Universidad de Concepción
Facultad de Ciencias Físicas y Matemáticas
Magíster en Matemática

Mori Cone of blow-ups of the plane at $s \geq 10$ points in very general position

Master's Thesis

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Date: January 2026

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Agradecimientos

Quiero agradecer al Profesor Antonio Laface y Profesor Luca Ugaglia [1] por sus aportes fundamentales a esta tesis. Esta investigación fue parcialmente financiada por el Proyecto Regular n. 1230287 de FONDECYT y por el “Piano strategico per il miglioramento della qualit‘a della ricerca e dei risultati della VQR 2020-2024 - Misura A” de la Universidad de Palermo. También a la Profesora Michela Artebani, cuyos comentarios hicieron legible este texto.

Hacer matemáticas es un compromiso diario, una disciplina que requiere mucha dedicación y yo no me considero una persona dedicada. Aprendí a tener paciencia conmigo misma y, una vez más, que las cosas toman tiempo, como dice Piet Hein en su poema *T.T.T.*

Agradezco a Dios por darme la salud y la paz necesaria para terminar este escrito, también a mi familia y a los amigos, nuevos y viejos, por su incondicional apoyo. Por último, quiero agradecer a todas las personas que pasaron por mi vida y dejaron un legado imborrable en mí, aún cuando ya han pasado los años. Esta tesis está dedicada a todos ellos.

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Introduction

A central theme in classical algebraic geometry is the study of plane linear systems with prescribed multiplicities at a finite set of points. In order to measure when these conditions behave as expected, one compares the actual dimension of a linear system with its *expected dimension*. Let X be a smooth projective surface and let D be a divisor on X. Its *expected dimension* is defined by

$$e(D) := \max\{-1,\ \chi(X, \mathcal{O}_X(D)) - 1\}.$$

We say that $|D|$ (or D) is *special* if $\dim |D| > e(D)$, and *non-special* otherwise. In this thesis we focus on the surfaces

$$X_s := \mathrm{Bl}_{p_1,\ldots,p_s} \mathbb{P}^2,$$

where $p_1, \ldots, p_s \in \mathbb{P}^2$ are points in *very general position* and $s \geq 10$. We denote by H the pull-back of the class of a line in $\mathbb{P}^2$ and by $E_1, \ldots, E_s$ the exceptional classes. Thus every divisor class can be written as

$$D = dH - \sum_{i=1}^{s} m_i E_i.$$

A divisor D on X_s is called *asymptotically special* if there exists an integer $n_0 \geq 1$ such that, for every $n \geq n_0$, the linear system $|nD|$ is special. In this thesis we prove the following equivalence with the (-1)-*curves conjecture* (see II.4.2).

Theorem 1. *For every $s \geq 10$, the following are equivalent:*

1. *the only irreducible curves $C \subset X_s$ with $C^2 < 0$ are (-1)-curves;*
2. *X_s does not contain nef and big asymptotically special divisors.*

A second key result shows that the *Segre–Harbourne–Gimigliano–Hirschowitz conjecture* (see II.4.3) can be reduced to a nefness criterion.

Theorem 2. *For every $s \geq 10$, the following are equivalent:*

1. *the Segre–Harbourne–Gimigliano–Hirschowitz conjecture for X_s;*

2. *every nef divisor class on X_s is non-special.*

To describe boundary phenomena, we consider the quadratic cones

$$Q(X_s) := \{ \alpha \in N^1(X_s)_{\mathbb{R}} \mid \alpha^2 \geq 0 \text{ and } \alpha \cdot H \geq 0 \},$$
$$Q_0(X_s) := \{ \alpha \in Q(X_s) \mid \alpha^2 = 0 \text{ and } \alpha \cdot K_s \geq 0 \},$$

where $K_s = -3H + \sum_{i=1} E_i$ is the canonical class divisor of X_s. Using (pseudo)standard classes and the action of the Weyl group $W(X_s)$, we obtain nefness results along the boundary of $Q(X_s)$, i.e. the classes in $Q(X_s)$ of self-intersection zero.

Theorem 3. *Let $s \geq 10$. Then:*

1. *every standard class in $Q_0(X_s)$ is nef;*
2. *every class in $E(X_s) \cap Q_0(X_s)$ is nef, where $E(X_s)$ denotes the region corresponding to (pseudo)standard classes.*

The case $s = 10$ is especially rigid and yields a global statement.

Corollary 4 (The case $s = 10$)**.** *Every class in $Q_0(X_{10})$ is nef. Consequently, every irrational class in $Q_0(X_{10})$ generates an irrational nef ray on the boundary of $Q(X_s)$.*

Inside the boundary $\partial Q(X_s)$ we distinguish two notable types of rays: *good rays*, typically rational but non-effective, and *wonderful rays*, generated by irrational nef classes on the boundary of $Q(X_s)$, originally defined in [11]. Both rays are extremal to the Mori Cone of X_s and the existence of wonderful rays prove that the Mori and nef cones are not rational polyhedral. The previous corollary produces a large explicit family of wonderful rays on X_{10}. For $s > 10$, the picture changes: there exist classes in $Q_0(X_s)$ which are not nef, and additional constructions are needed.

Finally, we adapt a collision/uncollision procedure that transports suitable classes from the case $s = 10$ to higher values of s and produces new irrational nef rays. In particular, the following statement provides a concrete construction on X_{13}.

Proposition 5. *Let $D \in Q_0(X_{10})$ be an irrational class with $D^2 = 0$. Then its transform* $\mathrm{Uncoll}(D) \in Q_+(X_{13})$ *generates a wonderful ray on X_{13}.*

Organization of the thesis

The thesis is divided into three chapters. In Chapter I we review the basic background: divisors, blow-ups, very general position, and the relevant cones (nef, pseudoeffective, and Mori). In Chapter II we establish the equivalences between conjectural statements, including Theorem 1 and the nef reduction of the Segre–Harbourne–Gimigliano–Hirschowitz conjecture (Theorem 2), and we introduce the Weyl group action and standard classes. Motivated in part by the aforementioned conjectures, in Chapter III we study the cones of X_s for $s \geq 10$, analyze the quadratic boundary (Theorem 3 and Corollary 4), and construct good and wonderful rays, including the uncollision method (Proposition 5). Finally, in the Appendix we have included the Magma codes used for some computer calculations in the sections II.7 and III.3.

Contents

I. Preliminaries

In the following, we will refer as *field* for an algebraically closed field. For a *ring* a commutative ring with unit.

1. Varieties

1.1. Affine varieties

Let k be a field. Let $\mathbb{A}^n$ be the affine $n-$space over k and $R = k[x_1, \dots, x_n]$ the polynomial ring over k in n variables. If $f \in k[x_1, \dots, x_n]$ and $P = (a_1, \dots, a_n) \in \mathbb{A}^n$, denote $f(P) = f(a_1, \dots, a_n)$, the set of zeros of f by $Z(f) = \{P \in \mathbb{A}^n : f(P) = 0\}$ and for a subset $T \subseteq R$, the zero set of T

$$Z(T) = \{P \in \mathbb{A}^n : f(P) = 0 \text{ for all } f \in T\}.$$

Since $k[x_1, \dots, x_n]$ is a noetherian ring, any ideal $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$ has a finite set of generators $f_1, \dots, f_r$, thus $Z(\mathfrak{a})$ can be expressed as the zero set of the finite set of polynomials $f_1, \dots, f_r$. For any subset $Y \subseteq \mathbb{A}^n$, define the **ideal** of Y in $k[x_1, \dots, x_n]$ by

$$I(Y) = \{f \in R : f(P) = 0 \text{ for all } P \in Y\}.$$

Definition 1.1.1.

- A subset Y of $\mathbb{A}^n$ is an **algebraic set** if there exists a subset $T \subseteq k[x_1, \dots, x_n]$ such that $Y = Z(T)$.
- The **Zariski topology** on $\mathbb{A}^n$ takes the open subsets to be the complements of algebraic sets,
- A nonempty subset Y of a topological space X is **irreducible** if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y. The empty set is not considered to be irreducible.
- An **affine variety** is an irreducible closed subset of $\mathbb{A}^n$ (with the induced topology).

The relation and properties between the functions $Z(\cdot)$ and $I(\cdot)$ are summarized in [25, I, Prop. 1.2]. In particular, there is an inclusion-reversing bijection

$$\begin{aligned} \{\text{affine varieties in } \mathbb{A}^n\} &\overset{1:1}{\longleftrightarrow} \{\text{radical ideals in } R\} \\ Y &\longmapsto I(Y) \\ Z(J) &\longleftarrow J \end{aligned}$$

Definition 1.1.2. If $Y \subseteq \mathbb{A}^n$ is an affine variety, we define the **affine coordinate ring** $A(Y) = k[x_1, \ldots, x_n]/I(Y)$.

From the earlier bijection, we have the following correspondence

$$\{\text{affine varieties in } \mathbb{A}^n\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{finitely generated } k\text{-algebras} \\ \text{which are an integral domain in } R \end{array} \right\}$$

where the direct map is $Y \mapsto A(Y)$. Conversely, any finitely generated k-algebra B which is a domain is the affine coordinate ring of some variety, i.e. $B = k[x_1, \ldots, x_n]/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of $k[x_1, \ldots, x_n]$. Then, the reversing map is $B \to Z(\mathfrak{a})$.

1.2. Projective varieties

Let k be a fixed algebraically closed field and $\mathbb{P}^n_k$ be the projective $n-$space over k, defined as the quotient of the set $\mathbb{A}^n - \{(0, \ldots, 0)\}$ under the equivalence relation which identifies points lying on the same line through the origin.

Consider the graded polynomial ring $R = k[x_0, \ldots, x_n]$ with S_d the set of all linear combinations of monomials of total degree d in $x_0, \ldots, x_n$. If $f \in S$ is a polynomial, we cannot use it to define a function on $\mathbb{P}^n$, because of the non-uniqueness of the homogeneous coordinates. However, if f is a homogeneous polynomial of degree d, then the property of f being zero or not depends only on the equivalence class of $(a_0, \ldots, a_n)$. Thus f gives a function from $\mathbb{P}^n$ to $\{0, 1\}$ by $f(P) = 0$ if $f(a_0, \ldots, a_n) = 0$, and $f(P) = 1$ if $f(a_0, \ldots, a_n) \neq 0$.

Similar to the case of affine varieties, if f is a homogeneous polynomial, the zeros of f is the set

$$Z(f) = \{P \in \mathbb{P}^n | f(P) = 0\}.$$

If T is any set of homogeneous elements of S, we define the *zero set of* T to be

$$Z(T) = \{P \in \mathbb{P}^n | f(P) = 0 \text{ for all } f \in T\}.$$

If $\mathfrak{a}$ is a homogeneous ideal of S, we define $Z(\mathfrak{a}) = Z(T)$, where T is the set of all homogeneous elements in $\mathfrak{a}$. Since S is a noetherian ring, any set of homogeneous elements T has a finite subset $f_1, \ldots, f_r$ such that $Z(T) = Z(f_1, \ldots, f_r)$.

Definition 1.2.1.

- A subset Y of $\mathbb{P}^n$ is an **algebraic set** if there exists a set T of homogeneous elements of S such that $Y = Z(T)$.
- We define the **Zariski topology** on $\mathbb{P}^n$ by taking the open sets to be the complements of algebraic sets.
- A **projective variety** is an irreducible algebraic set in $\mathbb{P}^n$, with the induced topology. An open subset of a projective variety is a **quasi-projective variety**.
- The **dimension** of a projective or quasi-projective variety is its dimension as a topological space.

Definition 1.2.2. If Y is any subset of $\mathbb{P}^n$, we define the **homogeneous ideal** of Y in S, denoted $I(Y)$, to be the set

$$I(Y) = \{f \in S | f \text{ is homogeneous and } f(P) = 0 \text{ for all } P \in Y\}.$$

If Y is an algebraic set, we define the **homogeneous coordinate ring** of Y to be $S(Y) = S/I(Y)$.

2. Sheaves and Schemes

The purpose of this section is to define sheaves, some important coherent sheaves and their associated cohomology in Section 3. For this, we will assume the definitions of some algebraic structures, such as rings, groups and modules.

2.1. Definition and categorical aspects

First, we introduce some definitions of category theory that will be important in this section and for Section 3. For a deeper understanding of the topic, we suggest you to read [26, Chapter 2] and [25, Chapters II, III].

Definition 2.1.1 (Abelian category)**.** A category $\mathscr{C}$ with a zero object 0 is called **abelian** if:

- for each A, B objects of $\mathscr{C}$, $\operatorname{Hom}(A, B)$ has a structure of an abelian group and the direct product $A \times B$ and the direct sum $A \oplus B$ exists;
- every morphism has a kernel and a cokernel;
- every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and
- every morphism can be factored into an epimorphism followed by a monomorphism.

Definition 2.1.2 (Exact sequence)**.** In an abelian category $\mathscr{C}$ a sequence of morphism

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called **exact** if the equality $\operatorname{Im} f = \operatorname{Ker} g$ holds.

Definition 2.1.3 (Exact functor)**.** Let $F : \mathscr{C} \to \mathscr{C}'$ a functor between two abelian categories and $0 \to A \to B \to C \to 0$ an exact sequence in $\mathscr{C}$. F is called an **exact functor** if the induced sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is again an exact sequence. If $0 \to F(A) \to F(B) \to F(C)$ is an exact sequence, we called F a **left exact functor** and if $F(A) \to F(B) \to F(C) \to 0$ is exact, we called F a **right exact functor**.

Definition 2.1.4 (Presheaf)**.** Let X be a topological space. A **presheaf** $\mathcal{F}$ (of rings) on X consists of the data:

- for every open set $U \subseteq X$ a ring $\mathcal{F}(U)$, with $\mathcal{F}(\emptyset) = \{0\}$,

- for every inclusion $U \subseteq V$ of open sets in X a ring homomorphism

$$\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U),$$

  called *restriction map*, such that
  - for all U, $\rho_{U,U}$ is the identity map on $\mathcal{F}(U)$,
  - for any inclusion $U \subseteq V \subseteq W$ of open sets in X, $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$.

The elements of $\mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over U, and for $s \in \mathcal{F}(U)$ the restriction $\rho_{V,U}(s)$ is often denoted $s|_U$ by analogy with restriction of functions.
Sometimes, we will use the notation $\Gamma(U, \mathcal{F})$ for the ring $\mathcal{F}(U)$.

Remark 2.1.5 (Presheaves for other categories)**.** Let $\mathscr{C}$ be an abelian category on a topological space X. A presheaf of objects in $\mathscr{C}$ on X, or $\mathscr{C}-$**presheaf**, is a contravariant functor $\mathcal{F} : Ouv_X \to \mathscr{C}$ such that $\mathcal{F}(\emptyset) = \{0\}$, where Ouv_X is the poset of open sets in X.

If $\mathscr{C}$ is the category of rings, we have the Definition 2.1.4. In the same way one can define presheaves of sets, abelian groups, $R-$modules (given a ring R) or other suitable categories, by requiring that all $\mathcal{F}(U)$ are objects and all restriction maps are morphisms in the corresponding category. Denote by Ab_X, Rings_X and Mod_R the categories of abelian groups, rings and $R-$modules on X. From now on, $\mathscr{C}$ is always one of these categories.

Definition 2.1.6 (Sheaf)**.** A $\mathscr{C}$-presheaf $\mathcal{F}$ is called $\mathscr{C}$-**sheaf** if it satisfies the ***sheaf axiom***: Let $\{U_i\}_{i\in I}$ be an arbitrary open cover of $U \subseteq X$ and $\{\varphi_i \in \mathcal{F}(U_i)\}_{i\in I}$ be a family of sections. If all pairs of sections agree on the overlap of their domains, that is, $\varphi_i|_{U_i\cap U_j} = \varphi_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there is a unique $\varphi \in \mathcal{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all i.

Example 2.1.7. A sheaf is a presheaf whose sections are, in a technical sense, uniquely determined by their restrictions.

1. Given a topological space X, the $\mathbb{R}$-modules

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} \text{ continuous}\}$$

   for open subsets $U \subseteq X$ and the restrictions maps $\rho_{V,U}$ given by restricting the domain of a function from U to V, form a sheaf $\mathcal{F}$.

2. Let $X = \mathbb{R}^n$ with the standard topology, the rings

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} \text{ constant function}\}$$

with the usual restriction maps form a presheaf, but not a sheaf. To prove this, note that if U_1 and U_2 are non-empty disjoint open subsets of X, and $f_1 \in \mathcal{F}(U_1)$ and $f_2 \in \mathcal{F}(U_2)$ are two constant functions with different values, then f_1 and f_2 trivially agree on $U_1 \cap U_2 = \emptyset$, but there is still no constant function on $U = U_1 \cup U_2$ that restricts to both f_1 on U_1 and f_2 on U_2. Hence $\mathcal{F}$ does not satisfy the sheaf axiom.

Definition 2.1.8 (Stalk and germs)**.** Let $\mathcal{F}$ a $\mathscr{C}$-presheaf on a topological space X and $p \in X$. The **stalk** of $\mathcal{F}$ at P is the direct limit over all open subsets of X containing the given point P

$$\mathcal{F}_P := \varinjlim_{P \in U} \mathcal{F}(U).$$

In other words, if $\sim$ is the equivalence relation defined as: $(U, s) \sim (U', s')$ if there is a open subset V such that $P \in V \subseteq U \cap U'$ and $s|_V = s'|_V$ then

$$\mathcal{F}_P = \{(U, s) : U \subseteq X \text{ open with } P \in U,\ s \in \mathcal{F}(U)\}/\sim .$$

The elements of $\mathcal{F}_P$ are called **germs** of $\mathcal{F}$ at P.

For every open subset $U \subseteq X$ containing P, there is a natural morphism $\mathcal{F}(U) \to \mathcal{F}_P$ that takes a section $s \in \mathcal{F}(U)$ to its germ s_P. The **support** of s is the set $\mathrm{supp}(s) = \{P \in U : s_P \neq 0\}$, where 0 is the zero of the abelian group or the $R-$module $\mathcal{F}_P$.

The **support of a sheaf** $\mathcal{F}$ is $\mathrm{Supp}(\mathcal{F}) = \{P \in U : \mathcal{F}_P \neq 0\}$.

Definition 2.1.9. A **morphism** $\Phi : \mathcal{F} \to \mathcal{G}$ of $\mathscr{C}$-(pre)sheaves on X is given by the data of maps $\Phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ (a morphism in $\mathscr{C}$) for all open subsets $U \subseteq X$ that are compatible with restrictions, i.e. for all $V \subseteq U$ the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\Phi_U} & \mathcal{G}(U) \\ {\scriptstyle \rho_{U,V}}\downarrow & & \downarrow{\scriptstyle \rho'_{U,V}} \\ \mathcal{F}(V) & \xrightarrow[\Phi_V]{} & \mathcal{G}(V) \end{array}$$

commutes. An **isomorphism** is a morphism Φ that has a two-sided inverse.

Remark 2.1.10. By [25, II, Prop. 1.1.], Φ is called an **isomorphism** if and only if Φ induces a isomorphism of stalks $\Phi_P : \mathcal{F}_P \to \mathcal{G}_P$ for all $p \in X$. If two sheaves have isomorphic stalks, they are not always isomorphic.

To each presheaf $\mathcal{F}$ in X can be associated a sheaf $\overline{\mathcal{F}}$, unique up to isomorphisms, called *sheafification* of $\mathcal{F}$.

Definition 2.1.11 (Sheafification)**.** Let $\mathcal{F}$ be a $\mathscr{C}$-presheaf on X. For every open subset $U \subseteq X$ we set

$$\begin{aligned}\overline{\mathcal{F}}(U) :=& \{s = \{s(P)\}_{P\in U} \in \prod_{P\in U} \mathcal{F}_P : \text{ for every } P \in U \text{ there exist an open} \\ & \text{neighborhood } V \text{ of } P \text{ and } t \in \mathcal{F}(V) \text{ satisfying } s(P) = t_Q \ \forall Q \in V\}.\end{aligned}$$

For open subsets $V \subseteq U$ let $\rho_{U,V} : \overline{\mathcal{F}}(U) \to \overline{\mathcal{F}}(V)$ be the canonical map induced from the projection $\prod_{P\in U} \mathcal{F}_P \to \prod_{P\in V} \mathcal{F}_P$. Thus, $\overline{\mathcal{F}}$ is a sheaf unique up to isomorphisms and there is a natural morphism $\theta : \mathcal{F} \to \overline{\mathcal{F}}$ of presheaves given by

$$\mathcal{F}(U) \to \overline{\mathcal{F}}(U), \ s \mapsto \{s(P)\}_{P\in U}.$$

Remark 2.1.12. For another equivalent definition of $\overline{\mathcal{F}}$ in terms of functions instead of sequences, see [25, II, Prop. 1.2].

Remark 2.1.13. The inclusion of categories $i : \mathit{Sheaves}(X) \to \mathit{Presheaves}(X)$ is a left exact functor. The sheafification functor is *left adjoint* to the inclusion i, that is, for every $\mathcal{F}$ presheaf on X and $\mathcal{G}$ sheaf on X $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\overline{\mathcal{F}}, \mathcal{G})$ [40, Def. 2.3.9].

Definition 2.1.14 (Subsheaf and quotient sheaf)**.** Let $\mathcal{F}$ be an Ab_X-sheaf on X and $\mathcal{G}$ be a $\mathscr{C}$-sheaf on X. $\mathcal{F}$ is called a **subsheaf** of $\mathcal{G}$ if there is a morphism $\Phi : \mathcal{F} \to \mathcal{G}$ of Ab_X-sheaves such that, for every open subset $U \subseteq X$,

$$\Phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

is the inclusion map of a subgroup.
If $\mathcal{F}$ is a subsheaf of $\mathcal{G}$, one can define an Ab_X-presheaf $\mathcal{H}$ with $\mathcal{H}(U) = \mathcal{G}(U)/\mathcal{F}(U)$ for every open subset $U \subseteq X$ and, for $V \subseteq U$, the homomorphism of groups $\mathcal{H}(U) \to \mathcal{H}(V)$, induced from $\mathcal{G}(U) \to \mathcal{G}(V)$. The sheafification of $\mathcal{H}$, denoted by $\mathcal{G}/\mathcal{F}$, is called the **quotient sheaf** of $\mathcal{G}$ by $\mathcal{F}$.

Definition 2.1.15 (Kernel and image subsheaf)**.** Let $\Phi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathscr{C}-$sheaves. For a open subset $U \subseteq X$, define

$$\begin{aligned}\mathcal{K}er\ \Phi(U) &:= \{s \in \mathcal{F}(U) : \Phi(s)_P = 0, \ \forall P \in U\} \\ \mathcal{I}m\ \Phi(U) &:= \{t \in \mathcal{G}(U) : \text{ for every } P \in U \text{ there are } V_P \subseteq U \text{ and } s \in \mathcal{F}(V_P) \\ & \qquad \text{such that } P \in V_P \text{ and } t_P = \Phi_P(s_P)\}.\end{aligned}$$

For any open subset $V \subseteq U$, respectively, the restriction maps are defined as the canonical map induced from the restriction maps of sheaves $\mathcal{F}$ and $\mathcal{G}$. Then, both $\mathcal{K}er\ \Phi$ and $\mathcal{I}m\ \Phi$ are subsheaves, called **kernel** of Φ and **image** of Φ, respectively. The **cokernel** of Φ is the quotient sheaf $\mathcal{G}/\operatorname{Im}_\Phi$, denoted $\mathcal{C}oker\,\Phi$.

Definition 2.1.16. A morphism of sheaves $\Phi : \mathcal{F} \to \mathcal{G}$ is **injective** if $\mathcal{K}er\ \Phi = 0$, or equivalently, if the induced map $\Phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open set U of X, and Φ is **surjective** if $\mathcal{I}m\ \Phi = \mathcal{G}$.

Remark 2.1.17. If $\Phi : \mathcal{F} \to \mathcal{G}$ is **surjective**, the maps $\Phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ on sections need to be surjective. Nevertheless, we can say that Φ is surjective if only if the maps $\Phi_P : \mathcal{F}_P \to \mathcal{G}_P$ on stalks are surjective for each P.

If $\mathscr{C}$ is the category Ab_X, Rings_X or Mod_R, the category of $\mathscr{C}-$sheaves on X is abelian category. Here, if $\mathscr{C} = \mathrm{Mod}_R$, the kernel, image and cokernel are also $\mathscr{C}-$sheaves.

Proposition 2.1.18 ([26, Prop. 2.2.9.])**.** For morphisms $\Phi : \mathcal{F} \to \mathcal{G}$, $\Psi : \mathcal{G} \to \mathcal{H}$ of Ab_X-sheaves, the following are equivalent:

1. $0 \longrightarrow \mathcal{F} \xrightarrow{\Phi} \mathcal{G} \xrightarrow{\Psi} \mathcal{H} \longrightarrow 0$ is an exact sequence.
2. For every $p \in X$ the sequence $0 \longrightarrow \mathcal{F}_P \xrightarrow{\Phi_P} \mathcal{G}_P \xrightarrow{\Psi_P} \mathcal{H}_P \longrightarrow 0$ is a exact sequence of abelian groups.

Definition 2.1.19 (Direct and inverse image sheaves)**.** Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, we define the **direct image** sheaf $f_*\mathcal{F}$ on Y by

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

for any open set $V \subseteq Y$. For any sheaf $\mathcal{G}$ on Y, we define the **inverse image** sheaf $f^{-1}\mathcal{G}$ on X to be the sheaf associated to the presheaf

$$U \mapsto \lim_{V \supseteq f(U)} \mathcal{G}(V)$$

where U is any open set in X, and the limit is taken over all open sets V of Y containing $f(U)$.

Definition 2.1.20 (Direct image functor)**.** Let $f : X \to Y$ be a continuous map of topological spaces, and let $\mathfrak{Ab}(\cdot)$ denote the category of sheaves of abelian groups on a topological space. The **direct image functor**

$$f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$$

sends a sheaf $\mathcal{F}$ on X to its direct image sheaf $f_*\mathcal{F}$ on Y. Since a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on X defines a morphism of sheaves $f_*(\varphi) : f_*(\mathcal{F}) \to f_*(\mathcal{G})$ on Y we indeed have that f_* is a functor.

Example 2.1.21. If Y is a point, and $f : X \to Y$ the unique continuous map, then $\mathfrak{Ab}(Y)$ is the category Ab_Y, and the direct image functor $f_* : \mathfrak{Ab}(X) \to \mathrm{Ab}_Y$ equals the **global sections functor**.

2.2. Ringed spaces

Definition 2.2.1. A **ringed space** $(X, \mathcal{O}_X)$ is a topological space X together with a sheaf of rings $\mathcal{O}_X$ on X. The sheaf $\mathcal{O}_X$ is called the **structure sheaf** of X.

Definition 2.2.2. A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that all stalks of $\mathcal{O}_X$ are local rings (i.e. they have unique maximal ideals).

Let k be a field. An arbitrary topological space X can be considered a locally ringed space by taking $\mathcal{O}_X$ to be the sheaf of $k-$valued continuous functions on open subsets of X. The stalk at a point $P \in X$ is a local ring with the unique maximal ideal consisting of those germs whose value at P is 0.

If X is an algebraic variety with the Zariski topology, we can define a locally ringed space by taking $\mathcal{F}(U)$ to be the ring of regular functions on U; in the case of affine varieties, see [21, Lemma 3.19]. More in general, the spectrum of any commutative ring R and schemes are likewise locally ringed spaces.

Definition 2.2.3. Let $f : X \to Y$ be a continuous map between topological space, and denoted by $f_*\mathcal{O}_X$ the direct image sheaf of $\mathcal{O}_X$ via f (2.1.19). A pair $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called **morphism of ringed spaces** if f is a continuous map and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings on Y.

Given a point $P \in X$, the morphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a homomorphism of stalks $f^\#_P : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$. We prove it as follows: first, $f^\#$ induces the morphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ for every open set V in Y. Then as V ranges over all open neighborhoods of $f(P)$, $f^{-1}(V)$ ranges over a subset of the neighborhoods of P, then, taking direct limits, we obtain a map

$$\mathcal{O}_{Y,f(P)} = \varinjlim_{f(P)\in V} \mathcal{O}_Y(V) \to \varinjlim_{f(P)\in V} \mathcal{O}(f^{-1}(V))$$

where the latter limit maps to the stalk $\mathcal{O}_{X,P}$.

Definition 2.2.4. A **morphism of locally ringed spaces** is a morphism $(f, f^\#)$ of ringed spaces such that, for each $P \in X$, the induced map of local rings $f^\#_P : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism of local rings, that is $(f^\#_P)^{-1}(I_P) = I_{f(P)}$, where I_P and $I_{f(P)}$ denote the maximal ideals in the local rings $\mathcal{O}_{X,P}$ and $\mathcal{O}_{Y,f(P)}$, respectively.

Definition 2.2.5 (Sheaf of $\mathcal{O}_X-$modules)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf $\mathcal{F}$ of abelian groups on X is called **sheaf of $\mathcal{O}_X$-modules** on X (or just $\mathcal{O}_X$-module) if

- for every open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and

- the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structure via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

A **morphism** of $\mathcal{O}_X-$modules is morphism of sheaves such that the map $\Phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)-$modules homomorphism for every open $U \subseteq X$ (see 2.1.9). A sequence of $\mathcal{O}_X$-modules is **exact** if it is exact as sequence of sheaves of abelian groups.

Remark 2.2.6. By Definition 2.2.5 we can define the category of $\mathcal{O}_X$-modules, which is abelian. We denote this category by $\mathrm{Mod}_{\mathcal{O}_X}$.

Definition 2.2.7. Let $\mathcal{F}, \mathcal{G}$ be two sheaves of $\mathcal{O}_X$-modules. We denote by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ the group of morphisms $\mathcal{F} \to \mathcal{G}$. If $U \subseteq X$ is a open subset, then $\mathcal{F}|_U$ is an sheaf of $\mathcal{O}_X|U$-modules. The functor

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U), \quad U \subseteq X \text{ open subset}$$

defines a sheaf of $\mathcal{O}_X$-modules called **sheaf Hom**, and denoted by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. We call $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ the **dual** of the $\mathcal{O}_X$-module $\mathcal{F}$ and denote it by $\mathcal{F}^\vee$.

Definition 2.2.8. We define the **tensor product** $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of two $\mathcal{O}_X$-modules to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. We will often write simply $\mathcal{F} \otimes \mathcal{G}$ when $\mathcal{O}_X$ is understood.

Definition 2.2.9 (Free and locally free sheaves)**.** A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is **free** if it is isomorphic to a finite direct sum of copies of $\mathcal{O}_X$. It is **locally free** if X can be covered by open subsets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. In the case, the **rank** of $\mathcal{F}$ is the number of free copies of the structure sheaf needed. The rank of a locally free sheaf is the same everywhere when X is connected. A locally free sheaf of rank 1 is called an **invertible sheaf**.

By [25, II, Prop. 6.12], if $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a ringed space X, so is $\mathcal{L} \otimes \mathcal{M}$. If $\mathcal{L}$ is any invertible sheaf on X, then there exists an invertible sheaf $\mathcal{L}^{-1}$ on X such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$. This gives the next definition.

Definition 2.2.10. For any ringed space X, we define the **Picard group** of X, $\mathrm{Pic}\, X$, to be the group of isomorphism classes of invertible sheaves on X, under the operation $\otimes$.

Definition 2.2.11. A **sheaf of ideals** on X is a sheaf of modules $\mathcal{F}$ which is a subsheaf of $\mathcal{O}_X$, that is, for every open set U, $\mathcal{F}(U)$ is an ideal in $\mathcal{O}_X(U)$.

Definition 2.2.12. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces (see 2.2.3) and $\mathcal{F}$ be an $\mathcal{O}_X$-module. The sheaf $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module and with a natural structure of $\mathcal{O}_Y$-module, given by $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$. $f_*\mathcal{F}$ is called the **direct image** of $\mathcal{F}$ by the morphism f.

Now let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module (see the Definition 2.1.19). Consider the morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on X (which is unique thanks to the adjoint property of f^{-1}). The tensor product

$$f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

is a $\mathcal{O}_X$-module, called the **inverse image** of $\mathcal{G}$ by the morphism f.

2.3. Schemes

A scheme is a locally ringed space that has an open cover by affine schemes and the morphism of schemes are just morphism as locally ringed spaces (see 2.3.4 and 2.2.4).

The affine and projective varieties, after a slight modification, can be regarded as schemes. In this subsection we show this in a very brief way and we stablish some properties of schemes that we will be using in the following.

Definition 2.3.1. Let R be a ring. We define the space $\operatorname{Spec} R$ to be the set of all prime ideals of R. For any ideal $\mathfrak{a}$ of R, define

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{a} \subseteq \mathfrak{p}\}.$$

By the [25, II, Lemma 2.1], we define a topology on $\operatorname{Spec} R$ taking the subsets of the form $V(\mathfrak{a})$ to be the closed subsets.

Notation 2.3.2. Let R be a ring.

- For each prime ideal $\mathfrak{p} \subseteq R$, let $R_\mathfrak{p}$ be the localization of R at $\mathfrak{p}$.
- For any $f \in R$, denote by (f) the ideal of R generated by f and $D(f)$ the open complement of $V((f))$.

Note that open sets $D(f)$ form a base for the topology of $\operatorname{Spec} R$. Now, we define a sheaf of rings $\mathcal{O}_{\operatorname{Spec} R}$ on $\operatorname{Spec} R$, which will result, by [25, II, Prop. 2.2. (a)], in $(\operatorname{Spec} R,\ \mathcal{O}_{\operatorname{Spec} R})$ being a locally ringed space, called **spectrum** of R.

Definition 2.3.3 (Structure sheaf on $\operatorname{Spec} R$)**.** For any open set $U \subseteq \operatorname{Spec} R$, define $\mathcal{O}_{\operatorname{Spec} R}(U)$ to be the set of functions

$$s : U \to \coprod_{\mathfrak{p} \in U} R_\mathfrak{p}$$

such that for each $\mathfrak{p}$, $s(\mathfrak{p}) \in R_{\mathfrak{p}}$, and for each $\mathfrak{p} \in U$, there exists a neighborhood V of $\mathfrak{p} \in U$ and $g, f \in R$ such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = \dfrac{g}{f} \in R_{\mathfrak{q}}$.

Definition 2.3.4. An **affine scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to the spectrum of some ring. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_X|_U$, is an affine scheme. A **morphism of schemes** is a morphism as locally ringed spaces. An **isomorphism** is a morphism with a two-sided inverse.

Notation 2.3.5. By abuse of notation we will often write simply X for the scheme $(X, \mathcal{O}_X)$.

Remark 2.3.6. Note that not all points of an affine scheme are closed. Given a ring R, the closure of a point $\mathfrak{p} \in \operatorname{Spec} R$ is

$$\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R : \mathfrak{q} \supseteq \mathfrak{p}\},$$

then $\mathfrak{p}$ is closed if and only if $\mathfrak{p}$ is a maximal ideal of R.

Example 2.3.7. Let k be a field, and consider the **affine plane** over k defined as $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$. The set of all closed points of $\mathbb{A}^2_k$, with induced topology, is homeomorphic to the affine variety $\mathbb{A}^2$. In addition to the closed points, the zero ideal of $k[x, y]$ corresponds to the **generic point** ξ, whose closure is the whole space. Also, for each irreducible polynomial $f(x, y)$, there is a point η (called **generic point** of the curve $f(x, y) = 0$) whose closure consist of itself together with all closed points (a, b) for which $f(a, b) = 0$.

Let S be a graded ring, i.e. $S = \bigoplus_{d \geq 0} S_d$ with S_d additive groups, such that $S_m S_n \subseteq S_{m+n}$ for all $m, n \geq 0$. In this section, we give the construction of the set Proj of S, which is analogous to the *spectrum of a ring* construction of affine schemes and produces objects with the typical properties of projective varieties. We endow this space with a topology and a structure sheaf which makes it a classic example of locally ringed space.

Definition 2.3.8. Let S be a graded ring. A homogeneous ideal is an ideal $I \subset S$ generated by homogeneous elements. The **irrelevant ideal** of S is the ideal of elements of positive degree

$$S_+ = \bigoplus_{i > 0} S_i.$$

The **Proj** of S is the set

$$\operatorname{Proj} S = \{\mathfrak{p} \subseteq S \text{ homogeneous prime ideal }, S_+ \not\subseteq \mathfrak{p}\}.$$

Notation 2.3.9. Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring. Let $f \in S_d$ with $d \geq 1$.

- Denote by $S_{(f)}$ the subring of S_d consisting of of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$.
- If M is a $S-$module, denoted by $M_{(f)}$ the sub module of M_f consisting of elements of the form x/f^n with x homogeneous of degree nd, which is a $S_{(f)}-$module.
- For $f \in S$ homogeneous of degree > 0 define

$$D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S : f \not\in \mathfrak{p}\}.$$

- For a homogeneous ideal $I \subset S$ define

$$V(I) = \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{a} \subseteq \mathfrak{p}\}.$$

From the Definition 2.3.8, $\operatorname{Proj} S$ is a subset of $\operatorname{Spec} S$. We endow $\operatorname{Proj} S$ with the induced topology (see 2.3.1) where the subsets $D_+(f)$, with f homogeneous and $\deg(f) > 0$, are open in $\operatorname{Proj} S$, whereas the closed subsets are those of the form of $V(I)$ with I a homogeneous ideal $I \subset S$.

Analogously to the definition spectrum of a ring, we define a sheaf of rings $\mathcal{O}_{\operatorname{Proj} S}$ on $\operatorname{Proj} S$, which will result, by [25, II, Prop. 2.5. (a)], in $(\operatorname{Proj} S,\ \mathcal{O}_{\operatorname{Proj} S})$ being a locally ringed space, called **homogeneous spectrum** of R, (see 2.3.3).

Definition 2.3.10 (Structure sheaf on $\operatorname{Proj} S$)**.** For any open set $U \subseteq \operatorname{Proj} S$, define $\mathcal{O}_{\operatorname{Proj} S}(U)$ to be the set of functions

$$s : U \to \coprod_{\mathfrak{p}\in U} S_{(\mathfrak{p})}$$

such that for each $\mathfrak{p}$, $s(\mathfrak{p}) \in S_{\mathfrak{p}}$, and for each $\mathfrak{p} \in U$, there exists a neighborhood V of $\mathfrak{p} \in U$ and homogeneous $g, f \in S$ with $\deg(g) = \deg(f)$ such that for all $\mathfrak{q} \in V, f \not\in \mathfrak{q}$ and $s(\mathfrak{q}) = \dfrac{a}{f} \in S_{(\mathfrak{p})}$. Then $\mathcal{O}_{\operatorname{Proj} S}$ is a sheaf with the natural restrictions and from the local nature of its definition.

From [25, II, Prop. 2.5 (b)], $\operatorname{Proj} S$ has a covering by affine patches of the form $D_+(f)$ with f is some homogeneous element of positive degree of S. Then, for each such open set, there is an isomorphism of locally ringed spaces

$$(D_+(f), \mathcal{O}|_{D,(f)}) \simeq \operatorname{Spec} S_{(f)}$$

where $S_{(f)}$ is the sub-ring of elements of degree 0 in the localized ring S_f. By [25, II, Prop. 2.5 (a)] $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a locally ringed space and, since it can be covered by open affine schemes, then it is a scheme.

Definition 2.3.11 (Projective $n-$space over a ring)**.** Let A be a ring. The scheme

$$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$$

is called the **projective** $n-$**space** over A. If $A = k$, with k an field, the $\mathbb{P}^n_k$ is a scheme whose subspace of closed points is naturally homeomorphic to the projective $n-$space.

Definition 2.3.12 (Projective $n-$space over a scheme)**.** If $A \to B$ is a homomorphism of rings and $\operatorname{Spec} B \to \operatorname{Spec} A$ is the corresponding morphism of affine schemes then

$$\mathbb{P}^n_B \cong \mathbb{P}^n_A \times_{\operatorname{Spec} A} \operatorname{Spec} B.$$

For any scheme Y, we define the **projective** n**-space** over Y as $\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} Y$. The **twisting sheaf** $\mathcal{O}_{\mathbb{P}^n_Y}(1)$ on $\mathbb{P}^n_Y$ is $g^*(\mathcal{O}(1))$ where $g : \mathbb{P}^n_Y \to \mathbb{P}^n_{\mathbb{Z}}$ is the natural map.

Notation 2.3.13. A closed subscheme X of $\mathbb{P}^n$ together with the inclusion map $i : X \to \mathbb{P}^n$ is called a **projective scheme**. By [25, II, Prop. 2.6], every projective variety is a projective scheme.

Definition 2.3.14. A scheme X is **reduced** if for every open set U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements. Equivalently (Ex. 2.3), X is reduced if and only if the local rings $\mathcal{O}_p$, for all $p \in X$, have no nilpotent elements.

Definition 2.3.15. A scheme X is **integral** if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Definition 2.3.16. A scheme is **normal** if all of its local rings are integrally closed domains.

In the case of a surface X, intuitively, this means that X has “minimal singularities” in a certain sense. For example, a normal surface doesn't have “cusps” or “self-intersections” like some singular surfaces.

Remark 2.3.17.

(i) A scheme is integral if and only if it is both reduced and irreducible [25, II, Prop. 3.1].

(ii) Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\tilde{A}$ be the integral closure of A in its quotient field, and let $\tilde{U} = \operatorname{Spec} \tilde{A}$. Gluing the schemes $\tilde{U}$ one can obtain a normal integral scheme $\tilde{X}$, called the **normalization** of X [25, II, Exercise 3.8].

(iii) For a projective variety $Y \subseteq \mathbb{P}^n$ with homogeneous coordinate ring $S(Y) = k[x_1, \ldots, x_n]/P$, Y is called **projectively normal** if $S(Y)$ is an integrally closed domain.

Definition 2.3.18 (Noetherian scheme)**.** A scheme X is **locally noetherian** if it can be covered by open affine subsets $\operatorname{Spec} A_i$, where each A_i is a noetherian ring. X is **noetherian** if it is locally noetherian and quasi-compact. Equivalently, X is noetherian if it can be covered by a finite number of open affine subsets $\operatorname{Spec} A_i$, with each A_i a noetherian ring.

Proposition 2.3.19. [25, II, Prop. 3.2] A scheme X is locally noetherian if and only if for every open affine subset $U = \operatorname{Spec} A$, A is a noetherian ring. In particular, an affine scheme $X = \operatorname{Spec} A$ is a noetherian scheme if and only if the ring A is a noetherian ring.

Definition 2.3.20 (Separated morphism and scheme)**.** Let $f : X \to Y$ be a morphism of schemes. The **diagonal morphism** is the unique morphism $\Delta : X \to X \times_Y X$ whose composition with both projection maps $p_1, p_2 : X \times_Y X \to X$ is the identity map of $X \to X$.
We say that the morphism f is **separated** if the diagonal morphism Δ is a closed immersion. In that case we also say X is **separated over** Y. A scheme X is **separated** if it is separated over $\operatorname{Spec} \mathbb{Z}$.

Definition 2.3.21. Let X be an arbitrary variety over an algebraically closed field k. X is **nonsingular** (or **smooth**) at a point $P \in X$ if the local ring $\mathcal{O}_{X,P}$ is a regular local ring. X is nonsingular or smooth if it is nonsingular at every point.

Definition 2.3.22. Let X be a scheme, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is **generated by global sections** if there is a family of global sections $\{s_i\}_{i \in I}, s_i \in \Gamma(X, \mathcal{F})$, such that for each $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$ generate that stalk as an $\mathcal{O}_X$-module.

Note that $\mathcal{F}$ is generated by global sections if and only if $\mathcal{F}$ can be written as a quotient of a free sheaf [38, Tag 01AL].

2.4. (Quasi)-coherent sheaves

Working with schemes and quasi-coherent sheaves simplifies several of the definitions given in the Section 2.2. On the other hand, they are also called by other names. For example, given a morphism of schemes $f : X \to Y$ and a quasi-coherent sheaf $\mathcal{F}$ over Y, the direct image $f_*\mathcal{F}$ and inverse image $f^*\mathcal{F}$ of the Definition 2.2.12, are called the pushforward and pullback of $\mathcal{F}$, respectively.

In this section, we define quasi-coherent sheaves in affine schemes and projective schemes using Proj. In addition, we will describe some important examples of these sheaves. The definitions and results are quoted from [25, Section II.5] and [21, Chapter 14].

Definition 2.4.1 (Sheaf associated to a module on $\operatorname{Spec} R$)**.** Let R be a ring and let M be a R-module. For each prime ideal $\mathfrak{p}$ in R, let $M_\mathfrak{p}$ be the localization of M at $\mathfrak{p}$. For any open set $U \subseteq \operatorname{Spec} R$ we define the group

$$\begin{aligned}\widetilde{M}(U) :=\{s = \{s(\mathfrak{p})\}_{\mathfrak{p}\in U} :\ & s(\mathfrak{p}) \in M_\mathfrak{p} \text{ for all } \mathfrak{p} \in U, \text{ and for all } \mathfrak{p} \\ & \text{there is an open neighborhood } V \text{ of } \mathfrak{p} \text{ in } U \text{and } g \in M, f \in R \\ & \text{such that for each } \mathfrak{q} \in V, f \notin \mathfrak{q}, \text{ and } s(\mathfrak{q}) = \frac{g}{f} \in M_\mathfrak{q}\}.\end{aligned}$$

Let be $X = \operatorname{Spec} R$. Using the obvious restriction maps, $\widetilde{M}$ is a sheaf on X and $\widetilde{M}(U)$ is, by construction, a module over $\widetilde{R}(U) = \mathcal{O}_X(U)$. Hence, $\widetilde{M}$ is a sheaf of modules on X called the **sheaf associated** to M.

Definition 2.4.2 (Quasi-coherent sheaves on a affine scheme)**.** Let $(X, \mathcal{O}_X)$ be an affine scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is **quasi-coherent** if X can be covered by open affine subsets $U_i = \operatorname{Spec} R_i$, such that for each i there is an R_i-module M_i with $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. If in addition each M_i can be taken to be a finitely generated R_i-module, we say that $\mathcal{F}$ is **coherent**.

In the case of $X = \operatorname{Proj} S$, with S a graded ring, we can define a sheaf associated to a module analogously to the Definition 2.4.1 taking in consideration the structure sheaf $\mathcal{O}_X = \mathcal{O}_{\operatorname{Proj} S}$ defined in 2.3.10.

Definition 2.4.3 (Sheaf associated to a module on a $\operatorname{Proj} S$)**.** Let S be a graded ring and let M be a graded $S-$module. For each $\mathfrak{p} \in \operatorname{Proj} S$, let $M_{(\mathfrak{p})}$ be the group of elements of degree 0 in the localization $T^{-1}M$ where T is the multiplicative system of homogeneous elements of S not contained in $\mathfrak{p}$.

$$\begin{aligned}\widetilde{M}(U) :=\{s = \{s(\mathfrak{p})\}_{\mathfrak{p}\in U} :\ & s(\mathfrak{p}) \in M_{(\mathfrak{p})} \text{ for all } \mathfrak{p} \in U, \text{ and for all } \mathfrak{p} \text{ there is an open} \\ & \text{neighborhood } V \text{ of } \mathfrak{p} \text{ in } U, m \in M \text{ and } f \in S \text{ of the same degree} \\ & \text{such that for every } \mathfrak{q} \in V, f \notin \mathfrak{q}, \text{ and } s(\mathfrak{q}) = \frac{m}{f} \in M_{(\mathfrak{q})}\}.\end{aligned}$$

Taking the obvious restriction maps, $\widetilde{M}$ is a sheaf. By [25, II, Prop. 5.11], $\widetilde{M}$ is a quasi-coherent sheaf $\mathcal{O}_X$-module. Moreover, if S is noetherian and M is finitely generated, then $\widetilde{M}$ is coherent.

Note that the Definition 2.4.3 can be rephrased using the open subsets of the form $D_+(f)$ ($f \in S$, homogeneous and $\deg(f) > 0$), whose form base of the topology of $\operatorname{Proj} S$. For a graded $S-$module M, define $\widetilde{M}$ by assigning $\widetilde{M}(D_+(f)) = M_{(f)}$, where $M_{(f)} \subseteq M_f$ denotes de group of degree 0 in the localized module M_f.

Notation 2.4.4. Given a morphism of schemes $f : X \to Y$ and a quasi-coherent sheaf $\mathcal{F}$ on Y, the direct image $f_*\mathcal{F}$ is usually called **pushforward** of $\mathcal{F}$ by f, while the inverse image $f^*\mathcal{F}$ is called **pullback** of $\mathcal{F}$.

We detail a little the construction of $f^*\mathcal{F}$ on X. If $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ are affine, then f correspond to a ring homomorphism $S \to R$. Moreover, as $\mathcal{F}$ is quasi-coherent, we have $\mathcal{F} = \widetilde{M}$ for an $S-$module M. Then M and R are S-modules, and hence we can form the tensor product $M \otimes_S R$ as an $R-$module. Its associated sheaf $(M \otimes_S R)^\sim$ on X is called the **pullback** $f^*\mathcal{F}$ of $\mathcal{F}$ along f.

For arbitrary schemes X, Y, $f^*\mathcal{F}$ is defined as in Definition 2.2.12 which, in the affine case, corresponds to the previous construction.

Remark 2.4.5.

(a) For any morphism $f : X \to Y$ of schemes, $f^*\mathcal{O}_Y = \mathcal{O}_X$. For any quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on Y, $f^*(\mathcal{F} \oplus \mathcal{G}) \cong f^*\mathcal{F} \oplus f^*\mathcal{G}$.

(b) For a closed point $P \in X$ and an quasi-coherent $\mathcal{F}$ on X, if $i : P \to X$ is the inclusion map, $i^*\mathcal{F}$ on P has only one non-trivial space of sections $i^*\mathcal{F}(P)$. If $X = \operatorname{Spec} R$ is affine, so that $P \trianglelefteq R$ is a maximal ideal and $\mathcal{F} = \widetilde{M}$ for an $R-$module, then we have

$$i^*\mathcal{F}(P) = M \otimes_R R/P = M/PM,$$

which is a vector space over the residue field $k(P) = R/P$ (because is a $k(P)$-module and $k(P) \cong P$). $i^*\mathcal{F}(P)$ is called the **fiber** of $\mathcal{F}$ at P.

Some authors denote the fiber of a sheaf by $\mathcal{F}(P)$ instead of using the pullback notation, defining $\mathcal{F}(P) = \mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} k(P)$ and it is interpreted as “the result of evaluating all of the germs at P”. Since $\mathcal{F}$ is a quasi-coherent sheaf, every section $\varphi \in \mathcal{F}$ over a open subset $U \subseteq X$ containing P determines a element $\varphi_P \in \mathcal{F}_P = M_P$, and hence, by taking it modulo PM, a **value** $\varphi(P) \in i^*\mathcal{F}$.

(c) From the Definition 2.2.9 and remark 2.4.5(b), a locally free sheaf of rank r on a scheme X defines an r-dimensional vector space over the field $k(P)$ for all P.

Some authors usually switch effortlessly between the notion of a locally free sheaf and a *vector bundle* (see [25, II, Ex. 5.18].

(d) From [25, II, Prop. 5.7.] the kernel, the cokernel and image of any morphism of quasi-coherent sheaves are quasi-coherent. Moreover, from commutative algebra, if the sheaves $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves on a scheme X, with $\operatorname{rk} \mathcal{F} = n$ and $\operatorname{rk} \mathcal{G} = m$, then

- the direct sum $\mathcal{F} \oplus \mathcal{G}$ is locally free of rank $n + m$
- the tensor product $\mathcal{F} \otimes \mathcal{G}$ is locally free of rank nm
- the dual $\mathcal{F}^\vee$ is locally free of rank n
- the pullback $f^*\mathcal{F}$ for a morphism $f : Y \to X$ is locally free of rank n.

Regarding the last remark, there is a result regarding short exact sequences that we will use later.

Lemma 2.4.6. [21, Lemma 14.20] Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of quasi-coherent sheaves on a scheme X.

(i) For another quasi-coherent sheaf $\mathcal{F}$ on X the sequence

$$0 \to \mathcal{F}_1 \otimes \mathcal{F} \to \mathcal{F}_2 \otimes \mathcal{F} \to \mathcal{F}_3 \otimes \mathcal{F} \to 0$$

is also exact on X if $\mathcal{F}$ is locally free or all $\mathcal{F}_i$ are locally free.

(ii) For any morphism $f : Y \to X$ of schemes the sequence

$$0 \to f^*\mathcal{F}_1 \to f^*\mathcal{F}_2 \to f^*\mathcal{F}_3 \to 0$$

is exact on Y if all $\mathcal{F}_i$ are locally free.

Proposition 2.4.7. [21, Lemma 14.15] If $i : Y \to X$ is the inclusion of a closed subscheme, then for all sheaves $\mathcal{F}$ on Y and $\mathcal{G}$ on X we have

$$i^*i_*\mathcal{F} \cong \mathcal{F} \quad \text{and} \quad i_*(\mathcal{F} \otimes i^*\mathcal{G}) \cong (i_*\mathcal{F}) \otimes \mathcal{G}.$$

The second equality is known as **projection formula**.

Definition 2.4.8 (Ideal sheaf)**.** Let Y be a closed subscheme of a scheme X, and let $i : Y \to X$ be the inclusion morphism. We define the **ideal sheaf** of Y, denoted $\mathcal{O}_Y$, to be the kernel of

$$i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Y.$$

Definition 2.4.9 (Twisting sheaves)**.** Let S be a graded ring, and let $X = \operatorname{Proj} S$. For any $n \in \mathbb{Z}$, define the sheaf $\mathcal{O}_X(n)$ to be $(S(n))^\sim$, where $S(n) = S_{m+n}$ for all $m \in \mathbb{Z}$. The **twisting sheaf of Serre** is $\mathcal{O}_X(1)$. For any sheaf of $\mathcal{O}_X$-modules, $\mathcal{F}$, define the **twisted sheaf** $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Example 2.4.10 (Twisting sheaves on $\mathbb{P}^n$)**.** Let $n \in \mathbb{N}$ and $d \in \mathbb{Z}$. For each non-empty open subset $U \subseteq \mathbb{P}^n$, the *twisting sheaf of* $\mathbb{P}^n$, $\mathcal{O}_{\mathbb{P}^n}(d)$, is defined by

$$(\mathcal{O}_{\mathbb{P}^n}(d))(U) := \left\{ \frac{g}{f} : f, g \in K[x_0, \dots, x_n] \text{ homogeneous with } \deg g - \deg f = d \text{ and } f(P) \neq 0 \text{ for all } P \in U \right\}$$

as a subset of the quotient field of $K[x_0, \dots, x_n]$. Together with setting $(\mathcal{O}_{\mathbb{P}^n}(d))(\emptyset) := \{0\}$ and considering the restriction maps given by the identity on the quotient field of $K[x_0, \dots, x_n]$, $\mathcal{O}_{\mathbb{P}^n}(d)$ is a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules.

Also, this sheaf is locally isomorphic to the structure sheaf on $\mathbb{P}^n$, which make it a locally free sheaf of a rank 1 (2.2.9), i.e. a *line bundle*. Note that for any open set $U_i = \{(x_0 : \cdots : x_n) : x_i \neq 0\}$ for $i = 0, \dots, n$, we can define the map $\alpha : \mathcal{O}_{\mathbb{P}^n}|_{U_i} \to \mathcal{O}_{\mathbb{P}^n}(d)|_{U_i}$ defined $s \mapsto s x_i^d$ with inverse

$$\alpha^{-1} : \mathcal{O}_{\mathbb{P}^n}(d)|_{U_i} \to \mathcal{O}_{\mathbb{P}^n}|_{U_i}, s \mapsto \frac{s}{x_i^d}.$$

For $n \in \mathbb{N}$ and $d, e \in \mathbb{Z}$, it can be proven that $\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_{\mathbb{P}^n}(e) \simeq \mathcal{O}_{\mathbb{P}^n}(d+e)$.
By Remark 2.4.5(d), the next sheaf is a line bundle as well.

Example 2.4.11 (Twisting sheaves on projective schemes)**.** Let X be a closed subscheme of $\mathbb{P}^n$. If $i : X \to \mathbb{P}^n$ is the inclusion map, for any $d \in Z$ we define the twisting sheaf $\mathcal{O}_X(d)$ on X by

$$\mathcal{O}_X(d) = i^* \mathcal{O}_{\mathbb{P}^n}(d).$$

2.5. Cotangent sheaf

Definition 2.5.1 (Kähler differentials)**.** Let A be a ring, let B be an $A-$algebra, and let M be a $B-$module. An **A-linear derivation** on B into M is an A-module homomorphism $d : B \to M$ such that

$$d(bb') = d(b)b' + bd(b') \quad \text{and} \quad da = 0 \text{ for all } a \in A.$$

We define the module of **relative differential forms** or **Kähler differentials** of B over A to be the $B-$module $\Omega_{B/A}$, for which there is a universal A-derivation $d : B \to \Omega_{B/A}$.

Definition 2.5.2 (Sheaf of differentials)**.** Let $f : X \to Y$ be a morphism of schemes. We consider the diagonal morphism $\Delta : X \to X \times_Y X$ 2.3.20. From [25, II, Cor. 4.2] Δ gives an isomorphism of X onto its image $\Delta(X)$ which is a locally closed subscheme of $X \times_Y Y$, i.e., a closed subscheme of an open subset W of $X \times_Y Y$. Let $\mathscr{I}$ the the sheaf of ideals of $\Delta(X)$ in W. Then we define **the sheaf of relative differentials** (or **cotangent sheaf**) of X over Y to be the sheaf

$$\Omega_{X/Y} = \Delta^*(\mathscr{I}/\mathscr{I}^2).$$

Remark 2.5.3. $\Omega_{X/Y}$ has a natural structure of $\mathcal{O}_X$-module and, from [25, II, Prop. 5.9], is a quasi-coherent sheaf.

Remark 2.5.4. In the case that X and Y are affine schemes, we can consider the open subsets $U = \operatorname{Spec} A$ of Y and $V = \operatorname{Spec} B$ of X such that $f(V) \subseteq U$. Note that B is a $A-$algebra and $V \times_U V$ is open affine subset of $X \times_Y X$ which is isomorphic to $\operatorname{Spec}(B \otimes_A B)$. $\Delta(X) \cap (V \times_U V)$ is the closed subscheme defined by the kernel of the diagonal homomorphism

$$g : B \otimes_A B \to B \qquad\qquad b \otimes b' \mapsto bb'.$$

Thus, if $I = \operatorname{Ker} g$, the sheaf $\mathscr{I}$ is the sheaf associated to the ideal I, and $\mathscr{I}/\mathscr{I}^2$ the sheaf associated to the B-module I/I^2. Now, considering the map $d : B \to I/I^2$ defined by $db = 1 \otimes b - b \otimes 1 \mod I^2$, $(I/I^2, d)$ is a module of Kähler differentials for B/A. It follows that $\Omega_{V/U} \cong (\Omega_{B/A})^{\sim}$.

The derivations $d : B \to \Omega_{B/A}$ glue together to give a map $\delta : \mathcal{O}_X \to \Omega_{X/Y}$ of sheaves of abelian groups on X, which is a derivation of the local rings at each point.

There is a important example when $A = k$ is a field and $B = k[x_1, \dots, x_n]$ is a polynomial ring over A. The module of Kälher differentials $\Omega_{B/k}$ is the free $B-$module of rank n generated by the $k-$derivations $dx_1, \dots, dx_n$ (see [25, II, Example 8.2.1]). Analogously, if $X = \mathbb{A}^n_k$ is the affine space over k, the cotangent sheaf $\Omega_{X/k}$ is a free $\mathcal{O}_X$-module of rank n generated by the global sections $dx_1, \dots, dx_n$, where $x_1, \dots, x_n$ are the affine coordinates fo $\mathbb{A}^n_k$ (see [25, II, Example 8.12.1]).

Example 2.5.5. From the [25, II, Thm. 8.13] the cotangent bundle of $\mathbb{P}^n_{\mathbb{C}}$ is determined by the following exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

where $\mathcal{O}_{\mathbb{P}^n}(-1)^{n+1}$ stands for the direct sum $n+1$ copies of the twisting sheaf $\mathcal{O}_{\mathbb{P}^n}(-1)$.

From Definition 2.3.21, the nonsingularity of a variety X over a field k is closely linked to the differentials on X.

Theorem 2.5.6. [25, II, Thm. 8.15] Let X be an irreducible separated scheme of finite type over an algebraically closed field k. Then $\Omega_{X/k}$ is a locally free sheaf of rank $\dim X = n$ if and only if X is a nonsingular variety over k.

Definition 2.5.7 (Tangent sheaf)**.** Let X be a smooth variety over k. The **tangent sheaf** of X is defined to be $\mathscr{T}_X = \Omega^{\vee}_{X/k}$ which is a vector bundle of rank n.

2.6. Canonical bundle

Given an R-module M, from the universal property of the tensor product, any bilinear function on $M \times M$ turns into a linear function out of $M \otimes M$. Likewise, a multilinear function on M^n turns into a linear function out of $\underbrace{M \otimes \cdots \otimes M}_{n \text{ times}}$ (or $M^{\otimes n}$). The *nth exterior power of* M is the module that universally linearizes the alternating multilinear functions on M^n.
The objective of this section is to define the canonical sheaf of a nonsingular variety X over k, which is the nth exterior power of the sheaf of differentials of X.

Definition 2.6.1. Let M, W be a two $R-$modules and $n \in \mathbb{N}$. A multilinear map

$$f : \underbrace{M \times \cdots \times M}_{n \text{ times}} \to W$$

is called **alternating** if $f(m_1, \ldots, m_n) = 0$ for all $m_1, \ldots, m_n \in V$ such that $m_i = m_j$ for some $i \neq j$.

Definition 2.6.2 (nth exterior power)**.** Let M be a $R-$module and let $n \in \mathbb{N} \cup \{0\}$. A n**th exterior power** of M is a R-module space T together with an alternating $n-$fold multilinear map $\tau : M^n \to T$ satisfying the following universal property: For every $n-$fold alternating multilinear map $f : M^n \to W$, where W is another R-module, there is a unique linear map $g : T \to N$ with $f = g \circ \tau$, i. e. such that the following diagram commutes.

$$\begin{array}{ccc} M^n & \xrightarrow{f} & W \\ \tau\big\downarrow & \nearrow_{g} & \\ T & & \end{array}$$

By [21, Prop. 8.7], τ exists and is unique up to isomorphism. We denote T by $\Lambda^n M$ and $\tau(m_1, \ldots, m_n)$ as $m_1 \wedge \cdots \wedge m_n$ for all $m_1, \ldots m_r \in M$.

If $\mathcal{F}$ is a sheaf on a scheme X, $\Lambda^r \mathcal{F}$ is defined as the sheaf associated to the presheaf $U \to \Lambda^r(\mathcal{F}(U))$.

This definition is the definition a nth tensor product of M except for the hypothesis of *alternating*. That is why some authors call a nth exterior power of M a *$n-$fold* *alternating tensor product* of M. The definition of $\Lambda^n M$ as a set is the following: for $n \in \mathbb{N}$, $\Lambda^n M = M^{\otimes n}/J_n$ where $J_1 = \{0\}$ and for $n \geq 2$ J_n is the submodule of $M^{\otimes n}$ spanned by $m_1 \otimes \cdots \otimes m_n$ with $m_i = m_j$ for some $i \neq j$. In the case of $n = 0$, set $\Lambda^0 M = R$.

Definition 2.6.3 (Canonical sheaf)**.** Let X be a smooth variety over k with $\dim X = n$. The **canonical sheaf** of X is $\omega_X = \Lambda^n \Omega_{X/k}$, the nth exterior power of the sheaf of differentials of X.

From the Theorem 2.5.6 and the fact that n-th exterior product of a free module of rank r is again free of rank $\binom{r}{n}$, ω_X is locally free sheaf of rank 1, then a line bundle.

Definition 2.6.4 (Geometric genus)**.** Let X be a projective and nonsingular variety over k. The **geometric genus** of X is $p_g = \dim_k \omega_X(X)$.

Theorem 2.6.5. [25, II, Thm. 8.19] Let X and X' be two birationally equivalent nonsingular projective varieties over k. Then $p_g(X) = p_g(X')$.

Example 2.6.6. Taking the highest exterior powers of the exact sequence of 2.5.5 and from the fact that $\wedge^n(\mathcal{O}_X(-1)^{n+1}) \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$ and $\wedge^n \mathcal{O}_X \cong \mathcal{O}_X$, we conclude

$$\mathcal{O}_{\mathbb{P}^n}(-n-1) \cong \omega_{\mathbb{P}^n} \otimes \mathcal{O}_X \cong \omega_{\mathbb{P}^n}.$$

Remark 2.6.7.

- Since $\mathcal{O}(l)$ has no global sections for $l < 0$, we find that $p_g(\mathbb{P}^n) = 0$ for any $n \geq 1$.
- Recall that a **rational variety** is defined as a variety birational to $\mathbb{P}^n$ for some n (for other equivalent definitions, see [24, p.78]). If X is any nonsingular projective rational variety, by Theorem 2.6.5 $p_g(X) = 0$.

3. Sheaf Cohomology

We recall some definitions of Section 2.1 and refer to some definitions of homological algebra.

Definition 3.0.1 (Complex)**.** Given an abelian category $\mathscr{C}$, a **complex** $A^{\cdot}$ is collection of objects $A^i \in Ob(\mathscr{C})$, $i \in \mathbb{Z}$, and morphisms $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all i. A **morphism of complexes** $f : A^{\cdot} \to B^{\cdot}$ is a set of morphisms $f^i : A^i \to B^i$ for each i, which commute with the coboundary maps d^i.

Definition 3.0.2 (Cohomology object)**.** The i**th cohomology object** $h^i(A^{\cdot})$ of a complex $A^{\cdot}$ is defined to be $\operatorname{Ker}(d^i)/\operatorname{Im}(d^{i-1})$. If $f : A^{\cdot} \to B^{\cdot}$ is a morphism of complexes, then f induce a natural map $h^i(f) : h^i(A^{\cdot}) \to h^i(B^{\cdot})$.

Definition 3.0.3 (Injective resolution)**.** An object I of an category $\mathscr{C}$ is called **injective** if the functor $\operatorname{Hom}(\cdot, I)$ is exact. An **injective resolution** of an object A of $\mathscr{C}$ is a complex $I^{\cdot}$ in $\mathscr{C}$, defined in degrees $i \geq 0$, together with a morphism $\varepsilon : A \to I^0$, such that I^i is an injective object of $\mathscr{C}$ for each i, and such that the sequence

$$0 \to A \xrightarrow{\varepsilon} I^0 \to I^1 \to \dots$$

is exact. If every object of $\mathscr{C}$ is isomorphic to a subobject of an injective object of $\mathscr{C}$, then we say $\mathscr{C}$ **has enough injectives**. In this case, every object has an injective resolution.

Let $(X, \mathcal{O}_X)$ be a ringed space. The category $\mathfrak{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules has enough injectives [25, III, Prop. 2.2]. For a topological space X, the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on X has enough injectives [25, III, Cor. 2.3].

Let X be a topological space. The **global section functor** Γ from $\mathfrak{Mod}(X)$ or $\mathfrak{Ab}(X)$ to an abelian category defined by $\mathcal{F} \mapsto \mathcal{F}(X)$ is covariant *left exact*, i.e. if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is exact, then

$$0 \to \mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \mathcal{F}_3(X)$$

is exact. The problem is that the last map in this sequence is in general not surjective, so that we cannot obtain much information about $\mathcal{F}_3(X)$ from this. To solve this inconvenience, it is useful to study the sequence of the cohomology groups of the exact sequence.

The **cohomology functor** on X are the *right derived functors* of Γ (see [40, Section 2.5])

$$H^i(X, \mathcal{F}) = R^i\Gamma(\mathcal{F}).$$

For any sheaf $\mathcal{F}$, the group $H^i(X, \mathcal{F})$ is called ith-**cohomology group** of $\mathcal{F}$. Note that even if X and $\mathcal{F}$ have some additional structure, e.g., X a scheme and $\mathcal{F}$ a quasi-coherent sheaf, we always take cohomology in this sense, regarding $\mathcal{F}$ simply as a sheaf of abelian groups on the underlying topological space X. On the other hand, [25, III, Prop. 2.6],

$$H^i(X, \mathcal{F}) = h^i(\Gamma(I^\cdot))$$

where $I^\cdot$ is the injective resolution of $\mathcal{F}$, that is, if the sequence of the resolution is

$$0 \to \mathcal{F} \to I_0 \to I_1 \to I_2 \to \cdots,$$

the sheaf cohomology groups $H^i(X, \mathcal{F})$ are the cohomology groups of the chain complex of abelian groups

$$0 \to I_0(X) \xrightarrow{d^0} I_1(X) \xrightarrow{d^1} I_2(X) \xrightarrow{d^2} \cdots.$$

Note that this sequence is in general not exact, as we only have an inclusion $\operatorname{Im} d^p \subseteq \operatorname{Ker} d^{p+1}$ for all p, which might not be an equality. However, this inclusion allows us to form the quotient spaces $\operatorname{Ker} d^{p+1}/\operatorname{Im} d^p$ that measure "by how much the sequence fails to be exact" and are usually called the cohomology of this complex.

Standard arguments in homological algebra imply that these cohomology groups are independent of the choice of injective resolution of $\mathcal{F}$.

Theorem 3.0.4. [25, III, Thm. 2.7] Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$.

Theorem 3.0.5. [25, III, Thm. 3.5] Let $X = \operatorname{Spec} A$ be the spectrum of a noetherian ring A. Then for all quasi-coherent sheaves $\mathcal{F}$ on X, and for all $i > 0$, we have $H^i(X, \mathcal{F}) = 0$.

3.1. Čech Cohomology

Let X be a topological space, and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X. Fix, once and for all, a well-ordering of the index set I. For any finite set of indices $i_0, \ldots, i_p \in I$ we denote the intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0, \ldots, i_p}$.

Now let $\mathcal{F}$ be a sheaf of abelian groups on X. We proceed to define a complex $C^\cdot(\mathfrak{U}, \mathcal{F})$ of abelian groups. For each integer $p \geq 0$, and indices $i_0, \ldots, i_p \in I$ let

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}).$$

Hence, an element $\varphi \in C^p(\mathcal{F})$ is just a collection of sections $\varphi_{i_0,\ldots,i_p} \in \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p})$ for all intersections of $p+1$ sets taken from the chosen affine open cover. These sections can be completely unrelated.

For every $p \geq 0$ we define the coboundary map $d^p : C^p(\mathcal{F}) \to C^{p+1}(\mathcal{F})$ by

$$(d^p\varphi)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1}(-1)^k \varphi_{i_0,\ldots,\hat{i_k},\ldots,i_{p+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}},$$

where the notation $\hat{i_k}$ means that the index i_k is to be left out. Note that this makes sense as $\varphi_{i_0,\ldots,\hat{i_k},\ldots,i_{p+1}}$ is a section of $\mathcal{F}$ on $U_{i_0} \cap \cdots \cap \widehat{U_{i_k}} \cap \cdots \cap U_{i_{p+1}}$, which contains $U_{i_0} \cap \cdots \cap U_{i_{p+1}}$ as an open subset for all k.

It can be easily proved that $d \circ d = 0$, so we have indeed defined a complex of abelian groups.

Definition 3.1.1. Let X be a topological space and let $\mathfrak{U}$ be an open covering of X. For any sheaf of abelian groups $\mathcal{F}$ on X, we define the **pth Čech cohomology group** of $\mathcal{F}$, with respect to the covering $\mathfrak{U}$, to be

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = H^p(C^{\cdot}(\mathfrak{U}, \mathcal{F})).$$

Remark 3.1.2. [25, III, Thm. 4.5] In the case X is a noetherian separated scheme, the sheaf $\mathcal{F}$ is quasi-coherent, and the covering of X is an finite open affine covering, then **Čech cohomology** groups coincide with the cohomology groups of $\mathcal{F}$, that is, there is an isomorphism

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) \xrightarrow{\simeq} H^p(X, \mathcal{F}).$$

Notation 3.1.3. Let A be a noetherian ring, let $S = A[x_0, \ldots, x_r]$, and let $X = \operatorname{Proj} S$ be the projective space $\mathbb{P}^r_A$ over A. Let $\mathcal{O}_X(1)$ be the twisting sheaf of Serre (definition 2.4.9). For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we denote by $\Gamma_*(\mathcal{F})$ the graded S-module $\bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n))$.

Theorem 3.1.4. [25, III, Thm. 5.1] Let A be a noetherian ring, and let $X = \mathbb{P}^r_A$, with $r \geq 1$. Then:

(i) the natural map $S \to \Gamma_*(\mathcal{O}_X) = \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{O}_X(n))$ is an isomorphism of graded S-modules;

(ii) $H^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$ and all $n \in \mathbb{Z}$;

(iii) $H^r(X, \mathcal{O}_X(-r-1)) \cong A$;

(iv) The natural map

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong A$$

is a perfect pairing of finitely generated free A-modules, for each $n \in \mathbb{Z}$.

3.2. Rational singularities

We want to state some vanishing theorems on varieties with rational singularities (for a study on surfaces, see [2]). First, recall the definition of resolution of singularities. We take the following from [26, Def. 4.3.2.]. For other version see [27, Thm. 0.2].

Let X be an algebraic variety or analytic space. A point which is not smooth is called a **singular point** (see Definition 2.3.21). Denote by $X_{\text{sing}} = \{x \in X : x \text{ is a singular point of } X\}$ the **singular locus** of X.

Definition 3.2.1. Let X be an algebraic variety (then $X \setminus X_{\text{sing}}$ is an open dense subset of X). A morphism $f : Y \to X$ (or Y) is called a **resolution of the singularities** of X, if the following hold:

1. The morphism f is proper.
2. The restriction $f|_{Y \setminus f^{-1}(X_{\text{sing}})} : Y \setminus f^{-1}(X_{\text{sing}}) \to X \setminus X_{\text{sing}}$ of f is isomorphic.
3. The variety Y is non-singular.

The right derived functors of the direct image of a map $f : X \to Y$ between topological spaces are called higher direct images and denoted $R^i f_*$ (see 2.1.20).

Definition 3.2.2. [27, Def. 5.8.] Let X be a variety over a field of characteristic 0 and $f : Y \to X$ a resolution of singularities. We say that f is a **rational resolution** if

1. $f_* \mathcal{O}_Y = \mathcal{O}_X$ (equivalently, X is normal), and
2. $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$.

We say that X has **rational singularities** if every resolution $f : Y \to X$ is rational.

Remark 3.2.3.

- By [15, Thm. 11.4.2] a normal toric variety has rational singularities .
- One of the hypothesis of Kawamata-Viehweg vanishing Theorem 4.4.21 is that the projective variety has rational singularities. This result is very important for our work in the sections II.5 and II.7.

4. Divisors

Throughout this chapter, a *surface* will mean a nonsingular projective variety of dimension 2 over an algebraically closed field k. $\mathbb{P}^n_k$ denote the projective n-space over the field k. For $k = \mathbb{C}$ we will simply use $\mathbb{P}^n$.

4.1. Basic definitions

For our work, we will restrict to the case where X is an irreducible complex variety. We recall the definition of Cartier divisor from [30, Def. 1.1.1]. Denote by $\mathcal{M}_X = \mathbb{C}(X)$ the (constant) sheaf of rational functions on X, which contains the structure sheaf $\mathcal{O}_X$ as a subsheaf, and so there is an inclusion $\mathcal{O}_X^* \subseteq \mathcal{M}_X^*$ of sheaves of multiplicative abelian groups.

Definition 4.1.1 (Cartier divisor)**.** A **Cartier divisor** on X is a global section of the quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$. We denote by $\mathrm{CDiv}(X)$ the group of all such, so that

$$\mathrm{CDiv}(X) = \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*).$$

A divisor $D \in \mathrm{Div}(X)$ is represented by data $\{(U_i, f_i)\}_{i\in I}$ consisting of an open covering $\{U_i\}_{i\in I}$ of X together with elements $f_i \in \Gamma(U_i, \mathcal{M}_X^*)$, having the property that on $U_{ij} = U_i \cap U_j$ one can write

$$f_i = g_{ij} f_j \quad \text{for some} \quad g_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^*).$$

The group operation of $\mathrm{CDiv}(X)$ is denoted additively: for two divisors $D, D' \in \mathrm{Div}(X)$ represented respectively by data $\{(U_i, f_i)\}$ and $\{(U_i, f_i')\}$, $D + D'$ is given by $\{(U_i, f_i f_i')\}$. The **support** of a divisor $D = \{(U_i, f_i)\}$ is the set of points $x \in U_i \subset X$ at which a local equation of D at x, f_i, is not a unit in $\mathcal{O}_{X,x}$.

For a definition of Cartier divisor on an arbitrary scheme, see [25, p. 140].

Definition 4.1.2. A divisor $D \in \mathrm{CDiv}(X)$ is called

- **principal** if there is f a global section of $\Gamma(X, \mathcal{M}_X^*)$ such that the divisor associated to f by the natural map

  $$\Gamma(X, \mathcal{M}_X^*) \to \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$$

  coincides with D.

- **effective**, denoted by $D \geq 0$, if it can be represented by $\{(U_i, f_i)\}$ where each $f_i \in \Gamma(U_i, \mathcal{O}_X)$ is regular on U_i. If $D' \in \mathrm{CDiv}(X)$, $D \geq D'$ indicates that $D - D'$ is effective.

Definition 4.1.3. Two divisor $D_1, D_2 \in \mathrm{CDiv}(X)$ are **linearly equivalent**, denoted by $D_1 \sim D_2$ if their difference is a principal divisor. The quotient group pf $\mathrm{CDiv}(X)$ by the subrgroup of principal divisors is called the **Cartier divisor class group** of X and denoted by $\mathrm{CaCl}(X)$.

Definition 4.1.4. A $\mathbb{Q}-$**divisor** ($\mathbb{R}$-divisor) D is a finite linear combination of Cartier divisors with rational (real, respectively) coefficients.

Definition 4.1.5 (Weil divisor)**.** A **prime divisor** on X is a closed subvariety Y of codimension one. A **Weil divisor** is an element of the free abelian group $\mathrm{Div}(X)$ generated by the prime divisors. We write a Weil divisor as $D = \sum n_i Y_i$, where the Y_i are prime divisors, the n_i are integers, and only finitely many n_i are different from zero. If all the $n_i \geq 0$, we say that D is **effective**.

Definition 4.1.6. Let Y be a prime divisor on X and $\eta \in Y$ be its generic point. As X is integral, the local ring $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field $\mathbb{C}(X)$, the function field of X. Note that since X is separated, Y is uniquely determined by its valuation, v_Y, see [25, §I.6].
Now let $f \in \mathbb{C}^*(X)$ be any nonzero rational function on X. Then $v_Y(f)$ is an integer. If it is positive, we say f has a zero along Y, of that order; if it is negative, we say f has a pole along Y, of order $-v_Y(f)$. We define the **divisor** of f, denoted (f), by

$$(f) = \sum v_Y(f) \cdot Y,$$

where the sum is taken over all prime divisors of X. By [25, II, Lemma 6.1], this is a finite sum, hence it is a Weil divisor. Any Weil divisor which is equal to the divisor of a function is called a **principal divisor**. The quotient group pf $\mathrm{Div}(X)$ by the subgroup of principal divisors is called the **divisor class group** of X and denoted by $\mathrm{Cl}(X)$.

Definition 4.1.7 (Cycles)**.** Let X be a variety or scheme of pure dimension n. A k-**cycle** on X is a $\mathbb{Z}$-linear combination of irreducible subvarieties of dimension k. The group of all such is written $Z_k(X)$.

Remark 4.1.8.

- A Weil divisor on X is an $(n-1)$-cycle, i.e. a formal sum of codimension one subvarieties with integer coefficients.

- Consider a Cartier divisor D represented by $\{(U_i, f_i)\}_{i\in I}$. Each local rational function f_i defines a Weil divisor (f_i) on U_i. By gluing these local divisors together over the whole variety, one obtains a single global Weil divisor, which is a formal sum of irreducible subvarieties of codimension 1. Thus $\mathrm{CDiv}(X) \subseteq \mathrm{Div}(X)$.

- If we assume that the complex variety X is normal, then there is an isomorphism between $\mathrm{Div}(X)$ and $\mathrm{CDiv}(X)$, where the principal and effective Weil divisors correspond to the principal and effective Cartier divisors (see [30, Remark 1.1.4], [25, II, Remark 6.17.1]).

- In Section II.5 we will consider X to be a normal $\mathbb{Q}$-factorial projective surface, where it is satisfied that any Weil divisor is $\mathbb{Q}-$Cartier, i.e. it has a multiple of a Cartier divisor.

Definition 4.1.9 (Sheaf associated to a divisor)**.** Let D be a Cartier divisor on X, represented by $\{(U_i, f_i)\}$ as above. We define a subsheaf $\mathcal{O}_X(D)$ of the sheaf of total quotient rings $\mathcal{M}_X$ by taking $\mathcal{O}_X(D)$ to be the sub-$\mathcal{O}_X$-module of $\mathcal{M}_X$ generated by f_i^{-1} on U_i. This is well-defined, since f_i/f_j is invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module. We call $\mathcal{O}_X(D)$ the **sheaf associated to** D.

From Definition 2.2.10 we denote by $\mathrm{Pic}(X)$ the Picard group of isomorphism classes of line bundles on X. From [25, II, Prop. 6.13], we have a canonical homomorphism of abelian groups

$$\mathrm{CDiv}(X) \longrightarrow \mathrm{Pic}(X)\,, \quad D \mapsto \mathcal{O}_X(D).$$

which induces the injection $\mathrm{CaCl} \to \mathrm{Pic}(X)$ noting that $D_1 \simeq D_2 \iff \mathcal{O}(D_1) \cong \mathcal{O}(D_2)$ for any pair of divisors D_1, D_2. More abstractly, one can view $\mathcal{O}_X(D)$ as the image of D under the connecting homomorphism

$$\mathrm{CDiv}(X) = \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*) \longrightarrow H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X)$$

determined by the exact sequence $0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}_X^* \longrightarrow \mathcal{M}_X^*/\mathcal{O}_X^* \longrightarrow 0$ of sheaves on X.

Definition 4.1.10. Let ω_X be the canonical line bundle on X. We call **canonical divisor** of X any divisor K_X such that $\mathcal{O}_X(K_X) = \omega_X$.

Remark 4.1.11.

- If X is a noetherian, integral, separated locally factorial scheme, then $\mathrm{CaCl}(X) \cong \mathrm{Cl}(X) \cong \mathrm{Pic}(X)$ (see [25, II, Cor. 6.16])

- If $X = \mathbb{P}^n_k$, $\mathrm{Pic}(X) \cong \mathbb{Z}$ by [25, II, Prop. 6.4] and every invertible sheaf on X is isomorphic to $\mathcal{O}_X(l)$ for some $l \in \mathbb{Z}$. In particular, if $l \geq 1$, $\mathcal{O}_{\mathbb{P}^2}(l)$ represents the class of plane curves of degree l.

- If H is a hyperplane in $\mathbb{P}^n$, then $\mathcal{O}_{\mathbb{P}^n}(d) = \mathcal{O}(dH)$. In particular, from Example 2.6.6, the canonical divisor of $\mathbb{P}^n$ is $K_{\mathbb{P}^n} = (-n-1)H$.

- Recall the definition of rational variety of 2.6.7. By [25, II, Prop. 8.20], if Y is a hypersurface of degree d in $\mathbb{P}^n$ with $n \geq 2$, $\omega_Y \cong \mathcal{O}_Y(d-n-1)$. In particular, if $n = 2$ and $d = 3$, Y is a nonsingular plane cubic curve and $\omega_Y = \mathcal{O}_Y$, then $p_g(Y) = \dim \Gamma(X, \mathcal{O}_Y) = 1$ so it is not rational. On the other hand, for $d \geq 4$, the geometric genus of Y is $p_g(Y) = \dfrac{1}{2}(d-1)(d-2)$ and one concludes that Y is not rational.

Definition 4.1.12. A rational surface X with effective canonical divisor $-K_X$ is called **anticanonical rational divisor**.

Definition 4.1.13 (Degree of a divisor)**.** Let X be a closed subvariety of $\mathbb{P}^n_k$ which is nonsingular in codimension one. For any divisor $D = \sum n_i Y_i$ on X, we define the **degree** of D to be $\sum n_i \deg Y_i$, where $\deg Y_i$ is the degree of Y_i, considered as a projective variety itself.

Remark 4.1.14. If X is a closed subvariety of $\mathbb{P}^n_k$, the **degree of an invertible sheaf** $\mathcal{F}$ on X can be defined as the degree of the associated divisor, i.e. if $\mathcal{F} = \mathcal{O}_X(D)$ for some D then $\deg \mathcal{F} := \deg D$.

4.2. Intersection number

The intersection theory between complex projective subvarieties depends on the ambient variety X. We present this theory in the case where X is a complex surface based on [6, Chapter 1] and, in less detail, for algebraic varieties based on [25, Appendix A] and [30, Sections 1.1, 1.4].

Definition 4.2.1. Let C, C' be two distinct irreducible curves on a smooth projective surface X, $p \in C \cap C'$, $\mathcal{O}_{X,x}$ the local ring of S at x. If f (resp. g) is an equation of C (resp. C') in $\mathcal{O}_{X,p}$, the **intersection multiplicity** of C and C' at p is defined to be

$$i_p(C \cap C') = \dim_{\mathbb{C}} \mathcal{O}_{X,x}/(f,g)\,.$$

By the Nullstellensatz [25, I, Thm. 1.3A] the ring $\mathcal{O}_{X,p}/(f,g)$ is a finite-dimensional vector space over $\mathbb{C}$.

Definition 4.2.2. Let C and C' be two distinct irreducible curves on a smooth projective surface X. The **intersection number** $(C \cdot C')$ is

$$(C \cdot C') = \sum_{p \in C \cap C'} i_p(C \cap C')\,.$$

Notation 4.2.3. In the following, we will use the next notation

- Given a quasi-coherent sheaf $\mathcal{F}$ on a smooth projective surface X, $h^i(X, \mathcal{F}) = \dim_{\mathbb{C}} H^i(X, \mathcal{F})$ (see 3.1.2).
- Given a divisor D on X, $H^i(D) = H^i(X, \mathcal{O}_X(D))$.

Definition 4.2.4. Let X be a smooth projective surface. For any coherent sheaf of $\mathcal{O}_X$-modules, $\mathcal{F}$, on X, let

$$\chi(X) = \sum_i (-1)^i h^i(X, \mathcal{F})$$

be the **Euler-Poincaré characteristic** of $\mathcal{F}$.

Theorem 4.2.5. [6, Thm. I.4] For L, L' in $\operatorname{Pic}(X)$, define

$$(L \cdot L') = \chi(\mathcal{O}_X) - \chi(L^{-1}) - \chi(L'^{-1}) + \chi(L^{-1} \otimes L'^{-1}) .$$

Then $(\cdot)$ is a symmetric bilinear form on $\operatorname{Pic}(X)$, such that if C and C' are two distinct irreducible curves on X then

$$(\mathcal{O}_X(C) \cdot \mathcal{O}_X(C')) = (C \cdot C') .$$

A useful and necessary lemma in the proof of the above theorem is the next one:

Lemma 4.2.6. [6, Lemma I.6] If C is a non-singular irreducible curve on a surface X, for a Cartier divisor D on X we have:

$$(D \cdot C) = \deg D|_C.$$

Here, $D|_C$ means the divisor on C such that $\mathcal{O}_C(D|_C) = \mathcal{O}_X(D) \otimes \mathcal{O}_C$.

Now, let X be a nonsingular quasi-projective variety over a fixed algebraically closed field k. There are several ways of defining intersection multiplicity. We just mention Serre's definition presented in [37], where it is proven that the following expression is an integer. Let Y, Z be two subvarieties of X. If Y and Z intersect properly, and if W is an irreducible component of $Y \cap Z$, we define

$$i(Y, Z; W) = \sum (-1)^j \operatorname{length} \operatorname{Tor}_j^A(A/\mathfrak{a}, A/\mathfrak{b}),$$

where A is the local ring $\mathcal{O}_{W,X}$ of the generic point of W on X, and $\mathfrak{a}$ and $\mathfrak{b}$ are the ideals of Y and Z in A.

By [25, Appendix A, Thm. 1.1] there is a unique intersection theory for cycles modulo rational equivalence on the varieties as X which satisfies seven axioms, among them, the following three that define the intersection product in X:

(A5) If Y and Z are cycles on X, and if $\Delta : X \to X \times X$ is the diagonal morphism, then

$$Y \cdot Z = \Delta^*(Y \times Z).$$

(A6) If Y and Z are subvarieties of X which *intersect properly* (meaning that every irreducible component of $Y \cap Z$ has codimension equal to $\operatorname{codim} Y + \operatorname{codim} Z$), then we can write

$$Y \cdot Z = \sum i(Y, Z; W_j) W_j,$$

where the sum runs over the irreducible components W_j of $Y \cap Z$, and where the integer $i(Y, Z; W_j)$ depends only on a neighborhood of the generic point of W_j on X. We call $i(Y, Z; W_j)$ the *local intersection multiplicity* of Y and Z along W_j.

(A7) If Y is a subvariety of X, and Z is an effective Cartier divisor meeting Y properly, then $Y \cdot Z$ is just the cycle associated to the Cartier divisor $Y \cap Z$ on Y, which is defined by restricting the local equation of Z to Y.

From this, we define numerical equivalence for divisor and one-cycles (see Definition 4.1.7):

Definition 4.2.7. Two divisors $D_1, D_2 \in \operatorname{CDiv}(X)$ are **numerically equivalent**, denoted by $D_1 \equiv D_2$, if $D_1 \cdot C = D_2 \cdot C$ for every irreducible curve C (or equivalently if $(D_1 \cdot \gamma) = (D_2 \cdot \gamma)$ for all one-cycles γ on X.

Numerical equivalence of line bundles is defined in the analogous manner. A divisor or line bundle is **numerically trivial** if it is numerically equivalent to zero, and $\operatorname{Num}(X) \subseteq \operatorname{CDiv}(X)$ is the subgroup consisting of all numerically trivial divisors.

Definition 4.2.8. Given two one-cycles $\gamma_1, \gamma_2 \in Z_1(X)$, γ_1 is **numerically equivalent** to γ_2 if $D \cdot \gamma_1 = D \cdot \gamma_2$ for every $\mathbb{R}-$divisor D.

Definition 4.2.9. The **Néron-Severi group** of X, $N^1(X)$, is the group of Cartier divisors modulo numerical equivalence, i.e.

$$N^1(X) = \operatorname{CDiv}(X)/\operatorname{Num}(X).$$

The rank of $N^1(X)$ is called the **Picard number** of X, denoted by $\rho(X)$.

Remark 4.2.10.

- In [25], the Néron-Severi group for a smooth complex projective variety X is defined as the the group of divisors modulo algebraic equivalence. These two definitions do not coincide for divisors with integer coefficients but they do for real or rational coefficients (see [30, Remark 1.1.21]).

- The Néron-Severi group is a free abelian group and $\rho(X) < \infty$ ([30, Prop. 1.1.16]).

Definition 4.2.11.

- The group of **numerical equivalence classes of $\mathbb{R}$-divisors** is the tensor product $N^1(X)_\mathbb{R} = N^1(X) \otimes_\mathbb{Z} \mathbb{R}$, a real vector space equipped with the Euclidean topology.
- We denote by $Z_1(X)_\mathbb{R}$ the real vector space of **real one-cycles**, consisting of all finite linear combinations with real coefficients of irreducible curves on X. The corresponding vector space of **numerical equivalence classes of one-cycles** is written $N_1(X)_\mathbb{R}$.

Definition 4.2.12 (Duality between $N_1(X)_\mathbb{R}$ and $N^1(X)_\mathbb{R}$)**.** The intersection product between divisors and one-cycles forms a duality of vector spaces $N_1(X)_\mathbb{R}$ and $N^1(X)_\mathbb{R}$ in the sense of [4, Def. 3.1]. Thus, by construction one has a perfect pairing

$$\begin{aligned} N^1(X)_\mathbb{R} \times N_1(X)_\mathbb{R} &\longrightarrow \mathbb{R} \\ (\,\delta\,,\gamma\,) &\longmapsto (\delta \cdot \gamma) \end{aligned}$$

In particular, $N_1(X)_\mathbb{R}$ is a finite dimensional real vector space on which we put the standard Euclidean topology.

4.3. Rational maps and linear systems

Let X, Y be two nonsingular projective varieties over an algebraically closed field k.

Definition 4.3.1. A **rational map** φ from X to Y, written $f : X \dashrightarrow Y$, is a morphism $\varphi : U \to Y$ from a non-empty open subset $U \subseteq X$ to Y. We say that two such rational maps $\varphi_1, \varphi_2 : X \dashrightarrow Y$ defined on U_1 resp. U_2 are the same if $f_1 = f_2$ on a non-empty open subset of $U_1 \cap U_2$. A rational map $\varphi : X \dashrightarrow Y$ defined on some open subset $U \subset X$ is **dominant** if $\varphi(U)$ is dense in Y.

Definition 4.3.2. A **birational map** $\varphi : X \dashrightarrow Y$ is a rational map which admits an inverse, namely a rational map $\psi : Y \dashrightarrow X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$ as rational maps. If there is a birational map from X to Y, we say that X and Y are **birationally equivalent**, or simply **birational**.

The degree of a rational map $\varphi : X \dashrightarrow Y$ is the number of points in the intersection of the graph of Γ_φ and the fiber $X \times \{q\}$ with a general point $q \in Y$, which is the number of preimages of a general point in Y counted with multiplicity. On the other hand, two varieties X and Y are birational if and only $k(X) \cong k(Y)$, with $k(X)$ and $k(Y)$ the function fields of X and Y, respectively. By [24, Prop. 7.16] we have an alternative definition:

Definition 4.3.3 (Degree of a rational map)**.** For a rational dominant and generically finite map $\varphi : X \dashrightarrow Y$ with $\dim X = \dim Y$, its **degree** is defined as

$$\deg(\varphi) = [K(X) : \varphi^*(K(Y))],$$

where $\varphi^* : K(Y) \to K(X)$ is the pullback of fields given by $f \mapsto f \circ \varphi$, and $\varphi^*(K(Y))$ is its image (a subfield of $K(X)$). This definition coincides with counting, with multiplicities, the points of the general fiber of φ.

Remark 4.3.4. The Definition 4.3.3 allows us to define the *degree* of a irreducible projective variety $X \subseteq \mathbb{P}^n$ of dimension m as the number of points of intersection of X with a general $(n-k)$-plane $\Lambda \subseteq \mathbb{P}^n$ (see [24, Def. 18.1]).

Definition 4.3.5 (Divisor of zeros)**.** Let $\mathcal{L}$ be an invertible sheaf on X, and let $s \in \Gamma(X, \mathcal{L})$ be a nonzero section of $\mathcal{L}$. Over any open set $U \subseteq X$ where $\mathcal{L}$ is trivial, let $\varphi : \mathcal{L}|_U \to \mathcal{O}_U$ be an isomorphism. Then $\varphi(s) \in \Gamma(U, \mathcal{O}_U)$. As U ranges over a covering of X, the collection $\{U, \varphi(s)\}$ determines an effective Cartier divisor on X, which we call **divisor of zeros** of s. we denoted it by $(s)_0$.

Definition 4.3.6 (Linear system)**.** Let $|D|$ be the set of all effective divisors on X linearly equivalent to D. By [25, II, Prop. 7.7], $|D|$ can be identify identified with the projective space of one-dimensional subspaces of $H^0(X, \mathcal{O}_X(D)) = \Gamma(X, \mathcal{O}_X(D))$.

- A subset $\mathfrak{d}$ of $|D|$ is called a **linear system** on X if it is defined by a vector subspace V of $H^0(X, \mathcal{O}_X(D))$, where

  $$V = \{s \in \Gamma(X, \mathcal{O}_X(D)) : (s)_0 \in \mathfrak{d}\} \cup \{0\}.$$

  We say $\mathfrak{d}$ is **complete** if $\mathfrak{d} = |D|$.

- The **dimension** of $\mathfrak{d}$ is by definition its dimension as a projective space. Hence $\dim \mathfrak{d} = \dim V - 1$. Note these dimensions are finite because $\Gamma(X, \mathcal{O}_X(D))$ is a finite-dimensional vector space.

- A point $P \in X$ is a **base point** of a linear system $\mathfrak{d}$ if $P \in \operatorname{Supp} D$ for all $D \in \mathfrak{d}$, where $\operatorname{Supp} D$ means the union of the prime divisors of D.

- By the [25, II, Lemma 7.8], if $\mathfrak{d}$ is a linear system on X corresponding to the subspace $V \subset \Gamma(X, \mathcal{O}_X(D))$, then $\mathfrak{d}$ is **base-point-free** if and only if $\mathfrak{d}$ has no base points, i.e. $\mathcal{O}_X(D)$ is generated by the global sections in V.

Remark 4.3.7. The definition of a linear system comes from *linear series* [30, Section 1.1.B]. Given a line bundle $\mathcal{L}$ on variety or scheme X, the **base locus** of $\mathcal{L}$, $\mathrm{Bs}(\mathcal{L})$, is the set of points at which all the sections in $H^0(X, \mathcal{L})$ vanish. If $\mathrm{Bs}(\mathcal{L})$ is empty, $\mathcal{L}$ is **free** or, equivalently, $\mathcal{L}$ is generated by its global sections or globally generated (see Definition 2.3.22).

To finish this section, we recall the correspondence between morphisms X to $\mathbb{P}^n_k$ and linear systems of dimension n without base points. To give a morphism $X \to \mathbb{P}^n_k$ is equivalent to give a linear system $\mathfrak{d}$ without base points on X, and a set of elements $s_0, \ldots, s_n \in V$, which span the vector space V of $\mathfrak{d}$. The set $s_0, \ldots, s_n$ should be chosen as a basis of V. If we choose a different basis, the corresponding morphism $\varphi_V : X \to \mathbb{P}^n$ defined as $p \mapsto [s_0(p) : \cdots : s_n(p)]$ would only differ by an automorphism of $\mathbb{P}^n$.

In the case that X is a smooth projective (complex) surface, we add other terms. We say that a curve C is a **fixed component** of $\mathfrak{d}$ if every divisor of $\mathfrak{d}$ contains C, i.e. all points of C are base points. The **fixed part** of $\mathfrak{d}$ is the biggest effective divisor F that is contained in every element of $\mathfrak{d}$. Then the linear system $\mathfrak{d} - F$ has no fixed part. Finally, we have the next $1:1$ correspondence (see [6, Section II.6])

$$\left\{ \begin{array}{c} \text{rational maps } \phi : X \dashrightarrow \mathbb{P}^n \\ \text{such that } \phi(X) \text{ is} \\ \text{contained in no hyperplane} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{linear system on } X \\ \text{without fixed part} \\ \text{and with dimension } n \end{array} \right\}.$$

4.4. Ample, nef and big divisors

Let $\mathcal{L}$ be an invertible sheaf (or line bundle) on a noetherian scheme X. We are interested in knowing when a given line bundle $\mathcal{L}$ on X has positivity properties such as ampleness, nefness and bigness. We recall the definitions 2.3.12 and 2.3.22. For the definition of (closed or open) immersion, see [38, Section 26.10] and [39, Def. 9.1.1].

Definition 4.4.1 (Very ample line bundle)**.** Let Y be a scheme. If X is any scheme over Y, an invertible sheaf $\mathcal{L}$ on X is called **very ample relative to** Y if there is a closed immersion $i : X \to \mathbb{P}^N_Y$ for some N, such that $i^*(\mathcal{O}_{\mathbb{P}^N_Y}(1)) \cong \mathcal{L}$.

Remark 4.4.2. If $Y = \operatorname{Spec} A$, the Definition 4.4.1 can be rewritten as follows. Suppose $\pi : X \to \operatorname{Spec} A$ is a morphism, and $\mathcal{L}$ is an invertible sheaf on X. We say that $\mathcal{L}$ is **very ample over** A or π**-very ample**, or **relatively very ample** if $X \cong \operatorname{Proj} S_\bullet$, where $S_\bullet$ is a finitely generated graded ring over A generated in degree 1, and $\mathcal{L} \cong \mathcal{O}_{\operatorname{Proj} S_\bullet}(1)$. This is equivalent to say that the line bundle $\mathcal{L}$ admits a set of global sections $s_0, \ldots, s_n$, with no common zeros, such that the morphism $X \to \mathbb{P}^n_A$ defined as $x \to [s_0(x) : \cdots : s_n(x)]$ is a closed immersion.

One often just says **very ample** if the structure morphism is clear from the context. Note that the existence of a very ample line bundle implies that X is projective.
The property of being generated by global sections is used to define notion of an ample invertible sheaf, which, by [25, II, Thm. 5.17], is more general than very ample definition.

Definition 4.4.3 (Ample line bundle)**.** An invertible sheaf $\mathcal{L}$ on a noetherian scheme X is said to be **ample** if for every coherent sheaf $\mathcal{F}$ on X, there is an integer $n_0 > 0$ (depending on $\mathcal{F}$) such that for every $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections.

Remark 4.4.4.

- By [25, II, Prop. 5.7], if X is a noetherian scheme and $\mathcal{L}$ an invertible sheaf on X, then $\mathcal{L}$ ample $\iff$ $\mathcal{L}^{\otimes m}$ is ample for all $m > 0$ $\iff$ $\mathcal{L}^{\otimes m}$ is ample for some $m > 0$.
- If $\mathcal{L}$ and $\mathcal{M}$ are ample invertible sheaves on noetherian scheme X, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

The next theorem is known as the Cartan-Serre-Grothendieck Theorem. The conclusion in (ii) is often referred to as **Serre's vanishing theorem**.

Theorem 4.4.5. [30, Thm. 1.2.6] Let $\mathcal{L}$ be a line bundle on a complete scheme X. The following are equivalent:

(i) $\mathcal{L}$ is ample.

(ii) Given any coherent sheaf $\mathcal{F}$ on X, there exists a positive integer $m_1 = m_1(\mathcal{F})$ having the property that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0 \quad \text{for all } i > 0, m \geq m_1(\mathcal{F}).$$

(iii) Given any coherent sheaf $\mathcal{F}$ on X, there exists a positive integer $m_2 = m_2(\mathcal{F})$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is generated by its global sections for all $m \geq m_2(\mathcal{F})$.

(iv) There is a positive integer $m_3 > 0$ such that $L^{\otimes m}$ is very ample for every $m > m_3$.

Definition 4.4.6. A Cartier divisor D on X is called

- **nef** if $D \cdot C \geq 0$ for all irreducible curves $C \subset X$;
- **ample** or **very ample** if the corresponding line bundle $\mathcal{O}_X(D)$ is so (see Definition 4.1.9).

Remark 4.4.7.

(i) If X is an irreducible curve, and $\mathcal{L}$ is a line bundle on X, then $\mathcal{L}$ is ample $\iff$ $\deg(\mathcal{L}) > 0$.

(ii) If $X = \mathbb{P}^n_k$, then $\mathcal{O}_X(1)$ is very ample by definition. Moreover, $\mathcal{O}_X(l)$ is ample $\iff$ very ample $\iff l > 0$ by [25, II, Example 7.6.1].

(iii) An ample line bundle $\mathcal{L}$ on a projective variety X has a positive degree on every curve in X.

(iv) If X is a projective variety with $\mathrm{Pic}(X) = \mathbb{Z}$, then any non-zero effective divisor on X is ample [30, Thm. 1.2.6].

(v) If $D_1, \dots, D_n$ are ample divisors on an n-dimensional projective variety X, then $(D_1 \cdot \dots \cdot D_n) > 0$.

Definition 4.4.8 (Amplitude for $\mathbb{Q}$-divisors)**.** Let X be a complete algebraic variety or scheme. A $\mathbb{Q}$-divisor on X is ample if any one of the following three equivalent conditions is satisfied:

(i) D is of the form $D = \sum c_i A_i$ where $c_i > 0$ is a positive rational number and A_i is an ample Cartier divisor.

(ii) There is a positive integer $r > 0$ such that $r \cdot D$ is integral and ample.

(iii) D satisfies the statement of Nakai's criterion for amplitude [30, Thm. 1.2.23], i.e.

$$(D^{\dim V} \cdot V) > 0$$

for every irreducible subvariety $V \subseteq X$ of positive dimension.

An $\mathbb{R}-$divisor D on X is **ample** if the first condition is satisfied with positive real numbers c_i's.

Proposition 4.4.9. [30, Prop. 1.3.7] Let X be a projective variety, H an ample $\mathbb{Q}$-divisor on X, and E be an arbitrary $\mathbb{Q}$-divisor. Then $H + \varepsilon E$ is ample for all sufficiently small rational numbers $0 \leq |\varepsilon| \ll 1$. More generally, given finitely many $\mathbb{Q}$-divisors $E_1, \dots, E_r$ on X,

$$H + \varepsilon_1 E_1 + \cdots + \varepsilon_r E_r$$

is ample for all sufficiently small rational numbers $0 \leq |\varepsilon_i| \ll 1$.

There is two important consequences of the Nakai's criterion which we present in the following.

Proposition 4.4.10. [30, Corollary 1.2.24] If $D_1, D_2 \in \mathrm{Div}(X)$ are numerically equivalent Cartier divisors on a projective variety or scheme X, then D_1 is ample if and only if D_2 is.

Proposition 4.4.11. [30, Corollary 1.2.28] Let $f : Y \longrightarrow X$ be a finite and surjective mapping of projective schemes, and let $\mathcal{L}$ be a line bundle on X. If $f^*\mathcal{L}$ is ample on Y, then $\mathcal{L}$ is ample on X.

The next theorem is a corollary of Kleiman's Theorem [30, Thm. 1.4.9].

Theorem 4.4.12. [30, Thm. 1.4.10] Let X be a projective variety or scheme, and D be a nef $\mathbb{R}$-divisor on X. If H is any ample $\mathbb{R}$-divisor on X, then

$$D + \varepsilon \cdot H$$

is ample for every $\varepsilon > 0$. Conversely, if D and H are any two divisors such that $D + \varepsilon H$ is ample for all sufficiently small $\varepsilon > 0$, then D is nef.

Now we focus on the important case that some power of $\mathcal{L}$ is free. By Theorem 4.4.16 enunciated below, semiample line bundles (or divisors) define semiample fibrations and this result will be very useful in the Section II.7.

Definition 4.4.13 (Semiample line bundles and divisors)**.** A line bundle $\mathcal{L}$ on a complete scheme is **semiample** if $\mathcal{L}^{\otimes m}$ is globally generated for some $m > 0$. A divisor D is **semiample** if the corresponding line bundle is so.

Definition 4.4.14 (Semigroup and exponent of a line bundle)**.** Let $\mathcal{L}$ be a line bundle on the irreducible projective variety X. The **semigroup** of $\mathcal{L}$ consists of those non-negative powers of L that have a non-zero section:

$$\mathbf{N}(\mathcal{L}) = \mathbf{N}(X, \mathcal{L}) = \{m \geq 0 \mid H^0(X, \mathcal{L}^{\otimes m}) \neq 0\}.$$

In particular, $\mathbf{N}(\mathcal{L}) = \{0\}$ if $H^0(X, \mathcal{L}^{\otimes m}) = 0$ for all $m > 0$. Assuming $\mathbf{N}(\mathcal{L}) \neq \{0\}$, we call **exponent** of $\mathcal{L}$ to the number $e_{\mathcal{L}} = \gcd(\mathbf{N}(\mathcal{L})) \geq 1$.

The semigroup $\mathbf{N}(X, D)$ and exponent e_D of a **Cartier divisor** D are defined analogously, or equivalently by passing to $\mathcal{L} = \mathcal{O}_X(D)$.

Fixing a semiample line bundle $\mathcal{L}$, for the purposes of this discussion we denote by $M(X, \mathcal{L}) \subseteq \mathbf{N}(X, \mathcal{L})$ the sub-semigroup

$$M(X, \mathcal{L}) = \{m \in \mathbb{N} \mid \mathcal{L}^{\otimes m} \text{ is free}\}.$$

Definition 4.4.15. An **algebraic fibre space** is a surjective projective mapping $f : X \longrightarrow Y$ of reduced and irreducible varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Theorem 4.4.16. [30, Thm. 2.1.27] Let X be a normal projective variety, and let $\mathcal{L}$ be a **semiample line bundle** on X. Then there is an algebraic fibre space

$$\phi : X \longrightarrow Y$$

having the property that for any sufficiently large integer $m \in M(X, \mathcal{L})$,

$$Y_m = Y \quad \text{and} \quad \phi_m = \phi.$$

Furthermore there is an ample line bundle A on Y such that $\phi^* A = \mathcal{L}^{\otimes g}$, where $g = \gcd(M(X, \mathcal{L}))$.

Remark 4.4.17. In other words, for $m \gg 0$ the mappings ϕ_m stabilize to define a fibre space structure on X (essentially characterized by the fact that $\mathcal{L}^{\otimes g}$ is trivial on the fibres). [30, Lemma 2.1.13] implies that $H^0(X, L^{\otimes kg}) = H^0(Y, A^{\otimes k})$ for all $k \geq 0$, while it follows from [30, Example 2.1.15] that Y is normal.

Theorem 4.4.18 (Zariski–Fujita [30, Remark 2.1.32])**.** Let $\mathcal{L}$ be a line bundle on a projective variety X with the property that the base locus $\mathrm{Bs}(\mathcal{L})$ is a finite set. Then $\mathcal{L}$ is semiample, i.e. $\mathcal{L}^{\otimes m}$ is free for some $m > 0$.

Below, we define a big divisor and give a characterization of bigness for nef divisors. We conclude this section by establishing the Kawamata-Viehweg vanishing theorem for projective variety with rational singularities (for a more general version, see [31, Thm. 9.1.18]).

Definition 4.4.19. Let D be a divisor on X. We denote by $\mathbf{N}(D) = \mathbf{N}(\mathcal{O}_X(D))$ the semigroup of $\mathcal{O}_X(D)$ of those non-negative powers of the line bundle $\mathcal{O}_X(D)$ that have a non-zero section. Then D is **big** if one of the following equivalent conditions holds:

(i) $\max_{m \in \mathbf{N}(D)} \{\dim(\Phi_{|mD|}(X)\} = \dim X$, where $\Phi_{|mD|}$ is the map associated to the linear system $|mD|$ and $\Phi_{|mD|}(X)$ is the image of this map.

(ii) [30, Lemma 2.2.3.] There exist a constant $\alpha > 0$ such that

$$\alpha m^{\dim X} \leq h^0(X, \mathcal{O}_X(mD))$$

for all sufficiently large $m \in \mathbf{N}(D)$.

Theorem 4.4.20. [30, Thm. 2.2.16] Let D be a nef divisor on an irreducible projective variety X of dimension n. Then D is big if and only if its top self-intersection is strictly positive, i.e. $(D^n) > 0$.

Theorem 4.4.21 (Kawamata-Viehweg vanishing theorem [28])**.** Let X be a projective variety over an algebraically closed field of characteristic zero with rational singularities and $\mathcal{L}$ a nef and big line bundle on X. Then

$$H^i(X, \omega_X \otimes \mathcal{L}) = 0 \quad \text{for } i > 0.$$

Example 4.4.22. Let X be a projective variety of dimension n having only rational singularities and D be a big and nef divisor on X. By the Serre Duality Theorem [25, III, Cor. 7.7], $H^i(X, \mathcal{O}(-D)) \cong H^{n-i}(X, \omega_X \otimes \mathcal{O}(D))^*$, then by the above theorem

$$H^i(X, \mathcal{O}_X(-D)) = 0 \quad \text{for } i < n.$$

4.5. The Riemann-Roch theorem

Let X be a smooth projective variety and consider a curve as an integral scheme of dimension 1, proper over $\mathbb{C}$, all of whose local rings are regular, which is necessarily projective [25, II, Prop. 6.7]. We recall our convention that a *surface* will mean nonsingular projective surface.

Definition 4.5.1. For a curve C in projective space, we define the **arithmetic genus** $p_a(C) = 1 - P_C(0)$, where P_C is the Hilbert polynomial of C (i.e. the *Hilbert polynomial* of the homogeneous coordinate ring $S(C)$ in the sense of [25, I, Thm. 7.5]). The **geometric genus** is $p_g(C) = \dim_{\mathbb{C}} \Gamma(X, \omega_C)$, where ω_C is the canonical sheaf on C.

Proposition 4.5.2. [25, IV, Prop. 1.1] If C is a curve, then

$$p_a(C) = p_g(C) = \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C),$$

so we call this number simply the **genus** of X, and denote it by g.

Theorem 4.5.3 (Adjunction Formula)**.** [25, V, Prop. 1.5] If C is a nonsingular curve of genus g on the smooth surface X, and if K is the canonical divisor on X, then

$$2g - 2 = C \cdot (C + K).$$

Definition 4.5.4.

- For any divisor D on the surface X, $l(D) := h^0(D) = |D| + 1$, where $|D|$ is the complete linear system of D.
- For any divisor D on the surface X, the **superabundance** is $s(D) := h^1(D)$.
- The **arithmetic genus** p_a of X is $p_a = \chi(\mathcal{O}_X) - 1$.

- The **arithmetic genus** of a divisor D on X is given by

$$p_a(D) := \frac{1}{2}\left((D + K_X)D\right) + 1 = 1 - \mathcal{O}_D.$$

Theorem 4.5.5 (Riemann-Roch)**.** [25, V, Thm. 1.6] With the notation of Definition 4.5.4, if D is any divisor on the surface X, then

$$l(D) - s(D) + l(K - D) = \frac{1}{2}D \cdot (D - K_X) + 1 + p_a.$$

Definition 4.5.6 ((−1)-curves)**.** A smooth reduced and irreducible curve C with genus $g(C) = 0$ on a surface X such that $C^2 = -1$ is called (−1)**-curve**.

Remark 4.5.7. From Adjunction Formula 4.5.3, we have that every curve C on a complex surface X such that $C^2 < 1$ and $K_X \cdot C = -1$ is a (−1)-curve.

Proposition 4.5.8. [8, Lemma 3.3] Let X be a surface, and let $E = E_1 \cup \cdots \cup E_n$ be a connected curve on X, with pairwise distinct integral components $E_1, \ldots, E_n$. Let $Z = \sum_{i=1}^{n} r_i E_i, r_i \geq 0$, be an effective divisor with support contained in E. the following conditions are equivalent:

(i) $H^1(Z, \mathcal{O}_Z) = 0$;

(ii) for every divisor Z' with $0 < Z' \leq Z$ we have $p_a(Z') \leq 0$.

Proposition 4.5.9. [8, Cor. 3.6] With the hypotheses of Proposition 4.5.8, assume also that $H^1(Z, \mathcal{O}_Z) = 0$. Then every invertible $\mathcal{O}_Z$-module $\mathcal{L}$ such that $d_i = \deg_{E_i}(\mathcal{L}) \geq 0, \forall i$, is generated by its global sections and has $H^1(\mathcal{L}) = 0$.

One of the applications of the Riemann-Roch theorem is the proof of **Hodge Index Theorem** which we enunciate in the following.

Theorem 4.5.10. [25, V, Thm. 1.9] Let H be an ample divisor on a surface X, and suppose that D is a divisor, $D \not\equiv 0$, with $D \cdot H = 0$. Then $D^2 < 0$.

There is a generalization of this theorem (and this is the version we will usually use) for any divisor D with $D^2 > 0$ and, in which case, we conclude that the intersection pairing on $D^\perp$ is negative definite.

Corollary 4.5.11. The signature of the intersection pairing on a surface X is $(1, \rho(X) - 1)$, where $\rho(X)$ is the Picard number.
In particular, for any two divisors D and E on X, if $D^2 > 0$ and $D \cdot E = 0$, then $E^2 \leq 0$ and $E^2 = 0$ if and only if E is numerical trivial.

Proof. It suffices to prove the particular case. Because the Néron-Severi group $N^1(X)_{\mathbb{R}}$ can be decompose into $D \oplus D^{\perp}$. Let A be an ample divisor. If $A \cdot E = 0$, by Theorem 4.5.10, we know that $E^2 \leq 0$. Assume that $A \cdot E \neq 0$. Note that $D \cdot A \neq 0$. Let $F = (A \cdot E)D - (A \cdot D)E$. Note that $A \cdot F = 0$ which implies that

$$F^2 = ((A \cdot E)D - (A \cdot D)E)^2 = (A \cdot E)^2 D^2 + (A \cdot D)^2 E^2 \leq 0.$$

Consequently, $E^2 \leq 0$.

If in addition $E^2 = 0$, it suffices to prove that E is numerically trivial. Assume on the contrary that E is not numerically trivial. There would exist a curve C such that $E \cdot C \neq 0$. Let $G = (D^2)C - (D \cdot C)D$. Then $D(nE + G) = 0$ for any n. Hence,

$$(nE + G)^2 = n^2 E^2 + 2nE \cdot G + G^2 = 2nE \cdot G + G^2 \leq 0$$

If $E \cdot C > 0$, then $E \cdot G = D^2 E \cdot C > 0$ and $2nE \cdot G + G^2 > 0$ for a sufficiently large n. That's a contradiction. If $E \cdot C < 0$, then $2nE \cdot G + G^2 > 0$ for a sufficiently large $-n$. Again, there is a contradiction.

Therefore, if $D \cdot E = 0$ and $E^2 = 0$, then E must be numerically trivial. □

The next theorem is a version of the Hodge Index Theorem in higher dimensions.

Theorem 4.5.12. [30, Thm. 1.6.1] Let X be a smooth projective variety of dimension n and $D_1, \ldots, D_k$ be nef divisors on X. If $n_1 + \cdots + n_k = n \geq 2$ and $n_i \geq 0$ for all i, then

$$(D_1 \cdots D_k)^n \geq (D_1^n)^{n_1} \cdots (D_k^n)^{n_k}.$$

5. The Pseudoeffective Cone and the Mori Cone

Let X be a smooth complex projective variety. In this section, we define the pseudoeffective cone in the vector space $N^1(X)_{\mathbb{R}}$ and the Mori cone in $N_1(X)_{\mathbb{R}}$.

Definition 5.0.1. A subset W of a vector space V is called a **cone** if $0 \in W$ and for every $x \in W$ λx belongs to W, for all $\lambda > 0$. A cone W is called **convex** if $0 \in W$ and if for any two points $x, y \in W$ and any two numbers $\alpha, \beta \geq 0$, the point $z = \alpha x + \beta y$ is also in W.

Definition 5.0.2. A closed and convex subcone $F \subseteq W$ is called **extremal face** of W if for all $u, v \in W$ such that $u + v \in F$, then necessarily $u, v \in F$. A $1-$dimensional extremal face is called an **extremal ray** which is contained in the boundary of W.

Definition 5.0.3. Let X be a smooth complex projective variety. We define the next convex cones of $N^1(X)_{\mathbb{R}}$:

- the **effective cone** of X, denoted by $\mathrm{Eff}(X)$, generated by the classes of all effective $\mathbb{R}$-divisors on X;
- the **pseudoeffective cone**, $\overline{\mathrm{Eff}}(X)$, is the closure in the Euclidean topology of $\mathrm{Eff}(X)$;
- the **ample cone**, $\mathrm{Amp}(X)$, is the cone of all ample $\mathbb{R}-$divisor classes on X;
- the **nef cone** of X, $\mathrm{Nef}(X)$, is the cone of all nef $\mathbb{R}$-divisors classes of X.

Remark 5.0.4.

- The ample cone can be defined as the cone spanned by the classes of all ample integral (or rational) divisors.
- The statement of Proposition 4.4.9 remains valid for $\mathbb{R}$-divisors. This implies that $\mathrm{Amp}(X)$ is a open cone. Moreover, by [30, Thm. 1.4.23], we have that $\overline{\mathrm{Amp}}(X) = \mathrm{Nef}(X)$.

Definition 5.0.5. For a curve $C \subset X$, we denote $[C]$ for its class in $N_1(X)_{\mathbb{R}}$. The **cone of curves**, denoted by $\mathrm{NE}(X)$, is the convex cone spanned by the classes in $N_1(X)_{\mathbb{R}}$ of all effective one-cycles on X. Concretely,

$$\mathrm{NE}(X) = \left\{ \sum a_i [C_i] : C_i \subset X \text{ an irreducible curve, } a_i \geq 0 \right\}.$$

The closure $\overline{\mathrm{NE}}(X) \subseteq N_1(X)_{\mathbb{R}}$ in the Euclidean topology is the **Mori cone** of X.

There is a duality relation between the nef cone and the Mori cone via the perfect pairing $N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$ defined in 4.2.12. By Remark 5.0.4, $\mathrm{Nef}(X)$ is a closed cone, moreover its dual $(\mathrm{Nef}(X))^*$ is known to be the Mori cone $\overline{\mathrm{NE}}(X)$.

The dual of the effective cone is the cone of *moveable curves*. The classes of big and ample divisors of X also define convex cones in $N^1(X)_{\mathbb{R}}$, but this thesis will not further explore into these cones, nor into the cone of moveable curves.

5.1. Cone Theorem for surfaces

The **Cone Theorem** is one of the central theorems of the *Mori's Program* or *Minimal Model Program* (MMP), which is part of the birational classification of algebraic varieties. We state this result as in [30, Thm. 1.5.33] adapted for surfaces. For an introduction of MMP and full proofs, see [27, Chapters 1, 3] and [32].

Let X be a smooth projective surface. Note that a one-cycle in X is the same thing as a divisor on X, then $N^1(X)_{\mathbb{R}} \simeq N_1(X)_{\mathbb{R}}$. In particular, $\overline{\mathrm{Eff}}(X) \simeq \overline{\mathrm{NE}}(X)$ and that is why some authors call $\overline{\mathrm{Eff}}(X)$ the Mori cone of X. Let K_X be the canonical divisor of X.

Remark 5.1.1. If X is a smooth projective surface $\mathrm{Nef}(X) \subseteq \overline{\mathrm{NE}}(X)$. Note that $\mathrm{Amp}(X) \subseteq \mathrm{NE}(X)$ and then it suffices to take the respective closures and conclude by 5.0.4. The equality holds if and only if $(C^2) \geq 0$ for every irreducible curve $C \subset X$.

Before stating the cone theorem, we establish the notation and some necessary definitions.

Notation 5.1.2. Given an irreducible curve C on X, we write

- $C_{\geq 0} = \{\delta \in N^1(X) : \delta \cdot [C] \geq 0\}$;
- $C^{\perp} = \{\delta \in N^1(X)_{\mathbb{R}} : \delta \cdot [C] = 0\}$;
- $\overline{\mathrm{NE}}(X)_{C\geq 0} = \overline{\mathrm{NE}}(X) \cap [C]_{\geq 0}$.

and similarly in the cases $> 0, \leq 0$ and < 0. Analogously, we adopt the notation $D_{\geq 0}$ $(> 0, \leq 0$ and $< 0)$ for a divisor class D on X.

Definition 5.1.3. A **ray** in $N_1(X)_{\mathbb{R}}$ is the set of all non-negative real multiples of a non-zero vector in $N_1(X)_{\mathbb{R}}$. If C is not numerically equivalent to 0, the set $R(C) := \mathbb{R}_{\geq 0}[C]$ is called **ray generated by** C. If C is a $(-1)-$curve, the ray generated by C is called $(-1)-$**ray**. If K_X is the canonical divisor of X, the ray generated by $-K_X$ is called **anticanonical ray**.

Theorem 5.1.4 (Cone Theorem)**.** Assume that K_X fails to be nef.

1. There are countably many rational curves $C_i \subset X$, with $-3 \leq C_i \cdot K_X \leq 0$, such that
$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i R(C_i)$$

2. Given an ample divisor H and an $\epsilon > 0$ there are only finitely many C_i such that $C_i \cdot (K_X + \epsilon H) \leq 0$.

Remark 5.1.5. By [32, Thm. 8-1-3], every extremal ray R of $\overline{\mathrm{NE}}(X)_{K_X<0}$ is associated to an *extremal contraction* [32, Def. 8-1-1], that is a proper morphism $X \to Y$ with connected fibers, with Y being normal and projective, that precisely contracts the curves on X whose classes belong to R.

6. Blow-up of an affine and projective variety

We are interested in the blow-up of a smooth complex projective variety along a finite set of points, but in this section, based on [21, Chapter 9] and [6, Chapter II], we also consider blow-ups along more large sets. For a definition of a blow-up of a scheme along a closed subscheme, see [39, Chapter 22].

6.1. Basic definitions

Construction 6.1.1 (Blowing up of an affine variety)**.** Let $X \subseteq \mathbb{A}^n$ be an affine variety. For some given polynomial functions $f_1, \ldots, f_r \in A(X)$ on X, we set $U = X - Z(f_1, \ldots, f_r)$. As $f_1, \ldots, f_r$ do not vanish simultaneously at any point of U, there is a well-defined morphism

$$f : U \to \mathbb{P}^{r-1}, \quad x \mapsto (f_1(x) : \cdots : f_r(x)).$$

We consider the graph of f $\Gamma_f = \{(x, f(x)) : x \in U\} \subseteq U \times \mathbb{P}^{r-1}$. It is closed in $U \times \mathbb{P}^{r-1}$ (by [21, Prop. 5.20 (a)]), but in general not in $X \times \mathbb{P}^{r-1}$. Denote by $\widetilde{X}$ the closure of Γ_f in $X \times \mathbb{P}^{r-1}$, then we have the next commutative map

$$\begin{array}{ccc} \widetilde{X} & \hookrightarrow & X \times \mathbb{P}^{r-1}_k \\ & \overset{\pi}{\searrow} & \downarrow \\ & & X \end{array}$$

The variety $\widetilde{X}$ is called the **blow-up** of X at $f_1, \ldots, f_r$. Note that there is a natural projection morphism $\pi : \widetilde{X} \to X$ to the first factor. Sometimes we will also say that this morphism π is the blow-up of X at $f_1, \ldots, f_r$. In this case, by [21, Lemma 9.14] it satisfies

$$\widetilde{X} \subseteq \{(x, y) \in X \times \mathbb{P}^{r-1} : y_i f_j(x) = y_j f_i(x) \text{ for all } i, j = 1, \ldots, r\}.$$

In the Construction 6.1.1, the graph Γ_f is isomorphic to U, with isomorphism $\pi|_{\Gamma_f} : \Gamma_f \to U$. By abuse of notation, one often uses this isomorphism to identify Γ_f with U, so that U becomes a dense open subset of $\widetilde{X}$.

Construction 6.1.2 (Generalizations of the blow-up construction)**.**

- For an ideal $J \subseteq A(X)$ we define the blow-up of X at J to be the blow-up of X at any set of generators of J, which is well-defined up to isomorphisms. If $Y \subset X$ is a closed subvariety the blow-up of X at $I(Y) \subseteq A(X)$ will also be called the blow-up of X at Y.

- Let X be a projective variety. If $f_1, \ldots, f_r \in S(X)$ are homogeneous of the same degree, the blow-up of X at $f_1, \ldots, f_r$ is defined as the closure of the graph

$$\Gamma = \{(x, (f_1(x) : \cdots : f_r(x))) : x \in U\} \subseteq U \times \mathbb{P}^{r-1}$$

(for $U = X - Z(f_1, \ldots, f_r)$) in $X \times \mathbb{P}^{r-1}$; by the Segre embedding [25, I, Exercise 2.14] it is again a projective variety.

Definition 6.1.3 (Exceptional sets)**.** The complement $\widetilde{X} - U = \pi^{-1}(Z(f_1, \ldots, f_r))$, on which π is usually not an isomorphism, is called the **exceptional set** of the blow-up.

Remark 6.1.4. If X is irreducible and $f_1, \ldots, f_r$ do not vanish simultaneously on all of X, then $U = X - Z(f_1, \ldots, f_r)$ is a non-empty and hence dense open subset of X. So its closure in the blow-up, which is all of $\widetilde{X}$ by definition, is also irreducible. We therefore conclude that X and $\widetilde{X}$ are birational in this case, with common dense open subset U.

Definition 6.1.5 (Strict transform)**.** Let Y be a closed subvariety of X. Then we can blow up Y at $f_1, \ldots, f_r$ as well. By construction, the resulting space $\widetilde{Y} \subseteq Y \times \mathbb{P}^{r-1} \subset X \times \mathbb{P}^{r-1}$ is then also a closed subvariety of $\widetilde{X}$. The subset $\widetilde{Y}$ of $\widetilde{X}$ is often called the **strict transform** of Y in the blow-up of X.

In particular, if $X = X_1 \cup \cdots \cup X_m$ is the irreducible decomposition of X then $\widetilde{X}_i \subset \widetilde{X}$ for $i = 1, \ldots, m$. Moreover, since taking closures commutes with finite unions it is immediate from Construction 6.1.1 that

$$\widetilde{X} = \widetilde{X}_1 \cup \cdots \cup \widetilde{X}_m,$$

i.e. that for blowing up X we just blow up its irreducible components individually. For many purposes it therefore suffices to consider blow-ups of irreducible varieties.

Example 6.1.6 (Blow-up of $\mathbb{A}^n$ at one point)**.** We consider $\mathbb{A}^n$ blown-up at the origin $O = (0, \ldots, 0)$ which is equivalent to blown-up at the coordinates functions $x_1, \ldots, x_n$. It can be shown that

$$\widetilde{\mathbb{A}^n} = \{(x, y) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : y_i x_j(x) = y_j x_i(x) \text{ for all } i, j = 1, \ldots, n\}.$$

By these equations, the morphism $\pi : \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$ is defined in $U_1 = \{(x, y) \in Y : y_1 \neq 0\}$ by

$$((x_1, \ldots, x_n), [y_1 : \cdots : y_n]) \mapsto \left(x_1, \frac{x_1 y_2}{y_1}, \ldots, \frac{x_1 y_n}{y_1}\right)$$

Similarly, π can be defined in every open set $U_i = \{(x, y) \in Y : y_i \neq 0\}$. Note that π is an isomorphism on $U = \mathbb{A}^n - \{0\}$ and the exceptional set of π is

$$\pi^{-1}(0) = \{(0, y) \in \mathbb{A}^n \times \mathbb{P}^{n-1}\} \cong \mathbb{P}^{n-1}.$$

Example 6.1.7 (Blow-up of $\mathbb{P}^2$ at one point.)**.** Let $p = [0:0:1]$ and the map $f : U = \mathbb{P}^2 - \{p\} \to \mathbb{P}^1$ defined by $[x_1 : x_2 : x_3] \mapsto [x_1 : x_2]$. Note that

$$\Gamma_f = \{(x, f(x)) : x \in U\} \subseteq \{[x_1 : x_2 : x_3], [y_1 : y_2]) : y_1 x_2 = y_2 x_1\} := Y \subseteq \mathbb{P}^2 \times \mathbb{P}^1.$$

We prove that closure of Γ_f is Y. Consider $U_i = \{(x, y) \in Y : y_j = 1\}$. The map $\mathbb{A}^2 \to U_1$ defined by

$$((x, z) \mapsto ([x : xz : 1], [1 : z])$$

is a isomorphism. By a similar way we can define a isomorphism $\mathbb{A}^2 \to U_2$. Whereas $\dim U_1 = \dim U_2 = 2$ and since $([1:1:1],[1:1]) \in U_1 \cap U_2$, it follows that Y is irreducible of dimension 2, which is the dimension of Y. The exceptional set of Y is

$$Y - \Gamma_f = \{([0:0:1], [x_1, x_2]) \in \mathbb{P}^2 \times \mathbb{P}^1\}.$$

Let's discuss this example in more detail. To do this, we will use the following results.

Lemma 6.1.8 ([6, Lemma II.2])**.** Let $\pi : \widetilde{X} \to X$ be the blow-up of a point p, and consider an irreducible curve C on S that passes through p with multiplicity m. The closure of $\pi^{-1}(C - \{p\})$ in $\widetilde{X}$ is an irreducible curve $\widetilde{C}$ on $\widetilde{X}$, called **strict transform** of C, and

$$\pi^* C = \widetilde{C} + mE.$$

Proposition 6.1.9 ([6, Prop. II.3])**.** Let X be a surface, $\pi : \widetilde{X} \to X$ the blow-up of a point $p \in X$, $E \subset \widetilde{X}$ the exceptional curve and let K_X and $K_{\widetilde{X}}$ the canonical divisor of X and $\widetilde{X}$, respectively.

(i) There is an isomorphism $\mathrm{Pic}(X) \oplus \mathbb{Z} \to \mathrm{Pic}(\widetilde{X})$ defined by $(D, n) \mapsto \pi^* D + nE$.

(ii) Let D, D' be divisors on X. Then $(\pi^* D).(\pi^* D') = D.D'$, $E.(\pi^* D) = 0$, $E^2 = -1$.

(iii) $N^1(\widetilde{X}) \cong N^1(X) \oplus \mathbb{Z} \cdot E$.

(iv) $K_{\widetilde{X}} = \pi^* K_X + E$.

Remark 6.1.10. By the Castelnuovo's contractibility criterion [6, Thm. II.17], given a curve E on a surface X, if E is isomorphic to $\mathbb{P}^1$ and $E^2 = -1$, then E is an exceptional curve of X (i.e. E is the exceptional curve of a blow-up $\pi : X \to X'$ with X' a smooth surface).

Example 6.1.11. Let $\pi : Y \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at s points.

- If $s = 1$, $\mathrm{Pic}(Y)$ is generated by the class of a hyperplane H, which is the pullback of a line that does not contain p, and the class of the exceptional divisor E, with $H \cdot H = 1$, $H \cdot E = 0$ and $E \cdot E = -1$, which is also the only one (-1)-curve of Y.

- Recall that $K_{\mathbb{P}^2} = -3H$. For $s = 1$, from Proposition 6.1.9 (iv), we have $K_Y = \pi^*(-3H)+E = -3H+E$. If $s > 1$ and E_i is the exceptional divisor at the $i-$point, repeating the previous argument s times, we obtain that $K_Y = -3H+E_1+\cdots+E_s$.

The example of the blow-up of $\mathbb{P}^2$ at a point p is the simplest of all, and from it we can deduce the blow-up of $\mathbb{P}^2$ at more points. One way to visualize the smooth surfaces obtained from this process is detailed in the paper *Interactive visualizations of blowups of the plane* [35]. If we consider the real affine plane, via the blow-up $\pi : \widetilde{\mathbb{A}^2} \to \mathbb{A}^2$, all the lines in the plane that pass through p with a slope M can be thought of as the projection of the lines that intersect the vertical axis at p at a height M. Since the blow-up is a smooth closed surface, it can be thought of as a Möbius strip around p, as in the Figure 1 of [35].

6.2. Points in very general position

Let $p_1, \ldots, p_s$ be s distinct points in $\mathbb{P}^2$ and denote by $\mathscr{L}_d(m_1, ..., m_s)$, with $d > 0$, the linear system of plane curves of degree d passing through these points with multiplicity at least m_i for each p_i. It is a classical multivariate polynomial interpolation problem to calculate the dimension of this linear system.

We provide a more rigorous definition in the following. Let $S := \mathbb{C}[x_0, x_1, x_2]$ and denote by S_d the subspace of homogeneous polynomials of degree d. Given a subset of points $p_1, \ldots, p_s \in \mathbb{P}^2$ and positive integers $m_1, \ldots, m_s$ denote by

$$\mathscr{L}_d(m_1, \ldots, m_s) := \mathbb{P}(\{f \in S_d : \operatorname{mult}_{p_i}(f) \geq m_i \text{ for any } i\})$$

the projectivization of the vector subspace $V_d(m_1, \ldots, m_s) \subseteq S_d$ consisting of polynomials which vanish with multiplicity at least m_i at p_i for each i.

Definition 6.2.1. The **dimension** of $\mathscr{L} = \mathscr{L}_d(m_1, \ldots, m_s)$ is

$$\dim(\mathscr{L}) = \dim_{\mathbb{C}}\langle f \in I_{p_1}^{m_1} \cap \cdots \cap I_{p_r}^{m_r} \,|\, \deg(f) = d\rangle - 1,$$

where $I_{p_i}^{m_i}$ is the m_i power of the maximal homogeneous ideal defining p_i.

The defining polynomial of a plane curve of degree d consists of $\binom{d+2}{2} - 1$ monomials and $\binom{m_i+2}{2}$ of these are monomials whose vanishing order in p_i is less than m_i. Then, by the condition of *multiplicity at least* m_i at each p_i, we have the next definition.

Definition 6.2.2. The **virtual dimension** of $\mathscr{L} = \mathscr{L}_d(m_1, \ldots, m_s)$ is

$$v(\mathscr{L}) := \binom{d+2}{2} - \sum_{i=1}^{s} \binom{m_i+2}{2} - 1$$

and its **expected dimension** $e(\mathscr{L}) := \max\{v(\mathscr{L}), -1\}$. $\mathscr{L}$ is called **special** if $\dim \mathscr{L} > e(\mathscr{L})$.

Remark 6.2.3.

- In general, $\dim \mathscr{L} \geq v(\mathscr{L})$. Note that $v(\mathscr{L}) = \dim \mathscr{L}$ only if the linear equations on the coefficients of a general curve in $\mathscr{L}$ are independent.
- If $d \geq \sum_{i=1}^{s} m_i - 1$ then $v(\mathscr{L}) = \dim \mathscr{L}$, which can be proved by putting all the points on a line.

The dimension $\dim \mathscr{L}_d(m_1, \ldots, m_s)$ depends on the position of the points p_i. For example, consider the points $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$ and $p_3 = [1 : 1 : 0]$, then

$$\begin{aligned}
\dim \mathscr{L}_1(1,1,1) &= \dim_{\mathbb{C}}\langle x_2\rangle - 1 = 0,\\
\dim \mathscr{L}_2(1,1,1) &= \dim_{\mathbb{C}}\langle x_0x_1, x_0x_2, x_2^2\rangle - 1 = 2,\\
\dim \mathscr{L}_2(2,1,1) &= \dim_{\mathbb{C}}\langle x_2^2, x_1x_2, x_0x_2\rangle - 1 = 2,\\
\dim \mathscr{L}_3(1,1,1) &= \dim_{\mathbb{C}}\langle x_2^3, x_0x_1x_2, x_0^2x_2\rangle - 1 = 2,\\
\dim \mathscr{L}_3(2,1,1) &= \dim_{\mathbb{C}}\langle x_2^3, x_0x_1x_2, x_0x_1^2, x_0x_2^2, x_0^2x_2\rangle - 1 = 4,\\
\dim \mathscr{L}_3(2,2,1) &= \dim_{\mathbb{C}}(\langle x_0^2x_1, x_0x_1^2\rangle \cup \mathscr{L}_3(2,1,1)) - 1 = 6.
\end{aligned}$$

It is straightforward that $\dim \mathscr{L}_d(m_1, m_2, m_3) \geq 2$ for all $d \geq 3$ and $m_i \geq 1$.

On the other hand, if we consider three non-collinear points we obtain for example $\dim \mathscr{L}_1(1,1,1) = 0$ and $\dim \mathscr{L}_d(1,1,1) = 2$, and for all $d \geq 2$. It is clear that

$$\dim \mathscr{L}_d(m_1, m_2, m_3) \geq 2 \quad \text{for all } d \geq 2,\ m_1, m_2, m_3 \geq 1.$$

Moreover, the dimension of the linear system $\mathscr{L}_d(m_1, m_2, m_3)$ at three non-collinear points is *minimal* for all $d, m_i \geq 1$.

Let $p_1, \ldots, p_r$ be distinct points of $\mathbb{P}^n$ and let $m \in \mathbb{N}^r$. Consider the Hilbert scheme $(\mathbb{P}^n)^{[r]}$ parametrizing r-tuples of points in $\mathbb{P}^n$. Let $\mathcal{P} \in (\mathbb{P}^n)^{[r]}$ be the point corresponding to the p_i's, see [20, §5] for the definition of Hilbert scheme.

Denote by $\mathcal{H}(d, m, \mathcal{P})$ the vector space of degree d homogeneous polynomials of $\mathbb{C}[x_0, \ldots, x_n]$ with multiplicity at least m_i at each p_i. Observe that $\dim \mathcal{H}(d, m, \mathcal{P})$ depends on $\mathcal{P}$ and that there is an open Zariski subset $\mathcal{U}(d, m) \subseteq (\mathbb{P}^n)^{[r]}$ where this dimension attains its minimal value. Let us denote by

$$\mathcal{U} := \bigcap_{(d,m)\in\mathbb{N}^{r+1}} \mathcal{U}(d, m),$$

which is the complement of a countable union of Zariski closed subspaces of the configuration space.

Definition 6.2.4. A set of points $p_1, \dots, p_r \in \mathbb{P}^n$ are in **very general position** if the corresponding $\mathcal{P}$ is in $\mathcal{U}$, this implies that three of these points are not collinear, no six lie on a conic and no cubic contains nine of this points, no eight on a cubic with a double point at one of them, and so on.

Remark 6.2.5. Consider $\pi : X_s \to \mathbb{P}^2$ the blow-up of $\mathbb{P}^2$ at the points $p_1, \dots, p_s$, let H be the class of the pullback of line in the plane and E_i be the class of the exceptional over the $i-$point.

(i) The strict transform of an element of the linear system $\mathscr{L} = \mathscr{L}_d(m_1, \dots, m_s)$ is an element of the linear system $|dH - m_1E_1 - \cdots - m_sE_s|$ of X_s. In particular the two linear systems have the same dimension:

$$\dim(\mathscr{L}) = \dim |dH - m_1E_1 - \cdots - m_sE_s|.$$

(ii) From the Proposition 6.1.9 (ii), the blow-up of $\mathbb{P}^2$ contains (-1)-curves, and if the points which are blown up are not in a very general position, X_s may contain rational curves with self-intersection less than -1. Otherwise, we will see in the Corollary II.6.3 that X_s cannot have (-2)-curves.

(iii) It is a long-standing problem to classify all special linear systems of the form of $\mathscr{L}$. One way speciality of linear systems can arise, is if they have as a multiple base curve a (-1)-curve. For example, the set of the conics passing double through two points, $\mathscr{L}_2(2, 2)$, has virtual dimension -1, but if $f \in S_1$ is the polynomial of a line through the two points, then $[f^2] \in \mathscr{L}$, thus $\mathscr{L}$ is special. The *Harbourne-Hirschowitz conjecture* says that this is the only way speciality can arise (see [9, Conjecture 3.1.] and the Conjecture II.4.3).

II. Equivalent Conjectures

In this chapter, the first four sections detail the motivations for the research, which is based on the papers [9], [11], [29] and [12]. The theorem II.5.2 provide a characterization of asymptotical speciality of a nef and big divisor D on an algebraic surface in terms of the arithmetic genus of curves in $D^{\perp}$ and as a consequence we prove that the SHGH conjecture for linear systems on X_s is equivalent to the fact that each nef class is non-special (Theorem II.6.6). Finally, in Section II.7 we prove that if $r < 2^n$ then any nef divisor of the blowing-up of the n-dimensional projective space at s points in very general position is asymptotically non-special.

The Chapter I of Preliminaries aims to cover most of the results needed for the last three sections; however, some results on anticanonical rational surfaces from [23], spectral sequences and elliptic fibrations are also required (see [26, Section 3.6] and [3, Chapter V, section 7]).

II.1. Notation

Notation II.1.1. Let $\pi : X_s \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at $p_1, \ldots, p_s$ points in very general position, with exceptional divisors $E_1, \ldots, E_s$ and let H be the pullback of a line.

- The Picard Group of X_s is $\text{Pic}(X_s) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E_1 \oplus \cdots \oplus \mathbb{Z} \cdot E_s$.
- From remark 4.2.10, we consider the real vector space $N^1(X)_{\mathbb{R}} = \text{Pic}(X_s) \otimes_{\mathbb{Z}} \mathbb{R}$ of dimension $s + 1$.
- Denote by K_s the canonical divisor of X_s where $K_s := -3H + \sum_{i=1}^{s} E_i$.
- The **effective cone** of X_s, $\text{Eff}(X_s) \subset \text{Pic}(X_s)$, is the cone generated by the classes of effective divisors of X_s.
- The **pseudoeffective cone** of X_s is the closure $\overline{\text{Eff}}(X_s)$.
- The **nef cone**, $\text{Nef}(X_s)$, generated by the classes of nef divisors of X_s.

- The **positive light cone** defined as the quadric

$$Q(X_s) := \{D \in \mathrm{Pic}(X_s)_{\mathbb{R}} \,:\, D^2 \geq 0 \text{ and } D \cdot H \geq 0\}.$$

- The **boundary of the light cone** of X_s, $\partial Q(X_s) = \{D \in Q(X_s) \,:\, D^2 = 0\}$.

Remark II.1.2.

- Because X_s is a surface, $\overline{\mathrm{Eff}}(X_s)$ coincides with the **Mori cone** of X_s, $\overline{\mathrm{NE}}(X_s)$, defined in Section 5 of Preliminaries. We will use both notations and both names to refer to the same cone.

- Abusing notation, we will sometimes refer to the classes of $\mathrm{Pic}(X_s)$ simply as divisors.

By Riemann-Roch's theorem one has $Q(X_s) \subseteq \overline{\mathrm{NE}}(X_s)$ and because $Q(X_s)$ is the dual space of itself, we have $\mathrm{Nef}(X_s) \subseteq Q(X_s)$.

By Hodge Index Theorem 4.5.10, $Q(X_s)$ is defined in suitable coordinates by the conditions $u_0^2 \geq u_1^2 + \cdots + u_s^2$ and $u_0 \geq 0$. Then, if $s \geq 2$, $Q(X_s)$ is circular.

Given a divisor $D \in \mathrm{Pic}(X_s)$, from the remark 6.2.5, we define the virtual and expected dimension of a linear system $|D|$ analogously to the Definition 6.2.2.

Definition II.1.3. Given a class divisor $D = dH - \sum_{i=1}^{s} m_i E_i$ on X_s, the **virtual dimension** of the linear system $|D|$ with

$$v(D) := \binom{d+2}{2} - \sum_{i=1}^{s} \binom{m_i+1}{2} - 1$$

and its **expected dimension** is $e(D) := \max\{v(D), -1\}$.

The divisor D is **special** if $\dim |D| = e(D)$. The divisor D is **non-special** if $\dim |D| = e(D)$ or equivalently if $h^0(X_s, \mathcal{O}_{X_s}(D)) \cdot h^1(X_s, \mathcal{O}_{X_s}(D)) = 0$.

Remark II.1.4.

- For simplicity, given a $D \in \mathrm{Pic}(X_s)$, we will refer to the (virtual, expected) dimension of the linear system $|D|$ as the dimension of D.

- It is straightforward to show that that the virtual dimension of a $D = dH - \sum_{i=1}^{n} m_i E_i$ is $v(D) = \frac{1}{2}(D^2 - D \cdot K_s)$.

- The Definition II.1.3 can be generalized to divisors on the blow up of $\mathbb{P}^n$ at s points in very general position (see II.7.1).

II.2. Weil Group and standard classes

We introduce the definition of standard divisor, which we will see in the sections II.4 and II.5 play an important rule in the formulation of conjectures for special divisors on X_s.

Definition II.2.1. Let $W(X_s)$ be the subgroup of isometries of $\text{Pic}(X_s)$ generated by the reflections

$$\sigma_F : D \mapsto D + (D \cdot F)F,$$

where F is one of the **fundamental roots** (classes of self-intersection -2): $E_1 - E_2, \ldots, E_{s-1} - E_s, H - E_1 - E_2 - E_3$.

It is not difficult to show that any such reflection preserves the nef cone, Mori cone and positive light cone of X_s.

Observe that $W(X_s)$ contains all the permutations of the s exceptional divisors. When the first three points are the fundamental ones of the projective plane, then the reflection by the last root is the map induced by the quadratic Cremona transformation

$$[x_0 : x_1 : x_2] \mapsto [x_0^{-1} : x_1^{-1} : x_2^{-1}].$$

The choice of a distinguished representative for an orbit of $W(X_s)$ contained in the effective cone leads to the following definition.

Definition II.2.2. A class $D := dH - \sum_{i=1}^{s} m_i E_i \in N^1(X_s)_{\mathbb{R}}$ is **pseudostandard** if

$$d \geq m_1 + m_2 + m_3 \quad \text{and} \quad m_1 \geq \cdots \geq m_s.$$

If in addition $m_s \geq 0$ then the class D is **standard**.

The first inequality in the definition is equivalent to $D \cdot (H - E_1 - E_2 - E_3) \geq 0$, while the ordering of the multiplicities comes from $D \cdot (E_i - E_{i+1}) \geq 0$.

Remark II.2.3.

- By [1, Thm. 5.7.3], $W(X_s)$ is infinite group if $s \geq 9$. In the paper [12], the authors define the *Cremona-Kantor group* (i.e. group generated by quadratic transformations on P^2 whose action is on linear systems of the form $\mathcal{L}_d(m_1, \ldots, m_s)$ and which coincides with $W(X_s)$) and use it to generate a sequence of rational rays of $\text{Nef}(X_s)$ whose limit is an irrational ray extremal to the Mori cone of X_s.
- Every divisor class of $N^1(X_s)_{\mathbb{R}}$ on the same $W(X_s)$-orbit have the same expected, virtual and true dimension.

Remark II.2.4. Let $\pi\colon X_s^n \to \mathbb{P}^n$ be the blowing-up of the projective space at s points in very general position with exceptional divisors $E_1, \dots, E_s$ and let H be the pullback of a hyperplane. The definitions of Weil group, pseudostandard and standard can be extend to $\mathrm{Pic}(X_s^n)_{\mathbb{R}}$ and elements of $W(X_s^n)$ are related to some birational maps of X_s^n. This is further detailed in the paper [29], and also we have following properties

(i) If $D = dH - \sum_{i=1}^{s} m_i E_i \in \mathrm{Pic}(X_s^n)$ with $d > 0$ and $m_i \geq 0$, then $h^i(D) = 0$ for any $i > 1$ [29, Prop. 1.2].

(ii) If $D = dH - \sum_{i=1}^{s} m_i E_i \in \mathrm{Pic}(X_s^n)$ with $d > 0$ and $m_i < 0$ for some i, then E_i is contained in the base locus of $|D|$ [29, Remark 1.3].

(iii) If $D \in \mathrm{Pic}(X_s^n)$ is a effective class then there exists a isometry $w \in W(X_s)$ such that $w(D)$ is pseudostandard [29, Prop. 3.2]. In the Appendix Item 1 we present the function `StdForm` in Magma software which transforms an effective divisor into its pseudostandard form.

(iv) If $D \in N^1(X_s)_{\mathbb{R}}$ is a standard then $w(D) \cdot E \geq 0$ for any (-1)-curve E and any $w \in W(X_s)$ [29, Thm. 4.4].

II.3. Description of the Mori cone of X_s

A well-studied class of surface are the (complex) **del Pezzo surfaces** defined as smooth projective complex surfaces with ample anticanonical line bundle. The degree of a del Pezzo surface is the self-intersection of its canonical divisor. The possible degrees are $1 \leq d \leq 9$, in which case we denote the surface S_d. For $d \neq 8$, S_d is the blow-up of $\mathbb{P}^2$ at $9-d$ points in very general position while S_8 is either isomorphic to X_1 or $\mathbb{P}^1 \times \mathbb{P}^1$.

For $s \leq 6$, the anticanonical divisor of X_s, $-K_s$,is very ample and defines an embedding $\varphi : X_s \hookrightarrow \mathbb{P}^{9-s}$. The linear system of cubics through $p_1, \ldots, p_s$ is the linear system $|-K_s|$ and by [6, Prop. IV.9], S_{10-s} is the image of X_s via φ.

If $s \leq 8$ X_s contains a finite number of (-1)-curves, which is the number of lines on S_{9-s} [1]: X_1 has just one (-1)-curve, X_2 has 3 (i.e. E_1, E_2 and the proper transform of the line joining the two points, $H - E_1 - E_2$), X_3 has $6 = 3 + \binom{3}{2}$, and so on until $s = 8$, where S_8 has 240 $(-1)-$curves. For $s \geq 9$, X_s has infinitely many (-1)-curves because the group $W(X_s)$ has infinite order for $s \geq 9$ which implies that its action on any E_i generates an infinite orbit.

The description of $\overline{\text{NE}}(X_s)$ will depend on the value of s as we will see below (we use the notation 5.1.2 of Preliminaries). First, note that if $s \geq 2$, $Q(X_s)$ is circular and every point on the boundary of $Q(X_s)$ generates an extremal ray, so there cannot be countably many of these rays. Then the curves $C_i \subset X_s$ of the Cone Theorem 5.1.4 must have negative self-intersection. Second, from the Genus Formula 4.5.3, since every C_i has genus 0, we obtain $C_i \cdot K_s = C_i^2 = -1$, i.e. C_i are (-1)-curves.

(i) The **Del Pezzo surface** X_s $(s \leq 8)$: in this case the anticanonical class $-K_s$ is ample because $K_s^2 = 9 - s > 0$ and then

$$\overline{\text{NE}}(X_s) = \sum R(C_i).$$

For $s \leq 8$ there is a finite number of (-1)-curves in X_s, then the preceding sum is finite and the Mori cone is said to be **polyhedral**.

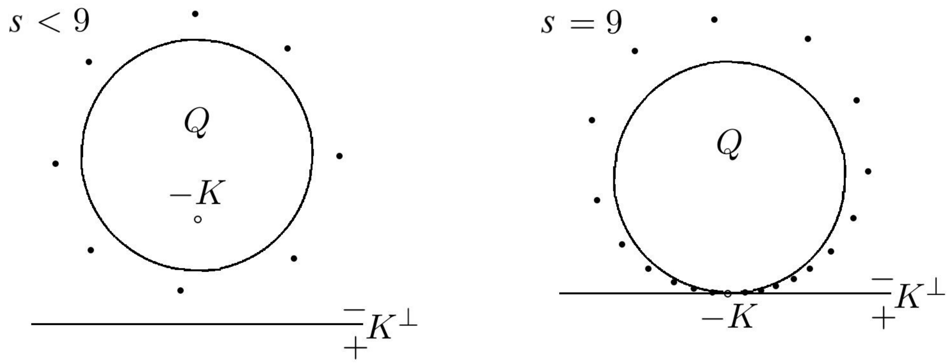
(ii) The **quasi Del Pezzo surface** X_9: $-K_9$ is a nef divisor with self-intersection 0, then the anticanonical ray $R(-K_9)$ is nef and it is contained on $\partial Q(X_9)$. The tangent hyperplane to $\partial Q(X_9)$ at $R(K_9)$ is the hyperplane of the classes in $K_9^\perp$. Then

$$\overline{\text{NE}}(X_9) = R(K_9) + \sum R(C_i).$$

$\overline{\text{NE}}(X_9)$ has infinitely many (-1)-rays (then it is not polyhedral) and $R(K_9)$ is the only limit ray of (-1)-rays (due to Nagata's work in [33]).

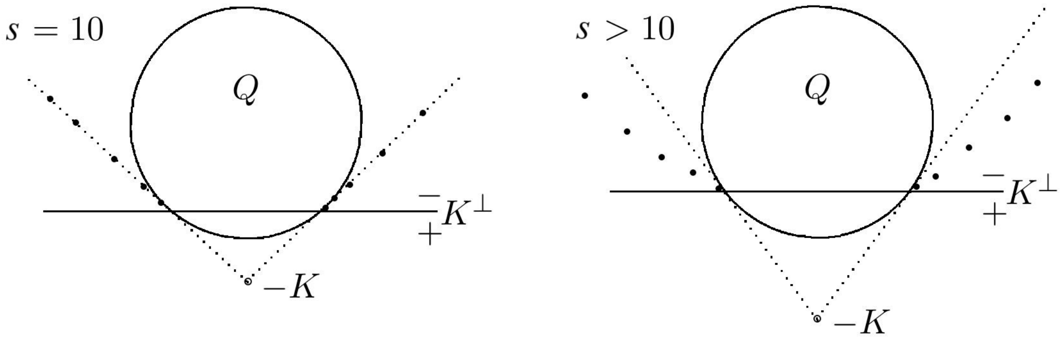
[1] This calculation is provided in [7].

The next figure illustrates [2] the positive light cone for $s \leq 9$, where the black dots represent the $(-1)-$rays and the circle is the boundary of the $Q(X_s)$.



(iii) The general case $s \geq 10$: $-K_s$ is not effective and has negative self-intersection $9 - s$. Hence $R(K_s)$ lies off $Q(X_s)$ which implies that has non-empty with $\overline{\mathrm{NE}}(X_s)_{K_s>0}$ and $\overline{\mathrm{NE}}(X_s)_{K_s<0}$. There are infinitely many (-1)-curves on X_s, whose rays lie in $\overline{\mathrm{NE}}(X_s)_{K_s<0}$ and their limit rays lie at the intersection of $\partial Q(X_s)$ with the hyperplane $K_s^\perp$.

By [11, Thm. 3.3], if $s > 10$, all (-1)-rays lie in the cone $Q(X_s) - R(F_s)$ where $F_s := \sqrt{s-1}H - \sum_{i=1}^{s} E_i$ is the *de Fernex class* and, in the case of $s = 10$, all (-1)-rays lie in the boundary of this cone, where $F_{10} = -K_{10}$.



Very little is known about $\overline{\mathrm{NE}}(X_s)_{K_s \geq 0}$ for $s \geq 0$. Up to now, no curve with self intersection less than -1 have been found so it is conjectured that the only *negative* curves of X_s are the (-1)-curves (Conjecture II.4.2). In Section III.1, we present some conjectures related to the ray $R(F_s)$.

[2] Taken from [17].

II.4. Known conjectures

In this section we state some famous conjectures about the divisors on X_s. The knowledge of the nef cone of X_s is thus of a fundamental importance for all the following conjectures.

Conjecture II.4.1 (Nagata [33])**.** The divisor class $N := \sqrt{s}H - \sum_{i=1}^{s} E_i$ is nef.

Nagata's conjecture holds if s is a square number (see [11, Prop. 2.2]) and can be stated as follows: if $s \geq 10$ and $D = dH - \sum_{i=1}^{s} m_i E_i$ is effective divisor on X_s, then

$$d \geq \frac{1}{\sqrt{s}} \sum_{i=1}^{s} m_i.$$

Conjecture II.4.2 ((−1)-curves)**.** The only negative curves of X_s are $(-1)-$curves.

By the Cone Theorem 5.1.4, this conjecture states that the part of the Mori cone of X_s whose intersection with the canonical divisor K_s is non-negative is contained in the positive light cone (see [17]).

The next one is known as the Segre–Harbourne–Gimigliano–Hirschowitz Conjecture (or SHGH Conjecture), because three authors give a equivalent formulation to the Segre Conjecture [36], which establishes that if a linear system $|D|$ is special then it has some multiple curve in its base locus. The equivalence between these conjectures is proved in [9].

Conjecture II.4.3 (SHGH [9])**.** An effective divisor D which has intersection product at least -1 with any (-1)-curve of X_s is non-special.

Remark II.4.4. The [29, Conjecture 0.1] is equivalent to II.4.3 which states that an effective class divisor on X_s in standard form is non-special.

The following implications are known:

$$\text{Conjecture } II.4.3 \implies \text{Conjecture } II.4.2 \implies \text{Conjecture } II.4.1$$

By the 5.1.4, every extremal ray R_i of $\overline{\mathrm{NE}}(X_s)$ with $R_i \cdot K_s < 0$ is generated by a (-1)-curve, then Conjecture II.4.3 implies that there are no negative self-intersecting curves in $\overline{\mathrm{NE}}(X_s)_{K\geq 0}$, so there are no negative self-intersecting curves other than the (-1)-curves, that is, Conjecture II.4.2 holds.

On the other hand, if Conjecture II.4.2 holds, then any divisor D of the boundary of $Q(X_s)$ whose intersection product with the (-1)-curves is positive is nef. If E is a (-1)-curve, by II.2.4 (iv) (because N is standard), we have that $\sigma(N) \cdot E \geq 0$ for any $\sigma \in W(X_s)$, in particular for $\sigma = \mathrm{id}$. Then II.4.1 holds.

Over the last two decades, more conjectures have been put forward assuming certain (-1)-curves Conjecture to be true, such as the Δ-*Conjecture* and the *Strong* Δ-*Conjecture* (their statements are in III.1.2 and III.1.3, respectively), which provides insight into the boundary of $Q(X_s)$. We will discuss these in more detail in the Chapter III.

II.5. Asymptotically special divisors

Definition II.5.1. Let X be a normal $\mathbb{Q}$-factorial projective variety and let D be a divisor of X. We say that D is **asymptotically special** if

$$h^1(X, \mathcal{O}_X(nD)) > 0, \quad \text{for } n \gg 0.$$

Similarly we say that D is **asymptotically non-special** if $h^1(X, \mathcal{O}_X(nD)) = 0$ for $n \gg 0$.

If X is a surface, K_X the canonical divisor of X and D a divisor such that $D^2 > 0$ then the quadratic function $D^\perp \to \mathbb{Q}$, defined by $E \mapsto p_a(E) := \frac{1}{2}(E^2 + E \cdot K_X) + 1$ is bounded from above. Indeed, by the corollary 4.5.11 of the Hodge index Theorem, the intersection form on $D^\perp$ is negative definite, so that the function p_a is concave. Thus the following number is well defined

$$p_a(D^\perp) := \begin{cases} 0 & \text{if } D \text{ is ample} \\ \max\{p_a(E) \,:\, E \neq 0 \text{ effective and } E \cdot D = 0\} & \text{otherwise.} \end{cases}$$

The main result of this section is the following.

Proposition II.5.2. Let X be a smooth projective surface and let D be a nef and big Cartier divisor of X.

- If $p_a(D^\perp) = 0$, then D is asymptotically non-special.
- If $p_a(D^\perp) = 1$, $D \geq 0$ and $\mathcal{O}_X(D)|_C \simeq \mathcal{O}_C$ for an irreducible curve C with $p_a(C) = 1$, then D is asymptotically special.
- If $p_a(D^\perp) \geq 2$, then D is asymptotically special.

Example II.5.3. In case $p_a(D^\perp) = 1$ the divisor D can be neither asymptotically special nor asymptotically non-special. As an example, consider the blow-up X of $\mathbb{P}^2$ at 10 points $p_1 \ldots, p_{10}$ on a smooth plane cubic. Denote by H the pull back of a line, by E_i the exceptional divisor over p_i and let C be the strict transform of the given cubic (i.e. C is the unique element in $|-K_X|$). We consider a divisor $D := 10H - \sum_{i=1}^{10} 3E_i$ which restricts to a 2-torsion class on C and we claim that

$$h^1(X, \mathcal{O}_X(nD)) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

First of all observe that by [23, Thm. III.1 (d)], C is contained in the base locus of nD if and only if the restriction of nD to C is non-trivial, i.e. if and only if n is odd.

If this is the case, $nD - C$ is effective and satisfies $(nD - C) \cdot (-K_X) = -C^2 = 1$, so that $h^1(X, \mathcal{O}_X(nD)) = 0$ by [23, Thm. III.1 (b)]. On the other hand, if n is even, the restriction of nD to C is trivial, and this implies that C is not contained in the base locus of nD. Therefore nD has no fixed part and by [23, Thm. III.1 (c)] we conclude that $h^1(X, \mathcal{O}_X(nD)) = 1$.

In order to prove the proposition II.5.2 we need some preliminary lemmas. We begin with the following result which is well known in literature.

Lemma II.5.4. Let X be a normal $\mathbb{Q}$-factorial projective surface and let $C_1, \dots, C_n \subseteq X$ be distinct irreducible and reduced curves whose intersection matrix is negative definite. Then the classes $[C_1], \dots, [C_n]$ are linearly independent in $N^1(X)$. In particular if D is a nef and big divisor of X, then $D^\perp$ contains finitely many classes of irreducible curves.

Proof. Let $D := \sum_i a_i C_i \sim 0$. Write $D = A - B$, where A is the sum over the positive coefficients of D and B over the negative ones. Then $0 = D^2 = A^2 - 2A \cdot B + B^2$, and the fact that the three summands are all non-positive, imply that both A and B are linearly equivalent to 0. Since both A and B are effective divisors and X is complete, it follows that $A = B = 0$, so that $a_i = 0$ for any i.

To prove the second statement, observe that by the Corollary 4.5.11 and the fact that $D^2 > 0$, the intersection form on $D^\perp$ is negative definite. The classes contained in $D^\perp$ are finite, because $\rho(X_s) < \infty$, so that the statement follows. □

A version of the following lemma is proved in [26, Thm. 7.2.1] for smooth surfaces. The proof works verbatim in the $\mathbb{Q}$-factorial case and we include it here for the sake of completeness of argument.

Lemma II.5.5. Let $C_1, \dots, C_n$ be irreducible curves on a normal projective $\mathbb{Q}$-factorial surface whose intersection matrix is negative definite. Then there exists an effective divisor $E := \sum_{i=1}^n a_i C_i$ such that $E \cdot C_i < 0$ for each i.

Proof. Since the intersection matrix is non-singular there exists a divisor $E = \sum_{i=1}^n a_i C_i$ such that $E \cdot C_i < 0$ for any i. Let $E = E_1 - E_2$, where E_1, E_2 are effective divisors with no common support. Then $0 \geq E \cdot E_2 = (E_1 - E_2) \cdot E_2 = E_1 \cdot E_2 - E_2^2 \geq 0$ implies $E_2 = 0$, so that E is effective. □

Lemma II.5.6. Let X be a normal $\mathbb{Q}$-factorial projective surface with rational singularities and let D be a nef and big Cartier divisor of X which is not ample. Then there exists an effective divisor E in $D^\perp$ such that

$$H^1(X, \mathcal{O}_X(nD)) \simeq H^1(E, \mathcal{O}_E(nD)).$$

holds for $n \gg 0$.

Proof. By Lemma II.5.4 there are only finitely many irreducible and reduced curves $C_1, \dots, C_s$ in $D^\perp$. By Lemma II.5.5 there exists an effective divisor E, supported at these curves, such that $E \cdot C_i < 0$ for any i. Up to replace E with a positive multiple, we can assume E to be Cartier and $(-E - K_X) \cdot C_i > 0$ for any i. For any $n > 0$, we have the following exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(nD - E) \longrightarrow \mathcal{O}_X(nD) \longrightarrow \mathcal{O}_E(nD) \longrightarrow 0.$$

By the above assumption on E the divisor $nD - E - K_X$ is nef and big for $n \gg 0$. Thus we conclude by passing to cohomology and using the Kawamata-Viehweg vanishing theorem for Cartier divisors (Theorem 4.4.21). □

Proof of Proposition II.5.2. If D is ample, then the statement follows from the Serre vanishing theorem 4.4.5. Assume now that D is not ample. If $p_a(D^\perp) = 0$ then $h^1(E, \mathcal{O}_E) = 0$ for any effective divisor E in $D^\perp$, by Proposition 4.5.8. Thus

$$h^1(E, \mathcal{O}_E(nD)) = 0$$

for any $n > 0$ by Theorem 4.5.9. We conclude that D is asymptotically non-special by Lemma II.5.6.

Assume now $p_a(D^\perp) = 1$, $D \geq 0$ and $\mathcal{O}_X(D)|_C \simeq \mathcal{O}_C$ for an irreducible curve C with $p_a(C) = 1$. For $n \gg 0$ the natural surjection of sheaves $\mathcal{O}_X(nD) \twoheadrightarrow \mathcal{O}_C$, together with the vanishing of $h^2(\mathcal{O}_X(nD - C))$, gives a surjection $H^1(\mathcal{O}_X(nD)) \twoheadrightarrow H^1(\mathcal{O}_C)$. Since $h^1(\mathcal{O}_C) = 1$, the divisor D is asymptotically special.

Assume now $p_a(D^\perp) > 1$. Let E' be an effective divisor in $D^\perp$ such that $p_a(E') = p_a(D^\perp)$. Since $\deg \mathcal{O}_{E'}(nD) = 0$ it follows that

$$\chi(E', \mathcal{O}_{E'}(nD)) = \chi(E', \mathcal{O}_{E'}) = 1 - p_a(E') < 0,$$

where the first equality is by [38, Tag 0BRZ][3] and the second is by the definition of arithmetic genus. Since $p_a(E') \geq 2$ we deduce $h^1(E', \mathcal{O}_{E'}(nD)) > 0$. The divisor E in Lemma II.5.6 can be chosen so that $E' \leq E$. If we denote by I be the ideal sheaf of E' in $\mathcal{O}_E$, we have an exact sequence of sheaves

$$0 \longrightarrow I(nD) \longrightarrow \mathcal{O}_E(nD) \longrightarrow \mathcal{O}_{E'}(nD) \longrightarrow 0.$$

Passing to cohomology and using the fact that we are in dimension one we get the surjection $H^1(E, \mathcal{O}_E(nD)) \to H^1(E', \mathcal{O}_{E'}(nD))$, which implies $h^1(E, \mathcal{O}_E(nD)) > 0$, so that D is asymptotically special by Lemma II.5.6. □

We ended this section with a important consequence of the Proposition II.5.2.

[3] Other version of Riemann-Roch formula for schemes.

Corollary II.5.7. Let X be a smooth projective surface and let $C \subseteq X$ be an irreducible and reduced curve with $C^2 < 0$ and $p_a(C) > 1$. Then there exists a nef and big divisor D on X such that $D \cdot C = 0$ and D is asymptotically special.

Proof. Let A be an ample divisor on X. Set $\lambda = \frac{C^2}{A \cdot C}$ and define

$$D := C - \lambda A.$$

We check that D is nef: by construction $D \cdot C = 0$, and for any other curve $E \neq C$ we have $D \cdot E = C \cdot E - \lambda A \cdot E > 0$, since A is ample and $\lambda < 0$. Moreover

$$D^2 = (C - \lambda A)^2 = C^2 - 2\lambda(C \cdot A) + \lambda^2 A^2 = -C^2 + \lambda^2 A^2 > 0,$$

so D is big. Finally, the orthogonal complement $D^{\perp}$ in the intersection pairing contains C. Hence $p_a(D^{\perp}) \geq p_a(C) > 1$. By Proposition II.5.2, this implies that D is asymptotically special. □

II.6. Main results for X_s

In this section we will establish new formulations of the (-1)-curves conjecture and SHGH-Conjecture in terms of asymptotically special divisors. This simplifies the statements II.4.2 and II.4.3 and motivates extending these conjectures to explosions of $\mathbb{P}^n$ at points s in a very general position. In the Section II.7 we study this.

Notation II.6.1. In what follows, with abuse of notation, we will denote by $-K_r$ the class $3H - \sum_{i=1}^{r} E_i$ on X_s for any $r \leq s$.

The next proposition characterizes standard classes within those with non-positive self-intersection and non-positive intersection with K_s.

Lemma II.6.2. Let $D \in N^1(X_s)_{\mathbb{R}}$ be a standard class. If $D^2 \leq 0$ and $D \cdot K \leq 0$ then D is a positive multiple of either $H - E_1$ or $-K_9$.

Proof. Let $D = dH - \sum_{i=1}^{r} m_i E_i$. Since D is standard we have that $d \geq m_1 + m_2 + m_3$ and $m_1 \geq \cdots \geq m_s \geq 0$. The inequalities $D^2 \leq 0$ and $D \cdot K_s \leq 0$ imply $D^2 + m_3 D \cdot K_s \leq 0$, that is

$$d(d-3m_3) - \sum_{i=1}^{r}(m_i^2 - m_3 m_i) = 0$$

$$\Rightarrow d(d-3m_3) - (m_1^2 - m_1 m_3) - (m_2^2 - m_2 m_3) = \sum_{i=3}^{r}(m_i^2 - m_3 m_i) \leq 0.$$

Since $d \geq m_1 + m_2 + m_3$ the left hand side is greater than or equal to $(m_1 + m_2 + m_3)(m_1 + m_2 + m_3 - 3m_3) - (m_1^2 - m_1 m_3) - (m_2^2 - m_2 m_3) = 2(m_1 m_2 - m_3^2)$, from which we deduce

$$0 \leq 2(m_1 m_2 - m_3^2) \leq \sum_{i=3}^{r}(m_i^2 - m_3 m_i) \leq 0.$$

Therefore either $m_2 = \cdots = m_s = 0$ or all the multiplicities are equal. In the first case $D^2 \leq 0$ imply $d^2 \leq m_1^2$, so that $d = m_1$ and D is a multiple of $H - E_1$. In the second case $D = dH - \sum_{i=1}^{r} mE_i$. The inequalities $0 \leq d^2 - 3md = D^2 + mD \cdot K_s \leq 0$ imply $d = 3m$ and $D^2 = D \cdot K_s = 0$, which proves the statement. □

Corollary II.6.3. The surface X_s does not contain (-2)-curves and the classes of (-1)-curves form an orbit for $W(X_s)$.

Proof. Let C be a (-1) or (-2)-curve. Since the action of $W(X_s)$ preserves the effective cone, we can assume the class $[C] := dH - \sum_{i=1}^{r} m_i E_i$ to be effective and pseudostandard.

By Lemma II.6.2 this class cannot be standard. This immediately implies $d = 0$, so that by the irreducibility of C one deduces $m_1 = \cdots m_{r-1} = 0$ and $m_s = -1$, which proves the statement. □

Lemma II.6.4. If C is an irreducible and reduced curve of X_r^2 with $C^2 < 0$ and $p_a(C) = 0$, then C is a (-1)-curve.

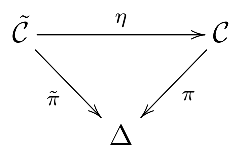
Proof. Since $p_a(C) = 0$, the curve C is smooth and rational. By Corollary II.6.3, it suffices to show that C^2 cannot be smaller than -2. Assume, by contradiction, that $C^2 < -2$.

Consider a flat family $\mathcal{X} \to \Delta$ whose general fiber is of type X_r^2 and whose special fiber Y over $0 \in \Delta$ is an anticanonical rational surface containing a smooth irreducible member $E \in |-K_Y|$. The existence of such a degeneration is guaranteed by specializing the r points to lie on a smooth plane cubic.

The smooth rational curve C degenerates to an effective divisor Γ on Y. Let $\pi : \mathcal{C} \to \Delta$ be a flat family with general fiber C and special fiber Γ. Since $p_a(C) = 0$, we have

$$\Gamma \cdot E = \Gamma \cdot (-K_Y) = C \cdot (-K_r) = C^2 + 2 < 0,$$

so E must be a component of Γ. Let $\eta : \tilde{\mathcal{C}} \to \mathcal{C}$ be a resolution of singularities. We obtain the commutative diagram



Since the base Δ has dimension 1, the morphism $\tilde{\pi}$ is still flat. Let $\tilde{\Gamma}$ be the special fiber of $\tilde{\pi}$. As $E \subset \Gamma$, the fiber $\tilde{\Gamma}$ contains a smooth genus–one curve $\tilde{E}$. By [34, Prop. 6.4.2], the morphism $\tilde{\pi}$ is cohomologically flat, hence $h^0(\mathcal{O}_{\tilde{\Gamma}}) = 1$. By invariance of the arithmetic genus, it follows that $h^1(\mathcal{O}_{\tilde{\Gamma}}) = 0$. The natural surjection $\mathcal{O}_{\tilde{\Gamma}} \twoheadrightarrow \mathcal{O}_{\tilde{\Gamma}_{\mathrm{red}}}$ then induces a surjection

$$H^1(\mathcal{O}_{\tilde{\Gamma}}) \twoheadrightarrow H^1(\mathcal{O}_{\tilde{\Gamma}_{\mathrm{red}}}),$$

showing that $p_a(\tilde{\Gamma}_{\mathrm{red}}) = 0$. Hence $\tilde{\Gamma}_{\mathrm{red}}$ consists only of smooth rational curves, contradicting the presence of the elliptic component $\tilde{E}$. This contradiction proves that $C^2 \geq -2$. □

Theorem II.6.5. Let s be a positive integer. Then the following are equivalent.

1. The only negative curves of X_s are (-1)-curves.

2. X_s does not contain nef and big divisors which are asymptotically special.

Proof. We prove (1) $\Rightarrow$ (2). Let D be a nef and big divisor of X_s. By hypothesis the only negative curves in $D^\perp$ are (-1)-curves. If $D^2 > 0$, by Corollary 4.5.11 the intersection matrix of these curves is negative definite. In particular these curves are disjoint, since $(E+E')^2 \geq 0$ if E, E' are (-1)-curves with $E \cdot E' > 0$. Contracting these curves we get a birational morphism $X_s \to X_r$, with $r \leq s$, and D is the pullback of an ample divisor of X_s, then D is ample by Proposition 4.4.11, implying that it is not asymptotically special.

We prove (2) $\Rightarrow$ (1). Assume there is a negative curve C on X_r^2 which is not a (-1)-curve. By Lemma II.6.4 we have $p_a(C) > 0$. If $p_a(C) > 1$, then by Corollary II.5.7 the surface X_r^2 would contain an asymptotically special divisor, a contradiction.

We may then assume that $p_a(C) = 1$. From the exact sequence

$$0 \to \mathcal{O}_{X_r}(K_r) \to \mathcal{O}_{X_r}(K_r + C) \to \mathcal{O}_C \to 0,$$

the fact that $h^i(\mathcal{O}_{X_r}(K_r)) = 1$ for $i = 2$ and 0 otherwise, and $h^2(\mathcal{O}_{X_r}(K_r + C)) = 0$ we deduce

$$K_r + C \sim D \geq 0, \qquad h^0(\mathcal{O}_{X_r}(D)) = 1, \qquad h^1(\mathcal{O}_{X_r}(D)) = 0.$$

Let $C_1, \ldots, C_s$ be the irreducible components of the support of D. Since $D - C \sim K_r$ is not effective, the divisor D cannot contain C in its support. Hence the condition $\mathcal{O}_{X_r}(D)|_C \simeq \mathcal{O}_C$ forces $C_i \cap C = \emptyset$ for all i. We may assume that every C_i has positive arithmetic genus: indeed, if $p_a(C_i) = 0$ and $C_i^2 \geq 0$, then $1 < h^0(\mathcal{O}_{X_r}(C_i)) \leq h^0(\mathcal{O}_{X_r}(D)) = 1$, a contradiction. Thus $C_i^2 < 0$, and by Lemma II.6.4 such a curve must be a (-1)-curve. Since the C_i are disjoint from C, we can contract all these (-1)-curves, obtaining a new surface X_s that still contains a copy of C with the same arithmetic genus and self-intersection.

So from now on we assume $p_a(C_i) > 0$ for any i. For any such C_i, Serre duality gives $H^2(\mathcal{O}_{X_r}(D - C_i)) \simeq H^0(\mathcal{O}_X(K_{X_r} - D + C_i)) = H^0(\mathcal{O}_{X_r}(C_i - C)) = 0$, where the last equality is due to $h^0(\mathcal{O}_{X_r}(C_i)) \leq h^0(\mathcal{O}_{X_r}(D)) = 1$. From the long exact sequence of

$$0 \to \mathcal{O}_{X_r}(D - C_i) \to \mathcal{O}_{X_r}(D) \to \mathcal{O}_{C_i}(D) \to 0$$

and $h^1(\mathcal{O}_{X_r}(D)) = 0$ it follows that $h^1(\mathcal{O}_{C_i}(D)) = 0$. Hence $D \cdot C_i \geq 0$ by the generalised Riemann-Roch Theorem for singular surfaces [38, Tag 0BRZ]. In particular D is nef. If D were big, it would be asymptotically special by Proposition II.5.2, a contradiction.

It remains to consider the case $D^2 = 0$. We have

$$D \cdot K_r = (K_r + C) \cdot K_r = (K_r + C)^2 = D^2 = 0.$$

Thus, by Lemma II.6.2, after applying an isometry ϕ of the Picard group which takes D to its standard form, we get $\phi(D) = -mK_9$, with $m > 0$. Moreover $\phi(K_r) = K_r$. Thus $K_r + \phi(C) \sim -mK_9$, i.e.

$$\phi(C) \sim -(m+1)K_9 - E'_{10} - \cdots - E'_r,$$

where E'_j are exceptional classes. Since $h^0(-(m+1)K_9) = 1$, imposing additional base points we obtain a divisor that can not be effective. Hence $\phi(C)$ (and therefore C) cannot exist. This contradiction completes the proof.

□

As a consequence of II.6.5 we can prove the following equivalence.

Theorem II.6.6. Let s be a positive integer. Then the following are equivalent.

1. The SHGH conjecture holds on X_s.
2. Each nef class of X_s is non-special.

Proof. The implication (1) ⇒ (2) is obvious.

We prove (2) ⇒ (1). Since every nef class is non-special, it is also asymptotically non-special, so that, by Theorem II.6.5, the only negative curves of X_s are (-1)-curves. Let D be a divisor such that $D \cdot E \geq -1$ for any (-1)-curve E. Then $D \sim M + F$, where F is the sum of all (-1)-curves which have intersection product -1 with D. Observe that if E, E' are two distinct (-1)-curves in F then both curves are in the base locus of $|D|$ so that $E \cdot E' = 0$. It follows that F is a reduced divisor (each curve in its support appears with coefficient 1) and $M \cdot F = 0$. Thus

$$\begin{aligned} v(D) = \frac{1}{2}(D^2 - D \cdot K_s) &= \frac{1}{2}((M+F)^2 - (M+F) \cdot K_s) \\ &= v(M) + v(F) + M \cdot F \\ &= v(M). \end{aligned}$$

Observe that M is nef because the only negative curves of X_s are (-1)-curves. Thus M is non-special and we deduce $\dim |D| = \dim |M| = v(M) = v(D)$, which shows that D is non-special as well. □

II.7. Higher dimension

Recall the notation of remark II.2.4. Let $\mathrm{Pic}(X_s^n)$ be the Picard group of X_s^n which is generated by H, $E_1, \dots, E_s$ and denote K_s^n the canonical divisor of X_s^n where $K_s^n := -(n+1)H + (n-1)\sum_{i=1}^{s} E_i$.

Definition II.7.1. Given a class divisor $D = dH - \sum_{i=1}^{s} m_i E_i$ on X_s^n:

- the **virtual dimension** of the linear system $|D|$ is
$$v(D) := \binom{d+n}{n} - \sum_{i=1}^{s} \binom{m_i + n - 1}{n} - 1.$$
and its **expected dimension** is $e(D) := \max\{v(D), -1\}$.
- D is called **special** if $\dim |D| = e(D)$ and D is **non-special** if $\dim |D| = e(D)$ or equivalently if $h^0(X_s^n, \mathcal{O}_{X_s}(D)) \cdot h^1(X_s^n, \mathcal{O}_{X_s}(D)) = 0$.

Recall definitions 4.4.13 and II.5.1. In this section we consider the asymptotical behavior of semiample divisors in X_s^n with the purpose of studying possible generalizations of the conjectures described II.4. A possible generalization of these conjectures could be:

Conjecture II.7.2. The blow-up X_s^n of $\mathbb{P}^n$ at s points in very general position does not contain nef divisors which are asymptotically special.

We are going to prove that Conjecture II.7.2 holds if $s < 2^n$ (Theorem II.7.6). In this proof, we use the next results taken from the paper [16].

We recall the notation of Theorem 4.4.16 of Preliminaries.

Definition II.7.3. Given a semiample Cartier divisor D on a normal variety X let
$$m_D := \gcd(n \in \mathbb{N} : nD \text{ is base point free}).$$

For any positive integer m denote by $\phi_m \colon X \to Y_m$ the rational map defined by the linear system $|mD|$. There is a morphism with connected fibers $f_D \colon X \to Y$ onto a normal variety Y, such that $\phi_m = f_D$ and $Y_m = Y$ for any sufficiently big multiple m of m_D. Moreover $m_D D = f_D^* A$ for some ample Cartier divisor A on Y.

Remark II.7.4. The semiample and big divisor D of Example II.5.3 has $m_D = 2$. Indeed any odd multiple of D restricts to a non-trivial degree zero class on the elliptic curve $C \in |-K_X|$, while the even multiples restrict trivially to C. The latter fact implies that C is not in the base locus of $2D$ and by [23, Thm. III.1] we conclude that $2D$ is base point free.

Proposition II.7.5. Let X be a normal projective variety and let D be a semiample big Cartier divisor of X with $m_D = 1$. If $R^1 f_{D*}\mathcal{O}_X$ is trivial then D is asymptotically non-special and otherwise it is asymptotically special.

Proof. Let $f := f_D$ and recall that $D = f^*A$, with A ample on Y. By Serre's vanishing Theorem 4.4.5 (ii) the higher cohomology groups of $\mathcal{O}_Y(mA)$ vanish for all $m \gg 0$. Since the Grothendieck-Leray spectral sequence

$$E_2^{p,q}(D) := H^p(Y, R^q f_*\mathcal{O}_X(D)) \Rightarrow H^{p+q}(X, \mathcal{O}_X(D))$$

is a first quadrant one, by [26, Prop. 3.6.2] we have the following five terms exact sequence:

$$0 \to E_2^{1,0}(mD) \to H^1(\mathcal{O}_X(mD)) \to E_2^{0,1}(mD) \to E_2^{2,0}(mD) \to H^2(\mathcal{O}_X(mD)).$$

By the projection formula (Proposition 2.4.7) we have $f_*\mathcal{O}_X(mD) = f_*\mathcal{O}_X \otimes \mathcal{O}_Y(mA) \simeq \mathcal{O}_Y(mA)$, so that the first and fourth cohomology groups $E_2^{i,0}(mD) = H^i(Y, f_*\mathcal{O}_X(mD))$ vanish for all $m \gg 0$. Thus the second and third cohomology groups are isomorphic. Again by the projection formula we have $R^1 f_*\mathcal{O}_X(mD) \simeq R^1 f_*\mathcal{O}_X \otimes \mathcal{O}_Y(mA)$ so that

$$H^1(X, \mathcal{O}_X(mD)) \simeq H^0(Y, R^1 f_*\mathcal{O}_X \otimes \mathcal{O}_Y(mA)) \qquad \text{for all } m \gg 0.$$

If $R^1 f_*\mathcal{O}_X$ is trivial we conclude that D is asymptotically non-special. On the other hand if $R^1 f_*\mathcal{O}_X$ is non-trivial, since it is a coherent sheaf and A is ample, by Serre's Theorem 4.4.5 (iii) the sheaf $R^1 f_*\mathcal{O}_X \otimes \mathcal{O}_Y(mA)$ is generated by global sections for all $m \gg 0$. We conclude that D is asymptotically special. □

Theorem II.7.6. If $s < 2^n$ then any nef divisor of X_s^n is asymptotically non-special.

Before proving Theorem II.7.6 we are now going to discuss an example of a nef and big divisor which is special but asymptotically non-special, on the blow up of $\mathbb{P}^4$ at 14 points.

Example II.7.7. Let D be the divisor $2H - \sum_{i=1}^{14} E_i$ on X_{14}^4. The virtual dimension of mD, for $m = 1, 2, 3$ is $0, -1, -1$ respectively, so that $2D$ and $3D$ are special. Moreover a computer calculation shows that $4D$ is non-special of dimension 4 and its base locus is zero-dimensional. Thus D is semiample by Theorem 4.4.18 and it is big because $D^4 = 2$. We now show that mD is non-special for $m \geq 5$ using a degeneration argument. Let $\mathcal{X} \to \mathbb{A}^1$ be a flat family whose general fiber is X_{14}^4 and whose special fiber Y is the blowing-up of $\mathbb{P}^4$ at 14 general points on the complete intersection of three general quadrics. Denote by $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq Y_4 = Y$ the strict transforms of the complete intersections of three, two and one quadrics respectively, plus the central fiber of the degeneration. By abuse of notation we denote by D the specialized divisor on Y. Observe that for any $2 \leq i \leq 4$, $Y_{i-1} \in |D|_{Y_i}|$, so that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{Y_i}((m-1)D) \longrightarrow \mathcal{O}_{Y_i}(mD) \longrightarrow \mathcal{O}_{Y_{i-1}}(mD) \longrightarrow 0.$$

Since Y_1 is a smooth curve of genus 5 and $\deg(D|_{Y_1}) = D \cdot Y_1 = 2$ it follows that $mD|_{Y_1}$ is non-special for $m \geq 5$. From the above exact sequence we deduce that $h^1(\mathcal{O}_{Y_2}(mD)) \leq h^1(\mathcal{O}_{Y_2}((m-1)D)) \cdots \leq h^1(\mathcal{O}_{Y_2}(5D))$. A computer calculation shows that $5D|_{Y_i}$ is non-special for $2 \leq i \leq 4$. A repeated use of the above exact sequence shows that for any $m \geq 5$, mD is non-special on Y_4 and, by semicontinuity of cohomology, it is non-special on X^4_{14} too.

Proof of Theorem II.7.6. Let $H, E_1, \ldots, E_r$ be a basis of $\mathrm{Pic}(X^n_r)$ consisting of the pullback of a hyperplane together with the exceptional divisors. If we denote by $h, e_1, \ldots, e_r$ the dual basis in $N_1(X^n_r)$, we have that the Mori cone of X^n_r is polyhedral, generated by the classes $h - e_i - e_j$ and e_k, for $i, j, k \in \{1, \ldots, r\}$ and $i < j$. (see [14]). The proof of this fact goes as follows: the cone generated by these classes is dual to the cone generated by the classes

$$H, \quad H - E_i, \quad 2H - \sum_{i \in I} E_i,$$

for any subset $I \subseteq \{1, \ldots, s\}$. These classes are all semiample (and thus nef) on the variety Y^n_s obtained by degenerating the s points to a complete intersection of n general smooth quadrics. Then one concludes by observing that the nef cone can only become smaller by degeneration.

By semicontinuity of cohomology it is now enough to show that any nef class of Y^n_s is asymptotically non-special. Observe that any $H - E_i$ is asymptotically non-special because it is pullback of a class of X^n_1 which is asymptotically non-special by Kawamata-Viehweg vanishing Theorem 4.4.21. All the nef classes of Y^n_s, which are not multiples of $H - E_i$, are big. Thus by Proposition II.7.5 it suffices to show that for any nef and big divisor class D of Y^n_s the sheaf $R^1 f_{D*}\mathcal{O}_{Y^n_s}$ is trivial. This is consequence of the fact that f_D can contract only exceptional divisors and strict transforms of lines through two points. In the first case one gets a smooth point while in the second case the image of f_D has a rational singularity because the morphism is locally toric and one can apply the Remark 3.2.3 of Preliminaries. □

Remark II.7.8. It is unclear to us if the statement of the Conjecture II.7.2 would be equivalent to one about the structure of the Mori cone of these blowups. On the other hand it is not true that any nef and big class on X^n_r is non-special, when $n > 2$, as shown in Example II.7.7. This makes a difference with the case $n = 2$ seen in the theorem II.6.6.

III. Cones of X_s

Recall the notation introduced in II.1.1 and 5.1.2 of Preliminaries: let $\pi : X_s \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at s points in very general position with $E_1, \dots, E_s$ the exceptional divisors, H the pullback of a line and $K_s := -3H + \sum_i E_i$ the canonical divisor class of X_s. If D is a divisor (class) let $R(D) := \mathbb{R}_{\geq 0}[D]$ be the ray generated by D.

In this chapter we construct an irrational and nef, with self–intersection zero rays (called *wonderful rays*) in $\overline{\mathrm{NE}}_{K_s \geq 0}$ using the *Uncollision* technique described in [12, Section 1]. This wonderful rays form orbits under the action of the Weil group $W(X_s)$, so knowing the fundamental regions of this group on the cones $Q(X_s)$ and $\overline{\mathrm{NE}}(X_s)$ would give us a number of orbits of such rays. On the other hand, the Mori cone, although is not polyhedral, could have a polyhedral region. In that way, recommend reading the paper [19].

In pursuit of this objective, several results obtained are also in the paper [13], as the III.2.4 where we determine quadrics on the boundary of the Mori cone for all $s \geq 10$.

III.1. Good and wonderful rays

From the Cone Theorem, we know that the negative part K_s of the Mori cone is bounded by rays generated by (-1)-curves, also called (-1)**-rays** as we see in the Section II.3.

By the Hodge Index theorem 4.5.10, if R is a ray supported on the boundary of the cone $Q(X_s)$, then $R^\perp \cap Q(X_s) = R$. This implies that if R is a ray supported on $\partial Q(X_s)$, then $R \subset \partial\overline{\mathrm{NE}}(X_s)$ if and only R is nef. Note that in this case R is a extremal ray of $\mathrm{Nef}(X_s)$, but not necessarily of $\overline{\mathrm{NE}}(X_s)$.

From now on, we assume that $s \geq 10$. We define the following extremal rays of $\overline{\mathrm{NE}}(X_s)$ contained in $\partial Q(X_s)$.

Definition III.1.1 ([11, p.193]). A **good ray** is a rational and non-effective ray in $\partial Q(X_s)$. A **wonderful ray** is a irrational nef ray in $\partial Q(X_s)$.

Both rays are nef and extremal to the nef cone and the Mori cone of X_s (see [11, Lemma 3.8, Theorem 5.1.1]). In [12, Thm. 1] the authors prove the existence of wonderful rays in $\overline{\text{NE}}(X_s)_{K_s>0}$ for all $s \geq 13$, which provide evidence for two conjectures stronger than what we described in Section II.4.

In the paper *On the Mori Cone of blow-ups of the plane* [17], de Fernex proves that conjecture of (-1)-curves II.4.2 in the case of $s = 10$ implies that

$$\overline{\text{NE}}(X_s)_{K_s\geq 0} = Q_{K_s\geq 0} \tag{III.1}$$

Moreover, if $s \geq 10$

$$\overline{\text{NE}}(X_s)_{(K_s-rH)\geq 0} = Q_{(K_s-rH)\geq 0} \tag{III.2}$$

where $r = \sqrt{s-1} - 3$. Thus, supposing that conjecture II.4.2 is true, a part of the positive side of the boundary of the Mori cone of X_s is circular.

We define the **de Fernex class divisor** $F_s := \sqrt{s-1}H - \sum_{i=1}^{s} E_i$ (the ray $R(F_s)$ is called de Fernex ray in [12]). A ray R is said to be **de Fernex positive, negative, or orthogonal** if $R \cdot R(F_s)$ is positive, negative, or zero, respectively. Note that if $R \cdot R(F_s) \leq 0$ then $R \cdot R(K_s) > 0$ and

$$K_s - rH = -\sqrt{s-1}H + \sum_{i=1}^{s} E_i = -F_s,$$

so the sub-indexes in Equation (III.2) can be rewritten in terms of de Fernex class divisor. From these observations, the following conjectures are stated, which are closely related to Nagata conjecture II.4.1:

Conjecture III.1.2 (Δ-Conjecture [11, Conj. 3.5])**.** If $s \geq 10$ one has $\partial Q(X_s)_{F_s\leq 0} \subset \text{Nef}(X_s)$.

If this conjecture holds, we have the equality (III.2) by [11, Proposition 3.6]). Furthermore, [11, Lemma 3.8] establishes that any rational non-effective ray on $\partial Q(X_s)$ is nef, so a stronger form this statement can be given.

Conjecture III.1.3 (Strong Δ-Conjecture [11, Conj. 3.10])**.** If $s > 10$, all rational rays in $\partial Q(X_s)_{F_s\leq 0}$ are non-effective. If $s = 10$, a rational ray in $Q(X_s)_{F_s\leq 0} = Q(X_s)_{K_s\geq 0}$ is non–effective, unless it is generated by a curve $W(X_s)$-orbit of $-K_9 = 3H - \sum_{i=1}^{9} E_i$.

The ray generated by Nagata class $N := \sqrt{s}H - \sum_{i=1}^{s} E_i$ belongs to $\partial Q(X_s)$ and

$$\left(\sqrt{s}H - \sum_{i=1}^{s} E_i\right) \cdot \left(\sqrt{s-1}H - \sum_{i=1}^{s} E_i\right) = \sqrt{s(s-1)} - s < 0,$$

then the Nagata ray is F_s-negative. Suppose that $s \geq 11$ and R then the Strong Δ-Conjecture would imply that an irrational de Fernex non-positive ray R with zero self-intersection is a wonderful ray, since any such ray would then be a limit of good rays.

Clearly, in the case of $s = 10$ N is not generated by $-K_9$. Thus, if Δ-Conjecture is true then N is nef.

The following implications are obtained

$$\text{Conjecture } III.1.3 \implies \text{Conjecture } III.1.2 \implies \text{Conjecture } II.4.1$$

The main theorem of [12] is the existence of wonderful rays in the Mori cone of X_s for all $s \geq 10$, and furthermore, that for certain values of s, there exist de Fernex negative wonderful rays.

Remark III.1.4. We do not know why the authors of [11] call the rays defined in III.1.1 good and wonderful, but we assume that it is because of their relationship with the Nagata conjecture and the possible existence of irrational Seshadri constants. Namely, there are many rays of self-intersection zero (like the Nagata ray) which, if proven to be wonderful, would imply that some Seshadri constant at a blow-up X_s is irrational.

The connection between these two famous problems is discussed in the paper [18]. For deeper study, we recommend [5].

III.2. Quadrics in Mori the Cone of X_s

Definition III.2.1. The boundary of the light cone of X_s, $\partial Q(X_s)$, consists of all divisors classes D such that $D^2 = 0$. Define the following subsets

$$Q_0(X_s) := \{D \in \partial Q(X_s) : D \cdot K_s = 0\} \qquad Q_+(X_s) := \{D \in \partial Q(X_s) : D \cdot K_s > 0\}.$$

We prove that $Q_0(X)$ is contained in $\mathrm{Eff}(X_s)$.

Lemma III.2.2. Let $D \in \mathrm{Pic}(X_s)$ be a divisor class of X_s such that $D \cdot H \geq 0$, $D^2 = 0$ and $D \cdot K_s = 0$. Then D is effective.

Proof. Let $D \in Q_0(X_s)$. By the Riemann-Roch Theorem 4.5.5

$$h^0(D) + h^0(K_s - D) - h^1(D) = \frac{1}{2} D \cdot (D - K_s) + 1$$
$$\Longrightarrow h^0(D) > 0 \ \vee \ h^0(K_s - D) > 0$$

then either D or $K_s - D$ is effective. This implies that either D or $-D$ is effective, since otherwise $2K_s = (K_s - D) + (K_s + D)$ would be a effective, which is a contradiction. Finally, $-D \cdot H \leq 0$ implies that D must be effective. □

Notation III.2.3. We denote by $E(X_s) \subseteq N^1(X_s)_{\mathbb{R}}$ the dual of the cone generated by classes of (-1)-curves, that is:

$$E(X_s) := \{D \in N^1(X_s)_{\mathbb{R}} \ : \ D \cdot E \geq 0 \text{ for any } (-1)\text{-curve } E\}.$$

Theorem III.2.4. For any $s \geq 10$ the following hold:

1. any standard class in $Q_0(X_s)$ is nef;
2. any class in $E(X_s) \cap Q_0(X_s)$ is nef.

Proof. Let D be a standard divisor class in $Q_0(X_s)$. Then by Lemma II.6.2 D is a positive multiple of $-K_9$, so it is nef.

Now let D be a primitive divisor class in $E(X_s) \cap Q_0(X_s) \cap \mathrm{Pic}(X_s)$. Observe that D is effective by lemma III.2.2. Thus, after possibly applying an element of $W(X_s)$, we can assume D to be standard, so that it is nef by the previous argument. This proves the statement for classes in $\mathrm{Pic}(X_s)$ and so it holds also for classes in $\mathrm{Pic}(X_s)_{\mathbb{Q}}$. In order to conclude it is enough to observe that $Q_0(X_s) \subseteq \mathrm{Pic}(X_s)_{\mathbb{R}} = N^1(X_s)_{\mathbb{R}}$ is a codimension two quadric which contains a rational point and thus rational points are dense in it and being a nef class is a closed property. □

III.3. The case $s = 10$

Proposition III.3.1. Let $s \geq 10$ and let $D \in \mathrm{Pic}(X_s)$ be an effective primitive divisor class with $D \cdot K_s \geq 0$. Then the number of (-1)-curves E such that $D \cdot E < 0$ is at most $s - 10$.

Proof. If E, E' are two distinct (-1)-curves with $E \cdot E' > 0$ then, by Riemann-Roch $\dim |E + E'| > 0$, so that the union of the two curves cannot be contained in the base locus of a linear system. Thus if $E_1, \ldots, E_r$ are distinct (-1)-curves contained in the base locus of a linear system, then $E_i \cdot E_j = 0$ for any $i \neq j$ and thus these curves span a sublattice of rank r in the Picard group. In particular r must be finite.

Assume $D \cdot E < 0$ for some (-1)-curve E. Then E is in the base locus of $|D|$ so that $D = D' + E$, with D' effective and $D' \cdot E = (D - E) \cdot E = D \cdot E + 1$. If D' has still negative intersection with E, we repeat the operation enough times with all the (-1)-curves which have negative intersection product with D. We get $D = D_0 + m_1 E_1 + \cdots + m_r E_r$, with D_0 effective, $D_0 \cdot E_i = 0$ and $m_i > 0$ for any i. If we denote by $\pi \colon X_s \to X_{s-r}$ the contraction of $E_1, \ldots, E_r$, then $D_0 = \pi^*(B)$ for an effective divisor class $B \in \mathrm{Pic}(X_{s-r})$. Thus

$$0 \leq D \cdot K_s = (D_0 + \sum_{i=1}^{s} m_i E_i) \cdot K_s = \pi^*(B) \cdot \pi^*(K_{X_{s-r}}) - \sum_{i=1}^{s} m_i$$
$$= B \cdot K_{X_{s-r}} - \sum_{i=1}^{s} m_i$$

implies that $B \cdot K_{X_{s-r}} \geq \sum_{i=1}^{s} m_i > 0$. On the other hand if $s - r \leq 9$, then $-K_{X_{s-r}}$ is nef, so that it cannot have negative intersection with an effective class. We conclude that $s - r \geq 10$. □

Corollary III.3.2. Any class of $Q_0(X_{10})$ is nef.

Proof. By proposition III.3.1 the inclusion $Q_0(X_{10}) \subseteq E(X_{10})$ holds. Thus we conclude by Theorem III.2.4. □

Remark III.3.3. If $s > 10$, then there are classes in $Q_0(X_s)$ which are not nef. For example the class $D := -K_8 - 3K_{10} + K_{11} = -2K_{11} - K_8 + 3E_{11}$ is in $Q_0(X_{11})$ but it is not nef because $D \cdot E_{11} = -1$ (see also Proposition III.4.5 and Example III.4.6).

The above corollary implies that any irrational class in $Q_0(X_{10})$ generates a wonderful ray of X_{10}. Equivalently, any irrational class $D = dH - \sum m_i E_i$ satisfying

$$3d - \sum_{i=1}^{10} m_i = d^2 - \sum_{i=1}^{10} m_i^2 = 0 \tag{III.3}$$

generates a wonderful ray.

Example III.3.4. Let $D := dH - \sum_{i=1}^{8} m_i E_i \in \mathrm{Pic}(X_8)$ be such that $(D \cdot K_8)^2 < 2D^2$. Let $a := -\frac{1}{2} D \cdot K_8$ and let $b := \frac{1}{2} D^2 - \frac{1}{4}(D \cdot K_8)^2$. Then the following class

$$dH - \sum_{i=1}^{8} m_i E_i - (a + \sqrt{b})E_9 - (a - \sqrt{b})E_{10}$$

belongs to $Q_0(X_{10})$. In particular, if b is not a square, such class generates a wonderful ray of X_{10}.

When the first eight multiplicities are equal we get the following. Let us fix 2 positive integers a and c, such that $a^2 - 4c^2 > 0$ and it is not a square. We claim that the following classes generate wonderful rays of X_{10}:

$$(6a \pm 8c)H - \sum_{i=1}^{8} (2a \pm 3c)E_i - (a - \sqrt{a^2 - 4c^2})E_9 + (a - \sqrt{a^2 - 4c^2})E_{10}.$$

Remark III.3.5. Corollary III.3.2 does not allow to prove the existence of irrational Seshadri constants because the first equation in (III.3) implies that if any of $d, m_1, \ldots, m_{10}$ is irrational, then there must be at least another such irrational number. In order to study these Seshadri constants one should understand which classes of $Q_+(X_{10})$ are nef.

III.4. The case $s > 10$

III.4.1. Uncollisions

We recall here the main construction from [12, Section 2]. It's called *collision* a degeneration of a linear system $\mathcal{L}_d(m_1, \ldots, m_s)$ where t^2 of the s points of equal multiplicity m, come together. The analysis of this situation for $t = 2$ was developed in [10]. *Uncollision* would be the reverse process.

Given a divisor class $D = dH - \sum_{i=1}^{s} m_i E_i \in \mathrm{Pic}(X_s)$, we can define the **uncollision** of D $\mathrm{Uncoll}_t(D, i)$ as the divisor replacing the term $m_i E_i$ by t^2 terms of the form m_i/tE'_j, with $E'_1, \ldots, E'_j$ the exceptional divisors classes of X_{s+t^2-1} and where $E_i \neq E'_j$. This makes sense at the level of (integral) divisors classes if m_i is divisible by t, but also makes sense as elements and rays of $N^1(X_s)_{\mathbb{R}} = \mathrm{Pic}(X_s)_{\mathbb{R}}$. In particular, if D_k is a sequence of divisors classes which generates a sequence of rays $R(D_k) \in N^1(X)_{\mathbb{R}}$, then

$$\lim_{k\to\infty} R(\mathrm{Uncoll}_t(D_k, i)) = R(\mathrm{Uncoll}_t(\lim_{k\to\infty}(D_k), i))$$

as rays in $N^1(X_s)_{\mathbb{R}}$. The uncollision process is given by rational parameters, and so preserves rationality and irrationality of rays, but is only available for actual divisors classes and no for irrational rays. For this reason, we uncollide each (integral) divisor class in the sequence and show that the limit of these uncollided divisors is wonderful.

In this section, we will only analyze the case $t = 2$. Given a divisor class $D := dH - \sum_{i=1}^{s} m_i E_i \in$ such that $m_1 = 2m$ is an even number, we consider the divisor

$$\mathrm{Uncoll}(D) := \mathrm{Uncoll}_2(D, 1) = dH - mE_1' - mE_2' - mE_3' - mE_4' - \sum_{i=2}^{s} m_i E_i.$$

Note that by uncolliding a divisor class D in $K_s^{\perp}$ we can obtain a divisor $D' = \mathrm{Uncoll}(D)$ such that $D' \cdot K_{s+3} > 0$ (this also holds for $t \geq 3$ by [12, Remark 9]). This observation and the next two lemmas allows us to construct wonderful rays in $Q_+(X_{13})$.

Lemma III.4.1 ([12, Lemma 10])**.** Let $D_k \in \mathrm{Pic}(X_s)$ be a sequence of divisor classes such that the corresponding sequence of rays converges to the irrational ray $R(D) \subseteq N^1(X_s)_{\mathbb{R}}$. If $\mathrm{Uncoll}(D) \notin \mathrm{Eff}(X_{s+3})$ for $k \gg 0$, then $\mathrm{Uncoll}(D)$ generates a wonderful ray of $\mathrm{Nef}(X_{s+3})$.

In order to construct such a sequence the authors make use of the following result.

Lemma III.4.2 ([12, Lemma 2(a)])**.** Let $D := dH - \sum_{i=1}^{s} m_i E_i \in \mathrm{Pic}(X_s)$. If $\dim |D| < m_1/2$, then $|\mathrm{Uncoll}(D)| = \emptyset$.

Proposition III.4.3. Let $D \in Q_0(X_{10})$ be an irrational class. Then $\mathrm{Uncoll}(D) \in Q_+(X_{13})$ generates a wonderful ray.

Proof. Since rational divisor classes are effective and dense in $Q_0(X_{10})$ it follows that these classes form a unique orbit under the action of the Weyl group $W(X_{10})$. By Lemma II.6.2 the only standard class of $Q_0(X_{10})$ is $-K_9$, so that we conclude that the above orbit is the one of $-K_9$. Thus, up to scalar multiples, the divisor class D is limit of the sequence of nef divisors $D_k := \sigma_k(-K_9)$, where $\sigma_k \in W(X_{10})$. Since $\dim |nD_k| = \dim |-nK_9| = 0$, for any positive integer n, the hypothesis of Lemma III.4.2 holds so that $\mathrm{Uncoll}(D_k) \notin \mathrm{Eff}(X_{13})$ for any k. Then the statement follows by Lemma III.4.1. □

III.4.2. Fundamental regions for the action of the Weyl group

Definition III.4.1. For any divisor class $D \in Q_0(X_s)$ and any $m \in \mathbb{Z}$ we define

$$\mathrm{Exc}_D(m) := \{(-1)\text{-curves } E \,:\, D \cdot E \leq m\}$$

i.e. the set of (-1)-curves whose intersection product with D is bounded by m.

Proposition III.4.2. Let $D := dH - \sum_{i=1}^s m_iE_i \in Q_0(X_s)$ be a standard class. Then

$$m_3 - 1 \leq \mu(D) < \max\{m_1, d - m_1 - m_2\}.$$

where $\mu(D) := \max\{m \in \mathbb{Z} : \mathrm{Exc}_D(m) \text{ contains only disjoint } (-1)\text{-curves}\}$.

Proof. Since D is standard, by [22, Lem. 1.4], we have $D = a_0H + a_1(H - E_1) + a_2(2H - E_1 - E_2) - a_3K_3 \cdots - a_rK_s$, with $a_i \in \mathbb{Z}_{\geq 0}$ for any i. As a consequence we have an equality

$$D + m_3K_s = D_2 + \sum_{i=3}^{s}(m_3 - m_i)E_i,$$

where D_2 is the pull-back of a class on X_2, in standard form and thus it is nef and effective. In particular $D + m_3K$ is effective, being sum of effective classes. If E is a (-1)-curve with $D \cdot E < m_3$, then $(D + m_3K) \cdot E < 0$, so that $E = E_i$ for some $i \geq 3$, since D_2 is nef. This proves $\mu(D) \geq m_3 - 1$. To prove the second inequality it suffices to observe that the (-1)-curves E_1 and $H - E_1 - E_2$ intersect non-trivially and their intersection products with D are m_1 and $d - m_1 - m_2$ respectively. □

Lemma III.4.3. Let $D := dH - \sum_{i=1}^s m_iE_i$ be a pseudostandard class of X_s. Then the minimum $\min\{D \cdot E : E \text{ is a } (-1)\text{-curve}\}$ is attained at E_s.

Proof. Let E be a (-1)-curve. Then $E = \sigma(E_s)$ for some $\sigma \in W(X_s)$. We know that σ is composition of reflections of the form σ_F as in Definition II.2.1. We begin by applying a reflection σ_F to E in such a way that either two multiplicities m_i, m_{i+1}, with $m_i < m_{i+1}$ are exchanged or the degree of $\sigma_F(E)$ is smaller than the degree of E. This is equivalent to say that $E \cdot R < 0$. From $\sigma_F(E) = E + (E \cdot R)R$ we deduce $D \cdot (\sigma_F(E) - E) = (E \cdot R)(D \cdot R) \leq 0$, where the last inequality is due to the fact that D is pseudostandard. We showed that $D \cdot \sigma_F(E) \leq D \cdot E$ and proceeding in this way one concludes that $D \cdot E_r \leq D \cdot E$. □

Lemma III.4.4. For any $s \geq 10$ the intersection form is negative definite on the subspace of $\mathrm{Pic}(X_s)$ generated by $-K_{10}, \ldots, -K_s$.

Proof. Let us proceed by induction on $s \geq 10$, the statement being trivial for $s = 10$. Assume now that the statement holds on X_{s-1}, and let us consider a class $D := \sum_{i=1}^s a_i(-K_i)$. Let us write $D = D_{s-1} + a_s(-K_s)$, with D_{s-1} the pullback of a class on X_{s+1}, and observe that $D_{s-1} \cdot (-K_s) = D_{s-1} \cdot (-K_{s-1})$, while $K_s^2 < K_{s-1}^2$, so that

$$\begin{aligned}
D^2 &= D_{s-1}^2 + 2a_sD_{s-1}(-K_s) + a_s^2K_s^2 \\
&< D_{s-1}^2 + 2a_sD_{s-1}(-K_{s-1}) + a_s^2K_{s-1}^2 \\
&= (D_{s-1} + a_s(-K_{s-1}))^2 < 0,
\end{aligned}$$

where the last inequality holds by the induction hypothesis. □

Proposition III.4.5. Let $m > 0$ be an integer. There are finitely many pseudostandard classes $D \in Q_0(X_s)$ such that

$$\min\{D \cdot E \,:\, E \text{ is a } (-1)\text{-curve of } X_s\} = -m.$$

Proof. Let $D = dH - \sum_{i=1}^{s} m_i E_i$ be a pseudostandard class in $Q_0(X_s)$. We can write $D = D_9 + R$, where D_9 is pullback of a standard class of X_9 and $R = \sum_{i=10}^{s} a_i(-K_i)$. Therefore

$$\begin{aligned}
D_9 \cdot R &= D_9 \cdot \sum_{i=10}^{s} a_i(-K_i) \\
&= (\sum_{i=10}^{s} a_i) D_9 \cdot (-K_s) && \text{due to } D_9 \cdot K_i = D_9 \cdot K_s \\
&= (\sum_{i=10}^{s} a_i) R \cdot K_s && \text{due to } D \cdot K_s = 0 \\
&= (\sum_{i=10}^{s} a_i)(\sum_{i=10}^{s} (i-9) a_i) && \text{due to } K_i \cdot K_s = K_i^2 = i - 9 \\
&= \sum_{i=10}^{s} (i-9) a_i^2 + \sum_{10 \le i < j \le s} (i + j - 18) a_i a_j.
\end{aligned}$$

On the other hand

$$R^2 = (\sum_{i=10}^{s} a_i K_i)^2 = -\sum_{i=10}^{s} (i-9) a_i^2 - 2 \sum_{10 \le i < j \le s} (i-9) a_i a_j.$$

From $D^2 = 0$ and $D_9^2 \ge 0$ (being D_9 nef) we get $2D_9 \cdot R + R^2 \le 0$, that is

$$\sum_{i=10}^{s} (i-9) a_i^2 + 2 \sum_{10 \le i < j \le s} (j-9) a_i a_j \le 0. \tag{III.4}$$

Observe that $m_i = a_i + \cdots + a_s$ for any $10 \le i \le s$, and being D pseudostandard, we deduce that $a_i \ge 0$ for any $i = 10, \ldots, s-1$, while $a_s = m_s = D \cdot E_s = -m$ by Lemma III.4.3. We can then write (III.4) as

$$\sum_{i=10}^{s} ((i-9) a_i^2 - 2(s-9) a_i m) + 2 \sum_{10 \le i < j < s} (j-9) a_i a_j \le 0.$$

Since the second sum is always non negative and $(i-9)a_i^2 - 2(s-9)a_i m \ge 0$ as soon as $a_i \ge 2(s-9)/(i-9)m$, it follows that there are finitely many integer solutions of inequality (III.4) satisfying $a_i \ge 0$ for any $i = 10, \ldots, s-1$ and $a_s = -m$. For each such solution the values of D_9^2 and $D_9 \cdot (-K_s)$ are fixed. There are finitely many such nef classes, unless D_9 is a positive multiple of $-K_9$. In this case we would have $0 = (-nK_9 + R)^2 = R^2$, so that, by Lemma III.4.4, $R \equiv 0$ and hence D nef, a contradiction. □

Example III.4.6. In the following table we provide a complete list of pseudostandard classes $D \in Q_0(X_{11})$ such that $\min\{D \cdot E : E \text{ is a } (-1)\text{-curve of } X_s\} = -m$, for values of $m \leq 3$.

m	D
1	$-K_8 - 3K_{10} + K_{11}$
2	$-K_8 - 3K_9 - 5K_{10} + 2K_{11}$
	$(H - E_1) - K_9 - 6K_{10} + 2K_{11}$
3	$-K_8 - 8K_9 - 7K_{10} + 3K_{11}$
	$-K_7 - 3K_9 - 8K_{10} + 3K_{11}$
	$-K_6 - K_9 - 9K_{10} + 3K_{11}$
	$(2H - E_1 - E_2) - 10K_{10} + 3K_{11}$

The list is produced by finding all the non-negative integer solutions of Equation (III.4) when $a_{11} = -m$ and then determining all the possibilities for D_9.

Remark III.4.7. If D is a nef divisor whose class $[D]$ lies in the cone spanned by (-1)-classes then $h^1(D) = 0$. Indeed, if we specialise the s points to a smooth elliptic curve, we obtain an anticanonical surface X'_s, i.e. where its anticanonical divisor $-K'_s$ is effective. The divisor D specialises to a D' which is still nef and satisfies $D' \cdot K'_s < 0$. By [23, Theorem 1.1] we have that $h^1(D') = 0$ and by semicontinuity we conclude that $h^1(D) = 0$ too.

This remark generalises [29, Theorem 5.6].

Appendix

In this section, we attach the Magma software codes for the calculations made in the sections II.7 and III.3.

1. **An algorithm for the pseudostandard form.** The following script transforms a effective divisor class on X_s into its pseudo-standard form.

```
StdForm := function(v)
 v := Eltseq(v);
 repeat
  v := [v[1]] cat Sort(v[2..#v]);
  k := v[1]+&+v[2..4];
  if k lt 0 then
   v[1] := v[1] + k;
   for i in [2..4] do
    v[i] := v[i] - k;
   end for;
  end if;
 until k ge 0 or v[1] le 0;
 return v;
end function;
```

   For example, the divisor $D := 5H - 2\sum_{i=1}^{6} E_i$ we get $\texttt{StdForm}(D) = H$, as we see in the Magma session:

```
> D:=[5,-2,-2,-2,-2,-2,-2];
> StdForm(D);
[ 1, 0, 0, 0, 0, 0, 0 ]
```

2. **Example II.7.7.** The next script verifies that the linear system $|4D|$ corresponding to the plane curves of degree 8 with 14 points of multiplicity 4,is non-special and has the expected dimension 4.

```
Q := GF(32003);
P<[x]> := ProjectiveSpace(Q,4);
pts := [P![Random(Q) : i in [1..5]] : n in [1..14]];
L8 := LinearSystem(P,8);
L := LinearSystem(L8,pts,[4 : i in [1..14]]);
```

To verify if the computed dimension equals the expected dimension (which is 4) we add the next assertion to above script, that will return "true" if equality is satisfied.

```
Dimension(L) eq Binomial(8+4,4)-14*Binomial(4+3,4)-1;
```

3. **Example II.7.7.**The following box shows the code used to verify that $5D|_{Y_i}$ is non-special at intersections Y_2, Y_3 and Y_4.

```
Q := GF(32003);
P<[x]> := ProjectiveSpace(Q,4);
L1 := LinearSystem(P,1);
L2 := LinearSystem(P,2);
L10 := LinearSystem(P,10);
ll := [Random(L2) : i in [1..3]];
Y3 := Scheme(P,ll[1]);
Y2 := Scheme(P,ll[1..2]);
Y1 := Scheme(P,ll[1..3]);

pts := {};
repeat
   pp := Points(Scheme(Y1,Random(L1)));
   if #pp ne 0 then pts := pts join {pp[1]}; end if;
until #pts eq 14;

pts := Setseq(pts);
Ls10 := LinearSystem(L10,pts,[5 : i in [1..14]]);
Dimension(Ls10) eq Binomial(10+4,4)-14*Binomial(5+3,4)-1;

J := IrrelevantIdeal(P);

for Y in [Y3,Y2,Y1] do
   I := &meet[Ideal(Cluster(P,p))^5+Ideal(Y) : p in pts] meet J^10;
   Ls := LinearSystemTrace(LinearSystem(P,[f : f in MinimalBasis(I)
                    | Degree(f) eq 10]),Y);
   Dimension(Ls) eq Dimension(LinearSystemTrace(L10,Y))
                    - 14*Binomial(Dimension(Y)+4,4);
end for;
```

4. **Example III.3.4.** The next script gives the (irrationals) coefficients of divisors classes in $Q_0(X_{10})$ which generates wonderful rays in $\overline{\mathrm{NE}}(X_{10})$ as seen in this example.

```
A<a,b> := AffineSpace(Rationals(),2);
R<x> := PolynomialRing(Rationals());
repeat
c := Random([10..20]);
m := [c-i: i in [1..8]];
d := Random([c..2*c]);
X := Scheme(A,[3*d-&+m-a-b,d^2-&+[u^2 : u in m]-a^2-b^2]);
f := Basis(EliminationIdeal(Ideal(X),{a}))[1];
Rre := Roots(Evaluate(f,[x,0]),RealField());
Rra := Roots(Evaluate(f,[x,0]));
until #Rra eq 0 and #Rre gt 0;
```

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