap P_2$, $(\bar{z}, \bar{\mu}, \bar{\gamma}_1)$ *is a solution to* (3.4.7), $\beta \doteq \mathrm{dist}(\nabla f(\bar{z}); -N(P_2; \bar{z}) - \mathrm{Img}(B^\top)) > 0$, *and there exists a scalar* $\alpha > 0$ *such that*

$$w^\top \nabla^2 f(\bar{z}) w \geq \alpha \|w\|^2,$$

for all $w \neq 0$ *satisfying* $Bw = 0$, $(H_1 w)_+ = 0$ *and* $(H_1 w)_0 \leq 0$, *then there exists a neighborhood* $\mathcal{N}$ *of* $\bar{z}$ *with the property that if* $z \in \mathcal{N}$ *and* (z, μ) *is a solution to* (3.4.5), *then* $z = \bar{z}$.

Proof. (*a*): By (*b*2) of Proposition 2.2, $N_M(P_1 \cup P_2; \bar{z}) = N_M(P_1; \bar{z}) = N(P_1; \bar{z})$, and therefore the desired conclusion is a consequence of [31, Proposition 1].
(*b*): It is similar to (*a*).

(c): Applying [31, Proposition 1], one concludes that there exists a neighborhood $\mathcal{N}$ of $\bar{z}$ such that both systems (3.4.6) and (3.4.8) have no solution (z, μ) with $z \in \mathcal{N}$ other than $z = \bar{z}$. Since $N_M(P_1 \cup P_2; \bar{z}) \subseteq N(P_1; \bar{z}) \cup N(P_2; \bar{z})$, we infer that (3.4.5) has no solution (z, μ) with $z \in \mathcal{N}$ other than $z = \bar{z}$.
(d): From (b) of Corollary 3.2 and continuity of ∇f around $\bar{z}$ it follows that there exists a neighborhood $\mathcal{N}_1$ of z such that for all $z \in \mathcal{N}_1$, one has $N(P_2; z) \subseteq N(P_2; \bar{z})$ and

$$\operatorname{dist}(\nabla f(z); -N(P_2; \bar{z}) - \operatorname{Img}(B^\top)) \geq \frac{\beta}{2} > 0. \tag{3.4.10}$$

We also obtain, for all $z \in \mathcal{N}_1$,

$$-N(P_2; z) - \operatorname{Img}(B^\top) \subseteq -N(P_2; \bar{z}) - \operatorname{Img}(B^\top). \tag{3.4.11}$$

Combining (3.4.11) and (3.4.10), we get

$$\operatorname{dist}(\nabla f(z); -N(P_2; z) - \operatorname{Img}(B^\top)) \geq \frac{\beta}{2} > 0, \quad \forall\, z \in \mathcal{N}_1.$$

or, equivalently

$$\nabla f(z) + B^\top \mu \notin -N(P_2; z), \quad \forall\, z \in \mathcal{N}_1, \ \forall\, \mu \in \mathbb{R}^k. \tag{3.4.12}$$

On the other hand, by assumption, $(\bar{z}, \bar{\mu}, \bar{\gamma}_1)$ is a solution to (3.4.7); thus, we apply [31, Proposition 1] to conclude that there exists a neighborhood $\mathcal{N}_2$ of z such that system (3.4.6) has no solution (z, μ) with $z \in \mathcal{N}_2$ other than $z = \bar{z}$. The latter together with (3.4.12) allows us to infer that both systems (3.4.6) and (3.4.8) have no solution (z, μ) with $z \in \mathcal{N} \doteq \mathcal{N}_1 \cap \mathcal{N}_2$ other than $z = \bar{z}$. Again, since $N_M(P_1 \cup P_2; \bar{z}) \subseteq N(P_1; \bar{z}) \cup N(P_2; \bar{z})$, we finally deduce that (3.4.5) has no solution (z, μ) with $z \in \mathcal{N}$ other than $z = \bar{z}$. □

We now deal with the parametric quadratic optimization problem:

$$\min \left\{ \frac{1}{2} z^\top Q z - \varphi^\top z : \ Bz = s, \ z \in P_1 \cup P_2 \right\}, \tag{3.4.13}$$

where $Q = Q^\top$ is an $n \times n$ symmetric matrix, $B \in \mathbb{R}^{k \times n}$ and $\varphi \in \mathbb{R}^n$ and $s \in \mathbb{R}^k$ are considered as parameters. Here, we use again M-stationary points, that is, we deal with the optimality condition

$$\Psi = \begin{pmatrix} \varphi \\ s \end{pmatrix} \in \mathcal{F}(z, \mu), \tag{3.4.14}$$

where

$$\mathcal{F}(z,\mu) \doteq \begin{pmatrix} Qz + B^\top \mu \\ Bz \end{pmatrix} - \begin{pmatrix} -N_M(P_1 \cup P_2; z) \\ 0 \end{pmatrix}.$$

As before, we consider the corresponding problems (3.4.3) and (3.4.4) for $f(z) = \frac{1}{2} z^\top Q z - \varphi^\top z$ and write their first-order necessary optimality conditions:

$$\exists \gamma_1 \in \mathbb{R}^{m_1}_+,\ Qz - \varphi + B^\top \mu + H_1^\top \gamma_1 = 0,\ Bz = s,\ H_1 z - d_1 \in N(\mathbb{R}^{m_1}_+; \gamma_1) \tag{3.4.15}$$

and

$$\exists \gamma_2 \in \mathbb{R}^{m_2}_+,\ Qz - \varphi + B^\top \mu + H_2^\top \gamma_2 = 0,\ Bz = s,\ H_2 z - d_2 \in N(\mathbb{R}^{m_2}_+; \gamma_2). \tag{3.4.16}$$

In view of Proposition 3.4, we only consider the interesting case $\bar{z} \in P_1 \cap P_2$.

Proposition 3.3. *Let $\bar{z} \in P_1 \cap P_2$ and consider any $\bar{\Psi} = (\bar{\varphi}, \bar{s})$. The following statements hold.*

(a) *If $(\bar{z}, \bar{\mu}, \bar{\gamma}_1)$ and $(\bar{z}, \bar{\mu}, \bar{\gamma}_2)$ are solutions to (3.4.15) and (3.4.16) respectively, corresponding to the same $\bar{\Psi}$, and there exists a scalar $\alpha > 0$ such that*

$$w^\top Q w \geq \alpha ||w||^2,$$

for all $w \neq 0$ such that $Bw = 0$, and either $(H_1 w)_+ = 0$ and $(H_1 w)_0 \leq 0$, or $(H_2 w)_+ = 0$ and $(H_2 w)_0 \leq 0$, then there exists a scalar $\beta > 0$, neighborhoods $\mathcal{N}_{\bar{z}}$ of $\bar{z}$ and $\mathcal{N}_{\bar{\Psi}}$ of $\bar{\Psi}$ such that if $z \in \mathcal{N}_{\bar{z}}$, (z, μ) solution to (3.4.14) corresponding *to* $\Psi \in \mathcal{N}_{\bar{\Psi}}$*, then*

$$||z - \bar{z}|| \leq \beta||\Psi - \bar{\Psi}||.$$

(*b*) *If* $\text{dist}(Q\bar{z} - \bar{\varphi}; -N(P_2; \bar{z}) - \text{Img}(B^\top)) > 0$*,* $(\bar{z}, \bar{\mu}, \bar{\gamma}_1)$ *is a solution to* (3.4.15) *corresponding to* $\bar{\Psi}$ *and there exists a scalar* $\alpha > 0$ *such that*

$$w^\top Q w \geq \alpha||w||^2,$$

for all $w \neq 0$ *such that* $Bw = 0$*,* $(H_1 w)_+ = 0$ *and* $(H_1 w)_0 \leq 0$*, then there exists a scalar* $\beta > 0$*, neighborhoods* $\mathcal{N}_{\bar{z}}$ *of* $\bar{z}$ *and* $\mathcal{N}_{\bar{\Psi}}$ *of* $\bar{\Psi}$ *such that if* $z \in \mathcal{N}_{\bar{z}}$*,* (z, μ) *solution to* (3.4.14) *corresponding to* $\Psi \in \mathcal{N}_{\bar{\Psi}}$*, then*

$$||z - \bar{z}|| \leq \beta||\Psi - \bar{\Psi}||. \tag{3.4.17}$$

Proof. First, we observe that under assumptions (*a*) or (*b*) we get local uniqueness by Theorem 3.4. From Theorem 3.3, it follows that $\mathcal{F}$ is a polyhedral multifunction in the sense that gph $\mathcal{F}$ is the finite union of polyhedra, and therefore so is $\mathcal{F}^{-1}$. Now consider the projection $\mathcal{P}(z, \mu) = z$, we know that $\mathcal{P} \circ \mathcal{F}^{-1}$ must also be polyhedral, since it is the composition of two polyhedral multifunctions. From the first lemma in [53], we get the following Lipschitz property for that polyhedral multifunction: there exist a neighborhood $\mathcal{N}_{\bar{\Psi}}$ of $\bar{\Psi}$ and a constant $\beta > 0$ such that

$$\mathcal{P} \circ \mathcal{F}^{-1}(\Psi) \subseteq \mathcal{P} \circ \mathcal{F}^{-1}(\bar{\Psi}) + \beta||\Psi - \bar{\Psi}||B(0; 1), \quad \forall\ \Psi \in \mathcal{N}_{\bar{\Psi}}. \tag{3.4.18}$$

In case $\mathcal{P} \circ \mathcal{F}^{-1}(\bar{\Psi}) = \{\bar{z}\}$, (3.4.17) trivially holds. Otherwise, by Proposition 3.4, there exists a neighborhood of $\bar{z}$, $\mathcal{N}_{\bar{z}}$, such that

$$(\mathcal{P} \circ \mathcal{F}^{-1})(\bar{\Psi}) \cap \mathcal{N}_{\bar{z}} = \{\bar{z}\}.$$

We can assume, by reducing $\mathcal{N}_{\bar{z}}$ if necessary, that

$$||z - \bar{z}|| < \frac{1}{3}\text{dist}(\bar{z}, (\mathcal{P} \circ \mathcal{F}^{-1})(\bar{\Psi}) \setminus \{\bar{z}\}), \quad \forall\ z \in \mathcal{N}_{\bar{z}}. \tag{3.4.19}$$

Now, we consider $\mathcal{N}_* \subseteq \mathcal{N}_{\bar{\Psi}}$ such that

$$\{\bar{z}\} + \beta||\Psi - \bar{\Psi}||B(0;1) \subseteq \mathcal{N}_{\bar{z}}, \quad \forall\ \Psi \in \mathcal{N}_*.$$

Thus, for all $\Psi \in \mathcal{N}_*$ and all $z \in \mathcal{P} \circ \mathcal{F}^{-1}(\Psi)$, (3.4.18) means

$$z \in (\{\bar{z}\} \cup ((\mathcal{P} \circ \mathcal{F}^{-1})(\bar{\Psi}) \setminus \{\bar{z}\}) + \beta||\Psi - \bar{\Psi}||B(0;1),$$

which says that the distance from z to either $\bar{z}$ or $((\mathcal{P} \circ \mathcal{F}^{-1})(\bar{\Psi}) \setminus \{\bar{z}\})$ must be less than $\beta||\Psi - \bar{\Psi}||$; then (3.4.19) gives

$$z \in \{\bar{z}\} + \beta||\Psi - \bar{\Psi}||B(0;1),$$

which amount to writing $||z - \bar{z}|| \leq \beta||\Psi - \bar{\Psi}||$, and the proof is complete. □

3.5 The general problem: M-stationarity and error estimates

Let us consider the following constrained optimization problem:

$$\min\{f(z) : \quad h(z) = 0, \quad z \in P_1 \cup P_2\}, \tag{3.5.1}$$

where f and h are functions twice continuously differentiable in a neighborhood of $\bar{z}$. As usual, the Lagrangian $\mathcal{L}$ associated to (3.5.1) is defined by:

$$\mathcal{L}(z, \mu) = f(z) + \mu^\top h(z).$$

We also need to define

$$T(z, \mu) \doteq \begin{pmatrix} \nabla_z \mathcal{L}(z, \mu) \\ h(z) \end{pmatrix}, \quad F(z) \doteq \begin{pmatrix} -N_M(P_1 \cup P_2; z) \\ 0 \end{pmatrix}.$$

Here, $\nabla_z \mathcal{L}(z, \mu) = \nabla f(z) + Dh(z)^\top \mu$, where $h = (h_1, \ldots, h_k)$ and $Dh(z) \in \mathbb{R}^{k \times n}$ is the Jacobian matrix of h at z, that is, the matrix whose rows are the transpose of the vector $\nabla h_i(z)$.

We continue to deal with M-stationarity with respect to problem (3.5.1), i.e., points z for which there is $\mu \in \mathbb{R}^k$ such that

$$T(z, \mu) \in F(z).$$

For a M-stationary point $\bar{z}$, define $\mathcal{M}(\bar{z}) \doteq \{\mu : \nabla_z \mathcal{L}(\bar{z}, \mu) \in -N_M(P_1 \cup P_2; \bar{z})\}$. In what follows, $\nabla_z^2 \mathcal{L}(\bar{z}, \bar{\mu})$ stands for the Hessian of $\mathcal{L}(\cdot, \bar{\mu})$ at $\bar{z}$. Observe again the importance of Corollary 3.2 in the following result.

Theorem 3.5. *Let f, g, h be functions as above with $\bar{z}$ being a M-stationary point to problem (3.5.1), $\bar{\mu} \in \mathcal{M}(\bar{z})$. Then, there exist a neighborhood $\mathcal{N}$ of $(\bar{z}, \bar{\mu})$, constants γ and $\delta > 0$ with the property that for every $(z, \mu) \in \mathcal{N}$ and for each $y \in \mathbb{R}^n$ with $\|y\| \leq \delta$ and*

$$T(z, \mu) + (y, 0) \in F(z), \ (0 \in \mathbb{R}^k), \tag{3.5.2}$$

we have

$$\|z - \bar{z}\| + \operatorname{dist}(\mu; \mathcal{M}(\bar{z})) \leq \gamma \|y\|,$$

under any of the following circumstances:

(*a*) *(Set $B = Dh(\bar{z})$). If $\bar{z} \in P_1 \cap P_2$, $(\bar{z}, \bar{\mu}, \bar{\gamma}_1)$ and $(\bar{z}, \bar{\mu}, \bar{\gamma}_2)$ are solutions to (3.4.7) and (3.4.9), respectively, and there exists $\alpha > 0$ such that $w^\top \nabla_z^2 \mathcal{L}(\bar{z}, \bar{\mu}) w \geq \alpha \|w\|^2$, for all $w \neq 0$ satisfying $Bw = 0$, and either $(H_1 w)_+ = 0$ and $(H_1 w)_0 \leq 0$, or $(H_2 w)_+ = 0$ and $(H_2 w)_0 \leq 0$.*

(*b*) *(Set $B = Dh(\bar{z})$). If $\bar{z} \in P_1 \cap P_2$, $(\bar{z}, \bar{\mu}, \bar{\gamma}_1)$ is a solution to (3.4.7),*

$$\operatorname{dist}(\nabla f(\bar{z}); -N(P_2; \bar{z}) - \operatorname{Img}(B^\top)) > 0,$$

and there exists $\alpha > 0$ such that $w^\top \nabla_z^2 \mathcal{L}(\bar{z}, \bar{\mu}) w \geq \alpha \|w\|^2$, for all $w \neq 0$ satisfying $Bw = 0$, $(H_1 w)_+ = 0$ and $(H_1 w)_0 \leq 0$.

Proof. Let us consider the optimization problem (3.4.13) with data $Q = \nabla_z^2 \mathcal{L}(\bar{z}, \bar{\mu})$ and $B = Dh(\bar{z})$ being the Jacobian matrix of h at $\bar{z}$, where $\Psi \doteq (\varphi, s)$ is the parameter. We put $\bar{\Psi} \doteq (\bar{\varphi}, \bar{s}) = (Q\bar{z} - \nabla f(\bar{z}), B\bar{z})$. Thus, Proposition 3.3 ensures the existence of $\beta > 0$ such that

$$||z - \bar{z}|| \le \beta ||\Psi - \bar{\Psi}||, \tag{3.5.3}$$

for all z close to $\bar{z}$ so that (z, μ) is solution to (3.4.14) for $\Psi = (\varphi, s)$ close to $\bar{\Psi}$. Take any Ψ of the form $L(z, \mu) - T(z, \mu) - (y, 0)$ with z, μ and y satisfying (3.5.2), where

$$L(z, \mu) \doteq \begin{pmatrix} Qz + B^\top \mu \\ Bz \end{pmatrix}.$$

Then

$$\Psi = \begin{pmatrix} Qz + B^\top \mu \\ Bz \end{pmatrix} - \begin{pmatrix} \nabla_z \mathcal{L}(z, \mu) + y \\ h(z) \end{pmatrix} \in \begin{pmatrix} Qz + B^\top \mu \\ Bz \end{pmatrix} - \begin{pmatrix} -N_M(P_1 \cup P_2; z) \\ 0 \end{pmatrix}$$

that is, (z, μ) is a solution to (3.4.14) for Ψ as above. Observe that $\bar{\Psi} = L(\bar{z}, \mu') - T(\bar{z}, \mu')$ for all μ' since $h(\bar{z}) = 0$, and obviously $\bar{\Psi} \in \mathcal{F}(\bar{z}, \bar{\mu})$.

From the continuous differentiability of the functions f, g and h, we know that $\Psi = L(z, \mu) - T(z, \mu) - y$ is close to $\bar{\Psi} = L(\bar{z}, \bar{\mu}) - T(\bar{z}, \bar{\mu})$ when (z, μ) is close to $(\bar{z}, \bar{\mu})$ and y is close to the origin.
By continuous differentiability, for any $0 < \epsilon$ there also exists a neighborhood $\mathcal{N}$ of $(\bar{z}, \bar{\mu})$ such that for all $(z, \mu) \in \mathcal{N}$ one has

$$||L(z, \mu) - T(z, \mu) - L(\bar{z}, \mu) + T(\bar{z}, \mu)|| = ||L(z - \bar{z}, 0) - T(z, \mu) + T(\bar{z}, \mu)|| \le \epsilon ||z - \bar{z}||.$$

By substituting it in (3.5.3), one obtains

$$||z - \bar{z}|| \le \beta ||\Psi - \bar{\Psi}|| \le \beta\epsilon ||z - \bar{z}|| + \beta ||y||.$$

By taking $0 < \epsilon$ such that $\beta\epsilon < 1$, we have $||z - \bar{z}|| \le \frac{\beta}{1 - \beta\epsilon} ||y||$. It turns out, by keeping the same notation for β, that

$$\|z - \bar{z}\| \le \beta \|y\|. \tag{3.5.4}$$

On the other hand, we can write

$$-N_M(P_1 \cup P_2; \bar{z}) = \bigcup_{j=1}^{p} C_j,$$

where

$$C_j = \{v \in \mathbb{R}^n : \quad \langle -a_{ij}, v\rangle \le b_{ij} \ \forall\ i = 1, \dots, L_j\}, \quad j = 1, \dots, p.$$

Here, $p, L_j \in \mathbb{N}$, $b_{ij} \in \mathbb{R}$ and $a_{ij} \in \mathbb{R}^n$. Thus, the inclusion

$$\nabla f(\bar{z}) + Dh(\bar{z})^\top \mu \in -N_M(P_1 \cup P_2; \bar{z})$$

becomes

$$\langle a_{ij}, \nabla f(\bar{z}) + Dh(\bar{z})^\top \mu\rangle \le b_{ij},$$

for some $j = 1, \dots, p$ and for all $i = 1, \dots, L_j$. Applying the Hoffman result in [33], to each system, we get

$$\operatorname{dist}(\mu; \mathcal{M}(\bar{z})) \le \beta \sum_{i \in I_+^j} \Big(\langle a_{ij}, \nabla f(\bar{z}) + Dh(\bar{z})^\top \mu\rangle - b_{ij}\Big), \tag{3.5.5}$$

where $I_+^j = \Big\{i \in \{1, \dots, L_j\} : \ \langle a_{ij}, \nabla f(\bar{z}) + Dh(\bar{z})^\top \mu\rangle - b_{ij} > 0\Big\}$ for any vector μ and any $j = 1, \dots, p$ (since the distance to $-N_M(P_1 \cup P_2; \bar{z})$ is smaller than the distance to any subset C_j).
From Part (b) of Corollary 3.2, it follows that $-N_M(P_1 \cup P_2; z) \subseteq -N_M(P_1 \cup P_2; \bar{z})$ for $z \in P_1 \cup P_2$ close enough to $\bar{z}$. Thus, for those z such that (z, μ) and y satisfy (3.5.2), i.e.,

$$\nabla f(z) + Dh(z)^\top \mu + y \in -N_M(P_1 \cup P_2; z),$$

one infers that for some $j \in \{1, \dots, p\}$,

$$\langle a_{ij}, \nabla f(z) + Dh(z)^\top \mu + y\rangle \le b_{ij}, \quad \forall\ i = 1, \dots, L_j.$$

Therefore,

$$\langle a_{ij}, \nabla f(\bar{z}) + Dh(\bar{z})^\top \mu\rangle - b_{ij} =$$

$$
\begin{aligned}
&= \langle a_{ij}, \nabla f(z) + Dh(z)^\top \mu + (\nabla f(\bar{z}) + Dh(\bar{z})^\top \mu - \nabla f(z) - Dh(z)^\top \mu) \\
&\quad + y - y\rangle - b_{ij} \\
&\leq \langle a_{ij}, \nabla f(z) + Dh(z)^\top \mu + y\rangle - b_{ij} + \alpha\big(\max_{i=1,\dots,L_j} \|a_{ij}\|\big)\epsilon_1 - \langle a_{ij}, y\rangle \\
&\leq \alpha\big(\max_{i=1,\dots,L_j} \|a_{ij}\|\big)\epsilon_1 - \langle a_{ij}, y\rangle \leq \alpha\big(\max_{i=1,\dots,L_j} \|a_{ij}\|\big)L||z - \bar{z}|| - \langle a_{ij}, y\rangle
\end{aligned}
$$

where $\alpha = \max\{1, \|\mu\|\}$, $\epsilon_1 := \|\nabla f(z) - \nabla f(\bar{z})\| + \|Dh(z) - Dh(\bar{z})\|$ and L is twice the maximum between the local Lipschitz constants of ∇f and Dh at $\bar{z}$. Combining (3.5.5) and the last inequality, we obtain

$$
\operatorname{dist}(\mu; \mathcal{M}(\bar{z})) \leq \beta \sum_{i \in I_+^j} \Big(\alpha\big(\max_{i=1,\dots,L_j} \|a_{ij}\|\big)L||z - \bar{z}|| - \langle a_{ij}, y\rangle\Big) \leq \alpha_1(\|z - \bar{z}\| + \|y\|).
$$

where α_1 is a fixed constant depending on β, a neighborhood $\mathcal{N}$ of $(\bar{z}, \bar{\mu})$, the vectors a_{ij} (which comes from $P_1 \cup P_2$) and the local Lipschitz constants of ∇f and Dh at $\bar{z}$. Combining this with (3.5.4), the desired result is obtained. □

Chapter 4

Limiting normal cone to quadric surfaces, local uniqueness and error bound in M-stationarity

4.1 Introduction

Necessary optimality conditions - stationarity - plays a central role in identifying optimal solutions to nonlinear optimization problems. Among those conditions we distinguish the so called KKT's, which involves the notion of contingent (Bouligand-Severi) cone; or M-stationarity expressed in terms of the limiting (Mordukhovich or basic) normal cone introduced in [42].

In order to gain insight into the mathematical difficulties raised by this type of problems, let us consider the special case under a single inequality constraint and a nonconvex geometric constraint set X:

$$\min\{f(x): \ g(x) \leq 0, \ x \in X\}. \tag{4.1.1}$$

By assuming differentiability assumptions, the KKT-condition for the above problem reads as follows

$$\exists\, \lambda \geq 0 : \ \nabla f(\bar{x}) + \lambda \nabla g(\bar{x}) \in [T(X;\bar{x})]^*, \ \lambda g(\bar{x}) = 0 \tag{4.1.2}$$

where $T(X;\bar{x})$ is the contingent cone of X at $\bar{x}$ and $[T(X;\bar{x})]^*$ denotes its polar cone, as described in Chapter 2; whereas M-stationarity (see [48, 47, 3, 15]) expresses

$$\exists\, \lambda \geq 0 : \ \nabla f(\bar{x}) + \lambda \nabla g(\bar{x}) \in -N_M(X;\bar{x}), \ \lambda g(\bar{x}) = 0. \tag{4.1.3}$$

Due to the inclusion $[T(X;\bar{x})]^* \subseteq -N_M(X;\bar{x})$ (see [44, Corollary 1.11]), clearly (4.1.2) provides more information than (4.1.3), once one knows that $\bar{x}$ satisfies (4.1.2). On the other hand, one of the advantages in using M-stationarity may rely on algorithmic purposes since the graph of the set-valued mapping $N_M(X;\cdot)$ is closed, (see page 11 in [44]) contrary to the graph of $[T(X;\cdot)]^*$. The interesting case occurs when $g(\bar{x}) = 0$ and $\nabla f(\bar{x}) \neq 0$. A second major advantage regards the validity of Fritz John optimality conditions without extra assumption (see [43]); whereas the corresponding optimality condition which involves contingent cones holds if, and only if a suitable system admits no solution in co $T(X;\bar{x})$, as shown in [16, Theorem 3.1].

Another advantage of M-stationarity is exhibited by the following instance: take the data $f(x) = 5x_1 + x_2 + x_3^2$, $g(x) = -2x_1 + x_2 \leq 0$, $X = \{(x_1, x_2, x_3) : \ x_1^2 = x_2^2\}$ and $\bar{x} = 0 \in \mathbb{R}^3$. Here, $T(X;0) = X$ which is the union to two polyhedra, and so (see Proposition 4.1) its polar cone reduces to $[T(X;0)]^* = \{0\}$. One verifies that $\bar{x} = 0$ is the only solution to problem (4.1.1) and there is no λ satisfying (4.1.2); even more, there is no pair $(\bar{x}, \lambda)$ with $\bar{x}$ being feasible and $\lambda \geq 0$ satisfying (4.1.2). So, KKT conditions are of no help to find a minimizer. However, (4.1.3), that is, M-stationarity, holds for $\lambda = 2$ and $\bar{x} = 0$. Moreover, we must point out that there is no $(x, \lambda, \mu) \in \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}$ with feasible x verifying the standard KKT condition $\nabla f(x) + \lambda \nabla g(x) + \mu \nabla q(x) = 0$ and the complementary slackness condition $\lambda g(x) = 0$. Characterizations for the fulfillment of (4.1.2) were established in [22] with no requirement of any constraint qualification.

Motivated by our previous example, the purpose of this chapter is twofold. Firstly, we compute the limiting normal cone to the union of two quadric surfaces, that is,

sets of the form $\{x \in \mathbb{R}^n : \; q_i(x) = 0\}$, $i = 1, 2$, where $q_i(x) \doteq \frac{1}{2}x^\top C_i x + d_i^\top x + \alpha_i$ and C_i is indefinite.
Secondly, we will establish local uniqueness to the problem

$$\min\{f(x) : \; Ax \leq a, \; Bx = b, x \in X\}, \tag{4.1.4}$$

where X is a quadric surface, A, B are given real matrices, and a, b are vectors. Certainly, our interest is when $T(X; \bar{x})$ is not convex and, X or $N_M(X; \bar{x})$, is the union of non-polyhedral sets (for instance $X = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = x_3^2\}$), and so most of the results existing in the literature are not applicable in our case, [32, 11, 24]. Furthermore, problem (4.1.4) cover situations where the Magasarian-Fromovitz constraint qualification fails.
As a side result, we identify three major assumptions under which error bound at M-stationary points is based.

4.2 Computing the limiting normal cone to quadric surfaces

This section deals with the computation of the limiting normal cone to quadrics, such formulae will be employed in the study of local uniqueness of M-stationary points.

4.2.1 The case of a single quadric surface

We deal with sets determined by a single quadric surface:

$$C = \{x \in \mathbb{R}^n : \; q(x) = 0\},$$

where $q(x) = \frac{1}{2}x^\top Ax + a^\top x + \alpha$ with $A \in \mathbb{R}^{n \times n}$ being a real symmetric matrix, $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Let $\bar{x} \in C$. It is straightforward to deduce

$$C = \{x \in \mathbb{R}^n : \; \nabla q(\bar{x})^\top (x - \bar{x}) + \frac{1}{2}(x - \bar{x})^\top A(x - \bar{x}) = 0\} = C_{00} + \bar{x},$$

where $C_{00} = \{v \in \mathbb{R}^n : \ \nabla q(\bar{x})^\top v + \frac{1}{2} v^\top A v = 0\}$.
Set

$$C_0 \doteq \{v \in \mathbb{R}^n : \ v^\top A v = 0\}.$$

In case $\nabla q(\bar{x}) = 0$, certainly $C_{00} = C_0$.

Some properties of C_0 are listed below.

Proposition 4.1. *Under the above data, the following statements hold:*

(*a*) $C_0 = -C_0$.

(*b*) C_0 *is closed.*

(*c*) $C_0 + \text{Ker } A = C_0$.

(*d*) *If A is indefinite, then* span $C_0 =$ co $C_0 = \mathbb{R}^n$.

(*e*) $A(C_0) = A(C_0 \cap (\text{Ker } A)^\perp) = \{0\} \cup A(C_0 \setminus \text{Ker } A)$ *is closed.*

Proof. (*a*), (*b*) and (*c*) are straightforward.
(*d*): It is Lemma 3.10 in [50].
(*e*): Let $y_k \in A(C_0)$ such that $y_k \to y$. Thus $y_k = Ac_k$ for some $c_k \in C_0$ and we can write $c_k = c'_k + c^+_k$ with $c'_k \in \text{Ker } A$ and $c^+_k \in (\text{Ker } A)^\perp$. By assumption, $c^+_k = -c'_k + c_k \in \text{Ker } A + C_0 \subseteq C_0$, that is, $c^+_k \in C_0$. Thus, $Ac_k = Ac^+_k$ and $c^+_k \in C_0$. We claim that $\sup_k \|c^+_k\| < +\infty$. If not, we can assume, up to a subsequence, that $\|c^+_k\| \to +\infty$ and $\frac{c^+_k}{\|c^+_k\|} \to v \in (\text{Ker } A)^\perp$. This implies $A\left(\frac{c^+_k}{\|c^+_k\|}\right) \to Av$, but

$$A\left(\frac{c^+_k}{\|c^+_k\|}\right) = \frac{1}{\|c^+_k\|} A(c^+_k) = \frac{y_k}{\|c^+_k\|} \to 0$$

since y_k is bounded. Hence $Av = 0$, which means that $v \in \text{Ker } A \cap (\text{Ker } A)^\perp = \{0\}$ yielding a contradiction. Therefore, $\sup_k \|c^+_k\| < +\infty$ and, up to a subsequence again, $c^+_k \to c_0 \in (\text{Ker } A)^\perp \cap C_0$, consequently $y = Ac_0 \in A(C_0 \cap (\text{Ker } A)^\perp)$, and so $y \in A(C_0)$. In other words, we actually we proved $\overline{A(C_0)} = A(C_0) = A(C_0 \cap (\text{Ker } A)^\perp)$. The equality $A(C_0) = \{0\} \cup A(C_0 \setminus \text{Ker } A)$ is straightforward. □

The following proposition collects most of the main formulae for the limiting normal cone to a quadric surface. These properties will be used in subsequent sections.

Proposition 4.2. *Under the above data, it holds:*

(a) *Assume that* $\nabla q(\bar{x}) \neq 0$. *Then*

$(a1)$ $T(C;\bar{x}) = \nabla q(\bar{x})^{\perp}$, $[T(C;\bar{x})]^* = \mathbb{R}\nabla q(\bar{x})$.

$(a2)$ $N_M(C;\bar{x}) = \hat{N}(C;\bar{x}) = \mathbb{R}\nabla q(\bar{x})$.

(b) *Assume that* $\nabla q(\bar{x}) = 0$. *Then*

$(b1)$ $T(C;\bar{x}) = \{v \in \mathbb{R}^n : \ v^{\top}Av = 0\} = C_0 = C - \bar{x}$.

$(b2)$ *If* A *is either positive or negative semidefinite, it holds*

$$T(C;\bar{x}) = \text{Ker } A, \ \hat{N}(C;\bar{x}) = N_M(C;\bar{x}) = A(\mathbb{R}^n).$$

$(b3)$ *If* A *is indefinite, then* $\hat{N}(C;\bar{x}) = \{0\} = \mathbb{R}\nabla q(\bar{x})$.

$(b4)$ *If* A *is indefinite, then*

$$N_M(C;\bar{x}) = A(C_0) = A(C) + a = \nabla q(C).$$

Proof. $(a1)$ may be found, for instance, in [12]; $(a2)$ follows easily from $(a1)$. $(b1)$ is directly obtained; the first part of $(b2)$ follows from $(b1)$ since $v^{\top}Av = 0$ if, only if $Av = 0$. Thus $-\hat{N}(C;\bar{x}) = (T(C;\bar{x}))^* = (\text{Ker } A)^{\perp} = A(\mathbb{R}^n)$. We now compute $N_M(C;\bar{x})$:

$$\begin{aligned} N_M(C;\bar{x}) &= \operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ x\in C}} \hat{N}(C;x) \\ &= \operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ x\in C\\ \nabla q(x)=0}} \hat{N}(C;x) \ \cup \ \operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ x\in C\\ \nabla q(x)\neq 0}} \hat{N}(C;x) \\ &= \operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ x\in C\\ \nabla q(x)=0}} A(\mathbb{R}^n) \ \cup \ \operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ x\in C\\ \nabla q(x)\neq 0}} \mathbb{R}\nabla q(x). \end{aligned}$$

But

$$\operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ x\in C\\ \nabla q(x)\neq 0}} \mathbb{R}\nabla q(x) = \operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ x\in C\\ \nabla q(x)\neq 0}} \mathbb{R}A(x-\bar{x}) \subseteq A(C_0) \subseteq A(\mathbb{R}^n).$$

Hence $N_M(C;\bar{x}) = A(\mathbb{R}^n)$.

$(b3)$: This is a consequence of the fact that co $C_0 =$ co $T(C;\bar{x}) = \mathbb{R}^n$ provided A is indefinite, see for instance [50, Lemma 3.10].

$(b4)$: In view of $(a2)$ and $(b3)$, and using the equality $\nabla q(x) = A(x-\bar{x}) + \nabla q(\bar{x})$ along with the change of variable $v = x - \bar{x}$, one obtains

$$\begin{aligned} N_M(C;\bar{x}) &= \operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ x\in C}} \hat{N}(C;x) = \operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ x\in C}} \mathbb{R}\nabla q(x) = \operatorname*{Lim\,sup}_{\substack{v\to 0\\ v\in C_0}} \mathbb{R}(Av) \\ &= \{u\in\mathbb{R}^n : \exists v_k \in C_0, v_k \to 0, \exists r_k\in\mathbb{R},\ r_kAv_k \to u\} \\ &= \{u\in\mathbb{R}^n : \exists v_k' \in C_0 \text{ with } Av_k' \to u\} \\ &= \overline{A(C_0)} = A(C_0). \end{aligned}$$

The last equality follows from Proposition 4.1. □

Remark 4.1. *Given any symmetric matrix M, the matrix $M^\dagger$ denotes its Moore-Penrose inverse, or simply, its pseudoinverse. It satisfies*

$$M^\dagger = M^\dagger M M^\dagger;\ \ MM^\dagger M = M.$$

Thus if $\mathcal{C}_0(M) \doteq \{x\in\mathbb{R}^n :\ x^\top Mx = 0\}$, then one immediately gets

$$x\in\mathcal{C}_0(M) \Longleftrightarrow Mx\in\mathcal{C}_0(M^\dagger);\ x\in\mathcal{C}_0(M^\dagger) \Longleftrightarrow M^\dagger x \in \mathcal{C}_0(M).$$

Hence

$$x\in\mathcal{C}_0(M) \Longleftrightarrow Mx\in\mathcal{C}_0(M^\dagger) \Longleftrightarrow M^\dagger Mx \in \mathcal{C}_0(M).$$

By applying these equivalences to $M = A$ with A satisfying $AA^\dagger = I$ (I is the identity matrix of order n), which implies $A^\dagger = A^{-1}$, and recalling that $C_0 = \mathcal{C}_0(A)$, one concludes

$$A(C_0) = \{v\in\mathbb{R}^n :\ v^\top A^{-1}v = 0\}.$$

4.2.2 Union of two quadric surfaces

We now consider the case of the union of two quadric surfaces:

$$C_i = \{x \in \mathbb{R}^n : \; q_i(x) \doteq \frac{1}{2} x^\top A_i x + a_i^\top x + \alpha_i = 0\}, \; i = 1, 2.$$

We now establish a formula for the limiting normal cone to $C_1 \cup C_2$ at $\bar{x} \in C_1 \cap C_2$, by starting firstly with the case $\nabla q_1(\bar{x}) \neq 0 \neq \nabla q_2(\bar{x})$.

Lemma 4.1. *Let $\bar{x} \in C_1 \cap C_2$ and assume that $\nabla q_1(\bar{x}) \neq 0 \neq \nabla q_2(\bar{x})$, then*

$$N_M(C_1 \cup C_2; \bar{x}) = \mathbb{R}\nabla q_1(\bar{x}) \cup \mathbb{R}\nabla q_2(\bar{x}).$$

Proof. First of all notice that for all x sufficiently close to $\bar{x}$, $\nabla q_1(x) \neq 0 \neq \nabla q_2(x)$. By Proposition 2.1, we have

$$\begin{aligned} -N_M(C_1 \cup C_2; \bar{x}) &= \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ x \in C_1 \setminus C_2}} [T(C_1; x)]^* \cup \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ x \in C_2 \setminus C_1}} [T(C_2; x)]^* \\ &\quad \cup \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ x \in C_1 \cap C_2}} [T(C_1 \cup C_2; x)]^*. \end{aligned} \tag{4.2.1}$$

To compute the first two Limsup of the right-hand side, we proceed as in the proof of Proposition 4.2. This leads to

$$\begin{aligned} \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ x \in C_1 \setminus C_2}} [T(C_1; x)]^* &= \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ x \in C_1 \setminus C_2}} \mathbb{R} A_1 x = \mathbb{R}\nabla q_1(\bar{x}), \\ \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ x \in C_2 \setminus C_1}} [T(C_2; x)]^* &= \mathbb{R}\nabla q_2(\bar{x}). \end{aligned}$$

For the last Limsup in (4.2.1), we obtain

$$\begin{aligned} \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ x \in C_1 \cap C_2}} [T(C_1 \cup C_2; x)]^* &= \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ x \in C_1 \cap C_2}} [T(C_1; x)]^* \cap [T(C_2; x)]^* \\ &\subseteq \mathbb{R}\nabla q_1(\bar{x}) \; \cap \; \mathbb{R}\nabla q_2(\bar{x}). \end{aligned}$$

This allows us to infer that $N_M(C_1 \cup C_2; \bar{x}) = \mathbb{R}\nabla q_1(\bar{x}) \cup \mathbb{R}\nabla q_2(\bar{x})$. □

The next lemma deals with the case when $\nabla q_1(\bar{x}) = \nabla q_2(\bar{x}) = 0$. This allows us to write, as stated in Chapter 2, the following

$$C_i - \bar{x} = C_{i0} \doteq \{x \in \mathbb{R}^n : \ x^\top A_i x = 0\} \quad i = 1, 2.$$

In what follows we set $\mathcal{S}^1 \doteq \{(\gamma_1, \gamma_2) \in \mathbb{R}^2 : \ \gamma_1^2 + \gamma_2^2 = 1\}$ and denote

$$\begin{aligned} K_0 \ &\doteq \ \Big\{ v \in C_{10} \cap C_{20} : \ \exists (\gamma_1, \gamma_2) \in \mathcal{S}^1, \ \gamma_1 A_1 v = \gamma_2 A_2 v \Big\} \\ &= \ \Big[\bigcup_{(\gamma_1,\gamma_2)\in\mathcal{S}^1} \mathrm{Ker}(\gamma_1 A_1 + \gamma_2 A_2) \Big] \cap (C_{10} \cap C_{20}). \end{aligned}$$

One deduces that: K_0 is a closed cone,

$$\overline{A_1(K_0)} = \overline{A_2(K_0)}, \ K_0 = -K_0, \ K_0 + (\mathrm{Ker}\ A_1 \cap \mathrm{Ker}\ A_2) \subseteq K_0. \tag{4.2.2}$$

Lemma 4.2. *Let C_1, C_2 be as above with A_1, A_2 being indefinite matrices. If $\bar{x} \in C_1 \cap C_2$ is such that $\nabla q_1(\bar{x}) = 0 = \nabla q_2(\bar{x})$, then*

$$N_M(C_1 \cup C_2; \bar{x}) = \overline{A_1(C_{10} \setminus C_{20})} \cup \overline{A_2(C_{20} \setminus C_{10})} \cup \overline{A_1(K_0)}.$$

Proof. By using the equalities $C_i - \bar{x} = C_{i0}$ for $i = 1, 2$, one obtains

$$N_M(C_1 \cup C_2; \bar{x}) = N_M(C_{10} \cup C_{20}; 0).$$

By Proposition 2.1, we have

$$\begin{aligned} -N_M(C_{10} \cup C_{20}; 0) \ &= \ \operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\setminus C_{20}}} [T(C_{10}; x)]^* \cup \operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{20}\setminus C_{10}}} [T(C_{20}; x)]^* \\ &\quad \cup \operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\cap C_{20}}} [T(C_{10} \cup C_{20}; x)]^*. \end{aligned} \tag{4.2.3}$$

To compute the first two Limsup of the right-hand side, we proceed as in the proof of Proposition 4.2. This leads to

$$\begin{aligned}
\operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\setminus C_{20}}}[T(C_{10};x)]^* &= \operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\setminus C_{20}}} \mathbb{R}A_1x = \overline{A_1(C_{10}\setminus C_{20})},\\
\operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{20}\setminus C_{10}}}[T(C_{20};x)]^* &= \overline{A_2(C_{20}\setminus C_{10})}.
\end{aligned}$$

It remains to compute the last Limsup in (4.2.3).

$$\begin{aligned}
\operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\cap C_{20}}}\ [T(C_{10}\cup C_{20};x)]^* &= \operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\cap C_{20}}}\ [T(C_{10};x)]^*\cap[T(C_{20};x)]^*\\
&= \operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\cap C_{20}}}\ \mathbb{R}A_1x\cap\mathbb{R}A_2x.
\end{aligned}$$

This Limsup reduces to consider those elements $x\in C_{10}\cap C_{20}$ for which A_1x and A_2x are linearly dependent (LD), in other words, those $x\in C_{10}\cap C_{20}$ such that there exist γ_1,γ_2 in $\mathbb{R}$ not both zero satisfying $v\in\operatorname{Ker}(\gamma_1A_1+\gamma_2A_2)$. This gives rise to the set K_0 satisfying (4.2.2). Thus

$$\operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\cap C_{20}}}\ [T(C_{10}\cup C_{20};x)]^* = \operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in K_0}} \mathbb{R}A_1x = \operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in K_0}} \mathbb{R}A_2x.$$

By virtue of (4.2.2), we proceed again as in the proof of Proposition 4.2 to conclude

$$\operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in K_0}} \mathbb{R}A_1x = \overline{A_1(K_0)} = \overline{A_2(K_0)}.$$

Hence

$$\begin{aligned}
N_M(C_1\cup C_2;\bar{x}) &= N_M(C_{10}\cup C_{20};0)\\
&= \overline{A_1(C_{10}\setminus C_{20})}\cup\overline{A_2(C_{20}\setminus C_{10})}\cup\overline{A_1(K_0)},
\end{aligned}$$

which completes the proof. □

Lastly, we deal with the remaining case.

Lemma 4.3. *Let $\bar{x}\in C_1\cap C_2$ with A_1 to be indefinite and $\nabla q_1(\bar{x})=0$, $\nabla q_2(\bar{x})\neq 0$.*

Then

$$C_1 \cap B(\bar{x};\epsilon) \nsubseteq C_2 \quad \forall \epsilon > 0.$$

Proof. Assume on the contrary that $C_1 \cap B(\bar{x};\epsilon) \subseteq C_2$ for some $\epsilon > 0$. Then $T(C_1;\bar{x}) \subseteq T(C_2;\bar{x})$, and therefore $\mathbb{R}\nabla q_2(\bar{x}) = [T(C_2;\bar{x})]^* \subseteq [T(C_1;\bar{x})]^* = \{0\}$, which is impossible. This proves the desired result. □

In view of the previous lemma, we split the case $\nabla q_1(\bar{x}) = 0 \neq \nabla q_2(\bar{x})$.

Lemma 4.4. *Let C_1, C_2 be as above with A_1 being an indefinite matrix. Assume that $\bar{x} \in C_1 \cap C_2$ and $\nabla q_1(\bar{x}) = 0 \neq \nabla q_2(\bar{x})$. The following assertions hold.*

(*a*) *If there exists $\epsilon > 0$ such that $C_2 \cap B(\bar{x};\epsilon) \subseteq C_1$, then*

$$N_M(C_1 \cup C_2;\bar{x}) = A_1(C_{10}).$$

(*b*) *If for every $\epsilon > 0$ we have $C_2 \cap B(\bar{x};\epsilon) \nsubseteq C_1$, then*

$$N_M(C_1 \cup C_2;\bar{x}) = \overline{A_1\Big(C_{10} \setminus (C_{20} \cap \nabla q_2(\bar{x})^{\perp})\Big)} \cup \mathbb{R}\nabla q_2(\bar{x}),$$

Proof. (*a*): By assumption, $T(C_2;x) \subseteq T(C_1;x)$ for all $x \in B(\bar{x};\epsilon) \cap C_2$. Thus $T(C_1 \cup C_2;x) = T(C_1;x) \cup T(C_2;x) = T(C_1;x)$ for all $x \in B(\bar{x};\epsilon) \cap C_2$. Then

$$-N_M(C_1 \cup C_2;\bar{x}) = \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ x \in C_1 \cup C_2}} [T(C_1 \cup C_2;x)]^* = \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ x \in C_1}} [T(C_1;x)]^* = A_1(C_{10}),$$

since $(C_1 \cup C_2) \cap B(\bar{x};\epsilon) = C_1 \cap B(\bar{x};\epsilon)$.

(*b*): We already know that $N_M(C_1 \cup C_2;\bar{x}) = N_M(C_{10} \cup (C_2 - \bar{x});0)$. By Proposition 2.1 again, we have

$$\begin{aligned} -N_M(C_{10} \cup (C_2 - \bar{x});0) \quad = \quad & \operatorname*{Lim\,sup}_{\substack{x \to 0 \\ x \in C_{10} \setminus (C_2 - \bar{x})}} [T(C_{10};x)]^* \cup \operatorname*{Lim\,sup}_{\substack{x \to 0 \\ x \in (C_2 - \bar{x}) \setminus C_{10}}} [T(C_2 - \bar{x};x)]^* \\ & \cup \operatorname*{Lim\,sup}_{\substack{x \to 0 \\ x \in C_{10} \cap (C_2 - \bar{x})}} [T(C_{10} \cup (C_2 - \bar{x});x)]^*. \end{aligned}$$

By assumption,

$$\operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in(C_2-\bar{x})\setminus C_{10}}} [T(C_2-\bar{x};x)]^* = \operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ x\in C_2}}[T(C_2;x)]^* = \mathbb{R}\nabla q_2(\bar{x}).$$

On the other hand, for all $x \in C_{10}\cap(C_2-\bar{x}) = (C_1\cap C_2)-\bar{x}$,

$$[T(C_{10}\cup(C_2-\bar{x});x)]^* = [T(C_{10};x)]^* \cap [T(C_2-\bar{x};x)]^* \subseteq [T(C_2-\bar{x};x)]^*.$$

It follows that

$$\operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\cap(C_2-\bar{x})}} [T(C_{10}\cup(C_2-\bar{x});\bar{x})]^* \subseteq \mathbb{R}\nabla q_2(\bar{x}).$$

Whence

$$\begin{aligned}\operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in(C_2-\bar{x})\setminus C_{10}}} [T(C_2-\bar{x};x)]^* \;\cup\; \operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\cap(C_2-\bar{x})}} [T(C_{10}\cup(C_2-\bar{x});x)]^*\\ = \mathbb{R}\nabla q_2(\bar{x}). \qquad (4.2.4)\end{aligned}$$

It remains to compute the following

$$\operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\setminus(C_2-\bar{x})}} [T(C_{10};x)]^* = \operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\setminus(C_2-\bar{x})}} \mathbb{R}A_1x.$$

The equality

$$\begin{aligned} C_2-\bar{x} &= \{y\in\mathbb{R}^n:\ q_2(y+\bar{x})=0\}\\ &= \{y\in\mathbb{R}^n:\ y^\top A_2 y = -2y^\top\nabla q_2(\bar{x})\}. \qquad (4.2.5)\end{aligned}$$

yields

$$C_{20}\cap\nabla q_2(\bar{x})^\perp \subseteq C_2-\bar{x}. \qquad (4.2.6)$$

We will prove

$$\operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\setminus(C_2-\bar{x})}} \mathbb{R}A_1x = \operatorname*{Lim\,sup}_{\substack{x\to 0\\ x\in C_{10}\setminus(C_{20}\cap\nabla q_2(\bar{x})^\perp)}} \mathbb{R}A_1x. \qquad (4.2.7)$$

The inclusion "$\subseteq$" follows from (4.2.6).

To prove the other inclusion, we proceed as follows. Let

$$y \in \operatorname*{Lim\,sup}_{\substack{x \to 0 \\ x \in C_{10} \setminus (C_{20} \cap \nabla q_2(\bar{x})^\perp)}} \mathbb{R}A_1 x.$$

Then, there exist $v_k \in C_{10} \setminus (C_{20} \cap \nabla q_2(\bar{x})^\perp$ and $r_k \in \mathbb{R}$ such that $v_k \to 0$ and $r_k A_1 v_k \to y$. Now, let us define the sequence $\tilde{v}_k$ by $\tilde{v}_k = v_k$ when $v_k \notin C_2 - \bar{x}$ and $\tilde{v}_k = \frac{1}{2} v_k$ when $v_k \in C_2 - \bar{x}$. We claim that $\tilde{v}_k \notin C_2 - \bar{x}$ for all $k \in \mathbb{N}$. Indeed, we need only to check the case when $v_k \in C_2 - \bar{x}$. If, on the contrary, $\tilde{v}_k = \frac{1}{2} v_k \in C_2 - \bar{x}$, (4.2.5) implies $\frac{1}{4} v_k^\top A_2 v_k = -v_k^\top A_2 \bar{x}$. This together with the fact that $v_k \in C_2 - \bar{x}$ (use (4.2.5) again), yields $v_k^\top A_2 v_k = 0$ and $v_k^\top \nabla q_2(\bar{x}) = 0$, which mean $v_k \in C_{20} \cap \nabla q_2(\bar{x})^\perp$. This cannot happen by the choice of v_k, proving the claim, that is, $\tilde{v}_k \in C_{10} \setminus (C_2 - \bar{x})$ for all $k \in \mathbb{N}$.

By choosing $\tilde{r}_k = r_k$ if $v_k \notin C_2 - \bar{x}$ and $\tilde{r}_k = 2r_k$ when $v_k \in C_2 - \bar{x}$, we obtain

$$\tilde{r}_k A_1 \tilde{v}_k = r_k A_1 v_k \to y.$$

This shows

$$y \in \operatorname*{Lim\,sup}_{\substack{v \to 0 \\ v \in C_{10} \setminus (C_2 - \bar{x})}} \mathbb{R}A_1 v,$$

and so the proof of (4.2.7) is complete.

Since the set $C_{20} \cap \nabla q_2(\bar{x})^\perp$ is a cone and $-(C_{20} \cap \nabla q_2(\bar{x})^\perp) = C_{20} \cap \nabla q_2(\bar{x})^\perp$, following similar arguments as before, one concludes

$$\operatorname*{Lim\,sup}_{\substack{v \to 0 \\ v \in C_{10} \setminus (C_2 - \bar{x})}} \mathbb{R}A_1 v = \operatorname*{Lim\,sup}_{\substack{v \to 0 \\ v \in C_{10} \setminus (C_{20} \cap \nabla q_2(\bar{x})^\perp)}} \mathbb{R}A_1 v = \overline{A_1 \Big(C_{10} \setminus (C_{20} \cap \nabla q_2(\bar{x})^\perp) \Big)}.$$

This along with (4.2.4) proves the desired result. $\square$

Remark 4.2. *Clearly $C_{10} \not\subseteq \nabla q_2(\bar{x})^\perp$ because of A_1 is indefinite and so* span $C_{10} = \mathbb{R}^n$*, see Lemma 3.10 in [50]. In case $\overline{C_{10} \setminus \nabla q_2(\bar{x})^\perp} = C_{10}$, the expression in (b) of*

Lemma 4.4 can be simplified. Indeed, in such a case, we obtain

$$\overline{A_1\Big(C_{10} \setminus (C_{20} \cap \nabla q_2(\bar{x})^{\perp})\Big)} = A_1(C_{10}),$$

since

$$A_1\Big(C_{10} \setminus (C_{20} \cap \nabla q_2(\bar{x})^{\perp})\Big) = A_1(C_{10} \setminus C_{20}) \cup A_1(C_{10} \setminus \nabla q_2(\bar{x})^{\perp}).$$

and

$$\overline{A_1(C_{10} \setminus \nabla q_2(\bar{x})^{\perp})} = \overline{A_1(\overline{C_{10} \setminus \nabla q_2(\bar{x})^{\perp}})} = A_1(C_{10}).$$

Recall that C_{10} and $A_1(C_{10})$ are closed.

The following notion plays an important role in the next section.

Definition 4.1. *We say that a set-valued mapping F is "outer contained at $\bar{x}$" if $F(x) \subseteq F(\bar{x})$ for all $x \in \mathrm{Dom}\ F$. It is "locally outer contained at $\bar{x}$" if there exists $\varepsilon > 0$ satisfying $F(x) \subseteq F(\bar{x})$ for all $x \in B(\bar{x}; \varepsilon) \cap \mathrm{dom}\ F$.*

In the following we state the fulfillment of (locally) outer contained property for some concrete situations.

Proposition 4.3. *If C_1 and C_2 are polyhedra, then $N_M(C_1 \cup C_2; \cdot)$ is locally outer contained at $\bar{x} \in C_1 \cap C_2$.*

Proof. It is Corollary 3.2. □

Proposition 4.4. *Let $C = \left\{x \in \mathbb{R}^n : \ q(x) \doteq \frac{1}{2}x^\top Ax + a^\top x + \alpha = 0\right\}$ with A indefinite and let $\bar{x} \in C$ be such that $\nabla q(\bar{x}) = 0$. Then $N_M(C; \cdot)$ is outer contained at $\bar{x}$.*

Proof. From Proposition 4.2(*b*), we know that $N_M(C;x) = A(C_0)$, with $C_0 = C - x = \{v \in \mathbb{R}^n : \ v^\top Av = 0\}$ for every $x \in C$ such that $\nabla q(x) = 0$. Therefore, $N_M(C;x) = N_M(C;\bar{x}) = A(C_0)$ for those points x. We now consider $x \in C$ such that $\nabla q(x) \neq 0$. By Proposition 4.2(*a*), we obtain

$$N_M(C;x) = \mathbb{R}\nabla q(x) = \mathbb{R}(Ax + a + a - a) = \mathbb{R}(Ax + a - A\bar{x} - a) = \mathbb{R}A(x - \bar{x}).$$

Thus $N_M(C;x) \subseteq \mathbb{R}A(C-\bar{x}) = A(C_0) = N_M(C;\bar{x})$. □

In case of two quadric surfaces, we need an extra assumption on the involved matrices.

Proposition 4.5. *Let* $C_i = \left\{x \in \mathbb{R}^n : \; q_i(x) \doteq \frac{1}{2}x^\top A_i x + a_i^\top x + \alpha_i = 0\right\}$ *with* A_i *being indefinite and non-singular for* $i = 1,2$. *If* $\bar{x} \in C_1 \cap C_2$ *satisfies* $\nabla q_1(\bar{x}) = 0 = \nabla q_2(\bar{x})$, *then* $N_M(C_1 \cup C_2; \cdot)$ *is outer contained at* $\bar{x}$.

Proof. From Lemma 4.2 it follows that

$$N_M(C_1 \cup C_2; \bar{x}) = \overline{A_1(C_{10} \setminus C_{20})} \cup \overline{A_2(C_{20} \setminus C_{10})} \cup \overline{A_1(K_0)},$$

with

$$K_0 = \left\{v \in C_{10} \cap C_{20} : \; \exists(\gamma_1,\gamma_2) \in \mathcal{S}^1, \; \gamma_1 A_1 v = \gamma_2 A_2 v\right\}.$$

Since A_1 and A_2 are non-singular, for any $x \neq \bar{x}$, $\nabla q_i(x) \neq \nabla q_i(\bar{x}) = 0$ for $i = 1,2$. Take any $x \in C_1 \setminus C_2$, then

$$N_M(C_1 \cup C_2; x) = N_M(C_1; x) = \mathbb{R}\nabla q_1(x) \subseteq \mathbb{R}A_1(x-\bar{x}) \subseteq \mathbb{R}A_1(C_{10} \setminus C_{20}),$$

since $C_1 \setminus C_2 = (C_{10}+\bar{x}) \setminus (C_{20}+\bar{x}) = (C_{10} \setminus C_{20}) + \bar{x}$. Similarly, for $x \in C_2 \setminus C_1$, one gets

$$N_M(C_1 \cup C_2; x) = N_M(C_2; x) = \mathbb{R}\nabla q_2(x) \subseteq \mathbb{R}A_2(x-\bar{x}) \subseteq \mathbb{R}A_2(C_{20} \setminus C_{10}).$$

It remains to consider $x \in C_1 \cap C_2$, $x \neq \bar{x}$. As above, $\nabla q_i(x) \neq \nabla q_i(\bar{x}) = 0$. Thus, Lemma 4.1 ensures that $N_M(C_1 \cup C_2; x) = \mathbb{R}\nabla q_1(x) \cup \mathbb{R}\nabla q_2(x)$. Now, we distinguish two cases. If $\{\nabla q_1(x), \nabla q_2(x)\}$ is linearly dependent, then $\mathbb{R}\nabla q_1(x) = \mathbb{R}\nabla q_2(x) \subseteq A_1(K_0)$. On the other hand, if $\{\nabla q_1(x), \nabla q_2(x)\}$ is linearly independent, then $N_M(C_1 \cup C_2; x) = \mathbb{R}\nabla q_1(x) \cup \mathbb{R}\nabla q_2(x)$. Since $T(C_1; x) \neq T(C_2; x)$, there exists $x_k \in C_1 \setminus C_2$ such that $x_k \to x$. Then $\nabla q_1(x_k) - \nabla q_1(\bar{x}) = A_1(x_k - \bar{x}) \in A_1(C_{10} \setminus C_{20})$, which implies $\nabla q_1(x) \in \overline{A_1(C_{10} \setminus C_{20})}$. Hence, as above,

$$\mathbb{R}\nabla q_1(x) \subseteq \overline{\mathbb{R}A_1(C_{10} \setminus C_{20})} \subseteq \overline{A_1(C_{10} \setminus C_{20})} \cup \{0\}.$$

Similarly, $\mathbb{R}\nabla q_2(x) \subseteq \overline{A_2(C_{20} \setminus C_{10})} \cup \{0\}$. Consequently, $N_M(C_1 \cup C_2; x) \subseteq N_M(C_1 \cup C_2; \bar{x})$ for all $x \in C_1 \cup C_2$, as desired. □

4.3 M-stationarity in nonconvex quadratic programming: local uniqueness

We start by considering the problem

$$\min\{f(x) : \; Ax \leq a, \; Bx = b, \; x \in X\}, \tag{4.3.1}$$

where $X = \{x \in \mathbb{R}^n : \; q(x) \doteq \frac{1}{2}x^\top Cx + d^\top x + \alpha = 0\}$ with $C = C^\top \in \mathbb{R}^{n \times n}$ being indefinite, $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{l \times n}$; $a \in \mathbb{R}^m$, $b \in \mathbb{R}^l$, $d \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. In this case, X may be the union of non-polyhedral sets. The polyhedral situation was discussed in Chapter 3 and [31].

The feasible set to problem (4.3.1) is denoted by K, and by $\tilde{K} \doteq \{x : \; Ax \leq a, \; Bx = b\}$. Clearly, $K = \tilde{K} \cap X$, and for $x \in K$, one obtains

$$T(\tilde{K}; x) = \{w : \; Bw = 0, \; (Aw)_i \leq 0 \; \forall \; i \in I(x)\}, \; I = I(x) \doteq \{i : \; (Ax - a)_i = 0\}.$$

Moreover, as stated in Proposition 4.2, $T(X; x) = X_0 \doteq \{v \in \mathbb{R}^n : \; v^\top Cv = 0\}$ whenever $\nabla q(x) = 0$. Hence

$$T(K; x) = T(\tilde{K} \cap X; \bar{x}) \subseteq T(\tilde{K}; \bar{x}) \cap T(X; \bar{x}).$$

We say that $x \in K$ is an M-stationary point if there exists (λ, μ), such that

$$\nabla f(x) + A^\top \lambda + B^\top \mu \in -N_M(X; x), \; \lambda^\top (Ax - a) = 0, \; \lambda \geq 0. \tag{4.3.2}$$

The set of those M-stationary points in K is denoted by $\mathcal{M}(K)$. Associated to $x \in \mathcal{M}(K)$ its set of multipliers is given by

$$\mathcal{L}_{\mathcal{M}}(x) \doteq \{(\lambda, \mu) : \; (\lambda, \mu) \text{ satisfies } (4.3.2)\}.$$

Set $J \doteq \{1, \ldots, m\}$. For fixed $\bar{x} \in \mathcal{M}(K)$ and $(\bar{\lambda}, \bar{\mu}) \in \mathcal{L}_{\mathcal{M}}(\bar{x})$, we define the following index sets:

$$I_+ = I_+(\bar{x}) = \{i \in J : \ \bar{\lambda}_i > 0\},$$

$$I_0 = I_0(\bar{x}) = \{i \in J : \ \bar{\lambda}_i = 0 = (A\bar{x} - a)_i\},$$

$$I_- = I_-(\bar{x}) = \{i \in J : \ (A\bar{x} - a)_i < 0\},$$

By the complementarity condition, $(A\bar{x} - a)_i = 0$ for $i \in I_+$ and $\lambda_i = 0$ for $i \in I_-$ and $I_+ \cup I_0 \cup I_- = J$. Notice that $I = I(\bar{x}) = I_+ \cup I_0$.

Set $\mathcal{D}_0(\bar{x}) \doteq \nabla f(\bar{x})^{\perp} \cap T_0(\tilde{K}; \bar{x}) \cap X_0$ and $\mathcal{D}(\bar{x}) \doteq \nabla f(\bar{x})^{\perp} \cap T(\tilde{K}; \bar{x}) \cap X_0$, where

$$T_0(\tilde{K}; \bar{x}) = \{w \in \mathbb{R}^n : \ Bw = 0, \ (Aw)_i = 0 \ \forall \ i \in I_+, \ (Aw)_i \leq 0 \ \forall \ i \in I_0\},$$

and denote $K_{\varepsilon}(\bar{x}) \doteq K \cap B(\bar{x}; \varepsilon)$. Clearly, $\mathcal{D}(\bar{x})$ and $\mathcal{D}_0(\bar{x})$ depend on the choice of $\bar{\lambda}$. Set, as usual, $L(x, \lambda, \mu) \doteq f(x) + \lambda^{\top}(Ax - a) + \mu^{\top}(Bx - b)$.

Theorem 4.1. *Let f be a function twice continuously differentiable in a neighborhood of a point $\bar{x} \in \mathcal{M}(K)$ with $\nabla q(\bar{x}) = 0$ and C being indefinite. Let $(\bar{\lambda}, \bar{\mu}) \in \mathcal{L}_{\mathcal{M}}(\bar{x})$ such that*

$$w^{\top} \nabla^2 f(\bar{x}) w > 0, \quad \forall \ w \in \mathcal{D}_0(\bar{x}), \ w \neq 0. \tag{4.3.3}$$

If, in addition, there exists $\varepsilon > 0$ satisfying $(\nabla f(\bar{x}) + A^{\top}\bar{\lambda})^{\top}(x - \bar{x}) \geq 0$ for all $x \in K_{\varepsilon}(\bar{x})$, then there exists a neighborhood $U_{\bar{x}}$ of $\bar{x}$ with the property:

$$x \in U_{\bar{x}} \cap \mathcal{M}(K) \Longrightarrow x = \bar{x}.$$

Proof. Suppose on the contrary that there exists $x_k \to \bar{x}$ with $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$, $x_k \in K$, multipliers (λ_k, μ_k) such that

$$\nabla f(x_k) + A^{\top}\lambda_k + B^{\top}\mu_k \in -N_M(X; x_k), \ \lambda_k \geq 0, \ \lambda_k^{\top}(Ax_k - a) = 0. \tag{4.3.4}$$

We can assume, up to a subsequence, that $\frac{x_k-\bar{x}}{\|x_k-\bar{x}\|} \to w$, which means

$$w \in T(K;\bar{x}) \subseteq T(\tilde{K};\bar{x}) \cap T(X;\bar{x}) = T(\tilde{K};\bar{x}) \cap X_0.$$

By the above remark, $Bw = 0$ and $(Aw)_i \leq 0$ for all $i \in I_+ \cup I_0$. Now, consider the functions

$$s_k(t) \doteq (\nabla f(x(t)) + A^\top \lambda(t))^\top (x_k - \bar{x}) + (a - Ax(t))^\top (\lambda_k - \bar{\lambda}),\ k \in \mathbb{N},\ t \in [0,1],$$

where $(x(t), \lambda(t)) = (1-t)(\bar{x}, \bar{\lambda}) + t(x_k, \lambda_k)$.
We now show that $s_k(1) - s_k(0) \leq 0$. By making some computation, and since $B(x_k - \bar{x}) = 0$, we get

$$s_k(1) = (\nabla f(x_k) + A^\top \lambda_k)^\top (x_k - \bar{x}) + (a - Ax_k)^\top (\lambda_k - \bar{\lambda}) \tag{4.3.5}$$

and

$$s_k(0) = (\nabla f(\bar{x}) + A^\top \bar{\lambda})^\top (x_k - \bar{x}) + (a - A\bar{x})(\lambda_k - \bar{\lambda}). \tag{4.3.6}$$

Since for all k sufficiently large, $(Ax_k - a)_i < 0$ for every $i \in I_-$, and so, $(\lambda_k)_i = 0$ for every $i \in I_-$, we get (use also $\lambda_k^\top (Ax_k - a) = 0$)

$$(x_k - \bar{x})^\top A^\top \lambda_k = \lambda_k^\top (Ax_k - a) + \lambda_k^\top (a - A\bar{x}) = 0, \tag{4.3.7}$$

and (4.3.5) reduces to

$$\begin{aligned} s_k(1) &= \nabla f(x_k)^\top (x_k - \bar{x}) + (a - Ax_k)^\top (\lambda_k - \bar{\lambda}) \\ &= \nabla f(x_k)^\top (x_k - \bar{x}) - (a - Ax_k)^\top \bar{\lambda}. \end{aligned}$$

In order to going on, two cases must be distinguished.
Case 1: There exist infinite indices $k \in \mathbb{N}$ such that $\nabla q(x_k) = 0$. By $(b4)$ in Proposition 4.2, $-N_M(X; x_k) = -C(X_0) = C(X_0)$, where $X = X_0 + \bar{x}$. From (4.3.4), it follows that

$$\nabla f(x_k) + A^\top \lambda_k + B^\top \mu_k = Cz_k,\ \text{for some } z_k \in X_0.$$

This implies, using (4.3.7) and $Bx_k = B\bar{x}$ again,

$$\begin{aligned}\nabla f(x_k)^\top (x_k - \bar{x}) &= (\nabla f(x_k) + A^\top \lambda_k + B^\top \mu_k)^\top (x_k - \bar{x}) \\ &= (Cz_k)^\top (x_k - \bar{x}) = z_k^\top C(x_k - \bar{x}) \\ &= z_k^\top (\nabla q(x_k) - \nabla q(\bar{x})) = 0,\end{aligned}$$

which gives $\nabla f(\bar{x})^\top w = 0$. This proves that $w \in \mathcal{D}(\bar{x})$.

On the other hand, by assumption, for all k sufficiently large, $(\nabla f(\bar{x}) + A^\top \bar{\lambda})^\top (x_k - \bar{x}) \geq 0$, which yields $(A^\top \bar{\lambda})^\top w \geq 0$. This allows us to infer that $(Aw)_i = 0$ for all $i \in I_+$. Hence $w \in T_0(\tilde{K}; \bar{x}) \cap X_0$, and therefore $w \in \mathcal{D}_0(\bar{x})$.

By M-stationarity on $\bar{x}$, $\nabla f(\bar{x}) + A^\top \bar{\lambda} + B^\top \bar{\mu} = Cz$ for some $z \in X_0$. Thus, (4.3.6) reduces to

$$\begin{aligned}s_k(0) &= z^\top C(x_k - \bar{x}) + (a - A\bar{x})(\lambda_k - \bar{\lambda}) \\ &= z^\top (\nabla q(x_k) - \nabla q(\bar{x})) + (a - A\bar{x})(\lambda_k - \bar{\lambda}) \geq 0.\end{aligned}$$

Hence

$$s_k(0) \geq 0 \geq -(a - Ax_k)^\top \bar{\lambda} = \nabla f(x_k)^\top (x_k - \bar{x}) - (a - Ax_k)^\top \bar{\lambda} = s_k(1),$$

and the claim is proved. By the mean value theorem, there exists $t_k \in (0, 1)$ such that

$$0 \geq s_k'(t_k) = (x_k - \bar{x})^\top \nabla f(x(t_k))(x_k - \bar{x}).$$

Observe that $x(t_k) = \bar{x} + t_k(x_k - \bar{x}) \to \bar{x}$. Thus, after dividing the last inequality by $\|x_k - \bar{x}\|^2$ and taking limits, we get $w^\top \nabla^2 f(\bar{x}) w \leq 0$, reaching a contradiction. This proves the desired result.

Case 2: It considers the situation when $\nabla q(x_k) \neq 0$ for infinitely many $k \in \mathbb{N}$.

By $(a2)$ in Proposition 4.2, $-N_M(X; x_k) = t_k \nabla q(x_k)$, for some $t_k \in \mathbb{R}$. Thus, the first relation in (4.3.4) reduces to

$$\nabla f(x_k) + A^\top \lambda_k + B^\top \mu_k = t_k \nabla q(x_k).$$

Again, by (4.3.7) and $Bx_k = B\bar{x}$, the last equality implies

$$\begin{aligned}\nabla f(x_k)^\top (x_k - \bar{x}) &= t_k \nabla q(x_k)^\top (x_k - \bar{x}) = t_k (Cx_k + d)^\top (x_k - \bar{x}) \\ &= t_k (C(x_k - \bar{x}) + C\bar{x} + d)^\top (x_k - \bar{x}) = 0 \qquad (4.3.8)\end{aligned}$$

since $x_k - \bar{x} \in X_0$ and $\nabla q(\bar{x}) = 0$, and so $\nabla f(\bar{x})^\top w = 0$. By proceeding as in Case 1, we conclude also that $(Aw)_i = 0$ for all $i \in I_+$. Hence, $w \in \mathcal{D}_0(\bar{x})$ since we have already proved that $w \in T_0(\tilde{K}; \bar{x}) \cap X_0$.

We now compute the difference $s_k(1) - s_k(0)$ for the same function s_k as in Case 1. By using (4.3.8), one gets

$$\begin{aligned}s_k(1) - s_k(0) &= -\bar{\lambda}^\top (a - Ax_k) - (\nabla f(\bar{x}) + A^\top \bar{\lambda})^\top (x_k - \bar{x}) \\ &\quad -(a - A\bar{x})(\lambda_k - \bar{\lambda}) \\ &= -\bar{\lambda}^\top (a - Ax_k) - (\nabla f(\bar{x}) + A^\top \bar{\lambda})^\top (x_k - \bar{x}) \leq 0.\end{aligned}$$

Thus there exists $t_k \in (0, 1)$ such that $s'(t_k) \leq 0$. Hence, we proceed as above to complete the proof. □

By looking carefully at the proof of Thorem 4.1, one can identify the next corollary. Recall that $\mathcal{D}(\bar{x}) \doteq \nabla f(\bar{x})^\perp \cap T(\tilde{K}; \bar{x}) \cap X_0$, which is a closed cone.

Corollary 4.1. *Let f be a function twice continuously differentiable in a neighborhood of a point $\bar{x} \in \mathcal{M}(K)$ with $\nabla q(\bar{x}) = 0$ and C being indefinite. Let $(\bar{\lambda}, \bar{\mu}) \in \mathcal{L}_{\mathcal{M}}(\bar{x})$ such that*

$$w^\top \nabla^2 f(\bar{x}) w > 0, \quad \forall\, w \in \mathcal{D}(\bar{x}),\ w \neq 0.$$

Then, there exists a neighborhood $U_{\bar{x}}$ of $\bar{x}$ with the property:

$$x \in U_{\bar{x}} \cap \mathcal{M}(K),\ \nabla q(x) = 0 \Longrightarrow x = \bar{x}.$$

Proof. We are in the situation of Case 1 discussed in the proof of Theorem 4.1. Notice that the condition on $\nabla f(\bar{x}) + A^\top \bar{\lambda}$ imposed in such a theorem was employed only to prove that $w \in \mathcal{D}_0(\bar{x})$. □

Different kind of stationarity notions are discussed in [57], and some relationships between local minimizer and those stationarity concepts are established, see for instance [57, Theorem 7].

An interesting remark arises.

Remark 4.3. *Suppose that $\mathcal{D}_0(\bar{x}) \neq \{0\}$. By taking*

$$\gamma \doteq \min\{w^\top \nabla^2 f(\bar{x}) w : \ w \in \mathcal{D}_0(\bar{x}), w^\top w = 1\},$$

one can prove that (4.3.3) is equivalent to $\gamma > 0$ and

$$w^\top \nabla^2 f(\bar{x}) w \geq \gamma \|w\|^2, \quad \forall \ w \in \mathcal{D}_0(\bar{x}).$$

One cannot avoid the condition imposed on $\nabla f(\bar{x}) + A^\top \bar{\lambda}$ in Theorem 4.1, as the next example shows.

Example 4.1. *Let us consider the problem:*

$$\mu_0 \doteq \min\{f(x) : \ Ax \leq 0, \ x \in X\},$$

where $f(x) = x_1 - 4x_2 - x_1^2 + 16x_2^2$, $A = (-1 \ \ 0) \in \mathbb{R}^{1\times 2}$ and $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - 16x_2^2 = 0\}$. Obviously, $K = \mathbb{R}_+(4,1) \cup \mathbb{R}_+(4,-1)$, $X = X_0 = \mathbb{R}(4,1) \cup \mathbb{R}(-4,1)$. Here, we can easily compute that $\mu_0 = 0$, $\operatorname{argmin}_K f = \mathbb{R}_+(4,1) = \mathcal{M}(K)$, $N_M(X;0) = \mathbb{R}(-1,4) \cup \mathbb{R}(1,4)$. By considering $\bar{x} = 0 \in \mathbb{R}^2 \in \mathcal{M}(K)$ and $\lambda = 2 \in \mathcal{L}_{\mathcal{M}}(0)$, the conclusion of Theorem 4.1 does not hold obviously. We will show that (4.3.3) holds true but the assumption on $\nabla f(\bar{x}) + A^\top \bar{\lambda}$ does not satisfy. By making some computations, we get

$$\nabla f(0)^\top (w_1, w_2) = 0 \Longleftrightarrow w_1 = 4w_2; I = \{1\} = I_+; A(w_1, w_2) = 0 \Longleftrightarrow w_1 = 0,$$

which implies $\mathcal{D}(0) = \{0\}$, and therefore (4.3.3) holds. However, since

$$(\nabla f(0) + A^\top \bar{\lambda})^\top u = ((1,-4) + (-2,0))^\top (u_1, u_2) = -u_1 - 4u_2,$$

the other assumption does not hold.

Before going on, we exhibit an instance where our Theorem 4.1 applies.

Example 4.2. *Take the objective function* $f(x) = (x_1+1)^2 + (x_2+1)^2 - 2x_1x_2$ *and as feasible set,* $K = \{(x_1, x_2) : -x_1 \le 0, -x_2 \le 0, q(x_1, x_2) \doteq x_1x_2 = 0\}$, *and* $\bar{x} = 0 \in \mathbb{R}^2$. *We identify the data*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and all the other data are null elements. It is easy to check that 0 *is the only minimizer of* f *on* K. *Let us compute* $\mathcal{D}(0)$. *To that end, we get* $\nabla f(0)^\top w = 0$ *if, and only if* $2w_1 + 2w_2 = 0$. *Since* $(w_2, w_2) \in X_0 = X$, *that is,* $w_1w_2 = 0$, *we obtain* $w_1 = 0 = w_2$, *and therefore* $\mathcal{D}(0) = \{0\}$. *By choosing* $\bar{\lambda} = (2, 2) \in \mathcal{L}_{\mathcal{M}}(0)$, *one concludes* $(\nabla f(0) + A^\top \bar{\lambda})^\top u = 0$ *for all* u.

4.4 Local error-bound in M-stationarity for a class of nonlinear optimization problems

In this last section, motivated by the previous one, we will establish a local error-bound at M-stationary points for a rather general class of nonlinear optimization problems. The main result to be established presently is based on three conditions: local uniqueness of a (reference) M-stationary point; validity of local outer contained property (as introduced in Definition 4.1) for the limiting normal cone to the geometric constraint set at the same point, and upper Lipschitzianity for the M-stationary points set-valued mapping.

It worth noting that our next result goes beyond the polyhedral setting. Indeed, all the above three conditions hold under polyhedrality, as shown in Chapter 3 with the aid of the works [33, 53]. The authors in [15] deal with a class of mathematical programs with equilibrium constraints via an equivalent formulation where the new geometric constraints set admits a limiting normal cone which is the union of finitely many convex polyhedra, see equation (27) in [15].

Thus, we shall discuss the existence of local estimates for the following nonlinear optimization problem:

$$\min\{f(x,\varphi):\ g(x,a)\le 0,\ h(x,b)=0,\ x\in X\}, \qquad (P_{\varphi,a,b})$$

where $f:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$, $g:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^r$, $h:\mathbb{R}^n\times\mathbb{R}^l\to\mathbb{R}^s$ and $X\subseteq\mathbb{R}^n$ is closed. The parameters are $(a,b,\varphi)\in\mathbb{R}^m\times\mathbb{R}^l\times\mathbb{R}^p$. As before, the feasible set to $(P_{\varphi,a,b})$ is denoted by $K(a,b)$. In this framework, given $x\in K(a,b)$, we say that it is an M-stationary point if there exists (λ,μ), such that

$$\nabla_x f(x,\varphi)+D_xg(x,a)^\top\lambda+D_xh(x,b)^\top\mu \ \in\ -N_M(X;x), \qquad (4.4.1)$$

$$\lambda^\top g(x,a)=0,\ \lambda\ge 0. \qquad (4.4.2)$$

Here, $D_xg(x,a)\in\mathbb{R}^{r\times n}$ stands for the Jacobian (with respect to x) matrix of g at x; similarly for $D_xh(x,b)\in\mathbb{R}^{s\times n}$. The set of those M-stationary points is denoted by $\mathcal{M}(\varphi,a,b)$, that is,

$$\mathcal{M}(\varphi,a,b)=\{x\in K(a,b):\ \exists(\lambda,\mu)\text{ satisfies }(4.4.1)-(4.4.2)\}.$$

Let $x\in\mathcal{M}(\varphi,a,b)$, its set of multipliers is given by

$$\mathcal{L}_{\mathcal{M}}(x)\doteq\{(\lambda,\mu):\ (\lambda,\mu)\text{ satisfies }(4.4.1)-(4.4.2)\}.$$

Assume that $N_M(X;\cdot)$ is locally outer contained at a point $\bar{x}\in K(\bar{\varphi},\bar{a},\bar{b})$, that is, there exists $U_{\bar{x}}$ such that $N_M(X;x)\subseteq N_M(X;\bar{x})$ for all $x\in U_{\bar{x}}$ and consider the following mapping

$$\begin{aligned}\tilde{\mathcal{M}}(\varphi,a,b) &= \{x\in K(a,b):\exists(\lambda,\mu)\in\mathbb{R}^r_+\times\mathbb{R}^s,\lambda^\top g(x,a)=0,\ \lambda\ge 0,\\ &\qquad \nabla_x f(x,\varphi)+D_xg(x,a)^\top\lambda+D_xh(x,b)^\top\mu\in -N_M(X,\bar{x})\}.\end{aligned}$$

It is clear that $\mathcal{M}(\varphi,a,b)\cap U_{\bar{x}}\subseteq\tilde{\mathcal{M}}(\varphi,a,b)$ for every (φ,a,b), since $N_M(X;x)\subseteq N_M(X;\bar{x})$ for all $x\in U_{\bar{x}}$.

Theorem 4.2. *Let us consider $(P_{\bar{\varphi},\bar{a},\bar{b}})$. Assume that $\bar{x}$ is a locally unique M-stationary* *point and* $N_M(X;\cdot)$ *is locally outer contained at* $\bar{x}$*. If there exist neighborhoods* $U_{\bar{x}}$*,* $U_{(\bar{\varphi},\bar{a},\bar{b})}$ *and a constant* $L > 0$ *such that for all* $(\varphi, a, b) \in U_{(\bar{\varphi},\bar{a},\bar{b})}$ *:*

$$\tilde{\mathcal{M}}(\varphi, a, b) \cap U_{\bar{x}} \subseteq \tilde{\mathcal{M}}(\bar{\varphi}, \bar{a}, \bar{b}) \cap U_{\bar{x}} + L||(\varphi, a, b) - (\bar{\varphi}, \bar{a}, \bar{b})||B(0; 1), \tag{4.4.3}$$

then there exist neighborhoods $\tilde{U}_{\bar{x}}$ *of* $\bar{x}$ *and* $\tilde{U}_{(\bar{\varphi},\bar{a},\bar{b})}$ *of* $(\bar{\varphi}, \bar{a}, \bar{b})$ *with the property that if* $(\varphi, a, b) \in \tilde{U}_{(\bar{\varphi},\bar{a},\bar{b})}$ *and* $x \in \mathcal{M}(\varphi, a, b) \cap \tilde{U}_{\bar{x}}$*, then*

$$||x - \bar{x}|| \leq L||(\varphi, a, b) - (\bar{\varphi}, \bar{a}, \bar{b})||.$$

Proof. Without loss of generality, assume that $\mathcal{M}(\bar{\varphi}, \bar{a}, \bar{b}) \cap U_{\bar{x}} = \{\bar{x}\}$. Then

$$\mathcal{M}(\bar{\varphi}, \bar{a}, \bar{b}) \cap U_{\bar{x}} = \tilde{\mathcal{M}}(\bar{\varphi}, \bar{a}, \bar{b}) \cap U_{\bar{x}} = \{\bar{x}\}.$$

Since $\mathcal{M}(\varphi, a, b) \cap U_{\bar{x}} \subseteq \tilde{\mathcal{M}}(\varphi, a, b)$, we obtain

$$\begin{aligned} \mathcal{M}(\varphi, a, b) \cap U_{\bar{x}} &\subseteq \tilde{\mathcal{M}}(\bar{\varphi}, \bar{a}, \bar{b}) \cap U_{\bar{x}} + L||(\varphi, a, b) - (\bar{\varphi}, \bar{a}, \bar{b})||B(0; 1) \\ &= \mathcal{M}(\bar{\varphi}, \bar{a}, \bar{b}) \cap U_{\bar{x}} + L||(\varphi, a, b) - (\bar{\varphi}, \bar{a}, \bar{b})||B(0; 1). \end{aligned}$$

We can assume, by taking $\tilde{U}_{\bar{x}} \subseteq U_{\bar{x}}$ if necessary, that

$$||x - \bar{x}|| < \frac{1}{3}\text{dist}(\bar{x}, \mathcal{M}(\bar{\varphi}, \bar{a}, \bar{b}) \setminus \{\bar{x}\}), \quad \forall\, x \in \tilde{U}_{\bar{x}}. \tag{4.4.4}$$

Now, we consider $\tilde{U}_{(\bar{\varphi},\bar{a},\bar{b})}$ such that

$$\{\bar{x}\} + L||(\varphi, a, b) - (\bar{\varphi}, \bar{a}, \bar{b})||B(0; 1) \subseteq U_{\bar{x}}, \quad \forall\, (\varphi, a, b) \in \tilde{U}_{(\bar{\varphi},\bar{a},\bar{b})}.$$

Take any $(\varphi, a, b) \in \tilde{U}_{(\bar{\varphi},\bar{a},\bar{b})}$. If $x \in \mathcal{M}(\varphi, a, b) \cap \tilde{U}_{\bar{x}}$, then

$$x \in (\{\bar{x}\} \cup \mathcal{M}(\bar{\varphi}, \bar{a}, \bar{b}) \setminus \{\bar{x}\}) + L||(\varphi, a, b) - (\bar{\varphi}, \bar{a}, \bar{b})||B(0; 1),$$

which says that the distance from x to either $\bar{x}$ or $\mathcal{M}(\bar{\varphi}, \bar{a}, \bar{b}) \setminus \{\bar{x}\}$ must be less than $L||(\varphi, a, b) - (\bar{\varphi}, \bar{a}, \bar{b})||$. Thus (4.4.4) gives

$$x \in \{\bar{x}\} + L||(\varphi, a, b) - (\bar{\varphi}, \bar{a}, \bar{b})||B(0; 1), \quad \forall\, x \in \tilde{U}_{\bar{x}}.$$

This amounts to writing $||x - \bar{x}|| \leq L||(\varphi, a, b) - (\bar{\varphi}, \bar{a}, \bar{b})||$, and the proof is complete. □

Some remarks are in order.

Remark 4.4. *(i) As previously mentioned, it is well known that polyhedral multifunctions satisfy the upper Lipschitz property. Notice that the multifunction $\tilde{\mathcal{M}}$ might be more easy to handle than $\mathcal{M}$ and, with the aid of Propositions 2.1 and 2.2 in [36], one can derive upper Lipschitzianity, besides the existence of an isolated point.*

(ii) It is well known that condition (4.4.3) holds for polyhedral multifunctions $\mathcal{M}$, which is always true in case the functions take the form $f(x, \varphi) = \frac{1}{2}x^\top Qx - \varphi^\top x$, $g(x, a) = Ax - a$, $h(x, b) = Bx - b$ and the set $N_M(X; \bar{x})$ is the union of finitely many convex polyhedra.

The next example shows an instance where our Theorem 4.2 applies. In particular, it shows also that condition (4.4.3) is satisfied even if $N_M(X; \bar{x})$ is non-polyhedral.

Example 4.3. *Let us consider the problem*

$$\min\{\frac{1}{2}x^\top Qx - \varphi^\top x : \; x_3 \leq a, \; x \in X\},$$

where $Q = \operatorname{diag}(0, 0, 2)$ and $X = \{x \in \mathbb{R}^3 : \; x^\top Cx \doteq x_1^2 + x_2^2 - x_3^2 = 0\}$. Here, $C = \operatorname{diag}(1, 1, -1) = C^{-1}$. The reference parameter is $\overline{\varphi} = 0 \in \mathbb{R}^3$, $\overline{a} = 0$. For these values, it is clear that $\bar{x} = (0, 0, 0)$ is the only minimizer, moreover, it is the only M-stationary point. Now, consider the mapping

$$\begin{aligned} \tilde{\mathcal{M}}(\varphi, a) \;\; &= \;\; \{x \in \mathbb{R}^3 : \; \exists \lambda \geq 0, \lambda(x_3 - a) = 0, \; x_3 \leq a, \; x \in X, \\ & \qquad Qx - \varphi + \lambda(0 \; 0 \; 1)^\top \in C(X) = X\}. \end{aligned}$$

By the previous remark, $\tilde{\mathcal{M}}((0, 0, 0), 0) = \{(0, 0, 0)\}$. Let $(\varphi, a) \neq (0, 0)$ and write

$\varphi = (\varphi_1, \varphi_2, \varphi_3)$. *Then*

$$\begin{aligned}\tilde{\mathcal{M}}(\varphi, a) &= \{x \in \mathbb{R}^3 : \ \exists \lambda \geq 0, \lambda(x_3 - a) = 0, x_3 - a \leq 0, x_1^2 + x_2^2 = x_3^2, \\ &\quad (0,0,x_3)^\top - \varphi + \lambda(0,0,1)^\top \in X\} \\ &= \{x \in \mathbb{R}^3 : \ \exists \lambda \geq 0, \lambda(x_3 - a) = 0, x_3 - a \leq 0, x_1^2 + x_2^2 = x_3^2, \\ &\quad \varphi_1^2 + \varphi_2^2 = (x_3 - \varphi_3 - \lambda)^2\}.\end{aligned}$$

We now distinguish two cases, if there exists $x \in \tilde{\mathcal{M}}(\varphi, a)$, *for which the only solution to* $\varphi_1^2 + \varphi_2^2 = (x_3 - \varphi_3 - \lambda)^2$ *if given when* $\lambda > 0$, *then* $x_3 = a$, *which implies that* $x_1^2 + x_2^2 = a^2$, *so clearly*

$$||x|| \leq L|a|,$$

for some $L > 0$. *Or, given* $x \in \tilde{M}(\varphi, a)$, *for which* $\varphi_1^2 + \varphi_2^2 = (x_3 - \varphi_3)^2$, *then*

$$|x_3| \leq L||\varphi|| \leq ||(\varphi, a)||,$$

for some $L > 0$. *Which, together with the fact that* $x_1^2 + x_2^2 = x_3^2$ *implies that*

$$||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{2x_3^2} \leq \tilde{L}||\varphi|| \leq \tilde{L}||(\varphi, a)||,$$

for $\tilde{L} = \sqrt{2}L > 0$. *In any case, it is clear that*

$$x \in B(0; \max\{L, \tilde{L}\}||(\varphi, a)||),$$

that is,

$$\begin{aligned}\tilde{\mathcal{M}}(\varphi, a) &\subseteq B(0; \max\{L, \tilde{L}\}||(\varphi, a)||) \\ &= \tilde{\mathcal{M}}(0,0) + \max\{L, \tilde{L}\}||(0,0) - (\varphi, a)||B(0;1).\end{aligned}$$

Chapter 5

Asymptotic analysis for a class of quasiconvex semi-infinite programming problems

5.1 Introduction

Research on the existence of solutions and on the lower semicontinuity of the value function (especially in relation to the zero duality gap property) has long been a central topic in optimization. This chapter addresses the general optimization problem

$$\mu := \inf\{f_0(x) : x \in P, f_l(x) \leq 0, \forall\ l \in L\}, \tag{5.1.1}$$

and its associated value function $\psi_L : \mathbb{R}^{(L)} \to \mathbb{R} \cup \{+\infty\}$, defined by

$$\psi_L(a) := \inf\{f_0(x) : x \in P, f_l(x) \leq a_l, \forall\ l \in L\},$$

where $f_0, f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ for $l \in L$ are quasiconvex and lower semicontinuous functions, L is an arbitrary (possibly infinite) index set, $\mathbb{R}^{(L)}$ denotes the dual of the locally convex product space $\mathbb{R}^L$; and $a = (a_l)_{l \in L} \in \mathbb{R}^{(L)}$. Note that we denote the functions in the constraints by f_l, rather than g_l, as in the previous chapters. This choice will simplify certain definitions later.

The main objectives of this work are to: (i) characterize certain global properties of quasiconvex functions based on their behavior at single points; (ii) prove the existence of solutions to problem (5.1.1); and (iii) establish the lower semicontinuity of the value function at 0.

Although the relation between local and global behavior of convex functions is well understood (e.g., local minima are global, and asymptotic behavior can often be inferred from a single point; see [2, 54]), quasiconvex functions are not as straightforward in this regard. Various global properties of quasiconvex functions have been studied via different asymptotic functions, see for instance [19, 20, 26, 29, 30, 34, 39].

Existence results available in the literature can be grouped according to the cardinality of L, distinguishing between the finite and infinite cases. In the finite case, when problem (5.1.1) is convex, extensive research has been conducted (see [2, 14, 49, 54] and references therein). For nonconvex optimization problem, [49] provides results for the class $\mathcal{F}$ of functions, a class introduced in [1] and recalled in Definition 2.2. Furthermore, [52] establishes existence results by looking at some special recession directions. In the context of quasiconvex functions, existence results have been characterized in [18] and [30]. For quasiconvex functions belonging to class $\mathcal{C}$, introduced in [20] and recalled in Definition 2.3, the papers [19] and [26] present an existence result under additional conditions. For the semi-infinite case, the key work [55] introduced the asymptotic regularity condition to establish the existence of solution and strong duality for convex programs.

In the convex case, the zero duality gap property is characterized by the lower semicontinuity of the value function at zero and the existence of a finite optimal value (see [2, 14, 17, 54], also [37] for parametric convex programs). For treatment of a more general setting we refer the reader to [41]. Several recent contributions have extended aspects of this theory to quasiconvex and semistrictly quasiconvex functions together with new classes of functions, like class $\mathcal{C}$, see for instance [19, 20, 26, 30].

We adopt an asymptotic analysis framework, replacing the standard compactness assumptions with the asymptotic regularity condition introduced in [55] to establish the existence of solution. To establish the lower semicontinuity of the value function, we reduce the original problem (5.1.1) to a family of finite subproblems. We then derive sufficient conditions under which the lower semicontinuity of the subproblems' value functions guarantees the lower semicontinuity of the original value function. A similar technique was used in [13], where the authors characterize other notions related to duality also by replacing the full constraint set with a finite subset.

In this chapter, we characterize certain global behaviors of quasiconvex functions by analyzing their behavior at single points, extending global convex properties to the quasiconvex setting, and complement the results presented in [19, 20, 26]. We focus on the class $\mathcal{C}$, demonstrating how the behavior of the function in one direction constraints behavior in the opposite direction, and we link this with the notions of ba- and ia-directions of recession (introduced in [14]). This notions have been further developed in [5] as a way to prove existence of saddle values in the convex case and in [38] to prove zero duality gap in a nonconvex setting.

Our existence results extend classical results, such as [55], to the quasiconvex framework and are not encompassed by previous results like those of [49], since the class $\mathcal{C}$ is not a subset of $\mathcal{F}$ (consider, for instance, the function $f(x) = e^{-x}$). Furthermore, our findings complement the contributions of [19, 26].

Finally, we strengthen and complement results such as those of [37] by establishing the lower semicontinuity of the value function under broader conditions, such as asymptotic regularity and a generalized consistency assumption, even in cases where the solution sets are unbounded.

5.2 Existence of Solution

Given the quasiconvex functions $f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, l \in L := I \cup J$ we define the problem

$$\inf\{f(x) : \; x \in P, \; f_j(x) \leq 0, \; \forall j \in J\}, \tag{B}$$

where $f(x) := \sup\{f_i(x) : i \in I\}$, L is a (finite or infinite) partitioned collection of quasiconvex lsc proper functions belonging to class $\mathcal{C}$ (with $I \cap J = \emptyset, I \neq \emptyset$) and P denotes a nonempty closed convex set in $\mathbb{R}^n$. We always assume dom $f_l = P$ for all $l \in L$. Of course, if $P \subseteq$ dom f_l for some $l \in L$, we can redefine f_l to take the value

$+\infty$ outside P.

The feasible set of problem (B) is defined as

$$K := \{x \in P : f_j(x) \le 0, \forall\, j \in J\}.$$

For each $\mu \in \mathbb{R}, \varepsilon \ge 0$ and each $L_1 \subseteq L$, it is also useful to define the sets

$$K^{\mu}_{L_1}(\varepsilon) := \{x \in P : f_i(x) \le \mu + \varepsilon, \forall\, i \in I \cap L_1, f_j(x) \le \varepsilon, \forall\, j \in J \cap L_1\}.$$

Note that L_1 may be a subset of either I or J, i.e., $L_1 \cap J = \emptyset$ or $L_1 \cap I = \emptyset$, respectively. In such cases, only the corresponding group of functions (either those in the definition of the cost function or in the constraints) is considered in the definition of $K^{\mu}_{L_1}(\varepsilon)$. In particular, if $\varepsilon = 0$, we just denote $K^{\mu}_{L_1} := K^{\mu}_{L_1}(0)$.

Definition 5.1. *The quasiconvex program (B) is weakly consistent (WC, for short) if there exists at least one $\mu \in \mathbb{R}$ such that, for all $\varepsilon > 0$ and for every $L_1 \subseteq L$ with $|L_1| \le n+1$, it holds that $K^{\mu}_{L_1}(\varepsilon) \neq \emptyset$.*

Consistency (nonempty feasible set and finite optimal value) implies weak consistency. Indeed, if we assume that $K \cap \mathrm{dom}\, f \neq \emptyset$ (in our case, since $\mathrm{dom}\, f = P$, $K \cap \mathrm{dom}\, f = K$) and μ is the optimal value of the problem, then (B) is weakly consistent, since

$$\emptyset \neq K \cap \mathrm{dom}\, f \subseteq K^{\mu}_{L_1}(\varepsilon) \cap \mathrm{dom}\, f, \forall\, L_1 \subseteq L = I \cup J, \forall\, \varepsilon > 0.$$

Now, the asymptotic directions along which the functions must be analyzed are determined. As shown in previous works (see [19, 26]), these directions lie in the following nonempty closed convex cone in $\mathbb{R}^n$, which will also play an important role in our analysis

$$R_0 := \{v \in P^{\infty} : (f_l)^{\infty}_q(v) \le 0, \forall\, l \in L\}.$$

Below, we present an adaptation of Rockafellar's result [55, Theorem 1] for quasiconvex functions, using asymptotic sets and asymptotic functions. For this, we first present the definitions of asymptotically regular condition given by Rockafellar, but now described using the terminology of asymptotic analysis.

Definition 5.2. *The quasiconvex program* (B) *is said asymptotically regular (AR, in brief) if there exists* $L_0 \subseteq L$, $I \not\subseteq L_0$, *such that* $|L_0| < +\infty$, f_l *is affine on* P *for all* $l \in L_0$ *and for each* $v \in R_0$ *we have* $f_l(x+tv) = f_l(x)$ *for all* $l \in L \setminus L_0$, *all* $t \geq 0$ *and all* $x \in \text{dom } f_l$.

Remark 5.1. (*i*) *Assume* $K = \mathbb{R}^n$ *and that the functions* f_l *are proper, lsc, and quasiconvex. By [19, Proposition 4.2(b)], we can conclude that a solution exists when* R_0 *is a subspace. Since in this case, the functions are constant along the directions in* R_0, *which is a particular case of the AR condition.*

(*ii*) *Whenever the problem* (B) *satisfies the AR condition, we can take* L_0 *to be as small as possible* ($I \not\subseteq L_0$), *meaning that for all* $l \in L_0$, *there must exist* $v \in P^\infty \cap (\cap_{l \in L \setminus L_0}[(f_l)_q^\infty \leq 0])$ *such that* f_l *strictly decreases along* v *(otherwise we could exclude said* l *from the set* L_0*), with* $f_l(x) = \langle b_l, x\rangle + \beta_l$, *for some* $b_l \in \mathbb{R}^n \setminus \{0\}$, $\beta_l \in \mathbb{R}$, $x \in P$ *and all* f_l *with* $l \in L \setminus L_0$ *are constant along* v, *that is, constant in* $x + \mathbb{R}_+ v$ *for all* $x \in P$.

The following is a modified version of Helly's Theorem, which appears in [6, Theorem 2.181].

Theorem 5.1. *Let* $A_i, i \in I$, *be a (possibly infinite) family of closed convex subsets of* $\mathbb{R}^n$. *Suppose that* $\cap_{i\in I}(A_i)^\infty = \{0\}$, *and that the intersection of any* $n+1$ *sets of this family is nonempty. Then the intersection of all sets of this family is nonempty.*

The following lemma represents a quasiconvex extension of [55, Lemma 4].

Lemma 5.1. *Let* $f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, l \in L$ *be proper, lsc, quasiconvex functions belonging to* $\mathcal{C}$ *and* $P \subseteq \mathbb{R}^n$ *be a nonempty closed convex set. Suppose* (B) *is WC, and let* $\bar{\mu}$ *be the infimum of the real numbers* μ *for which the weak consistency condition is satisfied with* $+\infty > \bar{\mu} \geq -\infty$. *Let* $\bar{L} = \{\bar{l} = (l, \mu, \varepsilon) : l \in L, \bar{\mu} < \mu, 0 < \varepsilon\}$, *and for each* $\bar{l} \in \bar{L}$, *define*

$$C_{\bar{l}} = P \cap [f_l \leq \mu + \varepsilon], \textit{if } l \in I \textit{ or } C_{\bar{l}} = P \cap [f_l \leq \varepsilon], \textit{ if } l \in J.$$

Then, the collection $\{C_{\bar{l}} : \bar{l} \in \bar{L}\}$ *consists of closed convex subsets of* $\mathbb{R}^n$ *such that every finite subcollection has a nonempty intersection. Moreover,* $\bar{x}$ *is a solution to*

(B) if and only if $\bar{x}$ belongs to all the $C_{\bar{l}}$. Also in this case, $\bar{\mu}$ is finite and corresponds to the minimum of f in (B).

Proof. The sets $C_{\bar{l}}$ are closed convex subsets of $\mathbb{R}^n$ because P is closed and convex, and the functions f_l are quasiconvex and lsc. From the definition of weakly consistent, for every $\mu \in \mathbb{R}$ with $\bar{\mu} < \mu$ and for all $\varepsilon > 0$ the set $K^{\mu}_{L_1}(\varepsilon) \neq \emptyset$, for all $L_1 \subseteq L, |L_1| \leq n+1$. So, by Helly's Theorem $K^{\mu}_{L_1}(\varepsilon) \neq \emptyset$ whenever $|L_1| < +\infty$. Note that $\cap_{l \in L_1} C_{l,\mu,\varepsilon} = K^{\mu}_{L_1}(\varepsilon)$.

On the other hand,

$$\bar{x} \in \cap_{\bar{l} \in \bar{L}} C_{\bar{L}} \Longleftrightarrow \bar{x} \in \cap_{l \in J}[f_l \leq 0] \cap P \text{ and } -\infty < f(\bar{x}) \leq \bar{\mu}.$$

But, if the point $\bar{x}$ is not the solution to problem (B), then there exists $x \in P$ that satisfies the constraints and $f(\bar{x}) > f(x) = \mu$, then the weakly consistency condition is satisfied for this μ, and hence $\mu \geq \bar{\mu}$, a contradiction. Therefore, $\bar{x}$ is a solution to problem (B) and $\bar{\mu}$ is the finite minimum. □

Now we proceed to prove the existence of solution.

Lemma 5.2. *Let $f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, l \in L$ be proper, lsc, quasiconvex functions belonging to $\mathcal{C}$, and let $P \subseteq \mathbb{R}^n$ be a nonempty closed convex set. If additionally $R_0 = \{0\}$ and (B) is WC, then the problem admits a solution.*

Proof. Let $\varepsilon > 0,\ \mu \in \mathbb{R}$ (μ given by the WC) and $C_{\bar{l}} = C_{l,\mu,\varepsilon}, l \in L$ be a family of nonempty sets defined by

$$\begin{aligned} C_{l,\mu,\varepsilon} &:= P \cap [f_l \leq \mu + \varepsilon], \quad \text{for } l \in I, \\ C_{l,\mu,\varepsilon} &:= P \cap [f_l \leq \varepsilon], \quad \text{for } l \in J, \end{aligned}$$

then its recession cones are $(C_{l,\mu,\varepsilon})^\infty = P^\infty \cap [(f_l)_q^\infty \leq 0]$, for any $l \in L$. Moreover,

$$\begin{aligned}\bigcap_{l,\mu,\varepsilon}(C_{l,\mu,\varepsilon})^\infty &:= \bigcap_{l\in L}\bigcap_{\mu<\tilde{\mu}}\bigcap_{\varepsilon>0}(C_{l,\tilde{\mu},\varepsilon})^\infty = \left(\bigcap_{l\in L}\bigcap_{\mu<\tilde{\mu}}\bigcap_{\varepsilon>0}[(f_l)_q^\infty \leq 0] \cap P^\infty\right) \\ &= \left(\bigcap_{l\in L}[(f_l)_q^\infty \leq 0] \cap P^\infty\right) = \{0\}. \quad \text{(by hypothesis)}\end{aligned}$$

Thus, $(\cap_{l,\mu,\varepsilon} C_{l,\mu,\varepsilon})^\infty = \{0\}$ and, by Lemma 5.1, the intersection of any $n+1$ sets in this family is nonempty. By Theorem 5.1, we conclude that the set $\bigcap_{l,\mu,\varepsilon} C_{l,\mu,\varepsilon}$ is nonempty. As a consequence, this set must be compact. Furthermore, using Lemma 5.1, the problem (B) admits a solution $\bar{x} \in \bigcap_{l,\mu,\varepsilon} C_{l,\mu,\varepsilon}$. □

Now we consider the case where the problem involves directions in which the functions remain constant along certain half lines. Rockafellar in [55, Lemma 6] presented a method, for the convex case, where it transforms the problem with this type of directions into a problem without them. In the following lemma this method is adapted to apply it to quasiconvex functions.

Lemma 5.3. *Let $f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, l \in L$ be proper, lsc, quasiconvex functions belonging to $\mathcal{C}$, and let P be a polyhedron. Suppose that (B) is WC and satisfies the AR condition with $L_0 = \emptyset$. Then, (B) admits a solution.*

Proof. By hypothesis, for all $v \in R_0$ and all $x \in P$ the following equality is fulfilled

$$f_l(x + tv) = f_l(x) \ \forall\, t \geq 0, \forall\, l \in L. \tag{5.2.1}$$

As we mentioned before, R_0 is a nonempty closed convex cone in $\mathbb{R}^n$. Let M be the subspace generated by R_0, i.e., $M := R_0 - R_0$, and $P' = \text{Proj}(M^\perp; P) = M^\perp \cap (M+P)$ (the orthogonal projection of P over $M^\perp$). Actually,

$$P' = \{x' \in M^\perp : x' + u \in P \text{ for some } u \in R_0\}, \tag{5.2.2}$$

by definition of M. Now, for each $l \in L$, we define the functions f_l' by

$$f_l'(x') = f_l(x' + u), \ \text{whenever } x' \in P', u \in R_0 \text{ and } x' + u \in P,$$

Otherwise, if $x' \notin P'$, we set $f'_l(x') = +\infty$. So, for all $l \in L, \text{dom } f'_l = P'$.

First, we show that f'_l is a well-defined function on P'. Fix $l \in L$ and any $x' \in P'$. If there exist $u_1, u_2 \in R_0$ such that $x' + u_1, x' + u_2 \in P$, then by (5.2.1) we get

$$f_l((x' + u_1) + u_2) = f_l(x' + u_1) \text{ and } f_l((x' + u_2) + u_1) = f_l(x' + u_2).$$

Since u_1 and u_2 are directions in which each f_l is constant; thus,

$$f'_l(x') = f_l(x' + u_1) = f_l(x' + u_2) = f_l(x' + u_1 + u_2). \tag{5.2.3}$$

This proves that the definition of f'_l does not depend on the particular u chosen.

In view of Lemma 2.2, to prove f'_l is quasiconvex lsc on P' we prove that each function f'_l is quasiconvex and lsc on each line segment on P'. Let $x', y' \in P'$. By (5.2.2), we can choose $u_1, u_2 \in R_0$ such that $x' + u_1, y' + u_2 \in P$. Let $u = u_1 + u_2 \in R_0$. Because R_0 is a convex cone, this implies that $x' + u = x \in P$ and $y' + u = y \in P$. Moreover, using definition (5.2.3) we get

$$f'_l(\alpha x' + (1 - \alpha)y') = f_l(\alpha x + (1 - \alpha)y) \text{ for } 0 \leq \alpha \leq 1. \tag{5.2.4}$$

Since the functions f_l are quasiconvex and lsc on P, the right-hand side of (5.2.4) is quasiconvex and lsc as a function of α; hence f'_l is quasiconvex and lsc.

With these new functions we consider the problem

$$\inf\{\sup\{f'_i(x) : i \in I\} : x \in P', f'_j(x) \leq 0, \forall\, j \in J\}. \tag{5.2.5}$$

Now we want to prove that problem (5.2.5) is WC using the fact that problem (B) is WC. For any $\varepsilon > 0$, any finite subset $L_1 \subseteq L$ and $u \in R_0$, we have the inclusion

$$K_{L_1}^{\mu_0}(\varepsilon) + u \subseteq K_{L_1}^{'\mu_0}(\varepsilon) := P' \cap (\cap_{l \in L_1 \cap I}[f'_l \leq \mu_0 + \varepsilon]) \cap (\cap_{l \in L_1 \cap J}[f'_l \leq \varepsilon])$$

and since $K_{L_1}^{\mu_0}(\varepsilon)$ is nonempty (by the WC property of problem (B)), this implies that $K_{L_1}^{'\mu_0}(\varepsilon) \neq \emptyset$. Therefore, problem (5.2.5) is WC.

On the other hand, since P is polyhedral and M is a subspace, it follows that $M+P$ is a polyhedron with $(M+P)^\infty = M+P^\infty$. This follows directly from the characterization of polyhedrons as closed convex sets with finitely many extreme points and directions; see, for instance, [54, Theorem 19.1]. Hence P' is a (nonempty) polyhedral convex set in $\mathbb{R}^n$ and

$$(P')^\infty = (M^\perp \cap (M+P))^\infty = M^\perp \cap (M+P^\infty) \subseteq M^\perp. \tag{5.2.6}$$

Let $R_0' := (P')^\infty \cap \bigcap_{l\in L}[(f_l')_q^\infty \le 0]$ (associated to problem (5.2.5)) and for any $u' \in (P')^\infty$ we have that

$$u' \in \cap_{l\in L}[(f_l')_q^\infty \le 0] \Longrightarrow u' \in \cap_{l\in L}[(f_l)_q^\infty \le 0] - u, \text{ for some } u \in R_0. \tag{5.2.7}$$

Indeed, given $x' \in P'$ and $u' \in R_0'$, we can choose $x \in P$, $u_0, u \in R_0$ such that $x - u_0 = x'$ and for all $\lambda \ge 0$ we get

$$(x' + \lambda u') + (u_0 + \lambda u) = x + \lambda(u' + u) \in P \text{ and } x' + (u_0 + \lambda u) = x + \lambda u \in P.$$

This, together with the definition of f_l', means that for all $l \in L$

$$f_l'(x' + \lambda u') = f_l(x + \lambda(u' + u)) \text{ and } f_l'(x') = f_l(x + \lambda u) = f_l(x), \quad \forall \lambda \ge 0.$$

Thus, $u' \in \cap_{l\in L}[(f_l')_q^\infty \le 0]$ implies $u' + u \in \cap_{l\in L}[(f_l)_q^\infty \le 0]$, for some $u \in R_0$, which proves (5.2.7). Moreover, for any $u \in R_0$ using (5.2.6), (5.2.7) and the definition of M, we obtain

$$R_0' \subseteq (P')^\infty \cap \left(\cap_{l\in L}[(f_l)_q^\infty \le 0] - u\right) \subseteq M^\perp \cap M = \{0\}.$$

From Lemma 5.2, the new problem (5.2.5) has a solution, let's say $\bar{x}'$, that is, $\bar{x}' \in P'$, $f_l'(\bar{x}') \le 0$, for every $l \in J$ and $f'(\bar{x}') \le f'(x')$ for every feasible point x' of (5.2.5). Using $\bar{x}'$ we construct an optimal solution to the original problem, for this consider $\bar{u} \in R_0$ such that $\bar{x} := \bar{x}' + \bar{u} \in P$, one obtains $f_l'(\bar{x}') = f_l(\bar{x}' + \bar{u}) = f_l(\bar{x})$ for every $l \in L$. Therefore the point $\bar{x}$ is feasible for (B). In addition, $\bar{x}$ is a solution to the original problem (B), because for any x' feasible point of (5.2.5), we have

$$f(\bar{x}) = f'(\bar{x}') \leq \sup_{i\in I} f_i(x'+u) = f(x'+u) \quad \text{(for some } u \in R_0)$$

and for every $x \in P$, there exists $x' \in P'$ and $u \in M$ such that $x = x' + u$. This completes the proof. □

Theorem 5.2. *Let $f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, l \in L$ be proper, lsc, quasiconvex functions belonging to $\mathcal{C}$, and let P be a polyhedron. Assume that* (B) *is WC and AR, then* (B) *admits a solution. Moreover, the minimum in* (B) *is the smallest real number μ for which the WC condition is satisfied.*

Proof. Without loss of generality, we suppose that problem (B) satisfies the AR condition taking L_0 as small as possible (see Remark 5.1(ii)).

Let $\mu_0 \in \mathbb{R}$ be the value for which WC is satisfied, and fix $\varepsilon_0 > 0$. For the set $L' := L \setminus L_0$ ($L := I \cup J$) we define the subproblems given by

$$\inf\{f'(x) := \sup\{f_i'(x) : i \in I'\} : x \in P',\ f_l'(x) \leq 0, \forall\, l \in J'\}, \tag{5.2.8}$$

where $P' := K_{L_0}^{\mu_0}(\varepsilon_0)$ (which is nonempty by WC), P' is a polyhedron, $I' := I \setminus L_0$, $J' := J \setminus L_0$ and $f_l' = f_l + \iota_{P'}$ for all $l \in L'$.

Given any $\varepsilon > 0$, the sublevel sets of f_l' are: $[f_l' \leq \mu_0 + \varepsilon] = [f_l \leq \mu_0 + \varepsilon] \cap P'$ and $[f_l' \leq \varepsilon] = [f_l \leq \varepsilon] \cap P'$ for $l \in I'$ and $l \in J'$, respectively. Moreover, by Lemma 5.1 the sublevels are all nonempty.

We claim that problem (5.2.8) is WC with the same μ_0. So, for any arbitrary subset $S \subseteq L', |S| < +\infty$ and $\varepsilon > 0$, we need to prove that

$$K_S'^{\mu_0}(\varepsilon) := P' \cap (\cap_{l\in S\cap J'}[f_l' \leq \varepsilon]) \cap (\cap_{l\in S\cap I'}[f_l' \leq \mu_0 + \varepsilon]) \neq \emptyset.$$

First, if $\varepsilon = \varepsilon_0$, $K_S'^{\mu_0}(\varepsilon_0) = P' \cap K_S^{\mu_0}(\varepsilon_0) \neq \emptyset$. Second, for every $\varepsilon > \varepsilon_0$, $K_S^{\mu_0}(\varepsilon_0) \subseteq K_S^{\mu_0}(\varepsilon)$ and by WC for problem (B) $K_S^{\mu_0}(\varepsilon_0) \neq \emptyset$, then we have $\emptyset \neq P' \cap K_S^{\mu_0}(\varepsilon_0) \subseteq P' \cap K_S^{\mu_0}(\varepsilon) = K_S'^{\mu_0}(\varepsilon)$. Finally, if $\varepsilon < \varepsilon_0$, then $K_{L_0}^{\mu_0}(\varepsilon) \subseteq P'$. Since problem (B) is WC for all $\varepsilon > 0$ we get $K_{L_0}^{\mu_0}(\varepsilon) \cap K_S^{\mu_0}(\varepsilon) \neq \emptyset$ (because $|S \cup L_0| < +\infty$) and the inclusion $K_{L_0}^{\mu_0}(\varepsilon) \cap K_S^{\mu_0}(\varepsilon) \subseteq P' \cap K_S^{\mu_0}(\varepsilon)$ implies that $K_S^{'\mu_0}(\varepsilon) := P' \cap K_S^{\mu_0}(\varepsilon) \neq \emptyset$. Therefore, (5.2.8) is WC.

To prove that (5.2.8) is AR, we use the fact that problem (B) is. First, we observe that

$$P' \subseteq P, (P')^\infty \subseteq P^\infty \text{ and } R_0' := (P')^\infty \cap (\cap_{l \in L'}[(f_l')_q^\infty \leq 0]) = R_0. \tag{5.2.9}$$

Moreover, for any direction $v \in R_0 \subseteq P^\infty$, all the functions f_l are constant along the ray $x + \mathbb{R}_+ v$ for $x \in P$. By (5.2.9), for all $v \in (P')^\infty$, the following property hold: $x + tv \in P'$ for all $t \geq 0$ and $x \in P'$; additionally, $f_l'(x + tv) = f_l(x + tv)$ and $f_l'(x) = f_l(x)$ for all $l \in L'$. This proves that problem (5.2.8) is AR.

Thus, problem (5.2.8) is AR and WC without affine functions. By Lemma 5.3, (5.2.8) has a solution, $\bar{x}'$ and a finite minimum $\bar{\mu}'$. The solution must now be adjusted to generate a solution for the original problem. The set L_0 is finite and following the proof of [55, Lemma 5] we obtain a solution $\bar{x}$ for the original problem (B) with optimal value $\bar{\mu}$.

Consider any $l_1 \in L_0$. There must exists $u_1 \in P^\infty \cap \left(\cap_{l \in L}[(f_l)_q^\infty \leq 0]\right)$ such that all f_l with $l \in L \setminus L_0$ are constant along u_1 and the function f_{l_1} decreases linearly along u_1, otherwise we could exclude l_1 from L_0 (as stated before). For this u_1 we have $\bar{x}' + \lambda u_1 \in P$ for all $\lambda \geq 0$ and $f_l(\bar{x}' + \lambda u_1) \leq f_l(\bar{x}')$ for all $l \in L$. Now, by taking $\lambda \geq 0$ big enough, with $\bar{x}_1' := \bar{x}' + \lambda u_1$, we can get $f_{l_1}(\bar{x}_1') \leq \bar{\mu}'$ if $l_1 \in I$ or $f_{l_1}(\bar{x}_1') \leq 0$ if $l_1 \in J$, where $\bar{\mu}'$ is the infimum of (5.2.8). Also, $\bar{x}_1' \in P$ and $f_l(\bar{x}_1') \leq f_l(\bar{x}')$ for all $l \in L$, therefore $\bar{x}_1'$ is an optimal solution of the subproblem with $\tilde{L}' := I' \cup J' \cup \{l_1\}$.

By starting at $\bar{x}_1'$ instead of $\bar{x}'$ and taking another $l_2 \in L_0 \setminus \{l_1\}$ and so on, we can repeat this process $|L_0| - 1$ times and obtain a solution $\bar{x} \in P$ such that $f_i(\bar{x}) \leq \bar{\mu}'$ for all $i \in I$ and $f_j(\bar{x}) \leq 0$ for all $j \in J$. Finally, since $[f_j \leq 0] \subseteq [f_j \leq \varepsilon_0]$ and $[f_i \leq \bar{\mu}'] \subseteq [f_i \leq \mu_0 + \varepsilon_0]$ for all $j \in J \cap L_0$ and all $i \in I \cap L_0$, respectively. The feasible set of (5.2.5) includes the feasible set of (B), therefore $\bar{\mu}' \leq \bar{\mu}$ and $\bar{x}$ is a optimal solution of (B).

Furthermore, Lemma 5.1 says that if $\bar{x}$ is a solution of the problem (B), then the minimum of the $\mu_0 \in \mathbb{R}$ that satisfies the WC is the optimal value of the original problem. □

Remark 5.2. *In Problem* (B)*, assuming that* $P = \mathbb{R}^n$*, the functions* f_l*, for* $l \in L$*, are quasiconvex, lsc, proper, and belong to the class* $\mathcal{C}$*, and that the index sets satisfy* $|I| = 1$*,* $|J| < +\infty$*, we are able to compare our findings with those available in the existing literature.*

- (i) *If* $R_0 = \{0\}$*, the existence of a solution follows from [19, Theorem 4.1] when* $J = \emptyset$*. If additionally,* $J \neq \emptyset$ *[26, Theorem 4.1] similarly guarantees existence. Together, these theorems provide alternative proofs for Lemma 5.2.*
- (ii) *When* R_0 *is a subspace,* $|I| = 1$ *and* $J = \emptyset$*, the existence result follows from [19, Proposition 4.2]. Even when* P *is any nonempty closed convex set, the result remains valid as established in [19, Theorem 4.7]. For a finite set* J*, [19, Corollary 4.8], combined with [19, Proposition 4.9], ensure the existence of a solution. Additionally, [26, Theorem 4.1] provides another useful existence result for a finite set* J*, offering alternative proofs to Lemma 5.3.*
- (iii) *Finally, when* I, J *are finite set and* R_0 *is not a subspace, under additional hypotheses, Theorems 4.1 and 4.2 of [26] can be applied to establish the existence of a solution.*

It is well known that problem (B) *can be equivalently formulated using the constraint function* $\sup_{l\in J} f_l(x) \leq 0$*. Associated with this equivalent problem, we define the nonempty convex cone*

$$\tilde{R}_0 := P^\infty \cap [f_q^\infty \leq 0] \cap [(\sup_{l\in J} f_l)_q^\infty \leq 0].$$

Then, when the problem has a finite number of constraints, the results obtained in [19, 26] can be applied using Proposition 2.5.

Our results complement those presented in [26]. This is illustrated by the following example, in which [26, Theorem 4.2] is not applicable, whereas Lemma 5.3 can be successfully applied.

Example 5.1. *Consider the problem* $\min\{f(x) : x \in \mathbb{R}\}$*, where* $f(x) = x + |x|$ *belongs to* $\mathcal{C}$ *and* $J = \emptyset$*. The problem does not satisfy the hypotheses of [26, Theorem*

4.2], however, $R_0 = [f_q^\infty \leq 0] = -\mathbb{R}_+$ and the linear space generated by R_0 is $M = \mathbb{R}$, so $M^\perp = \{0\}$. Then, the auxiliary problem

$$\inf\{f'(x') : x' \in M^\perp = \{0\}\}$$

where $f'(x') = f(x' + u)$, for all $u \in -\mathbb{R}_+$, satisfies the assumption in Lemma 5.3. Thus, $f'(0)$ is the optimal value and now we recover the solution of the original problem where $f'(0) = f(0 - u) = -u + |-u| = 0, \forall u \geq 0$. Since $\mathbb{R} = M \oplus M^\perp$, fix $u \in R_0 = -\mathbb{R}_+$, there exists $x \in \mathbb{R}$ such that $x - u \in M^\perp$. This implies $x = u$. Thus, the optimal value of the original problem is $f'(0) = f(x) = 0$, for $x \leq 0$. Consequently, all $x \leq 0$ minimize f.

The following examples demonstrate that Theorem 5.2 fails when any of its hypotheses (assumptions) is omitted.

Example 5.2 (Function not in class $\mathcal{C}$)**.** *Consider the problem* $\inf\{f(x) : x \in \mathbb{R},\ f_1(x) \leq 0\} = 0$, *where $f(x) = e^{-x}$ and $f_1(x) = |x| - 1$ when $x \leq 1$ and $f_1(x) = 0$ when $x > 1$. The function f_1 does not belong to class $\mathcal{C}$ and $[(f_1)_q^\infty(\cdot) \leq 0] = \{0\}$, while $[f_q^\infty(\cdot) \leq 0] = \mathbb{R}_+$, so $R_0 = \{0\}$. Therefore, the problem is AR and does not have a solution.*

Example 5.3 (No AR condition)**.** *Consider the functions $f(x_1, x_2) = e^{-x_1} + x_2$ and $f_l(x_1, x_2) = -l^2 x_1 - l x_2$ for $l \in [0, 1]$. Here we have again* $\inf\{f(x_1, x_2) : (x_1, x_2) \in \mathbb{R}^2,\ f_l(x_1, x_2) \leq 0, \forall\ l \in [0, 1]\} = 0$. *Each function is lsc and quasiconvex in class $\mathcal{C}$ (actually convex) and $R_0 = \mathbb{R}_+ \times \{0\}$. The problem is not AR, since f is not constant along $(1, 0) \in R_0$ and does not have a solution.*

Example 5.4 (No polyhedral structure)**.** *Let $P = \{(x_1, x_2) \in \mathbb{R}^2 :\ x_2 \geq x_1^2\}$ and the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = x_1$, then $P^\infty = \{0\} \times \mathbb{R}_+$ and f is constant along P^∞, but* $\inf\{f(x_1, x_2) :\ (x_1, x_2) \in P\} = -\infty$.

Corollary 5.1. *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, i \in I$ be a lsc quasiconvex and proper functions belonging to class $\mathcal{C}$. Under the AR and WC conditions the problem $\mu := \inf_{x \in \mathbb{R}^n} \sup_{i \in I} f_i(x)$, has at least one solution.*

Proof. Just consider the problem (B) with $J = \emptyset$ and $P = \mathbb{R}^n$. □

Corollary 5.2. *Let f be lsc convex proper function and P a polyhedron. If for all $v \in P^\infty \cap [f^\infty \leq 0]$ we have $f(x+tv) = f(x)$ for all $x \in P$ and all $t \geq 0$, then f achieves a minimum on P.*

Another common assumption used to establish the existence of solution is that the functions are bounded from below; see, for instance, [27]. However, our results do not remain valid if the asymptotic regularity (AR) condition is replaced with the boundedness-from-below assumption.

Example 5.5. *Let $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x) = e^x$ be a quasiconvex, lsc and proper function that belongs to class $\mathcal{C}$, additionally f is bounded from below. The associated problem* $\min\{f(x) : x \in \mathbb{R}\}$ *is WC, not AR (because $f(x+tv) \neq f(x)$ for any $v \in R_0 = -\mathbb{R}_+$ and any $t > 0$) and do not admit a solution.*

5.3 Existence result beyond the polyhedral case

It is of interest to consider a broader class of sets for geometric constraints. The polyhedral structure is used in the proof of Theorem 5.2, to conclude $(M+P)^\infty = M + P^\infty$, and also in Lemma 5.3, to define the auxiliary problem with geometric restriction given by $P' = K_{L_0}^{\mu_0}(\varepsilon_0)$, so that we can apply Lemma 5.2 to this new problem. Therefore, to deal with a different set P, we need for this new P to hold these two properties.

Given two nonempty closed convex sets $A, B \subseteq \mathbb{R}^n$, it is not always true that $(A+B)^\infty = A^\infty + B^\infty$. Take, for instance, the sets $A = \{0\} \times \mathbb{R}$ and $B = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 \geq x_1^2\}$. Our approach to extend the previous results will be done through a family of sets with a special decomposition.

Lemma 5.4. *Let $A, B \subseteq \mathbb{R}^n$ be two nonempty sets.*

(a) If A is compact and B is closed, then, $A+B$ is closed and $(A+B)^\infty = A^\infty + B^\infty = B^\infty$.

(b) If $A = C_0 + A^\infty$ and $B = C_1 + B^\infty$ with C_0, C_1 compact and $A^\infty + B^\infty$ is closed, then $A+B$ is closed and $(A+B)^\infty = A^\infty + B^\infty$.

Proof. (a): This is a well-known result.

(b): From the decomposition of A and B, and using item (a) we have $(A+B)^\infty = (C_0 + C_1 + A^\infty + B^\infty) = (A^\infty + B^\infty)^\infty$, and since $A^\infty + B^\infty$ is a closed cone, we conclude $(A+B)^\infty = (A^\infty + B^\infty)^\infty = A^\infty + B^\infty$. □

The next example appears in [58, Example 3.3]

Example 5.6. *The hypothesis "$A^\infty + B^\infty$ is closed" does not always hold. Indeed, we can consider the sets*

$$\begin{aligned} A &= \mathbb{R}(1,0,1) = A^\infty, \\ B &= \{(x,y,z) \in \mathbb{R}^3 : \; z \geq \sqrt{x^2+y^2}\} = B^\infty. \end{aligned}$$

For convex sets, the decomposition in (b) is called Motzkin decomposition (see for instance [28]).

Definition 5.3. *Let $A \subseteq \mathbb{R}^n$ be a nonempty convex set, A is said to be Motzkin decomposable (M-decomposable for short) if there exists a compact convex set $C \subseteq \mathbb{R}^n$ and a closed convex cone $D \subseteq \mathbb{R}^n$ such that $A = C + D$.*

The set D in the definition above always coincides with $A^{\mathrm{rec}} = A^\infty$. A particular case in which the sum is closed can be derived from Dieudonné's Theorem.

Corollary 5.3. *If A and B allow M-decompositions and $A^\infty \cap (-B^\infty)$ is a subspace. Then, $A+B$, $A^\infty + B^\infty$ are closed and $(A+B)^\infty = A^\infty + B^\infty$.*

In particular, all polyhedrons are M-decomposable, and the sum of their asymptotic cones is always a closed polyhedral cone, a direct consequence of [2, Proposition 1.1.18].

For the set P' (as defined in the proof of Theorem 5.2), we also require this representation. However, this property does not necessarily hold when P admits an M-decomposition. For example, consider an ice-cream cone intersected with a vertical plane that does not pass through the origin (as in [35]). The resulting set is a solid half-hyperbola, which fails to be M-decomposable. Specifically, let

$$\begin{aligned} A &= \{(x,y,z)\in\mathbb{R}^3:\ z\geq 0,\ z^2\geq x^2+y^2\},\\ B &= \{(1,y,z)\in\mathbb{R}^3:\ y,z\in\mathbb{R}\},\\ A\cap B &= \{(1,y,z)\in\mathbb{R}^3:\ z\geq 0,\ z^2\geq 1+y^2\}. \end{aligned}$$

With the above lemmas, we can extend Theorem 5.2.

Theorem 5.3. *Let $f_l:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}, l\in L$ be proper, lsc, quasiconvex functions belonging to $\mathcal{C}$, and let $P\subseteq\mathbb{R}^n$ be a M-decomposable set. Suppose the set $M := \big(\cap_{l\in L}[(f_l)^\infty_q\leq 0]\big)\cap P^\infty - \big(\cap_{l\in L}[(f_l)^\infty_q\leq 0]\big)\cap P^\infty$ is such that $P^\infty+M$ is closed. Additionally, if problem (B) satisfies the AR condition with $L_0=\emptyset$, then (B) admits a solution.*

Proof. We follow the previous proofs up to Lemma 5.3, using Lemma 5.4 when needed. □

5.4 Lower semicontinuity of the value function

Throughout this section, we address the following problem

$$\mu := \inf\{f_0(x) : x\in P,\ f_l(x)\leq 0, \forall\, l\in L\}, \tag{5.4.1}$$

where P is a closed convex set, $L=\{1,\cdots,m\}$ is a finite index set (note that this set L does not consider the cost function), each function is quasiconvex, lsc, proper, belongs to class $\mathcal{C}$, and $\operatorname{dom} f_l = P$ for all $l\in L$. The dual problem to (5.4.1) is

$$\nu := \sup_{\lambda\in\mathbb{R}^m_+}\inf_{x\in P}\{f_0(x)+\lambda_1 f_1(x)+\cdots+\lambda_m f_m(x)\}$$

and it is said that problem (5.4.1) has zero duality gap when $\mu=\nu$. To (5.4.1), associate the value function

$$\psi_L:\mathbb{R}^m\to\mathbb{R}\cup\{\pm\infty\}, a\mapsto \psi_L(a)=\inf\{f_0(x): x\in P, f_l(x)\leq a_l, \forall\, l\in L\}.$$

It is known that $\mu = \psi_L(0) = \nu$ holds whenever ψ_L is lsc at $a = 0$ and $\overline{(f_0, \ldots, f_m)(P) + \mathbb{R}^{m+1}_+}$ is convex (see [17, Theorem 4.1] for more details).

We study the lower semicontinuity of the value function at 0. For this, we proceed with a similar method to the one used to prove the existence of a solution, dealing with easier cases first and using auxiliary problems to reduce the more complicated cases to the simpler ones. For $\emptyset \neq L_1 \subseteq L$, we consider the subproblems

$$\mu_{L_1} := \inf\{f_0(x) : x \in P,\ f_l(x) \leq 0, \forall\, l \in L_1\}. \tag{5.4.2}$$

We apply the same approach to the value function, ψ_{L_1}, although it is important to note that function ψ_{L_1} is defined over $\mathbb{R}^{|L_1|}$. In particular, we just write $\mu = \mu_L$ and $\psi = \psi_L$.

The techniques used relate to those in [19, 26] by studying asymptotic properties, but using the AR condition instead of requiring for the set $R_0 = P^\infty \cap \bigcap_{l \in L \cup \{0\}} [(f_l)^\infty_q \leq 0]$ (or a similar set constructed by taking only some of the functions) to be a linear space. The AR condition covers cases where this does not happen, when all the involved functions are constant along some direction v, but $-v$ does not belong to R_0.

Lemma 5.5. *Let us consider problem (5.4.1), $f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, l \in L$ be proper, lsc, quasiconvex functions belonging to $\mathcal{C}$, and let $P \subseteq \mathbb{R}^n$ be a nonempty closed convex set. If $R_0 = \{0\}$ and the problem is WC, then $\overline{\psi}(0) = \psi(0) \in \mathbb{R}$.*

Proof. This is a direct consequence of [19, Theorem 5.2]. □

Lemma 5.6. *Let us consider problem (5.4.1), $f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, l \in L$ be proper, lsc, quasiconvex functions belonging to $\mathcal{C}$, and let $P \subseteq \mathbb{R}^n$ be a polyhedron. Assume that (5.4.1) is WC and satisfies the AR condition with $L_0 = \emptyset$. Then $\overline{\psi}(0) = \psi(0) \in \mathbb{R}$.*

Proof. We begin by recalling that AR with $L_0 = \emptyset$ means that for all $v \in R_0$ we have $f_l(x + tv) = f_l(x)$ for all $l \in L \cup \{0\}$, $t \geq 0$, $x \in P$. Under these hypotheses, Lemma 5.3 guarantees the existence of a solution to problem (5.4.1). Moreover, the auxiliary problem

$$\inf\{f'_0(x') : x' \in P', f'_l(x') \leq 0, \forall\, l \in L\}, \tag{5.4.3}$$

is WC and AR with $R_0' = \{0\}$ (where P', f_l' and R_0' are as defined in the proof of Lemma 5.3). From Lemma 5.5, the value function of problem (5.4.3) is lsc at 0. To prove the lower semicontinuity of the value function at 0 for problem (5.4.1), we use [17, Theorem 5.1].

Let $i_0 \in L$, consider the sequences $(x_k)_{k\in\mathbb{N}} \subseteq P$ and for all $l \in L \setminus \{i_0\}$, $(q_k^l)_{k\in\mathbb{N}} \subseteq \mathbb{R}_+$ such that $f_l(x_k) + q_k^l \to 0$, for all $l \in L \setminus \{i_0\}$; $0 < f_{i_0}(x_k) \to 0$ and $f_0(x_k) < \mu$. For each $x_k \in P$, there exists $u_k^1, u_k^2 \in R_0$ such that $x_k' := x_k + u_k^1 - u_k^2 \in M^\perp$ and this implies $x_k' + u_k^2 = x_k + u_k^1 \in P + R_0 \subseteq P + P^\infty = P$. Therefore $x_k' \in P'$ and for all $l \in L \cup \{0\}$, $f_l'(x_k') = f_l(x_k' + u_k^2) = f_l(x_k + u_k^1) = f_l(x_k)$, because $u_k^1 \in R_0$. Since for all $l \in L \setminus \{i_0\}$, we have $f_l'(x_k') + q_k^l = f_l(x_k) + q_k^l \to 0$, $0 < f_{i_0}'(x_k') = f_{i_0}(x_k) \to 0$ and $f_0'(x_k') = f_0(x_k) < \mu$.

By the lower semicontinuity at 0 of the value function associated with the problem (5.4.3) and the fact that the optimal value of (5.4.3) is the same as the one of the original problem, as was shown in the proof of Lemma 5.3, one has $\mu = \limsup_{k\to+\infty} f_0'(x_k') = \limsup_{k\to+\infty} f_0(x_k)$ and so we conclude $\overline{\psi}(0) = \psi(0) \in \mathbb{R}$. □

Theorem 5.4. *Let us consider problem (5.4.1), $f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, l \in L$ be proper, lsc, quasiconvex functions belonging to $\mathcal{C}$, and let $P \subseteq \mathbb{R}^n$ be a polyhedron. Suppose that problem (5.4.1) is AR and WC. Then, $\overline{\psi}(0) = \psi(0)$.*

Proof. Under these hypotheses, Theorem 5.2 ensures the existence of a solution to the problem. Let L_0 be minimal, $\mu_0 \in \mathbb{R}$ such that problem (5.4.1) satisfies WC. Fix $\varepsilon_0 > 0$. We know that the following problem

$$\mu' = \inf\{f_0'(x) : \; x \in P', f_l'(x) \leq 0, \forall\, l \in L' := L \setminus L_0\}, \tag{5.4.4}$$

is WC and AR with $L_0' = \emptyset$, $\mu' = \mu$, $P' := \{x \in P : \; f_l(x) \leq \varepsilon_0, \forall\, l \in L_0\}$ and $f_l' := f_l + \iota_{P'}$ (see the proof of Theorem 5.2).

Moreover, from Lemma 5.6 we conclude that its value function is lsc at 0. Using [17, Theorem 5.1] we know that for each $i_0 \in L'$,

$$\begin{aligned} f'_l(x'_k) + q^l_k \to 0, q^l_k \geq 0, l \in L' \backslash \{i_0\}, 0 < f'_{i_0}(x'_k) \to 0, f'_0(x'_k) \leq \mu' \\ \Rightarrow \limsup_k f'_0(x'_k) = \mu' = \mu. \end{aligned} \tag{5.4.5}$$

Since L_0 is minimal (see Remark 5.1(ii)), for each $l_1 \in L_0$ there exists $u_1 \in R_0 \subseteq (P')^\infty$ such that $f_{l_1}(x) = \langle b_{l_1}, x \rangle + \beta_{l_1}$ for some $b_{l_1} \in \mathbb{R}^n \setminus \{0\}$, $\beta_{l_1} \in \mathbb{R}$ with $\langle b_{l_1}, u_1 \rangle < 0$ and

$$f'_l(x'_k) = f'_l(x'_k + \lambda u_1), \forall\, \lambda \geq 0, \forall\, l \in L' \cup \{0\}.$$

Now we prove the lower semicontinuity of the value function associated with the new problem

$$\inf \left\{ f'_0(x) : x \in P'_1, f_{l_1}(x) \leq 0, f'_l(x) \leq 0, \forall\, l \in L' \right\},$$

where $P'_1 = P \cap \left(\cap_{l \in L_0 \backslash \{l_1\}} [f_l \leq \varepsilon_0] \right)$.

Let $i_0 \in L' \cup \{l_1\}$ and consider sequences $(x_k)_{k\in\mathbb{N}} \subseteq P'_1$ and $(q^l_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}_+$ for $l \in L' \cup \{l_1\} \setminus \{i_0\}$ such that

$$f'_l(x_k) + q^l_k \to 0,\; 0 < f_{i_0}(x_k) \to 0,\; f'_0(x_k) < \mu.$$

Note that $f'_l(x_k) + q^l_k \to 0$, implies that $x_k \in P'$ for k large enough and so $f'_l(x_k) + q^l_k = f_l(x_k) + q^l_k = f_l(x_k + \lambda u_1) + q^l_k \to 0$, where the second equality holds true for all $\lambda \geq 0$. We consider two cases:

Case 1: $i_0 \in L'$. In this case, $f_{l_1}(x_k) + q^{l_1}_k \to 0$ implies $f_{l_1}(x_k) \leq \varepsilon_0$ for k large enough, therefore $x_k \in P'$. So this x_k falls under (5.4.5) which means that $\limsup_{k\to+\infty} f'_0(x_k) = \mu$. And so $\overline{\psi}(0) = \psi(0)$.

Case 2: $i_0 = l_1$. We know that there exists $\lambda > 0$ such that $f_{l_1}(x_k + \lambda u_1) < 0$ and $f_l(x_k + \lambda u_1) = f_l(x_k)$ for all $l \in L' \cup \{0\}$, from our choice of L_0. Now let us study the sequences $f_l(x_k + \lambda u_1) + q^l_k = f_l(x_k) + q^l_k$ with $l \in L'$. First, suppose that for each $l \in L'$, the function satisfies $f_l(x_k) \leq 0$ for all sufficiently large k. In this case, $x_k + \lambda u_1$ becomes a feasible for problem (5.4.4), which contradicts the inequality $f'_0(x_k) < \mu$. Hence, this scenario cannot occur. Therefore, there exists at least one $l_2 \in L'$ such that $0 < f_{l_2}(x_k) \to 0$ for infinite many k and, so x_k falls under (5.4.5) which means that $\limsup_{k\to+\infty} f_0'(x_k) = \mu$. This implies $\overline{\psi}(0) = \psi(0)$. □

The previous result deals with problems for which only some functions remain constant along asymptotic directions, so it cover cases which [19, Theorem 5.2] cannot.

Now we introduce results that do not require the AR and WC hypothesis. For this, we recall [40, Proposition 1].

Proposition 5.1. *Let us consider problem (5.4.1), $f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, l \in L$ be proper, quasiconvex functions and let $P \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Assume that $n < m = |L|$. Then there exists $L_1 \subseteq L$ with $|L_1| = n$ such that $\psi_{L_1}(0) = \psi(0) = \mu$.*

Note that this result is originally stated for convex functions, but the same proof works for quasiconvex functions, since only the convexity of the level sets is considered and not the convexity of the functions themselves.

Using Proposition 5.1, together with [17, Theorem 5.1], we are able to study the lower semicontinuity of the value function via the subproblem with the same optimal value.

Lemma 5.7. *Let us consider problem (5.4.1), $f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, l \in L$ be proper, lsc, quasiconvex functions and let $P \subseteq \mathbb{R}^n$ be a closed convex set. Let $L_1 \subseteq L$ with $|L_1| = n < m = |L|$ be such that $\psi_{L_1}(0) = \psi(0)$. If $\psi_{L_1}(0) = \overline{\psi}_{L_1}(0)$, then also $\overline{\psi}(0) = \psi(0)$.*

Proof. From [17, Theorem 5.1], if ψ_{L_1} is lsc at 0, then for every $l_0 \in L_1$ and each x_k, q_k^l with $|x_k| \to +\infty$ and $q_k^l \geq 0$ such that $f_l(x_k) + q_k^l \to 0$ for all $l \in L_1 \setminus \{l_0\}$; $0 < f_{l_0}(x_k) \to 0$ and $f_0(x_k) < \psi_{L_1}(0) = \psi(0) = \mu$ it holds that $\limsup_k f_0(x_k) = \mu$.

We now consider $l_1 \in L$ and sequences x_k, q_k^l with $|x_k| \to +\infty$, $q_k^l \geq 0$ such that $f_l(x_k) + q_k^l \to 0$ for all $l \in L \setminus \{l_1\}$; $0 < f_{l_1}(x_k) \to 0$ and $f_0(x_k) < \psi(0)$. Our aim is to prove that, just as before, $\limsup_k f_0(x_k) = \mu$.

First, we note that if $l_1 \in L_1$, then the sequence x_k falls in the first case and so $\limsup_k f_0(x_k) = \mu$. Therefore, we are only interested in $l_1 \notin L_1$. Note that for each $k \in \mathbb{N}$ there must exist $l_k \in L_1$ such that $0 < f_{l_k}(x_k)$, otherwise x_k would be feasible for the subproblem (5.4.2) with $f_0(x_k) < \mu$, which is a contradiction. Therefore, since $|L_1| = n < +\infty$, there is at least one $l_2 \in L_1$ and a subsequence x_{k_r} such that $0 < f_{l_2}(x_{k_r})$ for all $r \in \mathbb{N}$. Finally, note that

$$0 < f_{l_2}(x_{k_r}) \leq f_{l_2}(x_{k_r}) + q_{k_r}^{l_2} \to 0.$$

Then $0 < f_{l_2}(x_{k_r}) \to 0$, and since $l_2 \in L_1$, this means that we are once again in the first case, so

$$\mu = \limsup_r f_0(x_{k_r}) \leq \limsup_k f_0(x_k) \leq \mu,$$

from which we conclude that $\limsup_k f_0(x_k) = \mu$ and ψ is lsc at 0. □

By following the work in [14], another instance in which we can prove the lower semicontinuity of the value function is obtained.

We say a direction v is a direction of recession for the problem (5.4.1) whenever $(f_0)_q^\infty(v) \leq 0$ and v is a direction of recession for the feasible set, i.e., $v \in P^\infty \cap (\cap_{l=1,\cdots,m}[f_l \leq 0]^\infty)$.

Theorem 5.5. *Assume that $P = \mathbb{R}^n$, each function f_0 and f_l with $l \in L = \{1, \cdots, m\}, m \geq 1$ is quasiconvex, proper, lsc and in $\mathcal{C}$. The duality gap is zero whenever at least one of the directions of recession for (5.4.1) is either an ia-direction of recession of f_0 or an ia-direction of recession of the feasible set.*

Proof. Knowing that for all $l \in L \cup \{0\}$, $[f_l \leq \lambda]^\infty = [(f_l)_q^\infty \leq 0]$ for any $\lambda \in \mathbb{R}$ such that $[f_l \leq \lambda] \neq \emptyset$ as long as f_l is lsc, proper and belongs to $\mathcal{C}$, using Proposition 2.9, we can follow the proofs of Theorems 4.1 and 4.2 in [14] to conclude in a similar manner as [14, Theorem 5.1]. □

The original result in [14] is stated for convex function. However, the concepts of ia- and ba-direction of recession has been further developed in [38] to deal with the quasiconvex case, but following a different approach, considering special recession direction of the Lagrangian.

Under much stronger assumptions, we can extend this result to problems with infinitely many constraints by leveraging the AR property. More precisely, we consider problem

(5.4.1) with an infinite index set L. In this setting, the value function ψ associated with (5.4.1) is

$$\psi : \mathbb{R}^{(L)} \to \mathbb{R} \cup \{\pm\infty\}, a \mapsto \psi(a) = \inf \{f_0(x) : x \in P,\ f_l(x) \leq a_l, \forall\ l \in L\}.$$

We then attempt to reduce the problem and study the lower semicontinuity of this value function via the value function of the subproblems, for this we consider standard hypothesis.

Theorem 5.6. *Let us consider problem (5.4.1), $f_l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, l \in L$ be proper, quasiconvex functions and let $P \subseteq \mathbb{R}^n$ be a polyhedron. Suppose that problem (5.4.1) is AR and WC, L is a (possibly infinite) compact set, the mapping $l \mapsto f_l$ is upper semicontinuous on L and all subproblems (5.4.2) with $L_1 \subseteq L$ and $|L_1| = n$ are AR. Then $\overline{\psi}(0) = \psi(0)$ and also, there exists $\bar{L} \subseteq L$ with $|\bar{L}| = n$ such that $\psi_{\bar{L}}(0) = \psi(0)$.*

Proof. From the definition of ψ, for any $\varepsilon > 0$

$$[f_0 \leq \psi(0) - \varepsilon] \cap (\cap_{l \in L}[f_l \leq 0]) \cap P = \emptyset.$$

Since the problem is AR, WC and each sublevel set is closed and convex, by [55, Corollary 1], for all $\varepsilon > 0$ there exists a nonempty set $L_\varepsilon \subseteq L$, with $|L_\varepsilon| \leq n$, such that

$$[f_0 \leq \psi(0) - \varepsilon] \cap (\cap_{l \in L_\varepsilon}[f_l \leq 0]) \cap P = \emptyset. \tag{5.4.6}$$

For each $L_\varepsilon \subseteq L$ with $|L_\varepsilon| = n$ (without loss of generality we assume $|L_\varepsilon| = n$, since we can take the intersection with extra $[f_l \leq 0]$ and the left-hand side remains empty), we fix an enumeration $L_\varepsilon = \{l_1, l_2, \ldots, l_n\}$ of distinct elements (where $l_i \neq l_j$ for $i \neq j$) and consider the corresponding subproblem

$$\inf\{f_0(x) : x \in P, f_{l_1}(x) \leq 0, \cdots, f_{l_n}(x) \leq 0\}. \tag{5.4.7}$$

We associate with this subproblem its respective value function $\psi_{L_\varepsilon} : \mathbb{R}^n \to \overline{\mathbb{R}}$, $a \mapsto \psi_{L_\varepsilon}(a) = \inf\{f_0(x) : x \in P, f_{l_i}(x) \leq a_i, \forall\ i \in \{1, \ldots, n\}\}$, where $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$.

To compare the value function ψ with ψ_{L_ε}, we introduce the auxiliary function $\tilde{\psi}_{L_\varepsilon} : \mathbb{R}^{(L)} \to \mathbb{R}\cup\{\pm\infty\}$, defined by $\tilde{\psi}_{L_\varepsilon}(a) = \psi_{L_\varepsilon}(a|_{L_\varepsilon})$, (just considers the constraints indexed by L_ε) where $a = (a_l)_{l\in L} \in \mathbb{R}^{(L)}$ and $a|_{L_\varepsilon} \in \mathbb{R}^n$ be the restriction of a to the index set L_ε, following the same ordering as before.

Note that for all $a \in \mathbb{R}^{(L)}, \psi(a) \geq \tilde{\psi}_{L_\varepsilon}(a)$, then $\overline{\psi}(0) \geq \overline{\tilde{\psi}}_{L_\varepsilon}(0)$. From (5.4.6), for any $x \in (\cap_{l\in L_\varepsilon}[f_l \leq 0]) \cap P$ we have $f_0(x) \geq \psi(0) - \varepsilon$ and, so for $0 \in \mathbb{R}^{(L)}$

$$\psi_{L_\varepsilon}(0|_{L_\varepsilon}) = \tilde{\psi}_{L_\varepsilon}(0) \geq \psi(0) - \varepsilon \geq \tilde{\psi}_{L_\varepsilon}(0) - \varepsilon.$$

Additionally, since the subproblem (5.4.7) is AR and retains the WC property from (5.4.1), Theorem 5.4 shows that $\overline{\psi}_{L_\varepsilon}(0) = \psi_{L_\varepsilon}(0)$.

Now note that any element $b \in \mathbb{R}^n$ can take the form $b = a|_{L_\varepsilon}$ for an appropriated $a \in \mathbb{R}^{(L)}$, so

$$\overline{\tilde{\psi}_{L_\varepsilon}}(0) = \liminf_{a\to 0} \tilde{\psi}_{L_\varepsilon}(a) = \liminf_{a\to 0} \psi_{L_\varepsilon}(a|_{L_\varepsilon}) = \liminf_{b\to 0} \psi_{L_\varepsilon}(b) = \overline{\psi}_{L_\varepsilon}(0)$$

then we can conclude

$$\bar{\psi}_{L_\varepsilon}(0) = \psi_{L_\varepsilon}(0) \geq \psi(0) - \varepsilon \geq \bar{\psi}(0) - \varepsilon \geq \bar{\tilde{\psi}}_{L_\varepsilon}(0) - \varepsilon = \bar{\psi}_{L_\varepsilon}(0) - \varepsilon$$

Note that the 0s involved belong to different spaces.

Suppose that for all $\varepsilon > 0$, $L_\varepsilon = \{l_\varepsilon^1, \ldots, l_\varepsilon^n\} \subseteq L$. Let $\varepsilon = \frac{1}{k}$, $k \in \mathbb{N}$ and take $k \to +\infty$, since $l_{\frac{1}{k}}^1 \in L$ for all $k \in \mathbb{N}$ and L is compact, necessarily $l_{\frac{1}{k}}^1$ must have a subsequence with a limit $\bar{l}^1$, we can continue to take subsequences for $l_{\frac{1}{k}}^r$ for $r = 2, \ldots, n$ and we end up with a limit $\bar{L} = \{\bar{l}^1, \ldots, \bar{l}^n\}$ for which

$$\overline{\psi}_{\bar{L}}(0) \geq \psi(0) \geq \overline{\psi}(0) \geq \overline{\psi}_{\bar{L}}(0).$$

Therefore, $\overline{\psi}(0) = \psi(0)$ and also $\psi(0) = \psi_{\bar{L}}(0)$. □

Chapter 6

Conclusions and future work

6.1 Conclusions

We have developed an algebraic representation of the limiting normal cone to the union of two polyhedra, as well as its graph, which we then used successfully to establish local uniqueness and stability properties involving the solution mapping of the M-stationary conditions.

We have also developed an algebraic representation of the limiting normal cone to a quadric surface, as well as to the union of two quadrics. Based on this, we derived a local uniqueness result for a problem with linear constraints and a geometric constraint defined by a single quadric surface set.

Through the application of asymptotic conditions and local properties of quasiconvex functions, we have sucessfully extended classical results from convex analysis to broader function classes.

Moreover, our work demonstrates the applicability of asymptotic techniques, previously used in the finite setting, to semi-infinite programming beyond the convex case.

6.2 Future Work

The research on this thesis has lead us to valuable results, but also gave rise to some open questions that deserve to be studied in future works. Some of them are described below:

- Further develope stability results by studying when the mappings $\mathcal{M}$ and $\tilde{\mathcal{M}}$ introduced in Chapter 4 are (locally) upper Lipschitz.
- Noting that the M-stationary condition, in case we have geometric constraint given by a quadric surface, can be expresed as

$$\nabla f(x) + \nabla g(x)^\top \lambda + \nabla h(x)^\top \mu + Dz = 0,$$

  for the problem

$$\min\{f(x) : \; g(x) \leq 0, h(x) = 0, x \in X \doteq \{x \in \mathbb{R}^n : \; q(x) = 0\}\},$$

  with $z \in X_0$ (X_0 as in Chapter 4), when $\nabla q(x) = 0$ and $z = tx$ when $\nabla q(x) \neq 0$. This can be expresed as $||tx - z||\nabla q(x) = 0$ and $z \in X$ for some $t \in \mathbb{R}$. This makes it seem that, when dealing with a geometric constraint given by a quadric surface, the M-stationary condition relates to a KKT condition with a special type of complementarity condition. This could be further explored.
- Continue the study of the ia- and ba-direction of recession in the quasiconvex setting and see if they can be used in the semi-infinite setting.
- Although we require strong hypotheses, the final section of Chapter 5 suggests that our approach could be useful for addressing duality even in the semi-infinite case.

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