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Symmetries in gravity :
from AdS to flat space and back

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Publications

I, José Figueroa, declare that this thesis titled "Symmetries in gravity: from AdS to Flat space and back" is based on the following publications or works in progress

1. S. Detournay, J. Figueroa and A. Vilar López, *Asymptotic T-duality in three dimensions*, *SciPost Phys.* **19**, 044 (2025) doi:10.21468/SciPostPhys.19.2.044 arXiv:2412.16136 [hep-th].

2. J. Figueroa, G. Giribet, A. Neira-Gallegos, J. Oliva and M. Oyarzo, *Overflying nilpotent horizons*, *Eur. Phys. J. C* **84**, no.11, 1215 (2024) doi:10.1140/epjc/s10052-024-13578-9 arXiv:2404.18378 [hep-th].

3. A. Anabalón, F. Correa, J. Figueroa, J. Oliva, M. Oyarzo, *Carrollian physics from the* $\ell \to 0$ *limit of AdS*. In preparation.

"Now, my excellent friend, said Herr C..., you are in possession of everything you need to know to understand my point. We see that the extent to which, as in the organic world, thought becomes dimmer and weaker, the grace within it emerges ever brighter and more powerful. Indeed just as when the intersection of two lines, on the one side of a point, after passing through infinity, suddenly presents itself again on the other side, or the image made by a concave mirror, after disappearing into infinity, suddenly reappears complete before us; so, when knowledge has, as it were, passed through an infinity, grace returns; and in such a manner, that it, simultaneously, appears most purely in that form of the human body that has either absolutely none, or infinite consciousness; that is to say, either in the form of a manikin, or a god."

On the Marionette Theatre, Heinrich von Kleist

Abstract

In this thesis we study symmetries of gravitational theories, focusing in exploiting (non-standard) relations between asymptotically AdS and flat spacetimes. This allows us to use what is already known in AdS to understand different aspects of flat holography.

First, we focus on a three dimensional gravity model arising as the low energy effective action of string theory. This model enjoys T-duality symmetry, inherited from string theory in the form of Buscher rules, which allows us to map solutions of the theory that are asymptotically AdS to other solutions that have vanishing curvature asymptotically. We use this propriety to map AdS boundary conditions to new ones that include asymptotically flat spacetimes. We study the symmetries and conserved charges in both cases: the later leading to a new phase space including the Horne-Horowitz black string, with a surprisingly large algebra of charges.

On the other hand, motivated by different physical intuitions and recent results, we explore the limit of the cosmological constant going to infinity in Anti de Sitter spacetime, finding that it captures Carrollian physics. We show how to obtain a Pseudo-Carrollian structure from this limit and find that its algebra of isometries differ from the standard Carroll algebra. We also study (interacting)-scalar field theories in this limit and its relation to recent results in the literature about Carrollian field theories. The realization of this degenerated spacetime as an homogeneous space associated to a group quotient is discussed.

Along the same line, and motivated by the physical insights gained through the study of symmetries, the final chapter of this thesis explores the symmetries of five-dimensional black holes solutions to the Einstein equations, whose event horizon is characterized by homogeneous anisotropic spaces, in particular the Nil geometry, which is one of the eight Thurston geometries. We study the near-horizon symmetries and the asymptotic symmetries for boundary conditions including these black hole solutions. We also study its thermodynamics and slowly rotating generalization.

Resumen

En esta tesis estudiamos las simetrías de teorías gravitacionales, centrándonos en explotar relaciones (no estándar) entre espacios-tiempo asintóticamente AdS y planos. Esto nos permite utilizar los conocimientos previos sobre AdS para comprender diferentes aspectos de holografía plana.

En primer lugar, nos centramos en un modelo de gravedad tridimensional que surge como acción efectiva de baja energía de teoría de cuerdas. Este modelo presenta la simetría T-duality, heredada de teoría de cuerdas en la forma de las reglas de Buscher, lo que nos permite mapear soluciones de la teoría que son asintóticamente AdS a otras soluciones con curvatura cero asintóticamente. Utilizamos esta propiedad para mapear las condiciones de borde para AdS a nuevas condiciones que incluyan espacio-tiempos asintóticamente planos. Estudiamos las simetrías y las cargas conservadas en ambos casos: el último conduce a un nuevo espacio de fases que incluye la cuerda negra de Horne-Horowitz, con un álgebra de cargas sorprendentemente grande.

Por otra parte, motivados por diferentes intuiciones físicas y resultados recientes, exploramos el límite de la constante cosmológica tendiendo a infinito en el espacio-tiempo anti-de-Sitter, encontrando que captura física carrolliana. Mostramos cómo obtener una estructura pseudo-carrolliana a partir de este límite y descubrimos que su álgebra de isometrías difiere del álgebra estándar de Carroll. También estudiamos teorías de campos escalares (interactuantes) en este límite y su relación con resultados recientes en la literatura sobre teorías de campos carrollianas. Se discute la realización de este espacio-tiempo degenerado como un espacio homogéneo asociado a un cociente de grupos.

En la misma línea, y motivado por los conocimientos de física adquiridos mediante el estudio de las simetrías, el capítulo final de esta tesis explora las simetrías de agujeros negros soluciones a las ecuaciones de Einstein en cinco dimensiones, cuyo horizonte de sucesos se caracteriza por espacios anisotrópicos homogéneos, en particular la geometría Nil, que es una de las ocho geometrías de Thurston. Estudiamos las simetrías en el horizonte y las simetrías asintóticas para condiciones de borde que incluyen estas soluciones de agujeros negros. También estudiamos su termodinámica y su generalización de rotación lenta.

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Chapter 1

Introduction

The quest to understand gravity within the framework of quantum mechanics remains one of the central challenges of modern theoretical physics. While general relativity has withstood extensive experimental tests at macroscopic scales, its classical nature becomes problematic when probing regimes where quantum effects become significant, such as near singularities or in the early universe. On the other hand, quantum field theory, the foundation of the Standard Model, lacks the geometric and background independent nature that defines gravitational phenomena. Bridging these two pillars into a consistent theory of quantum gravity has thus motivated decades of exploration.

String theory has emerged as a good candidate for such a unification. By replacing point particles with one-dimensional extended objects, it not only reproduces general relativity at low energies but also provides a UV-complete framework where gravitational and gauge interactions coexist. A key point in this context was the discovery of the AdS/CFT correspondence, or more generally, gauge/gravity duality. First proposed in the context of string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions [1], this duality posits an equivalence between a gravitational theory in asymptotically anti-de Sitter (AdS) spacetime and a conformal field theory (CFT) defined on its boundary.

Beyond its original formulation, the AdS/CFT correspondence has been largely developed and refined. It provides a powerful non-perturbative tool to study strongly coupled field theories, and also offers a concrete definition of quantum gravity in AdS backgrounds. Notably, the correspondence has shed light on longstanding puzzles such as black hole entropy, entanglement entropy [2], and hydrodynamics in holography [3]. In lower dimensions, particularly in three-dimensional gravity, the structure of asymptotic symmetries has played a crucial role in understanding the correspondence. The seminal work of Brown and Henneaux showed that the canonical realization of the asymptotic symmetry algebra of AdS_3 gravity is given by two copies of the Virasoro algebra, laying the foundation for the application of CFT methods to gravitational systems [4].

Motivated by the success of AdS/CFT, recent efforts have turned toward extending holographic principles to spacetimes with different asymptotics, particularly flat space. This direction, often referred to as "flat holography", aims to construct a dual description of gravity in asymptotically flat spacetimes, which are more realistic cosmological background. One of the key insights in this endeavor is the realization that the asymptotic symmetry group of flat spacetime at null infinity is the BMS group rather than the Poincaré group. In three dimensions, this group takes the form $\mathfrak{bms}_3$, which shares structural similarities with the conformal algebra [5].

The pursuit of a flat holographic dictionary has inspired a reevaluation of the role of asymptotic symmetries, soft theorems, memory effects, and their interrelations. These developments have opened new windows into the infrared structure of gravity and gauge theories, and have motivated studies of celestial holography and Carrollian dynamics. In particular, Carrollian geometries, defined as ultra relativistic limits of spacetime where the speed of light tends to zero, have gained increasing attention due to their natural emergence at null infinity and their connection to BMS symmetries [6]. The Carroll group, originally introduced as a contraction of the Poincaré group [7], now appears as a central structure in both asymptotic and near-horizon descriptions of gravitational systems.

The chapters of this thesis are embedded in this broader context, and each explores a distinct facet of the intricate relationship between gravity, symmetries, and holography. The first chapter investigates asymptotic symmetries in three-dimensional supergravity, with emphasis on T-duality and its ability to relate asymptotically AdS and asymptotically flat spacetimes, offering a non-trivial map between seemingly disconnected holographic regimes. The second chapter focuses on a Carrollian limit of AdS geometries, analyzing how a Carroll structure emerges in the limit $\ell \to 0$, and exploring Carroll-invariant field theories defined on this limit. Finally, the third chapter turns to higher-dimensional black hole solutions with non-homogeneous Nil horizon geometries, revealing novel aspects of their symmetry structures and thermodynamics. Together, these studies contribute to the ongoing effort to understand the symmetry content of gravitational theories and to uncover potential clues toward a unified framework for quantum gravity and holography.

Chapter 2

Asymptotic symmetries in 3D supergravity

The identification of simple, lower-dimensional gravitational models which capture the essence of physically relevant higher-dimensional spacetimes has been a very fruitful enterprise. Three-dimensional theories are a sweet spot for this approach, and over the years they have illuminated questions in a wide range of topics: the asymptotic symmetry structure of AdS gravity in the pre-holographic era [4], black hole microstate countings using the symmetries of the underlying dual [8–11], linear-response theory in black hole backgrounds [12], string theory embeddings [13–17], or the relation between asymptotic and worldsheet symmetries [18–22], to name a few. An early three-dimensional toy model, inaugurating the arXiv and even predating the BTZ black hole [23], was the Horne-Horowitz black string [24]. It is a $(2 + 1)$-dimensional solution to the string theory low-energy effective action which shares many features with higher-dimensional Reissner-Nordström black holes: it has a non-trivial causal structure with outer and inner horizons, a timelike curvature singularity, thermal behavior, and vanishing curvature at large radius (making it, in a certain sense, asymptotically flat as it will be defined below).

Despite all of this, or perhaps because of it, the three-dimensional black string has been less studied than its asymptotically AdS_3 black hole counterpart: the BTZ black hole, which has played a pivotal role in many developments of AdS/CFT. Still, the three-dimensional black string enjoys several remarkable properties and a relatively close relationship with BTZ black holes. It was introduced as an exact string theory background, via the construction of a gauged version of the $SL(2,\mathbb{R}) \times \mathbb{R}$ Wess-Zumino-Witten CFT [24]. It was later recognized that black strings could also be obtained as the target space of marginal deformations of the $SL(2,\mathbb{R})$ WZW worldsheet theory describing BTZ black holes [25]. Another relation came from the observation that a general class of black strings results from TsT transformations [26, 27] applied to BTZ black holes, and this resulted in the proposal that TsT-transformed AdS_3 backgrounds are holographically dual to single-trace $T\bar{T}$ deformations of the boundary CFT_2 [28, 29] (see also [30]).

In this chapter, we will exploit a different connection between three-dimensional black strings and BTZ black holes [13]: they are related by the low-energy manifestation of T-duality implemented by Buscher rules [31, 32]. These transformations map any solution of the low energy string equations with a translational Killing vector to another solution. Under certain conditions, the related backgrounds actually correspond to equivalent world-sheet theories [33], but we would like to emphasize from the start that in this work we will not be dealing with the underlying BTZ CFT [14, 22]. We will instead restrict ourselves to the (super)gravity realm in which Buscher rules provide a map generating new solutions from existing ones (with an isometry). Previous works used this perspective to discuss invariance of the thermodynamic properties of horizons under T-duality [34], even in the presence of α' corrections [35, 36]. We will now exploit a different property of the duality which makes it particularly interesting for the study of asymptotic structures: it can relate solutions with wildly different asymptotic behavior.

At a classical level, asymptotic boundary conditions for gravitational theories determine the symmetries present in the phase space [37–39]. These are in turn an essential hint towards a quantum formulation of the theory, as the well-known example of Brown-Henneaux boundary conditions in AdS_3 illustrates [4]: the asymptotic symmetry algebra should be realized in any putative dual quantum mechanical description. Given that BTZ black holes belong to the well-understood Brown-Henneaux phase space, and that they can be related via T-duality to the Horne-Horowitz black strings, our goal will be to use T-duality as a tool to obtain a phase space for the black strings. This is a novel approach, different to those used to define boundary conditions for the three-dimensional black strings in the past [40, 41]. The result of our method will be a phase space carrying a new asymptotic symmetry algebra (2.5.35)-(2.5.36), significantly larger than the ones previously discussed in the literature and including as subsets some well-known subalgebras: $\mathfrak{bms}_2$, $\mathfrak{bms}_3$ and a twisted warped conformal algebra. Notice that, since the black strings are asymptotically flat, this is a symmetry algebra for the three-dimensional low-energy string EFT with asymptotically flat boundary conditions (although, admittedly, the notion of asymptotic flatness is slightly non-standard due to the Killing fields having diverging norm at infinity, in spite of possessing vanishing curvature invariants; see section 2.5 for the details).

It is important to emphasize that this symmetry algebra is different from the one of the original Brown-Henneaux phase space, which as it is well-known, is canonically realized as two copies of the Virasoro algebra. This may seem puzzling at first, particularly if one thinks of T-duality in the stringy sense as relating equivalent backgrounds. However, we are here just applying Buscher rules to the low-energy (super)gravity fields, and we are doing so in a restricted and asymptotic sense (to be explained in section 2.5) which allows to generate new boundary conditions from existing ones, giving rise to a well-defined new phase space. As a consequence, solutions in the original Brown-Henneaux phase space are not one-to-one mapped to solutions in the black string phase space (it is only solutions having an exact isometry that get mapped between both sides). It is in this sense that we dub our procedure *asymptotic T-duality*. We believe similar ideas can be applicable to other sets of dual pair (not necessarily in three dimensions), thus making asymptotic T-duality an interesting way to generate new boundary conditions from well-understood ones.

This chapter is structured as follows. In section 2.3, we uplift the standard Brown-Henneaux analysis of asymptotically AdS_3 boundary conditions to the universal bosonic sector of the low energy effective field theory of strings – that is, we include a dilaton and a Kalb-Ramond two-form. Despite obtaining the same set of charges as in the original work, forming two Virasoro towers with the Brown-Henneaux central charge, the section sets up the notation and conventions that we will use later when applying Buscher rules. In section 2.4, as a warm-up, we consider a situation in which we have a complete phase space with an exact Killing vector: a chiral subset of the previous AdS_3 solution space. Upon dualizing in the direction of the exact Killing vector, we obtain a dual phase space which turns out to be exactly the one obtained by Compère, Song and Strominger (CSS) in [42]. Section 2.5 constitutes the bulk of this chapter, and it shows by means of an example how T-duality can be used to generate new boundary conditions from existing ones. From the full set of Brown-Henneaux boundary conditions, and dualizing along an angular direction which provides an asymptotic isometry, we obtain a new set of boundary conditions for the low energy string effective field theory. The non-extremal black strings, being T-dual to BTZ black holes, are included within the phase space defined by these boundary conditions. The asymptotic symmetry algebra has four infinite towers of charges, which combine in a way to include $\mathfrak{bms}_2$, $\mathfrak{bms}_3$, and a twisted version of the warped conformal algebra as subalgebras. On the way, in subsection 2.4.1, we also show that in a phase space with an exact Killing vector, asymptotic symmetry transformations are the same before and after applying a T-duality transformation. And in 2.6 we discuss in detail how black strings fit within the boundary conditions of section 2.5 and its thermodynamics.

2.1 The theory, solutions and T-duality

We start by reviewing the set up in which we will work. The theory corresponds to Type II Supergravity with pure NS-NS fields, that is, only the ten dimensional metric G_{AB}, the Kalb-Ramond two-form B_{AB} (which appears through its field strength $H = \mathrm{d}B$ in the term

$H^2 = H^{ABC}H_{ABC}$) and the Dilaton Φ. These fields appear as a set of massless excitations of a closed string. The action in string frame takes the form

$$S_{\text{II}} = \frac{1}{2\kappa_N^2}\int d^{10}x\sqrt{-g}e^{-2\Phi}\left(R - (\partial\Phi)^2 - \frac{1}{2}\frac{1}{3!}H^2\right) \tag{2.1.1}$$

For simplicity we would like to go to three dimensions, where we have a lot of control on the space of solutions. So we compactify the extra seven dimensions on a 3-sphere and a 4-torus. For this consider the following ansatz for the fields of (2.1.1)

$$\begin{aligned} ds_{10D}^2 &= ds_{3D}^2 + k\, ds^2(S^3) + ds^2(T^4), \\ H_{10D} &= dB_2 = H_3 + 2k\,\text{Vol}(S^3), \\ \Phi &= \Phi(x) \end{aligned} \tag{2.1.2}$$

where H_3 is a 3-form on the manifold ds_{3D}^2 and x are coordinates of that same space. Here k counts the number of NS_5-branes. We can replace this ansatz in (2.1.1) and get an effective three-dimensional action in string frame

$$S_{\text{3D}} = \frac{1}{2\kappa_N^2}\int d^3x\sqrt{-g}e^{-2\Phi}\left(\frac{4}{k} + R - 4(\partial\Phi)^2 - \frac{1}{2}\frac{1}{3!}H^2\right) \tag{2.1.3}$$

Now we switch to Einstein frame by doing $G_{MN} \to e^{4\Phi}G_{MN}$, we do this because it will make the expressions for the charges way simpler. We also change the notation to differential forms for simplicity. The Lagrangian in Einstein frame reads

$$L = \frac{1}{2\kappa_N^2}\mathcal{L}_0\,\epsilon\,, \qquad \mathcal{L}_0 = R - 2\Lambda_0 e^{4\Phi} - 4(\partial\Phi)^2 - \frac{1}{12}e^{-8\Phi}H^2\,, \tag{2.1.4}$$

where ϵ is the volume form of the manifold and κ_N^2 is the gravitational coupling. For the rest of this chapter, with the exception of the next subsection about T-duality transformations in 10D, we will work with this theory and notation. Varying the fields we obtain

$$\delta L = \frac{1}{2\kappa_N^2}\left[-\delta G_{MN}\,\mathcal{E}_G^{MN} + 2\delta\Phi\,\mathcal{E}_\Phi + \frac{1}{2}\delta B_{PQ}\,\mathcal{E}_B^{PQ}\right]\epsilon + \mathrm{d}\Theta\,, \tag{2.1.5}$$

with the equations of motions given by

$$\mathcal{E}_\Phi = 4\nabla^2\Phi - 2e^{-8\Phi}h^2 - 4\Lambda_0 e^{4\Phi} = 0\,, \tag{2.1.6a}$$

$$\mathcal{E}_B^{PQ} = \epsilon^{PQM}\nabla_M\left(e^{-8\Phi}h\right) = 0\,, \tag{2.1.6b}$$

$$\mathcal{E}_G^{MN} = R^{MN} - \frac{1}{2}G^{MN}\mathcal{L}_0 - 4\nabla^M\Phi\nabla^N\Phi + \frac{1}{2}e^{-8\Phi}h^2G^{MN} = 0\,, \tag{2.1.6c}$$

and the boundary contribution (relevant for the charges) is $\Theta = \theta\cdot\epsilon$ with

$$\theta^R = \frac{1}{2\kappa_N^2}\left[2G^{L[R}\nabla^{S]}\delta G_{SL} - 8\delta\Phi\nabla^R\Phi - \frac{1}{2}e^{-8\Phi}H^{RPQ}\delta B_{PQ}\right]. \tag{2.1.7}$$

We are using the notation for differential form calculus of [43], and we have often simplified expressions by taking the scalar dual of the 3-form H, $h = -\star H$ (so that $H = h\,\epsilon$).

2.1.1 T-duality

As we mentioned, the supergravity theory inherits the T-duality symmetry from String theory. Let's recap briefly T-duality at the level of bosonic String theory on $\mathcal{M} \times S^1_R$, where R is the radius of the compact direction.

T-duality is a symmetry of the String theory spectrum which interchanges momentum and winding modes along the circle S^1 for closed strings. In the case of open strings, it maps Dirichlet boundary conditions to Neumann ones, and vice versa, while also transforming the radius of the circle according to $R \to R' = \frac{\alpha'}{R}$. As a result, T-duality relates a string theory defined on $\mathcal{M} \times S^1_R$ to another theory formulated on $\mathcal{M} \times S^1_{R'}$. Importantly, the $U(1)$ isometry along the circle S^1_R is traded with a different $U(1)$ isometry along the $S^1_{R'}$ direction in the dual theory. Since T-duality connects two distinct $U(1)$ isometries it is also called *Abelian T-duality*.

This String theory duality is realized at the level of the effective 10D Type II supergravity via Buscher rules [31, 32], we now present the details of this transformation in the supergravity framework.

T-duality rules in 10D

Suppose that we can write a solution of the action (2.1.1) in the following form (we split the ten-dimensional coordinates as $\mathbf{x}^A = (x^a, y)$)

$$\begin{aligned} ds^2_{10D} &= ds^2_{9D} + e^{2C(x)} \left(dy + A_a(x)dx^a\right), \\ B_2 &= \frac{1}{2} B_{ab}(x) dx^a \wedge dx^b + B_a(x) dx^a \wedge dy, \\ \Phi &= \Phi(x) \end{aligned} \tag{2.1.8}$$

Note that ∂_y is an isometry (specifically a $U(1)$ isometry) of the background. Then we can find a new solution by replacing

$$\begin{aligned} &\tilde{C} = -C, \quad \tilde{A}_a = -B_a, \quad \tilde{B}_a = -A_a \\ &\tilde{B}_{ab} = B_{ab} + 2A_{[a}B_{b]}, \quad \tilde{\Phi} = \Phi - C \end{aligned} \tag{2.1.9}$$

that is, the background

$$\begin{aligned} ds^2_{10D} &= ds^2_{9D} + e^{-2C(x)} \left(dy - B_a dx^a\right), \\ B_2 &= \frac{1}{2} \left(B_{ab} + 2A_{[a}B_{b]}\right) dx^a \wedge dx^b - A_a dx^a \wedge dy, \\ \Phi &= \Phi(x) - C(x) \end{aligned} \tag{2.1.10}$$

is also a solution of the equations of motion (2.1.6). This operation corresponds to the Buscher rules for T-duality. Note that the $U(1)$ isometry is still present in the new background, so it maps $U(1) \to U(1)$.

T-duality rules in 3D

The Buscher Rules get simplified in 3D, simply because we can use the fact that requiring the $U(1)$ isometry restricts the structure of the 2-form. In 3D the Kaluza-Klein ansatz (2.1.8) takes the form

$$\begin{aligned} ds^2_{3D} &= ds^2_{2D} + e^{2C(x)} \left(dz + A_\mu(x)dx^\mu\right)^2 , \\ B_2 &= \frac{1}{2} B_{\mu\nu}(x)dx^\mu \wedge dx^\nu + B_\mu(x)dx^\mu \wedge dz, \\ \Phi &= \Phi(x) \end{aligned} \tag{2.1.11}$$

since $B_{\mu\nu}$ is a 2-form in $2D$, its field strength is identically zero, hence is pure gauge and we can set it to zero [1]. Then Kaluza-Klein the ansatz reduces to

$$\begin{aligned} ds^2_{3D} &= ds^2_{2D} + e^{2C(x)} \left(dz + A_\mu(x)dx^\mu\right)^2 , \\ B_2 &= B_\mu(x)dx^\mu \wedge dz, \\ \Phi &= \Phi(x) \end{aligned} \tag{2.1.12}$$

and after the Buscher procedure we get

$$\begin{aligned} ds^2_{3D} &= ds^2_{2D} + e^{-2C(x)} \left(dz - B_\mu dx^\mu\right)^2 , \\ B_2 &= -A_\mu dx^\mu \wedge dz, \\ \Phi &= \Phi(x) - C(x) \end{aligned} \tag{2.1.13}$$

This particularly simple way of seeing Buscher procedure (due to the simple interchange of the fields) it is useful to clearly see how it changes the asymptotic behavior of the solutions in a general way: without referring to a particular solution, but a set of solutions instead. We will see this later when we study the asymptotic symmetries of T-dual related phase spaces.

2.1.2 BTZ black hole

Consider the theory (2.1.4). This theory, as we mentioned, involves 3 different fields: the metric, the dilaton and the Kalb-Ramond two-form. Notice, although, that it is possible to set the dilaton and two-form to a particular value, consistently, in such a way that the equations of motion for the metric are the Einstein field equations with a negative cosmological constant.

Indeed, consider equations (2.1.6) and set the Dilaton to a constant, $\Phi = \Phi_0$, and the field strength of the two-form H to be proportional to the volume form of the spacetime,

[1] We have to be careful here since the gauge transformation needed to set these components to zero may have non-vanishing charges, so when gauge fixing we could lose part of the phase space. Luckily, in the cases presented in this chapter, this does not happen

which implies $h = h_0$. In doing this we see that the equation of motion for the Dilaton relates h_0 to Φ_0 and Λ_0. The equation for the B-field is automatically satisfied. And the equation for the metric recasts itself as Einstein equations with an effective (negative) cosmological constant.

This implies that all solutions of General Relativity with a negative cosmological constant in 3D are also solutions of the low-dimensional supergravity [13]. In particular, the BTZ black hole is a solution. Its metric is given by

$$ds^2 = \left(M - \frac{r^2}{l^2}\right) dt^2 - J dt d\varphi + r^2 d\varphi^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-1} dr^2 \tag{2.1.14}$$

where the constants M and J are related to the two horizons of the black hole as

$$M = \frac{r_+^2 + r_-^2}{l^2} \qquad J = \frac{2r_+ r_-}{l} \tag{2.1.15}$$

and, as it is well known, they parametrize the mass and angular momentum of the black hole.

The B-field supporting this solution looks, in a particular gauge, as $B_{\varphi t} = \frac{r^2}{l}$. Which has a field strength proportional to the volume form, as expected.

But, of course, notice that there are more solution to this theory than the General Relativity solutions. In particular, it has propagating degrees of freedom, which difficults the goal of obtaining a closed expression for the phase space like in usual General Relativity in 3D.

2.1.3 Horne-Horowitz black string

One interesting solution of this theory is the Horne-Horowitz black string. It was actually discovered before BTZ black hole [24], but here we will describe it as coming from BTZ by using T-duality as showed in [13].

Notice BTZ black hole has a $U(1)$ isometry along ∂_φ. Direct application of Buscher rules on BTZ along that isometry give us the following metric

$$\begin{gathered} \tilde{ds}^2 = \left(M - \tfrac{J^2}{4r^2}\right) dt^2 + \tfrac{2}{l} dt d\varphi + \tfrac{1}{r^2} d\varphi^2 + \left(\tfrac{r^2}{l^2} - M + \tfrac{J^2}{4r^2}\right)^{-1} dr^2 \\ \tilde{B}_{\varphi t} = -\tfrac{J}{2r^2} \qquad \phi = -\ln r \end{gathered} \tag{2.1.16}$$

we can make a change of coordinates in order to bring it to the usual form of the black string. Doing

$$t = \frac{l(\hat{x} - \hat{t})}{\left(r_+^2 - r_-^2\right)^{1/2}}, \qquad \varphi = \frac{r_+^2 \hat{t} - r_-^2 \hat{x}}{\left(r_+^2 - r_-^2\right)^{1/2}}, \qquad r^2 = l\hat{r} \tag{2.1.17}$$

we obtain

$$\begin{gathered}\tilde{ds}^2 = -\left(1-\frac{\mathcal{M}}{\hat{r}}\right)d\hat{t}^2 + \left(1-\frac{\mathcal{Q}^2}{\mathcal{M}\hat{r}}\right)d\hat{x}^2 + \left(1-\frac{\mathcal{M}}{\hat{r}}\right)^{-1}\left(1-\frac{\mathcal{Q}^2}{\mathcal{M}\hat{r}}\right)^{-1}\frac{l^2 d\hat{r}^2}{4\hat{r}^2} \\ \phi = -\frac{1}{2}\ln \hat{r}l, \quad B_{\hat{x}\hat{t}} = \frac{\mathcal{Q}}{\hat{r}}\end{gathered} \tag{2.1.18}$$

The solution is parameterized by the mass and charge, which are given by $\mathcal{M} = r_+^2/l$ and $\mathcal{Q} = J/2$ in terms of BTZ quantities.

Notice in particular the linear dilaton; a completely different asymptotic behavior compared to the dilaton for BTZ. This happens also at the level of the metric, for which the curvature goes to zero at infinity (although the metric does not looks asymptotically like Minkowski spacetime, and an appropriate coordinate transformation to bring it there will spoil the power fall-off of the metric at infinity). So we have just seen that T-duality (in this particular case) maps a solution with AdS asymptotics to another one with vanishing curvature at infinity, namely BTZ and the black string.

As stated in the introduction, the black string shares many proprieties with higher-dimensional Reissner-Nordström black holes: it has a non-trivial causal structure with outer and inner horizons, a timelike curvature singularity, thermal behavior, and, as already said, vanishing curvature at large radius (making it, in a certain sense, asymptotically flat). Actually its Penrose diagram looks the same.

One important point to make is that the coordinate transformation (2.1.17) breaks down for three important cases, for those a separate analysis has to be done. It turns out that the T-dual for those cases are

- Zero mass BTZ → pp-wave.
- Extremal BTZ → pp-wave on the extremal black string.
- AdS → Product of time and the dual of the 2D euclidean BH.

Along this chapter we will define consistent boundary conditions for the theory including this black string.

2.2 Conserved charges in diffeomorphism invariant theories

We are interested in studying the symmetries of the theory we just discussed, because it allow us to have two interesting different asymptotic behavior and relate them. The importance of the asymptotics and boundary conditions in spacetimes with boundaries arise because, even if the Lagrangian of the theory is diffeomorphism/gauge invariant, not all those transformations preserve the asymptotic behavior, then those transformations are not symmetries of the theory defined by the action and the given boundary conditions. Even more, some of the remaining symmetry transformations have conserved charges associated to them. In the quantum theory this would correspond to an observable, which label the different quantum states. Then, the transformations generated by those symmetries are physical and relate different states/solutions.

We are interested, then, in studying the conserved charges of a theory in order to understand the symmetries of its phase space. The Hamiltonian formalism of field theory is perfectly adapted for the study of symmetries, but at the cost of loosing the covariance. Although, there exist an elegant formalism, by Iyer, Lee, Wald, and Zoupas, that treats Hamiltonian definitions in a way diffeomorphism invariance is preserved: The covariant phase space formalism [44–48]. We will review quickly the formalism in this subsection, in particular the extensions considered in [43], and provide the important expressions which will be used during this chapter. We use the convenient language of differential forms.

Consider a Lagrangian field theory

$$S = \int_M L + \int_{\partial M} \ell. \tag{2.2.1}$$

along with some boundary condition for the dynamical fields. Boundary conditions are imposed at a spatial boundary. And we are leaving the fields unconstrained at the future/past boundaries, because fixing the solution at those boundaries correspond to choosing a particular state (if any), and we want allow for the different states. Then, the system, given a particular initial condition, evolves it into the final state by means of the equations of motion. To obtain them, we should look for configuration that make the action stationary under variation of the dynamical fields subjected to the boundary conditions. We however, should require that the action be stationary up to terms localized at the future and past boundaries, given the above discussion on boundary conditions. We make the following decomposition of the spacetime $\partial M = \Gamma \cup \Sigma_- \cup \Sigma_+$, with Γ the spatial boundary, Σ_- the past boundary, and Σ_+ is the future boundary, then we require

$$\delta S = \int_{\Sigma_+} \Psi - \int_{\Sigma_-} \Psi. \tag{2.2.2}$$

The variation of the Lagrangian, after integration by parts, is generically given by

$$\delta L = E_a \delta\phi^a + d\Theta, \tag{2.2.3}$$

where ϕ^a are the dynamical fields and $\delta\phi^a$ its variations. E_a correspond to the equations of motion and Θ, which depends on the field and their variations, is called symplectic potential. It is defined up to the addition of a total derivative dY. Then, the variation of the action is

$$\delta S = \int_M E_a \delta\phi^a + \int_{\partial M} (\delta\ell + \Theta)\,, \tag{2.2.4}$$

Given that we want stationary up to integrals in the future/past boundaries we ask that the term integrated in the spatial boundary vanishes there. One could ask that $(\delta\ell + \Theta)|_\Gamma = 0$, but given that Θ comes from an integration by parts procedure, it can be shifted by a boundary term. Then asking the above condition it is unnatural. We ask a more general condition

$$(\Theta + \delta\ell)|_\Gamma = dC, \tag{2.2.5}$$

then

$$\begin{aligned}\delta S &= \int_M E_a \delta\phi^a + \int_{\Sigma_+ - \Sigma_-} (\Theta + \delta\ell) + \int_\Gamma (\Theta + \delta\ell) \\ &= \int_M E_a \delta\phi^a + \int_{\Sigma_+ - \Sigma_-} (\Theta + \delta\ell) + \int_{\partial\Gamma} C \\ &= \int_M E_a \delta\phi^a + \int_{\Sigma_+ - \Sigma_-} (\Theta + \delta\ell - dC)\,,\end{aligned} \tag{2.2.6}$$

Then the action will be stationary (up to future/past boundary contributions) for configurations obeying the equations of motion.

Now, to switch to Hamiltonian formalism we need the pre-symplectic form. In order to do that, we introduce a notation that enables us to think of quantities that depend on variation of the fields as one-forms in a space called the jet space, in which the dynamical field fiber over the spacetime. We will view field variations $\delta\phi^a(x)$ as coordinate differential on this new space. δ will then be the exterior derivative for differential forms, and the action of a one-form $\delta\phi^a(x)$ on a vector field in field space is given by

$$\delta\phi^a(x)\left(\int d^dx' f^b(\phi, x')\frac{\delta}{\delta\phi^b(x')}\right) = f^a(\phi, x). \tag{2.2.7}$$

So, to take back $\delta\phi^a(x)$ from a one-form to a variation, we act on a vector with components given by those variations.

Settled the notation, we define the pre-symplectic current as

$$\omega \equiv \delta\Psi = \delta(\Theta - dC). \tag{2.2.8}$$

It is trivial to see that it is closed on field space, because $\delta^2 = 0$. Also ω is zero on Γ, because using (2.2.5) we have

$$\omega|_\Gamma = \delta(\Theta + \delta\ell - dC)|_\Gamma = 0. \tag{2.2.9}$$

And another propriety is that ω is closed as a form on spacetime

$$d\omega = d\delta(\Theta - dC) = \delta d\Theta = \delta(\delta L - E_a\delta\phi^a) = -\delta E_a \wedge \delta\phi^a = 0. \tag{2.2.10}$$

where in the last equality we have assumed that the equations of motion are satisfied (in particular for perturbations around solutions).

With this we can define the pre-symplectic form as

$$\Omega \equiv \int_\Sigma \omega. \tag{2.2.11}$$

Now, we want to obtain the canonical charges. They are the generators of the symmetries on the phase space. We want a function H_ξ for which

$$\delta H_\xi = -X_\xi \cdot \Omega, \tag{2.2.12}$$

with

$$X_\xi \equiv \int d^d x \mathcal{L}_\xi \phi^a(x) \frac{\delta}{\delta \phi^a} \tag{2.2.13}$$

the vector field in field space that acts on the one-forms and gives the lie derivative of the field.

Then we turn to compute the right hand side of (2.2.12). For that it is useful to introduce the Noether current

$$J_\xi \equiv X_\xi \cdot \Theta - \xi \cdot L. \tag{2.2.14}$$

which satisfy the propriety that is closed on the spacetime

$$\begin{aligned} dJ_\xi &= d(X_\xi \cdot \Theta) - d(\xi \cdot L) \\ &= X_\xi \cdot (\delta L - E_a \delta \phi^a) - \mathcal{L}_\xi L \\ &= \delta_\xi L - \mathcal{L}_\xi L - E_a \mathcal{L}_\xi \phi^a \\ &= 0. \end{aligned} \tag{2.2.15}$$

Then we compute

$$\begin{aligned} -X_\xi \cdot \omega &= -X_\xi \cdot \delta(\Theta - dC) \\ &= \delta \left(X_\xi \cdot (\Theta - dC)\right) - \mathcal{L}_{X_\xi}(\Theta - dC) \\ &= \delta J_\xi + \xi \cdot \delta L - \mathcal{L}_\xi \Theta + d\left(\delta_\xi C - \delta(X_\xi \cdot C)\right) \\ &= \delta J_\xi + \xi \cdot (d\Theta + E_a \delta \phi^a) - \mathcal{L}_\xi \Theta + d\left(\delta_\xi C - \delta(X_\xi \cdot C)\right) \\ &= \delta J_\xi + d\left(\delta_\xi C - \delta(X_\xi \cdot C) - \xi \cdot \Theta\right). \end{aligned} \tag{2.2.16}$$

And after integrating on a cauchy slice

$$\begin{aligned} -X_\xi \cdot \Omega &= \int_\Sigma \delta J_\xi + \int_{\partial\Sigma} \left(\mathcal{L}_\xi C - \delta(X_\xi \cdot C) - \xi \cdot \Theta\right) \\ &= \int_\Sigma \delta J_\xi + \int_{\partial\Sigma} \left(\xi \cdot (dC - \Theta) - \delta(X_\xi \cdot C)\right) \\ &= \delta \left(\int_\Sigma J_\xi + \int_{\partial\Sigma} (\xi \cdot \ell - X_\xi \cdot C) \right). \end{aligned} \tag{2.2.17}$$

Where we see that the charge is actually always integrable. This comes form the fact that we are not allowing symplectic flux at infinity when choosing boundary conditions that give us a well-posed variational problem, as we previously discussed.

The charge is given by

$$H_\xi \equiv \int_\Sigma J_\xi + \int_{\partial\Sigma} (\xi \cdot \ell - X_\xi \cdot C) + \text{constant}. \tag{2.2.18}$$

We can see that the charges are independent of the Cauchy surface. Consider two slices Σ' and Σ, such that $\partial\Sigma' - \partial\Sigma = \partial\Xi$, with $\Xi \subset \Gamma$, then the difference of the charges evaluated on those Cauchy surfaces is given by

$$\begin{aligned}\int_\Xi (J_\xi + d(\xi \cdot \ell - X_\xi \cdot C)) &= \int_\Xi (X_\xi \cdot (\Theta - dC) - \xi \cdot L + d(\xi \cdot \ell)) \\ &= \int_\Xi (-X_\xi \cdot \delta\ell + d(\xi \cdot \ell) - \xi \cdot L) \\ &= \int_\Xi (-\delta_\xi \ell + \mathcal{L}_\xi \ell - \xi \cdot L) \\ &= 0. \end{aligned} \tag{2.2.19}$$

We then conclude that the charges are indeed conserved.

For the special case of full invariance under diffeomorphisms (or local gauge transformations), J does not only satisfy $dJ_\xi = 0$, but also there will be a local covariant $(d-2)$-form Q_ξ called the Noether charge, that satisfy

$$J_\xi = dQ_\xi. \tag{2.2.20}$$

Then, for fully covariant theories, we can write the following expression for the charges

$$H_\xi = \int_{\partial\Sigma} (Q_\xi + \xi \cdot \ell - X_\xi \cdot C) + \text{constant} \tag{2.2.21}$$

Where we can see it is a pure boundary term, as expected.

2.3 Asymptotic symmetries with AdS boundary conditions

In this section, we will generalize the classical analysis of asymptotically AdS_3 boundary conditions by Brown and Henneaux [4] so that it applies to the low energy effective theory governing the universal massless NS-NS sector of string theory presented below. This problem has already been considered in previous works [49], and our results will be compatible with them. The role of this section is mainly to set the stage for a later application of T-duality transformations to the Brown-Henneaux boundary conditions, so in particular the notation and methods presented here will be used all along this chapter of the thesis.

Consider the theory (2.1.4). We will impose boundary conditions for the fields, inspired by those of [4], in a form adapted to connect with Bañados metrics [50, 51]. The proposed boundary conditions (or fall-offs) are

$$\mathrm{d}s^2 = \frac{\ell^2}{r^2}\mathrm{d}r^2 + r^2\left(\eta_{ab} + \frac{\ell^2}{r^2}Y_{ab} + \dots\right)\mathrm{d}x^a\,\mathrm{d}x^b\,, \tag{2.3.1a}$$

$$B = \left(\frac{r^2}{C_0} + b(x^a) + \frac{\beta(x^a)}{r^2} + \dots\right)\mathrm{d}x^+\wedge\mathrm{d}x^-\,, \tag{2.3.1b}$$

$$\Phi = \frac{1}{4}\log\left(\frac{2}{C_0}\right) + \frac{\tilde{Y}(x^a)}{r^2} + \frac{\phi(x^a)}{r^4} + \dots\,, \tag{2.3.1c}$$

where $a, b, \dots$ label coordinates in the two-dimensional space orthogonal to the radial direction, and η_{ab} is the Minkowski metric in that two-dimensional space. These boundary conditions are defined by two fixed parameters, a length scale ℓ setting the asymptotic AdS_3 radius and a dimensionless constant C_0.[2] This sets the charge of the Kalb-Ramond two-form field, B, as we will later discuss in more detail. An asymptotic analysis of the equations of motion (2.1.6) shows that, in order to have solutions, we must relate the parameters ℓ and C_0 to the cosmological constant Λ_0 as

$$\Lambda_0 = -\frac{C_0}{\ell^2}\,. \tag{2.3.2}$$

This is analogous to the usual $\Lambda_0 = -1/\ell^2$ value in Einstein gravity, but here modified by the B-field charge.[3] Note that we cannot set $C_0 = 0$, since the constant piece in the dilaton would blow up. From now on we will write our expressions in terms of C_0 and ℓ. We have also chosen to use null coordinates for the constant radius surfaces, but they can be traded for more standard ones via $x^\pm = t/\ell \pm \phi$. The choice of Fefferman-Graham gauge, setting $G_{rr} = \ell^2/r^2$ and $G_{ra} = 0$, is convenient but not essential.

The previous set of boundary conditions can be shown to define a well-posed variational problem if we add to our action the following boundary term

$$\mathcal{B} = \frac{1}{\kappa_N^2}\left(K - \frac{1}{\ell}\right)\epsilon_{\partial\mathcal{M}} + \frac{1}{2\kappa_N^2}\left(e^{-8\Phi}\star H\right)\,B\,, \tag{2.3.3}$$

so that when varying we consider the full action

$$S = \int_{\mathcal{M}} L + \int_{\partial\mathcal{M}} \mathcal{B}\,. \tag{2.3.4}$$

[2]One could think of generalizing the boundary conditions, allowing C_0 (and thus ℓ) to vary in the spirit of [52]. While such an extension may be interesting, it is not needed for our purposes.

[3]The asymptotic value of the dilaton is also fixed in terms of C_0 as written in the boundary conditions in order to have non-trivial solutions. One would typically set it to 0 ($C_0 = 2$) by conformally rescaling the metric by a constant, but we will keep it general here.

Indeed, a first order variation produces a boundary term of the form

$$(\Theta + \delta\mathcal{B})|_{\partial\mathcal{M}} = -\frac{\epsilon_{\partial\mathcal{M}}}{2\kappa_N^2}\left[\left(K^{MN} - \left(K - \frac{1}{\ell}\right)\gamma^{MN}\right)\delta G_{MN} + 8n^R\partial_R\Phi\,\delta\Phi\right] + \frac{1}{2\kappa_N^2}\delta\left(e^{-8\Phi}\star H\right)B|_{\partial\mathcal{M}} + \mathrm{d}C\,, \tag{2.3.5}$$

where

$$C = c\cdot\epsilon_{\partial\mathcal{M}}\,, \qquad c^M = -\frac{1}{2\kappa_N^2}\gamma^{MN}n^R\delta G_{NR}\,. \tag{2.3.6}$$

In these expressions, extrinsic curvatures are computed with respect to the outward pointing unit normal $n = (\ell/r)\mathrm{d}r$, and γ_{MN} refers to the metric induced at the boundary,

$$\gamma_{MN}\mathrm{d}x^M\mathrm{d}x^N = r^2\left(\eta_{ab} + \frac{\ell^2}{r^2}Y_{ab} + \dots\right)\mathrm{d}x^a\,\mathrm{d}x^b\,. \tag{2.3.7}$$

With the boundary conditions above, the term (2.3.5) behaves as

$$(\Theta + \delta\mathcal{B})|_{\partial\mathcal{M}} = \frac{1}{4\ell\kappa_N^2}\left(\ell^2\eta^{ab}\delta Y_{ab} + 16\delta\tilde{Y} + \dots\right)\mathrm{d}x^+\wedge\mathrm{d}x^- + \mathrm{d}C\,, \tag{2.3.8}$$

where the total derivative on the boundary $\mathrm{d}C$ plays no role since its integral over $\partial\mathcal{M}$ vanishes. We thus define a well-posed variational problem if we demand

$$\tilde{Y} = -\frac{\ell^2}{16}\eta^{ab}Y_{ab} \equiv -\frac{\ell^2}{16}Y\,. \tag{2.3.9}$$

Even though relating subleadings can in general be a dangerous restriction on a given space of solutions (potentially eliminating many or all of them), here we are justified in doing it because this is the condition required by conservation of the Kalb-Ramond charge,

$$\partial_M\left(e^{-8\Phi}\star H\right) = 0\,, \tag{2.3.10}$$

and this is one of our equations of motion. Note that the last term in the boundary piece (2.3.3) converts to a fixed charge ensemble, allowing a well posed variational problem by fixing just C_0, with $b(x^a)$ free (without that term, we would need to demand $\delta b = 0$). The requirement to work at fixed charge is motivated by the fact that the leading term in the B-field in (2.3.1) is the one giving the charge, and not the pure gauge piece at the boundary, $b(x^a)$.[4] This is similar to the situation with standard gauge fields in low dimensions (e.g., AdS_2), where the non-normalizable (leading) mode specifies the charge instead of the chemical potential.[5]

[4]Incidentally, adding the boundary piece to work at fixed charge also makes the action finite for *any* configuration satisfying the boundary conditions (2.3.1). This would be a desirable feature in a hypothetical definition of a quantum gravity path integral.

[5]See [53] for a nice recent discussion about ensemble choices and natural boundary conditions in gravitational theories.

We can now analyze the asymptotic symmetry transformations which preserve the previous boundary conditions. A general transformation is composed of a diffeomorphism and a gauge transformation of the B-field,

$$\delta_{\xi,\Lambda}G_{MN} = \mathcal{L}_\xi G_{MN}\,, \qquad \delta_{\xi,\Lambda}B_{MN} = \mathcal{L}_\xi B_{MN} + 2\partial_{[M}\Lambda_{N]}\,, \qquad \delta_{\xi,\Lambda}\Phi = \mathcal{L}_\xi\Phi\,. \tag{2.3.11}$$

Respecting the metric boundary conditions lands us in the Brown-Henneaux diffeomorphisms,

$$\xi[T^a] = -\frac{r}{2}\partial_a T^a\,\partial_r + \left(T^a + \frac{\ell^2}{2}\int^r \frac{\mathrm{d}r'}{(r')^3}h^{ab}\partial_b\partial_c T^c\right)\partial_a\,, \tag{2.3.12}$$

characterized by two chiral functions, $T^\pm(x^\pm)$. The subleadings in ∂_a just ensure we stay in Fefferman-Graham gauge, and h^{ab} denotes there the inverse of the (renormalized) metric induced at fixed r, $h_{ab} = \eta_{ab} + (\ell^2/r^2)Y_{ab} + \ldots$. These diffeomorphisms also preserve the boundary conditions in the other fields,[6] thus they constitute asymptotic Killing vectors of our theory. Regarding transformations of the B-field, our boundary conditions allow asymptotically

$$\Lambda = \lambda + \mathcal{O}(r^{-1})\,, \tag{2.3.13}$$

with λ a one-form on fixed-r surfaces. Here λ just acts as a boundary gauge transformation, $b \to b + \mathrm{d}\lambda$.

The equations of motion (2.1.6) can be solved perturbatively in the asymptotic expansion. The B-field equation is simply charge conservation, $e^{-8\Phi}\star H = -C_0/\ell$. The dilaton equation of motion demands

$$\eta^{ab}Y_{ab} = 0\,, \tag{2.3.14}$$

and the metric one then sets $\partial_- Y_{++} = \partial_+ Y_{--} = 0$. We thus obtain the standard two chiral functions $Y_{++} = Y_{++}(x^+)$, $Y_{--} = Y_{--}(x^-)$ of the Brown-Henneaux phase space. They transform under the asymptotic diffeomorphisms as

$$\delta_\xi Y_{\pm\pm} = T^\pm\partial_\pm Y_{\pm\pm} + 2Y_{\pm\pm}\partial_\pm T^\pm - \frac{1}{2}\partial_\pm^3 T^\pm\,. \tag{2.3.15}$$

Charges are then computed using the covariant phase space analysis of [37, 54], see also section 2.2. The general codimension-2 form relating charges between phase space solutions is

$$\begin{aligned} k_{BB} = \frac{\epsilon_{\partial\Sigma}}{\kappa_N^2}\tau_{[M}n_{N]}\Big[&2\xi^M\nabla^{[R}\delta G_R^{N]} + \xi^R\nabla^N\delta G_R^M + \frac{1}{2}\delta G_R^R\nabla^N\xi^M - \delta G_R^N\nabla^{[R}\xi^{M]} \\ &+ 8\delta\Phi\xi^N\nabla^M\Phi + \frac{e^{-8\Phi}}{4}\delta G_S^S H^{MNR}B_{RL}\xi^L + \frac{1}{2}\delta\left(e^{-8\Phi}H\right)^{MNR}B_{RL}\xi^L \\ &+ e^{-8\Phi}\left(H^{RM[N}\xi^{S]}\delta B_{RS} - \delta B^{MR}G^{NS}\mathcal{L}_\xi B_{SR}\right)\Big] - \frac{1}{2\kappa_N^2}\delta\left(e^{-8\Phi}\star H\right)\Lambda\,. \end{aligned} \tag{2.3.16}$$

[6] This is true up to the fact that the diffeomorphisms generate B_{ra} terms in the B-field, therefore taking us out of our chosen gauge. We must thus supplement the diffeomorphism by a compensating gauge transformation with parameter Λ_ξ adapted to cancel these terms. This can be done and the needed Λ_ξ is $\mathcal{O}(r^{-1})$, thus we neglect this technicality as it plays no role in what follows.

Here, n_N is the radial unit normal, τ_M the unit normal to a Cauchy slice (orthogonal to n_N), and $\epsilon_{\partial\Sigma}$ the volume form at the boundary of a Cauchy slice (these satisfy $\epsilon = \tau \wedge n \wedge \epsilon_{\partial\Sigma}$). As usual, δ denotes a variation between different solutions of the phase space. The term within square brackets corresponds to diffeomorphism charges, while the final piece is the gauge charge (clearly integrable). Evaluating in our phase space of solutions and for the asymptotic symmetry transformations, all charges turn out to be integrable and we get

$$L[T] = \frac{\ell}{\kappa_N^2} \int_0^{2\pi} \mathrm{d}\phi \left(Y_{++}(x^+)T^+(x^+) + Y_{--}(x^-)T^-(x^-) \right) , \tag{2.3.17}$$

for diffeomorphism charges and

$$N[\Lambda] = -\frac{C_0}{2\ell\kappa_N^2} \int_0^{2\phi} \mathrm{d}\phi \, \lambda_\phi \, , \tag{2.3.18}$$

for gauge transformations. For the gauge transformations we only get the global Kalb-Ramond charge (appearing for $\lambda_\phi = 1$, so $\Lambda = \mathrm{d}\phi + \dots$),[7] but we get two Virasoro towers from the diffeomorphisms, since the algebra computed via

$$\{L[T_1], L[T_2]\} = \delta_{T_2} \left(L[T_1] \right) , \tag{2.3.19}$$

is

$$i \left\{ L_m^{(\pm)}, L_n^{(\pm)} \right\} = (m-n) L_{m+n}^{(\pm)} + \frac{\ell\pi}{\kappa_N^2} m^3 \delta_{m+n,0} \, , \tag{2.3.20}$$

where $L_m^{(\pm)}$ is the charge associated to the mode $T^\pm(x^\pm) = e^{imx^\pm}$. As expected, we have obtained the well-known Brown-Henneaux result (with the same central charge).

2.4 Chiral Brown-Henneaux and CSS as T-dual phase spaces

In section 2.5, we will discuss how we can apply some notion of T-duality asymptotically to generate new boundary conditions from the ones of the previous section. Before doing so, however, we want to explore in this section a toy model of what T-duality can do to a certain set of boundary conditions and its associated solution space. In order to be as explicit as possible, we will work with a phase space which can be written in closed form, not only in an asymptotic expansion; and we will take it to have an exact Killing vector for all its field configurations. These conditions can be met starting from the well-known Bañados phase space for AdS_3 gravity [50, 51],

$$\mathrm{d}s^2 = \frac{\ell^2}{r^2}\mathrm{d}r^2 - r^2 \left(\mathrm{d}x^+ - \frac{\ell^2}{r^2} Y_{--}(x^-) \, \mathrm{d}x^- \right) \left(\mathrm{d}x^- - \frac{\ell^2}{r^2} Y_{++}(x^+) \, \mathrm{d}x^+ \right) . \tag{2.4.1}$$

[7] Clearly only the ϕ-independent part of λ_ϕ gives charges, but one may ask why we do not allow $\lambda_\phi = \lambda_\phi(t)$. The reason is that charges are not conserved in that case, as it is easy to check. This can be understood as follows. Conservation requires (off-shell) invariance of the action, which due to the form of Θ demands $\mathrm{d}\Lambda = 0$ at the boundary, so $\partial_\phi \lambda_t = \partial_t \lambda_\phi$. Expanding in modes, this equation forces the ϕ-independent part of λ_ϕ to be constant.

In order to extend these metrics to solutions of our theory (2.1.4), we must supplement them with an appropriate dilaton and Kalb-Ramond two-form [13]. To make contact with the previous section, we choose

$$B = \left(\frac{r^2}{C_0} + b(x^a) + \frac{\ell^4}{C_0 r^2} Y_{++}(x^+) Y_{--}(x^-)\right) \mathrm{d}x^+ \wedge \mathrm{d}x^- \,, \qquad \Phi = \frac{1}{4}\log\left(\frac{2}{C_0}\right) \,. \quad (2.4.2)$$

These are all solutions of our theory and they satisfy the boundary conditions (2.3.1), but notice that we are not claiming they are *all* the possible solutions satisfying such boundary conditions. In particular, matter fields being present, in the phase space of the previous section there will be backreacted solutions where the space is not everywhere locally AdS_3. These are not being considered now, so we are looking at a subset of the phase space in the previous section. As anticipated above, in order to apply T-duality we want to further restrict the field configurations so that we have an exact Killing vector throughout all of them, so we introduce a chiral version of the Bañados phase space in which we only keep left-moving excitations,

$$\mathrm{d}s^2 = \frac{\ell^2}{r^2}\mathrm{d}r^2 - r^2\left(\mathrm{d}x^+ - \frac{\ell^2}{r^2} y_{--}\,\mathrm{d}x^-\right)\left(\mathrm{d}x^- - \frac{\ell^2}{r^2} Y_{++}(x^+)\,\mathrm{d}x^+\right) \,, \qquad (2.4.3a)$$

$$B = \left(\frac{r^2}{C_0} + b(x^+) + \frac{\ell^4}{C_0 r^2} Y_{++}(x^+) y_{--}\right) \mathrm{d}x^+ \wedge \mathrm{d}x^- \,, \qquad (2.4.3b)$$

$$\Phi = \frac{1}{4}\log\left(\frac{2}{C_0}\right) \,. \qquad (2.4.3c)$$

where lowercase y_{--} emphasizes this is a constant now.

Since this is a subset of solutions within the phase space of the previous section, we can reuse most of the analysis done there to obtain the charges. Within the restricted phase space considered now, however, we will just get a single Virasoro tower, because we have frozen one of the chiral components. Let us be a bit more explicit. The asymptotic symmetry transformations are Brown-Henneaux diffeomorphisms (2.3.12) with $T^- = t^-$ a constant, as well as gauge transformations with asymptotic form $\lambda = \alpha(x^+)\mathrm{d}x^- + \tilde{\lambda}$, with $\tilde{\lambda}$ a closed one-form on the boundary. Notice that the algebra of these transformations mixes non-trivially. Indeed, expanding in modes using $x^+ \sim x^+ + 2\pi$,

$$T_n = e^{inx^+}\partial_+ + \dots \,, \qquad \alpha_m = e^{imx^+}\,\mathrm{d}x^- + \dots \,, \qquad (2.4.4)$$

symmetry transformations $\delta_{(\xi,\Lambda)}$ form the algebra

$$i[\delta_{(T_m,0)}, \delta_{(T_n,0)}] = (m-n)\delta_{(T_{m+n},0)} \,, \qquad (2.4.5a)$$

$$i[\delta_{(0,\alpha_m)}, \delta_{(0,\alpha_n)}] = 0 \,, \qquad (2.4.5b)$$

$$i[\delta_{(T_m,0)}, \delta_{(0,\alpha_n)}] = -n\delta_{(0,\alpha_{m+n})} \,. \qquad (2.4.5c)$$

At the level of the symmetry transformations, we have just found a Witt tower semi-directly with a $\mathfrak{u}(1)$ loop algebra (plus the zero mode ∂_-). It is interesting to observe though that not all of these are real symmetries of our theory, since the charges associated with gauge transformations vanish (other than the global charge). The argument is the same presented in the previous section: except for the zero mode, charges of the form (2.3.18) integrate to zero. We are thus left with a single Witt algebra associated to the left-moving sector.

The phase space has now an exact Killing vector, ∂_-, along which we can perform a T-duality transformation.[8] The simplest way to do this is by first going to string frame via $\tilde{G}_{MN} = e^{4\Phi} G_{MN}$, and then applying the standard Buscher rules presented in section 2.1.1. Finally, we go back to Einstein frame with the transformation $\widehat{G}_{MN} = e^{-4\widehat{\Phi}} \widehat{\tilde{G}}$.

Dropping tildes from now on, the result of this procedure in the chiral restriction of the Bañados phase space produces as dual solution space

$$\begin{aligned} \mathrm{d}s^2 = \frac{\hat{\ell}^2}{\rho^2}\mathrm{d}\rho^2 - \rho^2 \mathrm{d}x^+ \left(\mathrm{d}x^- - P(x^+)\mathrm{d}x^+\right) + \hat{\ell}^2 \left(L(x^+)(\mathrm{d}x^+)^2 + \Delta \left(\mathrm{d}x^- - P(x^+)\mathrm{d}x^+\right)^2\right) \\ - \frac{\Delta \hat{\ell}^4}{\rho^2} L(x^+)\, \mathrm{d}x^+ \left(\mathrm{d}x^- - P(x^+)\mathrm{d}x^+\right)\,, \end{aligned} \tag{2.4.6a}$$

$$B = \left(\frac{\rho^2}{\widehat{C}_0} + \frac{\Delta \hat{\ell}^4}{\widehat{C}_0}\frac{1}{\rho^2} L(x^+)\right) \mathrm{d}x^+ \wedge \mathrm{d}x^-\,, \tag{2.4.6b}$$

$$\Phi = \frac{1}{4}\log\left(\frac{2}{\widehat{C}_0}\right)\,, \tag{2.4.6c}$$

where we have rescaled the radial coordinate and defined some new constants and functions to ease the notation:

$$\begin{aligned} \rho &= \sqrt{\frac{2y_{--}}{C_0}}r\,, \qquad \hat{\ell}^2 = \frac{4\ell^2 y_{--}^2}{C_0^2}\,, \qquad \widehat{C}_0 = \frac{4y_{--}^2}{C_0}\,, \\ \Delta &= \frac{C_0^2}{4y_{--}}\,, \qquad P(x^+) = \frac{b(x^+)}{\ell^2}\,, \qquad L(x^+) = Y_{++}(x^+)\,. \end{aligned} \tag{2.4.7}$$

In this form, the above metrics can be easily recognized as those forming the CSS phase space [42] (supplemented with the appropriate B-field and dilaton to turn them into solutions of the theory (2.1.4)), so we have just shown that the chiral Bañados and CSS phase spaces are T-dual. Notice that $\Lambda = -C_0/\ell^2 = -\widehat{C}_0/\hat{\ell}^2$ and, to make contact with the standard CSS analysis, we are treating y_{--} (correspondingly, Δ) as a *fixed* constant. So, for each chiral Bañados phase space with a certain value of y_{--}, we get a corresponding CSS phase space with fixed Δ.[9]

[8] Technically, ∂_- may not be a spacelike direction throughout the whole spacetime, and it is certainly null asymptotically. It is then likely that a proper worldsheet definition of T-duality along this direction is not available. However, in this work we regard T-duality as a solution-generating technique in (super)gravity, so we will just apply Buscher rules, taking advantage of the fact that they relate solutions with an isometry to new solutions, irrespective of the string theory definition of the transformation.

[9] One could try to allow Δ to vary following the argument presented in the appendix of [42]. However, defining a phase space for all values of y_{--} and dealing with the whole set of duals still presents problems, because we would be varying also the B-field charge $\widehat{C}_0$ and AdS radius $\hat{\ell}$. We thus keep the example simple and discuss the duality at fixed y_{--} and Δ.

Asymptotic symmetry transformations are diffeomorphisms generated by vector fields of the form

$$\xi[\epsilon,\sigma] = -\frac{\rho}{2}\epsilon'(x^+)\partial_\rho + \left(\epsilon(x^+) + \mathcal{O}(\rho^{-2})\right)\partial_+ + \left(\sigma(x^+) + \mathcal{O}(\rho^{-2})\right)\partial_- \,, \tag{2.4.8}$$

and we do not have asymptotic gauge transformations of the B-field non-vanishing at the boundary because the duality has not produced a fluctuating $\mathcal{O}(\rho^0)$ term. The vector fields form a Witt algebra together with an Abelian loop algebra. In fact, the algebra is exactly (2.4.5), where T_m are modes of $\epsilon(x^+)$ and α_m are now modes of $\sigma(x^+)$, thus coming from diffeomorphisms instead of gauge transformations.[10] This is a manifestation of the well-known fact that gauge transformations of the B-field become diffeomorphisms after T-duality, and viceversa. It is intriguing however to note that now all transformations are real symmetries, since the charges computed using the covariant phase space method (2.3.16) become

$$\mathcal{L}[\epsilon] = \frac{\hat{\ell}}{\kappa_N^2}\int_0^{2\pi} \mathrm{d}\phi\, \epsilon(x^+)\left(L(x^+) - \Delta P^2(x^+)\right)\,, \tag{2.4.9a}$$

$$\mathcal{N}[\sigma] = 2\Delta\frac{\hat{\ell}}{\kappa_N^2}\int_0^{2\pi} \mathrm{d}\phi\, \sigma(x^+)P(x^+)\,, \tag{2.4.9b}$$

where, contrary to [42], we have not shifted the zero mode charge of σ. Using appropriate versions of (2.3.19), the algebra of charges becomes

$$i\left\{\mathcal{L}_m, \mathcal{L}_n\right\} = (m-n)\mathcal{L}_{m+n} + \frac{\hat{\ell}\pi}{\kappa_N^2}m^3\delta_{m+n,0}\,, \tag{2.4.10a}$$

$$i\left\{\mathcal{N}_m, \mathcal{N}_n\right\} = -4\Delta\frac{\hat{\ell}\pi}{\kappa_N^2}m\,\delta_{m+n,0}\,, \tag{2.4.10b}$$

$$i\left\{\mathcal{L}_m, \mathcal{N}_n\right\} = -n\,\mathcal{N}_{n+m}\,, \tag{2.4.10c}$$

where we have again expanded in modes $\epsilon_m(x^+) = e^{imx^+}$, $\sigma_m(x^+) = e^{imx^+}$.

Let us end this section with a brief summary and some comments about the result obtained. By restricting the phase space of solutions to a chiral subset possessing an exact Killing vector, (2.4.3), we can apply a T-duality transformation to all such solutions, obtaining an exact notion of T-dual phase space. This procedure has uncovered the surprising result that one (chiral) half of the Brown-Henneaux phase space is T-dual to the CSS phase space. Perhaps even more surprisingly, the algebra of symmetry transformations is the same in both cases, but this is not the case for the actual algebra of non-trivial charges. This can be traced back to the exchange between B-field gauge transformations and diffeomorphisms that T-duality produces. The Abelian tower comes from gauge transformations

[10] The invariance under T-duality of the asymptotic symmetry transformations we are observing here can be proven in full generality (without reference to any specific background, just assuming there is a $U(1)$ symmetry to dualize) by using the form of Buscher rules, as shown in the next subsection 2.4.1.

in the chiral Brown-Henneaux case, and most of these are trivial (i.e., they have vanishing charge). That same tower arises from diffeomorphisms in CSS which have non-trivial charges. This difference is particularly puzzling if we think of T-duality not as a solution generating technique in (super)gravity, but as an actual equivalence of stringy origin. In that philosophy, we would expect that the chiral Brown-Henneaux and CSS backgrounds define equivalent theories, and the symmetry algebras should then match. Of course, in a stringy setup one could question the consistency of unnaturally chopping off the allowed set of solutions to just a chiral half of the set of all asymptotically AdS_3 metrics (after all, the excluded backgrounds should also be valid stringy excitations), so we refrain from giving too much relevance to the mismatch in the toy model we just presented.

2.4.1 Asymptotic symmetries for an exact T-duality: The general case

We can, in spite of the previous discussion, ask ourselves what happens *in general* with the symmetries of a phase space with a $U(1)$ isometry after T-duality. In this subsection we advocate to this question: We will prove that the asymptotic symmetry transformations of a phase space with an exact $U(1)$ isometry are preserved under T-duality.

We start by recalling the general form of a metric, Kalb-Ramond and dilaton fields with a $U(1)$ isometry along the z direction using adapted coordinates (as shown in subsection 2.1.1).

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + e^{2C}\left(\mathrm{d}z + A_i\mathrm{d}x^i\right)^2\,, \tag{2.4.11a}$$

$$B_{ij} = \mathcal{B}_{ij} + B_{[i}A_{j]}\,, \qquad B_{iz} = B_i\,, \tag{2.4.11b}$$

$$\Phi = \phi + \frac{1}{2}C\,, \tag{2.4.11c}$$

where the coordinates are split as $x^N = (x^i, z)$. The fields g_{ij}, C, A_i, B_i, $\mathcal{B}_{ij}$ and ϕ depend only on x^i, and the seemingly unnatural definitions of ϕ and $\mathcal{B}_{ij}$ simplify later expressions. In particular, Buscher rules in string frame take a simple form. By interchanging

$$A_i \leftrightarrow B_i\,, \quad \text{and} \quad C \leftrightarrow -C\,, \tag{2.4.12}$$

we get a new solution of the equations of motion.

On the other hand, under a symmetry transformation generated by ξ and Λ the fields in (2.4.11) transform as

$$
\begin{aligned}
\delta_{\xi,\Lambda} g_{ij} =& \mathcal{L}_\xi g_{ij} - A_{(i}g_{j)l}\partial_z\xi^l \,, && (2.4.13a)\\
\delta_{\xi,\Lambda} A_i =& \mathcal{L}_\xi A_i + \partial_i\xi^z - A_i\left(\partial_z\xi^z + A_j\partial_z\xi^j\right) + e^{-2C}g_{ij}\partial_z\xi^j \,, && (2.4.13b)\\
\delta_{\xi,\Lambda} C =& \mathcal{L}_\xi C + \partial_z\xi^z + A_i\partial_z\xi^i \,, && (2.4.13c)\\
\delta_{\xi,\Lambda} B_i =& \mathcal{L}_\xi B_i + B_{ij}\partial_z\xi^j + B_i\partial_z\xi^z + \partial_i\Lambda_z - \partial_z\Lambda_i \,, && (2.4.13d)\\
\delta_{\xi,\Lambda} \mathcal{B}_{ij} =& \mathcal{L}_\xi \mathcal{B}_{ij} + B_{[i}\left(\partial_{j]}\xi^z - e^{-2C}g_{j]l}\partial_z\xi^l\right) + B_{[i}A_{j]}\left(\partial_z\xi^z + A_l\partial_z\xi^l\right) \\
& + 2\partial_{[i}\Lambda_{j]} - A_{[j}\left(B_{i]}\partial_z\xi^z + B_{i]l}\partial_z\xi^l + \partial_{i]}\Lambda_z - \partial_z\Lambda_{i]}\right) \,, && (2.4.13e)\\
\delta_{\xi,\Lambda} \phi =& \mathcal{L}_\xi\phi - \frac{1}{2}\left(\partial_z\xi^z + A_i\partial_z\xi^i\right) \,, && (2.4.13f)
\end{aligned}
$$

where we have split the reducibility parameters as $\xi = \xi^i\partial_i + \xi^z\partial_z$, $\Lambda = \Lambda_i \mathrm{d}x^i + \Lambda_z \mathrm{d}z$. Given that we are working on a phase space with a $U(1)$ isometry, we should impose that the parameters ξ^i and Λ have not dependence on z, and that $\xi^z = \zeta(x^i) + \alpha z$. This simplifies the transformation laws, obtaining

$$
\begin{aligned}
\delta_{\xi,\Lambda} g_{ij} =& \mathcal{L}_\xi g_{ij} \,, && (2.4.14a)\\
\delta_{\xi,\Lambda} A_i =& \mathcal{L}_\xi A_i + \partial_i\zeta - \alpha A_i \,, && (2.4.14b)\\
\delta_{\xi,\Lambda} C =& \mathcal{L}_\xi C + \alpha \,, && (2.4.14c)\\
\delta_{\xi,\Lambda} B_i =& \mathcal{L}_\xi B_i + \partial_i\Lambda_z + \alpha B_i \,, && (2.4.14d)\\
\delta_{\xi,\Lambda} \mathcal{B}_{ij} =& \mathcal{L}_\xi \mathcal{B}_{ij} + 2\partial_{[i}\Lambda_{j]} + B_{[i}\partial_{j]}\zeta + A_{[i}\partial_{j]}\Lambda_z \,, && (2.4.14e)\\
\delta_{\xi,\Lambda} \phi =& \mathcal{L}_\xi\phi - \frac{1}{2}\alpha \,, && (2.4.14f)
\end{aligned}
$$

At this point is easy to see that after applying the transformations (2.4.12) we get the same transformation laws for each of the fields, up to interchanging $\zeta \leftrightarrow \Lambda_z$ and $\alpha \leftrightarrow -\alpha$ (and recalling the fact the the dilaton has shift invariance).

The solution to the asymptotic symmetry parameters preserving some particular boundary conditions is then the same for one phase space and its T-dual, after the interchange of the gauge parameter and the z component of the diffeomorphism generator. It is worth noticing that this is only proven at the level of the reducibility parameters and not at the level of the charges. Indeed, symmetry transformations with vanishing charge can acquire a charge in the T-dual phase space, as it is shown in section 2.4.

2.5 Going flat: A phase space for the black string from T-duality

After having discussed how T-duality may affect the asymptotic structure of a theory in a controlled and simple setup, we aim now to use the well-understood Brown-Henneaux boundary conditions (2.3.1), together with some asymptotic notion of T-duality, to generate a new set of boundary conditions. At the very least, this will allow us to obtain boundary conditions defining a phase space which contains the three-dimensional black strings. Indeed, black strings are T-dual to BTZ black holes [13]. Since BTZ black holes are included within the Brown-Henneaux phase space, the dual black strings will be included in any notion of T-dual phase space we are able to define. More broadly, we believe the construction can serve as a blueprint for how to generate new boundary conditions using T-duality. This is interesting in its own right, since in many cases T-duality heavily affects the asymptotic structure, and thus we can use well understood boundary conditions (here, Brown-Henneaux AdS_3 asymptotics) to generate novel ones (here, the ones containing black strings, which are asymptotically flat in some sense to be made precise below).

2.5.1 T-dual boundary conditions

We start our journey from (2.3.1). These boundary conditions possess

$$\eta = \frac{\partial_+ - \partial_-}{\ell} = \frac{\partial_\phi}{\ell}\,, \tag{2.5.1}$$

as an exact Killing vector of the leading components. This is the condition we demand to apply our asymptotic notion of T-duality, expecting this will produce dual fields whose leading components also provide a solution (the situation is actually slightly subtler, as we will shortly discuss). A direct application of Buscher rules (**??**) gives us the following dual boundary conditions (in Einstein frame):

$$\mathrm{d}s^2 = \left(1 + \frac{F(x^a)}{\hat{r}} + \dots\right)\mathrm{d}\hat{r}^2 + \hat{r}^2\left(M_{ab} + \frac{1}{\hat{r}}Z_{ab} + \dots\right)\mathrm{d}x^a\,\mathrm{d}x^b\,, \tag{2.5.2a}$$

$$B = \frac{1}{2}\left(\frac{\tilde{\beta}(x^a)}{\hat{r}} + \dots\right)\mathrm{d}z\wedge\mathrm{d}w\,, \tag{2.5.2b}$$

$$e^{4\Phi} = \frac{\hat{r}_0^2}{\hat{r}^2}\left(1 + \frac{\psi(x^a)}{\hat{r}} + \dots\right)\,, \tag{2.5.2c}$$

where, to simplify the notation, we have redefined our coordinates as

$$\hat{r} = \frac{r^2}{C_0\ell}\,, \qquad \frac{z}{\sqrt{C_0}} = x^+ + x^- = 2\frac{t}{\ell}\,, \qquad \frac{w}{\sqrt{C_0}} = -x^+ + x^- = -2\phi\,, \tag{2.5.3}$$

and $\hat{r}_0 = \ell/\sqrt{2C_0}$. New fluctuations are related to the old ones in a way which will not be very relevant for us, e.g.,

$$F = \frac{2\ell}{C_0}\left(Y_{++} + Y_{--} - Y_{+-}\right)\,, \qquad \psi = -\frac{2\ell}{C_0}\left(Y_{++} + Y_{--} - \frac{3}{2}Y_{+-}\right)\,, \qquad \tilde{\beta} = \frac{\ell^3}{2C_0^2}\left(Y_{++} - Y_{--}\right)\,, \tag{2.5.4}$$

and these are all functions of $(x^a) = (z, w)$. However, two facts that naturally come out of the duality are crucial. The first one is the form of the fluctuating leading metric M_{ab},

$$(M_{ab}) = \begin{pmatrix} A(x^a) & -1/2 \\ -1/2 & 0 \end{pmatrix}, \qquad A = \frac{2b}{\ell^2} + \frac{2Y_{+-}}{C_0}. \tag{2.5.5}$$

Notice that subleading terms in the original boundary conditions became part of the leading pieces after dualizing, in particular through $A(z, w)$. Thus, it is not true that the leading pieces of (2.5.2) provide a valid solution, and going on-shell will later impose non-trivial restrictions on $A(z, w)$ – see (2.5.23). The other important fact is that Z_{ww}, the leading piece in $\mathrm{d}w^2$ (since $M_{ww} = 0$), is fixed to be a constant

$$(Z_{ab}) = \begin{pmatrix} Z_{zz}(x^a) & Z_{zw}(x^a) \\ Z_{zw}(x^a) & \ell/4 \end{pmatrix}. \tag{2.5.6}$$

Z_{zz} and Z_{zw} can be obtained as combinations of the old fluctuations from the Buscher transformations, although it will not be needed in the following discussion. Z_{zw} is found to be linear in the fluctuations Y_{ab}, while Z_{zz} is quadratic in Y_{ab} and includes also further subleading pieces from the Brown-Henneaux boundary conditions, (2.3.1). This different behavior can be traced back to the manifestly different treatment Buscher rules do of the dualizing coordinate (w or ϕ) versus the remaining directions.

As promised, the three-dimensional black string solutions introduced in [24] can be shown to satisfy these boundary conditions. However, due to the nature of our construction, they do so in their form obtained by T-dualizing the BTZ black holes [13]. We refer the reader to appendix **??** for details and we just quote here the main results. BTZ black holes correspond to solutions with constant $Y_{++} = L_+$ and $Y_{--} = L_-$ in the boundary conditions (2.3.1). Buscher rules transform them to

$$\mathrm{d}s^2 = g^4(\hat{r})\mathrm{d}\hat{r}^2 + \hat{r}^2 g^2(\hat{r}) \left[\left(\frac{2b}{\ell^2} + \frac{4\ell L_+ L_-}{C_0^2 \hat{r}} \right) \left(1 + \frac{b}{2\ell\hat{r}} \right) \mathrm{d}z^2 + \frac{\ell}{4\hat{r}} \mathrm{d}w^2 - \left(1 + \frac{b}{\ell\hat{r}} + \frac{\ell^2 L_+ L_-}{C_0^2 \hat{r}^2} \right) \mathrm{d}z\mathrm{d}w \right], \tag{2.5.7a}$$

$$B = \frac{\ell^3 (L_+ - L_-)}{4C_0^2 \hat{r} g^2(\hat{r})} \mathrm{d}z \wedge \mathrm{d}w, \tag{2.5.7b}$$

$$e^{4\Phi} = \frac{\hat{r}_0^2}{\hat{r}^2 g^4(\hat{r})}, \tag{2.5.7c}$$

where we are using coordinates (2.5.3), and $g^2(\hat{r})$ is

$$g^2(\hat{r}) = 1 + \frac{\ell}{C_0 \hat{r}}(L_+ + L_-) + \frac{\ell^2}{C_0^2 \hat{r}^2} L_+ L_-. \tag{2.5.8}$$

Note also that $b = b(t) = b(z)$, since any dependence on ϕ is forbidden by the requirement to have an exact angular Killing vector in the BTZ background, needed to apply T-duality.

All these solutions satisfy the dual boundary conditions (2.5.2), and they do so for constant subleading pieces satisfying $F = -\psi$. One can take these backgrounds to the more familiar black string form via the coordinate changes

$$\mathrm{d}w = \mathrm{d}W + \left(\frac{2b(z)}{\ell^2} - \frac{2(L_+ + L_-)}{C_0}\right) \mathrm{d}z \,, \tag{2.5.9}$$

and

$$R = \hat{r} g^2(\hat{r}) \,, \quad z = \frac{\sqrt{C_0}(T+X)}{2(L_+ L_-)^{1/4}} \,, \quad W = \frac{(\sqrt{L_+} + \sqrt{L_-})^2 T + (\sqrt{L_+} - \sqrt{L_-})^2 X}{\sqrt{C_0}(L_+ L_-)^{1/4}} \,, \tag{2.5.10}$$

after which the solution becomes

$$\mathrm{d}s^2 = \frac{\mathrm{d}R^2}{\left(1 - \frac{\mathcal{M}}{R}\right)\left(1 - \frac{\mathcal{Q}^2}{\mathcal{M}R}\right)} + R^2 \left[-\left(1 - \frac{\mathcal{M}}{R}\right) \mathrm{d}T^2 + \left(1 - \frac{\mathcal{Q}^2}{\mathcal{M}R}\right) \mathrm{d}X^2 \right] \,, \tag{2.5.11a}$$

$$B = -\frac{\hat{r}_0^2 \mathcal{Q}}{R} \mathrm{d}T \wedge \mathrm{d}X \,, \tag{2.5.11b}$$

$$e^{4\Phi} = \frac{\hat{r}_0^2}{R^2} \,, \tag{2.5.11c}$$

with

$$\mathcal{M} = \frac{\ell}{C_0} \left(\sqrt{L_+} + \sqrt{L_-}\right)^2 \,, \qquad \mathcal{Q}^2 = \frac{\ell^2}{C_0^2} \left(L_+ - L_-\right)^2 \,. \tag{2.5.12}$$

Note also that the solution with all fluctuations turned off in the boundary conditions (2.5.2) (i.e., the dual of a massless BTZ black-hole) is

$$\mathrm{d}s^2 = \mathrm{d}\hat{r}^2 + \hat{r}^2 \left(-\mathrm{d}z\mathrm{d}w + \frac{\ell}{4\hat{r}} \mathrm{d}w^2 \right) \,, \tag{2.5.13}$$

which is a plane wave in the presence of a dilaton $e^{4\Phi} = \hat{r}_0^2/\hat{r}^2$. The fact that this cannot be mapped to the $\mathcal{M} = \mathcal{Q} = 0$ black string was already noticed in [13], and the reason can be traced back to the transformation (2.5.10) being ill-defined for extremal solutions with either L_+ or L_- vanishing.[11] Therefore, the precise statement is that our boundary conditions include all solutions of the form (2.5.7), many of which can be mapped to non-extremal black strings in the standard form (2.5.11). Extremal black strings, defined by setting $\mathcal{M} = |\mathcal{Q}|$ in (2.5.11), are not part of our configuration space.

It is interesting to note that our boundary conditions (2.5.2) share some similarities with the CSS metrics discussed in the previous section, (2.4.6). In particular, the coordinate in which we dualize (x^- there, w here) has a vanishing leading piece in the metric, and the first subleading is fixed ($\hat{\ell}^2\Delta$ there, $\ell/4$ here). This is the first of many similarities with CSS we will encounter, but note that the different asymptotic behaviour of the Killing

[11] Another notable exceptional case is the solution obtained by dualizing empty AdS_3, for which the transformation (2.5.10) also does not make sense. Such dual, included in the boundary conditions (2.5.2), is a product of time and the two-dimensional Euclidean black hole [13, 55].

vector used to dualize ($R_\eta^2 \sim r^2$ now) has completely changed the asymptotic curvature. In particular, we are no longer dealing with locally AdS$_3$ metrics, as shown by the large-$\hat{r}$ expansion of the Ricci scalar

$$R = \frac{\mathcal{R}-2}{\hat{r}^2} + \mathcal{O}(\hat{r}^{-3})\,, \tag{2.5.14}$$

where $\mathcal{R} = 4\partial_w^2 A$ is the Ricci scalar of the 2d metric M_{ab}. The boundary conditions are asymptotically flat in the sense that this curvature scalar decays as $\hat{r}^{-2}$, but (2.5.2) are not standard asymptotically flat boundary conditions (in particular, for $A = 0$, the Killing vectors $\partial_z \pm \partial_w$ have diverging norm at large $\hat{r}$). It is also important to keep in mind for the forthcoming analysis that the fluctuating $A(x^a)$ forbids a clear identification of timelike and spacelike directions in the asymptotic region. Even though z comes from the time t, we have $\partial_z^2 \sim \hat{r}^2 A$, so z is actually a spacelike direction if $A > 0$. We will come back to this point later when integrating charges, since it will be relevant to pick our Cauchy surface of integration.

Let us briefly discuss the variational problem with the boundary conditions (2.5.2). The boundary term (2.3.3) must be slightly modified to

$$\hat{\mathcal{B}} = \frac{1}{\kappa_N^2}\left(K - \frac{e^{2\Phi}}{\hat{r}_0}\right)\epsilon_{\partial\mathcal{M}}, \tag{2.5.15}$$

so that varying the action with this modified boundary term produces

$$\begin{aligned}\left(\Theta + \delta\hat{\mathcal{B}}\right)\Big|_{\partial\mathcal{M}} = -\frac{\epsilon_{\partial\mathcal{M}}}{2\kappa_N^2}\Bigg[&\left(K^{MN} - \left(K - \frac{e^{2\Phi}}{\ell}\right)\gamma^{MN}\right)\delta G_{MN}\\ &+8\left(n^R\partial_R\Phi + \frac{e^{2\Phi}}{2\hat{r}_0}\right)\delta\Phi\Bigg] + \mathrm{d}C\,,\end{aligned} \tag{2.5.16}$$

with $\mathrm{d}C$ as in (2.3.5) an irrelevant total derivative.[12] Using the boundary conditions gives

$$\left(\Theta + \delta\hat{\mathcal{B}}\right)\Big|_{\partial\mathcal{M}} = \frac{\ell}{8\kappa_N^2}\delta A\,\mathrm{d}z\wedge\mathrm{d}w + \mathrm{d}C\,. \tag{2.5.17}$$

This structure is exactly the same appearing in the CSS construction [42]: the non-vanishing contribution can be written as $Z_{ab}\delta M^{ab}$, with M^{ab} the inverse of M_{ab}. In order to have a well-defined variational problem, we thus apply the same trick presented in [42] and include an extra, non-covariant piece in the action

$$S \longrightarrow S - \frac{1}{\kappa_N^2}\int_{\partial\mathcal{M}}\mathrm{d}^2x\,\sqrt{-\det M_{ab}}\,\frac{\ell}{4}A(z)\,, \tag{2.5.18}$$

such that now $\delta S = 0$ with the boundary conditions (2.5.2).

[12] It would be interesting to find a T-duality invariant boundary term able to produce a well-defined variational problem with both boundary conditions, (2.3.1) and (2.5.2). Also, we have dropped the piece in $\mathcal{B}$ fixing the charge, $(e^{-8\Phi}\star H)B$. While it can be included, we believe it is more natural to think of the boundary conditions (2.5.2) in a "grand canonical" ensemble, with B fixed to zero at the boundary. One could actually add to (2.5.2) a piece $B_{zw} = (\tilde{b} + \tilde{\beta}/\hat{r} + \ldots)/2$, such that B goes to a fixed $\tilde{b}$ at the boundary. The following analysis still holds, so we prefer to leave $\tilde{b} = 0$ as this value naturally comes out of the duality.

2.5.2 Asymptotic symmetry transformations and charges

The asymptotic symmetry transformations preserving the boundary conditions (2.5.2) are diffeomorphisms generated by

$$\begin{aligned}\xi[R,Q,T,S] =& \ell\left(f(z,w)+\dots\right)\partial_{\hat{r}} + \left(T(z) - \frac{2\ell}{\hat{r}}\partial_w f + \dots\right)\partial_z \\ &+ \left(S(z) - wT'(z) - \frac{2\ell}{\hat{r}}\left(\partial_z f + 2A\partial_w f\right) + \dots\right)\partial_w\,,\end{aligned} \tag{2.5.19}$$

where we have written the leading $\hat{r}$ component as

$$f(z,w) = R(z) + wQ(z) + \frac{w^2}{8}T'(z)\,, \tag{2.5.20}$$

and the form of $\xi^{\hat{r}}$ comes from imposing $\delta Z_{ww} = 0$. Since we are now fixing the boundary value of the B-field, the allowed gauge transformations have $\Lambda = \tilde{\lambda} + \mathcal{O}(\hat{r}^{-1})$ with $\tilde{\lambda}$ a closed one-form on the boundary. Much like with Brown-Henneaux boundary conditions, this will only produce the global charge, so we will not discuss it further.

Before computing the charges associated to these transformations and the corresponding symmetry algebra, we need to go on-shell and impose the conditions derived from the equations of motion, (2.1.6). From the dilaton equation of motion (note that the cosmological constant is $\Lambda_0 = -C_0/\ell^2 = -1/(2\hat{r}_0^2)$)

$$F + 2\psi + Z + M^{ab}D_aD_b\psi = 0\,, \tag{2.5.21}$$

where D_a is the covariant derivative associated to the 2d metric M_{ab}, and $Z = M^{ab}Z_{ab}$. The B-field equation simply requires to have constant charge, so $\partial_M\tilde{\beta} = 0$. Finally, the metric equation of motion gives at leading order in the radial part the requirement that M_{ab} must be locally flat

$$\mathcal{R} = 4\partial_w^2 A = 0\,, \tag{2.5.22}$$

so we get

$$A(z,w) = A_0(z) + wA_1(z)\,. \tag{2.5.23}$$

Subleading pieces give the following three conditions

$$D^2Z - D^aD^bZ_{ab} = F + 2\psi + Z\,, \tag{2.5.24a}$$

$$M^{cd}D_cZ_{da} = \partial_a(F + \psi + Z)\,, \tag{2.5.24b}$$

$$D_aD_bF = (D^2F)M_{ab}\,, \tag{2.5.24c}$$

where indices in derivatives D_a were raised with M^{ab}. Solving all these equations in full generality is fortunately not needed for our purposes. We can just impose them as conditions when we are evaluating on-shell quantities, such as the charges we are about to compute. In particular, the explicit form of the second equation in (2.5.24) is going to be relevant. The z-component is

$$2\partial_w\left(Z_{zz}+2AZ_{zw}\right)=-\partial_z\left(F+\psi-2Z_{zw}-\frac{\ell}{2}A\right)\,, \tag{2.5.25}$$

while the w-component imposes

$$\partial_w\left(F+\psi-2Z_{zw}+\frac{\ell}{2}A\right)=0\,. \tag{2.5.26}$$

There is a final point we need to address before obtaining the charges associated with the asymptotic symmetry transformations (2.5.19): we need to fix the Cauchy slice used to compute them (at least asymptotically close to the boundary, where charges will be evaluated). Naively, we could think that surfaces of constant z are the natural choice, since z is directly related to t before the duality transformation. However, as indicated before, these surfaces are asymptotically null, so not really well-suited for the standard Hamiltonian analysis. Since we are using T-duality merely as a solution generating technique, we will not relate the choices after the transformation with those made before: we regard the theory with the boundary conditions (2.5.2) as a well defined entity of its own, not tied to the structure before the duality. The form of the boundary conditions suggests a better choice of Cauchy slices: those with constant w, so that we will perform integrals over z. Compactifying this spatial coordinate to regulate IR issues, we will be able to expand functions such as $A_0(z)$ in Fourier modes, as it is conventional.

Note that there are still potential issues when $A(z,w)<0$, since in that case constant w surfaces are asymptotically timelike (alternatively, $\partial_z^2\sim\hat{r}^2A(z,w)$ becomes negative). Nevertheless, a choice must be made, because having a fluctuating leading boundary metric given by M_{ab} forbids the identification of spacelike surfaces consistently throughout the whole phase space, and we believe the results that will be presented below justify the choice of constant w surfaces as surfaces of integration. Incidentally, it is worth mentionning that this is also very similar to the choice made in the CSS construction [42]. There, the equivalent to (z,w) coordinates are light-cone coordinates on the boundary, and Cauchy slices are taken to have $z+w=$ constant. It is straightforward to generalize our discussion to this alternative choice of Cauchy slice, and our results still hold. But it must be noted that also in CSS the leading boundary metric has a fluctuating piece, $P(x^+)$ using our notation in (2.4.6), and the chosen $2t=x^++x^-=$ constant surfaces are spacelike if and only if $P(x^+)>-1$.

We can now compute the charges associated with the asymptotic symmetry transformations. Using the general form presented in (2.3.16), we obtain integrable diffeomorphism charges of the form:

$$\mathcal{T} = \frac{1}{2\kappa_N^2}\int \mathrm{d}z\,\left[(Z_{zz}+2AZ_{zw})\,T - \frac{w}{2}\left(F+\psi-2Z_{zw}-\frac{\ell}{2}A_0\right)T'\right], \tag{2.5.27a}$$

$$\mathcal{S} = \frac{1}{4\kappa_N^2}\int \mathrm{d}z\,\left(F+\psi-2Z_{zw}+\frac{\ell}{2}A\right)S\,, \tag{2.5.27b}$$

$$\mathcal{R} = -\frac{\ell}{\kappa_N^2}\int \mathrm{d}z\,A_1\,R\,, \tag{2.5.27c}$$

$$\mathcal{Q} = \frac{\ell}{\kappa_N^2}\int \mathrm{d}z\,A_0\,Q\,, \tag{2.5.27d}$$

where calligraphic letters denote the charges associated with $T(z)$, $S(z)$, $R(z)$, and $Q(z)$. Verifying that these charges are conserved (i.e., they do not depend on the fixed value of w used for the integrals) is a very non-trivial consistency check of our result. This is obvious for $\mathcal{R}$ and $\mathcal{Q}$, since they do not depend on w at all. For $\mathcal{S}$, the implicit dependence through the phase space functions combines to give a w-independent quantity on-shell, as equation (2.5.26) shows. For $\mathcal{T}$, the implicit and explicit dependences can be shown to cancel by virtue of equations (2.5.25), (2.5.26), and the fact that total derivatives integrate to zero over the full boundary cycle.

The symmetry algebra of these charges follows from

$$\{\mathcal{P}_1,\mathcal{P}_2\} = \delta_{\xi_2}\mathcal{P}_1\,, \tag{2.5.28}$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are any two of the charges in (2.5.27), and ξ_2 is the asymptotic Killing vector of the form (2.5.19) associated with the charge $\mathcal{P}_2$. In essence, brackets are given by the variation under the asymptotic Killing vectors of the charges.

These variations can be deduced from those of the basic phase space functions appearing in the expressions for the charges. The variation of such functions is

$$\delta_\xi A_0 = TA_0' + 2T'A_0 + SA_1 - S'\,, \tag{2.5.29a}$$

$$\delta_\xi A_1 = TA_1' + T'A_1 + T''\,, \tag{2.5.29b}$$

$$\delta_\xi F = T\partial_z F + (S-wT')\partial_w F\,, \tag{2.5.29c}$$

$$\delta_\xi \psi = T\partial_z\psi + (S-wT')\partial_w\psi - 2\ell R - 2\ell w Q - \frac{\ell}{4}w^2T'\,, \tag{2.5.29d}$$

$$\begin{aligned}\delta_\xi Z_{zz} &= T\partial_z Z_{zz} + 2T'Z_{zz} + (S-wT')\partial_w Z_{zz} + 2(S'-wT'')Z_{zw}\\ &\quad + 2\ell\partial_z^2 f - 2\ell\partial_w A\partial_z f + 2\ell\partial_z A\partial_w f - 4\ell A\partial_w A\partial_w f + 2\ell A f\,,\end{aligned} \tag{2.5.29e}$$

$$\delta_\xi Z_{zw} = T\partial_z Z_{zw} + (S-wT')\partial_w Z_{zw} + \frac{\ell}{4}S' + 2\ell\partial_w A\partial_w f + 2\ell\partial_z\partial_w f - \ell f\,, \tag{2.5.29f}$$

where primes denote derivatives with z, and $T(z)$, $S(z)$, $R(z)$ and $Q(z)$ are the functions defining the asymptotic Killing vectors (2.5.19). To compute the algebra of charges, it is useful to read the following combined variation:

$$\delta_\xi \left(F + \psi - 2Z_{zw} + \frac{\ell}{2} A \right) = T\partial_z \left(F + \psi - 2Z_{zw} + \frac{\ell}{2} A \right) + \ell T' A_0 - 4\ell Q A_1 - \ell S' - 4\ell Q' \,, \tag{2.5.30}$$

where we have simplified the result using equation (2.5.26). The reader may also find useful when computing $\{\mathcal{T}_1, \mathcal{T}_2\}$ the following variations under the action of an asymptotic Killing vector for which only $T(z)$ is turned on

$$\delta_T (Z_{zz} + 2AZ_{zw}) = T\partial_z (Z_{zz} + 2AZ_{zw}) + 2T' (Z_{zz} + 2AZ_{zw}) - wT'\partial_w (Z_{zz} + 2AZ_{zw}) + \frac{\ell}{2} wT'\partial_z A + \frac{\ell}{2} wT'' \left(A - \frac{w}{2} A_1 \right) + \frac{\ell}{4} w^2 T''' \,, \tag{2.5.31a}$$

$$\delta_T \left(F + \psi - 2Z_{zw} - \frac{\ell}{2} A_0 \right) = T\partial_z \left(F + \psi - 2Z_{zw} - \frac{\ell}{2} A_0 \right) - \ell T' \left(A - \frac{w}{2} A_1 \right) - \frac{1}{2} \ell w T'' \,. \tag{2.5.31b}$$

Now, expanding in modes the functions appearing in the vectors,[13] we get

$$i\{\mathcal{T}_m, \mathcal{T}_n\} = (m-n)\mathcal{T}_{m+n} \,, \qquad i\{\mathcal{T}_m, \mathcal{S}_n\} = -(m+n)\mathcal{S}_{m+n} + \frac{1}{4} m \mathcal{Q}_{m+n} \,, \tag{2.5.32a}$$

$$i\{\mathcal{T}_m, \mathcal{Q}_n\} = (m-n)\mathcal{Q}_{m+n} \,, \qquad i\{\mathcal{T}_m, \mathcal{R}_n\} = -n\mathcal{R}_{m+n} - i\frac{2\pi\ell}{\kappa_N^2} m^2 \delta_{m+n,0} \,, \tag{2.5.32b}$$

$$i\{\mathcal{S}_m, \mathcal{S}_n\} = -\frac{\pi\ell}{2\kappa_N^2} m \delta_{m+n,0} \,, \qquad i\{\mathcal{S}_m, \mathcal{Q}_n\} = i\mathcal{R}_{m+n} - \frac{2\pi\ell}{\kappa_N^2} m \delta_{m+n,0} \,, \tag{2.5.32c}$$

with the remaining brackets vanishing,

$$\{\mathcal{S}_m, \mathcal{R}_n\} = \{\mathcal{R}_m, \mathcal{R}_n\} = \{\mathcal{R}_m, \mathcal{Q}_n\} = \{\mathcal{Q}_m, \mathcal{Q}_n\} = 0 \,. \tag{2.5.33}$$

In order to interpret this algebra, note that the $\mathcal{T}_m$ generate a Witt tower (i.e., a Virasoro algebra with $c = 0$). The remaining charges can be associated with fields of definite weight if we replace $\mathcal{S}_m$ by

$$\bar{\mathcal{S}}_m = \mathcal{S}_m - \frac{1}{8} \mathcal{Q}_m \,, \tag{2.5.34}$$

in which case brackets with $\mathcal{T}_m$ become

$$i\{\mathcal{T}_m, \mathcal{Q}_n\} = (m-n)\mathcal{Q}_{m+n} \,, \tag{2.5.35a}$$

$$i\{\mathcal{T}_m, \mathcal{R}_n\} = -n\mathcal{R}_{m+n} - i\frac{2\pi\ell}{\kappa_N^2} m^2 \delta_{m+n,0} \,, \tag{2.5.35b}$$

$$i\{\mathcal{T}_m, \bar{\mathcal{S}}_n\} = -(m+n)\bar{\mathcal{S}}_{m+n} \,. \tag{2.5.35c}$$

[13] Mode expansions are $S_m \sim e^{imz}\partial_w$ and similarly for the other asymptotic Killing vectors. Note that this sets the periodicity $z \sim z + 2\pi$. This is not a restriction since rescaling z can be mapped to a rescaling of the parameters in our boundary conditions.

$\mathcal{Q}$, $\mathcal{R}$ and $\bar{\mathcal{S}}$ have thus weight 2, 1, and 0 respectively; and we get a central extension between $\mathcal{T}$ and $\mathcal{R}$. After the redefinition we have $i\{\bar{\mathcal{S}}_m, \bar{\mathcal{S}}_n\} = 0$, so the only remaining non-trivial bracket is

$$i\{\bar{\mathcal{S}}_m, \mathcal{Q}_n\} = i\mathcal{R}_{m+n} - \frac{2\pi\ell}{\kappa_N^2} m\delta_{m+n,0}\,. \tag{2.5.36}$$

The generators $\{\bar{\mathcal{S}}_m, \mathcal{Q}_m, \mathcal{R}_m\}$ themselves form a subalgebra which can be identified by looking at the zero modes, for which the only non-vanishing bracket is

$$i\{\bar{\mathcal{S}}_0, \mathcal{Q}_0\} = i\mathcal{R}_0\,. \tag{2.5.37}$$

The Hermitian generators $\{\bar{\mathcal{S}}_0, \mathcal{Q}_0, \mathcal{R}_0\}$ thus satisfy the commutation relations of the three-dimensional Heisenberg algebra, with $\bar{\mathcal{S}}_0$ and $\mathcal{Q}_0$ acting as "position" and "momentum" operators, and $\mathcal{R}_0$ being the central element. It is possible to build a loop algebra on top of this, following the standard procedure [56]. The algebra of the $\bar{\mathcal{S}}$, $\mathcal{Q}$ and $\mathcal{R}$ towers is a central extension of the result of this construction, namely

$$i\{J^a_m, J^b_n\} = if^{ab}{}_c J^c_{m+n} + \frac{2\pi\ell}{\kappa_N^2} g^{ab} m\delta_{m+n,0}\,, \tag{2.5.38}$$

where $a, b, c \in \{\bar{\mathcal{S}}, \mathcal{Q}, \mathcal{R}\}$ label the different towers of generators, $f^{ab}{}_c$ are the structure constants of the Heisenberg algebra (2.5.37) ($f^{\bar{\mathcal{S}}\mathcal{Q}}{}_{\mathcal{R}} = -f^{\mathcal{Q}\bar{\mathcal{S}}}{}_{\mathcal{R}} = 1$), and $g^{\bar{\mathcal{S}}\mathcal{Q}} = g^{\mathcal{Q}\bar{\mathcal{S}}} = -1$ are the non-zero components of g^{ab}. Notice that g^{ab} is *not* the Killing metric of the three-dimensional Heisenberg algebra (which would be trivial). This is a consequence of the algebra (2.5.37) not being semisimple: in those cases, the central extension is not necessarily proportional to the Killing metric (e.g., a $\mathfrak{u}(1)$ algebra has a trivial Killing metric, but the loop algebra built from it admits a non-trivial central extension).

To summarize, the asymptotic symmetry algebra derived from the boundary conditions (2.5.2) has the form of a Witt tower (without central extension) plus three towers of weights two, one and zero respectively, which together form a central extension of the loop algebra constructed from the Heisenberg algebra (2.5.38). This is a very large algebra which contains some well-known subalgebras within it. The $\{\mathcal{T}, \mathcal{Q}\}$ generators form a $\mathfrak{bms}_3$ subalgebra without any central extension [5]; similarly, the $\{\mathcal{T}, \bar{\mathcal{S}}\}$ give a non-centrally extended $\mathfrak{bms}_2$ [57, 58]. Finally, the $\{\mathcal{T}, \mathcal{R}\}$ assemble together into a centrally extended version of the warped Witt algebra [11], where the central extension is trivial in the $\{\mathcal{P}_m, \mathcal{P}_n\}$ but non-trivial in the mixed bracket, and is thus often referred to as twisted warped Witt algebra [57, 59].

2.6 Thermodynamics of the black string

This section is devoted to the study of the thermodynamics of the black string, in the context of our boundary conditions.

We will explicitly show how the three-dimensional black strings fit into the boundary conditions defined by (2.5.2). Obtaining the black strings from T-duality was already done in [13], so our task is mainly to transform the results to our current notation. We will also include a brief discussion regarding the thermodynamics of black strings, as it will help clarify the role played by our choice of Cauchy slice when integrating charges.

Let us start from the form of the BTZ metrics. Writing them in a way that fits the boundary conditions (2.3.1), and including a free pure gauge term in the B-field, we have

$$\mathrm{d}s^2 = \frac{\ell^2}{r^2}\mathrm{d}r^2 - r^2\left(\mathrm{d}x^+ - \frac{\ell^2 L_-}{r^2}\,\mathrm{d}x^-\right)\left(\mathrm{d}x^- - \frac{\ell^2 L_+}{r^2}\,\mathrm{d}x^+\right), \tag{2.6.1a}$$

$$B = \left(\frac{r^2}{C_0} + b(x^a) + \frac{\ell^4}{C_0 r^2}L_+L_-\right)\mathrm{d}x^+\wedge\mathrm{d}x^-\,, \tag{2.6.1b}$$

$$\Phi = \frac{1}{4}\log\left(\frac{2}{C_0}\right), \tag{2.6.1c}$$

where $L_\pm$ are now constants related to the mass and angular momentum of the BTZ black hole by

$$M = \frac{2\pi(L_+ + L_-)}{\kappa_N^2}\,, \qquad J = \frac{2\pi\ell(L_+ - L_-)}{\kappa_N^2}\,. \tag{2.6.2}$$

A direct application of the T-duality Buscher rules along the direction $\xi = \partial_\phi = (\partial_+ - \partial_-)/\ell$ produces the dual field configuration

$$\begin{aligned}\mathrm{d}s^2 = g^4(\hat{r})\mathrm{d}\hat{r}^2 + \hat{r}^2 g^2(\hat{r})\Bigg[&\left(\frac{2b}{\ell^2} + \frac{4\ell L_+L_-}{C_0^2\hat{r}}\right)\left(1 + \frac{b}{2\ell\hat{r}}\right)\mathrm{d}z^2 + \frac{\ell}{4\hat{r}}\mathrm{d}w^2 \\ &- \left(1 + \frac{b}{\ell\hat{r}} + \frac{\ell^2 L_+L_-}{C_0^2\hat{r}^2}\right)\mathrm{d}z\mathrm{d}w\Bigg],\end{aligned} \tag{2.6.3a}$$

$$B = \frac{\ell^3(L_+ - L_-)}{4C_0^2\hat{r}g^2(\hat{r})}\mathrm{d}z\wedge\mathrm{d}w\,, \tag{2.6.3b}$$

$$e^{4\Phi} = \frac{\hat{r}_0^2}{\hat{r}^2 g^4(\hat{r})}\,, \tag{2.6.3c}$$

in which we have already transformed to the appropriate coordinates for the dual using (2.5.3), and $g^2(\hat{r})$ is given by

$$g^2(\hat{r}) = 1 + \frac{\ell}{C_0\hat{r}}(L_+ + L_-) + \frac{\ell^2}{C_0^2\hat{r}^2}L_+L_-\,. \tag{2.6.4}$$

By comparing with the boundary conditions (2.5.2), it is immediate to read the form of the subleading terms in this solution. Of course, to actually have a solution we need ξ to be an exact Killing vector of the original spacetime, so $b = b(z)$ is a function only of z (directly related to t before the duality).

In order to make contact with the original literature [13, 24], let us use the fact that $b(z)$ can be removed by a diffeomorphism and introduce new coordinates

$$\mathrm{d}z = \mathrm{d}Z\,, \qquad \mathrm{d}w = \mathrm{d}W + \left(\frac{2b(z)}{\ell^2} - \frac{2(L_+ + L_-)}{C_0}\right)\mathrm{d}Z\,, \tag{2.6.5}$$

in terms of which the metric looks like (2.6.3) but with $b \to \ell^2(L_+ + L_-)/C_0$. This diffeomorphism after T-duality can be regarded as a gauge choice for b in the BTZ solution, (2.6.1). It turns out this value was the one used in [13], once we take into account that we are in Fefferman-Graham gauge (with the radial piece of the metric fixed to ℓ^2/r^2). The standard form of the black string appears if we introduce yet another set of coordinates,

$$R = \hat{r} g^2(\hat{r}) \,, \quad Z = \frac{\sqrt{C_0}(T+X)}{2(L_+L_-)^{1/4}} \,, \quad W = \frac{(\sqrt{L_+}+\sqrt{L_-})^2 T + (\sqrt{L_+}-\sqrt{L_-})^2 X}{\sqrt{C_0}(L_+L_-)^{1/4}} \,, \tag{2.6.6}$$

in terms of which the solution becomes

$$\mathrm{d}s^2 = \frac{\mathrm{d}R^2}{\left(1-\frac{\mathcal{M}}{R}\right)\left(1-\frac{\mathcal{Q}^2}{\mathcal{M}R}\right)} + R^2\left[-\left(1-\frac{\mathcal{M}}{R}\right)\mathrm{d}T^2 + \left(1-\frac{\mathcal{Q}^2}{\mathcal{M}R}\right)\mathrm{d}X^2\right] \,, \tag{2.6.7a}$$

$$B = -\frac{\hat{r}_0^2 \mathcal{Q}}{R}\mathrm{d}T \wedge \mathrm{d}X \,, \tag{2.6.7b}$$

$$e^{4\Phi} = \frac{\hat{r}_0^2}{R^2} \,. \tag{2.6.7c}$$

where we have introduced new parameters

$$\mathcal{M} = \frac{\ell}{C_0}\left(\sqrt{L_+}+\sqrt{L_-}\right)^2 \,, \qquad \mathcal{Q}^2 = \frac{\ell^2}{C_0^2}\left(L_+ - L_-\right)^2 \,. \tag{2.6.8}$$

Let us analyze the thermodynamics of this black string. This is easy in the (T, X) frame: there is a horizon at $R = \mathcal{M}$ generated by the Killing vector ∂_T,[14] under which the Hawking temperature is

$$T_H = \frac{\sqrt{1-\mathcal{Q}^2/\mathcal{M}^2}}{4\pi} = \frac{(L_+L_-)^{1/4}}{2\pi(\sqrt{L_+}+\sqrt{L_-})} \,. \tag{2.6.9}$$

Notice that the asymptotics of this solutions does not allow us to normalize the generator of the horizon in any canonical way at infinity, given that $(\partial_T)^2$ diverges as R^2. It is important to remember our choice in order to compare with the results in other frames. In the (T, X) frame, it is natural to compute charges at fixed T, in which case we obtain

$$H_T = \frac{\mathcal{M}}{2\kappa_N^2}\Delta X \,, \qquad H_X = 0 \,, \tag{2.6.10}$$

for the charges associated to the exact Killing vectors ∂_T and ∂_X (ΔX is an IR regulator for the integral over X, $H_T/\Delta X$ is the physically meaningful quantity giving the energy of the black string per unit length). The first law can now be shown to hold in the form

$$\delta H_T = T_H \delta S_{BH} + \Psi_B \delta Q_B \,, \tag{2.6.11}$$

[14] We assume $\mathcal{M} \geq |\mathcal{Q}|$ in the following analysis. See the original literature for a discussion of the spacetime structure in other situations [24].

with S_{BH} the Bekenstein-Hawking entropy at constant T

$$S_{BH} = \frac{2\pi A_H}{\kappa_N^2} = \frac{2\pi\mathcal{M}}{\kappa_N^2}\sqrt{1-\frac{\mathcal{Q}^2}{\mathcal{M}^2}}\,\Delta X\,, \tag{2.6.12}$$

and Ψ_B and Q_B the potential and charge associated to the B-field [54],

$$\Psi_B = \frac{\hat{r}_0^2\mathcal{Q}}{\mathcal{M}}\,, \qquad Q_B = \frac{\mathcal{Q}}{2\kappa_N^2\hat{r}_0^2}\Delta X\,. \tag{2.6.13}$$

If we want to think of the black string as embedded in our dual configuration space, satisfying the boundary conditions (2.5.2), we should stay in the coordinates (2.6.3) and work with Cauchy slices at fixed w. This is indeed what we did in Section 4. The properties of the horizon are of course independent of the coordinates used, so we still have a horizon at $\hat{r} = \hat{r}_h$ with $\hat{r}_h g^2(\hat{r}_h) = \mathcal{M}$ generated by

$$\xi = \partial_T = \frac{2(L_+L_-)^{1/4}}{\sqrt{C_0}}\partial_w + \frac{\sqrt{C_0}}{2(L_+L_-)^{1/4}}\left(\partial_z + \frac{2b(z)}{\ell^2}\partial_w\right)\,. \tag{2.6.14}$$

Note that ∂_w and the vector within parentheses are the exact Killing vectors of the metric (2.6.3). The corresponding charges follow from the general covariant phase space formalism described in the main text,

$$H_{(w)} \equiv H\left[\partial_w\right] = \frac{\hat{r}_0^2(L_+ + L_-)}{2\ell\kappa_N^2}\left(\Delta z + \frac{1}{2\hat{r}_0^2(L_+ + L_-)}\mathcal{B}(\Delta z)\right)\,, \tag{2.6.15a}$$

$$H_{(z)} \equiv H\left[\partial_z + \frac{2b(z)}{\ell^2}\partial_w\right] = \frac{8\hat{r}_0^4 L_+L_-}{\ell^3\kappa_N^2}\left(\Delta z + \frac{L_+ + L_-}{8\hat{r}_0^2 L_+L_-}\mathcal{B}(\Delta z)\right)\,, \tag{2.6.15b}$$

where

$$\mathcal{B}(\Delta z) = \int_{\Delta z} b(z)\,\mathrm{d}z\,, \tag{2.6.16}$$

is the integral of $b(z)$ over a piece Δz of the Cauchy slice. In the main text we took z compact to regulate, in which case taking $\Delta z = 2\pi$ so that we integrate over the whole circle, this integral just picks the zero mode of $b(z)$. We assume this is the case from now on and write $\mathcal{B} = 2\pi b_0$. Note also that the non-trivial charges are associated with T and S in the asymptotic Killing vectors (2.5.19) (for constant b, they are the corresponding zero-mode charges).

We can finally verify the first law in this frame. Of course, the different choice of Cauchy slice did not change the properties of the horizon, in particular the temperature is still

$$T_H = \frac{\sqrt{1-\mathcal{Q}^2/\mathcal{M}^2}}{4\pi} = \frac{(L_+L_-)^{1/4}}{2\pi(\sqrt{L_+}+\sqrt{L_-})}\,. \tag{2.6.17}$$

However, the different slicing does change the entropy, which taking horizon slices of constant w becomes

$$S_{BH} = \frac{16\pi\sqrt{2}\hat{r}_0^3\sqrt{L_+L_-}\left(\sqrt{L_+}+\sqrt{L_-}\right)}{\ell^2\kappa_N^2}\left(1 + \frac{b_0}{4\hat{r}_0^2\sqrt{L_+L_-}}\right)\,. \tag{2.6.18}$$

Similarly, the different slicing affects the B-field potential, which is given by $-\xi \cdot B$ pulled-back to a w-constant horizon slice now [54],

$$\Psi_B = \frac{\sqrt{2}\hat{r}_0^3}{\ell} \frac{(L_+ L_-)^{1/4}(L_+ - L_-)}{(\sqrt{L_+} + \sqrt{L_-})^2} \left(1 + \frac{1}{4\hat{r}_0^2 \sqrt{L_+ L_-}} \frac{b_0}{2\pi}\right) , \tag{2.6.19}$$

and the three-form charge becomes

$$Q_B = \frac{\mathcal{Q}}{2\kappa_N^2 \hat{r}_0^2} \Delta z = 2\pi \frac{L_+ - L_-}{\ell \kappa_N^2} \,. \tag{2.6.20}$$

Varying with respect to L_+ and L_-, one can verify that the first law holds,

$$\frac{2\sqrt{2}\hat{r}_0 (L_+ L_-)^{1/4}}{\ell} \delta H_{(w)} + \frac{\ell}{2\sqrt{2}\hat{r}_0 (L_+ L_-)^{1/4}} \delta H_{(z)} = T_H \, \delta S_{BH} + \Psi_B \, \delta Q_B \,, \tag{2.6.21}$$

which should not come as a surprise since the first law is a theorem which does not care about how we slice our spacetime.

Let us end with some brief comments and lessons we can extract from this computation for our phase space. All black strings in the form (2.6.3) are contained within the phase space we built and analyzed in section 2.5. However, the slicing used there to study the charges is different to the standard slicing that the form (2.6.7) naturally suggests. This is unavoidable if we want to write a phase space that includes all black strings as they arise from T-duality of BTZ black holes: the map $(\hat{r}, z, w) \to (R, T, X)$ involves the charges $L_\pm$, so we cannot diagonalize the metric in a uniform way for all black strings. It is also illuminating for the discussion to write the entropy (2.6.18) in terms of the charges,

$$S_{BH} = \frac{2\pi^2 \hat{r}_0^3}{\kappa_N^2 \ell^2} \left(\sqrt{\frac{\left(2q_B^2 + h_{(z)} - 8h_{(w)}^2\right)\left(2q_B^2 + h_{(z)}\right)}{4h_{(w)}^2} + 2h_{(z)}} - \frac{2q_B^2 + h_{(z)} - 8h_{(w)}^2}{2h_{(w)}} \right)$$
$$\times \left(\sqrt{\frac{2q_B^2 + h_{(z)}}{h_{(w)}} + 4q_B} + \sqrt{\frac{2q_B^2 + h_{(z)}}{h_{(w)}} - 4q_B} \right) , \tag{2.6.22}$$

where we have introduced rescaled charges (per unit length) to simplify the notation,

$$h_{(w)} = \frac{\ell \kappa_N^2}{2\pi \hat{r}_0^2} H_{(w)} \,, \qquad h_{(z)} = \frac{\ell^3 \kappa_N^2}{2\pi \hat{r}_0^4} H_{(z)} \,, \qquad q_B = \frac{\ell \kappa_N^2}{2\pi} Q_B \,. \tag{2.6.23}$$

If we repeat the construction in the $b_0 = 0$ case, this imposes the relation between the charges $2q_B^2 + h_{(z)} - 8h_{(w)}^2 = 0$, producing a simple-looking formula for the entropy in terms of the Killing charges. To make it even more suggestive, note that for $b_0 = 0$ the charges $\mathcal{S}_0$ and $\bar{\mathcal{S}}_0$ related as in (2.5.34) agree, so identifying $\bar{\mathcal{S}}_0 = H_{(w)}$ and $\mathcal{T}_0 = H_{(z)}$ we can write the entropy as

$$S_{BH} = 8\pi \sqrt{\mathcal{T}_0} \left(\sqrt{\mathcal{S}_0 + \sqrt{\bar{\mathcal{S}}_0^2 + \frac{\bar{k}}{8}\mathcal{T}_0}} + \sqrt{\bar{\mathcal{S}}_0 - \sqrt{\bar{\mathcal{S}}_0^2 + \frac{\bar{k}}{8}\mathcal{T}_0}} \right) . \tag{2.6.24}$$

where $\bar{k} = -2\pi\ell/\kappa_N^2$ characterizes the central extension in the $\{\bar{\mathcal{S}}_m, \mathcal{Q}_n\}$ bracket (alternatively, it is four times the central extension in the $\{\mathcal{S}_m, \mathcal{S}_n\}$ algebra before the redefinition).

2.7 Discussion and summary

From a (super)gravity perspective, T-duality is a transformation that takes as input solutions to the field equations and produces new solutions, possibly with wildly different asymptotic behavior. A paradigmatic example of this is the duality between asymptotically AdS_3 BTZ black holes and black strings, which have asymptotically vanishing curvature. Strictly speaking, the duality transformation can only be used whenever the backgrounds have an exact Killing vector. However, by means of the BTZ / black string example, our goal in this work has been to show that whenever we have a well-understood phase space in which one set of solutions can be embedded (namely, BTZ black holes in the Brown-Henneaux phase space of section 2.3), T-duality transformations can inform the construction of a dual phase space that includes the dual solutions (black strings in the phase space of section 2.5). In this sense, we could call a construction along the lines of the one presented in this work *asymptotic T-duality*. Notice that this procedure does not directly relate a phase space of solutions with its dual. As we exemplified in section 2.5, what T-duality can do is generate a set of boundary conditions from existing ones, but the construction of a consistent phase space has to be done in the usual way starting from those new boundary conditions.

The construction is intended to be meaningful only at the level of classical gravitational theories. Thus, we view it as a way to generate new boundary conditions from existing ones, in a process that can eventually lead to interesting and new asymptotic symmetry algebras.[15] The different algebras make manifest that no kind of equivalence is in general expected between the theory with the original and the dual boundary conditions. This contrasts with the situation in string theory whenever an exact isometry is present, since then backgrounds related by T-duality are known to define equivalent theories [33].

One of the main byproducts of our analysis has been the construction of a phase space containing the three-dimensional black strings of Horne and Horowitz [24], at least whenever they are away from the extremal limit. Previous works have also addressed the problem of constructing such a phase space [40, 41], but our proposal resulting from asymptotic T-duality is different and new. We obtain a much larger symmetry algebra (2.5.35)-(2.5.36), potentially allowing more control over the set of states in the phase space. As a downside, the present construction does not allow us to discuss black strings in the extremal limit, since these are not obtained by dualizing BTZ black holes.

It would be interesting to further explore the implications of the symmetry algebra (2.5.35)-(2.5.36) for the theory with boundary conditions (2.5.2). A quantum gravitational theory with those boundary conditions would provide a representation of the aforemen-

[15] We remark that, even though we introduced the construction using three-dimensional backgrounds, it can be generalized and used in any other dimension with minor or no modifications to its philosophy.

tioned symmetry algebra on its Hilbert space, thus the study of the representations of the algebra could inform us about the possible spectrum of such a theory. Another interesting avenue to extract consequences from the symmetry algebra would be to derive some Cardy-like formula [60] able to constrain the density of states at sufficiently high energies. If such a formula exists, it should reproduce the entropy of the black strings as derived in section 2.6, thus giving a microscopic, symmetry-based argument for its origin. Since it is simple enough to be suggestive, let us reproduce here the result (2.6.24) giving the entropy of the black strings (2.5.7) in terms of their Killing charges for the case $b(z) = 0$:

$$S_{BH} = 8\pi\sqrt{\mathcal{T}_0}\left(\sqrt{\bar{\mathcal{S}}_0 + \sqrt{\bar{\mathcal{S}}_0^2 + \frac{\bar{k}}{8}\mathcal{T}_0}} + \sqrt{\bar{\mathcal{S}}_0 - \sqrt{\bar{\mathcal{S}}_0^2 + \frac{\bar{k}}{8}\mathcal{T}_0}}\right), \tag{2.7.1}$$

where $\mathcal{T}_0$ and $\bar{\mathcal{S}}_0 = \mathcal{S}_0$ are the charges associated with the Killing vectors ∂_z and ∂_w, and $\bar{k} = -2\pi\ell/\kappa_N^2$. This formula thus gives S_{BH} as a function of the zero-mode charges $\mathcal{T}_0$ and $\bar{\mathcal{S}}_0$ and the central extension of the algebra, which is the typical form of a Cardy-like formula. Deriving it from properties of the algebra (2.5.35)-(2.5.36) would be an extremely interesting result. However, it is worth highlighting an immediate challenge in trying to complete this program. A naive attempt to derive the formula using the subalgebras for which Cardy formulas are known ($\mathfrak{bms}_3$ [9, 10, 61] and twisted warped conformal [11, 62]) fails, because the relevant zero-mode charges are $\mathcal{T}_0$ and $\bar{\mathcal{S}}_0$, which belong to the $\mathfrak{bms}_2$ subalgebra. We do not have a Cardy formula for $\mathfrak{bms}_2$, so obtaining one could also be useful in the present situation. Finally, as already noted at the end of section 2.5, the algebra (2.5.35)-(2.5.36) is particularly appealing since it unifies well-known algebras ($\mathfrak{bms}_2$, $\mathfrak{bms}_3$, and twisted warped Witt) into a single structure. This is yet another argument to grant it further study.

More broadly, our construction of asymptotic T-duality through the BTZ / black string example can in principle be generalized to other T-dual pairs in (super)gravity. Given that T-duality can heavily affect the asymptotic structure of a spacetime, and supported by the results obtained for the example developed in the present work, we believe that this can provide a way to obtain novel boundary conditions which lead to interesting symmetry algebras for a variety of asymptotic behaviors. It is also possible to explore similar ideas in other potentially interesting contexts. One example would be trying to implement an asymptotic notion of T-duality in the language of double field theory [63–65]. Given that this is a particularly well-suited formalism to discuss T-duality equivalent backgrounds, it is conceivable that the asymptotic analysis we have performed in this work has also an illuminating counterpart in the double field theory language. Another potential avenue would be to implement alternative solution-generating transformations in an asymptotic sense. TsT transformations are probably the first and more natural example to consider, and it would be interesting to explore the effect of TsT transformations applied to the

Brown-Henneaux boundary conditions. If one can make sense out of such a construction, the result can also impact our understanding of three-dimensional black strings, given that these can be obtained by TsT transformations of BTZ black holes. Recent works have addressed the question of the effect of such TsT transformations in the asymptotic symmetry algebra using a worldsheet perspective [66], and the results point towards a conservation of the two Virasoro towers. It would be interesting to reproduce such a result from a target space perspective, using methods similar to the ones developed in this work. Finally, it is known that for backgrounds allowing an exact worldsheet description, spacetime and worldsheet symmetries can sometimes be related (this is the case for AdS_3 and some of its deformations, and 3d flat space [19, 20, 49, 66–68]). It would be interesting to investigate whether such a construction can be performed for the 3d black string. One aspect to take into account is the fact that the black string metrics are only valid to first order in α', and higher order corrections might have to be taken into account along the lines of [69, 70].

Chapter 3

A Carrollian limit of anti-de Sitter

As stated in the introduction of this thesis, recent progress in theoretical physics has highlighted the fundamental role played by Carrollian symmetries. Originally introduced by Lévy-Leblond and Sen Gupta in the 1960s [71, 72], the Carroll group arises from a vanishing speed of light limit applied to the Poincaré group. Interest in this symmetry resurged when Duval, Gibbons, and Horvathy identified an isomorphism between the BMS_{d+1} algebra and the conformal Carroll algebra in d dimensions [73]. Carrollian geometry has been found to naturally describe structures appearing on generic null hypersurfaces, such as black hole event horizons [74–80]. It also captures the asymptotic symmetry behavior near spacelike singularities [81–85], and has gained attention for its relevance to holography in asymptotically flat spacetimes, particularly in four dimensions [86–98].

A natural question arises regarding the construction of field theories that are invariant under Carroll symmetries, and whether such theories can be derived as limits of Lorentz-invariant—or more generally, symmetric—theories. Investigations using Hamiltonian [99] and Lagrangian frameworks [100–102] have revealed the existence of two distinct limiting procedures, commonly referred to as the electric and magnetic limits. In addition to these limiting constructions, Carroll-invariant field theories have also been formulated from an intrinsic perspective. Notable examples include scalar fields exhibiting nontrivial dynamics [103], the so-called Carroll swiftons [104] which feature propagation beyond the Carroll lightcone, and models arising from finite deformations of conformal field theories [105–109].

In the same line, recently in [110], a set of observations suggested that the $l \to 0$ limit of physics in AdS might be Carrollian. The first observation is that a particle in global AdS of curvature radius l, when $l \to 0$ cannot move, due to the infinite gravitational barrier that such particle might overcome. In fact any radial geodesic thrown towards infinity from the origin will return to such point in a universal time πl, regardless the initial velocity. Therefore, as l approaches zero, the particle will remain close to the origin, as it occurs when the Minkowskian light cones collapses in the Carrollian limit $c \to 0$. Since AdS spacetime is homogeneous, this sort of "ultralocality" will occur at every point of the spacetime. The second observation comes from field theory. Due to the collapse of the Minkowskian light cone when $c \to 0$, field theories with interactions between degrees of freedom defined at different points of the spacetime are expected to possess a tachyonic behavior in the Carrollian limit. Nevertheless, recently in [104], it was shown that in spite of the former intuitive argument, it is actually possible to construct interacting Carrollian field theories with energy bounded from below, introducing more than one field. Indeed, there is an analogous behavior for the propagation of massive field in AdS, since for a field of spin s and squared mass m^2, it is know that the energy functional is positive definite even for fields that would be tachyonic on flat space, provided $m^2 \geq m^2_{BF} = -\frac{n(D,s)}{l^2}$ where the Breitenlohner-Freedman mass m^2_{BF} defined by the latter equality, depends on the spin and on the spacetime dimension. When $l \to 0$ a massive field could be arbitrarily tachyonic on AdS, remaining with a positive energy. A third argument in favour of the potential relation of the $l \to 0$ limit of AdS physics and $c \to 0$ limit of Minkowskian physics comes from analyzing the form of the Brown-Henneaux central charge c_{BH} in three-dimensional General Relativity [4]. In such setup $c_{BH} = \frac{3l}{2G}$ and since the central charge is dimensionless, the Newton's constant in the denominator actually stands for Planck's length. Therefore, restoring the $\hbar$ and c factors, the Brown-Henneaux central charge takes the form $c_{BH} = \frac{3lc^3}{2\hbar G}$. From this expression we can see that taking the Carrollian $c \to 0$ limit has the same effect on the central charge as taking the $l \to 0$ limit. The fourth final observation is more precise, let us remember that AdS$_D$ of radius l is defined as the locus $-X_0^2 + X_1^2 + ... + X_{D-1}^2 - X_{D+1}^2 = -l^2$ on the space with flat metric $ds^2 = -dX_0^2 + dX_1^2 + ... + dX_{D-1}^2 - dX_{D+1}^2$. In the limit $l \to 0$ the AdS$_D$ surface degenerates into a double-null cone, and it is know that Carrollian structures naturally emerge on null surfaces.

3.1 Limits of Kinematical Lie algebras

The principle of relativity is, as we believe, one of the fundamental principles in nature. Because of this, the transformations that relate different inertial frames are of big importance in the construction of physical theories, and we want such theories to be invariant under these transformations. These symmetries are part of what is called kinematical or space-time symmetry algebras. In the seminal work of Barcy and Lévy-Leblond [7], they classified all possible kinematical Lie algebras, which include translations, rotations and boost. Among them, there are included well-known algebras as Poincaré and (A)dS, but also Carroll and Galilei algebras, and their generalizations with cosmological constant.

They also studied the different possible contraction between them. One of such contractions is the well-known flat limit from A(dS) to Poincaré, and another, new at the time, is the Carrollian limit of Poincaré. In the next subsection we show how to treat these two limits at the same time, for the sake of gaining intuition on these procedure. We then apply this limit to the two-copies of Virasoro algebras obtained as the group of asymptotic symmetries of asymptotically AdS spacetimes in 3D, obtaining an infinite dimensional extension of the Carroll algebra in 3D.

3.1.1 Contraction of $\mathfrak{so}(2,3)$ to Carroll

We start analyzing the four-dimensional case. Let us consider the algebra of $\mathfrak{so}(2,3)$

$$[M^{MN}, M^{RS}] = M^{MS}\eta^{NR} - M^{NS}\eta^{MR} + M^{NR}\eta^{MS} - M^{MR}\eta^{NS}\,, \tag{3.1.1}$$

where

$$\eta^{MN} = \mathrm{diag}(-1,-1,1,1,1)\,, \tag{3.1.2}$$

with indices running into $M, N \in \{\bullet, 0, i\}$ where $i, j \in \{1,2,3\}$. The generators are splitted as

$$M^{MN} = \left(\begin{array}{c|c} 0 & M^{\bullet\mu} \\ \hline M^{\mu\bullet} & M^{\mu\nu} \end{array}\right)\,, \tag{3.1.3}$$

where $\mu, \nu \in \{0,1,2,3\}$. Now, in order to avoid writing the bulleted-index, let us define $K^\mu = M^{\bullet\mu}$. Then one obtains a well-known splitting of the $\mathfrak{so}(2,3)$ algebra, namely

$$[K^\mu, K^\nu] = M^{\mu\nu}\,, \tag{3.1.4}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = M^{\mu\sigma}\eta^{\nu\rho} - M^{\nu\sigma}\eta^{\mu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma}\,, \tag{3.1.5}$$

$$[K^\mu, M^{\rho\sigma}] = K^\sigma\eta^{\mu\rho} - K^\rho\eta^{\mu\sigma}\,. \tag{3.1.6}$$

When rescaling the generators K^μ as $K^\mu \to L\,K^\mu$ the algebra is written in a way which is suitable for taking the $L \to \infty$ contraction leading to the Poincare algebra. Here, we are actually interested in the opposite limit, $L \to 0$, if we interpret L as the radius of Anti de Sitter. Note, anyway, that the parameter L introduced as a rescaling of the generators, does not necessarily has a direct physical interpretation.

We are interested in the contraction of the Carroll algebra, in order to do so, it is useful to introduce a further splitting of the generators. Let us consider

$$K^\mu = \begin{pmatrix} K^0 \\ K^i \end{pmatrix}\,, \qquad M^{\mu\nu} = \left(\begin{array}{c|c} 0 & M^{0i} \\ \hline M^{i0} & M^{ij} \end{array}\right)\,, \tag{3.1.7}$$

leading to the commutators

$$[K^0, M^{0i}] = -K^i\,, \tag{3.1.8}$$
$$[K^0, M^{ij}] = 0\,, \tag{3.1.9}$$
$$[K^0, K^i] = M^{0i}\,, \tag{3.1.10}$$
$$[K^i, K^j] = M^{ij}\,, \tag{3.1.11}$$
$$[K^i, M^{jk}] = K^k\delta^{ij} - K^j\delta^{ik}\,, \tag{3.1.12}$$
$$[K^i, M^{0j}] = -K^0\delta^{ij}\,, \tag{3.1.13}$$
$$[M^{0i}, M^{jk}] = M^{0k}\delta^{ij} - M^{0j}\delta^{ik}\,, \tag{3.1.14}$$
$$[M^{ij}, M^{kl}] = M^{il}\delta^{jk} - M^{jl}\delta^{ik} + M^{jk}\delta^{il} - M^{ik}\delta^{jl}\,, \tag{3.1.15}$$
$$[M^{0i}, M^{0j}] = M^{ij}\ . \tag{3.1.16}$$

As usual, in four dimensions we can introduce the dual generators

$$M^{ij} = \epsilon_{ijk}J_k \qquad \Longleftrightarrow \qquad J_n = \frac{1}{2}\epsilon_{ijn}M^{ij}\,. \tag{3.1.17}$$

Finally, in order to simplify the notation is it useful to redefine the generators as $K^0 = H, M^{0i} = P_i$. Then, the algebra of $\mathfrak{so}(2,3)$ takes the form

$$[P_i, P_j] = \epsilon_{ijk}J_k\,, \tag{3.1.18}$$
$$[H, P_i] = -K^i\,, \qquad [K_i, P_j] = -H\delta_{ij}\,, \tag{3.1.19}$$
$$[H, J_i] = 0\,, \qquad [K_i, J_m] = -\epsilon_{imk}K_k\,, \tag{3.1.20}$$
$$[H, K_i] = P_i\,, \qquad [P_i, J_n] = \epsilon_{ikn}P_k \tag{3.1.21}$$
$$[K_i, K_j] = \epsilon_{ijk}J_k\,, \qquad [J_n, J_m] = \epsilon_{mnk}J_k\,. \tag{3.1.22}$$

We want introduce only one parameter in the redefinition of the generators, such that we could take the limit of this parameter going to zero and obtain the Carroll algebra. A suitable rescaling of the generators is required. Note that a rescaling of J_n is not allowed, otherwise we would spoil the last commutation relation, and we need to keep such relation if J_i is to be interpreted as the rotation generators. Let us define the rescaled generator T by $\mathbb{T} = L^{-N_T}T$. The algebra of such generators reads

$$L^{2N_Y}[\mathbb{Y}_i, \mathbb{Y}_j] = \epsilon_{ijk}J_k\,, \tag{3.1.23}$$
$$[\mathbb{Z}, \mathbb{Y}_i] = -L^{-N_Z-N_Y+N_K}\mathbb{K}^i\,, \qquad [\mathbb{K}_i, \mathbb{Y}_j] = -L^{-N_K-N_Y+N_Z}\mathbb{Z}\delta_{ij}\,, \tag{3.1.24}$$
$$L^{N_Z}[\mathbb{Z}, J_i] = 0\,, \qquad [\mathbb{K}_i, J_m] = -\epsilon_{imk}\mathbb{K}_k\,, \tag{3.1.25}$$
$$[\mathbb{Z}, \mathbb{K}_i] = L^{-N_Z-N_K+N_Y}\mathbb{Y}_i\,, \qquad [\mathbb{Y}_i, J_n] = \epsilon_{ikn}\mathbb{Y}_k \tag{3.1.26}$$
$$[\mathbb{K}_i, \mathbb{K}_j] = L^{-2N_K}\epsilon_{ijk}J_k\,, \qquad [J_n, J_m] = \epsilon_{mnk}J_k\,. \tag{3.1.27}$$

The conditions to have a good $L \to 0$ limit that connects with the Carroll algebra are

$$-N_H - N_P + N_K > 0 \tag{3.1.28}$$
$$-N_H - N_K + N_P > 0 \tag{3.1.29}$$
$$-2N_K > 0 \tag{3.1.30}$$
$$-N_K - N_P + N_H = 0 \tag{3.1.31}$$

The following particular choice fulfills the previous constraints

$$N_K = -1, N_P = -1, N_H = -2\,,$$

Such election permits taking the contraction of the $\mathfrak{so}(2,3)$ algebra by taking the $L \to 0$ limit, leading to the Carroll Algebra $\mathfrak{carr}(1+3)$, namely

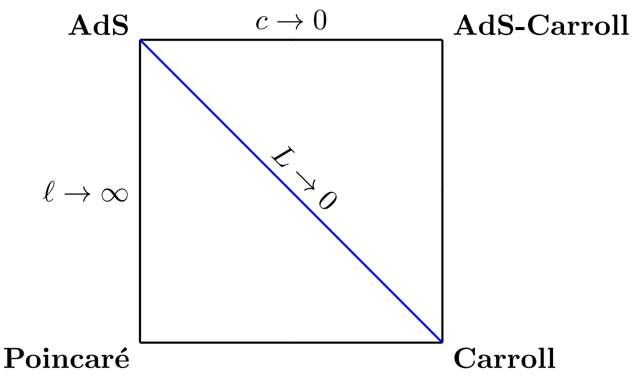
$$[\mathbb{P}_i, \mathbb{P}_j] = 0\,, \tag{3.1.32}$$
$$[\mathbb{H}, \mathbb{P}_i] = 0\,, \qquad [\mathbb{K}_i, \mathbb{P}_j] = -\mathbb{H}\delta_{ij}\,, \tag{3.1.33}$$
$$[\mathbb{H}, J_i] = 0\,, \qquad [\mathbb{K}_i, J_m] = -\epsilon_{imk}\mathbb{K}_k\,, \tag{3.1.34}$$
$$[\mathbb{H}, \mathbb{K}_i] = 0\,, \qquad [\mathbb{P}_i, J_n] = \epsilon_{ikn}\mathbb{P}_k \tag{3.1.35}$$
$$[\mathbb{K}_i, \mathbb{K}_j] = 0\,, \qquad [J_n, J_m] = \epsilon_{mnk}J_k\,. \tag{3.1.36}$$

We can see in this derivation that the conditions (3.1.28) tell us that $N_K + N_P = N_H$, for two arbitrary negative numbers N_K and N_P. Then, the limit can be interpreted as taking two limits at the same time, one corresponding to choosing $N_K = 0$ and $N_P = N_H$, and the second one to $N_P = 0$ and $N_K = N_H$. These limits correspond to the flat limit (but with the interpretation of L as the inverse of ℓ, the radius of AdS), and to the Carrollian limit of the Poincare obtained after the first limit. We have shown, however, how we can take this two limits at the same time using only one parameter.

The extension to higher dimension follows trivially. We can actually start from the algebra (3.1.8)-(3.1.16) with the lowercase Latin indices now running from 1 to D. Defining $K^0 = L^{-2}\mathbb{H},\ M^{0i} = L^{-1}\mathbb{P}^i,\ K^i = L^{-1}\mathbb{K}^i,\ M^{ij} = \mathbb{J}^{ij}$, after taking the limit $L \to 0$ leads to the Carroll algebra in $d+1$ dimensions, $\mathfrak{carr}(d+1)$, with the following non-vanishing commutators:

$$\left[\mathbb{K}^i, \mathbb{J}^j\right] = \mathbb{K}^k\delta^{ij} - \mathbb{K}^j\delta^{ik},\ \left[\mathbb{K}^i, \mathbb{P}^j\right] = -\mathbb{H}\delta^{ij},\ \left[\mathbb{P}^i, \mathbb{J}^{ij}\right] = \mathbb{P}^k\delta^{ij} - \mathbb{P}^j\delta^{ik},$$
$$\left[\mathbb{J}^{ij}, \mathbb{J}^{kl}\right] = \mathbb{J}^{il}\delta^{jk} - \mathbb{J}^{jl}\delta^{ik} + \mathbb{J}^{jk}\delta^{il} - \mathbb{J}^{ik}\delta^{jl} \tag{3.1.37}$$

In consequence, we have shown that the contraction of the AdS algebra $\mathfrak{so}(2, d-1)$ to the Carroll algebra $\mathfrak{carr}(1+d)$, can be taken introducing only one contraction parameter. This is captured by the following diagram



3.1.2 Infinite dimensional extension of Carroll from a contraction of Virasoro algebras

Given that the two Virasoro copies naturally arise as an infinite dimensional extension of the algebra of AdS in three dimensions, we apply what we learned in the previous subsection to this algebra and show that there is a contraction leading to an infinite dimensional extension of $\mathfrak{carr}(2+1)$.

Consider the two copies of the Virasoro algebra arising from the asymptotic symmetries of GR with negative cosmological constant $\Lambda = -\frac{1}{\ell^2}$ in three dimensions

$$i\left\{L_m^{\pm}, L_n^{\pm}\right\} = (m-n)L_{m+n}^{\pm} + \frac{c^{\pm}}{12}m\left(m^2-1\right)\delta_{m+n}^0, \quad \left\{L_m^{\pm}, L_n^{\mp}\right\} = 0 \tag{3.1.38}$$

with $c^{\pm} = \frac{3\ell}{2G}$.

Following the same spirit of reference [111] we redefine the generators as follows

$$P_m = L_m^+ + L_{-m}^-, \quad J_m = L_m^+ - L_{-m}^- \tag{3.1.39}$$

the algebra of the new generators is then given by

$$\begin{aligned} i\left\{J_m, J_n\right\} &= (m-n)J_{m+n} + \tfrac{c^+-c^-}{12}m\left(m^2-1\right)\delta_{m+n}^0 \\ i\left\{J_m, P_n\right\} &= (m-n)P_{m+n} + \tfrac{c^++c^-}{12}m\left(m^2-1\right)\delta_{m+n}^0 \\ i\left\{P_m, P_n\right\} &= (m-n)J_{m+n} + \tfrac{c^+-c^-}{12}m\left(m^2-1\right)\delta_{m+n}^0 \end{aligned} \tag{3.1.40}$$

On the other hand, we know that the generators of AdS are related to P and J as

$$p \sim P_1 \pm P_{-1}, \tag{3.1.41}$$

$$k \sim J_1 \pm J_{-1}. \tag{3.1.42}$$

and

$$h = P_0, \tag{3.1.43}$$

$$j = J_0 \tag{3.1.44}$$

and recalling that we go to Carroll via

$$\bar{p}_i = \ell p_i\,, \quad \bar{h} = \ell^2 h \tag{3.1.45}$$

$$\bar{k}_i = \ell k_i\,, \quad \bar{j} = j. \tag{3.1.46}$$

we propose the following rescaling for the generators J_n and P_n

$$\bar{P}_n = \ell P_n, \tag{3.1.47}$$

$$\bar{J}_n = \ell J_n \tag{3.1.48}$$

for $n \neq 0$, and

$$\bar{P}_0 = \ell^2 P_0 \tag{3.1.49}$$

$$\bar{J}_0 = J_0 \tag{3.1.50}$$

for $n = 0$.

Then, taking the limit $\ell \to 0$ after these rescalings, we obtain the following infinite dimensional extension of $\mathfrak{carr}(2+1)$

$$\begin{aligned} i\left\{\bar{J}_m, \bar{P}_n\right\} &= 2\bar{P}_0 \delta_{m,-n} \\ i\left\{\bar{J}_m, \bar{J}_0\right\} &= m\bar{J}_m \\ i\left\{\bar{P}_m, \bar{J}_0\right\} &= m\bar{P}_m \end{aligned} \tag{3.1.51}$$

where $n \neq 0$ and $i\left\{\bar{J}_m, \bar{J}_n\right\} = i\left\{\bar{P}_m, \bar{P}_n\right\} = 0$. It is composed of two Abelian currents that satisfy a Heisenberg algebra, with central extension $\bar{P}_0$ and a ladder operator $\bar{J}_0$.

This algebra already appeared in [112], where they studied asymptotic symmetries in 3D Carrollian theories of gravity, and it is a subalgebra of the result found in [113].

3.2 Embedding point of view

Let us consider $\mathbb{R}^{2,d-1}$ with metric

$$\eta_{MN} \mathrm{d}X^M \mathrm{d}X^N = -\mathrm{d}X_{-1}^2 - \mathrm{d}X_0^2 + \sum_{i=1}^{d-1} \mathrm{d}X_i^2\,. \tag{3.2.1}$$

We consider the constraint that defines AdS with $\ell = 0$

$$-X_{-1}^2 - X_0^2 + \sum_{i=1}^{d-1} X_i^2 = 0\,. \tag{3.2.2}$$

The constraint (3.2.2) defined a null hypersurface. To see that we consider the 1-form normal to the surface

$$\boldsymbol{n} = \frac{1}{2}\mathrm{d}\left(-X_{-1}^2 - X_0^2 + \sum_{i=1}^{3} X_i^2\right) = -X_{-1}\mathrm{d}X_{-1} - X_0\mathrm{d}X_0 + \sum_i X_i\mathrm{d}X_i \tag{3.2.3}$$

Note that this 1-form is null,

$$\boldsymbol{n}^2 = n_M n_N \eta^{MN} = -X_{-1}^2 - X_0^2 + \sum_i X_i^2 = 0\,. \tag{3.2.4}$$

Hence its only normal vector is null.

Now we will parameterize the surface (3.2.2) as follows

$$\begin{aligned} X_{-1} &= \xi \cos\lambda\,, \\ X_0 &= \xi \sin\lambda\,, \\ X_i &= \xi\mu_i\,. \end{aligned} \tag{3.2.5}$$

Replacing in the constraint we have that

$$\sum_{i=1}^{d-1} \mu_i^2 = 1\,, \tag{3.2.6}$$

for ξ different from zero. Let us call this induced manifold by $\mathcal{M}_c$. It is parameterized by the coordinates $(\xi, \lambda, \mu_i(\theta_\alpha))$ where the functions $\mu_i(\theta_\alpha)$ satisfy the constraint (3.2.6) with $\alpha = 1, \ldots, d-2$. The induced metric is given by

$$\begin{aligned} \mathrm{d}s_c^2 &= -(\mathrm{d}\xi\cos\lambda - \xi\sin\lambda\mathrm{d}\lambda)^2 - (\mathrm{d}\xi\sin\lambda + \xi\cos\lambda\mathrm{d}\lambda)^2 + \sum_{i=1}^{d-1}(\mathrm{d}\xi\mu_i + \xi\mathrm{d}\mu_i)^2\,, \\ &= -\mathrm{d}\xi^2 - \xi^2\mathrm{d}\lambda^2 + \sum_i \mu_i^2\mathrm{d}\xi^2 + \xi^2\sum_i \mathrm{d}\mu_i^2 + 2\sum_i \mu_i\mathrm{d}\mu_i\xi\mathrm{d}\xi\,, \\ &= -\xi^2\mathrm{d}\lambda^2 + \xi^2\sum_{i=1}^{d-1}\mathrm{d}\mu_i^2\,, \end{aligned} \tag{3.2.7}$$

where we used $\sum_i \mu_i\mathrm{d}\mu_i = 0$ which follows from the constraint (3.2.6). Observe that there exist a vector $\boldsymbol{v} = \frac{\partial}{\partial\xi} \in T\mathcal{M}_c$ which belongs to the kernel of the (Lorentzian) metric $g^c_{\mu\nu}$ defined by (3.2.7)

$$g^c_{\mu\nu}v^\nu = g^c_{\mu\nu}\delta^\nu_\xi = 0\,. \tag{3.2.8}$$

In particular for $d = 4$ we can parameterize the sphere as

$$\mu_1 = \cos\varphi\sin\theta\,, \qquad \mu_2 = \sin\varphi\sin\theta\,, \qquad \mu_3 = \cos\theta \tag{3.2.9}$$

Then the induced metric is

$$\mathrm{d}s_c^2 = \xi^2(-\mathrm{d}\lambda^2 + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\,. \tag{3.2.10}$$

3.3 The Carroll structure

3.3.1 Flash review of Carroll geometry

Let us first review some basic topics of Carrollian geometry. Carrollian structure is defined as a D-dimensional manifold M endowed with a degenerate metric $\boldsymbol{g} \in T^*M \otimes T^*M$ and a vector $n \in TM$ such that $\boldsymbol{g}(n,-) = \boldsymbol{g}(-,n) = 0$, namely $n \in \ker \boldsymbol{g}$. In order to construct Carroll invariant action functionals, it is useful to construct a putative volume form $\boldsymbol{\Omega} \in (T^*M)^{\otimes D}$, which can be done as follows. Consider an embedded surface of co-dimension 1, $\Sigma \subset M$ such that the pullback of $\boldsymbol{g}$ to Σ is non-degenerate. Let $\boldsymbol{\omega}$ be the volume form on Σ, then $\boldsymbol{\omega} \in (T^*M)^{\otimes(D-1)}$. Then we construct a 1-form $\theta \in T^*M$ dual to the vector n, such that it is normalized as $\theta(n) = n(\theta) = 1$. Then one can construct a D-form as

$$\boldsymbol{\Omega} = \boldsymbol{\omega} \wedge \theta \,. \tag{3.3.1}$$

Note that θ is not an intrinsic part of the Carrollian structure which leads to the following ambiguity. Let us put coordinates x^μ on a patch of the manifold M with $\mu = 1, \ldots, D$. In these coordinates one has $n = n^\mu \partial_\mu$, $\theta = \theta_\mu \mathrm{d}x^\mu$ and $\boldsymbol{g} = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. Let us define $\boldsymbol{G} \in TM \otimes TM$ such that its components $G^{\mu\nu}$ satisfy

$$G^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu - n^\mu \theta_\nu \,. \tag{3.3.2}$$

This equation does not fix $G^{\mu\nu}$ completely, since if we perform the transformation $G^{\mu\nu} \to G^{\mu\nu} + \lambda(x) n^\mu n^\nu$, equation (3.3.2) remains invariant, for any function $\lambda(x)$. One can fix this ambiguity by imposing $G^{\mu\nu}\theta_\mu\theta_\nu = 0$. Now, given a 1-form $v = v_\mu \mathrm{d}x^\mu$ one can construct a vector by acting with $G^{\mu\nu}$, namely

$$v^\mu \equiv G^{\mu\nu} v_\nu \,. \tag{3.3.3}$$

Then, if one goes tries to back to go the 1-form by acting with $g_{\mu\nu}$ one finds

$$g_{\mu\nu} v^\nu = v_\mu - \theta_\mu n^\nu v_\nu \,, \tag{3.3.4}$$

i.e. one does not get back v_μ, but instead we find an extra term proportional to θ_μ. If $v_\mu n^\mu$ vanishes, the ambiguity is removed.

3.3.2 The Carroll structure from the limit $\ell \to 0$ of AdS

We want to perform the limit $\ell \to 0$ of AdS and learn something from that. To do so we will start from the AdS metric and, allowing rescaling of the coordinates, take the limit $\ell \to 0$. Then we will try to take a free scalar field in AdS and, by considering the same rescaling of the coordinate that we consider in the previous step, we rescale the scalar field such that the limit $\ell \to 0$ is well defined.

The AdS metric in spherical coordinates is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2\,, \qquad f(r) = \frac{r^2}{\ell^2} + 1 \tag{3.3.5}$$

$$= -f(r)\left(\mathrm{d}t - \frac{\mathrm{d}r}{f(r)}\right)\left(\mathrm{d}t + \frac{\mathrm{d}r}{f(r)}\right) + r^2\mathrm{d}\Omega^2 \tag{3.3.6}$$

defining $\mathrm{d}u = \mathrm{d}t - \mathrm{d}r/f(r)$, the metric becomes

$$\mathrm{d}s^2 = -\left(\frac{r^2}{\ell^2} + 1\right)\mathrm{d}u^2 - 2\mathrm{d}r\mathrm{d}u + r^2\mathrm{d}\Omega^2\,, \tag{3.3.7}$$

Now considering the rescaling $u = \ell\tilde{u}$ we find that

$$\mathrm{d}s^2 = -(r^2 + \ell^2)\mathrm{d}\tilde{u}^2 - 2\ell\mathrm{d}r\mathrm{d}u + r^2\mathrm{d}\Omega^2\,. \tag{3.3.8}$$

Then the limit $\ell \to 0$ leads to

$$\boldsymbol{g} = r^2(-\mathrm{d}\tilde{u}^2 + \mathrm{d}\Omega^2)\,. \tag{3.3.9}$$

With this metric we can define a Carrollian structure because a general vector of the form $\boldsymbol{n} = z(x^\mu)\frac{\partial}{\partial r}$ belongs to the kernel of $\boldsymbol{g}$ defined as before.

Zero radius AdS Carroll structure

It is then sensible to define the following Carrollian structure. A D-dimensional manifold M with a degenerated Lorentzian metric, $\boldsymbol{g}$, with a vector in its kernel, $\boldsymbol{n}$. There is a coordinate patch (u, r, ϑ^i) where the metric and the vector are

$$\boldsymbol{g} = r^2(-\mathrm{d}u^2 + \gamma_{ij}\mathrm{d}\vartheta^i\mathrm{d}\vartheta^j)\,, \qquad \boldsymbol{n} = r\frac{\partial}{\partial r}\,, \tag{3.3.10}$$

and $\gamma_{ij}\mathrm{d}\vartheta^i\mathrm{d}\vartheta^j$ is the metric of a S^{D-2}. The reason to include the factor r in the kernel vector will be clear in what follows.

Symmetries of the Carroll structure

Now we can ask whether there exist a vector $\xi \in TM$ such that it preserves the Carrollian structure, namely

$$\mathcal{L}_\xi g_{\mu\nu} = 0\,,$$
$$\mathcal{L}_\xi n^\mu = 0\,.$$

To make the notation simple we split the indices as $\{\mu,\nu,\dots\} = \{r\}\cup\{a,b,\dots\}$ The second equation reads

$$\xi^\nu \partial_\nu n^\mu - n^\nu \partial_\nu \xi^\mu = 0 \implies \xi^r \partial_r r \delta^\mu_r - r\partial_r \xi^\mu = 0\,, \tag{3.3.11}$$

which implies two equations

$$\xi^r - r\partial_r \xi^r = 0 \implies \xi^r = \lambda(x^a) r \tag{3.3.12}$$

$$\partial_r \xi^a = 0\,. \tag{3.3.13}$$

while the equations for the metric reads

$$\begin{aligned} \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda}\partial_\nu \xi^\lambda + g_{\lambda\nu}\partial_\mu \xi^\lambda = 0 \implies \\ \xi^r \partial_r g_{\mu\nu} + \xi^c \partial_c g_{\mu\nu} + g_{\mu c}\partial_\nu \xi^c + g_{c\nu}\partial_\mu \xi^c = 0\,. \end{aligned} \tag{3.3.14}$$

where we used the fact that $g_{\mu r} = g_{r\mu} = 0$. The (r,r) equation is trivially satisfied since $g_{rr} = 0$. The equation (r,a) implies

$$g_{ca}\partial_r \xi^c = 0\,, \tag{3.3.15}$$

which is a consequence of (3.3.13). The last equation is

$$\xi^r \partial_r g_{ab} + \xi^c \partial_c g_{ab} + g_{ac}\partial_b \xi^c + g_{ca}\partial_b \xi^c = 0\,.$$

Observe that $\partial_r g_{ab} = \frac{2}{r} g_{ab}$ and $\xi^r = \lambda(x^a) r$ then

$$2\lambda(x^a) g_{ab} + \xi^c \partial_c g_{ab} + g_{ac}\partial_b \xi^c + g_{ca}\partial_b \xi^c = 0\,.$$

Consequently, defining $g_{ab} = r^2 h_{ab}$, where $h_{ab} = h_{ab}(x^a)$, this can be written as

$$\xi^c \partial_c h_{ab} + h_{ac}\partial_b \xi^c + h_{ca}\partial_b \xi^c = -2\lambda(x^a) h_{ab}\,, \tag{3.3.16}$$

where there is no r dependency and it corresponds to the conformal Killing equation on a manifold $\mathbb{R}\times S^{D-2}$. Consequently we have found that the diffeomorphisms preserving the (pseudo)-Carrollian structure (3.3.10) are given by the conformal killing vectors of the metric $h_{ab}(x^a)$, with conformal factor $\lambda(x^a)$

Interestingly enough, it can be shown that the limit $\ell \to 0$ of the *AdS* killing vectors, after an appropriate rescaling, match exactly with the isometries of the Carroll structure just discussed.

3.4 Carroll invariant field theories from AdS

We would like to construct Carrollian theories, and using the fact that a Carroll structure can arise from the zero AdS radius limit, then it could be useful to obtain those Carrollian theories from the limit of field theories defined on AdS.

Starting with a Carrollian structure: a manifold M with a degenerated metric $\boldsymbol{g}$ and a vector $\boldsymbol{n}$ in the kernel of $\boldsymbol{g}$. One can define a scalar field $\phi : M \to \mathbb{R}$ and its exterior derivative $\mathrm{d}\phi \in T^*M$. Given a coordinate patch x^μ, the derivatives of the scalars read $\mathrm{d}\phi = \partial_\mu \phi \mathrm{d}x^\mu$. With these ingredients one can construct the D-form $\boldsymbol{\Omega}$ and construct action principles, for instance

$$S = \frac{1}{2}\int (\boldsymbol{n}\phi)^2 \boldsymbol{\Omega} = \frac{1}{2}\int (n^\mu \partial_\mu \phi)^2 \boldsymbol{\Omega}$$

In the simple case of Minkowski Carrollian structure with coordinates (t, x^i), degenerated metric $\boldsymbol{g} = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ and vector in the kernel of the metric $\boldsymbol{n} = \frac{\partial}{\partial t}$, the above action principle reduces to $S = \frac{1}{2}\int \mathrm{d}^4x (\partial_t \phi)^2$ which equation of motion is $\partial_t^2 \phi = 0$, then it does not propagate in space.

To circumvent this problem, and following [104], one can try to add another scalar field χ and construct a 1-form $\boldsymbol{B} = B_\mu \mathrm{d}x^\mu$ such that $g_{\mu\nu}G^{\nu\rho}B_\rho = B_\mu$ without the extra term. For that end one needs to impose $B_\mu n^\mu = 0$. This constraint can be fulfilled by considering the 1-form B being defined by

$$B_\nu = n^\mu \partial_{[\mu}\phi \partial_{\nu]}\chi \iff B = i_n(\mathrm{d}\phi \wedge \mathrm{d}\chi)\,, \tag{3.4.1}$$

where i_n is the contraction operator. Hence one can construct a scalar $B^\mu B_\mu = B_\mu B_\nu G^{\mu\nu}$ and integrate it with our D-form. Namely construct an action principle of the form

$$S = \int \frac{1}{2}\left((n^\mu \partial_\mu \phi)^2 + (n^\mu \partial_\mu \chi)^2\right)\boldsymbol{\Omega} + \int B_\mu B^\mu \boldsymbol{\Omega}\,. \tag{3.4.2}$$

This is the two-swifton model constructed in [104], which lead to an interacting Carrollian field theory, with energy bounded from below.

3.4.1 Free scalar from $c \to 0$ limit

Let us consider the Minkowski metric $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ and a free scalar field. We want to take the limit $c \to 0$. The free action principle of the scalar fields is

$$\begin{aligned}\int \sqrt{-g}\mathrm{d}t\mathrm{d}^3\vec{x}\,\partial_\mu\phi\partial^\mu\phi &= \int c\mathrm{d}t\mathrm{d}^3\vec{x}\, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \int c\mathrm{d}t\mathrm{d}^3\vec{x}\left[-\frac{1}{c^2}(\partial_t\phi)^2 + \delta^{ij}\partial_i\phi\partial_j\phi\right],\\ &= \int \mathrm{d}t\mathrm{d}^3\vec{x}\left[-c^{-1+2N_\phi}(\partial_t\tilde{\phi})^2 + c^{1+2N_\phi}\delta^{ij}\partial_i\phi\partial_j\tilde{\phi}\right],\end{aligned} \tag{3.4.3}$$

where we have rescaled the scalar fields $\phi = c^{N_\phi}\tilde{\phi}$. Note that we can take the so called electric limit by setting $-1+2N_\phi = 0, 1+2N_\phi > 0$ which can be done by setting $N_\phi = 1/2$ and the $c \to 0$ which implies

$$S_{\text{elec}} = -\int \mathrm{d}t\mathrm{d}^3\vec{x}\,(\partial_t\tilde{\phi})^2\,. \tag{3.4.4}$$

This corresponds to the action of a Carrollian free scalar field obtained by the electric limit.

3.4.2 Free scalar from $\ell \to 0$

Now let us start (almost) from scratch by considering the AdS in coordinates $x^\mu = (u, r, \vartheta_i)$ where the line element is (same as before with $\mathrm{d}\Omega^2 = \gamma_{ij}\mathrm{d}\vartheta^i\mathrm{d}\vartheta^j$)

$$\mathrm{d}s^2 = -f(r)\mathrm{d}u^2 - 2\mathrm{d}r\mathrm{d}u + r^2\gamma_{ij}\mathrm{d}\vartheta^i\mathrm{d}\vartheta^j\,, \tag{3.4.5}$$

then the metric and its inverse are

$$g_{\mu\nu} = \left(\begin{array}{cc|c} -f(r) & -1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & r^2\gamma_{ij} \end{array}\right) \qquad g^{\mu\nu} = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ -1 & f(r) & 0 \\ \hline 0 & 0 & \frac{1}{r^2}\gamma^{ij} \end{array}\right) \tag{3.4.6}$$

the determinant of the metric that appears in the action is $\sqrt{-g} = r^{D-2}\sqrt{\det\gamma}$. The action principle of a free scalar field in AdS is

$$\begin{aligned} S &= \int \mathrm{d}u\mathrm{d}r\mathrm{d}^{D-2}\vartheta\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &= \int \mathrm{d}u\mathrm{d}r\mathrm{d}^{D-2}\vartheta r^{D-2}\sqrt{\det\gamma}\left[g^{rr}(\partial_r\phi)^2 + 2g^{ur}\partial_u\phi\partial_r\phi + g^{ij}\partial_i\phi\partial_j\phi\right]\,, \\ &= \int \mathrm{d}u\mathrm{d}r\mathrm{d}^{D-2}\vartheta r^{D-2}\sqrt{\det\gamma}\left[\left(1+\frac{r^2}{\ell^2}\right)(\partial_r\phi)^2 - 2\partial_u\phi\partial_r\phi + \frac{1}{r^2}\gamma^{ij}\partial_i\phi\partial_j\phi\right]\,, \end{aligned}$$

now let us rescalar $u = \ell\tilde{u}$ and $\phi = \ell^{N_\phi}\tilde{\phi}$ leading to (erasing the tilde)

$$\begin{aligned} S &= \int \ell\mathrm{d}u\mathrm{d}r\mathrm{d}^{D-2}\vartheta r^{D-2}\sqrt{\det\gamma}\left[\left(1+\frac{r^2}{\ell^2}\right)\ell^{2N_\phi}(\partial_r\phi)^2 - \frac{2}{\ell}\ell^{2N_\phi}\partial_u\phi\partial_r\phi + \frac{1}{r^2}\ell^{2N_\phi}\gamma^{ij}\partial_i\phi\partial_j\phi\right]\,, \\ &= \int \mathrm{d}^Dx r^{D-2}\sqrt{\det\gamma}\left[\left(\ell^{1+2N_\phi} + r^2\ell^{2N_\phi-1}\right)(\partial_r\phi)^2 - 2\ell^{2N_\phi}\partial_u\phi\partial_r\phi + \frac{1}{r^2}\ell^{2N_\phi+1}\gamma^{ij}\partial_i\phi\partial_j\phi\right]\,, \end{aligned}$$

Setting $N_\phi = \frac{1}{2}$ we can take the limit $\ell \to 0$ which leads

$$S_{\text{elec}} = \int \mathrm{d}^Dx r^D\sqrt{\det\gamma}(\partial_r\phi)^2\,. \tag{3.4.7}$$

At the same time by performing this limit on the metric we obtain

$$\boldsymbol{g} = r^2(-\mathrm{d}u^2 + \gamma_{ij}\mathrm{d}\vartheta^i\mathrm{d}\vartheta^j)\,. \tag{3.4.8}$$

Following our prescription to construct a 4-form we consider an embedded codim-1 surface with invertible induced metric $r = r_0$ and volume element $\boldsymbol{\omega} = r\mathrm{d}u \wedge r^{D-3}\boldsymbol{\omega}(S^{D-2}) = r^{D-2}\mathrm{d}u\wedge\boldsymbol{\omega}(S^{D-2})$ where $\boldsymbol{\omega}(S^{D-2})$ is the standard volume form of the S^{D-2}. Then we can construct a vector in the kernel of the metric of the form

$$\boldsymbol{n} = r^N\frac{\partial}{\partial r} \tag{3.4.9}$$

and a dual 1-form

$$\theta = r^{-N}\mathrm{d}r \tag{3.4.10}$$

such that $\boldsymbol{n}(\theta) = \theta(\boldsymbol{n}) = 1$, and where $N \in \mathbb{R}$. Then we define the volume element as

$$\boldsymbol{\Omega} = \boldsymbol{\omega} \wedge \theta = r^{D-1-N} \mathrm{d}u \wedge \boldsymbol{\omega}(S^{D-2}) \wedge \mathrm{d}r \equiv r^{D-1-N} \mathrm{d}^D x \sqrt{\det \gamma} \tag{3.4.11}$$

Then the covariant form of writing the action for a scalar is $\int \boldsymbol{\Omega}(n^\mu \partial_\mu \phi)^2$. We will fix the constant N such that this action is precisely (3.4.7), namely

$$\int \boldsymbol{\Omega}(n^\mu \partial_\mu \phi)^2 = \int r^{D-1-N} \mathrm{d}^D x \sqrt{\det \gamma} (r^N \partial_r \phi)^2 = \int r^{D-1+N} \mathrm{d}^D x \sqrt{\det \gamma} (\partial_r \phi)^2$$

then $D - 1 + N = D$, hence $N = 1$.

3.4.3 Interacting scalars with propagation on the Carroll structure

Now we would like to turn our attention into a field theory formulated on the Carroll structure (3.3.10) using the ingredients we have introduced. Let us consider the degenerated metric and vector in its kernel defined in (3.3.10). We define the 1-form $\theta = \frac{1}{r} \mathrm{d}r$ that is normalized with $\boldsymbol{n}$, namely $\theta(\boldsymbol{n}) = \boldsymbol{n}(\theta) = 1$. Then the D-dimensional volume form that we consider is

$$\boldsymbol{\Omega} = \boldsymbol{\omega} \wedge \theta = r^{D-2} \mathrm{d}u \wedge \boldsymbol{\omega}(S^{D-2}) \wedge \mathrm{d}r \equiv r^{D-2} \sqrt{\det \gamma} \mathrm{d}^D x \,. \tag{3.4.12}$$

Under this considerations the tensor $G^{\mu\nu}$ that satisfies (3.3.2) and $G^{\mu\nu}\theta_\mu\theta_\nu = 0$ is given by

$$G^{\mu\nu} = \left(\begin{array}{cc|c} -\frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{r^2}\gamma^{ij} \end{array} \right) . \tag{3.4.13}$$

Now let us consider two scalar fields ϕ, χ and construct the following tensors

$$W_{\rho\sigma} \equiv \partial_{[\rho} \phi \partial_{\sigma]} \chi \,, \tag{3.4.14}$$

$$B_\mu = n^\rho W_{\rho\mu} = n^\rho \partial_{[\rho} \phi \partial_{\sigma]} \chi \,. \tag{3.4.15}$$

An interacting Lagrangian that we can consider is

$$S_{\text{car}} = \int_M \boldsymbol{\Omega} \left[\frac{1}{2} (n^\mu \partial_\mu \phi)^2 + \frac{1}{2} (n^\mu \partial_\mu \chi)^2 + \beta G^{\mu\nu} B_\mu B_\nu \right] , \tag{3.4.16}$$

For our case the explicit form is

$$S_{\text{car}} = \int_M r^D \sqrt{\det \gamma} \mathrm{d}^D x \left[\frac{1}{2} (\partial_r \phi)^2 + \frac{1}{2} (\partial_r \chi)^2 \right] + \int_M r^{D-2} \sqrt{\det \gamma} \mathrm{d}^D x \beta G^{\mu\nu} B_\mu B_\nu \,. \tag{3.4.17}$$

Let us expand the interaction term

$$\begin{aligned} G^{\mu\nu} B_\mu B_\nu = G^{uu} B_u B_u + \frac{1}{r^2} \gamma^{ij} B_i B_j &= -\frac{1}{r^2} (n^\rho W_{\rho u})^2 + \frac{1}{r^2} \gamma^{ij} n^\rho W_{\rho i} n^\sigma W_{\sigma j} \\ &= -(W_{ru})^2 + \gamma^{ij} W_{ri} W_{rj} \,. \end{aligned} \tag{3.4.18}$$

Explicitly in terms of the scalars is given by

$$S_{\text{car-int}} = \beta \int_M r^{D-2} \sqrt{\det \gamma} \mathrm{d}^D x \left(-(\partial_{[r} \phi \partial_{u]} \chi)^2 + \gamma^{ij} \partial_{[r} \phi \partial_{i]} \chi \partial_{[r} \phi \partial_{j]} \chi \right) \tag{3.4.19}$$

3.4.4 Interacting scalars from $\ell \to 0$

Let us consider AdS$_D$ in the coordinates (u, r, ϑ^i) with metric given in (3.4.6), two scalar fields ϕ, χ and the following action principle for them

$$S[\phi,\chi] = \int_M \sqrt{-g}\mathrm{d}^D x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \alpha R^{\mu\nu\rho\sigma}\partial_\mu\phi\partial_\nu\chi\partial_\rho\phi\partial_\sigma\chi\right), \tag{3.4.20}$$

$$= \int_M r^{D-2}\sqrt{\det\gamma}\mathrm{d}u\mathrm{d}r\mathrm{d}^{D-2}\vartheta \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \alpha R^{\mu\nu\rho\sigma}\partial_\mu\phi\partial_\nu\chi\partial_\rho\phi\partial_\sigma\chi\right), \tag{3.4.21}$$

where α is dimension-full constant.[1] We know that taking the rescaling of the coordinates and fields

$$u = \ell\tilde{u}\,, \qquad \phi = \ell^{1/2}\tilde{\phi}\,, \qquad \chi = \ell^{1/2}\tilde{\chi}\,, \tag{3.4.22}$$

we can take the limit $\ell \to 0$ in the action and in the metric and we will get something finite. To make this computation explicit, let us consider only the interaction density and define the anti-symmetric tensor $W_{\mu\nu} = \partial_{[\mu}\phi\partial_{\nu]}\chi$, then

$$\mathcal{L}_{\text{int}} = \alpha R^{\mu\nu\rho\sigma}\partial_\mu\phi\partial_\nu\chi\partial_\rho\phi\partial_\sigma\chi = \alpha R^{\mu\nu\rho\sigma}W_{\mu\nu}W_{\rho\sigma}\,. \tag{3.4.23}$$

Note that for AdS

$$R^{\mu\nu}{}_{\rho\sigma} = -\frac{1}{\ell^2}(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho) \qquad \Longrightarrow \qquad R^{\mu\nu\rho\sigma} = -\frac{2}{\ell^2}g^{\rho[\mu}g^{\nu]\sigma}\,. \tag{3.4.24}$$

By running the indices we find that

$$\mathcal{L}_{\text{int}} = \frac{2\alpha}{\ell^2}\big(-2(\partial_{[u}\phi\partial_{r]}\chi)^2 - 4g^{ij}\partial_{[u}\phi\partial_{i]}\chi\partial_{[r}\phi\partial_{j]}\chi \tag{3.4.25}$$
$$+2f(r)g^{ij}\partial_{[r}\phi\partial_{i]}\chi\partial_{[r}\phi\partial_{j]}\chi + g^{ij}g^{kl}\partial_{[i}\phi\partial_{k]}\chi\partial_{[j}\phi\partial_{l]}\chi\big)\,.$$

Therefore the interaction term $S_{\text{int}} = \int\sqrt{-g}\mathrm{d}^D x\mathcal{L}_{\text{int}}$ under the rescaling (3.4.22) maps to

$$S_{\text{int}} = \int r^{D-2}\sqrt{\det\gamma}\ell\mathrm{d}u\mathrm{d}r\mathrm{d}^{D-2}\vartheta\frac{2\alpha}{\ell^2}\left(-2\ell^2\frac{1}{\ell^2}(\partial_{[u}\phi\partial_{r]}\chi)^2 - 4\ell^2\frac{1}{\ell}g^{ij}\partial_{[u}\phi\partial_{i]}\chi\partial_{[r}\phi\partial_{j]}\chi\right.$$
$$\left.+2\left(\frac{r^2}{\ell^2}+1\right)\ell^2 g^{ij}\partial_{[r}\phi\partial_{i]}\chi\partial_{[r}\phi\partial_{j]}\chi + \ell^2 g^{ij}g^{kl}\partial_{[i}\phi\partial_{k]}\chi\partial_{[j}\phi\partial_{l]}\chi\right), \tag{3.4.26}$$

where we have erased the tilde. Now expanding in the limit $\ell \to 0$ we find that

$$S_{\text{int}} = \frac{4\alpha}{\ell}\int r^{D-2}\sqrt{\det\gamma}\mathrm{d}u\mathrm{d}r\mathrm{d}^{D-2}\vartheta\left(-(\partial_{[u}\phi\partial_{r]}\chi)^2 + \gamma^{ij}\partial_{[r}\phi\partial_{i]}\chi\partial_{[r}\phi\partial_{j]}\chi\right) + o(1)\,, \tag{3.4.27}$$

We see that if we rescale $\alpha = \ell\tilde{\alpha}$, then the limit $\ell \to 0$ can be taken and then, the above interaction action reproduces the interaction term (3.4.19) constructed intrinsically on the Carrolian structure. We therfore conclude that the two-swifton interacting model of [104], on its pseudo-Carrollian formulation, can be obtained from the $\ell \to 0$ limit of an interacting model on AdS.

[1] If the coordiantes have length dimension $[x^\mu] = 1$, then $[g_{\mu\nu}] = 0$. Then the length dimension of the scalars is $[\phi] = (2-D)/2$ and consequently $[\alpha] = D+2$.

3.5 Summary

This chapter has investigated the emergence of Carrollian structures as a limit of AdS spacetime when the cosmological constant becomes infinitely large, i.e., when the AdS radius $\ell \to 0$. The study was motivated by both symmetry arguments and geometric constructions, leading to several significant insights. We began by demonstrating that the Carroll algebra can be obtained as a contraction of the AdS algebra $\mathfrak{so}(2,3)$, following the classical classification of Bacry and Lévy-Leblond [7], and extended this procedure to the infinite-dimensional case by applying a similar contraction to the two copies of the Virasoro algebra. The resulting structure aligns with recent developments concerning infinite-dimensional Carrollian extensions [112, 113]. From a geometric perspective, we analyzed the embedding of AdS_4 in $\mathbb{R}^{2,3}$ and showed that the limit $\ell \to 0$ yields a degenerate hypersurface that resembles a light cone. Through several parameterizations, we identified a Carroll structure on this surface (3.2.10). Furthermore, by taking the limit directly in AdS spacetime using coordinates adapted to the contraction, we found that the resulting Carrollian geometry is preserved not by the usual Carroll algebra, but surprisingly by the full AdS algebra $\mathfrak{so}(2,3)$, see section (3.3.2). In the three-dimensional case, this leads to an enhancement of the Carrollian symmetries to the full conformal group in two dimensions, manifesting as two copies of the Virasoro algebra. This unexpected symmetry structure raises an important conceptual question: whether the Carroll geometry obtained in the $\ell \to 0$ limit can be realized as a homogeneous space associated to the AdS group. While known Carrollian spacetimes such as AdS Carroll space do arise as such quotients, the recent classification in [114] suggests that no new homogeneous space arises from the AdS algebra, because the other possible group quotients are related by automorphisms. Determining whether our construction can be reconciled with this classification or represents a genuinely new type of Carrollian geometry remains an open and compelling problem. Lastly, the limiting procedure we employed allowed us to construct Carroll-invariant field theories as limits of AdS-invariant theories, providing a pathway to generate interacting scalar field models that remain dynamical in Carrollian spacetimes. This resonates with recent results showing the viability of propagating degrees of freedom in Carrollian field theories [104]. Taken together, these findings not only clarify the connection between AdS and Carrollian geometries but also suggest promising directions for future work, including the study of holographic dualities involving Carrollian boundaries, the classification of novel homogeneous structures, and the dynamics of interacting fields in ultra-relativistic regimes.

Chapter 4

Nil black holes: Symmetries at the horizon and infinity

Now we arrive to the final chapter of this thesis. Here, we explore asymptotics that are not AdS nor flat, but that naturally emerges as vacuum solutions of General Relativity. Indeed, one of the most interesting aspects of gravity in five or more dimensions is the existence of isolated event horizons with varied geometries, including spaces with non-trivial topology, spaces of non-constant curvature, disconnected horizons, among others. The most significant example is undoubtedly the black ring solution [115, 116], along with its generalizations [117]. There also exist generalizations of topological black holes [118], whose simplest versions have base manifolds composed of the direct product of homogeneous spaces. In five dimensions, there are also black hole solutions whose event horizons have constant-time sections given by other types of spaces, including homogeneous, non-Einstein spaces. Among these, there are black holes whose constant-time horizon foliations are anisotropic spaces that correspond to some of Thurston's geometries [119–121]; i.e. the eight geometries that appear in the geometrization problem of topological spaces in three dimension[1]. These solutions exhibit a richer geometric structure than topological black holes, as well as an asymptotic behavior that differ from those of maximally symmetric spacetimes. This is interesting as, typically, the black hole solutions studied in the literature have base manifolds that are either direct or warped product of spaces of constant

[1]Thurston geometrization theorem states that three-dimensional topological spaces admit a unique geometric structure associated with it. It can be regarded as the three-dimensional analogue of the uniformization theorem for two-dimensional surfaces. The latter states that simply connected Riemann surfaces can be given by one of three constant-curvature geometries. In three dimensions, while it is not always possible to assign a single geometry to a whole topological space, the geometrization theorem states that every closed three-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structure, one of which is the Nil geometry. The Nil geometry has been considered in physics in different context; in particular, in three-dimensional gravity [122, 123].

curvature. Studying horizon with richer structures is of interest, in particular because it permits investigating the applicability of different techniques that have so far been applied only to simple examples to the case of more abstruse horizon geometries. Due to the latter, studying physical properties of such black holes, such as their conserved charges or thermodynamic variables, requires a general framework that works for a wide class of spacetimes. This chapter of the thesis is devoted to address this problem. We will work with Einstein theory in five dimensions with negative cosmological constant normalized as $\Lambda = -1/\ell^2$. This is defined by the Einstein-Hilbert action

$$I = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left(R + \frac{2}{\ell^2} \right) + I_B \tag{4.0.1}$$

where I_B stands for boundary terms. This theory admits black hole solutions with diverse horizon structures. We will consider as a working example the case of black holes with horizons given by the so-called Nil space [119], the geometry associated to the Heisenberg group. For a solution of this type, we will extend the method of computing conserved charges associated with the infinite-dimensional symmetries that emerge near the horizons. This method was recently employed in [124] for the analysis of conserved charges and thermodynamic variables of the Black Ring. Here, we will adapt the computation to the case of black holes with Nil horizons. We will analyze their asymptotic symmetries both in the region near the horizon and in the asymptotic region, compute the associated charges, and the thermodynamic variables of the solution.

The chapter is based on the author's paper [125]. Its organized as follows: in section 4.1, the geometry of black holes with Nil horizon is introduced. In section 4.2, the symmetries of these spacetimes, both in the asymptotic far region and in the near horizon region, are presented. In section 4.3 the thermodynamics of these solutions is analyzed, which amounts to compute the near horizon charges. For then to conclude in section 4.4 by presenting a spinning generalization of the solution valid in the slowly rotating approximation.

4.1 Nilpotent black holes

Let us consider the black hole solution found by Cadeau and Woolgar in [119], whose metric is

$$ds^2 = -V(r)\, dt^2 + \frac{dr^2}{V(r)} + r^{4/3} \left(dx^2 + dy^2 \right) + r^{8/3} \left(dz - \frac{2}{3\ell}\, x\, dy \right)^2 . \tag{4.1.1}$$

with the function

$$V(r) = \frac{2}{11\ell^2} r^2 - \frac{2\mu}{r^{5/3}} \tag{4.1.2}$$

and where $t \in \mathbb{R}$, $r \in \mathbb{R}_+$. μ is an integration constant which we will take to be positive in order for an event horizon to exist. As we will see, μ turns out to be proportional to the mass of the solution. We will denote $x^1 = x$, $x^2 = y$, $x^3 = z$. Each of the space-like coordinates x^A on the constant-t, constant-r base manifold covers a domain D_A ($A = 1, 2, 3$). A compact base manifold is possible provided one considers a quotient by a discrete subgroup of its symmetry group. Locally, the base manifold is a fibration of the Nil geometry. In fact, if we define the rescaled coordinates $\hat{x} = (2/3\ell)^{\frac{1}{2}} x$, $\hat{y} = (2/3\ell)^{\frac{1}{2}} y$, $\hat{z} = z$ and evaluate at $r = (3\ell/2)^{\frac{4}{3}}$, we find the Nil metric

$$ds^2_{\text{Nil}} = \frac{9\ell^2}{4} \left(d\hat{x}^2 + d\hat{y}^2 + (d\hat{z} - \hat{x} d\hat{y})^2 \right). \tag{4.1.3}$$

The Nil manifold is one of the eight Thurston geometries. It fibers the 2-dimensional Euclidean space producing a twisted product $\mathbb{R}^2 \times \mathbb{R}$. It is the geometry of the Heisenberg group, and hence the name Nil, for "Nilpotent". It can be associated to other spaces relevant in physics: it is a section of the Bianchi type II solution to 4-dimensional Einstein equations. Since not only the base manifold but all constant-t, constant-r slices of the space (4.1.1) are locally Nil, we will refer to this solution as the Nil black hole.

The metric describes a black hole whose event horizon is located at $r_H = (11\ell^2\mu)^{\frac{3}{11}}$. The horizon is regular; however, the solution exhibits a curvature singularity at the origin $r = 0$, where the Kretschmann scalar is found to diverge as $R_{\mu\nu\sigma\eta}R^{\mu\nu\sigma\eta} \sim \mu/r^{\frac{22}{3}}$.

In order to study the properties of the solution (4.1.1), such as its asymptotic behavior and the conserved charges associated to it, it is convenient to redefine the radial coordinate as $\rho = r^{1/3}$. This yields

$$ds^2 = -\rho^4 \left(\frac{2}{11\ell^2}\rho^2 - \frac{2\mu}{\rho^9} \right) dt^2 + \frac{9\, d\rho^2}{\left(\frac{2}{11\ell^2}\rho^2 - \frac{2\mu}{\rho^9} \right)} + \rho^4 \left(dx^2 + dy^2 \right) + \rho^8 \left(dz - \frac{2}{3\ell}\, x\, dy \right)^2 . \tag{4.1.4}$$

In these coordinates the horizon is located at $\rho_H = (11\ell^2\mu)^{\frac{1}{11}}$. The ground state corresponds to the particular case $\mu = 0$, which has an extra killing vector and we will consider as the background configuration with respect to which we compute the conserved charges. In the asymptotic region, spacetime (4.1.4) exhibits anisotropic scale invariance, which we will discuss below. This makes it to be related to the solutions discussed in the literature in the context of non-relativistic holography [126]. In particular, it is easy to see that, if we define the new radial coordinate $\hat{r} = \rho^2$ and rescale t and z accordingly, the $dy = dz = 0$ slices of the manifold describe a three-dimensional Lifshitz black hole, which asymptotes to $ds^2_{\text{Lif}} \simeq -\hat{r}^{2(\text{z}+1)}dt^2 + \hat{r}^{-2}d\hat{r}^2 + \hat{r}^2 dz^2$ at large $\hat{r}$, with dynamical exponent $\text{z} = \frac{1}{2}$ (where we fixed $\ell^2 = 8/99$). Similarly, the $dx = dy = 0$ slices are diffeomorphic to a three-dimensional Lifshitz black hole with negative dynamical exponent $\text{z} = -\frac{1}{4}$. We will discuss the asymptotic scale invariance below.

4.2 Symmetries and Noether charges

The local isometry group of the Nil black hole solution has dimension 5, and in the case $\mu = 0$ gets enhanced to a 6-dimensional group generated by the Killing vectors

$$\xi_1 = -3\ell\partial_y \ , \qquad \xi_2 = 3\ell\partial_x + 2y\partial_z \ , \tag{4.2.1}$$

together with an anisotropic scale transformation

$$\xi_3 \quad = \quad 3t\partial_t - \rho\partial_\rho + 2x\partial_x + 2y\partial_y + 4z\partial_z \ , \tag{4.2.2}$$

the spacetime translations

$$\xi_4 = \partial_z \ , \qquad \xi_5 = \partial_t \, , \tag{4.2.3}$$

and the additional special transformation

$$\xi_6 = y\partial_x - x\partial_y - \frac{1}{3\ell}(x^2 - y^2)\partial_z \, . \tag{4.2.4}$$

These Killing vectors obey the following Lie algebra:

$$[\xi_1, \xi_2] = -6\xi_4 \, , \tag{4.2.5}$$

together with

$$[\xi_1, \xi_3] = 2\xi_1 \, , \qquad [\xi_2, \xi_3] = 2\xi_2 \, , \qquad [\xi_3, \xi_4] = -4\xi_4 \, , \qquad [\xi_3, \xi_5] = -3\xi_5 \tag{4.2.6}$$

and

$$[\xi_1, \xi_6] = -\xi_2 \, , \qquad [\xi_2, \xi_6] = \xi_1 \tag{4.2.7}$$

with the Lie products that are not shown here being zero. It might be convenient to rescale some generators, e.g. $\xi_4 \to 6\xi_4$, to write the algebra in a more familiar way. The isometries are globally well defined for $x, y, z \in \mathbb{R}$, i.e. $\cup_{A=1}^3 D_A = \mathbb{R}^3$.

The algebra above contains non-trivial Abelian ideals, e.g. the one generated by $\{\xi_4, \xi_5\}$. It also contains non-Abelian ideals. As said, the symmetry generated by ξ_3 breaks down in the case $\mu \neq 0$, while the other five isometries generated by $\{\xi_1, \xi_2, \xi_4, \xi_5, \xi_6\}$ remain unbroken. In the massive case ($\mu > 0$) the non-trivial part of the algebra reduces to the nilpotent Heisenberg algebra (4.2.5), with ξ_4 being a central element, in direct sum with the semisimple algebra (4.2.7), and both of them in direct sum with ξ_5. In the massless case ($\mu = 0$) the algebra contains additional non-Abelian dimension-2 nilpotent proper subalgebras (4.2.6). The six isometries persist asymptotically, at large ρ for all μ. The Killing vector ξ_3 generates an anisotropic scale transformation

$$t \to \lambda^3 t \, , \qquad \rho \to \lambda^{-1}\rho \, , \qquad x \to \lambda^2 x \, , \qquad y \to \lambda^2 y \, , \qquad z \to \lambda^4 z \, , \tag{4.2.8}$$

similar to those appearing in the Lifshitz spacetimes [126] and their generalizations. Notice that we can consistently assign length dimensions to the coordinates as follows: if we denote length dimensionality as $[\ell] = 1$, and so $[G] = 3$, then we have $[t] = 1$, $[x] = [y] = \frac{1}{3}$, $[z] = -\frac{1}{3}$, $[\rho] = \frac{1}{3}$.

The Nil black hole solution can be accommodated in the following asymptotic condition at infinity,

$$g_{tt} = -\frac{2}{11\ell^2}\rho^6 + \mathcal{O}\left(\rho^{-5}\right) , \qquad g_{\rho\rho} = \frac{99\ell^2}{2}\rho^{-2} + \mathcal{O}\left(\rho^{-13}\right) , \tag{4.2.9}$$

$$g_{11} = \rho^4 + \mathcal{O}\left(\rho^{-7}\right) , \qquad g_{12} = \mathcal{O}\left(\rho^{-2}\right) , \tag{4.2.10}$$

$$g_{13} = \mathcal{O}\left(\rho^{-7}\right) , \qquad g_{22} = \frac{1}{9\ell^2}x^2\rho^8 + \rho^4 + \mathcal{O}\left(\rho^{-7}\right) , \tag{4.2.11}$$

$$g_{23} = -\frac{2}{3\ell}x\rho^8 + \mathcal{O}\left(\rho^{-7}\right) , \qquad g_{33} = \rho^8 + \mathcal{O}\left(\rho^{-7}\right) , \tag{4.2.12}$$

with the gauge fixing conditions $g_{\rho t} = g_{\rho A} = 0$, and with the rest of the components being of order $\sim \mathcal{O}(\rho^{-5})$. One can relax these conditions to gather contributions $g_{1A} \sim \mathcal{O}(\rho^8)$; in fact, there exists a way of expressing the falling off conditions above that makes the symmetry in the (x, y) plane manifest; see (4.4.3)-(4.4.6) below. Notice that in (4.2.9)-(4.2.12) we are using the notation g_{AB} to label the components on the base manifold, e.g. $g_{23} = g_{yz} = g_{zy}$. These conditions yield finite conserved charges. It can also be shown that the symmetry group that preserves such large-ρ behavior is generated by a finite-dimensional Lie group which coincides with the exact isometry group of the background solution $\mu = 0$.

The mass of the Nil black hole (4.1.4) can be computed by different methods, the most efficient one being the phase space method. This yields a result proportional to μ; more precisely,

$$M = \frac{\mathrm{Vol}_3\,\mu}{3\pi G} \quad , \qquad \mathrm{Vol}_3 = \int_{\cup_A D_A} d^3x. \tag{4.2.13}$$

In the case the domains of integration of the coordinates x^A are $D_A \in \mathbb{R}$, the charges diverge due to the non-compactness of the base manifold, and so one has to interpret all physical quantities as densities per unit of 3-volume. Notice that Vol_3 has length dimension $\frac{1}{3}$. Below, we will discuss this value of the mass in relation to the thermodynamics.

4.3 Thermodynamics

The most efficient method to derive the thermodynamic variables of a solution like (4.1.4) is the near horizon computation. Near an isolated horizon we can always consider the following expansion in powers of η

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu}\,\eta + g^{(2)}_{\mu\nu}\,\eta^2 + \mathcal{O}(\eta^3) \tag{4.3.1}$$

where η measures the distance from the horizon: the horizon location is $\eta = 0$. $g^{(n)}_{\mu\nu}$ ($n \in \mathbb{Z}_{\geq 0}$) are functions of v and x^A ($A = 1, 2, 3$), and are independent of η. Next, we impose the following boundary conditions

$$g^{(0)}_{vv} = 0\,, \quad g^{(1)}_{vv} = -2\kappa\,, \quad g^{(0)}_{vA} = 0, \tag{4.3.2}$$

together with the gauge condition

$$g_{\eta v} = -1\,, \quad g_{\eta\eta} = 0\,, \quad g_{\eta A} = 0\,; \tag{4.3.3}$$

κ is the surface gravity. Boundary conditions (4.3.1)-(4.3.3) are the near horizon form studied in [127, 128] and they were considered in many different scenarios. It was shown in [128] that the asymptotic Killing vectors preserving the asymptotic conditions (4.3.1)-(4.3.3) form an infinite-dimensional algebra that includes supertranslations and superrotations, cf. [129]. The horizon supertranslation [130] symmetry corresponds to a local shift in the null coordinate on the horizon, namely $v \to v + f(x, y, z)$. These are generated by

$$\hat{\xi}_5 \;=\; P(x^A)\,\partial_v \tag{4.3.4}$$

which gives an infinite dimensional extension of ξ_5.

The Noether charges associated to the near horizon supertranslations (4.3.4) are given by [128]

$$Q[P\partial_v] = \frac{\kappa}{8\pi G}\int_{\bar{H}_+} d^3x\,\sqrt{\det g^{(0)}_{\mathrm{AB}}}\,P(x^C)\,, \tag{4.3.5}$$

where, again, we have used Latin indices $A, B = 1, 2, 3$ to denote the space-like coordinates on the base manifold. The charges are calculated as integrals over constant-v slices on the horizon, which we denote $\bar{H}_+$. The Dirac brackets of these charges yield the same algebra,

$$\{Q[P_1\partial_v], Q[P_2\partial_v]\} = 0, \tag{4.3.6}$$

and do not pick up any central extension.

Conservation and integrability of horizon charges like (23) have been proven in reference [128], where, among other aspects, a continuity equation was derived with the non-integrable part of the charges acting as a source. This source was shown to vanish for isolated horizons like the one considered here. It may also be instructive to analyze the conservation of horizon charges (23) in relation to a specific scattering process such as the one studied in [131], where the authors, by investigating a process involving a gravitational wave and a black hole, show that both supertranslation and superrotation charges happen to carry information of the BMS symmetries at null infinity.

In order to analyze the near horizon geometry of the Nil black hole, we define the Eddington type coordinates

$$v = t - \int^r \frac{\mathrm{dr}}{V(\mathrm{r})}\,, \qquad \eta = r - r_H\,, \tag{4.3.7}$$

This yields

$$g^{(0)}_{11} = \rho_H^4\,, \quad g^{(0)}_{22} = \rho_H^4 + \frac{4}{9\ell^2}\rho_H^8 x^2\,, \quad g^{(0)}_{23} = g^{(0)}_{32} = -\frac{2}{3\ell}\rho_H^8 x\,, \quad g^{(0)}_{33} = \rho_H^8\,, \qquad (4.3.8)$$

with the components $g^{(0)}_{\mu\nu}$ omitted here being zero, and with $\rho_H^{11} = 11\ell\mu$. From this, we have $\det g^{(0)}_{AB} = \rho_H^{16}$ or, more generally,

$$\sqrt{\det g_{AB}} = \rho^8\,. \qquad (4.3.9)$$

It is worth noticing that, under transformation (4.3.4), the metric (4.1.4) picks up a non-diagonal contribution

$$\delta g^{(0)}_{Av} = 0\,, \qquad \delta g^{(1)}_{Av} = -2\kappa\partial_A P(x^B) \qquad (4.3.10)$$

which yields terms of order $g_{Av} \sim \mathcal{O}(\eta)$ that vanish sufficiently fast near the horizon $\eta = 0$.

The Wald entropy corresponds to the charge

$$Q[\partial_v] = \frac{\kappa}{2\pi}\frac{A}{4G}\,, \qquad (4.3.11)$$

with the area $A = \rho_H^8 \mathrm{Vol}_3$ being the integral on the constant-v slices of the horizon at $\rho = \rho_H$ ($\eta = 0$). κ is the surface gravity: defining radial coordinates $\sigma^2 = 2\kappa^{-1}\eta$ and expanding the metric near the horizon, we get the Rindler space with Unruh temperature $\kappa/2\pi$ in direct product with the fibration of the Nil geometry, namely $ds^2 \simeq -(\kappa^2\sigma^2 dt^2 - d\sigma^2) + ds^2_{\mathrm{Nil}} + ...$, up to terms $\sim \mathcal{O}(\sigma^4)$. With all this, we can evaluate the charge (4.3.11), which gives the Hawking temperature and the Bekenstein-Hawking entropy of the Nil black hole. This yields

$$T = \frac{\kappa}{2\pi} = \frac{\rho_H^3}{6\pi\ell^2}\,, \qquad S = \frac{\rho_H^8 \mathrm{Vol}_3}{4G}\,, \qquad (4.3.12)$$

respectively. The mass of the black hole is given by

$$M = \frac{\rho_H^{11}\mathrm{Vol}_3}{33\pi G\ell^2}\,. \qquad (4.3.13)$$

which agrees with (4.2.13). This result for the mass is positive definite –something not entirely obvious in the case of solutions with event horizons of non-trivial topology– and is shown to obey the first law of black hole mechanics, $dM = T\,dS$.

The asymptotic Killing vectors that generate the symmetries at the Nil horizon yield an infinite-dimensional algebra that contains both supertranslations and superrotations. The generators of this algebra, of which (4.3.4) is an important part of, are given in equations (19) and (29) of reference [128], which here are guaranteed as it has been shown explicitly that it is possible to accommodate the Nil black hole solution in the form (4.3.1)-(4.3.3). Equation (4.3.4) are the generators of the horizon supertranslations.

Equation (4.3.9) shows that the area of the hyper-spheres of constant-ρ at fixed t grow as $\sim \rho^8$, while the Newtonian-like term in the metric (4.1.4) is $\sim 2\mu/\rho^9$. This is related to the fact that the potential function (4.1.2) can be written as

$$V(r) = -\frac{6\pi G}{r^{5/3}\text{Vol}_3}\left(M + \frac{8}{9}M_\Lambda(r)\right) \quad \text{with} \quad M_\Lambda(r) = \frac{\Lambda}{8\pi G}\int_{\cup_i D_i} d^3x \int_0^r dr\, \sqrt{\text{det}g^{(0)}_{AB}}.$$

That is, the constant-density contribution of the cosmological constant Λ can be understood as a gravitational energy contribution to the effective mass enclosed in a volume of radius r weighted with a geometrical Gauss factor $\frac{8}{9}$.

The expression for the mass (4.3.13) satisfies the Smarr type formula $TS = \frac{11}{8}M$. This implies that the Nil black hole has negative free energy $F = M - TS = -\frac{3}{8}M < 0$ for all values of the mass, with positive entropy $S = -\frac{\partial F}{\partial T} > 0$ and positive specific heat $c = \frac{\partial M}{\partial T} > 0$. This implies that the Nil black holes may be in thermal equilibrium with their own Hawking radiation; i.e. they are always *large* black holes, regardless their mass relative to the spacetime curvature.

4.4 Introducing spin

Before concluding, let us present a spinning generalization of the metric (4.1.4), which we derive in the slowly rotating approximation. In fact, it can be shown that the following metric satisfies Einstein equations in vacuum for small values of δa

$$\begin{aligned} ds^2 &= -\rho^4\left(\frac{2}{11\ell^2}\rho^2 - \frac{2\mu}{\rho^9}\right)dt^2 + 9\left(\frac{2}{11\ell^2}\rho^2 - \frac{2\mu}{\rho^9}\right)^{-1}d\rho^2 + \rho^4\left(dx^2+dy^2\right) \\ &\quad +\rho^8\left(dz + \frac{1}{3\ell}(ydx - xdy)\right)^2 + \frac{\delta a}{\rho^5}dt\left(dz + \frac{1}{3\ell}(ydx - xdy)\right). \end{aligned} \qquad (4.4.1)$$

This metric solves Einstein equation at first order in the parameter δa. Metric (4.4.1) satisfies the large ρ expansion (4.2.9)-(4.2.12). δa in the last term of (4.4.1) controls the rotational dragging; notice that the metric is treating now the coordinates x and y in equal footing, which is achieved by performing the change of coordinates $z \to z - xy/(3\ell)$, which induces the following change in the asymptotic Killing vectors:

$$\xi_1 \to \xi_1 = -3\ell\partial_y + x\partial_z\,, \qquad \xi_2 \to \xi_2 = 3\ell\partial_x + y\partial_z\,, \qquad \xi_6 \to \xi_6 = y\partial_x - x\partial_y\,, \qquad (4.4.2)$$

the latter being the rotation in the (x, y) plane. In this frame, the asymptotic conditions at infinity take the form

$$g_{tt} = -\frac{2}{11\ell^2}\,\rho^6 + \mathcal{O}\left(\rho^{-5}\right)\,, \qquad g_{\rho\rho} = \frac{99\ell^2}{2}\,\rho^{-2} + \mathcal{O}\left(\rho^{-13}\right)\,, \tag{4.4.3}$$

$$g_{11} = \frac{1}{9\ell^2}\,y^2\rho^8 + \rho^4 + \mathcal{O}\left(\rho^{-7}\right)\,, \qquad g_{12} = -\frac{1}{9\ell^2}\,xy\rho^8 + \mathcal{O}\left(\rho^{-2}\right)\,, \tag{4.4.4}$$

$$g_{13} = \frac{1}{3\ell}\,y\rho^8 + \mathcal{O}\left(\rho^{-7}\right)\,, \qquad g_{22} = \frac{1}{9\ell^2}x^2\rho^8 + \rho^4 + \mathcal{O}\left(\rho^{-7}\right)\,, \tag{4.4.5}$$

$$g_{23} = -\frac{1}{3\ell}\,x\rho^8 + \mathcal{O}\left(\rho^{-7}\right)\,, \qquad g_{33} = \rho^8 + \mathcal{O}\left(\rho^{-7}\right)\,, \tag{4.4.6}$$

which makes the rotational symmetry in the (x, y) plane manifest; cf. (4.2.9)-(4.2.12). The metric also obeys the requirement

$$g_{tA} \simeq \mathcal{O}(\rho^{-5})\,. \tag{4.4.7}$$

The angular momentum of the solution results proportional to δa. More precisely, the Noether charge associated to the Killing vector ξ_4 gives

$$Q[\xi_4] = \frac{13\,\delta a\,\mathrm{Vol}_3}{96\pi G}\,, \tag{4.4.8}$$

which corresponds to the density of momentum along the z direction; notice that, indeed, (4.4.8) has length dimension $\frac{1}{3}$. On the other hand, the charge associated to the Killing vector ξ_6 is

$$Q[\xi_6] = \frac{13\,\delta a\,I}{288\pi G\ell}\,, \qquad I = \int_{\cup_A D_A} d^3x\,(x^2 + y^2) \tag{4.4.9}$$

which depends on the domains of integration of the variables on the base manifold. (4.4.9) is actually dimensionless and corresponds to the angular momentum. The integral I can be thought of as the moment of inertia associated to rotations on the (x, y) plane, which is exactly what $\xi_6 = y\partial_x - x\partial_y$ generates. This invites to use polar coordinates $x = \mathrm{r}\cos\varphi$, $y = \mathrm{r}\sin\varphi$, with $\xi_6 = -2\partial_\varphi$. In these coordinates, the $dt = d\rho = 0$ sections of the metric take the form $ds^2_{\mathrm{Nil}} = \rho^4(d\mathrm{r}^2 + \mathrm{r}^2 d\varphi^2) + \rho^8(dz - \frac{1}{3\ell}\mathrm{r}^2 d\varphi)^2$.

4.5 Summary

In this chapter we studied solutions of Einstein equations with negative cosmological constant in five dimensions which describe black holes whose event horizons are homogeneous but anisotropic spaces. We focused on the case in which the constant-time slices of the horizon is the Nil geometry, the Thurston geometry associated to the Heisenberg group. For such spaces, we analyzed the symmetries both in the asymptotic region and in the near horizon region. We computed the associated conserved charges, which turned out to be finite and lead to a sensible physical interpretation; see (4.3.13). We analyzed the thermodynamic variables (4.3.12) of these black holes, and we derived a stationary spinning generalization of it in the slowly rotating approximation, which is given by (4.4.1) and carries angular momenta (4.4.8)-(4.4.9).

One of the reasons we have to be interested in the Nil black hole solution is its asymptotic behavior. As we have already pointed out, in the region of large ρ, Nil geometry exhibits an anisotropic scale invariance similar to that of the geometries considered in the context of non-relativistic holography. Therefore, the interpretation of a black hole solution with finite temperature embedded in such spaces could presumably have some application for the holographic description of strongly correlated systems with hyperscaling [132] at finite temperature. It would be interesting if the results for the charges and thermodynamic variables derived here, as well as the geometry (4.4.1), had applications in that context. The holographic interpretation of black hole solutions similar to those discussed here were studied in [133].

Chapter 5

Conclusions and further explorations

This thesis explored different forms of symmetries in diffeomorphism invariant theories. Our overarching objective was to uncover new aspects of symmetry structures in asymptotically flat spacetimes, particularly in contexts potentially relevant to flat space holography. To that end, we analyzed configurations that allow a controlled relation between asymptotically flat and asymptotically AdS geometries.

While previous works, such as [134, 135], have approached this problem by considering the vanishing cosmological constant limit of AdS spacetimes, our strategy deviated from this conventional route. Instead, we utilized symmetry transformations inherent to string theory, such as T-duality, which relate geometries with different asymptotic behaviors, or considered limits where the cosmological constant diverges.

In the first chapter we focused on a three-dimensional supergravity theory that descends from string theory and retains the T-duality symmetry. This duality maps solutions with a $U(1)$ isometry to other solutions with potentially distinct asymptotic structures. A well-known example is the duality between the BTZ black hole and the Horne-Horowitz black string. While the BTZ solution is asymptotically AdS, the black string approaches a flat geometry in a generalized sense—its curvature vanishes asymptotically, though the metric's falloffs differ from canonical Minkowski space.

We first proposed consistent AdS boundary conditions for this theory that include the BTZ black holes. These boundary conditions render the variational principle well-posed through the inclusion of appropriate boundary terms. Remarkably, we showed that the associated asymptotic symmetry algebra is given by two copies of the Virasoro algebra, as in three-dimensional general relativity. Interestingly, this holds despite the theory admitting a broader space of solutions than GR, as it includes propagating degrees of freedom. This suggests that the phase space is structured by the same symmetry even when the solution space is not fully integrable and has a richer dynamical content.

A natural question then arises: how do these boundary conditions behave under T-duality? While T-duality requires the presence of a $U(1)$ isometry, and thus cannot act on the full phase space, it may still be well-defined asymptotically. Motivated by this, we introduced the notion of asymptotic T-duality, applying the duality transformation only at the boundary. This leads to a new class of boundary conditions that include the T-duals of the original solutions and are themselves consistent.

Applying this framework to the Brown-Henneaux boundary conditions along the angular isometry ∂_ϕ, we derived a new set of boundary conditions that includes the Horne-Horowitz black string. We showed that these boundary conditions yield a stationary action and define a consistent phase space. An analysis of the asymptotic symmetries revealed a surprisingly large algebra of conserved charges, all of which are functionals of four arbitrary functions on the boundary circle. These charges form an algebra with a de Witt tower and three additional towers of weights two, one, and zero, forming a centrally extended loop algebra based on the Heisenberg algebra. This structure contains, as subalgebras, several known examples: the centrally extended warped Virasoro algebra, and non-centrally extended versions of $\mathfrak{bms}_3$ and $\mathfrak{bms}_2$.

Given the richness of the resulting algebra, it is pertinent to ask whether some of the charges are kinematical in the sense discussed by [136]. Kinematical charges exhibit three properties: (1) vanishing or unconstrained flux, (2) origin in a corner term of the symplectic potential, and (3) gauge dependence—they can be turned on or off in different gauges. While the charges we obtained indeed have vanishing flux at spatial infinity (a trivial result in this setup), they do not originate from a corner term in the symplectic potential since Θ vanishes at the boundary under the imposed conditions. The gauge dependence, however, remains to be investigated. A natural next step would be to study the same system in Bondi gauge, adapted to null asymptotics in flat spacetimes, potentially revealing more about the nature of these charges. One could, for instance, apply asymptotic T-duality to Brown-Henneaux boundary conditions expressed in Bondi gauge.

Our results also invite comparison with recent discoveries of new charges at spatial infinity in asymptotically flat 4D spacetimes [137]. There, two towers of Abelian charges with a central extension were found—suggesting a possible structural relation to the algebra we uncovered. Exploring the quantum implications of our algebra, such as applications of the Cardy formula, would also be a promising direction (see discussion of chapter 2).

As an unexpected byproduct of our analysis, we identified a novel relation between Brown-Henneaux and Compère–Song–Strominger (CSS) boundary conditions in AdS gravity. Specifically, we embedded the Bañados solution space into the supergravity theory and imposed that one of the chiral functions vanish, ensuring a common null $U(1)$ isometry. Performing a T-duality along this direction yielded a phase space that precisely matches the one defined by CSS boundary conditions in GR. This surprising connection between two AdS boundary conditions via a string-theoretic symmetry merits deeper exploration. Interestingly, some symmetry generators in the chiral Brown-Henneaux theory—corresponding to non-trivial gauge transformations of the Kalb-Ramond field B_{MN}—acquire non-vanishing charges in the CSS case. Motivated by this, we demonstrated that while the asymptotic symmetry transformations remain invariant under T-duality, the associated charges do not necessarily share this invariance.

A number of natural extensions and open questions arise from this work. One direction is to study other examples of spacetimes related by T-duality, particularly in higher dimensions, to see whether similar structures emerge. Another interesting possibility is to explore TsT transformations, which are known to relate BTZ black holes and black strings. Recent studies have investigated the asymptotic symmetries arising from such configurations from a worldsheet perspective, and it would be valuable to understand whether similar results can be obtained by applying TsT transformations to Brown-Henneaux boundary conditions. Lastly, it is worth considering the effect of varying the cosmological constant, especially in light of related theories where a conserved charge associated with this variation has been identified. Understanding how such a charge behaves under T-duality could provide new insights into the structure of the dual phase spaces.

The second chapter of this thesis investigates the limit in which the cosmological constant becomes infinite, effectively corresponding to the vanishing of the AdS radius. This analysis is motivated by both physical intuition and symmetry considerations.

We began by revisiting the work of Bacry and Lévy-Leblond on possible kinematical Lie algebras and their contraction limits [7]. In particular, we showed how the Carroll algebra can be obtained as a contraction of the $\mathfrak{so}(2,3)$ algebra—associated with AdS_4—using a single contraction parameter. This led us to consider an analogous contraction of the two copies of the Virasoro algebra, which are known to encode the symmetries of asymptotically AdS_3 spacetimes. The resulting limit yields an infinite-dimensional extension of the Carroll algebra, in line with recent developments in the literature [112, 113].

Next, we considered the geometric embedding of AdS_4 in a flat space $\mathbb{R}^{2,3}$ and studied the limit $\ell \to 0$, where ℓ denotes the AdS radius. This procedure results in a degenerate hypersurface reminiscent of a null cone or light cone in the embedding space. We examined different parameterizations of this surface and found that a Carroll structure naturally emerges in the limit.

After reviewing the fundamentals of Carroll geometry, we proceeded to analyze the $\ell \to 0$ limit directly in AdS spacetime using coordinates adapted to the limit. The resulting geometry matches one of the Carroll structures previously discussed. However, an intriguing feature arises: the symmetry algebra preserving this Carroll structure is not the Carroll algebra itself, but rather the full AdS algebra $\mathfrak{so}(2,3)$. And in particular, for the $D = 3$ case, the symmetries of the Carroll structure get enhanced to the full conformal group in two dimensions (two copies of the Virasoro algebra). This raises important conceptual questions. In general, homogeneous spacetimes such as Minkowski and AdS arise from quotienting their isometry group by appropriate subgroups. This is also the case for known Carrollian geometries like AdS Carroll space, which has Poincaré symmetry. One may thus wonder whether the Carrollian structure obtained from the $\ell \to 0$ limit can also be understood as a homogeneous space associated with $\mathfrak{so}(2,3)$. A recent classification of homogeneous spaces associated to kinematical Lie groups, presented in [114], suggests that no new homogeneous space should arise from the AdS group, because all the other possibilities are related by automorphisms. It remains an open and compelling problem to clarify whether the Carroll structure found in our construction can genuinely be realized as a homogeneous space or if it represents a fundamentally different object.

Finally, this limiting procedure enabled us to obtain Carroll-invariant field theories as the $\ell \to 0$ limit of AdS-invariant field theories. This connection allows the recovery of interacting Carrollian scalar field theories that support propagating degrees of freedom, in alignment with recent results in [104]. These findings offer a promising bridge between relativistic theories in AdS space and Carrollian dynamics, further enriching the interplay between geometry, symmetry, and field theory in gravitational settings.

In the last chapter of this thesis, we studied a set up which slightly differs from the study of AdS or flat space times, but that can still teach us things about the study of symmetries in gravitational theories, and on how general are expected to be some results in the literature. In this study we made the analysis of solutions to Einstein's equations with negative cosmological constant in five dimensions, describing black holes whose event horizons are homogeneous yet anisotropic. Special attention was given to the case where the spatial sections of the horizon are modeled by the Nil geometry, a non-trivial example among Thurston's three-dimensional homogeneous spaces associated with the Heisenberg group. We investigated the symmetry properties of these spacetimes both in the asymptotic region and near the horizon, and computed the corresponding conserved charges, which were found to be finite and physically well-defined. The thermodynamic behavior of these solutions was also analyzed, including the identification of relevant variables and the construction of a slowly rotating generalization carrying non-trivial angular momentum.

A interesting feature of the Nil black hole is its asymptotic behavior at large radial distance, where the geometry exhibits anisotropic scaling symmetries akin to those encountered in non-relativistic holography. This suggests a potential relevance of these solutions to the holographic description of strongly coupled systems with hyperscaling violation at finite temperature. In particular, the black hole backgrounds studied here could provide a useful laboratory for testing ideas related to gauge/gravity duality in anisotropic and scale-violating regimes. The possibility of applying the thermodynamic results and geometric properties of these solutions in such contexts presents an interesting direction for future research.

The work developed in this thesis has not only led to concrete results, but has also sharpened a number of open questions that I am eager to explore further. One of the most intriguing concerns lies in the nature of conserved charges in gravitational theories and the ambiguities inherent to their construction. In particular, the notion of kinematical charges, as discussed in [136], highlights the subtle interplay between boundary terms in the symplectic structure and the presence or absence of fluxes. A better understanding of how these ambiguities are constrained—especially in closed systems with well-posed variational principles—could provide new insights. While null infinity is naturally permeated by radiation and lacks a variational formulation with fixed energy content, spatial infinity allows for more controlled boundary conditions. It is conceivable that a clearer picture of the physics at null infinity could emerge by first solving the theory at spatial infinity and then extending it to the null region.

In this context, I am particularly interested in the emergence of new logarithmic and overleading $\mathcal{O}(r)$ charges at spatial infinity [137, 138], and their relation to the full asymptotic structure of flat spacetimes. This also ties into recent efforts to understand asymptotic infinities in terms of homogeneous kinematical spaces [139, 140], which offers a geometric framework to classify possible limits and contractions of spacetime symmetries. Given that a Carrollian structure emerges from the AdS spacetime in the limit $\ell \to 0$ and captures its symmetry algebra, including the asymptotic conformal symmetries in three dimensions, it is natural to ask whether similar homogeneous constructions might serve as a basis for defining new dual field theories. This could provide a fresh perspective on the holographic principle, especially in settings beyond the standard AdS/CFT framework, and may reveal novel types of dualities between gravity and Carroll-invariant quantum field theories.

Chapter 6

Conclusiones

Esta tesis exploró diferentes formas de simetrías en teorías invariantes bajo difeomorfismos. Nuestro objetivo general fue descubrir nuevos aspectos de las estructuras de simetría en espaciotiempos asintóticamente planos, particularmente en contextos potencialmente relevantes para la holografía en espacio plano. Para ello, analizamos configuraciones que permiten una relación controlada entre geometrías asintóticamente planas y asintóticamente AdS. Si bien trabajos previos, como [134, 135], han abordado este problema considerando el límite de la constante cosmológica a cero en los espaciotiempos AdS, nuestra estrategia se desvió de esta ruta convencional. En su lugar, utilizamos transformaciones de simetría inherentes a la teoría de cuerdas, como la T-dualidad, que relaciona geometrías con diferentes comportamientos asintóticos, o consideramos límites donde la constante cosmológica diverge.

El trabajo desarrollado en esta tesis no solo ha arrojado resultados concretos, sino que también ha agudizado una serie de preguntas abiertas que deseo explorar con mayor profundidad. Una de las más intrigantes reside en la naturaleza de las cargas conservadas en las teorías gravitacionales y las ambigüedades inherentes a su construcción. En particular, la noción de cargas cinemáticas, como se analiza en [136], destaca la sutil interacción entre los términos de borde en la estructura simpléctica y la presencia o ausencia de flujos. Una mejor comprensión de cómo se limitan estas ambigüedades, especialmente en sistemas cerrados con principios variacionales bien planteados, podría proporcionar nuevas perspectivas. Mientras que el infinito nulo está naturalmente permeado por la radiación y carece de una formulación variacional con un contenido energético fijo, el infinito espacial permite condiciones de contorno más controladas. Es concebible que se pueda obtener una visión más clara de la física en el infinito nulo resolviendo primero la teoría en el infinito espacial y luego extendiéndola a la región nula.

En este contexto, me interesa especialmente la aparición de nuevas cargas logarítmicas y lineales en la expansión radial, $\mathcal{O}(r)$, en el infinito espacial [137, 138], y su relación con la estructura asintótica de los espaciotiempos planos. Esto también se vincula con los esfuerzos recientes por comprender los infinitos asintóticos en términos de espacios cinemáticos homogéneos [139, 140], lo que ofrece un marco geométrico para clasificar posibles límites y contracciones de las simetrías del espaciotiempo. Dado que una estructura carrolliana emerge del espaciotiempo AdS en el límite $\ell \to 0$ y captura su álgebra de simetría, incluyendo las simetrías conformes asintóticas en tres dimensiones, es natural preguntarse si construcciones homogéneas similares podrían servir de base para definir nuevas teorías de campos duales. Esto podría proporcionar una nueva perspectiva sobre el principio holográfico, especialmente en entornos más allá del marco estándar AdS/CFT, y puede revelar nuevos tipos de dualidades entre gravedad y teorías cuánticas de campos invariantes de Carroll.

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