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Ciencias Físicas

MultiNUT-AdS spacetimes

A thesis submitted in fulfillment of the requirements for the degree of Master in science
by

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CONCEPCIÓN • CHILE

April 2026

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To Paloma

Acknowledgments

Escribir esta sección fue, quizás, una de las tareas más significativas y complicadas para mí. No porque la física carezca de estos atributos ya mencionados, sino más bien por el hecho de pensar en aquellas personas que han sido un gran apoyo para mí; es más, algunas de ellas lo siguen siendo. Más allá de eso, ontológicamente hablando, me permitió acceder a mi memoria y conectar con mi corazón, recordándome el amor y la admiración que siento por estos seres humanos.

Sin más dilación, me gustaría comenzar agradeciendo a mis padres, Gabriel Liborio y Silvana Jeannette, y a mis hermanos, Cristofer Gabriel y Natalia Catalina Jesús. Sin su apoyo durante todo este tiempo, nada de esto hubiese sido posible. Les estoy infinitamente agradecido y los amo con todo mi corazón. Además, me gustaría agradecer a mis cuñados, Elias Comen y Camila Sanhueza, por el amor que entregan a mis hermanos y también por su compañía en eventos familiares. Más aún, me gustaría agradecer a mis dos sobrinos, Adriel Elias y Mateo Enrique, por soportarme como tío y aguantar mis chistes fomes. Finalmente, agradezco a Graciela Uribe, más conocida como la “Mami Quela”, por abrirme las puertas de su casa en un momento en que más lo necesité y por darme amor a diario desde que tengo la dicha de compartir con ella.

Me gustaría continuar agradeciendo a mi director de tesis, Cristóbal Corral Badiola, por su infinita paciencia (no exagero con esto) al enseñarme física, por responder siempre mis dudas y por ordenar la forma caótica en la que pienso esta disciplina. Más aún, por hacerme ver la belleza de calcular y de apreciar la física mucho más allá del arduo trabajo que esta significa. Además, agradezco las conversaciones sobre política, música, vida personal, literatura y las “tallas” que aparecían mientras almorzábamos o tomábamos un café, entre muchas otras cosas.

También quisiera agradecer a mi profesor coguía de tesis, Julio Oliva Zapata, por motivarme en distintos cursos que impartió durante el pregrado y postgrado; por tener siempre la voluntad de contestar mis preguntas; y por mantener la puerta de su oficina abierta para ayudarme. Gracias por su infinita generosidad y ayuda constante con física, burocracia, gestiones y consejos de vida.

Me gustaría continuar agradeciendo a Cristián Erices Osorio por dedicar gran parte del verano de 2026 a conversar conmigo sobre física y enseñarme tópicos "avanzados" y muy interesantes sobre relatividad general, así como también sobre temas de investigación. Asimismo, agradezco la posibilidad de colaborar junto a él.

Adicionalmente, quiero extender mi gratitud a Daniel Flores-Alfonso, también por permitirme colaborar con él, por responder siempre las dudas que tenía y por enseñarme algunos tópicos de física que desconocía. Además, por mantener siempre un buen humor, contestar preguntas y no tener problemas en explicar algo muy complejo con palabras que cualquier ser humano podría entender.

Consecuentemente, me gustaría seguir extendiendo mi gratitud a profesores que también fueron una inspiración para mis estudios, así como por responder mis dudas en momentos en que necesitaba entender algún concepto o simplemente quería discutir sobre física: Andrés Anabalón, Adolfo Cisterna, Juan Crisóstomo, Fabrizio Canfora, Marcela Cárdenas, Aldo Delgado, Simón del Pino, Ernesto Frodden, Nicolas Grandi, Hernán González, Fernando Izaurieta, Marcela Lagos, Olivera Miskovic, Pedro Nuñez, Rodrigo Olea, Francisco Rojas, Guillermo Silva, Pablo Solano, Jorge Zanelli y muchos otros ¡Muchas gracias!.

Por otro lado, me gustaría agradecer a amigos y compañeros que conocí en la universidad, durante mi estadía en Concepción y también en conferencias; amigos con los cuales pude hablar de física, de la vida e incluso abrir mi corazón: Juan José Altamirano, Cielo R. de Arellano, Nicolás Cáceres, Marcos Canedo, Rafael Coquedan, Valentino Delle Rose, Borja Diez, José Figueroa, Leonardo Gajardo, Keanu Müller, Marcelo Oyarzo, Lucas Polymeris, Martín Quijada, Franco Sandoval, Leonardo Sanhueza, Fernanda Strotkötter, Ricardo Stuardo, Lilianne Tapia, Luis Urritia, Constanza Vargas, entre muchos otros. ¡Los quiero mucho!

También me gustaría agradecer a todos mis amigos de Puerto Montt, aquellos con quienes comparto el gusto por la música, las tocatas, las cosas bizarras y también parte de mi educación preuniversitaria: Renato Agüero, Maximiliano Anabalón, Boris Chamia, Christian Dörner, Martin González, Inti Loncon, Mirson Ojeda, Nicolas Olavarria, Valentina Osorio, William Oyarzo, Tomás Retamal, Bruno Salazar, César San Martin, Fernando Sanchez, Gabriel Toledo, Mizael Vargas, Matías Velásquez, Miguel Yañez, y muchos otros. ¡Los quiero mucho!

También me gustaría agradecer a los compañeros de equipo de Brazilian Jiu-Jitsu. Por una parte, a mi actual equipo, Cicero Costha de Viña del Mar, y a Alliance Puerto Montt, por transmitir el amor y el respeto a este hermoso arte marcial.

Para finalizar, me gustaría agradecer a una persona que ha aparecido hace poco en mi vida y que se ha ganado un lugar en mi corazón. Amaray Cuevas, muchas gracias por tu amor, apoyo, compañía y sentido del humor.

Resumen

Los agujeros negros suelen describirse como objetos “calvos”. De acuerdo con el teorema de no-pelo, quedan completamente caracterizados por solo tres parámetros: su masa (M), momento angular (J) y carga eléctrica (Q). Sin embargo, los teoremas de no-existencia adquieren especial relevancia cuando se intenta evadir esta conclusión mediante la modificación o relajación de algunas de las hipótesis subyacentes al teorema. En este contexto, las soluciones estacionarias —y en particular los agujeros negros en rotación— constituyen valiosos laboratorios teóricos para explorar estas posibilidades. En esta línea, Mann y Stelea generalizaron el espacio-tiempo de Taub–NUT introduciendo múltiples y distintas cargas NUT en dimensiones superiores, lo que conduce a soluciones que denominaremos multi-NUT. Estos parámetros adicionales introducen de manera efectiva planos de rotación extra en el espacio-tiempo. Mostraron que las soluciones multi-NUT en Relatividad General con constante cosmológica existen únicamente si se modifica la normalización de las variedades Einstein–Kähler asociadas. En este trabajo, evitamos las obstrucciones representadas por su estrategia. Para ello introducimos campos escalares mínimamente acoplados con perfiles axiónicos, los cuales sostienen la geometría preservando la normalización estándar de las variedades Einstein–Kähler. Además, mostramos que esta misma geometría surge naturalmente dentro de un sector particular de la gravedad con curvatura cuadrática. Concluimos con una versión de los monopolos de Kaluza-Klein con base planar en AdS con distintas cargas magnéticas.

Abstract

Black holes are often described as “bald” objects. According to the no-hair theorem, they are completely characterized by only three parameters: their mass (M), angular momentum (J), and electric charge (Q). However, no-go theorems become particularly relevant when one attempts to evade this conclusion by modifying or relaxing some of the assumptions underlying the theorem. In this context, stationary solutions—and in particular rotating black holes—provide valuable theoretical laboratories to explore such possibilities. In this direction, Mann and Stelea generalized the Taub–NUT spacetime by introducing multiple and independent NUT charges in higher dimensions, leading to solutions that we will refer to as multi-NUT geometries. These additional parameters effectively introduce extra rotation planes in the spacetime. They showed that multi-NUT solutions in General Relativity with a cosmological constant exist only if the normalization of the associated Einstein–Kähler manifolds is modified. In this work, we circumvent the obstructions inherent in their construction. To this end, we introduce minimally coupled scalar fields with axionic profiles, which support the geometry while preserving the standard normalization of the Einstein-Kähler manifolds. Furthermore, we show that the same geometry naturally arises within a particular sector of quadratic curvature gravity. We conclude by presenting a version of Kaluza-Klein monopoles with planar base in AdS, carrying distinct magnetic charges.

Chapter 1

Introduction

1.1 Motivations

General Relativity (GR) is one of the most influential physical theories developed in the twentieth century. It describes the gravitational interaction as a geometric theory emerging from a curved four-dimensional spacetime. The theory can be formulated through an action principle [1], introduced by Hilbert in a fully covariant form. The corresponding field equations were proposed by Albert Einstein in 1915 [2]. GR predicts several phenomena that cannot be explained within Newtonian gravity, such as gravitational redshift, black holes, and gravitational waves [3]. The first exact solution to Einstein's equations was found by Karl Schwarzschild in 1916 [4]. This solution describes the spacetime geometry outside a spherically symmetric, uncharged, non-rotating, and static mass distribution. An important result associated with this geometry is Birkhoff's theorem [5], which states that any spherically symmetric solution of the vacuum Einstein equations must necessarily be static and asymptotically described by the Schwarzschild metric. Consequently, the exterior gravitational field of any spherically symmetric body, independently of its internal structure or dynamical evolution, is given by this solution.

One of the first attempts to incorporate angular momentum into the Schwarzschild solution was carried out by Taub [6] and later extended by Newman, Tamburino, and Unti [7]. The resulting geometry is known as the Taub-NUT (TN) solution. This spacetime contains an additional parameter, called the NUT charge, which introduces gravitomagnetic effects often interpreted as a form of angular momentum or "magnetic mass" in the geometry [8]. This feature becomes apparent, for instance, in the geodesic motion of test particles, and the TN solution is frequently regarded as a gravitational analogue of the Dirac magnetic monopole. Moreover, it is continuously connected to the Schwarzschild spacetime in the limit where the NUT charge vanishes. In the Euclidean sector, obtained via analytic continuation, the TN solution exhibits further remarkable properties. In particular, as emphasized by Hawking [9], the Euclidean TN geometry closely resembles instanton solutions in Yang–Mills theory. Both configurations are characterized by non-trivial topology and finite action, and they can be interpreted as gravitational analogues of gauge theory instantons, contributing non-perturbatively in a semiclassical path integral framework.

The Schwarzschild solution was generalized to higher dimensions by Tangherlini in Ref. [10]. In the same decade, Roy Kerr found the first exact rotating black hole solution in four dimensions [11]. Unlike the Schwarzschild spacetime, which is fully characterized by its mass, the Kerr solution depends on two parameters: the mass and a rotational parameter associated with angular momentum. Several decades later, Myers and Perry [12] discovered the most general asymptotically flat rotating black hole solution in arbitrary dimensions, allowing for multiple independent rotation planes. The Kerr solution is recovered as a particular case when only one rotational parameter is present, in the four-dimensional limit. The extension of these solutions to include a cosmological constant was later obtained by Gibbons et al. [13]. Several years later, Cvetič et al. in Ref. [14] constructed charged rotating black hole solutions in five dimensions, including electric and scalar fields with multiple rotation parameters in Maxwell-Chern-Simons theory. Recently, Barrientos et al. in Ref. [15] studied five-dimensional rotating black holes with two rotation parameters minimally coupled to scalar fields, providing generalizations of Myers–Perry solutions in the presence of matter.

Following the same reasoning, one could ask about the generalization of the TN metric with multiple NUT parameters. This generalization of TN geometry with multiple NUT parameters is known as multi-NUT solutions. The multi-NUT, in GR, was reported by Mann and Stelea in Ref. [16]. Moreover, in the same work, they realized that if we supplement GR with the cosmological constant, the field equations gives a restriction that relates the NUT parameters and the cosmological constant. Which tells that the existence of the multi-NUT only exist in GR, i.e., when the cosmological constant vanishes. In the next year, the same authors in Ref. [17] showed that the multi-NUT exists in GR with cosmological constant, provided that a non-canonical normalization of the Einstein-Kähler base manifold is used. They realized that the rescaling of these solutions implies that the curvature of the base manifold must be non-trivial, excluding planar topologies.

Planar AdS black holes constitute one of the most physically relevant classes of solutions in modern gravitational physics, as they provide the natural setting for describing strongly coupled quantum systems via the AdS/CFT correspondence [18–20]. In particular, dyonic AdS black holes with planar horizons have been used to describe holographic superconductors, in which DC conductivities can be obtained from linear perturbations of bulk gauge fields [21–24]. Moreover, in contrast to spherical geometries, planar horizons are non-compact and exhibit translational invariance, allowing thermodynamic quantities to be interpreted as densities and thereby supply the appropriate framework for modeling extended systems at finite temperature. Additionally, planar AdS black holes are free from finite-size effects such as the Hawking–Page phase transition [25], making them the simplest and most universal gravitational backgrounds in this context.

In this work, we construct multi-NUT geometries in GR with a cosmological constant supplemented with minimally coupled scalar fields. The presence of matter fields allows us to obtain multi-NUT solutions with flat transverse sections, avoiding the Mann and Stelea no-go results [17]. In particular, these solutions can be interpreted as planar AdS black holes carrying multiple NUT parameters with axionic charge. Additionally, they arise when the scalar fields exhibit axionic profiles.

On the other hand, it is known that higher-curvature corrections appear naturally when GR is considered as an effective field theory from a Wilsonian viewpoint. We show that similar multi-NUT geometries appear within a particular sector of quadratic curvature gravity, where only an (R^2) correction is considered for a specific value of the coupling constant.

Kaluza–Klein monopoles (KK monopoles) were first constructed by Gross, Perry, and Sorkin [26, 27] in asymptotically flat spacetime within the four-dimensional Einstein–Maxwell–dilaton theory, using the Taub–NUT solution as a seed metric. Later, Onemli and Tekin [28] showed that there are no five-dimensional KK monopoles in AdS spacetime that smoothly reduce to the flat-space solution in the limit where the cosmological constant vanishes. Subsequently, Mann and Stelea [29] revisited this no-go result and demonstrated that KK monopoles can indeed be constructed in the presence of a cosmological constant. In their framework, the field equations impose a relation between the cosmological constant and the NUT charge. This, in turn, implies that in the flat limit not only must the cosmological constant vanish, but the NUT charge must vanish as well. Moreover, in the same work they studied KK monopoles in higher dimensions by considering multi-NUT deformations of the seed metric, obtaining a constraint similar to that found in [16], namely that either the cosmological constant must vanish or all NUT charges must be equal. Motivated by these results, and using the mechanism proposed by Cisterna and Oliva [30], we construct Kaluza–Klein multi-monopole solutions in the presence of a cosmological constant.

This thesis is organized as follows. In Chapter 2, we provide a brief review of General Relativity, the variational principle, then in Chapter 3 we discuss several methods used to compute conserved charges. In Chapter 4, we discuss relevant solutions such as the Taub–NUT geometry, the Kerr solution, and the Myers–Perry black holes. In chapter 5, we introduce the multi-NUT solutions and present our main results. Finally, in Chapter 6, we summarize our conclusions and discuss some further aspects and possible directions for future research.

Chapter 2

General Relativity

2.1 Einstein Hilbert action

The 20th century was one of the most important periods for the development of two physical theories that remain highly relevant today. One of them is General Relativity (GR), formulated by Albert Einstein in 1915 [2]. It is a theory describing the dynamics of spacetime and can be understood as a geometric theory of gravity. Einstein proposed the equations of motion that describe the dynamics of spacetime; however, he did not initially provide an action principle that reproduces these equations. Such a formulation was later obtained by Hilbert [1]. The corresponding action, known as the *Einstein–Hilbert* action, is given by

$$S_{EH}[g_{\mu\nu}] = \kappa \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{|g|}(R - 2\Lambda), \tag{2.1.1}$$

where $\kappa = (16\pi G_N)^{-1}$ is related with the Newton's gravitational constant G_N, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, and we consider the cosmological constant Λ.[1] The action principle (2.1.1) can be supplemented with matter fields, which can be included via their own action, i.e. $S_{matter}[g_{\mu\nu}, \Phi_i]$, where Φ_i denotes the matter fields labeled by i that accounts for a collection of matter fields. Then, the total action

[1] In the original work the cosmological constant was not considering. However, today is adding for the proposal to deal with spaces that are asymptotically (A)dS.

reads

$$S[g_{\mu\nu}, \Phi_i] = S_{EH}[g_{\mu\nu}] + S_{matter}[g_{\mu\nu}, \Phi_i]. \tag{2.1.2}$$

Varying the total action with respect to the metric $g_{\mu\nu}$ yields the *Einstein field equations* with matter,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2\kappa} T_{\mu\nu}, \tag{2.1.3}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor and $T_{\mu\nu}$ is the stress energy-momentum tensor associated with the matter content, which is given by,

$$T_{\mu\nu} := -\frac{2}{\sqrt{|g|}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}. \tag{2.1.4}$$

These equations imply the Bianchi identities by virtue of the diffeomorphism invariance

$$\nabla^{\mu} G_{\mu\nu} = 0, \tag{2.1.5}$$

that, using the metricity condition, i.e., $\nabla_{\lambda} g_{\mu\nu} = 0$, implies that the energy-momentum tensor is conserved, that is,

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{2.1.6}$$

2.2 Variation principle in General Relativity

For the sake of symplicity, let us consider (2.1.1) with $\Lambda = 0$, that is,

$$S_EH[g_{\mu\nu}] = \kappa \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}R. \tag{2.2.1}$$

Arbitrary variations with respect to the metric yields,

$$\begin{aligned} \delta S_{EH} &= \kappa \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}\left(-\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}R + R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right) & (2.2.2)\\ &= \kappa \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}\left(G_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right), & (2.2.3)\end{aligned}$$

where we have used

$$\delta(\sqrt{|g|}) = -\frac{1}{2}\sqrt{|g|}g_{\mu\nu}\delta g^{\mu\nu} \tag{2.2.4}$$

In the second term, we can use the *Palatini identity*, i.e., $\delta R^{\lambda}{}_{\rho\mu\nu} = 2\nabla_{[\mu}\delta\Gamma^{\lambda}_{\rho|\nu]}$. And rewrite as

$$\delta R_{\mu\nu} = \delta^{\alpha\beta}_{\lambda\nu}\nabla_\alpha\delta\Gamma^{\lambda}_{\mu\beta}\,. \tag{2.2.5}$$

Therefore, the last term can be written as

$$\begin{aligned} \kappa\int_M d^4x\sqrt{|g|}g^{\mu\nu}\delta R_{\mu\nu} &= \kappa\int_M d^4x\sqrt{|g|}g^{\mu\nu}\left(\delta^{\alpha\beta}_{\lambda\nu}\nabla_\alpha\delta\Gamma^{\lambda}_{\mu\beta}\right)\\ &= \kappa\int_M d^4x\sqrt{|g|}\nabla_\alpha\left(\delta^{\alpha\beta}_{\lambda\nu}g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\beta}\right)\\ &:= \kappa\int_{\mathcal{M}} d^4x\sqrt{|g|}\nabla_\alpha\Theta^\alpha \end{aligned}$$

Thus, the field equations are $G_{\mu\nu} = 0$. Moreover, using the Stokes's theorem we can conclude that, [2]

$$\delta S_{EH}\Big|_{on-shell} = \kappa\int_{\mathcal{M}} d^4x\sqrt{|g|}\nabla_\alpha\Theta^\alpha = \kappa\sigma\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}n_\alpha\Theta^\alpha, \tag{2.2.6}$$

where $n = n^\mu\partial_\mu$ is the unit normal vector to the boundary $\partial\mathcal{M}$, and σ is its norm, i.e., $n^\mu n_\mu = \sigma$. Moreover, h is the determinant of the induced metric of $\partial\mathcal{M}$ which

[2] $\kappa\int_M d^4x\sqrt{|g|}\nabla_\alpha\Theta^\alpha = \kappa\int_{\partial M} d^3x\sqrt{|h|}n_\alpha\Theta^\alpha$

defines as $h_{\mu\nu} = g_{\mu\nu} - \sigma n_\mu n_\nu$ proyected onto $\partial\mathcal{M}$. Finally, using (7.2.6) we can write

$$\Theta^\alpha = 2g^{\lambda\alpha}g^{\rho\nu}\nabla_{[\nu}\delta g_{\lambda]\rho} \tag{2.2.7}$$

(for more details see the Appedix 7).

Therefore, the on-shell variation of the Einstein-Hilbert actions reads,

$$\delta S_{EH}\Big|_{on-shell} = \kappa\int_{\partial M} d^3x\sqrt{|h|}\left(n^\sigma h^{\mu\beta} - n^\mu h^{\beta\sigma}\right)\nabla_\mu\delta g_{\sigma\beta}. \tag{2.2.8}$$

We thus conclude that the Einstein-Hilbert action does not admits a well-defined variational principle with Dirichlet boundary conditions for the metric, i.e., when $\delta g^{\mu\nu}\Big|_{\partial\mathcal{M}} = 0$, because the normal derivative of the metric variations does not vanish.

To address this issue, Gibbons and Hawking [31] and independently by York [32], realized that the action can be supplemented by a boundary term

$$S_{GHY} = 2\kappa\sigma\int_{\partial\mathcal{M}} d^3x\sqrt{|h|}\mathcal{K}, \tag{2.2.9}$$

where $\mathcal{K} = h^{\mu\nu}\mathcal{K}_{\mu\nu}$ is the trace of the extrisic curvature of the hypersurface $\partial\mathcal{M}$, defined as $\mathcal{K}_{\mu\nu} = h^\lambda_\mu\nabla_\lambda n_\nu$. Taking the on-shell arbitrary variation with respect to the metric for (2.1.1) supplemented with the *Gibbons-Hawking-York* term, the normal derivatives of the metrics variation cancel, giving

$$\delta(S_{EH} + S_{GHY})\Big|_{on-shell} = \kappa\sigma\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\left[(\mathcal{K}_{\mu\nu} - h_{\mu\nu}\mathcal{K})\delta h^{\mu\nu} - 2\mathcal{D}_\mu V^\mu\right], \tag{2.2.10}$$

where V^μ is a vector field on $\partial\mathcal{M}$, and $\mathcal{D}_\mu$ is covariant derivative compatible with $h_{\mu\nu}$, i.e., $\mathcal{D}_\lambda h_{\mu\nu} = 0$ (for more details see the Appedix 8). The boundary $\partial\mathcal{M}$ is a closed hypersurface, therefore the total divergence term $\mathcal{D}_\mu V^\mu$ vanish. Therefore, the complete action reads,

$$S_{EH} + S_{GHY} = \kappa\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}R + 2\kappa\sigma\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{|h|}\mathcal{K}, \tag{2.2.11}$$

which gives a well-posed variational principle under Dirichlet boundary conditions for the metric (for all intermediate computations see Appedix 7).

Chapter 3

Conserved Charges

3.1 Conserved Charges

An important, physically relevant, and interesting aspect of physics is the concept of conserved charges. In Ref. [33], Emmy Noether proved two fundamental theorems. The first establishes that global symmetries lead to conserved charges. The second theorem states that local gauge symmetries, which contain arbitrary functions of spacetime, are related to differential identities, involving the equations of motion. This result has profound implications in physics, since it allows us to describe the physical behavior of a system in terms of its symmetries.

Many prescriptions have been developed to study conserved charges. For instance, the Hamiltonian method [34,35], Komar charges [36], the Abbott–Deser–Tekin (ADT) [37–39] method, Wald's covariant phase space formalism [40], quasilocal methods [41, 42], among many others. Each of these approaches has its own advantages and limitations .

In this section, we discuss some relevant methods that we use to compute conserved charges. For each method we consider, the Schwarzschild black hole will be used as an illustrative example.

3.2 Brown-York charges and quasilocal stress-energy tensor

This method considers the Einstein-Hilbert action suplemented by the *Gibbons-Hawking-York*, to have a well-defined variational principle, which we introduced before (for more details see Chaper 2). We compute conserved quantities with respect to a background reference. Due to this, we can add a boundary term that measures the extrinsic curvature of a reference background, that is,

$$S_{BY} = \kappa \int_{\mathcal{M}} d^4x \sqrt{|g|} R + 2\kappa\sigma \int_{\partial\mathcal{M}} d^3x \sqrt{|h|} (\mathcal{K} - \mathcal{K}_0) \tag{3.2.1}$$

where $\mathcal{K}_0$ is referred to as the reference background and the variation of Eq. (3.2.1) with respect to the metric yields [for more details see Chapter 2 and Appendix 8]

$$\delta S_{BY} = \kappa \int_{\mathcal{M}} d^4x \sqrt{|g|} \delta g^{\mu\nu} G_{\mu\nu} - \frac{1}{2}\kappa\sigma \int_{\partial\mathcal{M}} d^3x \sqrt{|h|} \delta g^{\mu\nu} \tau_{\mu\nu}. \tag{3.2.2}$$

where $\tau_{\mu\nu}$ is the Brown-York stress-energy tensor [43], read as

$$\tau_{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} \frac{\delta S_{bdy}}{\delta h^{\mu\nu}} = -2\kappa\sigma [\mathcal{K}_{\mu\nu} - h_{\mu\nu}(\mathcal{K} - \mathcal{K}_0)] \tag{3.2.3}$$

Using the Codazzi-Mainardi relation (8.5.12), we can prove that [3]

$$D^{\mu} \tau_{\mu\nu} = -2\kappa\sigma R_{\lambda\rho} n^{\lambda} h^{\rho}_{\nu} \,. \tag{3.2.4}$$

Evaluating the latter on-shell, one obtains $\mathcal{D}_{\mu}\tau^{\mu\nu} = 0$ since $R_{\mu\nu} = 0$. Therefore, $\tau_{\mu\nu}$ is covariantly conserved in vacuum. The latter allows us to defined a conserved current as

$$J^{\mu} := \tau^{\mu\nu} \xi_{\nu}, \tag{3.2.5}$$

[3] (for more details see Appendix 8)

where $\xi = \xi^\mu \partial_\mu$ is a Killing vector. Hence, J^μ is conserved with respect to the intrinsic derivative,

$$\begin{aligned} D_\mu J^\mu &= D_\mu(\tau^{\mu\nu}\xi_\nu) \\ &= D_\mu \tau^{\mu\nu}\xi_\nu + \tau^{\mu\nu} D_\mu \xi_\nu \\ &= D_\mu \tau^{\mu\nu}\xi_\nu + \tau^{\mu\nu} h^\lambda_{(\mu} h^\rho_{\nu)} \nabla_\lambda \xi_\rho \\ &= 0. \end{aligned}$$

where in the third line we use the fact that $\mathcal{D}_\mu \tau^{\mu\nu} = 0$ on-shell, $\nabla_{(\mu}\xi_{\nu)} = 0$. Since ξ^μ is a Killing vector. Finally, the Brown-York charge associated with ξ^μ is defined as the integral of J^μ over a codimension-two Cauchy hypersurface, Σ, with a timelike unit normal vector u^μ, i.e., $u^\mu u_\mu = -1$,

$$Q[\xi] = \int_\Sigma d^2x \sqrt{|\sigma|} u_\mu \tau^{\mu\nu} \xi_\nu \,, \tag{3.2.6}$$

where $\sigma_{\mu\nu} = h_{\mu\nu} + u_\mu u_\nu$ is the induced metric on Σ.

3.3 Arnowitt-Deser-Misner formalism

One of the most powerful approaches to study the dynamics of General Relativity is the Arnowitt-Deser-Misner (ADM) formalism [44]. This method, widely used since the second half of the twentieth century, allows one to rewrite the Einstein-Hilbert action in Hamiltonian form. Since the Hamiltonian formulation requires the identification of a temporal evolution parameter, the method is not manifestly covariant and therefore a preferred coordinate, the temporal one, must be chosen. Thus, spacetime is foliated into a family of spatial hypersurfaces labeled by a time parameter t. The geometry is then described in terms of three fundamental quantities: the induced spatial metric $\gamma_{ab}(x)$, the lapse function $N(x)$, and the shift vector $N^a(x)$. This decomposition is commonly known as the ADM formalism, after Arnowitt, Deser, and Misner. This section is based on [34, 35, 45]. For further details we refer the reader to these references.

In the Lagrangian formulation, the spacetime metric depends on the coordinates.

In the ADM formalism, the spacetime line element is rewritten in terms of the variables γ_{ab}, N, and N^a, which correspond to the induced metric, the lapse function, and the shift vector, respectively. The line element between two spacetime events can then be decomposed as

$$\begin{aligned} \mathrm{d}s^2 &= \gamma_{ab}(\mathrm{d}x^a + N^a \mathrm{d}t)(\mathrm{d}x^b + N^b \mathrm{d}t) - (N\mathrm{d}t)^2 \\ &= (N^a N_a - N^2)\mathrm{d}t^2 + 2N_a \mathrm{d}x^a \mathrm{d}t + \gamma_{ab}\mathrm{d}x^a \mathrm{d}x^b, \end{aligned} \tag{3.3.1}$$

where $(\mathrm{d}x^a + N^a \mathrm{d}t)$ represents the displacement along the spatial hypersurface, while $N\,\mathrm{d}t$ corresponds to the proper time between two neighboring hypersurfaces. Here, Latin characters denote boundary indices.

From the general form of the metric written in ADM variables (3.3.1), one can identify the relation between the spacetime metric components $g_{\mu\nu}$ and the ADM variables as

$$g_{tt} = N^a N_a - N^2, \tag{3.3.2}$$

$$g_{ta} = N_a, \tag{3.3.3}$$

$$g_{ab} = \gamma_{ab}. \tag{3.3.4}$$

Similarly, the components of the inverse metric $g^{\mu\nu}$ are given by

$$\begin{aligned} g^{tt} &= -\frac{1}{N^2}, \\ g^{ta} &= \frac{N^a}{N^2}, \\ g^{ab} &= \gamma^{ab} - \frac{N^a N^b}{N^2}. \end{aligned} \tag{3.3.5}$$

Finally, the four-dimensional volume element takes the form

$$\sqrt{-g}\,\mathrm{d}^4x = N\sqrt{\gamma}\,\mathrm{d}^3x\,\mathrm{d}t. \tag{3.3.6}$$

We rewrite the gravitational action in terms of these new variables as

$$S[\gamma_{ab}, N, N^i] = \int \mathrm{d}t \int \mathrm{d}^3x \sqrt{\gamma} N \left(K_{ab}K^{ab} - K^2 + \mathcal{R} + (\Delta^\lambda)_{;\lambda} \right). \tag{3.3.7}$$

Then, defining the canonical conjugate momenta in terms of the induced metric γ_{ab} by

$$\pi^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma_{ab}}}, \tag{3.3.8}$$

where " $\cdot$ " denotes the derivative with respect to the parameter t. On the other hand, the momenta associated with the variables N^μ will be clearly equal to zero,

$$P^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}^\mu} = 0. \tag{3.3.9}$$

In order to calculate the momenta, we introduce the following object,

$$G^{abcd} \equiv \frac{1}{2}\sqrt{\gamma}\left[(\gamma^{ab}\gamma^{bd} + \gamma^{ad}\gamma^{bc}) - 2\gamma^{ad}\gamma^{cd} \right]. \tag{3.3.10}$$

This object is called a supermetric. This allow us to rewrite the Lagrangian in the following form

$$\mathcal{L} = N\left(G^{abcd}K_{ab}K_{cd} + \sqrt{\gamma}\mathcal{R} \right). \tag{3.3.11}$$

Moreover, we can construct the canonical momentum using the supermetric, giving

$$\pi^{ab} = -G^{abcd}K_{cd}. \tag{3.3.12}$$

Finally, we can rewrite the velocities,γ_{ab}, in the form

$$\dot{\gamma_{ab}} = 2(N_{(a|b)} + NG_{abcd}\pi^{cd}). \tag{3.3.13}$$

where $N_{a|b} := \mathcal{D}_b N_a$ with $\mathcal{D}_b$ being compatible with the induced metric. Then, the Hamiltonian of GR can be written as,

$$\mathbf{H} = \int \mathrm{d}^3x (\pi^{ab}\dot{\gamma_{ab}} - \mathcal{L}), \tag{3.3.14}$$

where the Hamiltonian density is given by

$$(\pi^{ab}\dot{\gamma_{ab}} - \mathcal{L}) = N\sqrt{\gamma}(K_{ab}K^{ab} - K^2 - \mathcal{R}) - 2\pi^{ab}N_{a|b}. \tag{3.3.15}$$

Finally, the Hamiltonian can be writen in terms of the constraints as

$$\mathbf{H} = \int \mathrm{d}^3x[N\mathcal{H}_0 + N^a\mathcal{H}_a], \tag{3.3.16}$$

where

$$\begin{aligned} \mathcal{H}_0 &= N\sqrt{h}(K_{ab}K^{ab} - K^2 - \mathcal{R})\,, \\ \mathcal{H}_a &= 2\pi^b_{a|b}, \end{aligned}$$

are the Hamiltonian constraints and N with N^a play the role of Lagrange mutipliers. From hereon, we follow the approach of Ref. [34], which is closely related to the Regge-Teitelboim prescription [35]. First, rewrite the Hamiltonian action explictly in terms of the conjugate momenta and the induced metric, that is,

$$\mathbf{H}_0 = \int \left\{Nh^{-1/2}\left(\pi_{ab}\pi^{ab} - \frac{1}{2}\pi^2\right) - Nh^{1/2}\mathcal{R} - 2N_a h^{1/2}(h^{-1/2}\pi^{ab})_{|b}\right\} \mathrm{d}^3x. \tag{3.3.17}$$

Now, the arbitrary variation with respect to h_{ij} and π^{ij}, yields

$$\delta_\pi\mathbf{H}_0 = \int_{\Sigma_t} 2[Nh^{-1/2}\left(\pi_{ab}\pi^{ab} - \frac{1}{2}\pi h_{ab}\right) + \mathcal{D}_{(b}N_{a)}]\delta\pi^{ab}\mathrm{d}^3x - 2\oint_{S_t} N_a h^{-1/2} r_b \delta\pi^{ab}\sqrt{\sigma}\mathrm{d}^2x\,. \tag{3.3.18}$$

On the other hand,

$$\begin{aligned} \delta_h\mathbf{H}_0 \;\;=\;\; & \int_{\Sigma_t}\left[-\frac{1}{2}Nh^{-1/2}\left(\pi^{cd}\pi_{cd} - \frac{1}{2}\pi^2\right)h^{ab} + 2Nh^{-1/2}\left(\pi^a{}_c\pi^{cb} - \frac{1}{2}\pi\pi^{ab}\right) + Nh^{1/2}\mathcal{G}^{ab}\right. \\ & \left. + 2\pi^{c(a}N^{b)}_{|b} + h^{1/2}\left(h^{ab}N^{;d}_{,d} - N^{,ab}\right) - h^{1/2}\left(h^{-1/2}\pi^{ab}N^d\right)_{|d}\right]\delta h_{ab}d^3x \\ & \left. + \oint_{S_t} Nh^{ab}\delta h_{ab,c}r^c\sqrt{\sigma}d^2x\right]. \end{aligned}$$

then the complete variation will be the sum of the previous variations, i.e.

$$\delta\mathbf{H}_0 = \int_{\Sigma_t} \left(\mathcal{H}_{ab}\delta\pi^{ab} + \mathcal{P}^{ab}\delta h_{ab}\right) \mathrm{d}^3x + \oint_{S_t} \left[N h^{ab}\delta_{ab,c}r^c - 2N_a h^{-1/2} r_b \delta\pi^{ab}\right] \, , \quad (3.3.19)$$

where

$$\begin{aligned}
\mathcal{H}_{ab} &= 2\left[N h^{-1/2}\left(\pi_{ab}\pi^{ab} - \frac{1}{2}\pi h_{ab}\right) + \mathcal{D}_{(b}N_{a)}\right] \, , \\
\mathcal{P}^{ab} &= -\frac{1}{2} N h^{-1/2}\left(\pi^{cd}\pi_{cd} - \frac{1}{2}\pi^2\right) h^{ab} + 2N h^{-1/2}\left(\pi^a{}_c\pi^{cb} - \frac{1}{2}\pi\pi^{ab}\right) + N h^{1/2}\mathcal{G}^{ab} \\
&\quad + 2\pi^{c(a}N^{b)}_{|b} + h^{1/2}\left(h^{ab}N^{,d}_{,d} - N^{,ab}\right) - h^{1/2}\left(h^{-1/2}\pi^{ab}N^d\right)_{|d} \, .
\end{aligned}$$

Hence, we see that the Einstein gravity does not have a good behaviour. This is because the arbitrary variations with respect to γ_{ij} and π^{ij} must yield only the field equations of the theory.. However, here the theory gives the field equations and surface integrals (boundary terms). This latter implies that GR does not have a well-posed variational principle. Regge and Teitelboim redefined the Hamiltonian of gravity, considering these surface charges from scratch. The latter is called the “True Hamiltonian". With this redefinition, they show that this Hamiltonian remains invariant under temporal translation, and these surface charges are identified with the energy.

Chapter 4

Exact solutions of Einstein Equations

Exact solutions of Einstein's field equations play a central role in the study of General Relativity, as they provide explicit spacetimes in which the physical and geometrical properties of the theory can be analyzed in detail. Through these solutions, one can investigate fundamental phenomena such as black holes, cosmological dynamics, gravitational collapse, and the propagation of gravitational waves within a fully nonlinear relativistic framework. In addition to their physical relevance, exact solutions also serve as important theoretical laboratories where new mathematical techniques and conceptual ideas can be developed and tested. For this reason, the systematic study and classification of exact solutions has become an essential part of research in gravitational physics and mathematical relativity.

In this section, we review some relevant solutions of Einstein's equations. We also discuss their generalizations to higher dimensions and their physical implications. This section is mainly based on the references [46–51]; for further details, we refer the reader to these works.

4.1 Schwazschild black hole

The first exact solutions to the Einstein field equations was found by Karl Schwazschild in Ref. [4] only a few months later that Einstein proposed its equations. It is widely applied both in astrophysics and in considerations of orbital motions about the Sun or the Earth. Moreover, it provides a model for a theory of strong gravitational fields that is widely applied in the final stages of stellar evolution and formation of black holes [49]. Due to the *Birkhoff's* theorem [5], the unique vacuum spherically symmetric solution is the Schwarzschild black hole. The line element is given by

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin\theta \mathrm{d}\phi^2)\,, \tag{4.1.1}$$

with $f(r) = 1 - \frac{2m}{r}$, where m is an integration constant which is associated with the mass (we compute this with the Brown-York method below). And we are using units such that $G_N = 1$. This solution reduces to Minkowski spacetime in spherical coordinate at $m \to 0$.

By virtue of being static, the spacetime admits $\xi = \partial_t$ as a timelike Killing vector; moreover, it is also invariant under the $SO(3)$ isometry group. The horizon of this geometry is located at $r_+ = 2m$, and it exhibits a curvature singularity at $r = 0$. The latter can be seen from the Kretschmann invariant, given by

$$R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = \frac{48m^2}{r^6}\,. \tag{4.1.2}$$

4.1.1 Brown-York charges of the Schwarzschild black hole

To ilustrate the Brown-York method, we compute the conserved charges for Schwazschild black hole in (4.1.1). To define the hypersurface at constant r and t, we define two normal vectors, i.e., $n = \sqrt{f(r)}\partial_r$ and $u = f(r)^{-1}\partial_t$. Another ingredient we need is the extrinsic curvature for this solution, given in equation (8.5.10), which for the Schwarzschild black hole gives,

$$\mathcal{K} = \frac{4f + rf'}{2\sqrt{f}r} = \frac{2}{\sqrt{r(2mG - r)}} - \frac{3mG}{r\sqrt{r(r - 2mG)}}. \tag{4.1.3}$$

where prime denotes differentiation with respect to r. Recall that, as we mentioned, the Schwarzschild solution is continuously connected with Minkowski as $m \to 0$. Thus, we consider, as a background, the Minkowski space, whose extrinsic curvature trace can be obtained as,

$$\mathcal{K}_0 = \lim_{m\to 0} \mathcal{K} = \frac{2}{r}. \tag{4.1.4}$$

Since the Schwarzschild's black hole is a static solution, it is endowed with a Killing vector $\xi = \partial_t$, which we use to compute the energy for the black hole. Replacing the previous ingredients in (3.2.6) yields

$$Q[\partial_t] = \lim_{r\to\infty} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{1}{4\pi} \left(2m - r + r\sqrt{1 - \frac{2m}{r}} \right) = m. \tag{4.1.5}$$

Therefore, we conclude that the black hole's energy is its mass.

4.2 Taub-NUT

A stationary one-parameter extension of the Schwarzschild solution was found by Taub [6] and Newman, Tamburino, and Unti [7]. Today, this solution is known as Taub-NUT (TN) and has been heavily studied since its discovery due to their close resemblance with gravitomagnetic monopoles and gravitational instantons in the Euclidean section. It is a solution of Einstein field equations with cosmological [cf.(2.1.3)]. The line element is usually parametrized as

$$\mathrm{d}s^2 = -f(r)\left(\mathrm{d}t + 2n\cos\theta \mathrm{d}\phi\right)^2 + \frac{\mathrm{d}r^2}{f(r)} + \left(r^2 + n^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{4.2.1}$$

The metric function that solves the Einstein field equations is,

$$f(r) = \frac{r^2 - n^2}{r^2 + n^2} - \frac{2mGr}{r^2 + n^2} - \frac{\Lambda}{3}\frac{\left(r^2 + 6r^2n^2 - 3n^4\right)}{\left(r^2 + n^2\right)} \tag{4.2.2}$$

where m is an integration constant related to the ADM mass [52–54] and n is the NUT charge, also known as the gravitomagnetic mass. Here, the range of the coordinates is $(t, r) \in \mathbb{R}$, $0 \leq \theta \leq \pi$, and $0 \leq \phi < 2\pi$. If the cosmological constant vanishes, i.e., $\Lambda = 0$ we reduce to solving the Einstein field equations, and the solution has two horizons; ther are

$$r_{\pm} = m \pm \sqrt{m^2 + n^2}. \tag{4.2.3}$$

This metric is free of curvature singularities for $r \in \mathbb{R}$ as it can be seen directly from the computation of the Kretschmann invariant. Additionally, it is continuously connected to the Schwarzschild metric if we consider the limit $n \to 0$.

Considering the magnetic and electric parts of the Weyl tensor, defined as

$$E_{ij} = W_{i\lambda j\rho} n^{\lambda} n^{\rho} \quad \text{and } B_{ij} = \tilde{W}_{i\lambda j\rho} n^{\lambda} n^{\rho} \tag{4.2.4}$$

where n^{μ} is the unit-normal spacelike vector that defines radial foliation, $W^{\mu\nu}_{\lambda\rho}$ and $\tilde{W}_{\mu\nu\lambda\rho}$ are the Weyl and dual Weyl tensor, respectively. The dual Weyl tensor is defined as,

$$\tilde{W}_{\mu\nu\lambda\rho} := \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} W^{\alpha\beta}_{\lambda\rho} \,. \tag{4.2.5}$$

In Ads, when $\Lambda = -3/\ell^2$ with ℓ being the AdS radius, the falloff of these tensor gives

$$B^i_j \sim \frac{2n(l^2 + 4n^2)}{l^2 r^3} - \frac{6MnG}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right) \tag{4.2.6}$$

$$E^i_j \sim \frac{2MG}{r^3} + \frac{6l^2 n^2 + 24n^4}{l^2 r^4} + \mathcal{O}\left(\frac{1}{r^5}\right) . \tag{4.2.7}$$

From these equations, we conclude that the leading term of the magnetic part of the Weyl tensor is proportional to the NUT charge, whereas the leading term of the electric part is dominated by the mass parameter, both being of the same order. Moreover, from the ADM charges, the NUT charge can be interpreted as a form of *"magnetic mass"* [8], and thus the Taub–NUT solution can be regarded as a gravitational dyon [55].

Despite that TN is a regular solution, it is well-known that it has a string-like topological defect that is analog to the *Dirac string* in the case of the magnetic monopole. To ilustrate this, let us define a vielbein basis,

$$e^0 = \sqrt{f}(\mathrm{d}t + 2n\cos\theta\mathrm{d}\phi) \tag{4.2.8}$$

$$e^1 = \frac{\mathrm{d}r}{\sqrt{f}} \tag{4.2.9}$$

$$e^2 = \sqrt{r^2+n^2}\mathrm{d}\theta \tag{4.2.10}$$

$$e^3 = \sqrt{r^2+n^2}\sin\theta\mathrm{d}\phi \tag{4.2.11}$$

where $f = f(r)$ and the vielbeins satisfy the equivalence principle, i.e., $g_{\mu\nu} := e^i_\mu e^j_\nu \eta_{ij}$ with $g_{\mu\nu}$ being the component of the TN metric given in (4.2.1), and $\eta_{ij} = diag(-1,+1,+1,+1)$ is the Minkowski metric. Then, solving for the dt 1-form from the vielbeins we obtain (for more details for this notation see [34, 56, 57])

$$\mathrm{d}t = \nabla_\mu t \mathrm{d}x^\mu = \frac{e^0}{\sqrt{f}} - \frac{2n\cot\theta e^3}{\sqrt{r^2+n^2}}.$$

The norm of the vector field that defines the Hamiltonian evolution is given by

$$-\nabla_\mu t \nabla^\mu t = \frac{1}{f(r)} - \frac{4n^2\cot^2\theta}{r^2+n^2}. \tag{4.2.12}$$

The norm is undefined when $r = r_+$, such as $f(r_+) = 0$ but it is a coordinate singularity and can be removed by a "good" patch of coordinates. Moreover, the string-like topological defect is present at $\theta = \{0, \pi\}$ the latter represents the effect of the *Misner String* [58]. The latter it is not possible to removed via a single coordinate patch

Similar to the magnetic monopole, the Misner string can be removed by a large gauge transformation. For instance, if we separate the metric in two patches, say S^+ and S^- with a temporal coordinate change $t_\pm \to t_\pm \pm 2n\phi$, then Eq. (4.2.1) can be written as

$$\mathrm{d}s^2_\pm = -f(r)\left[\mathrm{d}t_\pm \pm 2n(1\pm\cos\theta)\mathrm{d}\phi\right]^2 + \frac{\mathrm{d}r^2}{f(r)} + \left(r^2+n^2\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \tag{4.2.13}$$

where denotes the two different patches without the Misner string. We conclude that it becomes regular in $\theta = 0$ for the northern hemisphere (S^+) and for the southern hemisphere (S^-) when $\theta = \pi$. However, on the equator, these coordinates satisfy the relation $t_+ = t_- + 4n\phi$ and, since ϕ is an angular coordinate with period 2π, this implies that $t_\pm \sim t_\pm + 8\pi n$. Therefore, eliminating the Misner string implies the existence of closed timelike curves.

However, this is not the case when the transverse section is no longer spherical. For instance, we can consider different topologies for the transverse section of Eq. (4.2.1), that is

$$\mathrm{d}s^2 = -f(r)\left(\mathrm{d}t + 2nB_{(k)}\right)^2 + \frac{\mathrm{d}r^2}{f(r)} + \left(r^2+n^2\right)\mathrm{d}\Sigma^2_{(k)}\,, \tag{4.2.14}$$

where $\mathrm{d}\Sigma^2_k$ is the metric of a codimension-2 constant curvature space, which locally looks like S^2, H^2 or T^2 for $k = \pm 1, 0$, respectivelly, and

$$B_{(k)} = \begin{cases} \cos\theta \mathrm{d}\phi & \text{if } k=1 \quad \text{with } \mathrm{d}\Sigma^2_{(1)} = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \\ \cosh\theta \mathrm{d}\phi & \text{if } k=-1 \quad \text{with } \mathrm{d}\Sigma^2_{(-1)} = \mathrm{d}\theta^2 + \sinh^2\theta \mathrm{d}\phi^2 \\ x\mathrm{d}y & \text{if } k=0 \quad \text{with } \mathrm{d}\Sigma^2_{(0)} = \mathrm{d}x^2 + \mathrm{d}y^2 \end{cases} \tag{4.2.15}$$

is the Kähler 1-form associated to $\mathrm{d}\Sigma^2_k$. For these cases, the metric function that solves the Einstein field equations in vacuum, is given by

$$f(r) = k\left(\frac{r^2-n^2}{r^2+n^2}\right) - \frac{2mGr}{r^2+n^2} - \frac{\Lambda}{3}\frac{\left(r^4 - 6r^2n^2 - 3n^4\right)}{\left(r^2+n^2\right)}. \tag{4.2.16}$$

For instance, when $k = 0$, and using the equation (4.2.12), the norm of the vector field that defines the Hamiltonian evolution is

$$-\nabla_\mu t \nabla^\mu t = \frac{1}{f(r)} - \frac{4n^2x^2}{r^2+n^2} \tag{4.2.17}$$

which, for $r \neq r_+$, is regular $\forall x$. Therefore, one concludes that there is no Misner string in the planar case.

Let us consider the asymptotically locally flat case with planar transverse section ($k = 0$), that is, when the cosmological constant vanishes, i.e., $\Lambda = 0$. Then, the conserved charges with the Brown-York method, gives

$$Q[\partial_t] = m\,, \quad Q[\partial_\phi] = 2mn \tag{4.2.18}$$

As a final comment, historically Taub-NUT-AdS was study his analitic continuation, i.e., in the Euclidean signature. The latter can be do it by perfoming a double Wick rotation of the Lorentzian Taub-NUT, in the following way $t \to -i\tau$ and $n \to -in$. The minus is conventional. In this signature, TN present a curvature singularity, this can be seen by the Kretschmann invariant, which is,

$$R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = \frac{B[r,n,m,\Lambda]}{3(r^2-n^2)^6}, \tag{4.2.19}$$

where $B[r,n,m,\Lambda]$ is a monotonically increasing function of r, that depends on the power of n, m and Λ. It is clear that now the metric exhibits a singularity at $r = n$.

However, this metric is study in two different cases or geometric structures. These are known as the NUTs and Bolts, which correspond to fixed-point sets of the isometry group that are zero- and two-dimensional, respectively.

To obtain the NUT case, we require that the metric function $f(r)$ vanishes at $r = n$. This can be done by studying the series near $r = n$, yielding

$$f(r) \approx \frac{4\Lambda n^3 - 3m + 3n}{3(r-n)} + \frac{4\Lambda n^3 - 3m + 3n}{6n}\mathcal{O}(r-n). \tag{4.2.20}$$

From here, we can impose the condition,

$$m = \frac{4\Lambda n^3 + 3n}{3} := m_{NUT}. \tag{4.2.21}$$

Evaluating this new "mass", we can eliminate the divergence and the metric function is regular at $r = n$. Then, the metric function becomes

$$f(r)_{NUT} = \frac{r-n}{r+n} - \frac{\Lambda(r-n)^2(3n+r)}{3(r+n)}. \tag{4.2.22}$$

On the other hand, to eliminate the conical singularities, we must to demand the identification of the Euclidean time coordinate with a periodicity given by,

$$\beta_\tau = \left.\frac{4\pi}{f'(r)_{NUT}}\right|_{r=n} = 8\pi n\,, \tag{4.2.23}$$

where prime denotes derivative with respect to r.
For the Bolt condition, we made a similar procedure. In the absence of Λ, Taub-bolt was found by Page [59]. Assuming that the metric function vanishes for an arbitrary $r_b > n$. Expanding $f(r)$ near $r = r_b$ yields,

$$f(r) \approx \frac{r_b(6m - 3r_b + \Lambda r_b^3) - 3n^2(1 + 2\Lambda r_b^2) - 3\Lambda n^4}{3(n^2 - r_b^2)} + \mathcal{O}(r - r_b). \tag{4.2.24}$$

Then, imposing,

$$m = \frac{\Lambda(3n^4 + 6n^2 r_b^2) - r_b^2}{6r_b} + \frac{n^2 + r_b^2}{2r_b} := m_{Bolt}, \tag{4.2.25}$$

then the metric becomes [60, 61]

$$f(r)_{Bolt} = \frac{(r - r_b)(n^2 - r_b r)}{(n^2 - r^2)r_b} + \frac{\Lambda}{3}\frac{(r - r_b)(3n^4 - 6n^2 r_b r + r_b r^3 + r_b^2 r^2 + r_b^3 r)}{(n^2 - r^2)r_b}\,. \tag{4.2.26}$$

Again, to eliminate the conical singularities we demand the Euclidean time period given by

$$\beta_\tau = \left.\frac{4\pi}{f'(r)_{Bolt}}\right|_{r=r_b} = \frac{4\pi r_b}{1 + \Lambda(n^2 - r_b^2)}\,, \tag{4.2.27}$$

where prime denotes derivative with respect to r.

4.3 Kerr black hole

A relevant solution that describes a rotating black hole in vacuum is the Kerr solution, discovered by Kerr in Ref. [11]. It it is convenient to present this solution in terms of Boyer-Lindquist coordinates, that is

$$\mathrm{d}s^2 = -\frac{\Delta_r}{\varrho^2}(\mathrm{d}t - a\sin\theta\mathrm{d}\phi)^2 + \frac{\varrho^2}{\Delta_r}\mathrm{d}r^2 + \varrho^2\mathrm{d}\theta + \frac{\sin^2\theta}{\varrho^2}(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)^2, \quad (4.3.1)$$

where

$$\varrho^2 = r^2 + a^2\cos^2\theta \quad \text{and} \quad \Delta_r = r^2 - 2Mr + a^2,$$

are the metric functions that solve the Einstein field equations, with M and a beign integration constants. Here, M represents the ADM *mass* (or Komar energy), and $J = aM$ is the angular momentum of the solution [62]. The metric is continuously connected to the Schwarzschild black hole in the limit $a \to 0$. The parameter a is interpreted as a *rotation parameter*. Moreover, the Kerr solution is continuously connected to Minkowski spacetime in the limit $M \to 0$, although in non-standard coordinates. This metric is invariant under the simultaneous inversion of t and ϕ, i.e., $t \to -t$ and $\phi \to -\phi$. Furthermore, it belongs to the family of stationary and axisymmetric solutions, and therefore admits two Killing vectors, $\xi = \partial_t$ and $\chi = \partial_\phi$.

4.3.1 Horizons

Analysing the roots of $\Delta_r = 0$ yields

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \quad (4.3.2)$$

In the Schwarzschild limit, i.e., $a \to 0$, the root r_+ coincides with the event horizon of the Schwarzschild black hole, located at $r = 2M$. The other root, r_-, coincides with the curvature singularity at $r = 0$. The extremal limit of the Kerr black hole corresponds to the case in which the mass and rotation parameters satisfy $M = a$; in this case, both horizons coalesce, producing a degenerate event horizon. This configuration represents the limiting case separating regular black hole solutions from naked singularities, since for $a > M$ the spacetime no longer possesses an event horizon. Extremal black holes exhibit several remarkable properties. In particular, the surface gravity vanishes, implying that the Hawking temperature [63] is zero. Furthermore, the near-horizon geometry develops enhanced symmetries that make the extremal Kerr solution especially useful for analytical studies of black hole thermodynamics and quantum aspects of gravity [64]. For these reasons, extremal rotating black holes play an important role in the theoretical analysis of gravitational solutions and often arise as limiting configurations in higher-dimensional and String theory models [46, 47].

The Kretschmann invariant for the Kerr solution is given by

$$R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = \frac{48M^2(r^2 - a^2\cos^2\theta)(\varrho^4 - 16a^2r^2\cos^2\theta)}{\varrho^{12}}. \tag{4.3.3}$$

From this expression we observe that $\Delta_r = 0$ corresponds only to a coordinate singularity, whereas the spacetime becomes truly singular when $\varrho^{12} = 0$. Let us discuss this in more detail. From the definition $\varrho^2 = r^2 + a^2\cos^2\theta$, the singularity occurs only in the equatorial plane, i.e. $\theta = \pi/2$, at $r = 0$. Using the coordinate relations[4]

$$x^2 + y^2 = (r^2 + a^2)\sin^2\theta, \qquad z = r\cos\theta,$$

the surface $r = 0$ corresponds to the entire disk $x^2 + y^2 \leq a^2$ in the plane $z = 0$. The boundary of this disk is given by

$$x^2 + y^2 = a^2, \tag{4.3.4}$$

which lies in the equatorial plane. Therefore, the curvature singularity is located on a ring of radius a in the x–y plane, commonly known as the *ring singularity*.

4.3.2 Ergoregions and Ergosphere

For the purpose of studying the geometry of this solution, it is natural to ask where the Killing vector ξ becomes null. Let us consider static observers in the Kerr

[4] These relations follow from the Kerr–Schild coordinates; for more details see [34].

spacetime, i.e., observers whose four-velocity is proportional to ξ. Thus,

$$u^\mu = v\xi^\mu, \tag{4.3.5}$$

where $v \equiv (-g_{\mu\nu}\xi^\mu\xi^\nu)^{-1/2}$ ensures that the four-velocity is properly normalized. The motion of these observers is not geodesic, since they must be held in place by an external agent, for instance, a rocket engine.

Static observers cannot exist everywhere, because ξ is not timelike throughout the spacetime. It becomes null when $-g_{tt} = v^{-2} = 0$. Therefore, the *stationary limit* [5] is determined by the roots of $g_{tt} = 0$. These roots are

$$r_{sl}^{\pm} = M \pm \sqrt{M^2 - a^2\cos^2\theta}. \tag{4.3.6}$$

where *sl* denotes the *stationary limit*. Therefore, observers cannot remain static in the region $r_+ < r < r_{sl}^+$, where ξ becomes spacelike. This region is known as the *ergoregion*, and the stationary limit surface, i.e., $r = r_{sl}^+$, is called the *ergosphere*. Moreover, note that for $\theta = 0, \pi$ the static limit radius becomes $r_{sl}^{\pm} = r_\pm$, i.e., the stationary limit surface coincides with the event horizon.

4.3.3 Brown-York charges of the Kerr black hole

Using the Brown-York method, we compute the conserved charges. To define the hypersurface at constant r and t, we define two normal vectors, these are defined as

$$v = \sqrt{\frac{\Delta_r}{\varrho^2}}\partial_r\,, \tag{4.3.7}$$

$$u = \sqrt{\frac{\varrho^2}{a^2\cos^2\theta - 2Mr + r^2}}\partial_t\,, \tag{4.3.8}$$

[5] Some authors [49] refer to this surface as the static limit surface.

respectively. Another ingredient we need is the extrinsic curvature for this solution, which is given by (8.5.10)

$$\mathcal{K} = \sqrt{\frac{\Delta_r}{\varrho^6}}2r = \sqrt{\frac{r^2 - 2mr + a^2}{(a^2\cos^2\theta + r^2)^3}}2r\,. \tag{4.3.9}$$

Recall that, as we mentioned, the Kerr solutions is continuosly connected with Minkowski as $m \to 0$. Thus, we consider, as a background, the Minkowski space, whose extrinsic curvature trace can be obtained as

$$\mathcal{K}_0 = \lim_{m\to 0}\mathcal{K} = \sqrt{\frac{r^2 + a^2}{(a^2\cos^2\theta + r^2)^3}}2r\,. \tag{4.3.10}$$

Since the Kerr solution is a stationary and axisymmetric solution, it is endowed with a two Killing vector $\xi = \partial_t$ and $\chi = \partial_\phi$, as we mentioned before, which we use to compute the energy and angular momentum, respectively. Concretely, the quasi-local charges are

$$Q[\xi] = M, \qquad Q[\chi] = Ma. \tag{4.3.11}$$

4.4 Schwazschild-Tangh erlini

The generalization of Schwarzschild solution to $D \geq 4$ dimensions was found by Tangherlini [10] nowadays knowed as Schwarzschild-Tangherlini. The line element is given by,

$$\mathrm{d}s^2 = -h(r)\mathrm{d}t^2 + h(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-2}, \tag{4.4.1}$$

with

$$h(r) = 1 - \frac{\mu}{r^{D-3}}, \tag{4.4.2}$$

where μ is an integration constant and $\mathrm{d}\Omega^2_{D-2}$ denotes the line element on the unit $(D-2)$-sphere. The mass and angular momentum in D-dimensional spacetime was obtained for the ADM [65–67] formalism which examine the asymtotic structure of the metric. With this one finds that μ fixes the mass of the black hole, yielding,

$$M = \frac{(D-2)\Omega_{D-2}}{16\pi G}\mu\,, \tag{4.4.3}$$

where Ω_{D-2} is the area of a unit $(D-2)$ sphere. The latter has the following value,

$$\Omega_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}. \tag{4.4.4}$$

Similar that in $D = 4$ meanwhile the value of μ remains positive we will have an event horizon, which is located at the surface $r^{D-3} = \mu$. There is a curvature singularity at $r = 0$, this can be seen from the Kretschmann scalar, giving by

$$R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} \propto \frac{\mu^2}{r^{2(D-1)}}. \tag{4.4.5}$$

On the other hand, for the case $\mu < 0$ the spacetime has a naked singulatiry at $r = 0$. As a final remark the Schwazschild-Tangherlini metric is the most general solution and so any spherically symmetric solution of $R_{\mu\nu} = 0$ must also be static.

In the limit $D = 4$, one recovers the standard Schwarzschild solution. The Schwarzschild–Tangherlini geometry constitutes the simplest higher-dimensional black hole spacetime and serves as a starting point for more general solutions.

4.5 Taub-NUT-AdS in higher dimensions

The generalization of the Taub-NUT-AdS solution for a single rotational plane, i.e., with a single NUT charge n, was study mostly in the Euclidean signature. Awad and Chamblin studied the solution for a given dimension with different transverse sections in Ref. [68] but also gave a general form for the metric function that solves the field equations. Later, Clarkson et al. in Ref. [69] study the thermodynamics of Euclidean-Taub-NUT-AdS. In the Euclidean section, different authors studied the NUT and Bolt cases mentioned before.

The general form for the Taub-NUT/Bolt-AdS for a $U(1)$ fibration over $(S^2)^{\otimes(D-2)}$ is given by [68],

$$\mathrm{d}s^2 = f(r)(\mathrm{d}\tau + 2n\cos\theta_i \mathrm{d}\phi_i)^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 - n^2)(\mathrm{d}\theta_i^2 + \sin\theta_i \mathrm{d}\phi_i^2), \tag{4.5.1}$$

with i summed from 1 to $D-2$. The general form for $f(r)$ is

$$f(r) = \frac{r}{(r^2-n^2)^k}\int^r \left[\frac{(s^2-n^2)^k}{s^2} + \frac{2k+1}{\ell^2}\frac{(s^2-n^2)^{k+1}}{s^2}\right]\mathrm{d}s - \frac{2mr}{(r^2-n^2)^k}, \tag{4.5.2}$$

where $D = 2k+2$ and m is an integration constant and ℓ is the AdS radius. Which is related with the mass of the Taub-NUT/Bolt-AdS in any dimension [69]. For the conserved charges, the authors use the renormalized Euclidean on-shell action and the Noether method [70, 71].

Therefore, the mass [69] is given by

$$M = \frac{(D-1)(4\pi)^{n-2/2}m}{8\pi}\,. \tag{4.5.3}$$

Clarson et al. in Ref. [69] comment that they checked this expression for the mass up-to $D = 20$.

4.6 Myers and Perry solution

The Myers–Perry (MP) solution, introduced by Robert C. Myers and Malcolm J. Perry in 1986, generalizes the Kerr black hole to arbitrary spacetime dimensions ($D > 4$) [12]. Unlike in four dimensions, where a black hole has a single axis of rotation, a D-dimensional rotating black hole can possess $n = \lfloor \frac{D-1}{2} \rfloor$ independent rotation parameters, each is associated with a different rotational plane.

This solution provides important insights into the behavior of black holes in higher-dimensional spacetimes. In particular, it serves as a fundamental framework for studying black hole thermodynamics, stability, and the possible existence of naked singularities in dimensions greater than four [47]. Moreover, it plays a central role in several theoretical frameworks, including gauge/gravity duality and extra-dimensional gravity theories (e.g., the AdS/CFT correspondence) [18, 72].

In this section we analyze the Myers–Perry black hole, focusing on its mathematical formulation, physical properties, and implications for higher-dimensional gravity theories. Here we restrict our discussion to the basic aspects of this solution. For further details see Refs. [47, 51].

The rotating black hole in an even number of spacetime dimensions, i.e., $D = 2n + 2$ with $D \geq 4$, is described by the line element

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t^2 + \frac{\mu r}{\Pi F}\left(\mathrm{d}t + \sum_{i=1}^{n} a_i \mu_i^2 \mathrm{d}\phi_i\right)^2 + \frac{\Pi F}{\Pi - \mu r}\mathrm{d}r^2 \\ &+ \sum_{i=1}^{n}(r^2 + a_i^2)(\mathrm{d}\mu_i^2 + \mu_i^2 \mathrm{d}\phi_i^2) + r^2 \mathrm{d}\alpha^2, \end{aligned} \tag{4.6.1}$$

where

$$F = 1 - \sum_{i=1}^{n} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}, \tag{4.6.2}$$

$$\Pi = \prod_{i=1}^{n}(r^2 + a_i^2). \tag{4.6.3}$$

When only one rotation parameter is present, i.e., $n = 1$, the MP solution reduces to the Kerr solution. On the other hand, for an odd number of spacetime dimensions, $D = 2n + 1$ with $D \geq 5$, the metric becomes

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t^2 + \frac{\mu r^2}{\Pi F}\left(\mathrm{d}t + \sum_{i=1}^{n} a_i \mu_i^2 \mathrm{d}\phi_i\right)^2 + \frac{\Pi F}{\Pi - \mu r^2}\mathrm{d}r^2 \\ &+ \sum_{i=1}^{n}(r^2 + a_i^2)(\mathrm{d}\mu_i^2 + \mu_i^2 \mathrm{d}\phi_i^2), \end{aligned}$$

where the functions F and Π are given in Eq. (4.6.2). Studying the asymptotic structure of the metric using the ADM method [65–67] for a higher dimensions, one finds that the $n+1$ free parameters, i.e., μ and a_i determine the mass and angular momenta of the black holes and giving the following results.

$$M = \frac{(D-2)\Omega_{D-2}}{16\pi G}\mu, \tag{4.6.4}$$

$$J^{y_i x_i} = \frac{\Omega_{D-2}}{8\pi G}\mu a_i = \frac{2}{D-2} M a_i, \tag{4.6.5}$$

where Ω_{D-2} denotes the area of the unit S^{D-2} sphere.

4.6.1 Horizons and thermodynamic properties

The structure of the event horizon is one of the most important aspects of the MP solution. As mentioned previously, the metric takes different forms depending on whether the spacetime dimension is even or odd. Therefore, we analyze the location of the horizons case by case.

Let us begin with even spacetime dimensions, $D = 2n + 2$. The event horizon is located where the inverse radial component of the metric vanishes, i.e., when $g^{rr} = 0$. This condition leads to

$$\Pi - \mu r = 0. \tag{4.6.6}$$

This equation corresponds to a polynomial of degree $D - 2$ in the radial coordinate r, whose real positive roots determine the possible horizons of the spacetime. In general, analytic solutions for the location of the horizon exist only for low dimensions, such as $D = 4$ and $D = 6$. For higher dimensions, the roots must be determined numerically.

Nevertheless, some general observations can be made. If an event horizon exists, its topology must be that of a sphere S^{D-2}. Furthermore, to avoid a naked singularity the mass parameter μ must be positive. A closer inspection of Eq. (4.6.6) shows that three scenarios are possible: the spacetime may contain two horizons, a single degenerate (extremal) horizon, or no horizon at all.

On the other hand, in odd spacetime dimensions, $D = 2n + 1$, the location of the horizon is determined by

$$\Pi - \mu r^2 = 0. \tag{4.6.7}$$

To analyze this equation it is convenient to introduce the radial coordinate $\rho = r^2$, in terms of which Eq. (4.6.7) becomes

$$\prod_{i=1}^{n}(\rho + a_i^2) - \mu\rho = 0. \tag{4.6.8}$$

In this case, analytic expressions for the roots exist in dimensions $D = 5, 7, 9$. The condition $\rho > 0$ implies that $\mu > 0$. Moreover, the existence of a positive root requires that

$$\mu > \sum_i \prod_{j \neq i} a_j^2, \tag{4.6.9}$$

which guarantees that the coefficient of the linear term in Eq. (4.6.8) is negative. This condition is necessary, although not sufficient, for the absence of a naked singularity.

Once the event horizon radius r_+ is determined, one can define the Killing vector that generates the horizon,

$$\xi = \partial_t + \sum_{i=1}^{n} \Omega_i \partial_{\phi_i}, \tag{4.6.10}$$

where the angular velocities of the horizon [12] are given by

$$\Omega_i = \frac{a_i}{r_+^2 + a_i^2}. \tag{4.6.11}$$

The surface gravity associated with the Killing horizon determines the Hawking temperature [73] through

$$T = \frac{\kappa_s}{2\pi}. \tag{4.6.12}$$

where κ_s is the surface gravity given by,

$$\kappa_s^2 := -\frac{1}{2}\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu \,. \tag{4.6.13}$$

Together with the entropy, proportional to the horizon area, these quantities satisfy the first law of black hole thermodynamics [74].

An important qualitative feature arises in dimensions greater than five. Unlike the four-dimensional Kerr solution, which possesses an upper bound on the rotation parameter, MP black holes with a single rotation parameter in $D > 5$ do not exhibit such a bound. The angular momentum can grow arbitrarily large while the horizon still persists.

This regime is known as the *ultraspinning limit* [75]. In this limit, the horizon becomes increasingly flattened along the plane of rotation and begins to resemble a higher-dimensional black membrane. Consequently, these solutions are expected to develop dynamical instabilities analogous to the Gregory-Laflamme instability of black strings [76, 77].

These features illustrate the rich structure of higher-dimensional rotating solutions and motivate the search for new geometries with multiple rotational parameters and additional charges.

4.6.2 Ergoregion and Ergosphere

Analogous to the Kerr solution, rotating MP solutions possess an ergoregion. This region is bounded by the ergosurface, defined as the hypersurface where the norm of the asymptotically timelike Killing vector becomes null. In Boyer-Lindquist coordinates, this corresponds to the condition $g_{tt} = 0$.

To determine the ergoregion we therefore need to find the roots of

$$F\Pi - \mu r = 0, \qquad (4.6.14)$$

$$F\Pi - \mu r^2 = 0, \qquad (4.6.15)$$

for even and odd spacetime dimensions, respectively.

In general, these equations cannot be solved analytically for arbitrary dimensions and rotation parameters. Nevertheless, several general properties can be established. As in the case of the event horizon, the ergosurface has topology S^{D-2}. However, the presence of the factor F introduces a non-trivial dependence on the μ_i directions, leading to a more intricate angular structure than that of the horizon [12, 47].

For even spacetime dimensions, if m rotation parameters vanish, the ergosurface reduces locally to a $D = 2m$ dimensional sphere described by the constraint

$$\alpha^2 + \sum_{k=1}^{m} \mu_k^2 = 1, \qquad (4.6.16)$$

where the sum runs over the m indices for which $a_k = 0$. In this case, the geometry exhibits an enhanced rotational symmetry in the corresponding subspace.

On the other hand, when the rotation parameters are non-vanishing, the ergosurface and the event horizon generally do not coincide. Instead, the ergosurface lies outside the horizon except at special points where the Killing vector generating the horizon becomes aligned with the asymptotic time translation. In particular, the two surfaces touch only at the points where $\alpha = \pm 1$, which correspond to the rotation axis. This behaviour is directly analogous to that of the Kerr solution in four dimensions.

As emphasized in the analysis of higher-dimensional rotating black holes, the existence of an ergoregion implies that processes analogous to the Penrose process and superradiant scattering can occur. These phenomena allow the extraction of rotational energy from the black hole and play an important role in the dynamics and stability of rotating solutions in higher dimensions [47].

Chapter 5

MultiNUT solutions

5.1 Taub–NUT with Multiple NUT Charges

A deformation of the Taub-NUT solution in higher-dimensional Einstein gravity with a cosmological constant was studied by Mann and Stelea in [16, 17]. This deformation is obtained by introducing multiple and distinct NUT parameters. [6] Such geometries can be interpreted as higher-dimensional Taub-NUT spacetimes endowed with several independent gravitomagnetic sources.

In Ref. [16], the authors arrived at two important conclusions that can be drawn about these geometries. The first one is that the existence of multiNUT solutions in Einstein gravity with cosmological constant Λ is obstructed by the field equations.

To illustrate this, let us consider the case $D = 6$ with two NUT charges, namely n_1 and n_2, corresponding to a $U(1)$ fibration over $S^2 \times S^2$. The line element takes the form

$$\begin{aligned} \mathrm{d}s^2 \;&=\; -f(r)\left(\mathrm{d}t + 2n_1\cos\theta_1\mathrm{d}\phi_1 + 2n_2\cos\theta_2\mathrm{d}\phi_2\right)^2 + \frac{\mathrm{d}r^2}{f(r)} \\ &\quad + \left(r^2 + n_1^2\right)\left(\mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\phi_1^2\right) + \left(r^2 + n_2^2\right)\left(\mathrm{d}\theta_2^2 + \sin^2\theta_2\mathrm{d}\phi_2^2\right) . \end{aligned}$$

However, after integratint the tt components of the Einstein equations, the transverse components impose a constraint on this metric which, in the presence of a

[6] A generalization based on an analytic continuation of Taub-NUT solutions in Lovelock gravity was recently obtained by Corral et al. [78].

cosmological constant $\Lambda = -\frac{10}{\ell^2}$, takes the form

$$(n_{(1)}^2 - n_{(2)}^2)\Lambda = 0\,. \tag{5.1.1}$$

Therefore, different values of n_1 and n_2 are possible only if the cosmological constant vanishes. Moreover, the authors also analyzed other possible topologies for the transverse section, obtaining similar constraints to those given in (5.1.1).

Later, in Ref. [79], new solutions to the vacuum Einstein field equations (2.1.3) with different NUT charges were constructed, with and without a cosmological constant. This was achieved by adopting a different normalization of the constant-curvature spaces, namely

$$R_{ab}(\mathcal{M}_i) = k_{(i)} g_{ab},$$

where $R_{ab}(\mathcal{M}_i)$ denotes the Ricci curvature of the manifold associated with the transverse section $\mathcal{M}_i$. The general form of the line element is then given by

$$\mathrm{d}s_D^2 = -f(r)\left(\mathrm{d}t + \sum_{i=1}^{k} 2n_i B_{(i)}\right)^2 + f^{-1}(r)\,\mathrm{d}r^2 + \sum_{i=1}^{p}\left(r^2 + n_i^2\right) g_{M_i}\,, \tag{5.1.2}$$

where $k = (D-2)/2$ and $B_{(i)}$ is the Khäler 1-form, with $\Omega_{(i)} = \mathrm{d}B_{(i)}$ its the symplectic 2-form associated to the i-th Einstein–Kähler manifold $\mathcal{M}_{(i)}$.

The metric function that solves the Einstein field equations with $\Lambda = \pm\frac{(D-1)(D-2)}{2l^2}$ is given by

$$f(r) = \frac{r}{\prod_{i=1}^{p}\left(r^2+n_i^2\right)}\left[-m + \int^r \left(k_1 \mp \frac{D-1}{l^2}\left(s^2+n_1^2\right)\right)\frac{\prod_{i=1}^{p}\left(s^2+n_i^2\right)}{s^2}\mathrm{d}s\right]\,, \tag{5.1.3}$$

where M is an integration constant. In this case, the constraint relating the NUT parameters n_i and the cosmological constant Λ can be written as

$$\Lambda\left(n_{(j)}^2 - n_{(i)}^2\right) = k_{(j)} - k_{(i)}\,, \qquad \forall i,j. \tag{5.1.4}$$

For instance, in the six-dimensional case, the metric read as

$$
\begin{aligned}
ds^2 &= -f(r)\left(\mathrm{d}t - \frac{2n_1}{k_1}\cos\theta_1\mathrm{d}\phi_1 - \frac{2n_2}{k_2}\cos\theta_2\mathrm{d}\phi_2\right)^2 + \frac{\mathrm{d}r^2}{f(r)} \\
&\quad + \frac{r^2+n_1^2}{k_1}\left(\mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\phi_1^2\right) + \frac{r^2+n_2^2}{k_2}\left(\mathrm{d}\theta_2^2 + \sin^2\theta_2\mathrm{d}\phi_2^2\right) ,
\end{aligned}
$$

where $f(r)$ is given in Eq. (5.1.3) with $p = 2$. Then, the constraint is translated into a relation between the Gaussian curvatures of the transverse sections, which reads as

$$
\Lambda\left(n_1^2 - n_2^2\right) = k_1 - k_2 \,. \tag{5.1.5}
$$

Solving this equation for k_2 and replacing, we obtain

$$
\begin{aligned}
\mathrm{d}s^2 &= -f(r)\left(\mathrm{d}t - \frac{2n_1}{k_1}\cos\theta_1\mathrm{d}\phi_1 - \frac{2n_2}{k_1-\Lambda\left(n_1^2-n_2^2\right)}\cos\theta_2\mathrm{d}\phi_2\right)^2 + \frac{\mathrm{d}r^2}{f\left(r\right)} \\
&\quad + \frac{r^2+n_1^2}{k_1}\left(\mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\phi_1^2\right) + \frac{r^2+n_2^2}{k_1-\Lambda\left(n_1^2-n_2^2\right)}\left(\mathrm{d}\theta_2^2 + \sin^2\theta_2\mathrm{d}\phi_2^2\right) .
\end{aligned}
$$

Nonetheless, it is not applicable when the latter metric is flat.

5.2 MultiNUTs in Differential Forms language

In this section, we explore the multiNUTs in the language of the differential forms [for the details see the Appendix 7]. The line element of the multiNUTs is given by the following,

$$\mathrm{d}s^2 = -f\,(r)\left(\mathrm{d}t + \sum_{i=1}^{k} \mathcal{B}_{(i)}\right)^2 + \frac{\mathrm{d}r^2}{f\,(r)} + \sum_{i=1}^{k} N_{(i)} \mathrm{d}\Sigma^2_{(i)}\ , \tag{5.2.1}$$

where $\mathcal{B}_{(i)} = 2n_{(i)}B_{(i)}$ defines the Khäler potential, $N_{(i)}$ is the area coordinate and $\mathrm{d}\Sigma^2_{(i)}$ is the base manifold (or transverse section) of $D-2$-dimension. The dimension of manifold $\mathcal{M}$ is given by $D = 2k + 2$. Note that the index i is not an index of an internal group. It labels the Khäler potential, the NUT charges, and the base manifold.

Let us remember that the vielbeins satisfy,

$$\mathrm{d}s^2 = -e^0 \otimes e^0 + e^1 \otimes e^1 + \delta_{AB} \sum_{i=1}^{k} e^A_{(i)} \otimes e^B_{(i)}\ . \tag{5.2.2}$$

where $\delta_{AB} \in SO\,(2)$ and it is the Killing Cartan metric. We choose an orthonormal vielbein basis as

$$e^0 = \sqrt{f}\left(dt + \sum_{i=1}^{k} \mathcal{B}_{(i)}\right)\ , \quad e^1 = \frac{dr}{\sqrt{f}}\ , \quad e^A_{(i)} = \sqrt{N_{(i)}}\bar{e}^A_{(i)},$$

where $f = f(r)$ and $\bar{e}^A$ is related with the Kähler-Einstein manifold. The aim is to compute the two form curvature. We will proceed step by step, obtaining the main ingredients that we need. Let us start with the exterior derivate of the vielbein basis.

$$\begin{aligned}
\mathrm{d}e^0 &= \frac{f'}{2\sqrt{f}} e^1 \wedge e^0 + \sqrt{f} \sum_{i=1}^{k} \Omega_{(i)}\ , \\
\mathrm{d}e^1 &= 0\ , \\
\mathrm{d}e^A_{(i)} &= \frac{\sqrt{f} N'_{(i)}}{2N_{(i)}} e^1 \wedge e^A_{(i)} - \bar{\omega}^A_{(i)B} \wedge e^B_{(i)}\ .
\end{aligned}$$

where we defined

$$\mathrm{d}\mathcal{B}_{(i)} := 2n_{(i)}\Omega_{(i)} \tag{5.2.3}$$

with $\Omega_{(i)} = \frac{1}{2}\Omega^{(i)}_{AB}\bar{e}^{A}_{(i)} \wedge \bar{e}^{B}_{(i)}$.

Now, to compute the spin connection we will use the next expression,

$$\begin{aligned}
\omega_{ab} &= -i_{[a}de_{b]} + \frac{1}{2}\left(i_a i_b de_c\right)e^c, \\
&= -i_{[a}de_{b]} + \frac{1}{2}\left(i_a i_b de_0\right)e^0 + \frac{1}{2}\left(i_a i_b de_1\right)e^1 + \frac{1}{2}\left(i_a i_b de_C\right)e^C, \\
&= -i_{[a}de_{b]} + \frac{1}{2}\left(i_a i_b de_0\right)e^0 + \frac{1}{2}\left(i_a i_b de_1\right)e^1 + \frac{1}{2}\sum_{j=1}^{k}\left(i_a i_b de^{(j)}_C\right)e^C_{(j)}.
\end{aligned}$$

Note that we modify the expression and we introduce a new index j for this case j make the same role that i that we mention at the beggining. After some algebra (for more details see 7) the non-trivial components are,

$$\omega^{01} = \frac{f'}{2\sqrt{f}}e^0, \tag{5.2.4}$$

$$\omega^{0A}_{(i)} = \frac{\sqrt{f}n_{(i)}}{N_{(i)}}\Omega^{A}_{(i)B}e^{B}_{(i)}, \tag{5.2.5}$$

$$\omega^{1A}_{(i)} = -\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}e^{A}_{(i)}, \tag{5.2.6}$$

$$\omega^{AB}_{(i)} = \bar{\omega}^{AB}_{(i)} + \frac{\sqrt{f}n_{(i)}}{N_{(i)}}\Omega^{AB}_{(i)}e^0. \tag{5.2.7}$$

Finally, using these ingredients, we can compute the 2-form curvature defined in Eq.

49 of Ref. [57]. All the non-trivial components of the 2-form curvature are given by

$$
\begin{aligned}
R^{01} &= -\frac{f''}{2}e^0 \wedge e^1 + \sum_{i=1}^{k} \frac{n_{(i)}}{2}\left[\frac{f}{N_{(i)}}\right]' \Omega^{(i)}_{AB} e^A_{(i)} \wedge e^B_{(i)} \\
R^{0A} &= \sum_{i=1}^{k}\left(\frac{1}{2}\left[\frac{f}{N_{(i)}}\right]' n_{(i)}\Omega^A_{(i)B} e^1 \wedge e^B_{(i)} - \frac{1}{4}\left(\frac{f'N'_{(i)}N_{(i)} + 4fn^2_{(i)}}{N^2_{(i)}}\right) e^0 \wedge e^A_{(i)}\right. \\
&\left. + \frac{\sqrt{f}n_{(i)}}{N_{(i)}}\bar{D}^{(i)}\Omega^A_{(i)B} \wedge e^B_{(i)}\right) \\
R^{1A} &= \sum_{i=1}^{k}\left(\frac{1}{2}\left[\frac{-f'N_{(i)} + fN'_{(i)}}{N^2_{(i)}}\right] n_{(i)}\Omega^A_{(i)B} e^0 \wedge e^B_{(i)} + \left(\left[-\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}\right]' \sqrt{f} - \frac{fN'^2_{(i)}}{4N^2_{(i)}}\right) e^1 \wedge e^A_{(i)}\right) \\
R^{AB} &= \left(\sum_{i=1}^{k} d\bar{\omega}^{AB}_{(i)} + \sum_{i=1}^{k}\sum_{j=1}^{k} \bar{\omega}^A_{(i)C} \wedge \bar{\omega}^{CB}_{(j)} + \sum_{i=1}^{k}\left[\frac{f}{N_{(i)}}\right]' n_{(i)}\Omega^{AB}_{(i)} e^1 \wedge e^0\right. \\
&+ \sqrt{f}\sum_{i=1}^{k}\left(\frac{n_{(i)}}{N_{(i)}} d\Omega^{AB}_{(i)} + \sum_{j=1}^{k}\frac{n_{(j)}}{N_{(j)}}\bar{\omega}^A_{(i)C}\Omega^{CB}_{(j)} + \sum_{j=1}^{k}\frac{n_{(i)}}{N_{(i)}}\bar{\omega}^B_{(j)C}\Omega^{AC}_{(i)}\right) \wedge e^0 \\
&+ \sum_{i=1}^{k}\sum_{j=1}^{k}\frac{fn_{(i)}n_{(j)}}{N_{(i)}N_{(j)}}\Omega^{[A}_{(i)[C}\Omega^{B]}_{(j)D]} e^C_{(i)} \wedge e^D_{(j)} \\
&\left. - \sum_{i=1}^{k}\sum_{j=1}^{k}\frac{fN'_{(i)}N'_{(j)}}{4N_{(i)}N_{(j)}} e^A_{(i)} \wedge e^B_{(j)} + \sum_{i=1}^{k}\sum_{w=1}^{k}\frac{fn_{(i)}n_{(w)}}{N_{(i)}N_{(w)}}\Omega^{AB}_{(i)}\Omega^{(w)}_{CD} e^C_{(w)} \wedge e^D_{(w)}\right).
\end{aligned}
$$

for more details about the computation see the appendix 7 where we compute step by step all the latter results.

5.3 MultiNUTs with axionic fields

To avoid the constraint obtained by Mann and Stelea, we shaped the theory with free scalar fields. This latter allows one to circumvent the constraint imposed by the field equations, allowing to circunvent the obstruction imposed by the field equations. The flat plane metrics $\mathrm{d}\Sigma^2_{(i)}$ are Kähler manifolds with associated symplectic forms $\Omega_{(i)} = \mathrm{d}B_{(i)}$. Taking further advantage of their Kähler structure, we write them in terms of holomorphic and anti-holomorphic coordinates $z_{(i)} = x_{(i)} + i\, y_{(i)}$ and $\bar{z}_{(i)} = x_{(i)} - i\, y_{(i)}$, respectively, as follows

$$\mathrm{d}\Sigma^2_{(i)} = \tfrac{1}{2}\left(\mathrm{d}\bar{z}_{(i)}\mathrm{d}z_{(i)} + \mathrm{d}z_{(i)}\mathrm{d}\bar{z}_{(i)}\right), \tag{5.3.1}$$

$$2B_{(i)} = \tfrac{1}{2\mathrm{i}}\left(\bar{z}_{(i)}\mathrm{d}z_{(i)} - z_{(i)}\mathrm{d}\bar{z}_{(i)}\right), \tag{5.3.2}$$

where the sum over i relates the NUT charges with the differents base manifold.

Now, we consider GR with cosmological constant suplemented with a minimally coupled complex scalar field as a matter content in the theory. The action principle is giving by

$$S_{bulk} = \int \mathrm{d}^D x\sqrt{|g|}\left(\frac{R-2\Lambda}{16\pi} - \frac{1}{2}\sum_{i=1}^{k}\left|\nabla\varphi_{(i)}\right|^2\right), \tag{5.3.3}$$

where we are using units such that the Newton constant is $G_N = 1$. Arbitrary variations with respect to the metric and the scalar fields gives the field equation of the theory

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\, T^{(\varphi)}_{\mu\nu}, \tag{5.3.4a}$$

$$\Box\varphi_{(i)} = 0, \tag{5.3.4b}$$

respectively. Moreover, the energy-momentum tensor is given by

$$T^{(\varphi)}_{\mu\nu} = \frac{1}{2}\sum_{i=1}^{k}\left(\nabla_\mu\bar{\varphi}_{(i)}\nabla_\nu\varphi_{(i)} + \nabla_\mu\varphi_{(i)}\nabla_\nu\bar{\varphi}_{(i)} - g_{\mu\nu}\nabla_\alpha\bar{\varphi}_{(i)}\nabla^\alpha\varphi_{(i)}\right). \tag{5.3.4c}$$

If one relaxes the condition that the scalar field must remain invariant under the action of the isometry group of the metric, while keeping the stress-energy tensor invariant, one finds that the solution to the Klein-Gordon equation is given by [80,81]

$$\varphi_{(i)} = \lambda_{(i)} z_{(i)}, \tag{5.3.5}$$

where $\lambda_{(i)}$ denotes the axionic constant for the different base manifold, with $z_{(i)}$ being their coordinates. Note that these k complex scalar fields can be written as follows

$$\varphi_{(i)} = \phi_{(i)} + i\psi_{(i)}\,, \tag{5.3.6}$$

where $\phi_{(i)}$ and $\psi_{(i)}$ are real scalar fields.

The field equations for the metric can be integrated analytically, whose solution with multiple NUT parameters is found to be

$$f(r) = -\frac{r}{\prod_{i=1}^{k}\left(r^2+n_{(i)}^2\right)}\left[m + \int^r \frac{-8\pi\lambda_{(k)}^2 + \frac{\Lambda}{k}\left(\rho^2+n_{(k)}^2\right)}{\rho^2}\prod_{i=1}^{k}\left(\rho^2+n_{(i)}^2\right)\mathrm{d}\rho\right], \tag{5.3.7}$$

where the following constraint must be met,

$$\lambda_{(i)}^2 = \lambda_{(k)}^2 + \frac{\Lambda}{8\pi k}\left(n_{(k)}^2 - n_{(i)}^2\right)\,, \quad \forall i,k\,. \tag{5.3.8}$$

The latter equations are the same as in Ref. [17] for a multi-NUT spacetime whose base manifold is a product of k hyperbolic planes, each with Gaussian curvature $-8\pi\lambda_{(i)}^2$. It is worth mentioning that is well known that the axionic integration constants gravitate as an effective negative curvature [82]. This multi-NUT solution are continuosly connected with the well-known solution of Ref. [81] in the static limit, i.e., $n_{(i)} \to 0$.

5.4 Euclidean Action

In the case of Einstein-AdS gravity coupled with axions, holographic renormalization was studied in four dimensions in Ref. [83]. Specifically, for $D = d+1$ dimensions we consider

$$S = S_{bulk} + S_{GHY} + S_{ct}, \tag{5.4.1}$$

where,

$$S_{GHY} = \frac{1}{8\pi}\int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{|h|}\mathcal{K}\,, \tag{5.4.2}$$

$$\begin{aligned} S_{ct} &= \frac{1}{8\pi}\int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{|h|}\left[\frac{d-1}{\ell} + \frac{\ell}{2(d-2)}\mathcal{R} + \frac{\ell^3}{2(d-4)(d-2)^2}\left(\mathcal{R}_{ij}\mathcal{R}^{ij} - \frac{d}{4(d-1)}\mathcal{R}^2\right)\right. \\ &+ \left.\frac{\ell}{4(2\Delta-d-2)}\sum_{i=1}^{k} h^{mn}\mathcal{D}_m\bar{\varphi}_{(i)}\mathcal{D}_n\bar{\varphi}_{(i)} + \ldots\right]\,, \end{aligned} \tag{5.4.3}$$

with Δ being the conformal weight of the scalar fields. We are interested in the massless case, therefore we focus on the case $\Delta = d$.

Then, the free energy can be obtained from the renormalized Euclidean on-shell action to first order in the saddle point approximation. Here, we focus on the simplest nontrivial multi-NUT AdS solution at $d = 5$ with two NUT charges. Firsly, we will perform the Wick rotation $t \to -i\tau$ and $n_{(i)} \to -in_{(i)}$. To avoid the conical singularities at $r = r_h$, defined as the largest positive root of $f(r_h) = 0$ we demand that $\tau \sim \tau + \beta_\tau$ and β_τ defined the Euclidean time period, given by

$$\beta_\tau = \frac{4\pi\ell^2 r_h}{5(r_h^2 - n_2^2) - 8\pi\ell^2\lambda_2^2}\,, \tag{5.4.4}$$

where we use the axionic constant relation given by Eq. (5.3.8). The Euclidean time period is related with the Hawking temperature via

$$T_H = \beta_\tau^{-1}. \tag{5.4.5}$$

From hereon, we assume, without loss of generality, that $r_h > n_2 > n_1$. Regularity of the Euclidean on-shell action demands that the condition

$$\lambda_2^4 - \frac{(4n_1^2 - n_2^2)\lambda_{(2)}^2}{4\pi\ell^2} + \frac{5(n_1^2 - n_2^2)(7n_1^2 + 5n_2^2)}{256\pi^2\ell^4} = 0\,, \tag{5.4.6}$$

must be met. This condition is consistent with the planar Taub-NUT-AdS solution without the axionic fields when $n_{(1)} = n_{(2)} = n$. Then, the renormalized Euclidean on-shell action is given by

$$-I_E = \frac{\beta\Sigma}{16\pi}\left(m - 4\pi r_h(n_1^2 - n_2^2)(\lambda_1^2 - \lambda_2^2) - \frac{r_h[12r_h^4 - 20r_h^2(n_1^2 + n_2^2) + 15(n_1^2 + n_2^2)^2]}{6\ell^2}\right) \tag{5.4.7}$$

where Σ is the volume element of codimension-2 tranverse section of contant $t - r$. Similar to AdS black holes, the exitence of a horizon demands that there is a bound for the temperature of the multi-NUT configuration, i.e., the $T \geq T_{min}$, this is given by

$$T_{min} = \frac{\sqrt{25n_2^2 - 40\pi\ell^2\lambda_2^2}}{2\pi\ell^2}\,. \tag{5.4.8}$$

5.5 Hamiltonian energy of multi-NUTs

Considering the ADM formalism discussed in Chapter 3, we employ the surface integrals that arise in General Relativity to compute the energy of the system. However, since we supplement the theory with additional matter content, namely scalar fields, their presence modifies the value of the energy and must therefore be taken into account.

The Lagrangian density for a real scalar field is given by[7],

$$\mathcal{L}_\phi = \sqrt{|g|}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right)\,. \tag{5.5.1}$$

Then, we can expand the Lagrangian density as follows,

$$\mathcal{L}_\phi = \sqrt{|g|}\left(-\frac{1}{2}g^{tt}\dot{\phi}^2 - g^{ti}\dot{\phi}\nabla_i\phi - \frac{1}{2}g^{ij}\nabla_i\phi\nabla_j\phi\right)\,, \tag{5.5.2}$$

where $\dot{\phi} = \nabla_t\phi$. The conjugate momentum is given by,

$$\pi_\phi = \sqrt{|g|}(-\dot{\phi}g^{tt} - \nabla_i\phi g^{ti})\,. \tag{5.5.3}$$

[7] The extension to more than one scalar field is straightforward

From the latter, we can obtain the value of $\dot{\phi}$ which is given

$$\dot{\phi} = \frac{N}{\sqrt{h}}\pi_\phi + N^i \nabla_i \phi \,, \tag{5.5.4}$$

where h is the induced metric and N and N^i are the lapse and shift function, respectively.
Moreover, the Hamiltonian density reads

$$\mathbf{H}_\phi = \pi_\phi \dot{\phi} - \mathcal{L}_\phi \,. \tag{5.5.5}$$

Evaluating the Lagrangian density and the temporal derivative of the scalar field given by Eq. (5.5.2) and Eq. (5.5.4), respectively, yields

$$\begin{aligned} \mathbf{H}_\phi &= N\left(\frac{\pi_\phi^2}{2\sqrt{h}}\frac{1}{2}\sqrt{h}h^{ij}\nabla_i\phi\nabla_j\phi\right) + N^i(\pi_\phi \nabla_i \phi) \\ &= N\mathbf{H}_0^\phi + N^i \mathbf{H}_i^\phi \,. \end{aligned} \tag{5.5.6}$$

where

$$\mathbf{H}_0^\phi = \frac{\pi_\phi^2}{2\sqrt{h}} + \frac{1}{2}\sqrt{h}h^{ij}\nabla_i\phi\nabla_j\phi \,, \quad \mathbf{H}_i^\phi = \pi_\phi \nabla_i \phi \,. \tag{5.5.7}$$

In order to obtain well-defined equations of motion for the scalar field, we must carefully treat the boundary terms that arise from the arbitrary variation of $\mathcal{H}_\phi$ with respect to ϕ. It is worth noting that the variation with respect to π_ϕ vanishes identically. Then,

$$\delta_\phi \mathbf{H}_\phi = \left(\nabla_i N^i \pi_\phi - N^i \nabla_i \pi_\phi - \sqrt{h}h^{ij} N \nabla_i \nabla_j \phi - \nabla_i N \nabla_j \phi\right)\delta\phi \tag{5.5.8}$$

$$+ \; \nabla_i \left(\sqrt{h} N \nabla^i \phi \delta\phi + N^i \pi_\phi \delta\phi\right) . \tag{5.5.9}$$

Evidently, the first term in the latter equation contributes to the field equations, while the second is a boundary term and therefore contributes to the energy. Then, after applying Stokes' theorem, we obtain

$$\delta \mathbf{H}_\phi \Big|_{on-shell} = \oint_{S_t} \mathrm{d}^2 y \, r_i (\sqrt{\sigma} N \nabla^i \phi + N^i \pi_\phi) \delta\phi \,. \tag{5.5.10}$$

Therefore, the energy is given by the superposition of on-shell Eq. (3.3.19) and Eq. (5.5.10) that is

$$\begin{aligned}\delta(\mathbf{H}_0+\mathbf{H}_\phi)\Big|_{on-shell} &= \oint_{S_t}\sqrt{\sigma}\mathrm{d}^2y\left[Nh^{ab}\delta h_{ab,c}r^c-2N_a h^{-1/2}r_b\delta\pi^{ab}\right]\\ &- \oint_{S_t}\mathrm{d}^2y\, r_i(\sqrt{\sigma}N\phi^{;i}+N^i\pi_\phi)\delta\phi\,. \end{aligned}\tag{5.5.11}$$

To illustrate the latter we will consider the first non-trivial multi-NUTs case at $D=6$ dimensions. First, we need to compute the Lapse and Shift functions given by the ADM decomposition to the Lorentzian metric in Eq. (3.3.1). For the multi-NUTs reads as

$$N^2 = \frac{(r^2+n_1^2)(r^2+n_2^2)f(r)}{(r^2+n_1^2)(r^2+n_2^2)-4[r^2(x_1^2n_1^2+x_2^2n_2^2)+n_1^2n_2^2(x_1^2+x_2^2)]f(r)},\tag{5.5.12}$$

$$N_{(i)} = -2f(r)n_{(i)}Re(z_{(i)})\tag{5.5.13}$$

respectively. Where $f(r)$ is given by equation (5.3.7) when $k=2$. Then, the energy of the system is given by a modification of Eq. (5.5.11) which reads as,

$$\begin{aligned}\delta(\mathbf{H}_0+\mathbf{H}_\varphi+\mathbf{H}_{\varphi^*})\Big|_{on-shell} &= \oint_{S_t}\sqrt{\sigma}\mathrm{d}^2y\left[Nh^{ab}\delta h_{ab,c}r^c-2N_a h^{-1/2}r_b\delta\pi^{ab}\right]\\ &- \oint_{S_t}\mathrm{d}^2y\, r_i(\sqrt{\sigma}N\varphi^{;i}_{(i)}+N^i\pi_\varphi)\delta\varphi\\ &- \oint_{S_t}\mathrm{d}^2y\, r_i(\sqrt{\sigma}N\varphi^{*;i}_{(i)}+N^i\pi_{\varphi^*})\delta\varphi^* \end{aligned}\tag{5.5.14}$$

However, due to the complexity of the expressions, we do not present them here. It is worth noting that the presence of the cosmological constant introduces both cubic and linear divergences. Reassuringly, when all possible codimension-2 hypersurfaces used to compute the energy are taken into account, the superposition of their contributions cancels the cubic divergence. On the other hand, the linear divergence vanishes upon imposing the relation for the axionic constant given in Eq. (5.3.8), which furthermore ensures the integrability of the energy.

$$\mathbf{H}_0+\mathbf{H}_\varphi+\mathbf{H}_{\varphi^*}=m\,.\tag{5.5.15}$$

5.6 Higher curvature theory

The most general gravitational theory with quadratic curvature corrections is known as *Quadratic Curvature Gravity* (QCG). The action principle in $D \geq 4$ is given by

$$S_{QCG} = \int_{\mathcal{M}} \mathrm{d}^D x \sqrt{|g|} \left[\kappa(R - 2\Lambda) + \alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2 + \gamma \mathcal{G} \right] \tag{5.6.1}$$

where $\mathcal{G}$ is the Gauss-Bonnet term, defined as $\mathcal{G} := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$. The action reduces to Einstein-Gauss-Bonnet gravity with a cosmological constant when $\alpha = 0$ and $\beta = 0$. The action principle can be expressed in a more transparent way, via the generalized Kronecker delta contracted with the Riemman tensor. The latter can be written as,

$$\begin{aligned} S_{QCG} \;\; &= \;\; \int_{\mathcal{M}} d^D x \sqrt{|g|} \left[\kappa \left(\frac{1}{2} \delta^{\mu\nu}_{\lambda\rho} R^{\lambda\rho}_{\mu\nu} - 2\Lambda \right) + \alpha \delta^{\rho}_{\lambda} \delta^{\alpha}_{\beta} R^{\mu\lambda}_{\nu\rho} R^{\nu\beta}_{\mu\alpha} \right. && (5.6.2) \\ &\quad \left. + \beta \left(\frac{1}{4} \delta^{\mu\nu}_{\lambda\rho} \delta^{\alpha\beta}_{\sigma\tau} R^{\lambda\rho}_{\mu\nu} R^{\sigma\tau}_{\alpha\beta} \right) + \gamma \left(\frac{1}{4} \delta^{\mu\nu\alpha\beta}_{\lambda\rho\sigma\tau} R^{\lambda\rho}_{\mu\nu} R^{\sigma\tau}_{\alpha\beta} \right) \right] , && (5.6.3) \end{aligned}$$

where we use the generalized Kronecker delta defined in Eq. (7.1.2). The field equations of the theory are obtained by performing the arbitrary variation with respect to the metric, yielding,

$$\mathcal{E}_{\mu\nu} = \kappa G_{\mu\nu} + P_{\mu\nu} + \gamma H_{\mu\nu}, \tag{5.6.4}$$

with $G_{\mu\nu}$ the Einstein tensor, and $H_{\mu\nu}$ is the known as the Lanczos tensor, defined

$$\begin{aligned} H_{\mu\nu} \;\; &= \;\; -\frac{1}{2} g_{\mu\nu} \left(R^2 - 4R^{\alpha\beta} R_{\alpha\beta} + R^{\alpha\beta\lambda\rho} R_{\alpha\beta\lambda\rho} \right) \\ &\quad +2 \left(R R_{\mu\nu} - 2R_{\mu\lambda} R^{\lambda}{}_{\nu} - 2R_{\mu\alpha\nu\beta} R^{\alpha\beta} + R_{\mu\lambda\alpha\beta} R_{\nu}{}^{\lambda\alpha\beta} \right) , \end{aligned} \tag{5.6.5}$$

which gives the contribution of the Gauss-Bonnet term to the field equations of motion. Note that in dimension $D = 4$ the Gauss-Bonnet is a topological term, meaning that vanish identically and does not contribute with the field equations. On the other hand, the contributions that comes from the Ricci scalar-squared and the Ricci tensor-squared are captured in the tensor $P_{\mu\nu}$, which is defined as,

$$\begin{aligned} P_{\mu\nu} &= 2\beta R\left(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\right)+(\alpha+2\beta)(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)R \\ &\quad +\alpha\Box G_{\mu\nu}+2\alpha\left(R_{\mu\sigma\nu\lambda}-\frac{1}{4}g_{\mu\nu}R_{\sigma\lambda}\right)R^{\sigma\lambda}\,. \end{aligned} \tag{5.6.6}$$

5.7 R^2-corrected multi-NUT spacetime

Let us consider a particular sector of that theory. Specifically, when we have only the R^2 term. This sector allow one to introduce an asymptotic anisotropic scaling symmetry. Lifshitz black hole solutions have attracted considerable attention [84]. However, we focus on the isotropic scaling symmetry, because our multi-NUT solutions are generalizations of the AdS black holes of Ref. [84]. The action is given by,

$$S[g]=\int \mathrm{d}^Dx\sqrt{|g|}\left(R-2\Lambda+\beta R^2\right), \tag{5.7.1}$$

where β is a couplig constant. Performing the arbitrary variation we obtain the field equation of the action (5.7.1)

$$G_{\mu\nu}+\Lambda g_{\mu\nu}+\beta H_{\mu\nu}=0, \tag{5.7.2}$$

with

$$H_{\mu\nu}=2g_{\mu\nu}\Box R-2\nabla_\mu\nabla_\nu R+2RR_{\mu\nu}-\frac{1}{2}R^2g_{\mu\nu}\,, \tag{5.7.3}$$

which capture the higher-derivative terms. It is well known that in higher-curvature theories, vacuum states characterized by maximally symmetric spaces [85], have an effective curvature radius ℓ_{eff}. In this case, these are describes by

$$2D\beta(D-4)\Lambda_{\text{eff}}^2=(D-2)^2(\Lambda-\Lambda_{\text{eff}})\,, \tag{5.7.4}$$

where,

$$\Lambda_{\text{eff}}=-\frac{(D-1)(D-2)}{2\ell_{\text{eff}}^2}\,. \tag{5.7.5}$$

The multiNUTs are solutions when the coupling constant satisfies,

$$\beta = -\frac{1}{8\Lambda}. \tag{5.7.6}$$

Therefore, the metric function that solves (5.2.1) is giving by

$$f(r) = \frac{1}{\prod_{i=1}^{k}(r^2+n_i^2)}\left[-2\Lambda\sum_{i=0}^{k}\frac{e_{k-i}r^{2i+2}}{(i+1)(2i+1)} - mr + b\right], \tag{5.7.7}$$

where m and b are integration constants, and we have used the elementary symmetric polynomials e_i of k variables, the latter is define as follows,

$$e_i = e_i(n_{(1)}^2,\ldots,n_{(k)}^2) = \sum_{1\le a_1<a_2<\ldots a_i\le k} n_{(a_1)}^2 n_{(a_2)}^2 \ldots n_{(a_i)}^2 \,. \tag{5.7.8}$$

It is worth mentioning that in the limit where every NUT charges vanishes, we recover the AdS black holes of [84]. Moreover, at (5.7.6) point the on-shell action is identically zero when the curvature scalar satifies $R = 4\Lambda$. This can be seen from the evaluation of (5.7.6) in (5.7.1) yielding,

$$\begin{aligned} S[g] &= \int \mathrm{d}^D x\sqrt{|g|}(R - 2\Lambda - \frac{1}{8\Lambda}R^2) \\ &= -\frac{1}{8\Lambda}\int \mathrm{d}^D x\sqrt{|g|}(R-4\Lambda)^2\,. \end{aligned} \tag{5.7.9}$$

Nevertheless, if one consider the boundary term proposed in the Ref. [86], one can check that this also vanish on-shell. Moreover, the funtional derivative of the Lagrangian with respect to the Riemann is given by,

$$E^{\mu\nu}_{\lambda\rho} := \frac{\partial\mathcal{L}}{\partial R^{\lambda\rho}_{\mu\nu}} = -\frac{1}{8\Lambda}(R-4\Lambda)\delta^{\mu\nu}_{\lambda\rho}\,, \tag{5.7.10}$$

One can interpreted that the point $R = 4\Lambda$ the theory becomes critical.

5.8 Kaluza-Klein monopoles

Kaluza–Klein (KK) theories provide one of the earliest and most natural frameworks to unify gravity with other fundamental interactions by extending spacetime to higher dimensions. In the original proposal by Kaluza and Klein [87, 88], electromagnetism emerges from the extra-dimensional components of the metric, offering a geometric interpretation of gauge symmetries. Beyond this original motivation, KK constructions play a central role in modern theoretical physics, particularly in supergravity and string theory [89, 90], where extra dimensions are a fundamental ingredient. Moreover, they provide a natural setting to explore non-trivial gravitational configurations, such as monopoles and instantons [26, 27], and to investigate the interplay between geometry, topology, and field theory within a unified framework.

5.8.1 Kaluza-Klein reduction on S^1

To illustrate the mechanism of KK, we will assume that we start from $(D+1)$-dimensional Einstein-Hilbert Lagrangian given by, we follow the lectures notes of Pope [91]

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R}\,, \tag{5.8.1}$$

where $\hat{R}$ is the Ricci scalar and $\hat{g}$ denotes the determinant of the metric.

Assuming that the KK reduction ansatz is given by a $\hat{g}_{MN}(x,z)$, i.e., independent of z, which is the coordinate when we compactify on an S^1. The main point now is that from the D-dimensional point of view, the capital indices splits into a range lying in the D lover dimensions. Thus we can separate the components of $\hat{g}_{MN}$ into $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu z}$, and $\hat{g}_{zz}$. Hence, for the D-dimensional viewpoint these look as the metric, a 1-form gauge potential, and a scalar field, respectively.

For the latter, we can simplify the components of the $\hat{g}_{MN}$ as a D-dimensional fields, those are $g_{\mu\nu}$, $\mathcal{A}_\mu$, and ϕ respectively. Therefore, we can write the $D+1$-dimensional metric in terms of the D-dimensional fields such as

$$\mathrm{d}\hat{s}^2_{KK} = e^{2\alpha\phi}\mathrm{d}s^2 + e^{2\beta\phi}(\mathrm{d}z+\mathcal{A})^2\,, \tag{5.8.2}$$

where α and β are constant and $\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu$. Note, that the right-hand side of Eq. (5.8.2) is independent of z. Moreover, it is worth mentioning that the Eq.(5.8.2) is related with the higher-dimensional metric $\hat{g}_{MN}$ as,

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi}\mathcal{A}_\mu\mathcal{A}_\nu \,, \quad \hat{g}_{\mu z} = e^{2\beta\phi}\mathcal{A}_\mu \,, \quad \hat{g}_{zz} = e^{2\beta\phi} \,. \tag{5.8.3}$$

To obtain the reduced theory we need to obtain the Ricci scalar that emerges for the compactification. It is convenient to work in differential forms. For the following, we will choose the next vielbein basis,

$$\hat{e}^a = e^{\alpha\phi} e^a \,, \quad \hat{e}^z = e^{\beta\phi}(\mathrm{d}z + \mathcal{A}) \,. \tag{5.8.4}$$

Then, the spin connection we will obtain that

$$\hat{\omega}^{ab} = \omega^{ab} - 2\alpha e^{-\alpha\phi}\partial^{[a}\phi\hat{e}^{b]} - \frac{1}{2}\mathcal{F}^{ab} e^{(\beta-2\alpha)\phi}\hat{e}^z \,, \tag{5.8.5}$$

$$\hat{\omega}^{az} = -2\beta e^{-\alpha\phi}\partial^{[a}\phi\hat{e}^{z]} - \frac{1}{2}\mathcal{F}^a{}_b e^{(\beta-2\alpha)\phi}\hat{e}^b \,, \tag{5.8.6}$$

where $\mathcal{F} = \mathrm{d}\mathcal{A}$ and $\partial_a\phi$ means $E^\mu_a\partial_\mu\phi$ and E^μ_a is the inverse of the D-dimensional vielbein [for more details about differential forms and definition see Ref. [34, 56, 57]

To be in the Einstein frame and to have a canonical kinetic term for the dilaton field, we can conclude that the constants must be satisfy the following relations,

$$\alpha^2 = \frac{1}{2(D-1)(D-2)} \,, \quad \beta = -(D-2)\alpha \,. \tag{5.8.7}$$

Then, the Ricci tensor is given by [26, 27],

$$\hat{R}_{ab} = e^{-2\alpha\phi}\left(R_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\eta_{ab}\Box\phi\right) - \frac{1}{2}e^{-2D\alpha\phi}\mathcal{F}_a{}^c\mathcal{F}_{bc} \,, \tag{5.8.8}$$

$$\hat{R}_{az} = \frac{1}{2}e^{(D-3)\alpha\phi}\nabla^b\left(e^{-2(D-1)\alpha\phi}\mathcal{F}_{ab}\right) \,, \tag{5.8.9}$$

$$\hat{R}_{zz} = (D-2)\alpha e^{-2\alpha\phi}\Box\phi + \frac{1}{4}e^{-2D\alpha\phi}\mathcal{F}^2 \,, \tag{5.8.10}$$

where $\mathcal{F}^2 = \mathcal{F}_{ab}\mathcal{F}^{ab}$. With these components, we can obtaint the Ricci scalar as follows

$$\hat{R} = \eta^{AB}\hat{R}_{AB} = \eta^{ab}\hat{R}_{ab} + \hat{R}_{zz}\,, \tag{5.8.11}$$

then,

$$\hat{R} = e^{-2\alpha\phi}\left(R - \frac{1}{2}(\partial\phi)^2 + (D-3)\alpha\Box\phi\right) - \frac{1}{4}e^{-2D\alpha\phi}\mathcal{F}^2\,. \tag{5.8.12}$$

Finally, we need to compute the square root of the determinant of the metric $\hat{g}$ which is given by

$$\sqrt{-\hat{g}} = e^{(\beta+D\alpha)\phi}\sqrt{-g} = e^{2\alpha\phi}\sqrt{-g}\,. \tag{5.8.13}$$

Finally, plugging all the results before obtained,

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} = \sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 + (D-3)\alpha\Box\phi - \frac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}^2\right)\,. \tag{5.8.14}$$

Note that the third term of the Eq. (5.8.14) is a total derivative in $\mathcal{L}$,i.e., does not contribute to the field equations. The ϕ is commoly knows as the dilaton. Note that for the limit $\phi \to 0$, the theory reduces to Einstein-Maxwell Lagrangian in D-dimensions.

The field equations of Eq. (5.8.14) are given by,

$$G_{\mu\nu} = \frac{1}{2}\left(T^{\phi}_{\mu\nu} + e^{-2(D-1)\phi}T^{\mathcal{A}}_{\mu\nu}\right)\,, \tag{5.8.15}$$

$$\nabla_\mu\left(e^{-2(D-1)\phi}\mathcal{F}^{\mu\nu}\right) = 0\,, \tag{5.8.16}$$

$$\Box\phi = -\frac{1}{2}(D-1)\alpha e^{-2(D-1)\alpha\phi}\mathcal{F}^2\,, \tag{5.8.17}$$

where

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\,, \tag{5.8.18}$$

$$T^{\mathcal{A}}_{\mu\nu} = \mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho} - \frac{1}{2}g_{\mu\nu}\mathcal{F}^2\,. \tag{5.8.19}$$

An interesting class of configurations that can be explored within this framework are solutions belonging to the dimensionally reduced theory.

The first Kaluza–Klein monopole (KK monopole) was constructed by Gross, Perry, and Sorkin [26,27]. They start from the Taub–NUT metric given in Eq. (4.2.1) and oxidize it by adding a time-like direction, leading to

$$\mathrm{d}s^2_{TN} = f(r)(\mathrm{d}z + 2n\cos\theta\mathrm{d}\varphi)^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 - n^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) - \mathrm{d}t^2\,, \quad (5.8.20)$$

where $f(r) = r - n/(r+n)$. [8] This metric solves the five-dimensional Einstein equations in vacuum with $\Lambda = 0$. Then, by identifying the metric with the Kaluza-Klein ansatz given by Eq. (5.8.2).

Direct comparison between $\mathrm{d}\hat{s}^2_{KK} = \mathbf{d}s^2_{TN}$, where the latter solves $\hat{R}_{AB} = 0$. yields

$$\begin{aligned} e^{2\beta\phi} &= f(r) \to \phi(r) = \frac{1}{2\beta}\ln f(r)\,, && (5.8.21)\\ \mathcal{A} &= 2n\cos\theta\mathrm{d}\varphi\,, && (5.8.22)\\ e^{2\alpha\phi}\mathrm{d}s^2 &= -\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 - n^2)(\mathrm{d}\theta^2 + \sin\theta\mathrm{d}\varphi^2) && (5.8.23) \end{aligned}$$

therefore the four-dimensional metric can be written as

$$\begin{aligned} \mathrm{d}s^2 &= e^{-\frac{\alpha}{\beta}\ln f(r)}\left(-\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 - n^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\right)\\ &= f^{\frac{1}{(D-2)}}\left(-\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 - n^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\right) \end{aligned}$$

where we used the Eq. (5.8.7). The dimensionally reduced Einstein-Hilbert action is

$$\sqrt{|\hat{g}_{KK}|}\hat{R}_{KK} = \sqrt{|g|}\left(R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}^2\right)\,, \quad (5.8.24)$$

whose field equations are

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}\left(T^{(\phi)}_{\mu\nu} + e^{-2(D-1)\alpha\phi}T^{(\mathcal{A})}_{\mu\nu}\right)\,, && (5.8.25)\\ \nabla_\mu(e^{-2(D-1)\alpha\phi}\mathcal{F}^{\mu\nu}) &= 0\,, && (5.8.26)\\ \Box\phi &= -\frac{1}{2}(D-1)\alpha e^{-2(D-1)\alpha\phi}\mathcal{F}^2\,. && (5.8.27) \end{aligned}$$

[8] In the original construction, the authors fix $m = n$ in order to ensure regularity of the Euclidean Taub–NUT geometry at $r = n$.

From the four-dimensional perspective, this configuration describes a magnetic monopole solution of the Einstein-Maxwell-dilaton theory, where the gauge field originates from the Kaluza–Klein vector and the NUT charge plays the role of a magnetic charge. The resulting spacetime is asymptotically locally flat and free of singularities once the appropriate periodicity of the coordinate z is imposed, removing the Misner string. Attempts to generalize these solutions to asymptotically AdS spacetimes have proven to be highly non-trivial. In particular, the inclusion of a cosmological constant introduces additional constraints that obstruct a straightforward extension of the original Gross–Perry–Sorkin construction.

Our aim is to extend the notion of KK monopoles to Kaluza–Klein multi-monopole configurations in the presence of a cosmological constant.

5.9 Planar Kaluza-Klein multi-monopoles in AdS

To construct planar Kaluza-Klein multi-monopoles in AdS, we need to avoid the restriction imposed by the field equations, which require either that the cosmological constant vanish or that the NUT charges be equal. To do so, we take the solution in Sec. 5.3 as a seed metric and add a single flat direction, alongside an axion field that depends linearly on the coordinate that spans the extra dimension, following the strategy of Ref. [30]. This allows us to uplift the even-dimensional solution to odd dimensions, keeping the cosmological constant nonzero. Concretely, we consider the analytic continuation of the line element (5.2.1) by performing the Wick rotation $t \to -i\tau$ and $n_{(i)} \to -in_{(i)}$. The Kaluza-Klein reduction is performed along the periodic coordinate that generates the $U(1)$ fibration over the Einstein-Kähler manifold in the seed metric. This procedure yields

$$S = \int_{\mathcal{M}} \mathrm{d}^D x \sqrt{|g|} \Big[R - 2\Lambda e^{2\alpha\phi} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} \sum_{i=1}^{k} F_{(i)}^2 - \frac{1}{2}\sum_{i=1}^{k} |\nabla\varphi_{(i)}|^2 - \frac{1}{2}(\nabla\chi)^2 \Big] , \tag{5.9.1}$$

where $\alpha^{-2} = 2(D-1)(D-2)$, $A_{(i)} = 2n_{(i)}B_{(i)}$ are the Abelian gauge fields that represent the planar Kaluza-Klein multi-monopoles with $n_{(i)}$ being the different magnetic charges, and $B_{(i)}$ is the Kähler potential one-form defined in Eq. (5.3.2). Additionally, $F^2_{(i)} = F^{(i)}_{\mu\nu}F^{\mu\nu}_{(i)}$, with $F_{(i)} = A_{(i)}$ being the Abelian field strength. This is an Einstein-Maxwell-dilaton theory with a Liouville potential, minimally coupled to $k = (D-2)/2$ free complex scalar fields, $\varphi_{(i)}$, and one axion field χ. The field equations are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda e^{2\alpha\phi} = \frac{1}{2}\left(T^{(\varphi)}_{\mu\nu} + T^{(\phi)}_{\mu\nu} + T^{(\chi)}_{\mu\nu} + e^{-2(D-1)\alpha\phi}T^{(A)}_{\mu\nu}\right), \tag{5.9.2a}$$

$$\nabla_\mu\left(e^{-2(D-1)\alpha\phi}F^{\mu\nu}_{(i)}\right) = 0\,, \tag{5.9.2b}$$

$$\Box\phi - 4\alpha\Lambda e^{2\alpha\phi} = -\frac{1}{2}(D-1)\alpha e^{-2(D-1)\alpha\phi}\sum_{i=1}^{k}F^2_{(i)}\,, \tag{5.9.2c}$$

$$\Box\varphi_{(i)} = 0\,, \quad \Box\bar{\varphi}_{(i)} = 0\,, \quad \Box\chi = 0\,, \tag{5.9.2d}$$

where the stress energy tensor for the complex scalar fields is given in Eq. (5.3.4c), while for the dilaton and Abelian gauge fields, they are respectively given by[9]

$$T^{(\phi)}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\,, \quad \text{and} \quad T^{(A)}_{\mu\nu} = \sum_{i=1}^{k}\left(F^{(i)}_{\mu\lambda}F^{(i)\lambda}_{\nu} - \frac{1}{4}g_{\mu\nu}F^2_{(i)}\right). \tag{5.9.3a}$$

In order to solve the field equations of the dimensionally reduced theory, the axionic contribution χ must be taken into account. The latter is necessary to support the existence of a nontrivial Liouville potential for the dilaton, keeping a nonvanishing cosmological constant in the seed metric. This can be done by considering the following ansatz for the line element, free complex scalars, and axionic fields, that is,

$$\mathrm{d}s^2 = f(r)^{\frac{1}{D-2}}\left(\mathrm{d}z^2 + \frac{\mathrm{d}r^2}{f(r)} + \sum_{i=1}^{k}(r^2 - n^2_{(i)})\,\mathrm{d}\Sigma^2_{(i)}\right), \quad \varphi_{(i)} = \lambda_{(i)}\,z_{(i)}\,, \quad \chi = \lambda_{(0)}z\,, \tag{5.9.4}$$

[9] The stress tensor $T^{(\chi)}_{\mu\nu}$ can be obtained by replacing ϕ by χ in $T^{(\phi)}_{\mu\nu}$.

respectively, where $\mathrm{d}\Sigma^2_{(i)}$ is defined in Eq. (5.3.1), and $\lambda_{(0)}$ is a constant. The complex scalars and axionic field in Eq. (5.9.4) solve their associated field equations automatically. The remaining equations are solved by the metric function and the dilaton field

$$\begin{aligned} f(r) &= -\frac{r}{\prod_{i=1}^{k}\left(r^2+n^2_{(i)}\right)}\left[m-\int^r \frac{\frac{\lambda^2_{(k)}}{2}+\frac{2\Lambda}{2k+1}\left(\rho^2+n^2_{(k)}\right)}{\rho^2}\prod_{i=1}^{k}\left(\rho^2+n^2_{(i)}\right)\mathrm{d}\rho\right], \\ \phi(r) &= -\frac{\log f(r)}{2(D-2)\alpha}, \end{aligned} \tag{5.9.5}$$

where m is an integration constant, as long as the following constraints are met

$$\lambda^2_{(0)} = -\frac{4\Lambda}{D-1}, \qquad \text{and} \qquad \lambda^2_{(i)} = \lambda^2_{(k)} + \frac{4\Lambda}{2k+1}\left(n^2_{(k)} - n^2_{(i)}\right). \tag{5.9.6}$$

Reality conditions on the axion χ require that $\Lambda < 0$. This follows from the fact that the coordinate z is spacelike and that $\lambda_{(0)}$ must satisfy the constraint in Eq. (5.9.6). Nevertheless, the even-dimensional Euclidean metric in Eq. (5.9.4) can be analytically continued into the Lorentzian section by performing $z \to -it$. In such a case, one should perform the Wick rotation $\lambda_{(0)} \to i\lambda_{(0)}$ to keep the axion real and linear in the time coordinate, while the constraint in Eq. (5.9.6) yields $\Lambda > 0$.

In order to avoid a signature change in the metric (5.9.4), the radial coordinate should be restricted to $r \geq r_h \geq n_{(i)}$, where r_h is defined as the largest root of $f(r)$ in Eq. (5.9.5). Additionally, the coordinate z has to be extended along the real line. Otherwise, one cannot eliminate conical singularities at $r = r_h$, and the axion χ would have been multi-valued. On the other hand, the behavior of the dilaton is similar to that in Ref. [29]. However, the main difference is that the planar AdS Kaluza-Klein multi-instanton presented here has a smooth limit as $\Lambda \to 0$ which, in turn, is translated into $\chi \to 0$, while keeping $n_{(i)} \neq 0$, with or without complex scalar fields $\varphi_{(i)}$.

Chapter 6

Conclusions

In this thesis we constructed multi-NUTs geometries in asymptotically locally AdS, in two different scenarios. Firstly, we constructed multi-NUT when the tranverse section of the line element has a trivial Gaussian curvature. This solution was obtained by suplementing Einstein gravity with free complex scalar fields. The latter have axionic profiles. These solutions are continuously connected with planar black holes with axionic charge when all the NUT charges vanishes. The addition of multiple NUT parameters can be interpreted as different gravitomagnetic sources.

As an application, we use the multi-NUT-AdS solution to construct Kaluza-Klein multi-fluxes in Einstein-Maxwell-dilation-axion with a Liouville potential. We show that the planar AdS Kaluza-Klein multi-instanton presented here has a smooth limit when the cosmological constant vanishes.

This is another step to the study solutions that can be interpreted as a holographic fluid. However, all the machinery in four dimensions is not directly generalized in higher dimensions. Due to this, the motivation to develop a prescription to study holographic scenarios in higher dimensions is mandatory. It would be interesting to understand the role of the Cotton tensor in higher dimensions.

Secondly, we found the existence of multiNUTs with higher-curvature correction in the theory, without adding matter content to the theory. To the latter, all the conserved quantities vanish. This is due to the fact that the theory becomes critical at a particular value of the coupling constant. Furthermore, at this point it cannot be mapped to a scalar-tensor theory.

Interesting question remains open. Firstly, a detailed thermodynamic analysis of planar multi-NUT spaces is certainly worth exploring. This would unveil possible phase transitions of these planar configurations with thermal AdS, which are otherwise forbidden in the absence of NUT charges. Secondly, it is known that the NUT charge in four dimensions can be interpreted as magnetic mass [8, 9, 53, 92–94]. It would be interesting to study whether the multi-NUT spaces still admit the same type of interpretation in higher dimensions or if they induce some sort of angular momentum, similar to the Myers-Perry black hole. Thirdly, the role of multi-NUT parameters in fluid/gravity correspondence is certainly worth exploring, to see whether these configurations induce interesting effects in holographic fluid dynamics

Chapter 7

Appendix A

7.1 Differential geometry

General relativity is a geometric theory. To understand it is mandatory to learn and understand differential geometry, To do so, let us put some fundamental concepts that lead. In this Appendix we introduce some basics and fundamental concepts that we need it. This chapter is based on differents books [46, 95]

7.1.1 Manifold

We will say that a manifold $\mathcal{M}$ is an $D-$dimensional differentiable manifold if

1) $\mathcal{M}$ is a topological space;

2) $\mathcal{M}$ is provided with a family of pairs $\{(U_i, \varphi_i)\}$

3) $\{U_i\}$ is a family of open sets which covers $\mathcal{M}$, that is, $\cup_i U_i = \mathcal{M}$. φ_i is a homomorphism from U_i onto an open subset U_i' if $\mathbb{R}^D$; and

4) given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, the map $\psi_{ij} = \varphi_i o \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable.

With this definition, we assume in this work a $D-$ dimensional manifold $\mathcal{M}$ that are supplemented with a metric tensor $g_{\mu\nu}$, which is symmetric and covariant tensor of rank two. The signature of the metric is *Lorentzian*, which means that $(-, +, \ldots, +)$.

An arbitrary tensor T with (p,q)-rank can be expand as

$$T = T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} \partial_{\mu_1} \dots \partial_{\mu_p} \mathrm{d}x^{\nu_1} \dots \mathrm{d}x^{\nu_q}, \tag{7.1.1}$$

where $\partial_\mu := \frac{\partial}{\partial x^\mu}$ are the basis of the tangent vector fields to $\mathcal{M}$ at the point x, denoted by $T_x\mathcal{M}$ meanwhile $\mathrm{d}x^\mu$ is 1-form basis of the cotangent space, denoted by $T_x^*\mathcal{M}$. Note that both bases are dual. They satisfies an orthonormal relation, given by $\partial_\nu(\mathrm{dx}^\mu) = \delta^\mu_\nu$ with δ^μ_ν the Kronecker delta.

The generalized Kronecker delta is defined as

$$\delta^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_p} := p! \delta^{[\mu_1}_{[\nu_1} \dots \delta^{\mu_p]}_{\nu_p]} . \tag{7.1.2}$$

Therefore it is a completely antisymmetric tensor. Satisfying,

$$\delta^{\nu_1 \dots \nu_p}_{\mu_1 \dots \mu_p} \delta^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_k} = \frac{(D-p+k)!}{(D-p)!} \delta^{\nu_{k+1} \dots \nu_p}_{\mu_{k+1} \dots \mu_p} , \tag{7.1.3}$$

where D is the dimension of $\mathcal{M}$ and $k \leq p \leq D$.

The covariant derivative of a tensor T with (p,q)-rank on its components is defined as

$$\begin{aligned} \nabla_\lambda T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} &= \partial_\lambda T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} + \Gamma^{\mu_1}{}_{\rho\lambda} T^{\rho \dots \mu_p}{}_{\nu_1 \dots \nu_q} + \dots + \Gamma^{\mu_p}{}_{\rho\lambda} T^{\mu_1 \dots \mu_{p-1}\rho}{}_{\nu_1 \dots \nu_q} \\ &- \Gamma^\rho{}_{\nu_1\lambda} T^{\mu_1 \dots \mu_p}{}_{\rho \dots \nu_q} - \dots - \Gamma^\rho{}_{\nu_q\lambda} T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_{q-1}\rho}, \end{aligned} \tag{7.1.4}$$

where $\Gamma^\rho_{\mu\nu}$ is the *Christoffel connection*, defines as

$$\Gamma^\rho_{\mu\nu} := \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{7.1.5}$$

With these ingredients, we can define the *Riemann curvature tensor*,

$$[\nabla_\mu, \nabla_\nu] V^\lambda = R^\lambda{}_{\rho\mu\nu} V^\rho \tag{7.1.6}$$

where $R^\lambda{}_{\rho\mu\nu} V^\rho$ is defined via the Christoffel connection,

$$R^\lambda{}_{\rho\mu\nu} := \partial_\mu \Gamma^\lambda_{\rho\nu} - \partial_\nu \Gamma^\lambda_{\rho\mu} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\rho\nu} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\rho\mu}. \tag{7.1.7}$$

Note that we assumed a torsion-free connection.

7.2 Computations for the variational principle on GR

To compute the arbitrary variation with respect to the metric of the (2.1.1) we need some properties to make the work easier. Hence, we proceed to vary the action with respect to the inverse metric $g^{\alpha\beta}$ (we are assuming that the metric is invertible). Note that the action depends on second-order derivatives of the metric, i.e., $(\partial g)^2$. For the invariant volume element, we will use the next properties. For an arbitrary invertible matrix, denoted by M, satisfy

$$MM^{-1} = \mathbb{1} \quad /\delta \tag{7.2.1}$$

$$\Rightarrow \delta M M^{-1} = -M\delta M^{-1}, \tag{7.2.2}$$

where in the second line we take the arbitrary variation. On the other hand, the determinant of M is defined by

$$\det M = e^{\mathrm{Tr}(\ln M)}. \tag{7.2.3}$$

Then, the variation of the determinant reads

$$\begin{aligned}\delta \det M &= \delta e^{\mathrm{Tr}(\ln M)} \\ &= e^{\mathrm{Tr}(\ln M)}\,\mathrm{Tr}(\delta \ln M) \\ &= e^{\mathrm{Tr}(\ln M)}\,\mathrm{Tr}(M^{-1}\delta M) \\ &= -\det M\,\mathrm{Tr}(M\delta M^{-1}).\end{aligned}$$

Therefore, with this relation, we can express the square root of the determinant of the metric as

$$\delta\sqrt{-g} := -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta}\,. \tag{7.2.4}$$

Hence, taking the arbitrary variations of the equation (2.1.1), then

$$
\begin{aligned}
\delta S_{EH} &= \kappa \int_{\mathcal{M}} \delta\left(\sqrt{-g}R\right) \mathrm{d}^4x \\
&= \kappa \int_{\mathcal{M}} \left(\delta\sqrt{-g}R + \sqrt{-g}\delta R\right) \mathrm{d}^4x \\
&= \kappa \int_{\mathcal{M}} \left(-\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}R + \sqrt{-g}\delta\left(g^{\alpha\beta}R_{\alpha\beta}\right)\right) \mathrm{d}^4x \\
&= \kappa \int_{\mathcal{M}} \left(-\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}R + \sqrt{-g}\left(\delta g^{\alpha\beta}R_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}\right)\right) \mathrm{d}^4x \\
&= \kappa \int_{\mathcal{M}} \left(-\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}R + \sqrt{-g}\delta g^{\alpha\beta}R_{\alpha\beta} + \sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta}\right) \mathrm{d}^4x \\
&= \kappa \int_{\mathcal{M}} \left(\sqrt{-g}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)\delta g^{\alpha\beta} + \sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta}\right) \mathrm{d}^4x \\
&= \kappa \int_{\mathcal{M}} \left(\sqrt{-g}G_{\alpha\beta}\delta g^{\alpha\beta} + \sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta}\right) \mathrm{d}^4x
\end{aligned}
$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ it is identified as the Einstein tensor. The second term on the last expression corresponds to the variation of the Ricci tensor. Let us work out that term,

$$
\begin{aligned}
\delta R_{\alpha\beta} &= \delta R^{\lambda}_{\alpha\lambda\beta} \\
&= \delta\left(\partial_\sigma\Gamma^{\sigma}_{\beta\alpha} - \partial_\beta\Gamma^{\sigma}_{\sigma\alpha} + \Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\beta\alpha} - \Gamma^{\sigma}_{\beta\lambda}\Gamma^{\lambda}_{\sigma\alpha}\right) \\
&= \partial_\sigma\delta\Gamma^{\sigma}_{\beta\alpha} + \delta\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\beta\alpha} - \delta\Gamma^{\sigma}_{\beta\lambda}\Gamma^{\lambda}_{\sigma\alpha} - \Gamma^{\sigma}_{\beta\lambda}\delta\Gamma^{\lambda}_{\sigma\alpha} - \left(\partial_\beta\delta\Gamma^{\sigma}_{\sigma\alpha} - \delta\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\beta\alpha}\right), \quad \sigma \to \mu \\
&= \nabla_\mu\delta\Gamma^{\mu}_{\alpha\beta} - \nabla_\beta\delta\Gamma^{\mu}_{\mu\alpha}
\end{aligned}
$$

Therefore, the variation of the Ricci tensor is given in terms of covariant derivatives of the variation of the connection. This is known as the *Palatini identity*. Additionally, in a more elegant way, we can express this lemma using the Kronecker delta (7.1.2) as follows,

$$\delta R_{\mu\nu} := \delta^{\alpha\beta}_{\lambda\nu}\nabla_\alpha\delta\Gamma^{\lambda}_{\mu\beta} \tag{7.2.5}$$

The variation of the *Christoffel connection* is given by

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(\nabla_\beta\delta g_{\nu\alpha} + \nabla_\alpha\delta g_{\nu\beta} - \nabla_\nu\delta g_{\alpha\beta}), \tag{7.2.6}$$

which is a covariant expression, mean that it is valid in all frames. Hence, replacing (7.2.6) in the second term of the Einstein-Hilbert action yields,

$$\begin{aligned} g^{\alpha\beta}\delta R_{\alpha\beta} &= g^{\alpha\beta}(\nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - \nabla_\beta \delta\Gamma^\mu_{\mu\alpha}) \\ &= \nabla_\mu (g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\delta\Gamma^\beta_{\alpha\beta}) \\ &= \nabla_\mu V^\mu, \end{aligned}$$

where, in the second line, we relabeled indices and used the metricity condition to construct a total covariant derivative. Additionaly we define a vector V^μ that defines,

$$V^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\delta\Gamma^\beta_{\alpha\beta}. \tag{7.2.7}$$

Hence, we conclude that the variation of the Ricci tensor is given by a total derivative, which means that it is a boundary term for the action.

Replacing this result on the arbitrary variation yields,

$$\delta S_{EH} = \kappa \int_{\mathcal{M}} \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} \mathrm{d}^4x + \kappa \int_{\mathcal{M}} \sqrt{-g} \nabla_\mu V^\mu \mathrm{d}^4x. \tag{7.2.8}$$

The second integral can be rewritten as a surface integral for a codimension one hypersurface using Stokes' theorem (citar y que venga de la parte de hipersuperficies), which implies that the equation (7.2.8) can be written as,

$$\delta S_{EH} = \kappa \int_{\mathcal{M}} \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} \mathrm{d}^4x + \kappa\sigma \int_{\partial\mathcal{M}} \sqrt{|h|} n_\mu V^\mu \mathrm{d}^3x. \tag{7.2.9}$$

where σ is the norm of the unit-normal vector $n = n^\mu \partial_\mu$ and $h = \det(h_{ij})$ where h_{ij} is the induced metric of the codimension-1 hypersurface (for more details see the Appendix 8).

Finally, let us work more with the boundary term. To do that, we write the vector field V^μ of the second integral of (7.2.9). Then,

$$\begin{aligned}
\kappa \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|} n_\mu V^\mu &= \kappa \int_{\partial\mathcal{M}} d^3x \sqrt{|h|} n_\alpha \delta^{\alpha\beta}_{\lambda\nu} g^{\mu\nu} \delta\Gamma^\lambda_{\mu\beta} \\
&= \frac{\kappa}{2} \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|} n_\alpha \delta^{\alpha\beta}_{\lambda\nu} g^{\mu[v} g^{\lambda]\sigma} \left(\nabla_\mu \delta g_{\sigma\beta} + \nabla_\beta \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\beta}\right) \\
&= \frac{\kappa}{2} \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|} n_\alpha \delta^{\alpha\beta}_{\lambda\nu} g^{\mu\nu} g^{\lambda\sigma} \cdot 2\nabla_{[\mu} \delta g_{\sigma]\beta} \\
&= \kappa \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|} n_\alpha \delta^{\alpha\beta}_{\lambda\nu} g^{\mu\nu} g^{\lambda\sigma} \nabla_\mu \delta g_{\sigma\beta} \\
&= \kappa \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|} n_\alpha \left(\delta^\alpha_\lambda \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\lambda\right) g^{\mu\nu} g^{\lambda\sigma} \nabla_\mu \delta g_{\sigma\beta} \\
&= \kappa \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|} n_\alpha \left(g^{\mu\beta} g^{\alpha\sigma} \nabla_\mu \delta g_{\sigma\beta} - g^{\mu\alpha} g^{\beta\sigma} \nabla_\mu \delta g_{\sigma\beta}\right) \\
&= \kappa \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|} \left(n^\sigma g^{\mu\beta} \nabla_\mu \delta g_{\sigma\beta} - n^\mu g^{\beta\sigma} \nabla_\mu \delta g_{\sigma\beta}\right).
\end{aligned}$$

Therefore, plugging the latter in the arbitrary variation with respect to the metric gives,

$$\delta S_{EH} = \kappa \int_{\mathcal{M}} \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} \mathrm{d}^4x + \kappa \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|} \left(n^\sigma g^{\mu\beta} \nabla_\mu \delta g_{\sigma\beta} - n^\mu g^{\beta\sigma} \nabla_\mu \delta g_{\sigma\beta}\right). \tag{7.2.10}$$

Now, if we impose Dirichlet boundary conditions for the metric, i.e., $\delta g_{\alpha\beta}\big|_{\partial\mathcal{M}} = 0$ we can not cancel all the contributions for the boundary term. To be more explicit we will obtain,

$$\delta S_{EH} = \kappa \int_{\mathcal{M}} \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} \mathrm{d}^4x - \kappa \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|} n^\mu g^{\beta\sigma} \nabla_\mu \delta g_{\sigma\beta}. \tag{7.2.11}$$

By virtue of this is that we need to add a boundary term that cancels this. The term that we need to add is the Gibbons-Hawking-York [31, 32]. This implies to suplement the (2.1.1) with the next term,

$$S_{GHY}[g_{\mu\nu}] = 2\kappa\sigma \int_{\partial\mathcal{M}} \mathrm{d}^3x \sqrt{|h|} \mathcal{K}, \tag{7.2.12}$$

the arbitrary variation of latter gives,

$$\delta S_{GHY} = 2\kappa\sigma \int_{\partial\mathcal{M}} \mathrm{d}^3x \left[\delta\sqrt{|h|}\mathcal{K} + \sqrt{|h|}\delta\mathcal{K}\right]. \tag{7.2.13}$$

Let us work step by step. First, let us concentrate on the variation of the square root of the determinant of the metric, that is,

$$\delta\sqrt{|h|} = -\frac{1}{2}\sqrt{|h|}h_{\mu\nu}\delta h^{\mu\nu}\,, \tag{7.2.14}$$

where we used the Eq. (7.2.4). On the other hand, the variation of $\mathcal{K} = g^{\mu\nu}\mathcal{K}_{\mu\nu} = \nabla_\mu n^\mu$ where $\mathcal{K}_{\mu\nu}$ is the extrinsic curvature defines in Eq. (8.5.10). Then,

$$\delta(\nabla_\mu n^\mu) = \nabla_\mu \delta n^\mu + \delta\Gamma^\mu_{\lambda\mu} n^\lambda. \tag{7.2.15}$$

Using the equation (7.2.6), we can compute the second term, yielding

$$\delta\Gamma^\mu_{\lambda\mu} = -\frac{1}{2}\nabla_\lambda(g_{\mu\nu}\delta g^{\mu\nu}). \tag{7.2.16}$$

The first term in Eq. (7.2.15) can be simplified using the norm of the normal vector, i.e., $\sigma = n_\mu n^\mu$ this is,

$$n_\mu \delta n^\mu = -n^\mu \delta n_\mu. \tag{7.2.17}$$

Then, we can conclude that,

$$\delta n^\lambda = \sigma n^\lambda n_\nu \delta n^\nu. \tag{7.2.18}$$

Now, taking the arbitrary variation of the induced metric, yields

$$\begin{aligned} \delta h^{\mu\nu} &= \delta g^{\mu\nu} - 2\sigma n^{(\mu}\delta n^{\nu)} \,/\, \cdot\, n_\mu n_\nu && (7.2.19)\\ 0 &= n_\mu n_\nu \delta g^{\mu\nu} - 2\sigma^2 n_\nu \delta n^\nu && (7.2.20)\\ n_\nu \delta n^\nu &= \frac{1}{2} n_\mu n_\nu \delta g^{\mu\nu}. && (7.2.21) \end{aligned}$$

Replacing the latter in (7.2.18) yields,

$$
\begin{aligned}
\delta n^{\lambda} &= \sigma n^{\lambda} n_{\nu} \delta n^{\nu} \\
&= \frac{\sigma}{2} n^{\lambda} n_{\mu} n_{\nu} \delta g^{\mu\nu} \\
&= \frac{1}{2} (\delta^{\lambda}_{\mu} - h^{\lambda}_{\mu}) n_{\nu} \delta g^{\mu\nu} \\
&= \frac{1}{2} n_{\nu} \delta g^{\lambda\nu} - \frac{1}{2} h^{\lambda}_{\mu} n_{\nu} \delta g^{\mu\nu} \\
\delta n^{\lambda} &= \frac{1}{2} n_{\nu} \delta g^{\lambda\nu} ,
\end{aligned}
$$

where we used the fact $h^{\lambda}_{\mu} = \delta^{\lambda}_{\mu} - \sigma n^{\lambda} n_{\mu}$ and $h^{\lambda}_{\mu} n_{\nu} \delta g^{\mu\nu} = 0$ for orthogonality. Then, plugging all these ingredients in (7.2.15) yields

$$
\begin{aligned}
\delta \mathcal{K} &= \nabla_{\mu} \left(\frac{1}{2} n_{\nu} \delta g^{\mu\nu} \right) - \frac{1}{2} n^{\lambda} \nabla_{\lambda} (g_{\mu\nu} \delta g^{\mu\nu}) \\
&= \frac{1}{2} \nabla_{\mu} n_{\nu} \delta g^{\mu\nu} + \frac{1}{2} n_{\nu} \nabla_{\mu} \delta g^{\mu\nu} - \frac{1}{2} g_{\mu\nu} n^{\lambda} \nabla_{\lambda} \delta g^{\mu\nu} \\
&= \frac{1}{2} \delta^{\lambda}_{\mu} \nabla_{\lambda} n_{\nu} \delta g^{\mu\nu} + \frac{1}{2} n_{\nu} \nabla_{\mu} \delta g^{\mu\nu} - \frac{1}{2} g_{\mu\nu} n^{\lambda} \nabla_{\lambda} \delta g^{\mu\nu} \\
&= \frac{1}{2} \mathcal{K}_{\mu\nu} \delta g^{\mu\nu} + \frac{1}{2} n_{\nu} \nabla_{\mu} \delta g^{\mu\nu} - \frac{1}{2} g_{\mu\nu} n^{\lambda} \nabla_{\lambda} \delta g^{\mu\nu},
\end{aligned}
$$

where we used orthogonality arguments and the definition of (8.5.10). Therefore we conclude that the boundary terms of (7.2.11) vanish at Dirichlet boundary conditions.

7.3 *Curvature* and *Lorentz connection* computations for multiNUTs

In this section, we will made the explicit computations for the *Lorentz connection* and *Curvature* for the multiNUTs. We use the notation and definition of [34, 56, 57] The geometry that we are considering is *torsionless*. The aim of this is to try to plug this results in the field equations. Let us start by computing the non-trivial component of the *Lorentz connection* and then the *Curvature*.

The first non-trivial spin connection component is given by

$$\begin{aligned}
\omega_{01} &= -\frac{1}{2}\left(i_0 de_1 - i_1 de_0\right) + \frac{1}{2}\left(i_0 i_1 de_0\right) e^0 + \frac{1}{2}\left(i_0 i_1 de_1\right) e^1 + \frac{1}{2}\left(i_0 i_1 de_A^{(i)}\right) e^A_{(i)} \\
&= -\frac{1}{2}\left(\sqrt{f}\right)' e_0 - \frac{1}{2}\left(\sqrt{f}\right)' e_0 \\
&= -\frac{f'}{2\sqrt{f}} e_0 \ .
\end{aligned}$$

For the following and all next components, we will introduce an *"index"* related to the NUTs charges and the base manifold to the Lorentz connection. Another non-trivial component is given by,

$$\begin{aligned}
\omega^{(i)}_{0A} &= -\frac{1}{2}\left(i_0 de^{(i)}_A + i^{(i)}_A de_0\right) + \frac{1}{2}\left(i_0 i^{(i)}_A de_0\right) e^0 + \frac{1}{2}\left(i_0 i^{(i)}_A de_1\right) e^1 + \frac{1}{2}\sum_{j=1}^{k}\left(i_0 i^{(i)}_A de^{(j)}_B\right) e^B_{(j)} \\
&= -\frac{1}{2} i^{(i)}_A \left[\frac{\sqrt{f}}{2}\sum_{j=1}^{k}\frac{\Omega^{(j)}_{BC}}{N_{(j)}} e^B_{(j)} \wedge e^C_{(j)}\right] \ , \\
&= -\frac{\sqrt{f}}{2}\sum_{j=1}^{k}\frac{\Omega^{(j)}_{BC}}{2N_{(j)}} 2\delta^{(i)}_{(j)}\delta^{[B}_{A} e^{C]}_{(j)} \ , \\
\omega^{(i)}_{0A} &= -\frac{\sqrt{f}}{2N_{(i)}}\Omega^{(i)}_{AC} e^C_{(i)} \ ,
\end{aligned}$$

One more,

$$
\begin{aligned}
\omega^{(i)}_{1A} &= -\frac{1}{2}\left(i_1 de^{(i)}_A - i^{(i)}_A de_1\right) + \frac{1}{2}\left(i_1 i^{(i)}_A de_c\right) e^c \\
&= -\frac{1}{2}\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}} e_{A(i)} + \frac{1}{2}\left(i_1 i^{(i)}_A de_0\right) e^0 + \frac{1}{2}\left(i_1 i^{(i)}_A de_1\right) e^1 + \frac{1}{2}\sum_{j=1}^{k}\left(i_1 i^{(i)}_A de^{(j)}_B\right) e^B_{(j)} \\
&= -\frac{1}{2}\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}} e_{A(i)} - \frac{1}{2}\cdot\frac{\sqrt{f}}{2}\sum_{j=1}^{k}\frac{N'_{(j)}}{N_{(j)}}\delta^{(i)}_{(j)}\delta_{AB}\wedge e^B_{(j)} \\
\omega^{(i)}_{1A} &= -\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}} e^{(i)}_A \ .
\end{aligned}
$$

Finally, the most laborious component is given by,

$$
\begin{aligned}
\omega^{(i)}_{AB} &= -\sum_{j=1}^{k} i^{(i)}_{[A} de^{(j)}_{B]} + \frac{1}{2}\left(i^{(i)}_A i^{(i)}_B de_0\right) e^0 + \frac{1}{2}\left(i^{(i)}_A i^{(i)}_B de_1\right) e^1 + \frac{1}{2}\sum_{j=1}^{k}\left(i^{(i)}_A i^{(i)}_B de^{(j)}_C\right) e^C_{(j)} \\
&= -\frac{1}{2}\sum_{j=1}^{k}\left(i^{(i)}_A de^{(j)}_B - i^{(i)}_B de^{(j)}_A\right) + \frac{1}{2}\left(i^{(i)}_A i^{(i)}_B de_0\right) e^0 + \frac{1}{2}\sum_{j=1}^{k}\left(i^{(i)}_A i^{(i)}_B de^{(j)}_C\right) e^C_{(j)} \\
&= \bar{\omega}^{(i)}_{ABD}\bar{e}^D_{(i)} + \sqrt{f}\frac{n_{(i)}\Omega^{(i)}_{AB}}{N_{(i)}} e^0 \\
\omega^{(i)}_{AB} &= \bar{\omega}^{(i)}_{AB} + \sqrt{f}\frac{n_{(i)}\Omega^{(i)}_{AB}}{N_{(i)}} e^0 .
\end{aligned}
$$

Therefore, all the non-trivial components are,

$$
\begin{aligned}
\omega^{01} &= \frac{f'}{2\sqrt{f}} e^0 \ , \\
\omega^{0A}_{(i)} &= -\sqrt{f}\frac{n_{(i)}\Omega^{A}_{(i)B}}{N_{(i)}} e^B_{(i)} \ , \\
\omega^{1A}_{(i)} &= -\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}} e^A_{(i)} \ , \\
\omega^{AB}_{(i)} &= \bar{\omega}^{AB}_{(i)} + \sqrt{f}\frac{n_{(i)}\Omega^{AB}_{(i)}}{N_{(i)}} e^0 \ .
\end{aligned}
$$

Now, let us compute the 2-form curvature using the previous results. Considering the definition given by the eq. (**??**) the non-trivial components are,

$$
\begin{aligned}
R^{01} &= \mathrm{d}\omega^{01} + \sum_{i=1}^{k} \omega^{0}_{(i)A} \wedge \omega^{A1}_{(i)} \\
&= \mathrm{d}\omega^{01} - \sum_{i=1}^{k} \omega^{0}_{(i)A} \wedge \omega^{1A}_{(i)} \\
&= \left[\frac{f'}{2\sqrt{f}}\right]' \sqrt{f} e^1 \wedge e^0 + \left[\frac{f'}{2\sqrt{f}}\right] de^0 - \sum_{i=1}^{k} \left(-\frac{\sqrt{f}}{2}\frac{\Omega^{(i)}_{AB}}{N_{(i)}} e^{B}_{(i)}\right) \wedge \left(-\frac{\sqrt{f} N'_{(i)}}{2N_{(i)}} e^{A}_{(i)}\right) \\
&= \left[\frac{f'}{2\sqrt{f}}\right]' \sqrt{f} e^1 \wedge e^0 + \frac{f'}{2\sqrt{f}} \left(\frac{f'}{2\sqrt{f}} e^1 \wedge e^0 + \frac{\sqrt{f}}{2} \sum_{i=k}^{k} \frac{\Omega^{(i)}_{AB}}{N_{(i)}} e^{A}_{(i)} \wedge e^{B}_{(i)}\right) \\
&\quad + \frac{f}{4} \sum_{i=1}^{k} \frac{N'_{(i)}}{N^2_{(i)}} \Omega^{(i)}_{AB} e^{A}_{(i)} \wedge e^{B}_{(i)} \\
&= \left(\left[\frac{f'}{2\sqrt{f}}\right]' \sqrt{f} + \left[\frac{f'}{2\sqrt{f}}\right]^2\right) e^1 \wedge e^0 + \frac{1}{4} \sum_{i=1}^{k} \left(\frac{f'}{N_{(i)}} + \frac{f\ N'_{(i)}}{N^2_{(i)}}\right) \Omega^{(i)}_{AB}\ e^{A}_{(i)} \wedge e^{B}_{(i)} \\
R^{01} &= -\frac{f''}{2} e^0 \wedge e^1 + \sum_{i=1}^{k} \frac{n_{(i)}}{2} \left[\frac{f}{N_{(i)}}\right]' \Omega^{(i)}_{AB} e^{A}_{(i)} \wedge e^{B}_{(i)}.
\end{aligned}
$$

Moreover, another non-trivial component is given by,

$$
\begin{aligned}
R^{0A}_{(i)} &= d\omega^{0A}_{(i)} + \omega^{0}{}_{1} \wedge \omega^{1A}_{(i)} - \omega^{0}_{(i)B} \wedge \omega^{AB}_{(i)} \\
&= -d\left[\frac{\sqrt{f}}{2}\frac{\Omega^{A}_{(i)B}}{N_{(i)}} e^{B}_{(i)}\right] + \left(\frac{f'}{2\sqrt{f}} e^{0}\right) \wedge \left(-\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}\right) \\
&\quad -\frac{\sqrt{f}}{2}\sum_{i=1}^{k}\frac{\Omega^{A}_{(i)B}}{N_{(i)}} e^{B}_{(i)} \wedge \left(\bar{\omega}^{AB}_{(i)} + \frac{\sqrt{f}}{2}\sum_{i=1}^{k}\frac{\Omega^{AB}_{(i)}}{N_{(i)}} e^{0}\right) \\
&= \frac{1}{2}\sum_{i=1}^{k}\left(\left[\frac{\sqrt{f}}{N_{(i)}}\right]' \sqrt{f}\Omega^{A}_{(i)B} e^{1} \wedge e^{B}_{(i)} + \left[\frac{\sqrt{f}}{N_{(i)}}\right] \partial_{C}\Omega^{A}_{(i)B} e^{C}_{(i)} \wedge e^{B}_{(i)} + \frac{\sqrt{f}}{N_{(i)}}\Omega^{A}_{(i)B} de^{B}_{(i)}\right) \\
&\quad -\frac{f'N'_{(i)}}{4N_{(i)}} e^{0} \wedge e^{A}_{(i)} - \frac{\sqrt{f}}{2}\sum_{i=1}^{k}\frac{\Omega^{A}_{(i)B}}{N_{(i)}} e^{B}_{(i)} \wedge \bar{\omega}^{AB}_{(i)} - \frac{\sqrt{f}}{2}\sum_{i=1}^{k}\frac{\Omega^{A}_{(i)B}}{N_{(i)}} e^{B}_{(i)} \wedge \frac{\sqrt{f}}{2}\sum_{i=1}^{k}\frac{\Omega^{AB}_{(i)}}{N_{(i)}} e^{0} \\
&= \sum_{i=1}^{k}\left(\left[\frac{\sqrt{f}}{2N_{(i)}}\right]' \sqrt{f} - \frac{fN'_{(i)}}{4N^{2}_{(i)}}\right)\Omega^{A}_{(i)B} e^{B}_{(i)} \wedge e^{1} + \sum_{i=1}^{k}\frac{1}{4N_{(i)}}\left(f - f'N'_{(i)}\right) e^{0} \wedge e^{A}_{(i)} \\
&\quad + \sum_{i=1}^{k}\frac{\sqrt{f}}{2N_{(i)}}\left(\partial_{C}\Omega^{(i)}_{AB} e^{B}_{(i)} \wedge e^{C}_{(i)} - \Omega^{(i)}_{AB}\bar{\omega}^{B}_{(i)C} \wedge e^{C}_{(i)} + \Omega^{(i)}_{CB}\bar{\omega}^{AB}_{(i)} \wedge e^{C}_{(i)}\right) \\
R^{0A}_{(i)} &= \sum_{i=1}^{k}\left(\frac{1}{2}\left[\frac{f}{N_{(i)}}\right]' n_{(i)}\Omega^{A}_{(i)B} e^{1} \wedge e^{B}_{(i)} - \frac{1}{4}\left(\frac{f'N'_{(i)}N_{(i)} + 4fn^{2}_{(i)}}{N^{2}_{(i)}}\right) e^{0} \wedge e^{A}_{(i)} \right. \\
&\quad \left. + \frac{\sqrt{f}n_{(i)}}{N_{(i)}}\bar{D}^{(i)}\Omega^{A}_{(i)B} \wedge e^{B}_{(i)}\right),
\end{aligned}
$$

where we define the covariant derivative $\bar{D}^{(i)}$ which is definen with the spin connection. Moreover, another non-trivial component follows,

$$\begin{aligned}
R^{1A}_{(i)} &= d\omega^{1A}_{(i)} + \omega^{1}{}_{0} \wedge \omega^{0A}_{(i)} - \omega^{1}_{(i)B} \wedge \omega^{AB}_{(i)} \\
&= d\left[-\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}} e^{A}_{(i)}\right] + \frac{f'}{2\sqrt{f}} e_0 \wedge \frac{\sqrt{f}}{2} \frac{\Omega^{A}_{(i)B}}{N_{(i)}} e^{B}_{(i)} + \frac{\sqrt{f}N'_{(i)}}{2N_{(i)}} e_{(i)B} \wedge \left(\bar{\omega}^{AB}_{(i)} + \frac{\sqrt{f}}{2}\frac{\Omega^{AB}_{(i)}}{N_{(i)}} e^0\right) \\
&= -\left[\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}\right]' \sqrt{f} e^1 \wedge e^{A}_{(i)} - \left[\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}\right]^2 e^1 \wedge e^{A}_{(i)} + \frac{\sqrt{f}N'_{(i)}}{2N_{(i)}} \bar{\omega}^{A}_{(i)B} \wedge e^{B}_{(i)} \\
&\quad -\frac{f'}{4}\frac{\Omega^{A}_{(i)B}}{N_{(i)}} e^0 \wedge e^{B}_{(i)} - \frac{\sqrt{f}N'_{(i)}}{2N_{(i)}} \bar{\omega}^{A}_{(i)B} \wedge e^{B}_{(i)} - \frac{fN'_{(i)}}{4N_{(i)}}\frac{\Omega^{A}_{(i)B}}{N_{(i)}} e^0 \wedge e^{B}_{(i)} \\
&= -\left(\left[\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}\right]' \sqrt{f} + \left[\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}\right]^2\right) e^1 \wedge e^{A}_{(i)} - \left(\frac{f'}{4}\frac{\Omega^{A}_{(i)B}}{N_{(i)}} + \frac{fN'_{(i)}}{4N_{(i)}}\frac{\Omega^{A}_{(i)B}}{N_{(i)}}\right) e^0 \wedge e^{B}_{(i)} \\
R^{1A}_{(i)} &= -\left(\left[\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}\right]' \sqrt{f} + \left[\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}\right]^2\right) e^1 \wedge e^{A}_{(i)} - \frac{1}{4N_{(i)}}\left(f' + \frac{fN'_{(i)}}{N_{(i)}}\right) \Omega^{A}_{(i)B} e^0 \wedge e^{B}_{(i)}
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
R^{AB}_{(i)} &= \mathrm{d}\omega^{AB}_{(i)} + \omega^{A}_{(i)0}\wedge\omega^{0B}_{(i)} + \omega^{A}_{(i)1}\wedge\omega^{1B}_{(i)} + \omega^{A}_{(i)C}\wedge\omega^{CB}_{(i)} \\
&= \mathrm{d}\left(\bar{\omega}^{AB}_{(i)} + \frac{\sqrt{f}}{2}\frac{\Omega^{AB}_{(i)}}{N_{(i)}}e^{0}\right) + \left(-\frac{\sqrt{f}}{2}\frac{\Omega^{A}_{(i)C}}{N_{(i)}}e^{C}_{(i)}\right)\wedge\left(-\frac{\sqrt{f}}{2}\frac{\Omega^{B}_{(i)D}}{N_{(i)}}e^{D}_{(i)}\right) \\
&\quad + \left(\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}e^{A}_{(i)}\right)\wedge\left(-\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}e^{B}_{(i)}\right) + \bar{\omega}^{A}_{(i)C}\wedge\bar{\omega}^{CB}_{(i)} \\
&= \mathrm{d}\bar{\omega}^{AB}_{(i)} + \bar{\omega}^{A}_{(i)C}\wedge\bar{\omega}^{CB}_{(i)} + \mathrm{d}\left(\frac{\sqrt{f}}{2}\frac{\Omega^{AB}_{(i)}}{N_{(i)}}e^{0}\right) + \frac{f}{4N^{2}_{(i)}}\Omega^{A}_{(i)C}\Omega^{B}_{(i)D}e^{C}_{(i)}\wedge e^{D}_{(i)} - \frac{fN'^{2}_{(i)}}{4N^{2}_{(i)}}e^{A}_{(i)}\wedge e^{B}_{(i)} \\
&= \bar{R}^{AB}_{(i)} + \left(\left[\frac{\sqrt{f}}{2N_{(i)}}\right]'\sqrt{f}\Omega^{AB}_{(i)}e^{1}\wedge e^{0} + \left[\frac{\sqrt{f}}{2N_{(i)}}\right]\partial_{C}\Omega^{AB}_{(i)}e^{C}_{(i)}\wedge e^{0} + \left[\frac{\sqrt{f}}{2N_{(i)}}\right]\Omega^{AB}_{(i)}de^{0}\right) \\
&\quad + \frac{f}{4N^{2}_{(i)}}\Omega^{A}_{(i)C}\Omega^{B}_{(i)D}e^{C}_{(i)}\wedge e^{D}_{(i)} - \frac{fN'^{2}_{(i)}}{4N^{2}_{(i)}}e^{A}_{(i)}\wedge e^{B}_{(i)} \\
&= \bar{R}^{AB}_{(i)} + \left(\left[\frac{\sqrt{f}}{2N_{(i)}}\right]'\sqrt{f}\Omega^{AB}_{(i)}e^{1}\wedge e^{0} + \left[\frac{\sqrt{f}}{2N_{(i)}}\right]\partial_{C}\Omega^{AB}_{(i)}e^{C}_{(i)}\wedge e^{0}\right. \\
&\quad \left. + \left[\frac{\sqrt{f}}{2N_{(i)}}\right]\Omega^{AB}_{(i)}\left(\frac{f'}{2\sqrt{f}}e^{1}\wedge e^{0} + \frac{\sqrt{f}}{2}\sum_{j=1}^{k}\Omega^{(j)}_{CD}e^{C}_{(j)}\wedge e^{D}_{(j)}\right)\right) + \frac{f}{4N^{2}_{(i)}}\Omega^{A}_{(i)C}\Omega^{B}_{(i)D}e^{C}_{(i)}\wedge e^{D}_{(i)} \\
&\quad - \frac{fN'^{2}_{(i)}}{4N^{2}_{(i)}}e^{A}_{(i)}\wedge e^{B}_{(i)} \\
&= \bar{R}^{AB}_{(i)} + \left(\left[\frac{\sqrt{f}}{2N_{(i)}}\right]'\sqrt{f}\Omega^{AB}_{(i)}e^{1}\wedge e^{0} + \left[\frac{\sqrt{f}}{2N_{(i)}}\right]\partial_{C}\Omega^{AB}_{(i)}e^{C}_{(i)}\wedge e^{0} + \frac{f'}{4N_{(i)}}\Omega^{AB}_{(i)}e^{1}\wedge e^{0}\right. \\
&\quad \left. + \sum_{j=1}^{k}\left[\frac{f}{4N_{(i)}}\right]\frac{\Omega^{AB}_{(i)}\Omega^{(j)}_{CD}}{N_{(j)}}e^{C}_{(j)}\wedge e^{D}_{(j)}\right) + \frac{f}{4N^{2}_{(i)}}\Omega^{A}_{(i)C}\Omega^{B}_{(i)D}e^{C}_{(i)}\wedge e^{D}_{(i)} - \frac{fN'^{2}_{(i)}}{4N^{2}_{(i)}}e^{A}_{(i)}\wedge e^{B}_{(i)}
\end{aligned}
$$

Therefore, all the components of the 2-form curvature are given by,

$$\begin{aligned}
R^{01} &= -\frac{f''}{2}e^0\wedge e^1+\sum_{i=1}^{k}\frac{n_{(i)}}{2}\left[\frac{f}{N_{(i)}}\right]'\Omega^{(i)}_{AB}e^A_{(i)}\wedge e^B_{(i)}\\
R^{0A} &= \sum_{i=1}^{k}\left(\frac{1}{2}\left[\frac{f}{N_{(i)}}\right]'n_{(i)}\Omega^A_{(i)B}e^1\wedge e^B_{(i)}-\frac{1}{4}\left(\frac{f'N'_{(i)}N_{(i)}+4fn^2_{(i)}}{N^2_{(i)}}\right)e^0\wedge e^A_{(i)}\right.\\
&\left.+\frac{\sqrt{f}n_{(i)}}{N_{(i)}}\bar{D}^{(i)}\Omega^A_{(i)B}\wedge e^B_{(i)}\right)\\
R^{1A} &= \sum_{i=1}^{k}\left(\frac{1}{2}\left[\frac{-f'N_{(i)}+fN'_{(i)}}{N^2_{(i)}}\right]n_{(i)}\Omega^A_{(i)B}e^0\wedge e^B_{(i)}+\left(\left[-\frac{\sqrt{f}N'_{(i)}}{2N_{(i)}}\right]'\sqrt{f}-\frac{fN'^2_{(i)}}{4N^2_{(i)}}\right)e^1\wedge e^A_{(i)}\right)\\
R^{AB} &= \left(\sum_{i=1}^{k}d\bar{\omega}^{AB}_{(i)}+\sum_{i=1}^{k}\sum_{j=1}^{k}\bar{\omega}^A_{(i)C}\wedge\bar{\omega}^{CB}_{(j)}+\sum_{i=1}^{k}\left[\frac{f}{N_{(i)}}\right]'n_{(i)}\Omega^{AB}_{(i)}e^1\wedge e^0\right.\\
&+\sqrt{f}\sum_{i=1}^{k}\left(\frac{n_{(i)}}{N_{(i)}}d\Omega^{AB}_{(i)}+\sum_{j=1}^{k}\frac{n_{(j)}}{N_{(j)}}\bar{\omega}^A_{(i)C}\Omega^{CB}_{(j)}+\sum_{j=1}^{k}\frac{n_{(i)}}{N_{(i)}}\bar{\omega}^B_{(j)C}\Omega^{AC}_{(i)}\right)\wedge e^0\\
&+\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{fn_{(i)}n_{(j)}}{N_{(i)}N_{(j)}}\Omega^{[A}_{(i)[C}\Omega^{B]}_{(j)D]}e^C_{(i)}\wedge e^D_{(j)}\\
&\left.-\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{fN'_{(i)}N'_{(j)}}{4N_{(i)}N_{(j)}}e^A_{(i)}\wedge e^B_{(j)}+\sum_{i=1}^{k}\sum_{w=1}^{k}\frac{fn_{(i)}n_{(w)}}{N_{(i)}N_{(w)}}\Omega^{AB}_{(i)}\Omega^{(w)}_{CD}e^C_{(w)}\wedge e^D_{(w)}\right).
\end{aligned}$$

7.4 Computations for the QGC field equations

A shortcut to compute the field equation for the QCG can be obtained using the following equation [10]

$$\mathcal{E}^\mu_\nu=E^{\mu\lambda}_{\rho\sigma}R^{\rho\sigma}_{\nu\lambda}-\frac{1}{2}\delta^\mu_\nu\mathcal{L}-2\nabla_\lambda\nabla^\rho E^{\mu\lambda}_{\rho\nu}. \tag{7.4.1}$$

Where $E^{\mu\lambda}_{\rho\sigma}\equiv\frac{\partial\mathcal{L}}{\partial R^{\rho\sigma}_{\mu\lambda}}$. Computing the E-tensor for the Lagrangian yields

$$E^{\mu\lambda}_{\rho\sigma}=\frac{1}{2\kappa}\delta^{\mu\lambda}_{\rho\sigma}+2\alpha\delta^{[\mu}_{[\rho}R^{\lambda]}_{\sigma]}+\beta R\delta^{\mu\lambda}_{\rho\sigma}+\frac{\gamma}{2}\delta^{\mu\lambda\alpha\beta}_{\rho\sigma\epsilon\tau}R^{\epsilon\tau}_{\alpha\beta}. \tag{7.4.2}$$

[10] For more details see the equation (3) of Ref. [96] and their reference therein

Working with the first term of the (7.4.1) equation

$$\begin{aligned} E^{\mu\lambda}_{\rho\sigma}R^{\rho\sigma}_{\nu\lambda} &= \left[\frac{1}{2\kappa}\delta^{\mu\lambda}_{\rho\sigma} + 2\alpha\delta^{[\mu}_{[\rho}R^{\lambda]}_{\sigma]} + \frac{\beta}{2}\delta^{\mu\lambda}_{\rho\sigma}\delta^{\alpha\beta}_{\epsilon\tau}R^{\epsilon\tau}_{\alpha\beta} + \frac{\gamma}{2}\delta^{\mu\lambda\alpha\beta}_{\rho\sigma\epsilon\tau}R^{\epsilon\tau}_{\alpha\beta}\right]R^{\rho\sigma}_{\nu\lambda} \\ &= \left[\frac{1}{\kappa}R^{\mu}_{\nu} + \alpha\left(R^{\lambda}_{\rho}R^{\mu\rho}_{\nu\lambda} + R^{\mu}_{\rho}R^{\rho}_{\nu}\right) + 2\beta R^{\mu}_{\nu}\right. \qquad (7.4.3) \\ &\left. +2\gamma\left(RR^{\mu}_{\nu} + 2R^{\lambda}_{\alpha}R^{\mu\alpha}_{\lambda\nu} - 2R^{\mu}_{\alpha}R^{\alpha}_{\nu} + R^{\mu\lambda}_{\alpha\beta}R^{\alpha\beta}_{\nu\lambda}\right)\right]. \qquad (7.4.4) \end{aligned}$$

Now, the second term yields

$$\frac{1}{2}\delta^{\mu}_{\nu} = \frac{1}{2}\delta^{\mu}_{\nu}\left[\frac{1}{\kappa}(R-2\Lambda) + \alpha R^{\alpha}_{\beta}R^{\beta}_{\alpha} + \beta R^2 + \gamma\left(R^2 - 4R^{\alpha}_{\beta}R^{\beta}_{\alpha} + R^{\alpha\beta}_{\sigma\tau}R^{\sigma\tau}_{\alpha\beta}\right)\right] \qquad (7.4.5)$$

Finally, third term

$$\begin{aligned} -2\nabla_\lambda\nabla^\rho E^{\mu\lambda}_{\rho\nu} &= -2\nabla_\lambda\nabla^\rho\left(\underbrace{\frac{1}{2\kappa}\delta^{\mu\lambda}_{\rho\nu}}_{0} + 2\alpha\delta^{[\mu}_{[\rho}R^{\lambda]}_{\nu]} + \frac{\beta}{2}\delta^{\mu\lambda}_{\rho\nu}\delta^{\alpha\beta}_{\epsilon\tau}R^{\epsilon\tau}_{\alpha\beta} + \underbrace{\frac{\gamma}{2}\delta^{\mu\lambda\alpha\beta}_{\rho\nu\epsilon\tau}R^{\epsilon\tau}_{\alpha\beta}}_{=0\because DR=0}\right) \\ &= 2\nabla_\lambda\nabla^\rho\left(2\alpha\delta^{[\mu}_{[\rho}R^{\lambda]}_{\nu]}\right) - 2\nabla_\lambda\nabla^\rho\left(\frac{\beta}{2}\delta^{\mu\lambda}_{\rho\nu}\delta^{\alpha\beta}_{\epsilon\tau}R^{\epsilon\tau}_{\alpha\beta}\right) \\ &= 2\nabla_\lambda\nabla^\rho\left(2\alpha\delta^{[\mu}_{[\rho}R^{\lambda]}_{\nu]}\right) + \beta\delta^{\mu\lambda}_{\rho\nu}\delta^{\alpha\beta}_{\epsilon\tau}\left(\nabla_\lambda\nabla^\rho R^{\epsilon\tau}_{\alpha\beta}\right) - \beta\nabla_\lambda\nabla^\rho\left(\delta^{\mu\lambda}_{\rho\nu}\delta^{\alpha\beta}_{\epsilon\tau}R^{\epsilon\tau}_{\alpha\beta}\right) \\ &= -\alpha\left(\nabla_\lambda\nabla^\mu R^{\lambda}_{\nu} - \delta^{\mu}_{\nu}\nabla_\lambda\nabla^\rho R^{\lambda}_{\rho} - \Box R^{\mu}_{\nu} + \nabla_\nu\nabla^\rho R^{\mu}_{\rho}\right) + 2\beta\left(\delta^{\mu}_{\nu}\Box - \nabla_\mu\nabla^\nu\right)R \\ &= \alpha\left(-\nabla_\lambda\nabla^\mu R^{\lambda}_{\nu} + \delta^{\mu}_{\nu}\nabla_\lambda\nabla^\rho R^{\lambda}_{\rho} + \Box R^{\mu}_{\nu} - \nabla_\nu\nabla^\rho R^{\mu}_{\rho}\right) + 2\beta\left(\delta^{\mu}_{\nu}\Box - \nabla_\nu\nabla^\mu\right)R \\ &= \alpha\left(\frac{1}{2}\delta^{\mu}_{\nu}\Box R + \Box R^{\mu}_{\nu} - \nabla_\nu\nabla^\mu R\right) + 2\beta\left(\delta^{\mu}_{\nu}\Box - \nabla_\nu\nabla^\mu\right)R \end{aligned}$$

In the latter we were us the fact that,

$$\begin{aligned} \delta^{[\mu}_{[\rho}R^{\lambda]}_{\nu]} &= \frac{1}{2}\left(\delta^{\mu}_{[\rho}R^{\lambda}_{\nu]} - \delta^{\lambda}_{[\rho}R^{\mu}_{\nu]}\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\left[\delta^{\mu}_{\rho}R^{\lambda}_{\nu} - \delta^{\mu}_{\nu}R^{\lambda}_{\rho}\right] - \frac{1}{2}\left[\delta^{\lambda}_{\rho}R^{\mu}_{\nu} - \delta^{\lambda}_{\nu}R^{\mu}_{\rho}\right]\right) \\ &= \frac{1}{2}\left(\frac{1}{2}\left\{\left[\delta^{\mu}_{\rho}R^{\lambda}_{\nu} - \delta^{\mu}_{\nu}R^{\lambda}_{\rho}\right] - \left[\delta^{\lambda}_{\rho}R^{\mu}_{\nu} - \delta^{\lambda}_{\nu}R^{\mu}_{\rho}\right]\right\}\right) \\ &= \frac{1}{4}\left(\delta^{\mu}_{\rho}R^{\lambda}_{\nu} - \delta^{\mu}_{\nu}R^{\lambda}_{\rho} - \delta^{\lambda}_{\rho}R^{\mu}_{\nu} + \delta^{\lambda}_{\nu}R^{\mu}_{\rho}\right) \end{aligned}$$

also, we use the Contracted Bianchi identity,

$$\nabla_\rho R^\rho_\mu = \frac{1}{2}\nabla_\mu R$$

By combining the three terms,

$$\begin{aligned}
\mathcal{E}^\mu_\nu &= 0 = \left(\frac{1}{\kappa}R^\mu_\nu + \alpha\left(R^\lambda_\rho R^{\mu\rho}_{\nu\lambda} + R^\mu_\rho R^\rho_\nu\right) + 2\beta R R^\mu_\nu + 2\gamma\left(R R^\mu_\nu + 2R^\lambda_\alpha R^{\mu\alpha}_{\lambda\nu} - 2R^\mu_\alpha R^\alpha_\nu + R^{\mu\lambda}_{\alpha\beta}R^{\alpha\beta}_{\nu\lambda}\right)\right) \\
&\quad -\frac{1}{2}\delta^\mu_\nu\left[\frac{1}{\kappa}\left(R - 2\Lambda\right) + \alpha R^\alpha_\beta R^\beta_\alpha + \beta R^2 + \gamma\left(R^2 - 4R^\alpha_\beta R^\beta_\alpha + R^{\alpha\beta}_{\lambda\rho}R^{\lambda\rho}_{\alpha\beta}\right)\right] \\
&\quad +\alpha\left(\frac{1}{2}\delta^\mu_\nu\Box R + \Box R^\mu_\nu - \nabla_\nu\nabla^\mu R\right) + 2\beta\left(\delta^\mu_\nu\Box - \nabla_\nu\nabla^\mu\right)R \\
&= \underbrace{\frac{1}{\kappa}\left(R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R + \delta^\mu_\nu\Lambda\right)}_{G^\mu_\nu} + \\
&\quad \gamma\underbrace{\left(2\left(R R^\mu_\nu + 2R^\lambda_\alpha R^{\mu\alpha}_{\lambda\nu} - 2R^\mu_\alpha R^\alpha_\nu + R^{\mu\lambda}_{\alpha\beta}R^{\alpha\beta}_{\nu\lambda}\right) - \frac{1}{2}\delta^\mu_\nu\left(R^2 - 4R^\alpha_\beta R^\beta_\alpha + R^{\alpha\beta}_{\lambda\rho}R^{\lambda\rho}_{\alpha\beta}\right)\right)}_{H^\mu_\nu} \\
&\quad +\alpha\left[R^\lambda_\rho R^{\mu\rho}_{\nu\lambda} + R^\mu_\rho R^\rho_\nu - \frac{1}{2}\delta^\mu_\nu R^\alpha_\beta R^\beta_\alpha + \Box R^\mu_\nu + \left(\frac{1}{2}\delta^\mu_\nu\Box - \nabla_\nu\nabla^\mu\right)R\right] \\
&\quad +2\beta\left(R R^\mu_\nu - \frac{1}{4}\delta^\mu_\nu R^2 + \left(\delta^\mu_\nu\Box - \nabla_\nu\nabla^\mu\right)R\right) \\
&= \frac{1}{\kappa}G^\mu_\nu + \gamma H^\mu_\nu + 2\beta R\left(R^\mu_\nu - \frac{1}{4}\delta^\mu_\nu R\right)
\end{aligned}$$

7.4.1 Intermediate computations

$$\begin{aligned}
E^{\mu\lambda}_{\rho\sigma}(\alpha) &= \frac{\partial\left(\alpha\delta^{\phi}_{\chi}\delta^{\epsilon}_{\beta}R^{\kappa\beta}_{\xi\epsilon}R^{\xi\chi}_{\kappa\phi}\right)}{\partial R^{\rho\sigma}_{\mu\lambda}} \\
&= \alpha\delta^{\phi}_{\chi}\delta^{\epsilon}_{\beta}\left[\frac{\partial R^{\kappa\beta}_{\xi\epsilon}}{\partial R^{\rho\sigma}_{\mu\lambda}}R^{\xi\chi}_{\kappa\phi} + R^{\kappa\beta}_{\xi\epsilon}\frac{\partial R^{\xi\chi}_{\kappa\phi}}{\partial R^{\rho\sigma}_{\mu\lambda}}\right] \\
&= \alpha\delta^{\phi}_{\chi}\delta^{\epsilon}_{\beta}\left[\delta^{[\kappa}_{\rho}\delta^{\beta]}_{\sigma}\delta^{\mu}_{[\xi}\delta^{\lambda}_{\epsilon]}R^{\xi\chi}_{\kappa\phi} + R^{\kappa\beta}_{\xi\epsilon}\delta^{[\xi}_{\rho}\delta^{\chi]}_{\sigma}\delta^{\mu}_{[\kappa}\delta^{\lambda}_{\phi]}\right] \\
&= \alpha\left[\delta^{[\kappa}_{\rho}\delta^{\beta]}_{\sigma}\delta^{\mu}_{[\xi}\delta^{\lambda}_{\beta]}R^{\xi\phi}_{\kappa\phi} + R^{\kappa\beta}_{\xi\beta}\delta^{[\xi}_{\rho}\delta^{\phi]}_{\sigma}\delta^{\mu}_{[\kappa}\delta^{\lambda}_{\phi]}\right] \\
&= \alpha\left[\delta^{[\kappa}_{\rho}\delta^{\beta]}_{\sigma}\delta^{\mu}_{[\xi}\delta^{\lambda}_{\beta]}R^{\xi}_{\kappa} + R^{\kappa}_{\xi}\delta^{[\xi}_{\rho}\delta^{\phi]}_{\sigma}\delta^{\mu}_{[\kappa}\delta^{\lambda}_{\phi]}\right] \\
&= 2\alpha\delta^{[\kappa}_{\rho}\delta^{\beta]}_{\sigma}\delta^{\mu}_{[\xi}\delta^{\lambda}_{\beta]}R^{\xi}_{\kappa} \\
&= 2\alpha\delta^{[\kappa}_{\rho}\delta^{\beta]}_{\sigma}\frac{1}{2}\left(\delta^{\mu}_{\xi}\delta^{\lambda}_{\beta} - \delta^{\mu}_{\beta}\delta^{\lambda}_{\xi}\right)R^{\xi}_{\kappa} \\
&= \alpha\left(\delta^{[\kappa}_{\rho}\delta^{\lambda]}_{\sigma}R^{\mu}_{\kappa} - \delta^{[\kappa}_{\rho}\delta^{\mu]}_{\sigma}R^{\lambda}_{\kappa}\right) \\
&= \alpha\left(\frac{1}{2}\left(\delta^{\kappa}_{\rho}\delta^{\lambda}_{\sigma}R^{\mu}_{\kappa} - \delta^{\lambda}_{\rho}\delta^{\kappa}_{\sigma}R^{\mu}_{\kappa}\right) - \frac{1}{2}\left(\delta^{\kappa}_{\rho}\delta^{\mu}_{\sigma}R^{\lambda}_{\kappa} - \delta^{\mu}_{\rho}\delta^{\kappa}_{\sigma}R^{\lambda}_{\kappa}\right)\right) \\
&= \frac{\alpha}{2}\left(\delta^{\lambda}_{\sigma}R^{\mu}_{\rho} - \delta^{\lambda}_{\rho}R^{\mu}_{\sigma} - \delta^{\mu}_{\sigma}R^{\lambda}_{\rho} + \delta^{\mu}_{\rho}R^{\lambda}_{\sigma}\right) \\
&= \frac{\alpha}{2}4!R^{[\mu}_{[\rho}\delta^{\lambda]}_{\sigma]} \\
&= 2\alpha R^{[\mu}_{[\rho}\delta^{\lambda]}_{\sigma]}
\end{aligned}$$

It is important to mention that if we compute this object, the latter must have the same properties as the E-tensor, i.e., the same symmetries as the Riemann tensor

Another important computation is given by,

$$\begin{aligned}
2\alpha\delta^{[\mu}_{[\rho}R^{\lambda]}_{\sigma]}R^{\rho\sigma}_{\nu\lambda} &= 2\alpha\delta^{[\mu}_{\rho}R^{\lambda]}_{\sigma}R^{\rho\sigma}_{\nu\lambda} \\
&= 2\alpha\left(\frac{1}{2}\left(\delta^{\mu}_{\rho}R^{\lambda}_{\sigma}R^{\rho\sigma}_{\nu\lambda} - \delta^{\lambda}_{\rho}R^{\mu}_{\sigma}R^{\rho\sigma}_{\nu\lambda}\right)\right) \\
&= \alpha\left(R^{\lambda}_{\sigma}R^{\mu\sigma}_{\nu\lambda} - R^{\mu}_{\sigma}R^{\lambda\sigma}_{\nu\lambda}\right) \\
&= \alpha\left(R^{\lambda}_{\sigma}R^{\mu\sigma}_{\nu\lambda} + R^{\mu}_{\sigma}R^{\sigma}_{\nu}\right)
\end{aligned}$$

Chapter 8

Appendix B

8.1 Hypersurface

Considering a manifold D-simensional manifold $\mathcal{M}$ endowed by a metric $g_{\mu\nu}$. A hypersurface is a $(D-1)$-dimensional space, typically denoted as $\Sigma = \partial\mathcal{M}$ which can be selected by putting a restriction on the coordinates, i.e.,

$$\Phi(x^\mu) = 0, \tag{8.1.1}$$

or by giving a parametric equation of the form (embedding)

$$x^\mu = x^\mu(y^i), \tag{8.1.2}$$

where y^i are the coordinates on Σ. The latter describes curves that are entirely contained on the hypersurface Σ. To illustrate this, let us consider an example. Considering the two-sphere (S^2) in flat three-dimensional space ($\mathbb{R}^3$) with coordinates $\{x, y, z\}$ with the constraint $\Phi(x^\mu) = x^2 + y^2 + z^2 - R^2 = 0$, where R is the radius of the sphere. Then, considering the embedding

$$x = R\cos\phi\sin\theta, \tag{8.1.3}$$

$$y = R\sin\phi\sin\theta, \tag{8.1.4}$$

$$z = R\cos\theta. \tag{8.1.5}$$

where θ and ϕ are coordinates of S^2. Thus we can write the differential as

$$\begin{aligned} \mathrm{d}x &= R\cos\theta\cos\phi\mathrm{d}\theta - R\sin\phi\sin\theta\mathrm{d}\phi && (8.1.6)\\ \mathrm{d}y &= R\sin\phi\cos\theta\mathrm{d}\theta + R\cos\phi\sin\theta\mathrm{d}\phi && (8.1.7)\\ \mathrm{d}z &= -R\sin\theta\mathrm{d}\theta\,. && (8.1.8)\end{aligned}$$

The infinitesimal displacement on R^3 can be written as

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,, \tag{8.1.9}$$

replacing the change coordinate given by (8.1.6) yields

$$\mathrm{d}s^2 = R^2(\mathrm{d}\theta^2 + \sin\theta\mathrm{d}\phi^2), \tag{8.1.10}$$

which is the round S^2 with radius R.

8.2 Normal unit vector

The vector $\nabla_\mu\Phi$ is normal to Σ. The value of $\Phi(y^i)$ changes only in the orthogonal direction of Σ. We can define the normal unit vector as

$$n^\mu = \frac{v^\mu}{\sqrt{||v^\mu v_\mu||}}. \tag{8.2.1}$$

The analysis of these hypersurface need to be divided into three cases: when Σ is either timelike or spacelike and when is null. In this work, we will focus on when Σ is not null. Therefore, we can define the norm of the vectors as,

$$\sigma = n_\mu n^\mu = \begin{cases} -1 & \text{if } \Sigma \text{ is spacelike} \\ +1 & \text{if } \Sigma \text{ is timelike} \end{cases} \tag{8.2.2}$$

8.3 Induced metric

The *induced metric* on Σ is obtained by restricting the line element to displacement on the hypersurface entirely on Σ. For parametric equations of the form,

$$x^\mu = x^\mu(y^i), \tag{8.3.1}$$

we can defined tangent to the curves contained in Σ as

$$e^\mu_i = \frac{\partial x^\mu}{\partial y^i}, \tag{8.3.2}$$

by virtue that n^μ has not have tangential components we can conclude that $e^\mu_i n_\mu = 0$, means that are orthonormal to the normal vector. Moreover, let us remark that x^μ are coordinates of the D-dimensional $\mathcal{M}$ manifold and y^i are the coordinates of the hypersurface Σ. Hereon, we will use the Greek indices for the complete manifold $\mathcal{M}$ and latin indices for the hypersurface Σ. On Σ we can define an infinitesimal displacement as

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu && (8.3.3)\\ &= g_{\mu\nu}\left(\frac{\partial x^\mu}{\partial y^i}\mathrm{d}y^i\right)\left(\frac{\partial x^\nu}{\partial y^j}\mathrm{d}y^j\right) && (8.3.4)\\ &= g_{\mu\nu}e^\mu_i e^\nu_j \mathrm{d}y^i \mathrm{d}y^j && (8.3.5)\\ &= h_{ij}\mathrm{d}y^i \mathrm{d}y^j, && (8.3.6) \end{aligned}$$

where

$$h_{ij} := g_{\mu\nu}e^\mu_i e^\nu_j \tag{8.3.7}$$

is the **first fundamental form** or more typically the *induced metric* of the hypersuface Σ. This metric transform as an scalar on $\mathcal{M}$ under $x^\mu \to x^{\mu'}$ but a tensor on Σ under $y^i \to y^{i'}$. Additionally, for the orthogonality between the basis on Σ, i.e., the e's and the normal vector we can write (8.3.7) as

$$h_{ij} := e^\mu_i e^\nu_j h_{\mu\nu}, \tag{8.3.8}$$

where

$$h_{\mu\nu} = g_{\mu\nu} - \sigma n_\mu n_\nu, \tag{8.3.9}$$

known as the *projector*.

8.4 Tangent tensor fields

Let us consider an arbitrary tensor, for instance, $A^{\mu\nu}$ that are only defined in Σ. For these tensors we can write

$$A^{\mu\nu} := A^{ij} e_i^\mu e_j^\nu, \tag{8.4.1}$$

where h's are basis vectors on Σ that we defined before (8.3.2). As we know by definition $e_i^\mu n_\mu =$. Therefore, this decomposition shows directly that $A^{\mu\nu}$ is tangent, i.e,

$$A^{\mu\nu} n_\mu = A^{ij} e_i^\mu e_j^\nu n_\mu = 0 \tag{8.4.2}$$

The (8.4.1) can be generalized for a completely arbitrary tensor as

$$A^{\mu_1 \dots \mu_p}{}_{\nu_i \dots \nu_q} = A^{i_1 \dots i_p}{}_{j_1 \dots j_q} e^{\mu_1}_{i_1} \dots e^{\mu_p}_{i_p} e^{j_1}_{\nu_1} \dots e^{j_q}_{\nu_q}. \tag{8.4.3}$$

On the other side, an arbitrary tensor, i.e., $T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}$ defined on $\mathcal{M}$ can be projected on Σ, hence only its tangential components survive. This can be do it using the projector (8.4.1) yielding

$$T^{\lambda_1 \dots \lambda_p}{}_{\rho_1 \dots \rho_q} = h^{\lambda_1}_{\mu_1} \dots h^{\lambda_p}_{\mu_p} h^{\nu_1}_{\rho_1} \dots h^{\nu_q}_{\rho_q} T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}. \tag{8.4.4}$$

Depite that we used Greek indices, this projection is tangential to Σ, cause we show that all the contraction with the normal vector vanish, for the orthogonality. Therefore the latter admits a decomposition as in (8.4.3).

8.5 Intrinsic covariant derivative and Extrinsic curvature

The intrinsic derivative of a tensor A_i on Σ is defined as the projection of $\nabla_\mu A_\nu$ on Σ, that is

$$\mathcal{D}_i A_j = h_i^\mu h_j^\nu \nabla_\mu A_\nu, \tag{8.5.1}$$

where $\mathcal{D}$ symbolice the derivative that is compatible with h_{ij}. Integrating by parts we can conclude that

$$\begin{aligned} \nabla_\mu (A_\nu h_j^\nu) &= h_j^\nu \nabla_\mu A_\nu + A_\nu \nabla_\mu h_j^\nu \\ h_j^\nu \nabla_\mu A_\nu &= \nabla_\mu (A_\nu h_j^\nu) - A_\nu \nabla_\mu h_j^\nu \\ &= \partial_\mu A_j - A_\nu \nabla_\mu h_j^\nu, \end{aligned}$$

rewritting (8.5.1) as

$$\begin{aligned} \mathcal{D}_i A_j &= \partial_\mu A_j h_i^\mu - \nabla_\mu h_j^\nu A_\nu h_i^\mu \\ &= \partial_i A_j - h_k^\gamma \nabla_\mu h_{\gamma j} h_i^\mu A^k \\ &= \partial_i A_j - \Gamma_{kij} A^k \end{aligned}$$

where

$$\Gamma_{kij} = h_k^\gamma \nabla_\mu h_{\gamma j} h_i^\mu. \tag{8.5.2}$$

Therefore, we construct the covariant derivative on Σ

$$\mathcal{D}_i A_j = \partial_i A_j - \Gamma_{ij}^k A_k, \tag{8.5.3}$$

where Γ is the Christoffel connection that is constructed with the induced metric h_{ij}. Moreover, this derivative can be generalized by a general tensor (p,q)-rank as

$$\mathcal{D}_k T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q} \equiv h^{i_1}_{\lambda_1} \ldots h^{i_p}_{\lambda_p} h^{\rho_1}_{j_1} \ldots h^{\rho_q}_{j_q} h^{\tau}_{k} \nabla_\tau T^{\lambda_1 \ldots \lambda_p}{}_{\rho_1 \ldots \rho_q} \tag{8.5.4}$$

Finally, we can prove that this derivative is compatible with h_{ij} straightforwardly. Let us show this,

$$\begin{aligned}
\mathcal{D}_k h_{ij} &= h^{\mu}_{i} h^{\nu}_{j} h^{\lambda}_{k} \nabla_\lambda h_{\mu\nu} \\
&= h^{\mu}_{i} h^{\nu}_{j} h^{\lambda}_{k} \nabla_\lambda [g_{\mu\nu} - \sigma n_\mu n_\nu] \\
&= h^{\mu}_{i} h^{\nu}_{j} h^{\lambda}_{k} [\nabla_\lambda g_{\mu\nu} - \sigma \nabla_\lambda (n_\mu n_\nu)] \\
&= -\sigma h^{\mu}_{i} h^{\nu}_{j} h^{\lambda}_{k} [n_\nu \nabla_\lambda n_\mu + n_\mu \nabla_\lambda n_\nu] \\
&= 0.
\end{aligned}$$

Now, we will define the extrinsic curvature from the scratch using the latter results. Let us consider a tangent vector on Σ as $A^\mu = h^\mu_\nu A^\nu$, then the intrinsic derivative is given by

$$\begin{aligned}
h^{\lambda}_{\mu} \nabla_\lambda A^\nu &= h^{\lambda}_{\mu} \delta^{\nu}_{\rho} \nabla_\lambda A^\rho && (8.5.5) \\
&= h^{\lambda}_{\mu} (h^{\nu}_{\rho} + \sigma n^\nu n_\rho) \nabla_\lambda A^\rho && (8.5.6) \\
&= h^{\lambda}_{\mu} h^{\nu}_{\rho} \nabla_\lambda A^\rho + \sigma h^{\lambda}_{\mu} n^\nu n_\rho \nabla_\lambda A^\rho && (8.5.7)
\end{aligned}$$

where we use the fact $\delta^\nu_\rho = h^\nu_\rho + \sigma n^\nu n_\rho$ and we know that $A^\mu n_\mu = n_\mu h^\mu_\lambda A^\lambda = 0$. Hence, from this we can conclude that

$$\nabla_\lambda (n_\mu A^\mu) = A^\mu \nabla_\lambda n_\mu + n_\mu \nabla_\lambda A^\mu = 0. \tag{8.5.8}$$

Therefore, we can substitute this on (8.5.5) yielding

$$\begin{aligned}
h^{\lambda}_{\mu} \nabla_\lambda A^\nu &= h^{\lambda}_{\mu} h^{\nu}_{\rho} \nabla_\lambda A^\rho h^{\lambda}_{\mu} h^{\nu}_{\rho} \nabla_\lambda A^\rho + \sigma h^{\lambda}_{\mu} n^\nu n_\rho \nabla_\lambda A^\rho \\
&= h^{\lambda}_{\mu} h^{\nu}_{\rho} \nabla_\lambda A^\rho - \sigma h^{\lambda}_{\mu} h^{\rho}_{\sigma} n^\nu \nabla_\lambda n_\rho A^\sigma \\
&= \mathcal{D}_\mu A^\nu - \sigma \mathcal{K}_{\mu\sigma} A^\sigma n^\nu,
\end{aligned}$$

where the first term is the intrinsic covariant derivative. The second term corresponds to the **second fundamental form**, commonly called *extrinsic curvature*, which is defined as

$$\mathcal{K}_{\mu\nu} = h^{\lambda}_{\mu} h^{\rho}_{\nu} \nabla_{\lambda} n_{\rho}. \tag{8.5.9}$$

Due to the idempotence property of the projector h^{μ}_{ν}, the extrinsic curvate can be rewritten as

$$\mathcal{K}_{\mu\nu} = h^{\lambda}_{\mu} \nabla_{\lambda} n_{\nu}. \tag{8.5.10}$$

On the other side, we can defined the curvature of the hypersurface Σ which is analogous of the curvature tensor, i.e., Riemann tensor on $\mathcal{M}$, defined on (7.1.6). This is called *intrinsic curvature* and the definition is obtained with the commutator of the intrinsic derivative, defined in (8.5.4). Considering an arbitrary vector v^{ρ} which is tangential to Σ, then

$$[\mathcal{D}_{\alpha}, \mathcal{D}_{\beta}] v^{\lambda} = \mathcal{R}^{\lambda}{}_{\rho\mu\nu} v^{\rho} \tag{8.5.11}$$

where "$[,]$" is the commutator of the derivatives and $\mathcal{R}^{\lambda}{}_{\rho\mu\nu}$ is the *intrinsic curvare* and its defines as

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = h^{\rho}_{\alpha} h^{\beta}_{\sigma} h^{\gamma}_{\mu} h^{\lambda}_{\nu} R^{\alpha}{}_{\beta\gamma\lambda} + 2\sigma \mathcal{K}^{\rho}_{[\mu} \mathcal{K}_{\nu]\sigma}. \tag{8.5.12}$$

This last equation is known as the *Gauss-Codazzi* relation.

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