Universidad de Concepción
Facultad de Ciencias Físicas y Matemáticas

Generators of the Cox ring of an embedded hypersurface

Magíster en Matemática

CRISTÓBAL JAVIER HERRERA VALENZUELA
2026

Profesor Guía: Antonio Laface
Departamento de Matemática,
Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción

Contents

1 Introduction

Given an algebraic, projective, irreducible and normal variety Z over an algebraically closed field $\mathbb{K}$ of characteristic 0, with finitely generated and free divisor class group $\mathrm{Cl}(Z)$, its Cox ring $\mathcal{R}(Z)$ is the algebra

$$\mathcal{R}(Z) = \bigoplus_{D \in K} \Gamma(Z, \mathcal{O}_Z(D)),$$

where K is a subgroup of $\mathrm{Div}(Z)$ that projects isomorphically onto $\mathrm{Cl}(Z)$ through the morphism

$$K \to \mathrm{Cl}(Z), \qquad D \mapsto [D] = D + \mathrm{PDiv}(Z),$$

and, writing $E \geq 0$ for an effective divisor E,

$$\Gamma(Z, \mathcal{O}_Z(D)) := \{f \in \mathbb{K}(Z)^* : \mathrm{div}(f) + D \geq 0\} \cup \{0\}.$$

This object turns out to be a graded algebra over $\mathrm{Cl}(Z)$ related canonically to the variety and the interaction of divisors over it. It is an important invariant: when it is finitely generated, it allows one to realize Z as a quotient of a large open subset of an affine variety by the action of the quasitorus $\mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(Z)])$, in analogy with the homogeneous coordinate ring of projective space. We have the diagram

$$\begin{tikzcd} \widehat{Z} \arrow[r, hook] \arrow[d, "p_Z"'] & \overline{Z} \\ Z & \end{tikzcd}$$

Here $\widehat{Z}$ is the relative spectrum of the Cox sheaf of Z, while $\overline{Z} = \mathrm{Spec}(\mathcal{R}(Z))$ is the spectrum of the Cox ring. Here, the complement of $\widehat{Z}$ is a codimension ≥ 2 closed subset of $\overline{Z}$ and the morphism p_Z is a good quotient for the action of the quasi-torus $\mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(Z)])$.

When David Cox introduced the concept of the Cox ring in 1993, it was for the particular case where Z is a toric variety [4]. Since then, the computation of Cox rings for this type of varieties has become one of the fundamental pillars in the study of these graded algebras, as in that same work Cox provided a method to obtain $\mathcal{R}(Z)$ explicitly, thereby motivating the search for concrete descriptions of the Cox ring for other kinds of varieties.

This program of determining Cox rings was not limited to the study of specific varieties. As mentioned by M. Artebani and A. Laface in [2], if we consider the inclusion $i : X \to Z$ of an irreducible, normal, and closed subvariety X of Z such that the restrictions of Weil divisors to X are well defined, there exists a natural homomorphism $i_{\mathcal{R}} : \mathcal{R}(Z) \to \mathcal{R}(X)$. We would then like to find some explicit relation between these two Cox rings by analyzing when this homomorphism is surjective, or by examining its kernel, as J. Hausen did in [8] for the case where Z is a smooth toric variety. In that work, Hausen proved that if the inclusion $i : X \to Z$ is "*neat*", that is, if the pullback $i^* : \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ is well defined and an isomorphism, then $\mathcal{R}(X)$ is isomorphic to a quotient of $\mathcal{R}(Z)$ via the homomorphism $i_{\mathcal{R}}$. This motivated Artebani and Laface to extend the hypothesis by considering Z as a factorial Mori dream space.

This thesis is devoted to determining the Cox ring of X, where Z is a smooth Fano toric variety of dimension ≥ 4 and Picard rank 2, and X is a general anticanonical hypersurface of Z by the following results:

Theorem 1 (Theorem 6.1.1)**.** *Let Z be a normal variety with finitely generated Cox ring $\mathcal{R}(Z)$, and let $X \subseteq Z$ be a normal subvariety with $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$. Assume that:*

(1) X_0 is big in X;

(2) $p_Z^{-1}(X_0)$ is big in $\widehat{X}$;

(3) $\widetilde{X}_1$ is normal;

(4) the pullback $\iota^* : \mathrm{Cl}(Z) \xrightarrow{\sim} \mathrm{Cl}(X)$ is an isomorphism.

Set

$$R := \mathcal{R}(Z)/I(\widetilde{X}).$$

Let $m_1, \ldots, m_s \in R$ be homogeneous elements generating the ideal $I(\widetilde{X} \setminus \widetilde{X}_1)$. Then

$$\mathcal{R}(X) = R_{m_1} \cap \cdots \cap R_{m_s},$$

where the intersection is taken inside the common fraction field.

When Z is smooth and $\widetilde{X}$ is normal, hypotheses (1)–(3) are automatic. Theorem 1 generalizes the results in [8, §2] and the main result of [2]. A coarser version, obtained by localizing at homogeneous generators of the whole irrelevant ideal of Z, is recorded later in Remark 6.1.2.

Theorem 1 generalizes results in [8, §2] and the main result of [2]. As a consequence, we produce Algorithm 1, which computes the Cox ring of X iteratively by adding new generators at each step, and which terminates if and only if the Cox ring is finitely generated (See Proposition 5.1.4). Our first application is the following:

Theorem 2 (Theorem 6.1.3)**.** *Let $(\mathbb{P}, H)$ be a polarized pair, where $\mathbb{P}$ is a normal projective toric variety of dimension $n \geq 3$ and H is an ample Cartier divisor. Consider the Cox ring $\mathcal{R}(\mathbb{P}) = \mathbb{K}[T_1, \dots, T_r]$ and let $X \in |H|$ be a hypersurface defined by a homogeneous polynomial $f \in \mathcal{R}(\mathbb{P})$. If $n \geq 4$, we assume that X is a general element of the linear system $|H|$. If $n = 3$, we assume instead that X is a very general element of $|H|$ and that the divisor $H + K_{\mathbb{P}}$ is base point free. Assume that the irrelevant locus of $\mathbb{P}$ has a unique codimension-two irreducible component, namely $V(T_1, T_2)$. Let $d > 0$ be the multiplicity of f along this component, so that f admits a decomposition*

$$f = \sum_{i=0}^{d} (-1)^i f_i T_1^{d-i} T_2^i, \quad \text{with} f_0, \dots, f_d \in \mathbb{K}[T_1, \dots, T_r] \text{ homogeneous.}$$

Suppose further that the algebraic subset $V(T_1, T_2, f_0, \dots, f_d) \subseteq \mathbb{K}^r$ has codimension $d + 3$. Then Cox ring of X is isomorphic to

$$\mathcal{R}(X) \cong \frac{\mathbb{K}[T_1, \dots, T_r, S_1, \dots, S_d]}{\left\langle \begin{array}{l} f_0 + T_2 S_1, \\ f_i + T_1 S_i + T_2 S_{i+1},\ 1 \leq i \leq d-1, \\ f_d + T_1 S_d \end{array} \right\rangle}$$

where $S_1, \dots, S_d$ are new homogeneous variables of appropriate multidegrees.

Then, we apply Theorem 2 to the special case where X is a general ample hypersurface in a smooth projective toric variety of Picard rank 2:

Corollary 3 (Corollary 6.1.5)**.** *Let $\mathbb{P}$ be a smooth projective toric variety of dimension $n \geq 4$ and Picard rank 2, and let X be a smooth general ample hypersurface of $\mathbb{P}$. Then the Cox ring of X is as given in Theorem 6.1.3 in the following cases:*

(1) The grading matrix and the irrelevant ideal of $\mathbb{P}$ are

$$\begin{bmatrix} 1 & 1 & 0 & -a_1 & \dots & -a_{n-1} \\ 0 & 0 & 1 & 1 & \dots & 1 \end{bmatrix}, \qquad \langle T_1, T_2 \rangle \cap \langle T_3, \dots, T_{n+2} \rangle,$$

where $0 \leq a_1 \leq \dots \leq a_{n-1}$ and X has degree $[a, b]$, with $0 < a \leq \max\{k : a_k = 0\}$ and $b > 0$. In this case, $d = a$.

(2) The grading matrix and the irrelevant ideal of $\mathbb{P}$ are

$$\begin{bmatrix} 1 & \dots & 1 & 0 & -a_1 \\ 0 & \dots & 0 & 1 & 1 \end{bmatrix}, \qquad \langle T_1, \dots, T_n \rangle \cap \langle T_{n+1}, T_{n+2} \rangle,$$

where $a_1 \geq 0$ and X has degree $[a, b]$, with $a > 0$ and $0 < b \leq n - 1$. In this case $d = b$.

This criterion allows us to compute the Cox rings of general anticanonical hypersurfaces in smooth toric Fano varieties of Picard rank two. We also examine ambient toric varieties of Picard rank three to provide examples of hypersurfaces whose Cox rings are finitely generated but not complete intersections.

Relation with joint work. The principal results on embedded Cox rings, the localization algorithm, and the explicit presentation for toric hypersurfaces are joint work of Cristobal Herrera, Antonio Laface, and Luca Ugaglia. This thesis presents those results in a unified form and develops the background and computational material used in the applications.

2 Preliminaries

2.1 Topology

Definition 2.1.1. Let X be an affine variety. We define the **Zariski topology** on X to be the topology whose closed sets are exactly the affine subvarieties of X, that is, the subsets of the form $V(S)$ for some $S \subset A(X)$.

Definition 2.1.2. Let X be a topological space.

(a) We say that X is **reducible** if it can be written as $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$. Otherwise X is called **irreducible**.

(b) The space X is called **disconnected** if it can be written as $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$ with $X_1 \cap X_2 = \emptyset$. Otherwise X is called connected.

Definition 2.1.3. A topological space X is called **Noetherian** if there is no infinite strictly decreasing chain

$$X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \dots$$

of closed subsets of X.

Definition 2.1.4. Let X be a non-empty topological space.

(a) The **dimension** of X is

$$\dim X := \sup \{n : \; Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq X, \text{ with every } Z_i \text{ irreducible and closed}\}.$$

(b) If $Y \subseteq X$ is a non-empty irreducible closed subset, its **codimension** in X is

$$\operatorname{codim}_X(Y) := \sup \{n : \; Y = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subseteq X, \text{ with every } Z_i \text{ irreducible and closed}\}.$$

We write $\dim_{\mathbb{K}} V$ for the dimension of a vector space and reserve $\dim X$ for the dimension of a topological space or variety.

Definition 2.1.5.

(a) A Noetherian topological space X is said to be of **pure dimension** n if every irreducible component of X has dimension n

(b) An affine variety is called...

- a **curve** if it is of pure dimension 1;
- a **surface** if it is of pure dimension 2;
- a **hypersurface** in a pure-dimensional affine variety Y if it is an affine subvariety of Y of pure dimension $\dim Y - 1$.

2.2 Some Affine Structure

Definition 2.2.1. Let X be an affine variety and let $U \subseteq X$ be open. A **regular function** on U is a map $\varphi : U \to \mathbb{K}$ such that, for every $a \in U$, there exist polynomial functions $f, g \in A(X)$ and an open neighbourhood $U_a \subseteq U$ of a satisfying $f|_{U_a} \neq 0$ and

$$\varphi(x) = \frac{g(x)}{f(x)}$$

for every $x \in U_a$. The ring of regular functions on U is denoted by $\mathcal{O}_X(U)$.

Remark 2.2.2. Let U be an open subset of an affine variety X, and let $\varphi \in \mathcal{O}_X(U)$ be a regular function on U. Then

$$V(\varphi) := \{x \in U : \varphi(x) = 0\}$$

is closed in U.

Definition 2.2.3. For an affine variety X and a polynomial function $f \in A(X)$ on X we call

$$D(f) := X \setminus V(f) = \{x \in X : f(x) \neq 0\}$$

the **distinguished open subset** of f in X.

Definition 2.2.4. A **presheaf** $\mathcal{F}$ (of rings) on a topological space X consists of the data:

- for every open set $U \subset X$ a ring $\mathcal{F}(U)$
- for every inclusion $U \subset V$ of open sets in X a ring homomorphism $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ called the restriction map,

such that

- $\mathcal{F}(\emptyset) = 0$,
- $\rho_{U,U}$ is the identity map on $\mathcal{F}(U)$ for all U,
- for any inclusion $U \subset V \subset W$ of open sets in X we have $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$.

The elements of $\mathcal{F}(U)$ are usually called **sections** of $\mathcal{F}$ over U, and the restriction maps $\rho_{V,U}$ are written as $\varphi \mapsto \varphi|_U$.

A presheaf $\mathcal{F}$ is called a **sheaf** of rings if it satisfies the following gluing property: If $U \subset X$ is an open set, $\{U_i : i \in I\}$ an arbitrary open cover of U and $\varphi_i \in \mathcal{F}(U_i)$ sections of all i such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is a unique $\varphi \in \mathcal{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all i.

Construction 2.2.5. Let $\mathcal{F}$ be a presheaf on a topological space X, and let $a \in X$. Then the **stalk** of $\mathcal{F}$ at a is defined as

$$\mathcal{F}_a := \{(U, \varphi) : U \subset X \text{ open with } a \in U, \text{ and } \varphi \in \mathcal{F}(U)\}/\sim$$

where $(U, \varphi) \sim (U', \varphi')$ if there is an open subset V with $a \in V \subset U \cap U'$ and $\varphi|_V = \varphi'|_V$.

The elements of $\mathcal{F}_a$ are called **germs** of $\mathcal{F}$ at a. In the case of the sheaf $\mathcal{O}_X$ on an affine variety X, it is customary to write its stalk at a point $a \in X$ as $\mathcal{O}_{X,a}$ instead of as $(\mathcal{O}_X)_a$.

2.3 Varieties

Definition 2.3.1. A **prevariety** is a ringed space X that has a finite open cover by affine varieties. Morphisms of prevarieties are morphisms of ringed spaces. In accordance with 2.2.1, the elements of $\mathcal{O}_X(U)$ for an open subset $U \subset X$ will be called **regular functions** on U.

Construction 2.3.2. Let X_1, X_2 be two prevarieties, and let $U_{1,2} \subset X_1$ and $U_{2,1} \subset X_2$ be open subsets. Moreover, let $f : U_{1,2} \to U_{2,1}$ be an isomorphism. Then we can define a prevariety X obtained by *gluing* X_1 and X_2 along f. For more details see the Construction 5.4 in [7]

Definition 2.3.3. A prevariety X is called a **variety** (or **separated**) if the **diagonal**

$$\Delta_X := \{(x, x) : x \in X\}$$

is closed in $X \times X$.

Definition 2.3.4. Analogously to Definition 2.1.5, a variety of pure dimension 1 or 2 is called a **curve** or a **surface** respectively. If X is a pure-dimensional variety and Y a pure-dimensional closed subvariety of X with $\dim Y = \dim X - 1$ we say that Y is a **hypersurface** in X.

2.4 Gradings and quasitorus actions

Definition 2.4.1. An (affine) **algebraic group** is an (affine) variety G together with a group structure such that the group laws are morphisms of varieties:

$$G \times G \to G,\ (g_1, g_2) \mapsto g_1 g_2, \qquad G \to G,\ g \mapsto g^{-1}.$$

A *homomorphism* of two algebraic groups G and G' is a morphism $G \to G'$ of the underlying varieties that moreover is a homomorphism of groups.

Definition 2.4.2.

(a) A **character** of an algebraic group G is a homomorphism $\chi : G \to \mathbb{K}^*$ of algebraic groups, where $\mathbb{K}^*$ is the multiplicative group of the ground field $\mathbb{K}$.

(b) The **character group** of G is the set $\mathbb{X}(G)$ of all characters of G together with pointwise multiplication. Note that $\mathbb{X}(G)$ is an abelian group, and, given any homomorphism $\varphi : G \to G'$ of algebraic groups, one has a pullback homomorphism

$$\varphi^* : \mathbb{X}(G') \to \mathbb{X}(G), \qquad \chi' \mapsto \chi' \circ \varphi.$$

Definition 2.4.3. A **quasitorus** is an affine algebraic group H whose algebra of regular functions $\Gamma(H, \mathcal{O})$ is generated as a $\mathbb{K}$-vector space by the characters $\chi \in \mathbb{X}(H)$. A **torus** is a connected quasitorus.

Notation 2.4.4. The *standard n-torus* is the n-fold direct product $\mathbb{T}^n := (\mathbb{C}^*)^n$.

2.5 Projective Varieties

Definition 2.5.1. Let $n \in \mathbb{N}$. We define the **projective n-space** over $\mathbb{K}$, denoted $\mathbb{P}^n_{\mathbb{K}}$ or simply $\mathbb{P}^n$, to be the set of all 1-dimensional linear subspaces of the vector space $\mathbb{K}^{n+1}$.

Notation 2.5.2. A 1-dimensional linear subspace of $\mathbb{K}^{n+1}$ is uniquely determined by a non-zero vector in $\mathbb{K}^{n+1}$, with two such vectors spanning the same linear subspace if and only if they are scalar multiples of each other. In other words, we have

$$\mathbb{P}^n = (\mathbb{K}^{n+1} \setminus \{0\}) / \sim$$

with equivalence relation

$$(x_0, \dots, x_n) \sim (y_0, \dots, y_n) \qquad \Leftrightarrow \qquad x_i = \lambda y_i \text{ for some } \lambda \in \mathbb{K}^* \text{ and all } i$$

This is usually written as $\mathbb{P}^n = (\mathbb{K}^{n+1} \setminus \{0\})/\mathbb{K}^*$, and the equivalence class of $(x_0, \dots, x_n)$ will be denoted by $[x_0 : \dots : x_n] \in \mathbb{P}^n$. They are called the **homogeneous coordinates** of the point.

Definition 2.5.3. Let K be an abelian group.

(a) A **K-graded ring** is a ring R together with additive subgroups $R_w \subseteq R$, for $w \in K$, such that
$$R = \bigoplus_{w \in K} R_w, \qquad R_w R_{w'} \subseteq R_{w+w'}.$$
A non-zero element of R_w is called **homogeneous of degree** w.

(b) If R is a $\mathbb{K}$-algebra and every R_w is a $\mathbb{K}$-vector subspace, then R is a **K-graded $\mathbb{K}$-algebra**.

When $K = \mathbb{Z}_{\geq 0}$, this recovers the usual non-negatively graded case used for homogeneous coordinate rings. No degree is assigned to a non-homogeneous element.

Definition 2.5.4. An ideal in a graded ring is called **homogeneous** if it can be generated by homogeneous elements.

Definition 2.5.5. Let $n \in \mathbb{N}$.

(a) Let $S \subset \mathbb{K}[x_0, \dots, x_n]$ be a set of homogeneous polynomials. Then the (projective) **zero locus** of S is defined as
$$V(S) := \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in S\} \ \subset \mathbb{P}^n.$$
Subsets of $\mathbb{P}^n$ that are of this form are called **projective varieties**. For $S = (f_1, \dots, f_k)$ we will write $V(S)$ also as $V(f_1, \dots, f_k)$.

(b) For a homogeneous ideal $J \trianglelefteq \mathbb{K}[x_0, \dots, x_n]$ we set

$$V(J) := \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all homogeneous } f \in J\} \ \subset \mathbb{P}^n.$$

Clearly, if J is the ideal generated by a set S of homogeneous polynomials then $V(J) = V(S)$.

(c) If $X \subset \mathbb{P}^n$ is any subset, we define its **ideal** to be

$$I(X) := \langle f \in \mathbb{K}[x_0, \dots, x_n] \text{ homogeneous } : f(x) = 0 \text{ for all } x \in X\rangle \ \trianglelefteq \mathbb{K}[x_0, \dots, x_n].$$

Definition 2.5.6. Let $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n, (x_0, \dots, x_n) \mapsto [x_0 : \dots : x_n]$.

(a) An affine variety $X \subset \mathbb{A}^{n+1}$ is called a **cone** if $0 \in X$, and $\lambda x \in X$ for all $\lambda \in \mathbb{K}$ and $x \in X$. In other words, it consists of the origin together with a union of lines through 0.

(b) For a cone $X \subset \mathbb{A}^{n+1}$ we call

$$\mathbb{P}(X) := \pi(X \setminus \{0\}) = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : (x_0, \dots, x_n) \in X \setminus \{0\}\} \ \subset \mathbb{P}^n$$

the **projectivization** of X.

(c) For a projective variety $X \subset \mathbb{P}^n$ we call

$$C(X) := \{0\} \cup \pi^{-1}(X) = \{0\} \cup \{(x_0, \dots, x_n) : [x_0 : \dots : x_n] \in X\} \ \subset \mathbb{A}^{n+1}$$

the **cone** over X.

Definition 2.5.7. The (radical homogeneous) ideal

$$I_0 := \langle x_0, \dots, x_n\rangle \trianglelefteq \mathbb{K}[x_0, \dots, x_n]$$

is called the **irrelevant ideal**.

Construction 2.5.8. Let $X \subset \mathbb{P}^n$ be a projective variety. Then

$$S(X) := \mathbb{K}[x_0, \dots, x_n]/I(X)$$

is the **homogeneous coordinate ring** of X.

Definition 2.5.9. The **Zariski topology** on a projective variety X is the topology whose closed sets are exactly the projective subvarieties of X, that is, the subsets of the form $V(S)$ for some set $S \subset S(X)$ of homogeneous elements.

Definition 2.5.10. Let U be an open subset of a projective variety X. A **regular function** on U is a map $\varphi : U \to \mathbb{K}$ such that, for every $a \in U$, there exist $d \geq 0$, homogeneous elements $f, g \in S(X)_d$, and an open neighbourhood $U_a \subseteq U$ of a for which $f|_{U_a} \neq 0$ and

$$\varphi(x) = \frac{g(x)}{f(x)}$$

for every $x \in U_a$.

Definition 2.5.11. Let X and Y be irreducible varieties. A **rational map** f from X to Y written $f : X \dashrightarrow Y$, is a morphism $f : U \to Y$ from a non-empty open subset $U \subset X$ to Y. We say that two such rational maps $f_1, f_2 : X \dashrightarrow Y$ defined on U_1, U_2 respectively are the same if $f_1 = f_2$ on a non-empty open subset of $U_1 \cap U_2$.

Construction 2.5.12. Let X be an irreducible variety. A rational map $\varphi : X \dashrightarrow \mathbb{A}^1 = \mathbb{K}$ is called a **rational function** on X. In other words, a rational function on X is given by a regular function $\varphi \in \mathcal{O}_X(U)$ for some non-empty open subset $U \subset X$, with two such regular functions defining the same rational function if and only if they agree on a non-empty open subset. The set of all rational functions on X will be denoted $\mathbb{K}(X)$. Note that $\mathbb{K}(X)$ is a field: For $\varphi_1 \in \mathcal{O}_X(U_1)$ and $\varphi_2 \in \mathcal{O}_X(U_2)$ we can define $\varphi_1 + \varphi_2$ and $\varphi_1\varphi_2$ on $U_1 \cap U_2 \neq \emptyset$, the additive inverse $-\varphi_1$ on U_1, and for $\varphi_1 \neq 0$ the multiplicative inverse φ_1^{-1} on $U_1 \setminus V(\varphi_1)$. We call $\mathbb{K}(X)$ the **function field** of X.

Definition 2.5.13. An open subset $U \subseteq Y$ is *big* in Y if its complement $Y \setminus U$ has codimension at least 2.

Definition 2.5.14. Let X be an affine variety. We say that X is **normal** if its coordinate ring $A(X)$ is integrally closed in its field of fractions. More generally, a variety X is normal if every local ring $\mathcal{O}_{X,x}$ is an integrally closed domain.

Remark 2.5.15. If X is locally Noetherian, Serre's criterion states that X is normal if and only if it satisfies R_1 and S_2. In particular, a Cohen–Macaulay variety is normal if and only if it is regular in codimension one.

Example 2.5.16. An example of **non**-normal affine variety is given by any singular curve, since the first condition of normality is not satisfied.

Example 2.5.17. An example where the second condition is not satisfied is the affine variety X defined by the subalgebra $A \subseteq \mathbb{C}[x, y]$ generated by all the monomials but x. It is easy to see that A is in fact generated by x^2, x^3, y, xy, so that X is the image in $\mathbb{C}^4$ of the map $(x, y) \mapsto (x^2, x^3, y, xy)$. If u_1, u_2, u_3, u_4 are coordinates of $\mathbb{C}^4$ then on the open subset $U := X \setminus V(u_1, u_3)$ the function $u_2/u_1 = u_4/u_3$ is regular but it is not restriction of a regular function on X.

3 Toric varieties

3.1 Main elements

Definition 3.1.1. A **toric variety** is an irreducible, normal variety X together with an algebraic torus action $\mathbb{T} \times X \to X$ and a base point $x_0 \in X$ such that the orbit map $\mathbb{T} \to X$, $t \mapsto t \cdot x_0$ is an open embedding. In this setting, we refer to $\mathbb{T}$ as to the *acting torus* of the toric variety X. If we want to specify notation, we sometimes denote a toric variety X with acting torus $\mathbb{T}$ and base point x_0 as a triple $(X, \mathbb{T}, x_0)$.

Definition 3.1.2. Let X and X' be toric varieties. A **toric morphism** from X to X' is a pair $(\varphi, \widetilde{\varphi})$, where $\varphi : X \to X'$ is a morphism with $\varphi(x_0) = x_0'$ and $\widetilde{\varphi} : \mathbb{T} \to \mathbb{T}'$ is a homomorphism of the respective acting tori such that $\varphi(t \cdot x) = \widetilde{\varphi}(t) \cdot \varphi(x)$ holds for all $t \in \mathbb{T}$ and $x \in X$.

Definition 3.1.3. A **lattice** is a $\mathbb{Z}$-module $N \cong \mathbb{Z}^n$. The rational vector space associated with a lattice $N \cong \mathbb{Z}^n$ is $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^n$.

Definition 3.1.4. A (convex, polyhedral) **cone** in a finite-dimensional rational vector space V is a subset of the form

$$\sigma = \operatorname{cone}(v_1, \dots, v_r) := \{\sum a_i v_i : a_i \in \mathbb{Q}_{\geq 0} \subseteq V\},$$

where $v_1, \dots, v_r \in V$. This is similar in structure to Definition 2.5.6.

Remark 3.1.5. If $V = N_{\mathbb{Q}}$ with lattice N, then we call the pair (σ, N) a **lattice cone** or say that σ is a *cone in* N.

Definition 3.1.6. A **morphism of lattice cones** $(\sigma, N) \to (\sigma', N')$ is a homomorphism $F : N \to N'$ such that $F(\sigma) \subset \sigma'$. As before, F also denotes the induced linear map of rational vector spaces $N_{\mathbb{Q}} \to N'_{\mathbb{Q}}$. To each lattice cone, one can functorially associate a toric variety.

Notation 3.1.7. A subset σ of a finite-dimensional rational vector space V is a cone if and only if

$$\sigma = \operatorname{posloc}(u_1, \dots, u_s) := \{v \in V; u_1(v) \geq 0, \dots, u_s(v) \geq 0\}$$

holds with linear forms $u_1, \dots, u_s \in U := \mathrm{Hom}(V, \mathbb{Q})$. In this case, the linear forms $u_1, \dots, u_s$ generate the **dual cone** of σ:

$$\sigma^\vee := \{u \in U; u|_\sigma \geq 0\} = \mathrm{cone}(u_1, \dots, u_s) \subseteq U.$$

Definition 3.1.8. A **face** of a cone $\sigma \subseteq V$ is a subset $\tau \subseteq \sigma$ cut out by a linear form, that means that there is a $u \in \sigma^\vee$ with

$$\tau = u^\perp \cap \sigma = \{v \in \sigma : u(v) = 0\}.$$

We write $\tau \preccurlyeq \sigma$ if $\tau \subseteq \sigma$ is a face of the cone $\sigma \subseteq V$. A cone $\sigma \subseteq V$ is called *pointed*, if $\{0\} \preccurlyeq \sigma$ holds.

Definition 3.1.9. A **fan** in a finite dimensional vector space V is a finite set Σ of pointed cones in V such that

(a) for each $\sigma \in \Sigma$, also every $\tau \preccurlyeq \sigma$ belongs to Σ,

(b) for any two $\sigma, \sigma' \in \Sigma$ we have $\sigma \cap \sigma' \preccurlyeq \sigma, \sigma'$.

If $V = N_\mathbb{Q}$ holds with a lattice N, then we call the pair (Σ, N) a *lattice fan* or refer to Σ as a *fan in N*.

3.2 Tori and lattices

A natural first step in the combinatorial description of toric varieties is to relate affine toric varieties to *lattice cones*.

Definition 3.2.1. A **one parameter subgroup** (1-psg) of a torus $\mathbb{T}$ is a homomorphism $\lambda : \mathbb{C}^* \to \mathbb{T}$ of algebraic groups.

Remark 3.2.2. The set $\Lambda(\mathbb{T})$ of all 1-psg of a torus $\mathbb{T}$ becomes a lattice by defining the addition as pointwise multiplication:

$$(\lambda + \lambda')(t) := \lambda(t)\lambda'(t).$$

In order to see that this is indeed a lattice, it suffices to look at $\mathbb{T} = \mathbb{T}^n$. There we have mutually inverse isomorphisms

$$\begin{array}{ccc} \mathbb{Z}^n & \longleftrightarrow & \Lambda(\mathbb{T}^n) \\ v & \mapsto & [\lambda_v : t \mapsto (t^{v_1}, \dots, t^{v_n})], \\ \frac{\partial \lambda}{\partial t}(1) & \leftarrow\!\shortmid & \lambda. \end{array}$$

Definition 3.2.3. A 1-psg $\lambda : \mathbb{C}^* \to \mathbb{T}$ *converges* in $(X, \mathbb{T}, x_0)$ if the map $t \mapsto \lambda(t) \cdot x_0$ extends to a morphism $\overline{\lambda} : \mathbb{C} \to X$. In this case,

$$\lim(\lambda) := \lim_{t\to 0} \lambda(t) \cdot x_0 = \overline{\lambda}(0) \in X$$

is the *limit* of the 1-psg λ in X. The *convergency cone* of such a limit point $x \in X$ is the lattice cone $\sigma(x)$ in $\Lambda(\mathbb{T})$ given by

$$\sigma(x) := \overline{\operatorname{cone}(\lambda \in \Lambda(\mathbb{T}) : \lim(\lambda) = x)} \subseteq \Lambda_{\mathbb{Q}}(\mathbb{T}).$$

Theorem 3.2.4. *Let $(X, \mathbb{T}, x_0)$ be a normal toric variety. Then the set $\Sigma(X)$ of all convergency cones of X is a fan in the lattice $\Lambda(\mathbb{T})$.*

Remark 3.2.5. With any matrix $A = (a_{ij}) \in \operatorname{Mat}(m, n; \mathbb{Z})$, we associate a homomorphism of standard tori by

$$\varphi_A : \mathbb{T}^n \to \mathbb{T}^m, \ t \mapsto (t^{A_{1*}}, \dots, t^{A_{m*}}), \ t^{A_{i*}} := t_1^{a_{i1}} \cdots t_n^{a_{in}}.$$

So, φ_A is the monomial map having the rows of A as its exponent vectors. We have

$$\varphi_{AB} = \varphi_A \circ \varphi_B$$

whenever the integral matrices A and B fit together. Moreover, there are mutually inverse bijections

$$\begin{array}{ccc} \operatorname{Hom}(\mathbb{T}^n, \mathbb{T}^m) & \longleftrightarrow & \operatorname{Mat}(m, n; \mathbb{Z}), \\ \varphi & \mapsto & J_\varphi(\mathbb{1}_n), \\ \varphi_A & \leftarrow\!\shortmid & A. \end{array}$$

Here $\mathbb{1}_n := (1, \dots, 1) \in \mathbb{C}^n$, and $J_\varphi(\mathbb{1}_n)$ denotes the Jacobian of φ at $\mathbb{1}_n \in \mathbb{T}^n$.

Theorem 3.2.6. *We have a covariant equivalence of categories:*

$$\begin{array}{ccc} \{\textit{normal toric varieties}\} & \longleftrightarrow & \{\textit{lattice fans}\}, \\ (X, \mathbb{T}, x_0) & \mapsto & (\Sigma(X), \Lambda(\mathbb{T})), \\ (\varphi, \widetilde{\varphi}) & \leftarrow\!\shortmid & \widetilde{\varphi}_*. \end{array}$$

Definition 3.2.7. Analogously to Definition 2.4.2, and restricting to $\mathbb{K} = \mathbb{C}$, a **character of a torus** $\mathbb{T}$ is a homomorphism of algebraic groups $\chi : \mathbb{T} \to \mathbb{C}^*$. The set $\mathbb{X}(\mathbb{T})$ of all characters of a torus $\mathbb{T}$ becomes a lattice by defining the addition as pointwise multiplication:

$$(\chi + \chi')(t) = \chi(t)\chi'(t).$$

As for the 1-psg, we see that this is indeed a lattice by looking at $\mathbb{T} = \mathbb{T}^n$. There we have mutually inverse isomorphisms

$$\begin{array}{ccc} \mathbb{Z}^n & \longleftrightarrow & \mathbb{X}(\mathbb{T}^n), \\ u & \mapsto & [\chi^u : t \mapsto t_1^{u_1} \cdots t_n^{u_n}], \\ \operatorname{grad}_\chi(\mathbb{1}_n) & \leftarrow\!\shortmid & \chi. \end{array}$$

Construction 3.2.8. Let N be a lattice and set $M := \mathrm{Hom}(N, \mathbb{Z})$. The group algebra associated with M is defined by

$$\mathbb{C}[M] := \bigoplus_{u \in M} \mathbb{C}\chi^u, \quad \chi^u \chi^{u'} := \chi^{u+u'}.$$

Note that $\mathbb{C}[M]$ is isomorphic to a Laurent monomial algebra. We obtain a torus by passing to the spectrum

$$T_N := \mathrm{Spec}\, \mathbb{C}[M],$$

where the unit element $\mathbb{1}_N \in \mathbb{T}_N$ is determined by $\chi^u(\mathbb{1}_N) = 1$ for all $u \in M$ and the multiplication is given by its comorphism

$$\mathbb{C}[M] \to \mathbb{C}[M] \otimes \mathbb{C}[M], \qquad \chi^u \mapsto \chi^u \otimes \chi^u.$$

Remark 3.2.9. Let $F : N \to N'$ be a lattice homomorphism. The dual homomorphism $F^* : M' \to M$ yields a homomorphism

$$\psi_F : \mathbb{C}[M'] \to \mathbb{C}[M], \quad \chi^{u'} \mapsto \chi^{F^*(u')} = \chi^{u' \circ F}.$$

Passing to spectra gives a morphism $\varphi_F : \mathbb{T}_N \to \mathbb{T}_{N'}$, which turns out to be a homomorphism of algebraic groups.

Theorem 3.2.10. *We have covariant functors being essentially inverse to each other:*

$$\begin{array}{ccc} \{\textit{lattices}\} & \longleftrightarrow & \{\textit{tori}\} \\ (N; F) & \mapsto & (\mathbb{T}_N; \varphi_F) \\ (\Lambda(\mathbb{T}); \varphi_*) & \leftarrow\!\shortmid & (\mathbb{T}; \varphi). \end{array}$$

3.3 Varieties and lattices

Definition 3.3.1. The **dual** of a lattice cone (σ, N), is the lattice cone $(\sigma^\vee, M)$, where

$$M = \mathrm{Hom}(N, \mathbb{Z}), \quad \sigma^\vee = \{u \in M_\mathbb{Q} : u|_\sigma \geq 0\} \subseteq M_\mathbb{Q}.$$

Definition 3.3.2. Let (ω, M) be a lattice cone. The associated **cone monoid** is the additive submonoid $\omega \cap M \subseteq M$.

Remark 3.3.3. Let (ω, M) be a lattice cone. Then the cone monoid $\omega \cap M$ is finitely generated. If ω is of full dimension in $M_\mathbb{Q}$, then $\omega \cap M$ generates M as a group.

Definition 3.3.4. Let (ω, M) be a lattice cone. The associated **monoid algebra** is defined by

$$\mathbb{C}[\omega \cap M] := \bigoplus_{u \in \omega \cap M} \mathbb{C}\chi^u, \quad \chi^u \chi^{u'} := \chi^{u+u'}.$$

Remark 3.3.5. Let (ω, M) be a lattice cone. Then $\mathbb{C}[\omega \cap M]$ is an integral, normal, finitely generated subalgebra of $\mathbb{C}[M]$. If ω is of full dimension in $M_{\mathbb{Q}}$, then $\mathbb{C}[\omega \cap M]$ and $\mathbb{C}[M]$ have the same quotient field.

Construction 3.3.6. Let (σ, N) be a *pointed* lattice cone. Then the dual lattice cone $(\sigma^\vee, M)$ is of full dimension. Consider

$$T_N = \operatorname{Spec} \mathbb{C}[M], \qquad X_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$$

and the point $x_0 \in X_\sigma$ defined by $\chi^u(x_0) = 1$ for all $u \in \sigma^\vee \cap M$. Then

$$\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[M] \otimes \mathbb{C}[\sigma^\vee \cap M], \qquad \chi^u \mapsto \chi^u \otimes \chi^u$$

is the comorphism of an action $\mathbb{T}_N \times X_\sigma \to X_\sigma$ and $(X_\sigma, \mathbb{T}_N, x_0)$ is a normal *affine* toric variety.

Remark 3.3.7. Let $(\sigma, \mathbb{Z}^n)$ be a pointed lattice cone and suppose that we know generators for the monoid associated with the dual cone:

$$\sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0} u_1 + \cdots + \mathbb{Z}_{\geq 0} u_r \subseteq \mathbb{Z}^n.$$

Then the variety $X_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$ can be concretely realized via the closed embedding

$$X_\sigma \to \mathbb{C}^r, \qquad x \mapsto (\chi^{u_1}(x), \dots, \chi^{u_r}(x)).$$

Here, the base point x_0 is mapped to $\mathbb{1}_r$ and the action of $\mathbb{T}^r$ on X is explicitly given by

$$t \cdot z = (\chi^{u_1}(t) z_1, \dots, \chi^{u_r}(t) z_r).$$

Theorem 3.3.8. *We have covariant functors being essentially inverse to each other:*

$$\{\text{pointed lattice cones}\} \quad \longleftrightarrow \quad \{\text{normal affine toric varieties}\}$$

$$\begin{array}{ccc} (\sigma, N) & \mapsto & (X_\sigma, \mathbb{T}_N, x_0) \\ F & \mapsto & (\varphi_F, \widetilde{\varphi}_F) \\ & & \\ (\sigma(X), \Lambda(\mathbb{T})) & \leftarrow\!\shortmid & (X, \mathbb{T}, x_0) \\ \widetilde{\varphi}_* & \leftarrow\!\shortmid & (\varphi, \widetilde{\varphi}). \end{array}$$

Construction 3.3.9. Let (Σ, N) be a lattice fan. Then for any two cones $\sigma, \sigma' \in \Sigma$, we have the open inclusions

$$X_\sigma \supseteq X_{\sigma \cap \sigma'} \subseteq X_{\sigma'}.$$

We define X_Σ to be the gluing of all the X_σ and $X_{\sigma'}$ along the common open subsets $X_{\sigma \cap \sigma'}$. By construction, we have

$$X_\sigma \subseteq X_\Sigma, \qquad X_{\sigma \cap \sigma'} = X_\sigma \cap X_{\sigma'} \subseteq X_\Sigma,$$

where the affine variety X_σ is open in X_Σ and the intersection $X_\sigma \cap X_{\sigma'}$ is taken in X_Σ.

Theorem 3.3.10. *We have covariant functors being essentially inverse to each other:*

$$\begin{array}{ccc}
\{\textit{lattice fans}\} & \longleftrightarrow & \{\textit{normal toric varieties}\} \\
 & & \\
(\Sigma, N) & \mapsto & (X_\Sigma, \mathbb{T}_N, x_0) \\
F & \mapsto & (\varphi_F, \widetilde{\varphi}_F) \\
 & & \\
(\Sigma(X), \Lambda(\mathbb{T})) & \leftarrow\!\shortmid & (X, \mathbb{T}, x_0) \\
\widetilde{\varphi}_* & \leftarrow\!\shortmid & (\varphi, \widetilde{\varphi}).
\end{array}$$

3.4 Faces and orbits

Theorem 3.4.1. *Consider the toric variety $(X_\sigma, \mathbb{T}_N, x_0)$ arising from a pointed lattice cone (σ, N). Then we have mutually inverse bijections*

$$\begin{array}{ccc}
\{\textit{faces of}\,\sigma\} & \longleftrightarrow & \{\mathbb{T}_N\textit{-orbits of}\,X_\sigma\} \\
\tau & \mapsto & \mathbb{T}_N \cdot x_\tau \\
\omega(x)^\perp \cap \sigma & \leftarrow\!\shortmid & \mathbb{T}_N \cdot x.
\end{array}$$

Moreover, this correspondence is order reversing in the sense that any two faces $\tau, \tau' \preccurlyeq \sigma$ we have

$$\tau \preccurlyeq \tau' \iff \mathbb{T}_N \cdot x_{\tau'} \subseteq \overline{\mathbb{T}_N \cdot x_\tau}.$$

Remark 3.4.2. Consider a toric variety $(X_\Sigma, \mathbb{T}_N, x_0)$. Every $\sigma \in \Sigma$ gives a well-defined *limit point*, also called **distinguished point**:

$$x_\sigma = \lim_{t \to 0} \lambda_v(t) \cdot x_0 \in X_\sigma \subseteq X_\Sigma, \quad v \in \sigma^\circ \cap N.$$

Theorem 3.4.3. *In a similar way to Theorem 3.4.1, consider the toric variety $(X_\Sigma, \mathbb{T}_N, x_0)$ arising from a lattice fan (Σ, N). Then we have a bijection*

$$\Sigma \to \{\mathbb{T}_N\textit{-orbits of}\,X_\Sigma\}, \qquad \sigma \mapsto \mathbb{T}_N \cdot x_\sigma.$$

And, as before, the above map is order reversing in the sense that for any two cones $\sigma, \sigma' \in \Sigma$ we have

$$\sigma \preccurlyeq \sigma' \iff \mathbb{T}_N \cdot x_{\sigma'} \subseteq \overline{\mathbb{T}_N \cdot x_\sigma}.$$

Remark 3.4.4. Consider a toric variety $(X_\Sigma, \mathbb{T}_N, x_0)$ and let $\sigma \in \Sigma$. Then, for the isotropy group of $\mathbb{T}_N$ at x_σ, we have

$$(\mathbb{T}_N)_{x_\sigma} = \mathbb{T}_{N_\sigma}, \qquad \dim(\mathbb{T}_N)_{x_\sigma} = \dim(\sigma).$$

Moreover, $\mathbb{T}_N \cdot x_\sigma \cong \mathbb{T}_N / \mathbb{T}_{N_\sigma}$ holds. Thus, the dimensions of the orbit and its closure are given as

$$\dim\left(\mathbb{T}_N \cdot x_\sigma\right) = \dim(\mathbb{T}_N \cdot x_\sigma) = \dim(N_\mathbb{Q}) - \dim(\sigma).$$

3.5 Homogeneous coordinates on toric varieties

Definition 3.5.1. The **primitive generators** of a lattice fan (Σ, N) are the shortest non-zero vectors $v_1, \ldots, v_r \in N$ of the rays (one dimensional cones) $\varrho_1, \ldots, \varrho_r \in \Sigma$.

Definition 3.5.2. A lattice fan (Σ, N) is **non-degenerate** if its primitive generators span $N_{\mathbb{Q}}$ as a vector space.

Construction 3.5.3. Let (Σ, N) be a non-degenerate lattice fan with primitive generators $v_1, \ldots, v_r$. Define a linear map

$$P : \mathbb{Z}^r \to N, \qquad e_i \to v_i.$$

Moreover, write $\delta^r := \mathbb{Q}^r_{\geq 0}$ for the *positive orthant* and define a sub fan $\widehat{\Sigma}$ of the fan $\overline{\Sigma}$ of faces of δ^r by

$$\widehat{\Sigma} := \{\delta \preccurlyeq \delta^r : P(\delta) \subseteq \sigma \,\text{for some}\, \sigma \in \Sigma\}.$$

Then $\widehat{\Sigma}$ and $\overline{\Sigma}$ are fans in the lattice $\mathbb{Z}^r$. In fact, $P : \mathbb{Z}^r \to N$ is a map of fans from $(\widehat{\Sigma}, \mathbb{Z}^r)$ to (Σ, N).

Remark 3.5.4. Consider a non-degenerate fan Σ in N and the fans $\widehat{\Sigma}$ and $\overline{\Sigma}$ in $\mathbb{Z}^r$. The associated toric varieties respectively are

$$(X, \mathbb{T}_N, x_0), \qquad (\widehat{X}, \mathbb{T}^r, \mathbb{1}_r), \qquad (\overline{X}, \mathbb{T}^r, \mathbb{1}_r).$$

The toric morphism arising from the map of fans $P : \mathbb{Z}^r \to N, e_i \mapsto v_i$ is denoted by $(p, \widetilde{p})$. The situation is summarized as follows:

$$\begin{tikzcd} \widehat{X} \arrow[d, "p"'] & \subseteq & \overline{X} & = & \mathbb{C}^r \\ X & & & & \end{tikzcd}$$

Here $\widehat{X} \subseteq \overline{X}$ is an open toric embedding, and $p : \widehat{X} \to X$ is constant on the orbits of $H := \ker(\widetilde{p}) \subseteq \mathbb{T}^r$.

Definition 3.5.5. For every cone $\sigma \in \Sigma$, there is a unique cone $\widehat{\sigma} \in \widehat{\Sigma}$ with $P(\widehat{\sigma}) = \sigma$, namely

$$\widehat{\sigma} = \operatorname{cone}(e_i : v_i \in \sigma) \preccurlyeq \delta^r.$$

Every face $\delta \preccurlyeq \delta^r$ with $P(\delta) \subseteq \sigma$ is necessarily a face of $\widehat{\sigma}$. In fact, we have an injection

$$\Sigma \to \widehat{\Sigma}, \qquad \sigma \mapsto \widehat{\sigma}.$$

This is a bijection if and only if each $\sigma \in \Sigma$ is **simplicial**, that means, $(v_i : v_i \in \sigma)$ is linearly independent.

Definition 3.5.6. Let $(X_\Sigma, \mathbb{T}_N, x_0)$ be non-degenerate. The presentation of a point $x \in X := X_\Sigma$ in **homogeneous coordinates** is

$$x = [z] = [z_1 : \cdots : z_r],$$

where $x \in \mathbb{T}_N \cdot x_\sigma \subseteq X$ with a unique cone $\sigma \in \Sigma$, and $z \in \mathbb{T}^r \cdot x_{\widehat{\sigma}} \subseteq \widehat{X}$ is any point with $p(z) = x$.

3.6 Divisors

3.6.1 Main elements

For the following we will assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero. Let X be an irreducible prevariety.

Definition 3.6.1. A **prime divisor** of X is an irreducible subvariety $D \subset X$ of codimension 1 over X.

Definition 3.6.2. The group of **Weil divisors** of X is the free abelian group $\mathrm{WDiv}(X)$ generated by all the *prime divisors*.

Definition 3.6.3. Let X be a normal irreducible variety, let $D \subseteq X$ be a prime divisor, and let η_D be its generic point. The local ring $\mathcal{O}_{X,\eta_D}$ is a discrete valuation ring. For $f \in \mathbb{K}(X)^*$, the **order of** f **along** D, denoted $\mathrm{ord}_D(f)$, is the corresponding discrete valuation. Equivalently, if $f = g/h$ with $g, h \in \mathcal{O}_{X,\eta_D} \setminus \{0\}$, then

$$\mathrm{ord}_D(f) = \mathrm{length}\big(\mathcal{O}_{X,\eta_D}/\langle g\rangle\big) - \mathrm{length}\big(\mathcal{O}_{X,\eta_D}/\langle h\rangle\big).$$

Construction 3.6.4. Via the *order* $\mathrm{ord}_D(f)$ along a prime divisor D, one associates with every nonzero rational function $f \in \mathbb{K}(X)^*$ the Weil divisor

$$\mathrm{div}(f) := \sum_{D \text{ prime}} \mathrm{ord}_D(f) \cdot D.$$

The assignment $f \mapsto \mathrm{div}(f)$ is a homomorphism $\mathbb{K}(X)^* \to \mathrm{WDiv}(X)$, and its image $\mathrm{PDiv}(X) \subseteq \mathrm{WDiv}(X)$ is called the subgroup of **principal divisors**. In other words, a Weil divisor D on a normal variety X is **principal at** $x \in X$, if there are an open set $U \subseteq X$ and $f \in \mathbb{K}(X)$ with

$$x \in U, \qquad D|_U = \mathrm{div}(f) \in \mathrm{WDiv}(U).$$

Construction 3.6.5. We say that $D_1, D_2 \in \mathrm{WDiv}(X)$ are *linearly equivalent* ($D_1 \sim D_2$ for short), if $D_2 - D_1$ is a principal divisor. The **divisor class group** of X is the factor group

$$\mathrm{Cl}(X) := \mathrm{WDiv}(X)/\,\mathrm{PDiv}(X)$$

Definition 3.6.6. A Weil divisor $D = a_1 D_1 + \cdots + a_s D_s$ with *prime divisors* D_i is called **effective**, denoted as $D \geq 0$, if $a_i \geq 0$ for all $i \in \{1, \ldots, s\}$.

Remark 3.6.7. With every divisor $D \in \mathrm{WDiv}(X)$, one associates a sheaf $\mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules by defining its sections over an open $U \subset X$ as

$$\Gamma(U, \mathcal{O}_X(D)) := \{f \in \mathbb{K}(X)^*; (\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\},$$

where the restriction map $\mathrm{WDiv}(X) \to \mathrm{WDiv}(U)$ is defined for a prime divisor D as $D|_U = D \cap U$ if it intersects U and $D|_U := 0$ otherwise. Note that for any two functions $f_1 \in \Gamma(U, \mathcal{O}_X(D_1))$ and $f_2 \in \Gamma(U, \mathcal{O}_X(D_2))$ the product $f_1 f_2$ belongs to $\Gamma(U, \mathcal{O}_X(D_1 + D_2))$.

3.6.2 Divisors and toric varieties

Definition 3.6.8. Let $(X, \mathbb{T}, x_0)$ be a toric variety arising from a lattice fan (Σ, N) with primitive generators $v_1, \ldots, v_r$. Then we have prime divisors

$$D_i := D_{\varrho_i} := \overline{\mathbb{T} \cdot x_{\varrho_i}} \subseteq X, \quad \varrho_i = \mathrm{cone}(v_i), \quad i \in \{1, \ldots, r\}.$$

Observe that $\mathbb{T} \cdot D_i = D_i$ and the primitive divisors form the boundary of $\mathbb{T}$ in X in the following sense:

$$X \setminus \mathbb{T} \cdot x_0 = D_1 \cup \cdots \cup D_r.$$

The prime divisors $D_1, \ldots, D_r$ generate the subgroup of **invariant Weil divisors** of X:

$$\mathrm{WDiv}^{\mathbb{T}}(X) = \mathbb{Z}D_1 + \cdots + \mathbb{Z}D_r \subseteq \mathrm{WDiv}(X).$$

Remark 3.6.9. Consider $(X, \mathbb{T}, x_0)$ the toric variety arising from a lattice fan (Σ, N) and a character function $\chi^u \in \mathbb{K}(X)$. Then we have

$$\mathrm{ord}_{D_i}(\chi^u) = \langle u, v_i \rangle.$$

for the order of χ^u along the toric prime divisor D_i defined by a primitive generator v_i of Σ. Thus, the divisor of χ^u is given as

$$\mathrm{div}(\chi^u) = \langle u, v_1 \rangle D_1 + \cdots + \langle u, v_r \rangle D_r \in \mathrm{WDiv}^{\mathbb{T}}(X).$$

Remark 3.6.10. Consider a toric variety $(X, \mathbb{T}, x_0)$ arising from a lattice fan (Σ, M). As $\mathcal{O}(\mathbb{T})$ is a factorial ring (it is a localization of a polynomial ring), every Weil divisor on $\mathbb{T} \cdot x_0 \cong \mathbb{T}$ is principal. Thus

$$\mathrm{Cl}(X) = \mathrm{WDiv}^{\mathbb{T}}(X)/\,\mathrm{PDiv}^{\mathbb{T}}(X)$$

holds for the divisor class group, where the group $\mathrm{PDiv}^{\mathbb{T}}(X)$ of invariant principal divisors consists precisely of the divisors $\mathrm{div}(\chi^u)$ of the character functions χ^u with $u \in M$. In other words, in a toric variety the open torus has trivial divisor class group, and all divisor classes come from the boundary of the torus action on X.

Theorem 3.6.11. *Let $(X, \mathbb{T}, x_0)$ be the toric variety arising from a lattice fan (Σ, N). Then we have a commutative diagram with exact rows:*

$$\begin{tikzcd}
0 & K \arrow[l] \arrow[d] & \mathbb{Z}^r \arrow[l, "Q"'] \arrow[d, "a \mapsto \sum a_i D_i"'] & M \arrow[l, "P^*"'] \arrow[d, "u \mapsto \chi^u"] \\
0 & \mathrm{Cl}(X) \arrow[l] & \mathrm{WDiv}^{\mathbb{T}}(X) \arrow[l, "{[D] \leftarrow\!\shortmid D}"] & \mathbb{X}(\mathbb{T}) \arrow[l, "\mathrm{div}(\chi^u) \leftarrow\!\shortmid \chi^u"]
\end{tikzcd}$$

3.6.3 The associated polyhedra

Definition 3.6.12. Let X be a normal variety and let D be a Weil divisor on X. The space of **sections** of D is the vector subspace

$$\Gamma(X, \mathcal{O}(D)) = \{0\} \cup \{f \in \mathbb{K}(X)^* : \mathrm{div}(f) + D \geq 0\} \subseteq \mathbb{K}(X).$$

Construction 3.6.13. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N) with primitive generators $v_1, \ldots, v_r$. Given $a \in \mathbb{Z}^r$, we obtain the divisor

$$D = a_1 D_1 + \cdots + a_r D_r \in \mathrm{WDiv}^{\mathbb{T}}(X),$$

where $D_1, \ldots, D_r$ are the prime divisors associated with the primitive generators. The **divisorial polyhedron** defined by D is

$$B_D := \{u \in M_{\mathbb{Q}} : \langle u, v_i \rangle \geq -a_i, i \in \{1, \ldots, r\}\}.$$

Construction 3.6.14. Let the toric variety $(X, \mathbb{T}, x_0)$ arise from a non degenerate lattice fan (Σ, N) with primitive generators $v_1, \ldots, v_r$. Then we have the linear map

$$P : \mathbb{Z}^r \to N, \quad e_i \mapsto v_i.$$

Let $K = \mathbb{Z}^r / P^* M$ be the factor group by the image of the dual map and $Q : \mathbb{Z}^r \to K$ the projection. The **homogeneous coordinate ring** of X is the K-graded polynomial ring

$$\mathcal{R}(X) = \mathbb{C}[z_1, \ldots, z_r], \qquad \deg(z_i) := Q(e_i) \in K.$$

Definition 3.6.15. Let $(X, \mathbb{T}, x_0)$ arise from a non-degenerate lattice fan (Σ, N). For $w \in K$, the **class polyhedron** is

$$B_w = \operatorname{aff}(Q^{-1}(w)) \cap \mathbb{Q}^r_{\geq 0} \subseteq \mathbb{Q}^r.$$

The lattice points inside the class polyhedron correspond to the monomials of degree w in the homogeneous coordinate ring:

$$\mathcal{R}(X) = \bigoplus_{w \in K} \mathcal{R}(X)_w, \qquad \mathcal{R}(X)_w = \bigoplus_{\mu \in B_w \cap \mathbb{Z}^r} \mathbb{C} z^\mu$$

Remark 3.6.16. Let $(X, \mathbb{T}, x_0)$ be the toric variety arising from a non-degenerate lattice fan (Σ, N). Fix

$$a \in \mathbb{Z}^r, \qquad w := Q(a) \in K.$$

Then a defines an invariant Weil divisor D on X having w as its associated divisor class:

$$D = a_1 D_1 + \cdots + a_r D_r \in \operatorname{WDiv}^{\mathbb{T}}(X), \operatorname{Cl}(X) \ni [D] = w \in K.$$

Consider the affine embedding $M \to \mathbb{Z}^r$, $u \mapsto P^* u + a$. Then, for every $u \in M_{\mathbb{Q}}$ and any primitive generator v_i of Σ, we have

$$\langle u, v_i \rangle \geq -a_i \iff \langle P^* u + a, e_i \rangle \geq 0.$$

Thus, the above injection identifies the divisorial polyhedron $B_D \subseteq M_{\mathbb{Q}}$ with the class polyhedron $B_w \subseteq \mathbb{Q}^r$:

$$P^* B_D + a = B_w \subseteq \mathbb{Q}^r.$$

Furthermore, the space of sections of D is isomorphic to the component of degree w of the homogeneous coordinate ring via

$$\Gamma(X, \mathcal{O}(D)) \to \mathcal{R}(X)_w, \quad \chi^u \mapsto z^{P^* u + a}.$$

Finally, for a section χ^u of D and the corresponding monomial $z^{P^* u + a}$ of $\mathcal{R}(X)_w$, the divisors are related to each other via

$$\operatorname{div}(\chi^u) + D = p_* \operatorname{div}(z^{P^* u + a}),$$

where $p : \widehat{X} \to X$ is the toric morphism associated with $P : \mathbb{Z}^r \to N$ and p_* is the push forward, sending $V(z_i)$ to D_i.

3.6.4 Cones and classification of divisors

Definition 3.6.17. A Weil divisor D on a normal variety X is a **Cartier divisor** if it is *principal* near every $x \in X$.

Definition 3.6.18. The Cartier divisors form a subgroup

$$\mathrm{CDiv}(X) \subseteq \mathrm{WDiv}(X),$$

containing the group PDiv of principal divisors. The **Picard group** of X is the factor group

$$\mathrm{Pic}(X) = \mathrm{CDiv}(X)/\,\mathrm{PDiv}(X) \subseteq \mathrm{WDiv}(X)/\,\mathrm{PDiv}(X) = \mathrm{Cl}(X).$$

Remark 3.6.19. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N) and fix $\sigma \in \Sigma$. Then $D \in \mathrm{WDiv}^{\mathbb{T}}(X)$ is principal near x_σ if and only if

$$D|_{X_\sigma} = \mathrm{div}(\chi^u)$$

holds for some $u \in M$. In this case, look at $D' := D - \mathrm{div}(\chi^u)$. We have $D' = a_1' D_1 + \cdots + a_r' D_r$ with $a_i' = 0$ if $v_i \in \sigma$. Thus,

$$[D] = [D'] \in Q(\mathrm{lin}(\widehat{\sigma}^*) \cap \mathbb{Z}^r) \subseteq K = \mathrm{Cl}(X)$$

with $\widehat{\sigma}^* = \mathrm{cone}(e_j : v_j \notin \sigma) \preccurlyeq \mathbb{Q}^r_{\geq 0}$. Conversely, if $[D] \in Q(\mathrm{lin}(\widehat{\sigma}^*) \cap \mathbb{Z}^r)$ holds, then D is principal at $x_\sigma \in X$.

Theorem 3.6.20. *Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). Then the Picard group of X is given as*

$$\mathrm{Pic}(X) = \bigcap_{\sigma \in \Sigma} Q(\mathrm{lin}(\widehat{\sigma}^*) \cap \mathbb{Z}^r) \subseteq K = \mathrm{Cl}(X).$$

Remark 3.6.21. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). The following diagram allows us to transfer information on cones $\sigma \in \Sigma$ from $N = \Lambda(\mathbb{T})$ to $K = \mathrm{Cl}(X)$:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & \mathbb{Z}^r \arrow[r, "P: e_i \mapsto v_i"] & N \\
 & & \delta^r \succcurlyeq \hat{\sigma} \arrow[d] & \sigma \arrow[l] \\
 & Q(\hat{\sigma}^*) & \hat{\sigma}^* \preccurlyeq \gamma^r \arrow[l] & \\
0 & K \arrow[l] & \mathbb{Z}^r \arrow[l] & M \arrow[l, "P^*"']
\end{tikzcd}$$

where the orthants $\delta^r = \mathbb{Q}^r_{\geq 0}$ and $\gamma^r = \mathbb{Q}^r_{\geq 0}$ are regarded as dual to each other, and $\hat{\sigma}^* = \mathrm{cone}(e_j : v_j \notin \sigma) = \hat{\sigma}^\perp \cap \gamma^r \preccurlyeq \gamma^r$ corresponds to $\hat{\sigma} = \mathrm{cone}(e_i : v_i \in \sigma) \preccurlyeq \delta^r$. Moreover, $Q(\hat{\sigma}^*)$ lives in $K_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} K$.

Theorem 3.6.22. *Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). Given any $D \in \mathrm{WDiv}(X)$, set*

$$w := [D] \in \mathrm{Cl}(X) = K.$$

Then the base locus and the stable base locus of the divisor D are given as

$$\mathrm{Bs}(D) := \bigcup_{\substack{\sigma\in\Sigma \\ w\notin Q(\widehat{\sigma}^*\cap\mathbb{Z}^r)}} \mathbb{T}\cdot x_\sigma, \qquad \boldsymbol{B}(D) := \bigcup_{\substack{\sigma\in\Sigma \\ w\notin Q(\widehat{\sigma}^*)}} \mathbb{T}\cdot x_\sigma.$$

Definition 3.6.23. Let X be a normal variety and let D be a Weil divisor.

(a) The divisor class $[D]$ is **effective** if $H^0(X, \mathcal{O}_X(D)) \neq 0$, equivalently if D is linearly equivalent to an effective Weil divisor.

(b) The divisor D is **movable** if $\mathrm{Bs}(D)$ has codimension at least two.

(c) The divisor D is **semiample** if its stable base locus is empty.

(d) The divisor D is **ample** if X is covered by affine open subsets of the form

$$X_f = X \setminus \mathrm{Supp}(\mathrm{div}(f) + kD),$$

with $k \geq 1$ and $0 \neq f \in \Gamma(X, \mathcal{O}_X(kD))$.

Theorem 3.6.24. *Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). Then the cones generated by the effective, movable, semiample and ample divisor classes of X in $K_\mathbb{Q} = \mathrm{Cl}_\mathbb{Q}(X)$ are given by*

$$\mathrm{Eff}(X) = Q(\gamma^r), \qquad \mathrm{Mov}(X) = \textstyle\bigcap_{i=1}^r Q(\mathrm{cone}(e_j : j \neq i)),$$
$$\mathrm{SAmple}(X) = \textstyle\bigcap_{\sigma\in\Sigma} Q(\widehat{\sigma}^*), \qquad \mathrm{Ample}(X) = \textstyle\bigcap_{\sigma\in\Sigma} Q(\widehat{\sigma}^*)^\circ.$$

Remark 3.6.25. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). The variety X is projective if and only if Σ is complete, that is, $N_\mathbb{Q}$ is covered by the cones of Σ, and $\mathrm{Ample}(X)$ is non-empty.

4 Cox Rings and Localizations

4.1 Cox Sheaf and Cox Rings

Definition 4.1.1. Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. To any subgroup $K \subseteq \mathrm{WDiv}(X)$ with X a normal variety such that $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$, one can associate the following *sheaf of divisorial algebras*:

$$\mathcal{S} := \bigoplus_{D \in K} \mathcal{O}_X(D).$$

Now, assume that the quotient map $K \mapsto \mathrm{Cl}(X)$ is surjective, and let $K^0 \subseteq \mathrm{PDiv}(X)$ be its kernel. Fix a homomorphism $\chi : K^0 \to \mathbb{K}(X)^*$ such that $\mathrm{div}(\chi(D)) = D$ for any $D \in K^0$. Let $\mathcal{I} \subseteq \mathcal{S}$ be the ideal sheaf locally generated by element of the form $1 - \chi(D)$, where D ranges over K^0. From now on, assume that $\mathrm{Cl}(X)$ is finitely generated. The **Cox sheaf** is the $\mathrm{Cl}(X)$-graded sheaf of algebras defined as

$$\mathcal{R}_X := \mathcal{S}/\mathcal{I}.$$

Choosing a different K or a different χ does not change the isomorphism class of the Cox Sheaf. If X is $\mathbb{Q}$-factorial, the the Cox sheaf is locally of finite type [3, 1.6.1]. In this case, we denote its relative spectrum by

$$\widehat{X} := \mathrm{Spec}_X \mathcal{R}_X.$$

It can be shown that $\widehat{X}$ is an irreducible, normal, quasiaffine variety [3, Construction 1.6.1.3].

Definition 4.1.2. Under the same assumptions as in the previous definition, the **Cox ring** of X is defined as the ring of global sections of the Cox sheaf:

$$\mathcal{R}(X) := \Gamma(X, \mathcal{R}_X).$$

It can be shown that X can be covered by open subsets of the form $X_{[D],f} := X \setminus \mathrm{Supp}(\mathrm{div}(\widetilde{f}) + D)$, where D is any representative for $[D]$, and $\widetilde{f} \in \mathcal{O}_X(D)$ is a representative for f (the open subset does not depend on the choice of D). We can see that the map

$$\Gamma(X, \mathcal{R}_X)_f \to \Gamma(X_{[D],f}, \mathcal{R}_X), \qquad \frac{g}{f^n} \mapsto \frac{g}{f^n}$$

is well defined and can be shown to be an isomorphism of graded algebras. As a result, we obtain an embedding $\widehat{X} \to \overline{X}$, whose complement is a subset of codimension at least two, called **the irrelevant locus** of X [3, Construction 1.6.3.1].
The inclusion $\mathcal{O}_X \to \mathcal{R}_X$, as the degree-zero part of the Cox sheaf, induces a morphism $p_X : \widehat{X} \to X$, which is a good quotient for the action of the quasitorus $H_X := \operatorname{Spec} \mathbb{K}[\operatorname{Cl}(X)]$. This situation is summarized in the following diagram:

$$\begin{tikzcd} \widehat{X} \arrow[r, hook] \arrow[d, "p_X"'] & \overline{X} \\ X & \end{tikzcd}$$

The variety $\widehat{X}$ is called the **characteristic space** of X, and p_X is the characteristic morphism. The affine variety $\overline{X}$ is referred to as the *total coordinate space*. The *irrelevant locus* $\overline{X} \setminus \widehat{X}$ is the vanishing set of a distinguished homogeneous ideal $\mathscr{I}_{\mathrm{irr}}(X) \subseteq \mathcal{R}(X)$, called the **irrelevant ideal** of X (see [3, Def. 1.6.3.2]).

Remark 4.1.3. Recall that whenever X is a projective variety and H is any ample divisor on X we have

$$\mathscr{I}_{\mathrm{irr}}(X) = \sqrt{\langle \bigoplus_{n>0} \mathcal{R}(X)_{[nH]} \rangle},$$

that is, the irrelevant ideal is the radical of the ideal generated by all homogeneous pieces of $\mathcal{R}(X)$ corresponding to positive multiples of H (see [3, Def. 1.6.3.6]). In particular, if $f \in \mathcal{R}(X)_{[nH]}$ for some $n > 0$, then f lies in every prime ideal $\mathfrak{p} \subseteq \mathcal{R}(X)$ that contains $\mathscr{I}_{\mathrm{irr}}(X)$.
When X is a toric variety with Cox ring $\mathcal{R}(X) = \mathbb{K}[T_1, \dots, T_r]$, let Σ be its defining fan and let $\rho_1, \dots, \rho_r$ be its rays. The irrelevant ideal is

$$\mathscr{I}_{\mathrm{irr}}(X) = \left\langle \prod_{\rho_i \notin \sigma(1)} T_i : \sigma \in \Sigma_{\max} \right\rangle.$$

It is a square-free monomial ideal. Its minimal prime ideals are

$$\langle T_i : i \in I \rangle ,$$

where I ranges over the inclusion-minimal subsets of $\{1, \dots, r\}$ for which the rays $\{\rho_i : i \in I\}$ are not contained in any cone of Σ; see [5, Prop. 5.1.6].

4.2 Embedded Varieties

We recall some basic facts about the Cox ring of an embedded variety [3, §4.1.1]. Let Z be a normal variety with a finitely generated Cox ring, and let $p_Z : \widehat{Z} \to Z$ be its characteristic space. Denote by Z_0 the subset of smooth points of Z. Given a subvariety $X \subseteq Z$, define $X_0 := X \cap Z_0$, let $\widehat{X}$ to be the closure of $p_Z^{-1}(X_0)$ in $\widehat{Z}$, and let $p_X : \widehat{X} \to X$ denote the restriction of p_Z. The situation is summarized in the following diagram:

$$\begin{tikzcd} p_Z^{-1}(X_0) \arrow[r, hook] \arrow[d] & \widehat{X} \arrow[r, hook] \arrow[d, "p_X"'] & p_Z^{-1}(X) \arrow[r, hook] \arrow[d] & \widehat{Z} \arrow[d, "p_Z"] \\ X_0 \arrow[r, hook] & X \arrow[r, Rightarrow, no head] & X \arrow[r, hook] & Z \end{tikzcd}$$

For the rest of this section we will denote by $K \subseteq \mathrm{WDiv}(Z)$ a subgroup which surjects onto $\mathrm{Cl}(Z)$ and such that X is not contained in the support of any element of K. Since Weil divisors on a normal variety are completely determined by their restriction to the smooth locus, every $D \in K$ admits a canonical pullback

$$i^*(D) := \overline{D|_{X_0}} \in \mathrm{WDiv}(X),$$

where the bar denotes Zariski closure in X. Because no component of X is contained in $\mathrm{Supp}(D)$, this defines a well-defined group homomorphism

$$K \xrightarrow{i^*} \mathrm{WDiv}(X),$$

which maps principal divisors to principal divisors and, consequently, induces a pullback homomorphism

$$i^* : \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$$

on divisor class groups. We recall the following results and proof from [3, Cor. 4.1.1.2].

Proposition 4.2.1. *Let Z be a normal variety with finitely generated Cox ring, and let $X \subseteq Z$ be a normal subvariety with $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$. Let Z_0 be the smooth locus of Z and set $X_0 := X \cap Z_0$. Let $\widehat{X}$ be the closure of $p_Z^{-1}(X_0)$ in $\widehat{Z}$, and let $p_X : \widehat{X} \to X$ be the restriction of p_Z. Assume that:*

(1) $p_Z^{-1}(X_0)$ is big in $\widehat{X}$;

(2) $\widehat{X}$ is normal;

(3) $\iota^* : \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ is an isomorphism.

Then, for any Cox sheaf $\mathcal{R}_X$ on X, there is an isomorphism of $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebras

$$\mathcal{R}_X \cong (p_X)_* \mathcal{O}_{\widehat{X}}.$$

In particular, $p_X : \widehat{X} \to X$ is a characteristic space for X.

Proof. This is [3, Cor. 4.1.1.2]. In the notation of that result, X' is the open subset $X_0 = X \cap Z_0$. Moreover,

$$\widehat{X} \setminus p_X^{-1}(X_0) = \widehat{X} \setminus p_Z^{-1}(X_0),$$

and this has codimension at least two in $\widehat{X}$ by assumption. The normality of $\widehat{X}$ and the isomorphism $\iota^* : \mathrm{Cl}(Z) \to \mathrm{Cl}(X)$ are also among the hypotheses. Therefore [3, Cor. 4.1.1.2] gives the desired isomorphism

$$\mathcal{R}_X \cong (p_X)_* \mathcal{O}_{\widehat{X}}.$$

The same corollary also gives that $p_X : \widehat{X} \to X$ is a characteristic space for X.

□

Construction 4.2.2. Let Z be a variety with finitely generated Cox ring and let $X \subseteq Z$ satisfy the hypotheses of Proposition 4.2.1. Let $\widetilde{X} \subseteq \overline{Z}$ be the closure of $\widehat{X}$. Define $\widetilde{X}_1 \subseteq \widetilde{X}$ by removing all codimension-one irreducible components of $\widetilde{X} \setminus \widehat{X}$. Thus

$$\widehat{X} \subseteq \widetilde{X}_1 \subseteq \widetilde{X},$$

$\widehat{X}$ is big in $\widetilde{X}_1$, and $\widetilde{X} \setminus \widetilde{X}_1$ is a union of hypersurfaces.

Remark 4.2.3. In 6.1.1, if the coordinate ring of $\widetilde{X} \subseteq \overline{Z}$ is Cohen-Macaulay, and in particular if it is a complete intersection, then Serre's condition S_2 is automatically satisfied [13, Tag 0342]. Since this condition is local [13, Tag 033Q], we conclude that in this case $\widetilde{X}_1$ is normal if and only if it is smooth in codimension one.

5 Main algorithm

5.1 The program

In this section, R is a finitely generated normal domain over a field, $K := \mathrm{Frac}(R)$, and $m_1, \dots, m_s \in R$ are non-zero. All intersections of localizations are taken inside K. For every intermediate ring

$$R \subseteq A \subseteq R_{m_1} \cap \dots \cap R_{m_s}$$

and every $k > 0$, set

$$I_k(A) := \bigcap_{i=2}^{s} \left(\langle m_1^k \rangle_A : \langle m_i \rangle_A^\infty \right).$$

Thus $f \in I_k(A)$ if and only if $f/m_1^k \in A_{m_i}$ for every $i = 2, \dots, s$.

Input : A finitely generated normal domain R over a field and non-zero elements $m_1, \dots, m_s \in R$
Output: $R_{m_1} \cap \dots \cap R_{m_s}$, if the algorithm terminates
while *$V(m_1, \dots, m_s)$ has codimension 1 in* $\mathrm{Spec}(R)$ **do**
Search for the smallest integer $n > 0$ such that

$$I_n(R) \not\subseteq \langle m_1^n \rangle_R.$$

Choose a finite generating set of $I_n(R)$, and let B_n be the subset of generators which do not belong to $\langle m_1^n \rangle_R$.;
Define

$$R' := \frac{R[S_f \mid f \in B_n]}{(\langle S_f m_1^n - f \mid f \in B_n \rangle : \langle m_1 \rangle^\infty)}.$$

Replace R by the normalization of R' in K.;
end
return R;

Algorithm 1: Computing the intersection of localizations

5.1.1 Prior to the proof

Remark 5.1.1. When R is a graded ring, as in the case of Cox rings, and the elements $m_1, \dots, m_s$ are homogeneous, the ideals $\langle m_1^k \rangle_R$ and $\langle m_i \rangle_R$ are homogeneous. Since colon ideals and saturations of homogeneous ideals are homogeneous, each ideal $I_k(R)$ is homogeneous.

Therefore one may choose a homogeneous generating set for $I_k(R)$. With this choice, the elements of B_n are homogeneous. If one gives S_f the degree $\deg(f) - n\deg(m_1)$, then the relation $S_f m_1^n - f$ is homogeneous. Thus the quotient appearing in Algorithm 1 is graded. Moreover, the normalization of a finitely generated graded domain inside its homogeneous fraction field is again graded. Hence the algorithm is compatible with the Cox ring grading.

Lemma 5.1.2. *Let A be a Noetherian normal domain and let $m_1, \dots, m_s \in A$ be non-zero. Then*

$$A = \bigcap_{i=1}^{s} A_{m_i} \quad \Longleftrightarrow \quad \operatorname{codim}_{\operatorname{Spec}(A)} V(m_1, \dots, m_s) \geq 2.$$

Equivalently, if $R \subseteq A \subseteq R_{m_1} \cap \cdots \cap R_{m_s}$ and A is a Noetherian normal domain, then

$$A = R_{m_1} \cap \cdots \cap R_{m_s}$$

if and only if $V(m_1, \dots, m_s)$ has codimension at least 2 in $\operatorname{Spec}(A)$.

Proof. Let $K = \operatorname{Frac}(A)$. Since A is a Noetherian normal domain, it is a Krull domain, and hence

$$A = \bigcap_{\operatorname{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$$

inside K. Moreover,

$$A_m = \bigcap_{\substack{\operatorname{ht}(\mathfrak{p})=1 \\ m \notin \mathfrak{p}}} A_{\mathfrak{p}}.$$

Therefore $\bigcap_i A_{m_i}$ is the intersection of the height-one localizations $A_{\mathfrak{p}}$ for which $\mathfrak{p}$ does not contain all of $m_1, \dots, m_s$.

If $V(m_1, \dots, m_s)$ has codimension at least 2, no height-one prime contains all the m_i, and the intersection is therefore A.

Conversely, assume that $V(m_1, \dots, m_s)$ has codimension 1. Then some height-one prime $\mathfrak{p}$ contains all the m_i. The approximation theorem for Krull domains gives $x \in K$ with negative valuation along $\mathfrak{p}$ and non-negative valuation along every other height-one prime. Thus $x \notin A$. Since $m_i \in \mathfrak{p}$, the prime $\mathfrak{p}$ disappears after localization at m_i, and hence $x \in A_{m_i}$ for every i. Consequently $A \subsetneq \bigcap_i A_{m_i}$.

For the final statement, localizing $R \subseteq A \subseteq R_{m_1} \cap \cdots \cap R_{m_s}$ at m_i gives $A_{m_i} = R_{m_i}$ for every i. □

Lemma 5.1.3. *Let A be an integral domain, let $m \in A$ be non-zero, and let $f_1, \ldots, f_t \in A$ and $r_1, \ldots, r_t > 0$. Set*

$$A' := A\left[\frac{f_1}{m^{r_1}}, \ldots, \frac{f_t}{m^{r_t}}\right] \subseteq A_m.$$

Then

$$A' \cong \frac{A[S_1, \ldots, S_t]}{(\langle S_1 m^{r_1} - f_1, \ldots, S_t m^{r_t} - f_t\rangle : \langle m\rangle^\infty)}.$$

Proof. Consider the surjective homomorphism

$$\varphi : A[S_1, \ldots, S_t] \longrightarrow A', \qquad S_j \longmapsto f_j/m^{r_j}.$$

After localization at m, its kernel is generated by $S_j - f_j/m^{r_j}$, or equivalently by $S_j m^{r_j} - f_j$, for $j = 1, \ldots, t$. The kernel of φ is the contraction of this localized kernel to $A[S_1, \ldots, S_t]$, and this contraction is precisely the stated saturation. □

5.1.2 Proof of the algorithm

Let R_0 be the initial ring and set

$$T := (R_0)_{m_1} \cap \cdots \cap (R_0)_{m_s} \subseteq K.$$

We prove inductively that every current ring A is a finitely generated normal domain contained in T and satisfies $A_{m_i} = (R_0)_{m_i}$ for every i. This is clear for $A = R_0$.

If $V(m_1, \ldots, m_s)$ has codimension at least 2 in $\mathrm{Spec}(A)$, Lemma 5.1.2 gives $A = T$. Suppose instead that the codimension is 1. Then $A \neq T$. Choose $x \in T \setminus A$. Since $T \subseteq A_{m_1}$, write $x = f/m_1^n$ with $f \in A$ and $n > 0$. The condition $x \notin A$ gives $f \notin \langle m_1^n\rangle_A$. For $i \geq 2$, the condition $x \in A_{m_i}$ is equivalent to

$$f \in \langle m_1^n\rangle_A : \langle m_i\rangle_A^\infty.$$

Hence $f \in I_n(A) \setminus \langle m_1^n\rangle_A$, so the integer searched for by the algorithm exists.

Let B_n be obtained from a finite generating set of $I_n(A)$ by retaining only the generators outside $\langle m_1^n\rangle_A$. Adjoining the fractions f/m_1^n, for $f \in B_n$, also adjoins h/m_1^n for every $h \in I_n(A)$. By Lemma 5.1.3, the saturated quotient in Algorithm 1 is exactly the subring of A_{m_1} generated by these fractions. Every such fraction belongs to T, so the new ring is contained in T. Since each $(R_0)_{m_i}$ is normal and all intersections are taken in K, the ring T is normal. The normalization of the new ring in K therefore remains contained in T.

Finally, the normalized new ring A' contains A and is contained in A_{m_i} for every i. Thus

$$A_{m_i} \subseteq A'_{m_i} \subseteq A_{m_i},$$

so $A'_{m_i} = A_{m_i} = (R_0)_{m_i}$. The inductive properties are preserved, and the algorithm is correct whenever it terminates.

Proposition 5.1.4. *Algorithm 1 terminates if and only if $R_{m_1} \cap \cdots \cap R_{m_s}$ is a finitely generated R-algebra.*

Proof. Set $T := R_{m_1} \cap \cdots \cap R_{m_s}$. If the algorithm terminates, its final ring is T. At every step one adjoins finitely many fractions and then normalizes a finitely generated algebra over a field. Such a normalization is finite, so T is a finitely generated R-algebra.

Conversely, assume

$$T = R[\theta_1, \ldots, \theta_t].$$

Let A be an intermediate ring. The same elements generate T as an A-algebra. Write

$$\theta_j = \frac{a_j}{m_1^{r_j}}, \qquad a_j \in A,$$

with r_j minimal, and put $\rho(A) := \max_j r_j$. If $\rho(A) = 0$, then $A = T$, and the algorithm stops by Lemma 5.1.2.

Assume $\rho(A) > 0$, and let n be the integer chosen at the current step. If $r_j > 0$, then $\theta_j \notin A$, so $a_j \notin \langle m_1^{r_j} \rangle_A$. Since $\theta_j \in T$, we have $a_j \in I_{r_j}(A)$, and the minimality of n gives $n \leq r_j$. The new ring contains every fraction h/m_1^n with $h \in I_n(A)$. Therefore each θ_j with $r_j > 0$ can be written in the next intermediate ring with denominator exponent at most $r_j - n$. Consequently

$$\rho(A_{\text{next}}) \leq \max\{0, \rho(A) - n\} < \rho(A).$$

The non-negative integer $\rho(A)$ can decrease only finitely many times, so the algorithm terminates. ☐

Corollary 5.1.5. *Let R be a normal finitely generated $\mathbb{K}$-algebra, let $m_1, \ldots, m_k \in R$ be non-zero, and assume that*

$$q_j := \frac{f_j}{m_1^{r_j}} \in R_{m_1} \cap \cdots \cap R_{m_k}, \qquad j = 1, \ldots, t.$$

Let $P := R[s_1, \ldots, s_t]$ *and*

$$I := \langle s_1 m_1^{r_1} - f_1, \ldots, s_t m_1^{r_t} - f_t \rangle.$$

Assume that an ideal $J \subseteq P$ *satisfies*

$$I \subseteq J \subseteq I : \langle m_1 \rangle^\infty.$$

Set $A_0 := P/J$ *and suppose that*

$$\operatorname{codim}_{\operatorname{Spec}(A_0)} V(m_1, \ldots, m_k) \geq 2.$$

Let $\overline{A}$ *be the image of* A_0 *in* R_{m_1} *under* $s_j \mapsto q_j$*, and let* $\overline{A}^\nu$ *be its normalization in* $\operatorname{Frac}(R)$*. Then*

$$\overline{A}^\nu = R_{m_1} \cap \cdots \cap R_{m_k}.$$

In particular, if $\overline{A}$ *is normal, then it is the desired intersection.*

Proof. The homomorphism $P \to R_{m_1}$ defined by $s_j \mapsto q_j$ has kernel $I : \langle m_1 \rangle^\infty$ by Lemma 5.1.3. Hence

$$\overline{A} \cong P/(I : \langle m_1 \rangle^\infty) = R[q_1, \ldots, q_t].$$

The kernel of $A_0 \to \overline{A}$ is annihilated by a power of m_1. Therefore passing from A_0 to $\overline{A}$ discards only components contained in $V(m_1)$, and the locus defined by $m_1, \ldots, m_k$ still has codimension at least 2 in $\operatorname{Spec}(\overline{A})$.

For each i, the inclusions $R \subseteq \overline{A} \subseteq R_{m_i}$ give $\overline{A}_{m_i} = R_{m_i}$. Normalization commutes with localization, so $(\overline{A}^\nu)_{m_i} = R_{m_i}$. Since $\overline{A}$ is a finitely generated $\mathbb{K}$-algebra, its normalization is finite; hence the inverse image of $V(m_1, \ldots, m_k)$ in $\operatorname{Spec}(\overline{A}^\nu)$ also has codimension at least 2. Lemma 5.1.2 now yields

$$\overline{A}^\nu = \bigcap_{i=1}^{k} (\overline{A}^\nu)_{m_i} = \bigcap_{i=1}^{k} R_{m_i}.$$

□

6 Applications

6.1 For general hypersurfaces on toric varieties

Before showing concrete examples, we present a fundamental result that allows us to explicitly describe the Cox ring of certain hypersurfaces.

Theorem 6.1.1. *Let Z be a normal variety with finitely generated Cox ring $\mathcal{R}(Z)$, and let $X \subseteq Z$ be a normal subvariety with $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$. Assume that:*

(1) X_0 is big in X;

(2) $p_Z^{-1}(X_0)$ is big in $\widehat{X}$;

(3) $\widetilde{X}_1$ is normal;

(4) $\iota^* : \mathrm{Cl}(Z) \xrightarrow{\sim} \mathrm{Cl}(X)$ is an isomorphism.

Set

$$R := \mathcal{R}(Z)/I(\widetilde{X}).$$

Let $m_1, \ldots, m_s \in R$ be homogeneous elements generating $I(\widetilde{X} \setminus \widetilde{X}_1)$. Then

$$\mathcal{R}(X) = R_{m_1} \cap \cdots \cap R_{m_s},$$

where the intersection is taken inside the common fraction field.

Proof. Since $\widetilde{X}_1$ is normal and $\widehat{X}$ is open in $\widetilde{X}_1$, the variety $\widehat{X}$ is normal. Hence Proposition 4.2.1 applies. We obtain

$$
\begin{aligned}
\mathcal{R}(X) &= \Gamma(X, \mathcal{R}_X) \\
&= \Gamma\big(X, (p_X)_*\mathcal{O}_{\widehat{X}}\big) && \text{by Proposition 4.2.1} \\
&= \Gamma(\widehat{X}, \mathcal{O}) \\
&= \Gamma(\widetilde{X}_1, \mathcal{O}) && \text{because } \widehat{X} \text{ is big in the normal variety } \widetilde{X}_1 \\
&= \Gamma\big(\widetilde{X} \setminus V(m_1, \dots, m_s), \mathcal{O}\big) \\
&= \Gamma\left(\bigcup_{i=1}^{s} \widetilde{X}_{m_i}, \mathcal{O}\right) \\
&= \bigcap_{i=1}^{s} \Gamma(\widetilde{X}_{m_i}, \mathcal{O}) \\
&= \bigcap_{i=1}^{s} R_{m_i}.
\end{aligned}
$$

The last equality uses $\Gamma(\widetilde{X}, \mathcal{O}) \cong \mathcal{R}(Z)/I(\widetilde{X})$. □

Remark 6.1.2. There is also a coarser version. Let $m_1, \dots, m_s \in R$ be the images of homogeneous generators of the irrelevant ideal of Z. Since

$$\widehat{X} = \widetilde{X} \cap \widehat{Z} = \widetilde{X} \setminus V(m_1, \dots, m_s),$$

the same argument gives

$$\mathcal{R}(X) = \bigcap_{i=1}^{s} R_{m_i}.$$

This version uses the whole irrelevant ideal. Theorem 6.1.1 is sharper, because it localizes only along the divisorial part of $\widetilde{X} \setminus \widehat{X}$.

Observe that when Z is smooth the hypotheses (1) and (2) are automatically satisfied. Theorem 6.1.1 generalizes results in [8, §2] (see also [3, Cor 4.1.1.3]) and, in the case of hypersurfaces, the main result of [2]. Indeed , in these cases, one has $\widetilde{X}_1 = \widetilde{X}$. As a consequence of this description, we produce Algorithm 1, which computes the Cox ring of X iteratively by adding new generators at each step, and which terminates if and only if the Cox ring is finitely generated. Our first application is the following:

Theorem 6.1.3 (Herrera-Laface-Ugaglia, 2024)**.** *Let $(\mathbb{P}, H)$ be a polarized pair, where $\mathbb{P}$ is a normal projective toric variety of dimension $n \geq 3$ and H is an ample Cartier divisor. Consider the Cox ring $\mathcal{R}(\mathbb{P}) = \mathbb{K}[T_1, \dots, T_r]$ and let $X \in |H|$ be a hypersurface defined by a homogeneous polynomial $f \in \mathcal{R}(\mathbb{P})$. If $n \geq 4$, we assume that X is a general element of the linear system $|H|$. If $n = 3$, we assume instead that X is a very general* *element of $|H|$ and that the divisor $H + K_{\mathbb{P}}$ is base point free. Assume that the irrelevant locus of $\mathbb{P}$ has a unique codimension-two irreducible component, namely $V(T_1, T_2)$. Let $d > 0$ be the multiplicity of f along this component, so that f admits a decomposition*

$$f = \sum_{i=0}^{d} (-1)^i f_i T_1^{d-i} T_2^i, \quad \text{with } f_0, \dots, f_d \in \mathbb{K}[T_1, \dots, T_r] \text{ homogeneous.}$$

Suppose further that the algebraic subset $V(T_1, T_2, f_0, \dots, f_d) \subseteq \mathbb{K}^r$ has codimension $d+3$. Then the Cox ring of X is isomorphic to

$$\mathcal{R}(X) \cong \frac{\mathbb{K}[T_1, \dots, T_r, S_1, \dots, S_d]}{\left\langle \begin{array}{l} f_0 + T_2 S_1, \\ f_i + T_1 S_i + T_2 S_{i+1},\ 1 \le i \le d-1, \\ f_d + T_1 S_d \end{array} \right\rangle}$$

where $S_1, \dots, S_d$ are new homogeneous variables of appropriate multidegrees.

Proof. Set

$$A := \mathbb{K}[T_1, \dots, T_r]/\langle f \rangle.$$

We first verify the hypotheses of Theorem 6.1.1. Since H is an ample Cartier divisor on a projective toric variety, $|H|$ is base-point-free. A normal toric variety is Cohen–Macaulay. Since X is an effective Cartier divisor, it satisfies S_2. Bertini's theorem shows that a general member is smooth on $\mathbb{P}_{\mathrm{reg}}$, and generality also ensures that no component of $\mathbb{P}_{\mathrm{Sing}}$ is contained in X. Thus the singular locus of X has codimension at least 2 in X, so X satisfies R_1 and is normal.

The isomorphism

$$\mathrm{Cl}(\mathbb{P}) \xrightarrow{\sim} \mathrm{Cl}(X)$$

follows from [10, Thm. 1] when $n \ge 4$. When $n = 3$, it follows from [11, Thm. 1], applied to a very general member under the hypothesis that $H + K_{\mathbb{P}}$ is base-point-free.

Let $\widetilde{X} = V(f)$ in the total coordinate space. The irrelevant locus has the unique codimension-two component $V(T_1, T_2)$. Every other component has codimension at least 3, and a general f contains none of them. Their intersections with $\widetilde{X}$ therefore have codimension at least 2 in $\widetilde{X}$. Consequently the divisorial part of $\widetilde{X} \setminus \widehat{X}$ is $V(T_1, T_2) \cap \widetilde{X}$, and

$$\widetilde{X}_1 = \widetilde{X} \setminus V(T_1, T_2) = D(T_1) \cup D(T_2).$$

We now construct an affine model for this open subset. Let

$$P := \mathbb{K}[T_1, \dots, T_r, S_1, \dots, S_d]$$

and let $I \subseteq P$ be generated by

$$
\begin{aligned}
g_0 &:= f_0 + T_2 S_1, \\
g_i &:= f_i + T_1 S_i + T_2 S_{i+1}, \quad 1 \leq i \leq d-1, \\
g_d &:= f_d + T_1 S_d.
\end{aligned}
$$

Set $C := P/I$. After localizing at T_2, the relations $g_0, \ldots, g_{d-1}$ eliminate $S_1, \ldots, S_d$ recursively, and the remaining relation is equivalent to $f = 0$. Hence

$$C_{T_2} \cong A_{T_2}.$$

Similarly, elimination in the opposite direction gives $C_{T_1} \cong A_{T_1}$.

We prove that C is normal. The ideal I is generated by $d+1$ elements. No minimal component of $V(I)$ is contained in $V(T_1, T_2)$, because modulo T_1, T_2 its generators become $f_0, \ldots, f_d$, while

$$\operatorname{codim} V(T_1, T_2, f_0, \ldots, f_d) = d + 3.$$

On $D(T_1) \cup D(T_2)$ the preceding elimination shows that I has height $d+1$. Therefore I has height $d+1$ everywhere. Since P is a polynomial ring, the $d+1$ generators form a regular sequence. Thus C is a complete intersection, hence Cohen–Macaulay and S_2.

Moreover,

$$C/\langle T_1, T_2 \rangle \cong P/\langle T_1, T_2, f_0, \ldots, f_d \rangle,$$

so $V(T_1, T_2)$ has codimension 2 in $\operatorname{Spec}(C)$. Every height-one point of $\operatorname{Spec}(C)$ lies in $D(T_1) \cup D(T_2)$, where C is identified with the corresponding localization of A. This open set is the pullback of X over the smooth locus of the ambient toric variety and is regular at its codimension-one points. Hence C satisfies R_1. Serre's criterion now implies that C is normal.

Since $V(T_1, T_2)$ has codimension 2 in $\operatorname{Spec}(C)$, regular functions on $D(T_1) \cup D(T_2)$ extend uniquely, and therefore

$$C = C_{T_1} \cap C_{T_2} = A_{T_1} \cap A_{T_2}.$$

The same normality statement shows that $\widetilde{X}_1$ is normal. Theorem 6.1.1, with $m_1 = T_1$ and $m_2 = T_2$, gives

$$\mathcal{R}(X) = A_{T_1} \cap A_{T_2} = C,$$

which is the presentation in the statement. $\square$

The above theorem generalizes results of Ottem [9], who studied hypersurfaces in products of projective spaces, and recent findings by Denisi [6] on hypersurfaces in products of weighted projective spaces. For this result our technique resembles the unprojection methods developed by Miles Reid [12] and aligns with the approach taken by Abban in [1, Sec. 3.3].

Remark 6.1.4. Let $(\mathbb{P}, H)$ be a polarized pair consisting of a normal projective toric variety of dimension at least 4 and an ample Cartier divisor. Let $X \in |H|$ be general, defined by a homogeneous polynomial $f \in \mathcal{R} = \mathbb{K}[T_1, \ldots, T_r]$ in Cox coordinates. Let $R := \mathbb{K}[T_1, \ldots, T_r]/\langle f \rangle$. Assume that the irrelevant ideal $\mathscr{I}(\mathbb{P}) = \langle m_1, \ldots, m_s \rangle$ of $\mathbb{P}$ has a component of codimension 2, say $\langle T_1, T_2 \rangle$, so that each m_i is divisible by either T_1 or T_2. Let $d > 0$ be the multiplicity of f along this component. We can write

$$f = \sum_{i=0}^{d} (-1)^i f_i T_1^{d-i} T_2^i.$$

Then the Cox ring of X contains the homogeneous elements

$$S_1 := -\frac{f_0}{T_2}, \quad S_2 := -\frac{f_1 + S_1 T_1}{T_2}, \quad \ldots \quad S_d := -\frac{f_{d-1} + S_{d-1} T_1}{T_2}$$

because, as showed in the proof of Theorem 6.1.3, each of them belong to $R_{T_1} \cap R_{T_2} \subseteq R_{m_1} \cap \cdots \cap R_{m_s}$.

6.1.1 For Picard rank 2

Corollary 6.1.5. *Let $\mathbb{P}$ be a smooth projective toric variety of dimension $n \geq 4$ and Picard rank 2, and let X be a smooth general ample hypersurface of $\mathbb{P}$. Then the Cox ring of X is as given in Theorem 6.1.3 in the following cases:*

(1) The grading matrix and the irrelevant ideal of $\mathbb{P}$ are

$$\begin{bmatrix} 1 & 1 & 0 & -a_1 & \ldots & -a_{n-1} \\ 0 & 0 & 1 & 1 & \ldots & 1 \end{bmatrix}, \qquad \langle T_1, T_2 \rangle \cap \langle T_3, \ldots, T_{n+2} \rangle,$$

where $0 \leq a_1 \leq \cdots \leq a_{n-1}$ and X has degree $[a, b]$, with $0 < a \leq \max\{k : a_k = 0\}$ and $b > 0$. In this case, $d = a$.

(2) The grading matrix and the irrelevant ideal of $\mathbb{P}$ are

$$\begin{bmatrix} 1 & \ldots & 1 & 0 & -a_1 \\ 0 & \ldots & 0 & 1 & 1 \end{bmatrix}, \qquad \langle T_1, \ldots, T_n \rangle \cap \langle T_{n+1}, T_{n+2} \rangle,$$

where $a_1 \geq 0$ and X has degree $[a, b]$, with $a > 0$ and $0 < b \leq n - 1$. In this case $d = b$.

Proof. Let f be a defining polynomial of X.

In case (1), write

$$f = \sum_{i=0}^{a} (-1)^i f_i T_1^{a-i} T_2^i,$$

where $f_0, \dots, f_a$ have degree $[0, b]$. Put $m := \max\{k : a_k = 0\}$. After setting $T_1 = T_2 = 0$, the polynomials $f_0, \dots, f_a$ become $a+1$ general forms of degree b in the $m+1$ variables $T_3, \dots, T_{m+3}$. Since $a+1 \le m+1$, general such forms constitute a regular sequence. Hence

$$\operatorname{codim} V(T_1, T_2, f_0, \dots, f_a) = a + 3.$$

The hypotheses of Theorem 6.1.3 are satisfied, with $d = a$.

In case (2), write

$$f = \sum_{i=0}^{b} (-1)^i f_i T_{n+1}^{b-i} T_{n+2}^{i},$$

where f_i is a general form of degree $a + ia_1$ in $T_1, \dots, T_n$. Since $a > 0$ and $b+1 \le n$, these $b+1$ general forms constitute a regular sequence. Therefore

$$\operatorname{codim} V(T_{n+1}, T_{n+2}, f_0, \dots, f_b) = b + 3.$$

Again Theorem 6.1.3 applies, now with $d = b$. □

Example 6.1.6. Let $Bl_p\mathbb{P}^4$ be the blowing-up of $\mathbb{P}^4$ in the invariant point p with exceptional divisor E and H the pull-back of a general hyperplane. Let $X \subseteq \mathrm{Bl}_p\mathbb{P}^4$ be a smooth hypersurface in $|5H - 3E|$. An equation that represents X is

$$f_3T_5^2 - f_4T_5T_6 + f_5T_6^2 = 0,$$

where $f_i \in \mathbb{K}[T_1, T_2, T_3, T_4]$ are homogeneous of degree i, for $i = 3, 4, 5$. In the general case, by Theorem 6.1.3 the Cox ring and the grading matrix of X are

$$\frac{\mathbb{K}[T_1, \dots, T_6, S_1, S_2]}{\langle f_3 + T_6S_2, f_4 + T_5S_2 + T_6S_1, f_5 + T_5S_1\rangle} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 4 & 3 \\ -1 & -1 & -1 & -1 & 0 & 1 & -5 & -4 \end{bmatrix}$$

If $f_4 = 0$, the defining equation is

$$f_3T_5^2 + f_5T_6^2 = 0.$$

The Cox ring and its grading matrix are

$$\frac{\mathbb{K}[T_1, \dots, T_6, S_1]}{\langle f_3 + T_6^2S_1,\ f_5 - T_5^2S_1\rangle} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 3 \\ -1 & -1 & -1 & -1 & 0 & 1 & -5 \end{bmatrix}.$$

The opposite signs in the two relations are necessary in order to recover $f_3T_5^2 + f_5T_6^2 = 0$ after eliminating S_1.

Proposition 6.1.7. *Let $\mathbb{P}$ be a smooth projective toric variety of Picard rank two. Then the grading matrix of the Cox ring $\mathbb{K}[T_1, \ldots, T_{n+2}]$ of $\mathbb{P}$ and the irrelevant ideal are*

$$\begin{bmatrix} \underbrace{\begin{matrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{matrix}}_{n-k+1} & \begin{matrix} 0 & -a_1 & \cdots & -a_k \\ 1 & 1 & \cdots & 1 \end{matrix} \end{bmatrix}, \qquad \langle T_1, \ldots, T_{n-k+1} \rangle \cap \langle T_{n-k+2}, \ldots, T_{n+2} \rangle$$

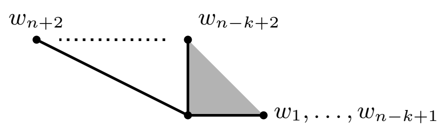
where $1 \leq k \leq n-1$, and $0 \leq a_1 \leq \cdots \leq a_k$. Moreover $\mathbb{P}$ is Fano if and only if $\sum_i a_i < n-k+1$.

Proof. Let $w_i = \deg(T_i)$ for any i and let v_i be the corresponding primitive generator of the one dimensional cone of the fan $\Sigma = \Sigma(\mathbb{P})$. Let $p_\sigma \in \mathbb{P}$ be the invariant point defined by a maximal cone $\sigma \in \Sigma$. Since p_σ is a smooth point of $\mathbb{P}$ it is a factorial point, that is the local class group $\mathrm{Cl}(\mathbb{P}, p_\sigma)$ is trivial (in a toric variety being smooth and being factorial is the same). By [3, Proposition 3.3.1.5] we have

$$\mathrm{Cl}(\mathbb{P}, p_\sigma) \simeq \mathrm{Cl}(\mathbb{P})/\langle w_i \,:\, v_i \notin \sigma(1) \rangle.$$

The free abelian group in the above denominator is generated by the two classes w_i, w_j such that $\mathbb{Q}_{\geq 0} \cdot v_i$ and $\mathbb{Q}_{\geq 0} \cdot v_j$ are not one dimensional rays of σ. Thus the factoriality of $\mathbb{P}$ at p_σ is equivalent to ask for the matrix (w_i, w_j) to have determinant ± 1.

Since $\mathbb{P}$ is smooth projective its nef cone is full dimensional. Let w be an ample class. Then by [3, Corollary 1.6.3.6] the irrelevant ideal is generated by the monomials $T_i T_j$ such that $w \in \mathrm{cone}(w_i, w_j)$. Equivalently the complement $\{1, \ldots, n+2\} \setminus \{i, j\}$ of any such pair of indices i, j defines a maximal cone of Σ. Thus by the previous argument we have that each such matrix (w_i, w_j) has determinant ± 1. In particular we can assume without loss of generality that the nef cone of $\mathbb{P}$ is the first quadrant. If there are other rays different from $(1, 0)$ and $(0, 1)$, by smoothness hypothesis they must be of the form $(-a, 1)$ or $(1, -b)$, for $a, b > 0$. Since the variety is complete the effective cone is pointed so that, if there exists at least one ray $(-a, 1)$, there can be no rays of the form $(1, -b)$. So up to exchanging the axes we can assume that there are no rays of the form $(1, -b)$ and in particular the ray $(1, 0)$ is an extremal ray of the effective, moving and nef cone at the same time. The situation is summarized in the following picture.



Since $(1, 0)$ is an extremal ray of the moving cone, by [3, Proposition 3.3.2.3] we deduce that $n-k+1 \geq 2$. For the same reason we have $k \geq 1$. Finally $\mathbb{P}$ is Fano if and only if the sum $\sum_i w_i$ lies in the interior of the nef cone, equivalently if $\sum_i a_i < n-k+1$. $\square$

Observe that a smooth projective toric variety $\mathbb{P}$ has a component of the irrelevant locus of codimension 2 when either $k = n-1$ or $k = 1$, with the same notation of Proposition 6.1.7. The grading matrices for these two cases are those which appear in the statement of Corollary 6.1.5.

Remark 6.1.8. If all the a_i vanish in case (1) of Corollary 6.1.5, or if $a_1 = 0$ in case (2), then

$$\mathbb{P} \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}.$$

Consequently, the Cox ring of a general hypersurface of degree $[a, b]$ in $\mathbb{P}^1 \times \mathbb{P}^{n-1}$, with $a \leq n-1$, is described by Theorem 6.1.3. This is also the result of [9, Thm. 1.1]. In the remaining range treated in [9, Thm. 5.6], the effective cone is not closed and the Cox ring is not finitely generated.

For a hypersurface of degree $[d, 1]$, one obtains the more precise description in Proposition 6.1.9.

Proposition 6.1.9. *Let*

$$X = V(f) \subseteq \mathbb{P}^1 \times \mathbb{P}^{n-1}, \qquad n \geq 4,$$

where f is a general homogeneous polynomial of degree $[d, 1]$. Let $a \in \{0, \dots, n-2\}$ be the residue of d modulo $n-1$. Then X is the scroll

$$S(\underbrace{0, \dots, 0}_{n-a-1}, \underbrace{1, \dots, 1}_{a}).$$

In particular, if $(n-1) \mid d$, then $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$.

Proof. Write

$$f = \sum_{i=0}^{d} (-1)^i f_i T_1^{d-i} T_2^i,$$

where $f_0, \dots, f_d$ are general linear forms in $T_3, \dots, T_{n+2}$.

Assume first that $d \leq n-1$. By Remark 6.1.8 and Theorem 6.1.3,

$$\mathcal{R}(X) = \mathbb{K}[T_1, \dots, T_{n+2}, S_1, \dots, S_d]/I,$$

where I is generated by

$$f_0 + T_2 S_1, \quad f_i + T_1 S_i + T_2 S_{i+1} \ (1 \leq i \leq d-1), \quad f_d + T_1 S_d.$$

After a linear change of the variables $T_3, \dots, T_{n+2}$, the forms $f_0, \dots, f_d$ can be used to eliminate $T_3, \dots, T_{d+3}$. Hence X is toric with Cox grading

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & -1 & \cdots & -1 \\ 0 & 0 & \underbrace{1 & \cdots & 1}_{n-d-1} & \underbrace{1 & \cdots & 1}_{d} \end{bmatrix}.$$

The unimodular change of basis sending $(1,0)$ to $(1,0)$ and $(-1,1)$ to $(0,1)$ identifies this grading with that of $S(0,\dots,0,1,\dots,1)$, with d entries equal to 1. If $d=n-1$, this is $S(1,\dots,1)$, which is isomorphic to $S(0,\dots,0)\cong\mathbb{P}^1\times\mathbb{P}^{n-2}$ after twisting the underlying vector bundle. Thus the assertion agrees with the residue $a\in\{0,\dots,n-2\}$.

Now assume $d\geq n$. By Remark 6.1.4, the Cox ring contains homogeneous elements $S_1,\dots,S_n$ satisfying

$$f_0+T_2S_1,\quad f_1+T_1S_1+T_2S_2,\quad \dots,\quad f_{n-1}+T_1S_{n-1}+T_2S_n.$$

Substitution in f, followed by division by T_2^n, gives

$$T_1^{d-n+1}S_n+f_nT_1^{d-n}-f_{n+1}T_1^{d-n-1}T_2+\cdots+(-1)^{d-n}f_dT_2^{d-n}. \tag{6.1.1}$$

By a linear change of variables, assume

$$f_0=T_3,\dots,f_{n-1}=T_{n+2},\qquad f_i=a_{i,3}T_3+\cdots+a_{i,n+2}T_{n+2}\quad (n\leq i\leq d).$$

The preceding relations give

$$T_3=-T_2S_1,\quad T_4=-T_1S_1-T_2S_2,\quad \dots,\quad T_{n+2}=-T_1S_{n-1}-T_2S_n.$$

Thus, for $n\leq i\leq d$,

$$\begin{aligned} f_i = & -(a_{i,4}S_1+\cdots+a_{i,n+2}S_{n-1})T_1\\ & -(a_{i,3}S_1+\cdots+a_{i,n+2}S_n)T_2. \end{aligned}$$

Substituting these expressions into (6.1.1) produces a polynomial $g=g(T_1,T_2,S_1,\dots,S_n)$. In the inherited grading,

$$\deg(T_1)=\deg(T_2)=(1,0),\qquad \deg(S_i)=(-1,1),$$

so g has degree $[d-n,1]$. After the unimodular change of basis

$$(1,0)\longmapsto(1,0),\qquad (-1,1)\longmapsto(0,1),$$

the same toric variety is another copy of $\mathbb{P}^1\times\mathbb{P}^{n-1}$, the variables S_i have degree $[0,1]$, and g has degree $[d-n+1,1]$. Write

$$g=\sum_{i=0}^{d-n+1}(-1)^i g_i T_1^{d-n+1-i}T_2^i,$$

where the g_i are linear forms in $S_1,\dots,S_n$.

The coefficient matrix of $g_0,\dots,g_{d-n+1}$ is a $(d-n+2)\times n$ matrix whose entries depend linearly on the parameters $a_{i,k}$. The parameters can be chosen so that the first $\min\{d-n+2,n\}$ rows contain an identity block. Hence maximal rank holds on a non-empty Zariski open subset, and the g_i are general in the sense needed to repeat the construction.

One step therefore replaces degree $[d,1]$ by $[d-(n-1),1]$. Iterating finitely many times gives degree $[a,1]$, where $a\in\{0,\dots,n-2\}$ is congruent to d modulo $n-1$. The first part of the proof identifies the resulting hypersurface with the stated scroll. □

Corollary 6.1.10. *Let $\mathbb{P}$ be a smooth toric Fano variety of dimension $n \geq 4$ and Picard rank two. Let $X \subseteq \mathbb{P}$ be a general anticanonical hypersurface whose homogeneous equation is $f = 0$. Then the Cox ring of X and its grading matrix are given in the following table.*

Cox ring	*Grading matrix*
$\mathbb{K}[T_1, \dots, T_{n+2}] / \langle f \rangle$	$\left[\begin{array}{ccccc} \underbrace{\begin{matrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{matrix}}_{n-k+1} & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} -a_1 \\ 1 \end{matrix} & \begin{matrix} \cdots \\ \cdots \end{matrix} & \begin{matrix} -a_k \\ 1 \end{matrix} \end{array}\right]$
where $2 \leq k \leq n-2$, $0 \leq a_1 \leq \cdots \leq a_k$, *and* $\sum_i a_i < n-k+1$.	
$\mathbb{K}[T_1, \dots, T_{n+2}, S]/I$ *with* I *generated by* $f_1 + ST_1$, $f_0 + ST_2$	$\left[\begin{array}{ccccc\|c} \begin{matrix} 1 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 0 \end{matrix} & \underbrace{\begin{matrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{matrix}}_{n-1} & \begin{matrix} -1 \\ 1 \end{matrix} & & \begin{matrix} -1 \\ n \end{matrix} \end{array}\right]$
where $f = f_0 T_1 - f_1 T_2$	
$\mathbb{K}[T_1, \dots, T_{n+2}, S_1, S_2]/I$ *with* I *generated by* $f_0 + T_{n+2} S_2$, $f_1 + T_{n+1} S_2 + T_{n+2} S_1$, $f_2 + T_{n+1} S_1$	$\left[\begin{array}{ccc\|cc} \underbrace{\begin{matrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{matrix}}_{n} & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} -a_1 \\ 1 \end{matrix} & \begin{matrix} n+a_1 \\ -1 \end{matrix} & \begin{matrix} n \\ -1 \end{matrix} \end{array}\right]$
where $a_1 \geq 0$ *and* $f = f_0 T_{n+1}^2 - f_1 T_{n+1} T_{n+2} + f_2 T_{n+2}^2$	

Proof. The grading matrix and irrelevant ideal of $\mathbb{P}$ are given in Proposition 6.1.7, and in what follows we use the same notation.

When $2 \leq k \leq n-2$ the irrelevant locus of $\mathbb{P}$ has no components of codimension 2 so that the Cox ring is a hypersurface (first line of the above table).

When $k = n-1$, the ambient toric variety is Fano exactly when $a_1 = \cdots = a_{n-2} = 0$ and $0 \leq a_{n-1} \leq 1$. If $a_{n-1} = 1$, then the anticanonical hypersurface X has degree $[1, n]$ and, by Corollary 6.1.5, the Cox ring is the one given in Theorem 6.1.3 with $d = 1$. The case $a_{n-1} = 0$ corresponds to the ambient toric variety $\mathbb{P}^1 \times \mathbb{P}^{n-1}$, which is analyzed in the next paragraph.

When $k = 1$, the ambient toric variety is Fano exactly when $a_1 \leq n-1$. The anticanonical hypersurface X has degree $[n - a_1, 2]$ and, by Corollary 6.1.5, the Cox ring is the one given in Theorem 6.1.3 with $d = 2$.

□

We now consider the case of a surface X embedded into a smooth projective toric threefold $\mathbb{P}$ of Picard rank 2. Since the Cox ring of $\mathbb{P}$ has 5 generators it follows that the irrelevant locus is always the union of a codimension 2 with a codimension 3 linear subspaces. In what follows we compute the Cox ring of X in some cases, using the same notation of Proposition 6.1.7.

Corollary 6.1.11. *Let $\mathbb{P}$ be a smooth projective toric threefold of Picard rank two. Let $X \subseteq \mathbb{P}$ be a very general ample surface whose homogeneous equation is $f = 0$. In the following table we compute a presentation for the Cox ring of certain such X.*

	Cox ring	*Grading matrix*
$k=2$, $a_2 > 0$, $\deg(f) = [1,b]$, $b \geq 3$	$\mathbb{K}[T_1,\dots,T_5,S]/I$ *with I generated by* $f_1 + ST_1$, $f_0 + ST_2$ *where* $f = f_0T_1 - f_1T_2$	$\left[\begin{array}{ccccc\|c} 1 & 1 & 0 & 0 & -a_2 & -1 \\ 0 & 0 & 1 & 1 & 1 & b \end{array}\right]$
$k=1$, $a_1 \geq 0$, $\deg(f) = [a,2]$, $a \geq \max\{1, 3-a_1\}$	$\mathbb{K}[T_1,\dots,T_5,S_1,S_2]/I$ *with I generated by* $f_0 + T_5S_2$, $f_1 + T_4S_2 + T_5S_1$, $f_2 + T_4S_1$ *where* $f = f_0T_4^2 - f_1T_4T_5 + f_2T_5^2$	$\left[\begin{array}{ccccc\|cc} 1 & 1 & 1 & 0 & -a_1 & a+2a_1 & a+a_1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$

Proof. By Proposition 6.1.7, the grading matrix of the Cox ring of $\mathbb{P}$ is one of the following (corresponding to the cases $k=2$ and $k=1$ respectively):

$$\begin{bmatrix} 1 & 1 & 0 & -a_1 & -a_2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & -a_1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

where $0 \leq a_1 \leq a_2$. The nef cone of $\mathbb{P}$ is the positive quadrant in both cases thus f has one of the following form:

$$f = \sum_{i=0}^{d} (-1)^i f_i T_1^i T_2^{d-i}, \qquad f = \sum_{i=0}^{d} (-1)^i f_i T_4^i T_5^{d-i},$$

respectively. Observe that since the spectrum of the Cox ring of $\mathbb{P}$ has dimension 5, the codimension condition given in Theorem 6.1.3 implies $d \leq 2$.

We proceed as in Theorem 6.1.3, whose proof applies verbatim here once the isomorphism at the level of divisor class groups is established. We use here the Lefschetz-type theorem [11, Thm. 1] whose hypotheses are: the class H of X must be ample and base point free, $H + K_{\mathbb{P}}$ must be base point free, and $X \in |H|$ must be very general. Since $\mathbb{P}$ is smooth and toric, the first two conditions are equivalent to requiring that H is just ample (i.e. the class of H is $[a, b]$, with $a, b > 0$) and $H + K_{\mathbb{P}}$ is nef.

Putting all this together, in case $k = 2$, since $d = a$ we obtain that

$$0 < a \leq 2, \qquad b > 0, \qquad a + a_1 + a_2 - 2 \geq 0, \qquad b - 3 \geq 0.$$

When we evaluate $f_0, \ldots, f_a$ in $T_1 = T_2 = 0$, we get $a + 1$ general homogeneous polynomials of degree b in the variables T_3 (if $a_1 > 0$), T_3, T_4 (if $a_1 = 0$ and $a_2 > 0$) or T_3, T_4, T_5 (if $a_1 = a_2 = 0$). In the first case the codimension condition cannot be satisfied. In the second case it can be satisfied only if $a = d = 1$. The third case has been included in the analysis of $k = 1$, after a change of variables.

Analogously, when $k = 1$ we have $d = b$, and we get the conditions:

$$a > 0, \qquad 0 < b \leq 2, \qquad a + a_1 - 3 \geq 0, \qquad b - 2 \geq 0,$$

which are the ones displayed in the statement. We conclude observing that under these conditions the locus $V(T_4, T_5, f_0, f_1, f_2)$ has the required codimension 5. □

6.2 A family of hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^n$

According to [9], a very general hypersurface of $\mathbb{P}^1 \times \mathbb{P}^n$ of bidegree (a, b), with $a \geq n+1$ and $b \geq 2$, is not a Mori dream space. We consider below a structured family of special hypersurfaces and use the algorithm to test finite generation in explicit instances. The construction provides candidates with controlled equations; no uniform finite-generation statement for the whole family is asserted here.

These polynomials can be constructed using the following MAGMA function:

```
Poli := function(a, b, n)
    if a lt 0 or b lt 0 then
        error "Se requiere: a >= 0 y b >= 0";
    end if;
    a1 := a;
    if n lt a then
        a1 := n;
    end if;
    P1 := ProjectiveSpace(Rationals(),1);
```

```
    Pn := ProjectiveSpace(Rationals(),n);
    P<[x]> := P1 * Pn;
    S := &+[u^b : u in x[3..n+3]];
    F := 0;
    for i in [0..a1] do
        gi := S - x[i+3]^b;
        term := x[1]^(a - i) * x[2]^i * gi;
        F +:= term;
    end for;
    return F + x[2]^a*S, P;
end function;
```

where $S = \sum_{j=0}^{n} y_j^b$. This family generates hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^n$ with a structure symmetric and controlled, and it allows us to describe in an effective way the conditions of the algorithm.

Example 6.2.1. In particular, we consider the case $(a, b, n) = (4, 2, 3)$, that is, it has bidegree $(4, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^3$, and we apply the algorithm to compute the Cox ring of the associated hypersurface defining the following Magma function using the main algorithm:

```
CoxPoli := function(a,b,n);
F,P<[x]> := Poli(a,b,n);
I := ideal<Parent(F) | F>;
lis := [Parent(F).1,Parent(F).2];
h := Transpose(Matrix(Gradings(P)));
return ComputeCox(I, lis, h);
end function;
```

which returns:

```
> CoxPoli(4,2,3);
[
-3/2*T[1]^2*T[9] - 3/4*T[1]*T[2]*T[9] + T[1]*T[8] - 3/4*T[2]^2*T[9] +
T[6]^2,
3/4*T[1]^2*T[9] + 3/2*T[1]*T[2]*T[9] + T[1]*T[7] - T[1]*T[8] +
3/4*T[2]^2*T[9] - T[2]*T[8] + T[5]^2,
3/4*T[1]^2*T[9] - 3/4*T[1]*T[2]*T[9] - T[1]*T[7] - T[2]*T[7] + T[2]*T[8] +
T[4]^2,
3/4*T[1]^2*T[9] - 3/4*T[1]*T[2]*T[9] + T[2]*T[7] + T[3]^2
]
```

```
[ 1  1  0  0  0  0 -1 -1 -2]
[ 0  0  1  1  1  1  2  2  2]
```

The Magma computation is performed over $\mathbb{Q}$. It yields the displayed presentation of a graded $\mathbb{Q}$-algebra. To identify its scalar extension to $\mathbb{C}$ with the Cox ring of the corresponding complex hypersurface, one must additionally verify that the normality and codimension certificates used by the algorithm remain valid after base extension. Accordingly, the output above should first be stated as a computation over $\mathbb{Q}$.

$$\mathcal{R}_{\mathbb{Q}}(X) \cong \mathbb{Q}[T_1, \dots, T_9]/I,$$

with grading matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{bmatrix}$$

and ideal I defined by the relations:

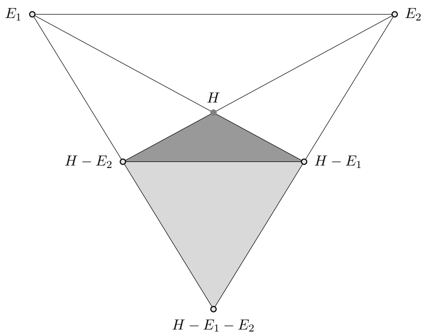
$$\begin{aligned} -\tfrac{3}{2}T_1^2T_9 - \tfrac{3}{4}T_1T_2T_9 + T_1T_8 - \tfrac{3}{4}T_2^2T_9 + T_6^2 &= 0, \\ \tfrac{3}{4}T_1^2T_9 + \tfrac{3}{2}T_1T_2T_9 + T_1T_7 - T_1T_8 + \tfrac{3}{4}T_2^2T_9 - T_2T_8 + T_5^2 &= 0, \\ \tfrac{3}{4}T_1^2T_9 - \tfrac{3}{4}T_1T_2T_9 - T_1T_7 - T_2T_7 + T_2T_8 + T_4^2 &= 0, \\ \tfrac{3}{4}T_1^2T_9 - \tfrac{3}{4}T_1T_2T_9 + T_2T_7 + T_3^2 &= 0. \end{aligned}$$

6.3 Cases with Picard rank different from 2.

Example 6.3.1. Let $Z := \mathrm{Bl}_2\,\mathbb{P}^4$ be the blowing up of $\mathbb{P}^4$ in two points, and let us denote by H the class of the pull back of a hyperplane and by E_1, E_2 the exceptional divisors. The grading matrix of the Cox ring $\mathcal{R}(Z) = \mathbb{K}[T_1, \dots, T_7]$ is the following

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

with of the $\mathrm{Pic}(Z) = \{H, E_1, E_2\}$, and the irrelevant ideal is $\mathscr{I}_{\mathrm{irr}}(Z) = \langle T_6T_7, T_4T_6, T_5T_7\rangle = \langle T_4, T_7\rangle \cap \langle T_5, T_6\rangle \cap \langle T_6, T_7\rangle$. In the following picture we display the effective cone of $\mathbb{P}$, the nef cone (in dark grey), and the movable one (the union of the nef cone and the light grey triangle).



Let us consider a general hypersurface $X \subseteq Z$ in an ample degree, for instance $3H - E_1 - E_2$. An equation f of X can be written as

$$T_4^2T_5T_6 + a_1T_4^2T_6^2 + T_4T_5^2T_7 + b_1T_4T_5T_6T_7 + b_2T_4T_6^2T_7 + c_1T_5^2T_7^2 + a_2T_5T_6T_7^2 + a_3T_6^2T_7^2$$

where $a_i, b_i, c_i \in \mathbb{K}[T_1, T_2, T_3]$ are homogeneous of degree i. Applying Algorithm 1 to the first two powers of $m_1 := T_6T_7$ we get the following new elements of the Cox ring [1]

$$\begin{aligned} S_1 &:= \frac{a_1T_6 + T_5}{T_7}, & S_2 &:= \frac{c_1T_7 + T_4}{T_6}, \\ S_3 &:= \frac{a_2 - b_1c_1 + c_1^2 + S_1S_2}{T_6}, & S_4 &:= \frac{a_1^2 - a_1b_1 + b_2 + S_1S_2}{T_7}. \end{aligned}$$

Their ideal of relations in $\mathbb{K}[T_1, \ldots, T_7, S_1, \ldots, S_4]$ is the following complete intersection

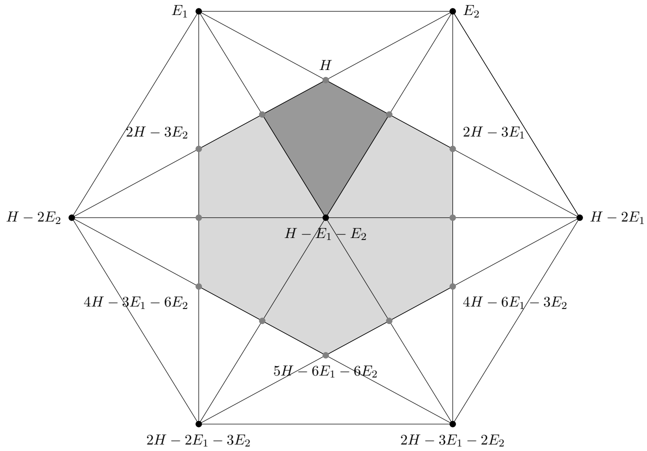
$$\tilde{J} = \left\langle \begin{array}{cc} a_1T_6 + T_5 - T_7S_1, & c_1T_7 + T_4 - T_6S_2 \\ -T_6S_3 + a_2 - b_1c_1 + c_1^2 + S_1S_2, & -T_7S_4 + a_1^2 - a_1b_1 + b_2 + S_1S_2 \\ \multicolumn{2}{c}{-a_1T_6S_3 + T_6S_2S_4 + T_7S_1S_3 + a_1c_1^2 - a_1S_1S_2 + a_3 + b_1S_1S_2 - b_2c_1 - 2c_1S_1S_2} \end{array} \right\rangle.$$

A direct calculation shows that each component $V(T_4, T_7)$, $V(T_5, T_6)$ and $V(T_6, T_7)$ has codimension 2 in $V(\tilde{J})$, so that the new elements generate the Cox ring by Lemma 5.1.2. Eliminating the variables T_4 and T_5, by means of the first two equations, and putting $J := \tilde{J} \cap \mathbb{K}[T_1, T_2, T_3, T_6, T_7, S_1, \ldots, S_4]$ the Cox ring $\mathcal{R}(X)$ and the grading matrix are

$$\mathbb{K}[T_1, T_2, T_3, T_6, T_7, S_1, \ldots, S_4]/J, \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 2 \\ -1 & -1 & -1 & 1 & 0 & -2 & -3 & 0 & -2 \\ -1 & -1 & -1 & 0 & 1 & 0 & -2 & -2 & -3 \end{bmatrix}.$$

In particular $\mathcal{R}(X)$ is a complete intersection. In the following picture we display the cones $\mathrm{Eff}(X)$, $\mathrm{Mov}(X)$ and $\mathrm{Nef}(X)$.

[1] The Magma program to find the four elements is available at https://github.com/alaface/Cox-subvarieties/blob/main/Example%206.1.



Example 6.3.2. We consider the projective toric variety $Z = \mathcal{X}_6$, obtained from the smooth Fano polytopes database of the MAGMA library (in particular, we choose the number 6). This variety has dimension 4, is smooth and its Cox ring is given by

$$R(Z) = \mathbb{F}_{32003}[x_1, \ldots, x_6],$$

where the grading matrix is obtained by the function `Gradings(FanoX(6))`.

We take a general section of the anticanonical divisor using the function `GenHyp(6)`, and define $X \subset Z$ as the hypersurface defined by the polynomial.

We apply the algorithm to compute the Cox ring of X. We use as input the ideals of codimension 2 given by `IdeCodim2(6)` and we generate the candidates to rational generators by `NewGen`. Finally, we verify that the obtained set does indeed generate the Cox ring with the `IsCox` function.

The resulting set of rational generators is the following:

$$\begin{aligned}
&T_1 = x_1, \quad T_2 = x_2, \quad T_3 = x_3, \quad T_4 = x_4, \quad T_5 = x_5, \quad T_6 = x_6,\\
&T_7 = \frac{x_1^4 + 31300\, x_2^4 x_5^4 + 31442\, x_4^4}{x_6},\\
&T_8 = \frac{x_1 + 27956\, x_2 x_5 + 26107\, x_4}{x_3}.
\end{aligned}$$

The computation produces the displayed rational generators for the graded algebra associated with the chosen hypersurface over $\mathbb{F}_{32003}$. To claim a presentation, the complete ideal of relations I must also be included, together with the normality and codimension checks used by the algorithm.

Subject to those checks, the computation proves finite generation for this specific graded algebra in characteristic 32003. It does not, by itself, imply finite generation for a characteristic-zero lift: such a conclusion would require a separate lifting or good-reduction argument. Moreover, the section used here is random over a finite field and should not be described as “very general”.

The example therefore illustrates the computational procedure outside the range of Theorem 6.1.3, but its conclusion is restricted to the stated finite-field calculation.

Command summary

For reproducibility, every computation cited in this chapter should record the Magma version, the source-file version or repository commit, the random seed, the exact input, and the complete output used to certify saturation, codimension, primality, and normality.

Some commands

We briefly describe below the Magma functions implemented and used to construct explicit examples and to apply the algorithm for computing the Cox ring of a general hypersurface in a toric variety.

Fano polytopes database

- `FanoP(ID)`: Given an identifier number ID, this function retrieves the corresponding smooth Fano polytope from the database. The vertices are reconstructed from a compressed encoding.
- `FanoX(ID)`: This function constructs the toric variety associated to the polytope obtained with `FanoP`, using the fan generated by its vertices. The variety is defined over the finite field $\mathbb{F}_{32003}$.
- `FanoDualX(ID)`: Variant of `FanoX` that constructs the toric variety from the dual fan of the polytope.

Tools for hypersurfaces

- `GenHyp(ID)`: Constructs a general section of the anticanonical divisor of the toric variety `FanoX(ID)`, using random linear combinations of a Riemann–Roch basis.
- `IdeCodim2(ID)`: Returns a basis of the irrelevant ideal of the variety `FanoX(ID)` in codimension 2, which is useful as a localization set for applying the intersection algorithm.

- `IdeCodim2aux(ID)`: An extended version including all components of the irrelevant ideal of codimension at least 2.

Some functions

- `IsGen(f, lis, I, Q)`: Given a quotient f, a list of previously obtained generators `lis`, an ideal I, and a degree matrix Q, this function decides whether f is a new generator of the Cox ring, that is, whether it does not belong to the subring generated by the previous ones.

- `NewGen(lis, I, B, Q)`: Computes new rational generators from an initial list `lis`, an ideal `I`, a set of powers `B` of the generators of the irrelevant ideal, and the degree matrix Q.

- `IsCox(lis, I, B)`: Verifies whether a set of rational quotients `lis` completely generates the Cox ring of a hypersurface defined by the ideal `I`, with respect to the localizations determined by `B`.

These tools make it possible to generate examples systematically and to apply the algorithm developed in Chapter 5 to families of hypersurfaces in concrete toric varieties.

Bibliography

[1] Hamid Ahmadinezhad and Francesco Zucconi, *Mori dream spaces and birational rigidity of Fano 3-folds*, Adv. Math. **292** (2016), 410–445. MR3464026 ↑39

[2] Michela Artebani and Antonio Laface, *Hypersurfaces in Mori dream spaces*, J. Algebra **371** (2012), 26–37. MR2975386 ↑4, 37

[3] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015. MR3307753 ↑27, 28, 29, 30, 37, 42

[4] David A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50. MR1299003 ↑3

[5] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.MR2810322 ↑28

[6] Francesco Antonio Denisi, *Birational geometry of hypersurfaces in products of weighted projective spaces*, 2024. ↑39

[7] Andreas Gathmann, *Algebraic geometry*, Class Notes TU Kaiserslautern, available at agag-gathmann.math.rptu.de/class/alggeom-2021/alggeom-2021.pdf, 2021/22. ↑9

[8] Jürgen Hausen, *Cox rings and combinatorics. II*, Mosc. Math. J. **8** (2008), no. 4, 711–757, 847. MR2499353 ↑4, 37

[9] John Christian Ottem, *Birational geometry of hypersurfaces in products of projective spaces*, 2014. ↑39, 43, 47

[10] G. V. Ravindra and V. Srinivas, *The Grothendieck-Lefschetz theorem for normal projective varieties*, J. Algebraic Geom. **15** (2006), no. 3, 563–590. MR2219849 ↑38

[11] ———, *The Noether-Lefschetz theorem for the divisor class group*, J. Algebra **322** (2009), no. 9, 3373–3391. MR2567426 ↑38, 47

[12] Miles Reid, *Graded rings and birational geometry*, 2000. Accessed: 2024-11-08. ↑39

[13] The Stacks project authors, *The stacks project*, 2024. ↑30