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WEIGHTED ORTHOGONAL POLYNOMIALS ON THE SIMPLEX

por

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Para $\mathrm{B}^{\perp}$

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Contents

Abstract

In this thesis we study weighted orthogonal polynomials on the d-simplex associated with a class of weights of the form,

$$\mathrm{W}_{\boldsymbol{\gamma}}(x) := \prod_{i=1}^{d} x_i^{\gamma_i} (1 - |x|)^{\gamma_{d+1}}$$

where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{d+1}) \in (-1, \infty)^{d+1}$. We obtain integral inequalities for polynomials in $\mathrm{L}^2_{\boldsymbol{\gamma}}$ norms and exploit them to obtain approximation properties of the respective $\mathrm{L}^2_{\boldsymbol{\gamma}}$-orthogonal projector measured in Sobolev norms on the d-simplex.

In addition, we characterize a certain family of second-order Sobolev-type weighted orthogonal polynomials in the triangle which generalize certain orthogonal polynomials introduced by Yuan Xu in [21] in his construction of projectors that are quasi-optimal with respect to the unweighted $\mathrm{W}^{1,2}$ norm.

Resumen

En esta tesis estudiamos polinomios ortogonales ponderados en el d-simplex asociados a una clase de pesos de la forma,

$$\mathrm{W}_{\boldsymbol{\gamma}}(x) := \prod_{i=1}^{d} x_i^{\gamma_i}(1-|x|)^{\gamma_{d+1}}$$

donde $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_{d+1}) \in (-1, \infty)^{d+1}$. Obtenemos desigualdades integrales para polinomios en normas $\mathrm{L}^2_{\boldsymbol{\gamma}}$, que explotamos para obtener propiedades de aproximación del respectivo proyector $\mathrm{L}^2_{\boldsymbol{\gamma}}$-ortogonal medidas en normas Sobolev en el d-simplex.

Además, caracterizamos una familia de polinomios ortogonales ponderados tipo Sobolev de segundo orden en el triángulo que generalizan ciertos polinomios ortogonales introducidos por Yuan Xu en [21] en su construcción de proyectores que son cuasioptimales respecto a la norma $\mathrm{W}^{1,2}$ sin ponderar.

List of Symbols

Symbol	Description	First mention
b_γ	Normalization constant for the weight W_γ.	Section 3.3
B^γ	Bilinear form associated with the operator $\mathcal{L}^\gamma$.	(2.2.11)
$\mathfrak{B}^\gamma$	Bilinear form involving second-order derivatives.	(3.1.1)
$\mathrm{C}^m(\overline{\mathrm{T}^d})$	Space of m-times continuously differentiable functions on the closure of T^d.	Section 2.1
d_j^γ, $d_{i,j}^\gamma$	First-order differential operators.	(2.2.5), (2.2.6)
$d^{\alpha,\beta}$	Univariate first-order differential operator.	(3.2.5)
$\mathcal{D}_j^\gamma$	Auxiliary first-order differential operator.	Definition 3.4.4
$\{e_i\}_{i=1}^d$	Standard basis vectors in $\mathbb{R}^d$.	(2.2.4)
$h_{n,i}^\gamma$	Auxiliary polynomials of degree n $(i = 1, 2, 3)$.	Definition 3.5.1
$\mathbb{H}_\gamma^2$	Topological completion of $\mathrm{C}^2(\overline{\mathrm{T}^2})$ w.r.t. $\|\cdot\|_{\mathbb{H}_\gamma^2}$.	Chapter 3
$\mathbb{H}_{\alpha,\beta}^\circ$	Weighted Sobolev space on I.	Section 3.2
$\mathcal{H}_n^\gamma$	Auxiliary space of polynomials of degree n.	Definition 3.5.1
H_γ^m	Topological completion of $\mathrm{C}^m(\overline{\mathrm{T}^d})$ w.r.t. $\|\cdot\|_{\gamma;m}$.	Definition 2.6.1
I	Interval $[-1, 1]$.	Section 3.2
I	Identity operator.	Theorem 3.3.5
$J_{n,k}^{\alpha,\beta,\gamma}$	Koornwinder polynomial of degree n on the triangle.	Section 2.4
$\mathrm{J}(\alpha, \beta)$	Jacobi weight function.	Section 2.3
$\mathcal{J}_n^\gamma$	Suitable space of Sobolev orthogonal polynomials.	Theorem 3.1.1

Symbol	**Description**	**First mention**
$\mathcal{L}^{\gamma}$	Sturm–Liouville operator associated with $\mathcal{V}_n^{\gamma}$.	(2.2.7)
$\tilde{\mathcal{L}}^{\gamma}$	Sturm–Liouville operator associated with $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$.	Theorem 3.1.2
L^2_{γ}	Weighted Lebesgue space with weight W_{γ}.	Section 2.1
$\mathrm{L}^2_{\gamma}(\breve{\mathrm{W}})$	Weighted Lebesgue space with respect to the depleted weight $\breve{\mathrm{W}}_{\gamma}$.	Section 2.5
$\mathrm{L}^2_{\alpha,\beta}$	Weighted Lebesgue space with respect to the Jacobi weight.	Section 3.2
m^{γ}	Auxiliary second-order differential operator.	(3.1.4)
M^{γ}	Auxiliary third-order differential operator.	(3.1.4)
$\mathrm{M}^{\alpha,\beta}$	Univariate second-order differential operator.	Definition 3.2.4
$\mathbb{N}$	Set of natural numbers.	Proposition 2.2.1
$\mathbb{N}_0$	Set of non-negative integers.	(2.2.3)
$\mathbb{N}_{\geq r}$	Set of integers greater than or equal to r.	Theorem 2.1.1
$\mathcal{N}_n^{\gamma}$	Auxiliary space of polynomials of degree n.	Definition 3.6.2
$\mathrm{P}_n^{(\alpha,\beta)}$	Jacobi polynomial of degree n.	Section 2.3
$\breve{\mathrm{P}}_{n,j}^{\gamma}$	Basis polynomials for the depleted space $\breve{\mathcal{V}}_n^{\gamma}$.	Definition 2.5.5
proj_n^{γ}	Orthogonal projector onto the space $\mathcal{V}_n^{\gamma}$.	Section 2.2
$\breve{\mathrm{proj}}_n^{\gamma}$	Orthogonal projector onto the depleted space $\breve{\mathcal{V}}_n^{\gamma}$.	(2.5.3)
$\mathrm{proj}_n^{\mathrm{J}(\alpha,\beta)}$	Orthogonal projector onto the space of Jacobi polynomials.	Section 3.2
$\mathbb{R}$	Set of real numbers.	Section 2.1
$\mathbb{R}^d$	d-dimensional Euclidean space.	Section 2.1
$\mathbb{R}^{d\times d}$	Set of $d \times d$ real matrices.	Proposition 2.5.20
R^{γ}	Projection-like operator involving derivatives.	Definition 3.3.2
$\mathcal{R}_n^{\gamma}$	Auxiliary space of polynomials of degree n.	Definition 3.6.5
S_n^{γ}	Orthogonal projector onto the polynomial space Π_n^d.	Section 2.1
$\mathcal{S}_{d+1}$	Group of permutations on $[d+1]$.	Section 2.2
T^d	Standard d-simplex in $\mathbb{R}^d$.	Section 2.1

Symbol	**Description**	**First mention**
T_σ	Affine map associated with the permutation σ.	Section 2.2
$T_\sigma^\star$	Pullback operator defined by $T_\sigma^\star f := f \circ T_\sigma$.	Section 2.2
$\mathcal{T}_{l,k}^{\gamma}$	Tensorization operator from Π_k^{d-1} to Π_l^d.	Definition 2.5.12
$V_n^{\gamma}(\mathfrak{B})$	Space of pseudo-orthogonal polynomials with respect to the bilinear form $\mathfrak{B}^{\gamma}$.	Section 3.3
$\mathcal{V}_n^{\gamma}$	Space of orthogonal polynomials with respect to $\mathrm{W}_{\boldsymbol{\gamma}}$.	(2.2.1)
$\breve{\mathcal{V}}_n^{\gamma}$	Space of orthogonal polynomials with respect to $\breve{\mathrm{W}}_{\boldsymbol{\gamma}}$.	(2.5.1)
$\mathcal{V}_n^{\gamma;\mathrm{Sob}}$	Space of Sobolev orthogonal polynomials of degree n.	(3.1.3)
$\mathcal{V}_n^{\mathrm{J}(\alpha,\beta)}$	Space of univariate Jacobi polynomials of degree n.	Section 3.2
$\mathcal{V}_n^{\alpha,\beta;\circ}$	Space of univariate Sobolev orthogonal polynomials.	Section 3.2
$\mathbb{V}_d$	Set of vertices of the simplex T^d.	Section 2.2
$\mathrm{W}_{\boldsymbol{\gamma}}$	Weight function on the simplex T^d.	Chapter 1
$\breve{\mathrm{W}}_{\boldsymbol{\gamma}}$	Depleted weight function on T^d.	Definition 2.5.1
$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$	Vectors in $\mathbb{R}^{d+1}$.	Chapter 1
$\boldsymbol{\delta}$	Vectorial function.	Definition 3.6.11
ε_i^{γ}	Second-order differential operator.	Definition 3.6.11
$\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}}$	Truncated vectors (first d components) of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$.	Proposition 2.5.4
$\Theta_{i,j}^{\gamma}$	Auxiliary third-order differential operator.	Definition 3.4.8
Λ_i^{γ}	Auxiliary second-order differential operator.	Definition 3.4.6
λ_n^{γ}	Eigenvalues associated with the operator $\mathcal{L}^{\gamma}$.	(2.2.9)
$\tilde{\lambda}_n^{\gamma}$	Eigenvalues associated with the operator $\tilde{\mathcal{L}}^{\gamma}$.	(3.1.2)
ν	Unit outward normal vector to the boundary of T^d.	Proposition 2.2.2
Π^d	Space of multivariate polynomials.	Proposition 3.4.21
Π_n^d	Space of multivariate polynomials of total degree at most n.	Section 2.1
$\varrho_{n,i}^{\gamma}$	Auxiliary polynomials of degree n ($i = 1, 2, 3$).	Definition 3.5.9
Σ_d	Specific subset of permutations in $\mathcal{S}_{d+1}$.	Definition 2.2.8
σ_j	Elements of the set Σ_d.	Definition 2.2.8

Symbol	Description	First mention
$\phi^{\gamma}_{n,i}$	Auxiliary polynomials of degree n ($i = 1, 2, 3$).	Definition 3.5.14
Φ	Map from the triangle to the reference square.	Section 3.5
$\Phi^{\gamma}_{i,j}$	Auxiliary fourth-order differential operator.	Definition 3.4.10
Ψ, Ψ_d	Maps from the reference square/cube to the simplex.	Subsection 2.5.1
Ψ^{γ}_{j}	Auxiliary second-order differential operator.	Definition 3.4.6
Ξ^{γ}_{j}	Auxiliary second-order differential operator.	Proposition 3.5.17
$[n]$	Set $\{1, 2, \dots, n\}$.	Proposition 2.2.1
$[n]_0$	Set $\{0, 1, 2, \dots, n\}$.	Proposition 2.3.3
$\|x\|$	Sum of components of the vector $x \in \mathbb{R}^d$.	Section 2.1
$\langle \cdot, \cdot \rangle^{\circ}_{\alpha,\beta}$	Inner product on the univariate Sobolev space $\mathbb{H}^{\circ}_{\alpha,\beta}$.	Section 3.2
$\langle \cdot, \cdot \rangle_{\mathbb{H}^2_\gamma}$	Inner product on the Sobolev space $\mathbb{H}^2_\gamma$.	(3.1.2)
∂_i	Partial derivative with respect to the i-th variable.	Section 2.2
$\partial_{i,j}$	Oblique partial derivative $\partial_{i,j} := \partial_j - \partial_i$.	Section 2.2
∇_k	k-fold gradient operator.	Section 2.1
$\\|\cdot\\|_\gamma$	Norm of the weighted Lebesgue space L^2_γ.	Section 2.1
$\vert\vert\vert\cdot\vert\vert\vert_\gamma$	Norm of the depleted weighted space $\mathrm{L}^2_\gamma(\breve{\mathrm{W}})$.	Section 2.5
$\\|\cdot\\|_{\gamma;m}$	Sobolev-like norm of order m on T^d.	Section 2.1
$\\|\cdot\\|_{\mathrm{J}(\alpha,\beta)}$	Norm of the weighted Lebesgue space $\mathrm{L}^2_{\alpha,\beta}$.	Section 3.2

Chapter 1

Introduction

Orthogonal polynomials have been extensively studied due to their remarkable properties and the wide range of applications they possess, both within and beyond mathematics. Among the various families of orthogonal polynomials are the Sobolev orthogonal polynomials, defined by means of an inner product that involves derivatives. Although their construction is often more involved and they exhibit fewer properties than classical orthogonal polynomials, they have the advantage that, in general, they provide better approximation results in norms involving derivatives. See [16] for a detailed survey on Sobolev orthogonal polynomials. In this thesis, we focus on the study of multivariate orthogonal polynomials on the simplex, associated with the weight function

$$\mathrm{W}_{\boldsymbol{\gamma}}(x) := \prod_{i=1}^{d} x_i^{\gamma_i}(1-|x|)^{\gamma_{d+1}},$$

where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{d+1}) \in (-1, \infty)^{d+1}$. Our objectives are twofold:

(i) To obtain approximation results for the associated orthogonal projector, measured in Sobolev norms on the simplex; and

(ii) To characterize a certain family of weighted Sobolev orthogonal polynomials on the triangle and derive an associated Sturm–Liouville operator.

The works [22] and [10] study approximation properties in the unit disk for certain weighted

L^2 orthogonal projectors, measured in Sobolev norms. In both cases, the analysis relies on exploiting specific properties of orthogonal polynomial bases. In contrast, [9] adopts a different approach, based on structural properties of Sobolev orthogonal polynomial spaces. Our first objective is inspired by this latter approach, adapting the techniques proposed in [9] to the simplex setting.

On the other hand, [21] studies approximation properties in the triangle for a second-order Sobolev orthogonal projector, measured in Sobolev norms. This work relies on the explicit construction of orthogonal polynomial bases and the exploitation of their properties. In contrast, [11] addresses an analogous problem in the unit ball, following an approach similar to that in [9], based on characterizing the Sobolev orthogonal polynomial spaces in terms of known Lebesgue orthogonal polynomial spaces. This characterization facilitates both the analysis and the extension of results. Our second objective combines the idea of defining a space analogous to that in [21] with the characterization techniques developed in [11].

The thesis is organized as follows:

- **Chapter 2** is devoted to the study of the L^2-projection error of orthogonal polynomials measured in Sobolev norms on the simplex. This chapter introduces the orthogonal polynomial spaces and their properties, analyzes Jacobi polynomials in the simplex setting, develops key integral inequalities (including Schur- and Markov-type inequalities), and finally presents the approximation results.

- **Chapter 3** This section addresses the characterization of certain Sobolev orthogonal polynomials on the triangle. By defining auxiliary differential operators and introducing bivariate polynomials, we characterize a family of weighted second-order Sobolev orthogonal polynomials. This result enables us to derive a second-order Sturm–Liouville problem associated with our Sobolev orthogonal polynomials.

- **Chapter 4** contains the conclusions of this work and briefly outlines possible directions for future research.

Chapter 2

Orthogonal polynomial projection error in Sobolev norms in the simplex

2.1 Introduction

Let T^d be the simplex defined by $\mathrm{T}^d := \{x \in \mathbb{R}^d \mid x_1, \ldots, x_d \geq 0,\ 1 - |x| > 0\}$, where $|x| = x_1 + \cdots + x_d$. Given $\boldsymbol{\gamma} \in (-1, \infty)^{d+1}$, let the weight function $\mathrm{W}_{\boldsymbol{\gamma}} \colon \mathrm{T}^d \to \mathbb{R}$ be defined by

$$\mathrm{W}_{\boldsymbol{\gamma}}(x) := x_1^{\gamma_1} \cdots x_d^{\gamma_d} (1 - |x|)^{\gamma_{d+1}} \tag{2.1.1}$$

and let $\mathrm{L}^2_{\boldsymbol{\gamma}}$ be the weighted Lebesgue space $\mathrm{L}^2(\mathrm{T}^d, \mathrm{W}_{\boldsymbol{\gamma}}) := \{\mathrm{W}_{\boldsymbol{\gamma}}^{-1/2} f \mid f \in \mathrm{L}^2(\mathrm{T}^d)\}$, whose natural squared norm is $\|u\|^2_{\boldsymbol{\gamma}} := \int_{\mathrm{T}^d} |u(x)|^2 \, \mathrm{W}_{\boldsymbol{\gamma}}(x) \, \mathrm{d}x$. Given an integer $n \geq 0$, let Π^d_n be the space of multivariate polynomials of total degree no higher than n (we adopt the convention $\Pi^d_n = \{0\}$ for $n < 0$). Let $S^{\boldsymbol{\gamma}}_n$ be the orthogonal projector mapping $\mathrm{L}^2_{\boldsymbol{\gamma}}$ onto Π^d_n. Given an integer $m \geq 0$, we define the Sobolev space $\mathrm{H}^m_{\boldsymbol{\gamma}}$ as the topological completion of $\mathrm{C}^m(\overline{\mathrm{T}^d})$ with respect to the norm $\|u\|^2_{\boldsymbol{\gamma};m} := \sum_{k=0}^m \|\nabla_k u\|^2_{\boldsymbol{\gamma}}$. The main result of this chapter is

Theorem 2.1.1. *Given $d \in \mathbb{N}_{\geq 2}$, let $\boldsymbol{\gamma} \in (-1, \infty)^{d+1}$. Then, for all integers l and r such that $1 \leq r \leq l$ there exists $C := C(d, \boldsymbol{\gamma}, l, r) > 0$ such that*

$$\left(\forall u \in \mathrm{H}^l_{\boldsymbol{\gamma}}\right) \quad \|u - S^{\boldsymbol{\gamma}}_n(u)\|_{\boldsymbol{\gamma};r} \leq C n^{2r+1/2-l} \|u\|_{\boldsymbol{\gamma};l} .$$

In [9], a result analogous to Theorem 2.1.1 is proven, but in the setting of the unit ball with the weight $\mathrm{W}_\alpha(x) = (1 - \|x\|^2)^\alpha$. The techniques used in the proof of this result [9, Th. 1.1] rely on the properties of orthogonal polynomial *spaces*, rather than depending on the specific properties of their bases. This approach is particularly useful in our context, as it allows us to extend approximation results to arbitrary dimensions without the need to deal explicitly with bases or spectral differentiation formulas (cf. [10, Th. 3.11]). Although Theorem 2.1.1 follows the ideas proposed in [9], the particular features of our orthogonal polynomial spaces lead to different results. In our case, we obtain an approximation rate involving a higher power on n than that found in [9, Th. 1.1], albeit numerical experiments suggest that there is room for improvement in the power on n.

2.2 Orthogonal polynomial spaces

Let $\mathcal{V}_n^\gamma$ be the space of the orthogonal polynomials of degree n with respect to the weight W_γ; i.e.,

$$\mathcal{V}_n^\gamma := \{p \in \Pi_n^d | \ (\forall q \in \Pi_{n-1}^d) \ \langle p, q \rangle_\gamma = 0\}. \tag{2.2.1}$$

For $n < 0$ we adopt the convention $\Pi_n^d = \{0\}$ and so $\mathcal{V}_n^\gamma = \{0\}$. There holds (cf. [6, Sec. 3.1])

$$\dim(\mathcal{V}_n^\gamma) = \binom{n+d-1}{n}. \tag{2.2.2}$$

We remark that this dimension of orthogonal polynomial spaces remains the same if we switch to any other well-defined inner product on polynomials. Let proj_n^γ denote the orthogonal projection from L_γ^2 onto $\mathcal{V}_n^\gamma$. From [6, Th. 3.2.18], $\Pi_n^d = \bigoplus_{k=0}^n \mathcal{V}_k^\gamma$ and $\mathrm{L}_\gamma^2 = \bigoplus_{k=0}^\infty \mathcal{V}_k^\gamma$, whence

$$(\forall n \in \mathbb{N}_0) \quad S_n^\gamma = \sum_{k=0}^n \mathrm{proj}_k^\gamma \quad \text{and} \quad (\forall u \in \mathrm{L}_\gamma^2) \quad u = \sum_{k=0}^\infty \mathrm{proj}_k^\gamma(u). \tag{2.2.3}$$

In what follows we will make heavy use of the definition (2.1.1) so that, e.g.,

$$\mathrm{W}_{e_j+e_{d+1}}(x) = x_j(1 - |x|). \tag{2.2.4}$$

The following proposition collects several properties of the orthogonal polynomial spaces that will be useful in what follows.

Proposition 2.2.1. *Let* $d \in \mathbb{N}$, $\boldsymbol{\gamma} \in (-1, \infty)^{d+1}$ *and* $j \in [d]$.

(i) Let $p_k \in \mathcal{V}_k^{\boldsymbol{\gamma}+e_j+e_{d+1}}$. *Then,* $\mathrm{W}_{e_j+e_{d+1}} p_k \in \mathcal{V}_k^{\boldsymbol{\gamma}} \oplus \mathcal{V}_{k+1}^{\boldsymbol{\gamma}} \oplus \mathcal{V}_{k+2}^{\boldsymbol{\gamma}}$.

(ii) Let $q_k \in \mathcal{V}_k^{\boldsymbol{\gamma}}$. *Then,* $q_k = \mathrm{proj}_{k-2}^{\boldsymbol{\gamma}+e_j+e_{d+1}}(q_k) + \mathrm{proj}_{k-1}^{\boldsymbol{\gamma}+e_j+e_{d+1}}(q_k) + \mathrm{proj}_{k}^{\boldsymbol{\gamma}+e_j+e_{d+1}}(q_k)$.

(iii) Let $u \in \mathrm{L}^2_{\boldsymbol{\gamma}}$. *Then,*

$$\mathrm{proj}_{k}^{\boldsymbol{\gamma}+e_j+e_{d+1}}(u) = \mathrm{proj}_{k}^{\boldsymbol{\gamma}+e_j+e_{d+1}}\left(\mathrm{proj}_{k}^{\boldsymbol{\gamma}}(u) + \mathrm{proj}_{k+1}^{\boldsymbol{\gamma}}(u) + \mathrm{proj}_{k+2}^{\boldsymbol{\gamma}}(u)\right).$$

(iv) Let $u \in \mathrm{L}^2_{\boldsymbol{\gamma}}$. *Then,*

$$\begin{aligned}\mathrm{proj}_{k}^{\boldsymbol{\gamma}+e_j+e_{d+1}}(u) = \mathrm{proj}_{k}^{\boldsymbol{\gamma}}(u) + \mathrm{proj}_{k}^{\boldsymbol{\gamma}+e_j+e_{d+1}} \circ \mathrm{proj}_{k+1}^{\boldsymbol{\gamma}}(u) - \mathrm{proj}_{k-1}^{\boldsymbol{\gamma}+e_j+e_{d+1}} \circ \mathrm{proj}_{k}^{\boldsymbol{\gamma}}(u)\\ + \mathrm{proj}_{k}^{\boldsymbol{\gamma}+e_j+e_{d+1}} \circ \mathrm{proj}_{k+2}^{\boldsymbol{\gamma}}(u) - \mathrm{proj}_{k-2}^{\boldsymbol{\gamma}+e_j+e_{d+1}} \circ \mathrm{proj}_{k}^{\boldsymbol{\gamma}}(u).\end{aligned}$$

Proof. Given $q_{k-1} \in \Pi_{k-1}$, $\langle \mathrm{W}_{e_j+e_{d+1}} p, q \rangle_{\boldsymbol{\gamma}} = \langle p, q \rangle_{\boldsymbol{\gamma}+e_j+e_{d+1}} = 0$ by definition (2.2.1). Therefore part (i) stems from (2.2.3). An analogous argument accounts for part (ii). Part (iii) comes from the fact that, given $p_k \in \mathcal{V}_k^{\boldsymbol{\gamma}+e_j+e_{d+1}}$,

$$\begin{aligned}\langle \mathrm{proj}_{k}^{\boldsymbol{\gamma}+e_j+e_{d+1}}(u), p_k \rangle_{\boldsymbol{\gamma}+e_j+e_{d+1}} = \langle u, p_k \rangle_{\boldsymbol{\gamma}+e_j+e_{d+1}} &\overset{(i)}{=} \langle u, \mathrm{W}_{e_j+e_{d+1}} p_k \rangle_{\boldsymbol{\gamma}}\\ &= \langle \mathrm{proj}_{k}^{\boldsymbol{\gamma}}(u) + \mathrm{proj}_{k+1}^{\boldsymbol{\gamma}}(u) + \mathrm{proj}_{k+2}^{\boldsymbol{\gamma}}(u), p_k \rangle_{\boldsymbol{\gamma}+e_j+e_{d+1}}.\end{aligned}$$

Part (iv) is obtained from adding and substracting the terms $\mathrm{proj}_{k-2}^{\boldsymbol{\gamma}+e_j+e_{d+1}}(\mathrm{proj}_{k}^{\boldsymbol{\gamma}}(u))$ and $\mathrm{proj}_{k-1}^{\boldsymbol{\gamma}+e_j+e_{d+1}}(\mathrm{proj}_{k}^{\boldsymbol{\gamma}}(u))$ to the right hand side of the part (iii) and using part (ii). ☐

We now present another set of results, this time concerning differentiation. Given $\boldsymbol{\gamma} \in \mathbb{R}^{d+1}$ and $i, j \in [d]$, we introduce two types of first-order differentiation operators, denoted by $d_i^{\boldsymbol{\gamma}}$ and $d_{i,j}^{\boldsymbol{\gamma}}$, defined as follows

$$d_j^{\boldsymbol{\gamma}} q := -\mathrm{W}_{\boldsymbol{\gamma}}^{-1} \partial_j (\mathrm{W}_{\boldsymbol{\gamma}+e_j+e_{d+1}} q) = \left[(\gamma_{d+1}+1)\mathrm{W}_{e_j} - (\gamma_j+1)\mathrm{W}_{e_{d+1}}\right] q - \mathrm{W}_{e_j+e_{d+1}} \partial_j q \qquad (2.2.5)$$

and

$$d_{i,j}^{\gamma} q := -\mathrm{W}_{\gamma}^{-1} \partial_{i,j} (\mathrm{W}_{\gamma+e_i+e_j} q) = \Big[(\gamma_i + 1) \mathrm{W}_{e_j} - (\gamma_j + 1) \mathrm{W}_{e_i} \Big] q - \mathrm{W}_{e_i+e_j} \partial_{i,j} q, \tag{2.2.6}$$

where ∂_i denotes the partial derivative with respect to x_i, and $\partial_{i,j} := \partial_j - \partial_i$. From expanded forms in (2.2.5) and (2.2.6), we observe that both d_j^{γ} and $d_{i,j}^{\gamma}$ depend on two parameters. Specifically, the former depends on γ_j and γ_{d+1}, while the latter depends on γ_i and γ_j.

We introduce the second-order differential operator

$$\mathcal{L}^{\gamma} q := \sum_{i=1}^{d} d_i^{\gamma} \partial_i q + \sum_{1 \le i < j \le d} d_{i,j}^{\gamma} \partial_{i,j} q. \tag{2.2.7}$$

For $\boldsymbol{\gamma} \in (-1, \infty)^{d+1}$, members of the space of orthogonal polynomials $\mathcal{V}_n^{\gamma}$ satisfy the second-order Sturm–Liouville problem (cf. [6, Sec. 5.3], [12, Sec. 2.1]).

$$(\forall p_n \in \mathcal{V}_n^{\gamma}) \quad \mathcal{L}^{\gamma}(p_n) = \lambda_n^{\gamma} p_n, \tag{2.2.8}$$

where

$$\lambda_n^{\gamma} = n(n + |\boldsymbol{\gamma}| + d). \tag{2.2.9}$$

From (2.2.3), the linearity of $\mathcal{L}^{\gamma}$ and the fact that the λ_n^{γ} are strictly increasing with respect to n, it is easy to check the reciprocal statement: Any polynomial that satisfies (2.2.8) belongs to $\mathcal{V}_n^{\gamma}$.

On the other hand, it is straightforward to check that (cf. [12, Eq. (2.2)]) for all $f \in \mathrm{C}^2(\overline{\mathrm{T}^d})$ and $g \in \mathrm{C}^1(\overline{\mathrm{T}^d})$

$$\langle \mathcal{L}^{\gamma}(f), g \rangle_{\gamma} = \sum_{i=1}^{d} \langle \partial_i f, \partial_i g \rangle_{\gamma+e_i+e_{d+1}} + \sum_{1 \le i < j \le d} \langle \partial_{i,j} f, \partial_{i,j} g \rangle_{\gamma+e_i+e_j} =: B^{\gamma}(f, g) \tag{2.2.10}$$

Thus, the Sturm–Liouville problem (2.2.8) satisfied by L^2_{γ}-orthogonal polynomials can be immediately recast into the weak form:

$$(\forall p_n \in \mathcal{V}_n^{\gamma}) \left(\forall q \in \mathrm{C}^1(\overline{\mathrm{T}^d}) \right) \quad B^{\gamma}(p_n, q) = \lambda_n^{\gamma} \langle p_n, q \rangle_{\gamma} \tag{2.2.11}$$

Let us note that the bilinear form B^γ defined in (2.2.10) still makes sense if its arguments lie in H^1_γ.

Proposition 2.2.2. *Let $d \in \mathbb{N}$, $\boldsymbol{\gamma} \in (-1,\infty)^{d+1}$ and $j \in [d]$.*

(i) d^γ_j *maps* Π^2_k *into* Π^d_{k+1}.

(ii) *Given* $p, q \in \mathrm{C}^1(\overline{\mathrm{T}^d})$, $\langle \partial_j p, q\rangle_{\gamma+e_j+e_{d+1}} = \langle p, d^\gamma_j q\rangle_\gamma$.

(iii) *Let* $r_k \in \mathcal{V}^{\gamma+e_j+e_{d+1}}_k$. *Then,* $d^\gamma_j r_k \in \mathcal{V}^\gamma_{k+1}$.

(iv) *Let* $p_k \in \mathcal{V}^\gamma_k$. *Then,* $\partial_j p_k \in \mathcal{V}^{\gamma+e_j+e_{d+1}}_{k-1}$.

(v) *Let* $u \in \mathrm{C}^1(\overline{\mathrm{T}^d})$. *Then,* $\partial_j \operatorname{proj}^\gamma_k(u) = \operatorname{proj}^{\gamma+e_j+e_{d+1}}_{k-1}(\partial_j u)$.

Proof. Part (i) is straightforward. Part (ii) is obtained by integration by parts and by noticing that no boundary term appears because $\mathrm{W}_{\gamma+e_j+e_{d+1}}$ vanishes on the set $\{x \in \overline{\mathrm{T}^d} : x_j = 0 \ \vee \ |x| = 1\}$, and because $\nu_j = 0$ on the rest of the boundary. Given $r_k \in \mathcal{V}^{\gamma+e_j+e_{d+1}}_k$, by part (i), $d^\gamma_j r_k \in \Pi^d_{k+1}$, and, on account of part (ii), it is L^2_γ-orthogonal to Π^d_k, whence part (iii). An analogous argument accounts for part (iv). Given $u \in \mathrm{C}^1(\overline{\mathrm{T}^d})$, by part (iv), $\partial_j \operatorname{proj}^\gamma_k(u) \in \mathcal{V}^{\gamma+e_j+e_{d+1}}_{k-1}$. Part (v) then comes about from the fact that for all $r \in \mathcal{V}^{\gamma+e_j+e_{d+1}}_{k-1}$,

$$\langle \partial_j \operatorname{proj}^\gamma_k(u), r\rangle_{\gamma+e_j+e_{d+1}} \overset{(ii)}{=} \langle \operatorname{proj}^\gamma_k(u), d^\gamma_j r\rangle_\gamma \overset{(iii)}{=} \langle u, d^\gamma_j r\rangle_\gamma \overset{(ii)}{=} \langle \partial_j u, r\rangle_{\gamma+e_j+e_{d+1}}.$$

□

Proposition 2.2.3. *Let $d \in \mathbb{N}$, $\boldsymbol{\gamma} \in (-1,\infty)^{d+1}$ and $i, j \in [d]$.*

(i) $d^\gamma_{i,j}$ *maps* Π^2_k *into* Π^d_{k+1}.

(ii) *Given* $p, q \in \mathrm{C}^1(\overline{\mathrm{T}^d})$, $\langle \partial_{i,j} p, q\rangle_{\gamma+e_i+e_j} = \langle p, d^\gamma_{i,j} q\rangle_\gamma$.

(iii) *Let* $r_k \in \mathcal{V}^{\gamma+e_i+e_j}_k$. *Then,* $d^\gamma_{i,j} r_k \in \mathcal{V}^\gamma_{k+1}$.

(iv) *Let* $p_k \in \mathcal{V}^\gamma_k$. *Then,* $\partial_{i,j} p_k \in \mathcal{V}^{\gamma+e_i+e_j}_{k-1}$.

(v) *Let* $u \in \mathrm{C}^1(\overline{\mathrm{T}^d})$. *Then,* $\partial_{i,j} \operatorname{proj}^\gamma_k(u) = \operatorname{proj}^{\gamma+e_i+e_j}_{k-1}(\partial_{i,j} u)$.

Proof. Part (i) is straightforward. Part (ii) is obtained by integration by parts and noticing that no boundary term appears because $\mathrm{W}_{\gamma+e_i+e_j}$ vanishes on the set $\{x \in \overline{\mathrm{T}^d} : \ x_i = 0 \ \vee x_j = 0\}$, because $(\nu_j - \nu_i) = 0$ on the rest of the boundary. Given $r_k \in \mathcal{V}_k^{\gamma+e_i+e_j}$, by part (i), $d_{i,j}^{\gamma} r_k \in \Pi_{k+1}^d$, and, on account of part (ii), it is L^2_γ-orthogonal to Π_k^d, whence part (iii). An analogous argument accounts for part (iv). Given $u \in \mathrm{C}^1(\overline{\mathrm{T}^d})$, by part (iv), $\partial_{i,j} \operatorname{proj}_k^\gamma(u) \in \mathcal{V}_{k-1}^{\gamma+e_i+e_j}$. Part (v) then comes about from the fact that for all $r \in \mathcal{V}_{k-1}^{\gamma+e_i+e_j}$,

$$\langle \partial_{i,j} \operatorname{proj}_k^\gamma(u), r \rangle_{\gamma+e_i+e_j} \overset{(ii)}{=} \langle \operatorname{proj}_k^\gamma(u), d_{i,j}^\gamma r \rangle_\gamma \overset{(iii)}{=} \langle u, d_{i,j}^\gamma r \rangle_\gamma \overset{(ii)}{=} \langle \partial_{i,j} u, r \rangle_{\gamma+e_i+e_j}.$$

□

An important property of the W_γ-orthogonal polynomials on the simplex is the covariance between the permutations of their parameters $(\gamma_1, \ldots, \gamma_{d+1})$ and of their set of vertices $\mathbb{V}_d := \{v_1, \ldots, v_{d+1}\}$, where $v_i = e_i$ for all $i \in [d]$ and $v_{d+1} = 0$. These properties will be exploited using suitable affine maps which are directly related to permutations of the set $\mathbb{V}_d$.

Let $(\mathcal{S}_{d+1}, \circ)$ denote the group of permutations on $[d+1]$. Given $\sigma \in \mathcal{S}_{d+1}$, we denote by $T_\sigma : \mathrm{T}^d \longrightarrow \mathrm{T}^d$ the affine map

$$T_\sigma(x) = \sum_{i=1}^{d} x_i v_{\sigma(i)} + (1 - |x|) v_{\sigma(d+1)}$$

which satisfies $T_\sigma(v_i) = v_{\sigma(i)}$ for all $i \in [d+1]$. Given any function f on T^d, we define $T_\sigma^\star f := f \circ T_\sigma$. Now, we will explore the action of $T_\sigma^\star$ on the weight function W_γ and on the orthogonal polynomial space $\mathcal{V}_n^\gamma$.

Proposition 2.2.4. *Let $d \in \mathbb{N}$ and let $\sigma, \tau \in \mathcal{S}_{d+1}$. Then, $T_\sigma \circ T_\tau = T_{\sigma\circ\tau}$.*

Proof. Let $x \in \mathrm{T}^d$. By definition, the affine map T_τ can be written as

$$T_\tau(x) = \sum_{i=1}^{d} x_i v_{\tau(i)} + (1 - |x|) v_{\tau(d+1)},$$

so that $T_\tau(x)$ is an affine combination of the vertices $v_{\tau(1)}, \ldots, v_{\tau(d+1)}$. Since T_σ is also an affine map, it preserves affine combinations, and therefore we have

$$T_\sigma \circ T_\tau(x) = \sum_{i=1}^{d} x_i T_\sigma(v_{\tau(i)}) + (1-|x|)T_\sigma(v_{\tau(d+1)}) = \sum_{i=1}^{d} x_i v_{\sigma\circ\tau(i)} + (1-|x|)v_{\sigma(\tau(d+1))} = T_{\sigma\circ\tau}(x)$$

as desired. □

Proposition 2.2.5. *Given $d \in \mathbb{N}$ and $\sigma \in \mathcal{S}_{d+1}$, the map T_σ is a bijection from T^d onto itself and $T_\sigma^{-1} = T_{\sigma^{-1}}$.*

Proof. To show that T_σ is injective, let $x, y \in \mathrm{T}^d$, such that $T_\sigma(x) = T_\sigma(y)$. Then,

$$\sum_{i=1}^{d}(x_i - y_i)v_{\sigma(i)} + |y - x|\, v_{\sigma(d+1)} = 0$$

if $\sigma(i) \neq d+1$ for all $i \in [d]$, then the set $\{v_{\sigma(i)}\}_{i=1}^{d}$ is linearly independent and $v_{\sigma(d+1)} = 0$, so $x = y$. On the other hand, if there is one $j \in [d]$ such that $\sigma(j) = d+1$, then the set $\mathbb{V}_d \backslash \{v_{\sigma(j)}\}$ is linearly independent. In this case, we obtain that $x_i = y_i$ for all $i \in [d]\backslash\{j\}$ and moreover, since $|x| = |y|$, it follows that $x_j = y_j$. Thus, in all cases, we conclude that $x = y$.

To show that T_σ is surjective, let $y \in \mathrm{T}^d$ be given. As there exists $\sigma^{-1} \in \mathcal{S}_{d+1}$ such that $\sigma \circ \sigma^{-1} = id$, we take $x = T_{\sigma^{-1}}(y)$ and by Proposition 2.2.4 we have that

$$T_\sigma(x) = T_\sigma \circ T_{\sigma^{-1}}(y) = T_{\sigma\circ\sigma^{-1}}(y) = T_{id}(y) = y.$$

To show that $T_\sigma^{-1} = T_{\sigma^{-1}}$, it suffices to compute $T_\sigma \circ T_{\sigma^{-1}}(x)$ using Proposition 2.2.4. □

Proposition 2.2.6. *Given $\boldsymbol{\gamma} \in (-1, \infty)^{d+1}$ and $\sigma \in \mathcal{S}_{d+1}$, the following hold:*

(i) $T_\sigma^\star \mathrm{W}_{\boldsymbol{\gamma}} = \mathrm{W}_{\sigma(\boldsymbol{\gamma})}$*, where* $\sigma(\boldsymbol{\gamma}) := (\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(d+1)})$.

(ii) For all $f, g \in \mathrm{C}(\overline{\mathrm{T}^d})$*, we have that* $\langle T_\sigma^\star f, T_\sigma^\star g\rangle_{\sigma(\boldsymbol{\gamma})} = \langle f, g\rangle_{\boldsymbol{\gamma}}$.

(iii) $T_\sigma^\star \mathcal{V}_n^{\boldsymbol{\gamma}} = \mathcal{V}_n^{\sigma(\boldsymbol{\gamma})}$.

Proof. To prove part (i), we define the coordinates $\xi(x) = (x_1, \ldots, x_d, 1 - |x|)$. From this definition, we deduce the identities

$$T_\sigma(x) = \sum_{i=1}^{d+1} \xi_i(x) v_{\sigma_i} \overset{j=\sigma(i)}{=} \sum_{j=1}^{d+1} \xi_{\sigma^{-1}(j)}(x) v_j = (\xi_{\sigma^{-1}(1)}, \dots, \xi_{\sigma^{-1}(d)}) \quad \text{and} \quad \mathrm{W}_\gamma(x) = \prod_{i=1}^{d+1} \xi_i^{\gamma_i}(x).$$

It follows that $\xi(T_\sigma(x)) = (\xi_{\sigma^{-1}(1)}(x), \dots, \xi_{\sigma^{-1}(d+1)}(x))$. Using these results, the transformation of the weight function is given by

$$T_\sigma^\star \mathrm{W}_\gamma(x) = \prod_{i=1}^{d+1} \xi_i^{\gamma_i}(T_\sigma(x)) = \prod_{i=1}^{d+1} \xi_{\sigma^{-1}(i)}^{\gamma_i}(x) \overset{i=\sigma(j)}{=} \prod_{j=1}^{d+1} \xi_j^{\gamma_{\sigma(j)}}(x) = \mathrm{W}_{\sigma(\gamma)}(x).$$

On the other hand, we observe that

$$\langle T_\sigma^\star f, T_\sigma^\star g \rangle_{\sigma(\gamma)} = \int_{\mathrm{T}^d} f(T_\sigma(x)) g(T_\sigma(x)) \mathrm{W}_{\sigma(\gamma)}(x) \, \mathrm{d}x.$$

Performing the change of variable $z = T_\sigma(x)$, and using the fact that $|\det(\mathrm{D}\, T_\sigma(x))| = 1$ and Proposition 2.2.5, we obtain

$$\langle T_\sigma^\star f, T_\sigma^\star g \rangle_{\sigma(\gamma)} = \int_{\mathrm{T}^d} f(z) g(z) T_{\sigma^{-1}}^\star \mathrm{W}_{\sigma(\gamma)}(z) \, \mathrm{d}z \overset{(i)}{=} \langle f, g \rangle_\gamma.$$

For part (iii), let $p \in \mathcal{V}_n^\gamma$. Given any $q \in \Pi_{n-1}^d$, using part (ii) and the fact that $T_{\sigma^{-1}}^\star$ preserves the total degree of polynomials,

$$\langle T_\sigma^\star p, q \rangle_{\sigma(\gamma)} = \langle p, T_{\sigma^{-1}}^\star q \rangle_\gamma = 0.$$

This shows that $T_\sigma^\star \mathcal{V}_n^\gamma \subset \mathcal{V}_n^{\sigma(\gamma)}$. As $T_\sigma^\star$ is and injective linear map between $\mathcal{V}_n^\gamma$ and $\mathcal{V}_n^{\sigma(\gamma)}$, because T_σ is a bijection, we have that $\dim \mathcal{V}_n^\gamma = \dim \mathcal{V}_n^{\sigma(\gamma)}$, so $T_\sigma^\star \mathcal{V}_n^\gamma = \mathcal{V}_n^{\sigma(\gamma)}$. □

Proposition 2.2.7. *Let $\sigma \in \mathcal{S}_{d+1}$ and $k \in [d]$. Then:*

(i) $\partial_k T_\sigma = v_{\sigma(k)} - v_{\sigma(d+1)}$.

(ii) For all $f \in \mathrm{C}^1(\overline{\mathrm{T}^d})$, we have that

$$\partial_k T_\sigma^\star f(x) = \sum_{l=1}^{d} T_\sigma^\star \partial_l f(x) (v_{\sigma(k)} - v_{\sigma(d+1)})_l.$$

Proof. For part (i), we have that

$$\partial_k T_\sigma(x) = \sum_{i=1}^{d} \partial_k x_i v_{\sigma(i)} + \partial_k(1-|x|)v_{\sigma(d+1)} = \sum_{i=1}^{d} \delta_{ik} v_{\sigma(i)} - v_{\sigma(d+1)} = v_{\sigma(k)} - v_{\sigma(d+1)}.$$

On the other hand, for part (ii), using the chain rule and (i) we have

$$\partial_k T_\sigma^\star f(x) = \sum_{l=1}^{d} T_\sigma^\star \partial_l f(x) \partial_k (T_\sigma)_l(x) = \sum_{l=1}^{d} T_\sigma^\star \partial_l f(x)(v_{\sigma(k)} - v_{\sigma(d+1)})_l,$$

which completes the proof. □

In Subsection 2.5.2, we will find it convenient to have affine maps with the property that, given $i \in [d]$, there exists $\sigma_i \in \mathcal{S}_{d+1}$ such that $\partial_{1,2} T_{\sigma_i}^\star f = T_{\sigma_i}^\star \partial_i f$ or $\partial_{1,2} T_{\sigma_i}^\star f = -T_{\sigma_i}^\star \partial_i f$. To this end, we construct the set Σ_d as follows.

Definition 2.2.8. *Let $d \in \mathbb{N}$. We define the set $\Sigma_d := \{\sigma_1, \sigma_2, \ldots, \sigma_d\}$, where*

(i) If $i \in \{1,2\}$, $\sigma_i \in \mathcal{S}_{d+1}$ is defined such that $\sigma_i(d+1) = i$, $\sigma_i(\frac{2}{i}) = d+1$, $\sigma_i(i) = \frac{2}{i}$ and $\sigma_i(k) = k$ for all $k \in \{3, \ldots, d\}$.

(ii) If $i \in \{3, \ldots, d\}$, $\sigma_i \in \mathcal{S}_{d+1}$ is defined such that $\sigma_i(2) = i$, $\sigma_i(i) = 2$, $\sigma_i(1) = d+1$, $\sigma_i(d+1) = 1$ and $\sigma_i(k) = k$ for all $k \in \{3, \ldots, d\} \backslash \{i\}$.

Proposition 2.2.9. *The following hold:*

(i) $\partial_1 T_{\sigma_1}^\star f = T_{\sigma_1}^\star \partial_{1,2} f, \quad \partial_2 T_{\sigma_1}^\star f = -T_{\sigma_1}^\star \partial_1 f \quad \text{and} \quad \partial_{1,2} T_{\sigma_1}^\star f = -T_{\sigma_1}^\star \partial_2 f.$

(ii) $\partial_1 T_{\sigma_2}^\star f = -T_{\sigma_2}^\star \partial_2 f, \quad \partial_2 T_{\sigma_2}^\star f = -T_{\sigma_2}^\star \partial_{1,2} f \quad \text{and} \quad \partial_{1,2} T_{\sigma_2}^\star f = T_{\sigma_2}^\star \partial_1 f.$

(iii) For each $i \in \{3, \ldots, d\}$, $\partial_1 T_{\sigma_i}^\star f = -T_{\sigma_i}^\star \partial_1 f$, $\partial_2 T_{\sigma_i}^\star f = T_{\sigma_i}^\star \partial_{i,1} f$ and $\partial_{1,2} T_{\sigma_i}^\star f = T_{\sigma_i}^\star \partial_i f$.

Proof. Direct from (ii) in Proposition 2.2.7 and Definition 2.2.8. □

In Chapter 3, we will use the subgroup $(\langle \Sigma_2 \rangle, \circ)$ to characterize the orthogonal polynomials on the triangle. Some properties of the affine maps associated with the elements of Σ_2 are as follows:

Proposition 2.2.10. *Let* $\boldsymbol{\gamma} := (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$. *Then, the following hold:*

(i) $\langle \Sigma_2 \rangle = \{id, \sigma_1, \sigma_2\}$.

(ii) $d_1^{\gamma} T_{\sigma_1}^{\star} = T_{\sigma_1}^{\star} d_{1,2}^{\sigma_2(\gamma)}$, $d_2^{\gamma} T_{\sigma_1}^{\star} = -T_{\sigma_1}^{\star} d_1^{\sigma_2(\gamma)}$ *and* $d_{1,2}^{\gamma} T_{\sigma_1}^{\star} = -T_{\sigma_1}^{\star} d_2^{\sigma_2(\gamma)}$.

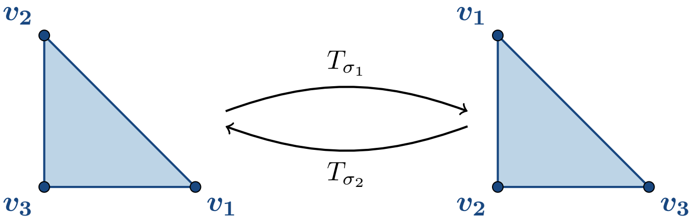
(iii) $d_1^{\gamma} T_{\sigma_2}^{\star} = -T_{\sigma_2}^{\star} d_2^{\sigma_1(\gamma)}$, $d_2^{\gamma} T_{\sigma_2}^{\star} = -T_{\sigma_2}^{\star} d_{1,2}^{\sigma_1(\gamma)}$ *and* $d_{1,2}^{\gamma} T_{\sigma_2}^{\star} = T_{\sigma_2}^{\star} d_1^{\sigma_1(\gamma)}$.

Proof. Part (i) is straightforward, using the fact that $\sigma_1 \circ \sigma_2 = id$. Part (ii) and (iii) follow from (i), (ii) and the definition of d_1^{γ}, d_2^{γ} and $d_{1,2}^{\gamma}$. □

Remark 2.2.11. When $d = 2$, the affine maps T_{σ_1} and T_{σ_2} are of the form

$$T_{\sigma_1}(x) = (1 - x_1 - x_2, x_1), \quad \text{and} \quad T_{\sigma_2}(x) = (x_2, 1 - x_1 - x_2).$$

Graphically, these maps can be represented as follows:



2.3 Jacobi polynomials

For $\alpha, \beta > -1$, let the weight function $\mathrm{J}(\alpha, \beta) : (-1, 1) \longrightarrow \mathbb{R}$ be defined by

$$\mathrm{J}(\alpha, \beta)(t) = (1 - t)^{\alpha}(1 + t)^{\beta}. \tag{2.3.1}$$

The Jacobi polynomials $\mathrm{P}_n^{(\alpha,\beta)}$ are the orthogonal polynomials on $(-1, 1)$ with respect to the weight $\mathrm{J}(\alpha, \beta)$, normalized according to [20, Eq. (4.1.1)]

$$\mathrm{P}_n^{(\alpha,\beta)}(1) = \binom{n + \alpha}{n}, \tag{2.3.2}$$

and given by the explicit representation [20, Eq. (4.1.2)]

$$\mathrm{P}_n^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\alpha+\beta+n+k+1)}{\Gamma(\alpha+k+1)} \left(\frac{1-t}{2}\right)^k . \tag{2.3.3}$$

Furthermore, they satisfy the following differential equation [20, Eq. (4.2.2)]

$$-\mathrm{J}(\alpha,\beta)^{-1}(t)\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{J}(\alpha+1,\beta+1)(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{P}_n^{(\alpha,\beta)}(t)\right] = n(n+\alpha+\beta+1)\mathrm{P}_n^{(\alpha,\beta)}(t). \tag{2.3.4}$$

Their weighted square norm is given by [20, Eq. (4.3.3)]

$$\begin{aligned}\left\|\mathrm{P}_n^{(\alpha,\beta)}\right\|^2_{\mathrm{J}(\alpha,\beta)} &:= \int_{-1}^{1} \left[\mathrm{P}_n^{(\alpha,\beta)}(t)\right]^2 \mathrm{J}(\alpha,\beta)(t)\ \mathrm{d}t \\ &= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!},\end{aligned} \tag{2.3.5}$$

where for $n = 0$ the product $(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)$ must be replaced by $\Gamma(\alpha+\beta+2)$. Some well-known properties of the Jacobi polynomials are a (see [20, Eq. (4.1.3) and Eq. (4.21.7)]):

$$\mathrm{P}_n^{(\alpha,\beta)}(-t) = (-1)^n \mathrm{P}_n^{(\beta,\alpha)}(t) \tag{2.3.6}$$

and, for $n \geq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{P}_n^{(\alpha,\beta)}(t) = \frac{n+\alpha+\beta+1}{2}\mathrm{P}_{n-1}^{(\alpha+1,\beta+1)}(t). \tag{2.3.7}$$

In what follows, we present two properties useful for the upcoming sections.

Proposition 2.3.1. *Let $\alpha, \beta > -1$ and $n \in \mathbb{N}_0$. Then,*

$$\left\|\mathrm{P}_n^{(\alpha+1,\beta)}\right\|^2_{\mathrm{J}(\alpha,\beta)} = \frac{2n+\alpha+\beta+2}{2(\alpha+1)}\left\|\mathrm{P}_n^{(\alpha+1,\beta)}\right\|^2_{\mathrm{J}(\alpha+1,\beta)}.$$

Proof. The polynomials $(1+\cdot)P_n^{(\alpha+1,\beta)\prime}$ and $n\,\mathrm{P}_n^{(\alpha+1,\beta)}$ share the same leading term in their respective expansions with respect to the basis $\left\{(1+t)^k\right\}_{k=0}^{n}$, so their difference is a polynomial of degree less than or equal to $n-1$. Therefore,

$$\begin{aligned}
2n\left\|\mathrm{P}_n^{(\alpha+1,\beta)}\right\|^2_{\mathrm{J}(\alpha+1,\beta)} &= 2\left\langle \mathrm{P}_n^{(\alpha+1,\beta)}, n\,\mathrm{P}_n^{(\alpha+1,\beta)}\right\rangle_{\mathrm{J}(\alpha+1,\beta)} \\
&= 2\left\langle \mathrm{P}_n^{(\alpha+1,\beta)}, \mathrm{J}(0,1)\mathrm{P}_n^{(\alpha+1,\beta)\prime}\right\rangle_{\mathrm{J}(\alpha+1,\beta)} \\
&= \int_{-1}^{1} 2\mathrm{P}_n^{(\alpha+1,\beta)}(t)\mathrm{P}_n^{(\alpha+1,\beta)\prime}(t)(1-t)^{\alpha+1}(1+t)^{\beta+1}\,\mathrm{d}t \\
&= \int_{-1}^{1} \left[\mathrm{P}_n^{(\alpha+1,\beta)}(t)^2\right]'(1-t)^{\alpha+1}(1+t)^{\beta+1}\,\mathrm{d}t \\
&\overset{\text{IBP}}{=} -\int_{-1}^{1} \mathrm{P}_n^{(\alpha+1,\beta)}(t)^2\left[(1-t)^{\alpha+1}(1+t)^{\beta+1}\right]'\,\mathrm{d}t.
\end{aligned}$$

Combining the above with the identity

$$-\left[(1-t)^{\alpha+1}(1+t)^{\beta+1}\right]' = 2(\alpha+1)(1-t)^{\alpha}(1+t)^{\beta} - (\alpha+\beta+2)(1-t)^{\alpha+1}(1+t)^{\beta},$$

we find that

$$2n\left\|\mathrm{P}_n^{(\alpha+1,\beta)}\right\|^2_{\mathrm{J}(\alpha+1,\beta)} = 2(\alpha+1)\left\|\mathrm{P}_n^{(\alpha+1,\beta)}\right\|^2_{\mathrm{J}(\alpha,\beta)} - (\alpha+\beta+2)\left\|\mathrm{P}_n^{(\alpha+1,\beta)}\right\|^2_{\mathrm{J}(\alpha+1,\beta)}.$$

The equality above implies the desired result. □

Proposition 2.3.2. *Let $\alpha,\beta>-1$ and $n\in\mathbb{N}_0$. Then,*

$$\left\|\mathrm{P}_n^{(\alpha+1,\beta+1)}\right\|^2_{\mathrm{J}(\alpha,\beta)} = \frac{(2n+\alpha+\beta+3)(\alpha+\beta+2)}{4(\alpha+1)(\beta+1)}\left\|\mathrm{P}_n^{(\alpha+1,\beta+1)}\right\|^2_{\mathrm{J}(\alpha+1,\beta+1)}$$

Proof. We start by noticing that

$$\begin{aligned}
-\Big((\beta-\alpha)+(\alpha+\beta+2)t\Big)\,&\mathrm{J}(\alpha+1,\beta+1)'(t) \\
&= 4(\alpha+1)(\beta+1)\,\mathrm{J}(\alpha,\beta)(t) \\
&\qquad - (\alpha+\beta+2)^2\,\mathrm{J}(\alpha+1,\beta+1)(t). \qquad (2.3.8)
\end{aligned}$$

Next, as the polynomials $n\,\mathrm{P}_n^{(\alpha+1,\beta+1)}$ and

$$\left(\frac{\beta-\alpha}{\alpha+\beta+2}+t\right)\mathrm{P}_n^{(\alpha+1,\beta+1)\prime}(t)$$

share the same leading term in their respective expansions with respect to the basis

$$\left\{1,\left(\frac{\beta-\alpha}{\alpha+\beta+2}+t\right),\dots,\left(\frac{\beta-\alpha}{\alpha+\beta+2}+t\right)^n\right\},$$

their difference is of degree less than or equal to $n-1$. Therefore,

$$\begin{aligned}2n\left\|\mathrm{P}_n^{(\alpha+1,\beta+1)}\right\|^2_{\mathrm{J}(\alpha+1,\beta+1)} &= 2\left\langle \mathrm{P}_n^{(\alpha+1,\beta+1)}, n\,\mathrm{P}_n^{(\alpha+1,\beta+1)}\right\rangle_{\mathrm{J}(\alpha+1,\beta+1)}\\ &= 2\int_{-1}^{1}\mathrm{P}_n^{(\alpha+1,\beta+1)}(t)\left(\frac{\beta-\alpha}{\alpha+\beta+2}+t\right)\mathrm{P}_n^{(\alpha+1,\beta+1)\prime}(t)\mathrm{J}(\alpha+1,\beta+1)(t)\,\mathrm{d}t.\end{aligned}$$

Using that

$$\left[\left(\frac{\beta-\alpha}{\alpha+\beta+2}+t\right)\mathrm{P}_n^{(\alpha+1,\beta+1)}(t)^2\right]' = \left(\frac{\beta-\alpha}{\alpha+\beta+2}+t\right)2\mathrm{P}_n^{(\alpha+1,\beta+1)}(t)\mathrm{P}_n^{(\alpha+1,\beta+1)\prime}(t)+\mathrm{P}_n^{(\alpha+1,\beta+1)}(t)^2.$$

It follows that

$$(2n+1)\left\|\mathrm{P}_n^{(\alpha+1,\beta+1)}\right\|^2_{\mathrm{J}(\alpha+1,\beta+1)} = \int_{-1}^{1}\left[\left(\frac{\beta-\alpha}{\alpha+\beta+2}+t\right)\mathrm{P}_n^{(\alpha+1,\beta+1)}(t)^2\right]'\mathrm{J}(\alpha+1,\beta+1)(t)\,\mathrm{d}t.$$

By integration by parts and using (2.3.8) this results in

$$\begin{aligned}(2n+1)\left\|\mathrm{P}_n^{(\alpha+1,\beta+1)}\right\|^2_{\mathrm{J}(\alpha+1,\beta+1)} &= \frac{4(\alpha+1)(\beta+1)}{\alpha+\beta+2}\left\|\mathrm{P}_n^{(\alpha+1,\beta+1)}\right\|^2_{\mathrm{J}(\alpha,\beta)}\\ &\quad-(\alpha+\beta+2)\left\|\mathrm{P}_n^{(\alpha+1,\beta+1)}\right\|^2_{\mathrm{J}(\alpha+1,\beta+1)}.\end{aligned}$$

The equality above implies the desired result. □

Proposition 2.3.3. *Let $\alpha,\beta>-1$, $n\in\mathbb{N}_0$ and $l\in[n]_0$. Then,*

$$\left\langle \mathrm{P}_n^{(\alpha+1,\beta)}, \mathrm{P}_l^{(\alpha,\beta)}\right\rangle_{\mathrm{J}(\alpha,\beta)} = 2^{\alpha+\beta+1}\frac{\Gamma(n+\beta+1)\Gamma(l+\alpha+1)}{l!\Gamma(n+\alpha+\beta+2)}.$$

Proof. From [19, Thm. 3.18], we have that

$$\mathrm{P}_n^{(\alpha+1,\beta)}(t) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \sum_{k=0}^{n} \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)} \mathrm{P}_k^{(\alpha,\beta)}(t).$$

Then, using the above and the orthogonality of the Jacobi polynomials we obtain

$$\left\langle \mathrm{P}_n^{(\alpha+1,\beta)}, \mathrm{P}_l^{(\alpha,\beta)} \right\rangle_{\mathrm{J}(\alpha,\beta)} = \frac{(2l+\alpha+\beta+1)\Gamma(n+\beta+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+2)\Gamma(l+\beta+1)} \left\| \mathrm{P}_l^{(\alpha,\beta)} \right\|^2_{\mathrm{J}(\alpha,\beta)}.$$

Finally, from (2.3.5) we deduce the desired result. □

Proposition 2.3.4. *Let $\alpha, \beta > -1$, $n \in \mathbb{N}_0$ and $l \in [n]_0$. Then,*

$$\left\langle \mathrm{P}_n^{(\alpha+2,\beta)}, \mathrm{P}_l^{(\alpha,\beta)} \right\rangle_{\mathrm{J}(\alpha+1,\beta)} = 2 \left(\frac{\alpha+1}{n+\beta+1} \right) \left\langle \mathrm{P}_{n+1}^{(\alpha+1,\beta)}, \mathrm{P}_l^{(\alpha,\beta)} \right\rangle_{\mathrm{J}(\alpha,\beta)}$$

Proof. From [2, eq. (6.4.21)] we know that

$$\mathrm{P}_l^{(\alpha,\beta)}(t) = \left(\frac{l+\alpha+\beta+1}{2l+\alpha+\beta+1} \right) \mathrm{P}_l^{(\alpha+1,\beta)}(t) - \left(\frac{l+\beta}{2l+\alpha+\beta+1} \right) \mathrm{P}_{l-1}^{(\alpha+1,\beta)}(t).$$

Then, we deduce that

$$\begin{aligned} \left\langle \mathrm{P}_n^{(\alpha+2,\beta)}, \mathrm{P}_l^{(\alpha,\beta)} \right\rangle_{\mathrm{J}(\alpha+1,\beta)} = &\left(\frac{l+\alpha+\beta+1}{2l+\alpha+\beta+1} \right) \left\langle \mathrm{P}_n^{(\alpha+2,\beta)}, \mathrm{P}_l^{(\alpha+1,\beta)} \right\rangle_{\mathrm{J}(\alpha+1,\beta)} \\ &- \left(\frac{l+\beta}{2l+\alpha+\beta+1} \right) \left\langle \mathrm{P}_n^{(\alpha+2,\beta)}, \mathrm{P}_{l-1}^{(\alpha+1,\beta)} \right\rangle_{\mathrm{J}(\alpha+1,\beta)}. \end{aligned}$$

Using Proposition 2.3.3 on the right-hand side of the above equality and performing some algebra, we have

$$\left\langle \mathrm{P}_n^{(\alpha+2,\beta)}, \mathrm{P}_l^{(\alpha,\beta)} \right\rangle_{\mathrm{J}(\alpha+1,\beta)} = 2^{\alpha+\beta+2} \frac{(\alpha+1)\Gamma(n+\beta+2)\Gamma(l+\alpha+1)}{l!\Gamma(n+\alpha+\beta+3)(n+\beta+1)}.$$

Finally, applying Proposition 2.3.3 again, we obtain the desired result. □

2.4 Koornwinder polynomials

Let $\alpha, \beta, \gamma > -1$ and $n \in \mathbb{N}_0$, in the case of the triangle, the Jacobi polynomials defined in Section 2.3 allow us to construct an orthogonal basis for the space $\mathcal{V}_n^{\alpha,\beta,\gamma}$ of (2.2.1), given by (cf. [6, Section 2.4]), $\left\{J_{n,k}^{\alpha,\beta,\gamma} :\ 0 \leq k \leq n\right\}$, where

$$J_{n,k}^{\alpha,\beta,\gamma}(x) := (x_1 + x_2)^k \mathrm{P}_k^{(\alpha,\beta)} \left(\frac{x_2 - x_1}{x_1 + x_2}\right) \mathrm{P}_{n-k}^{(2k+\alpha+\beta+1,\gamma)}(1 - 2x_1 - 2x_2), \quad 0 \leq k \leq n. \tag{2.4.1}$$

These polynomials and their variants have received various names, such as Jacobi polynomials on the triangle [21], Dubiner polynomials after [4] and (type IV) Koornwinder polynomials after [13].

Recalling the affine maps in Remark 2.2.11, the polynomials

$$K_{n,k}^{\alpha,\beta,\gamma}(x) := (1 - x_2)^k \mathrm{P}_k^{(\gamma,\alpha)} \left(\frac{2x_1}{1 - x_2} - 1\right) \mathrm{P}_{n-k}^{(2k+\alpha+\gamma+1,\beta)}(2x_2 - 1), \quad 0 \leq k \leq n, \tag{2.4.2}$$

and

$$L_{n,k}^{\alpha,\beta,\gamma}(x) := (1 - x_1)^k \mathrm{P}_k^{(\beta,\gamma)} \left(1 - \frac{2x_2}{1 - x_1}\right) \mathrm{P}_{n-k}^{(2k+\beta+\gamma+1,\alpha)}(2x_1 - 1), \quad 0 \leq k \leq n, \tag{2.4.3}$$

satisfy the relations

$$K_{n,k}^{\alpha,\beta,\gamma}(x) = T_{\sigma_1}^{\star} J_{n,k}^{\gamma,\alpha,\beta}(x) \quad \text{and} \quad L_{n,k}^{\alpha,\beta,\gamma}(x) = T_{\sigma_2}^{\star} J_{n,k}^{\beta,\gamma,\alpha}(x). \tag{2.4.4}$$

Thus, using (iii) in Proposition 2.2.6 and (2.4.4), it is straightforward to verify that the sets $\{K_{n,k}^{\alpha,\beta,\gamma} :\ 0 \leq k \leq n\}$ and $\{L_{n,k}^{\alpha,\beta,\gamma} :\ 0 \leq k \leq n\}$ each form an orthogonal basis for $\mathcal{V}_n^{\alpha,\beta,\gamma}$.

Let $f \in \mathrm{C}^1(\bar{I})$. By the chain rule, we have $\partial_{1,2} f(1 - 2x_1 - 2x_2) = 0$. In particular, this implies

$$\partial_{1,2} J_{n,0}^{\alpha,\beta,\gamma}(x) = \partial_{1,2} \mathrm{P}_n^{(\alpha+\beta+1,\gamma)}(1 - 2x_1 - 2x_2) = 0. \tag{2.4.5}$$

Consequently, combining (2.4.4) with Proposition 2.2.9, we obtain

$$\partial_1 K_{n,0}^{\alpha,\beta,\gamma} = 0 \quad \text{and} \quad \partial_2 L_{n,0}^{\alpha,\beta,\gamma} = 0. \tag{2.4.6}$$

On the other hand, from (2.3.7) and (2.4.1), it follows that

$$\partial_1 J_{n,0}^{\alpha,\beta,\gamma} = -(n+\alpha+\beta+\gamma+2) J_{n-1,0}^{\alpha+1,\beta,\gamma+1}.$$

Iterating this recurrence and using the fact that $J_{n,0}^{\alpha,\beta,\gamma}$ and all its derivatives are functions of $1-2x_1-2x_2$ we obtain

$$(\forall j \in [n]_0) \quad \partial_1^n J_{n,0}^{\alpha,\beta,\gamma} = \partial_1^{n-j} \partial_2^j J_{n,0}^{\alpha,\beta,\gamma} = (-1)^n \prod_{i=2}^{n+1} (n+i+\alpha+\beta+\gamma). \tag{2.4.7}$$

Let

$$Q_{n,k}^{\alpha,\beta,\gamma} := (-1)^k L_{n,k}^{\alpha,\beta,\gamma}, \quad 0 \le k \le n.$$

In view of (2.4.4), it is straightforward to verify that

$$J_{n,k}^{\alpha,\beta,\gamma} = (-1)^k T_{\sigma_1}^{\star} Q_{n,k}^{\gamma,\alpha,\beta}. \tag{2.4.8}$$

The $Q_{n,k}^{\alpha,\beta,\gamma}$ are precisely the polynomials obtained by setting $d=0$ in the four-parameter family of polynomials introduced in [18, Eq. (1)]. Furthermore, [18, Cor. 1] provides the following identities:

$$\begin{gathered}\partial_1 Q_{n,n}^{\alpha,\beta,\gamma} = (n+\beta) Q_{n-1,n-1}^{\alpha+1,\beta,\gamma+1}, \quad \partial_2 Q_{n,n}^{\alpha,\beta,\gamma} = (n+\beta+\gamma+1) Q_{n-1,n-1}^{\alpha,\beta+1,\gamma+1} \\ \text{and} \quad \partial_{1,2} Q_{n,n}^{\alpha,\beta,\gamma} = -(n+\gamma) Q_{n-1,n-1}^{\alpha+1,\beta+1,\gamma}.\end{gathered} \tag{2.4.9}$$

Combining (2.4.8), part (i) of Proposition 2.2.9 and (2.4.9) we have

$$\begin{aligned}\partial_1 J_{n,n}^{\alpha,\beta,\gamma} &= (-1)^n T_{\sigma_1}^{\star} \partial_{1,2} Q_{n,n}^{\gamma,\alpha,\beta} = -(n+\beta)(-1)^{n-1} T_{\sigma_1}^{\star} Q_{n-1,n-1}^{\gamma+1,\alpha+1,\beta} = -(n+\beta) J_{n-1,n-1}^{\alpha+1,\beta,\gamma+1} \\ \partial_2 J_{n,n}^{\alpha,\beta,\gamma} &= (-1)^{n+1} T_{\sigma_1}^{\star} \partial_1 Q_{n,n}^{\gamma,\alpha,\beta} = (n+\alpha)(-1)^{n-1} T_{\sigma_1}^{\star} Q_{n-1,n-1}^{\gamma+1,\alpha,\beta+1} = (n+\alpha) J_{n-1,n-1}^{\alpha,\beta+1,\gamma+1}.\end{aligned} \tag{2.4.10}$$

2.5 Integral inequalities on the simplex

In this section, we will prove three integral inequalities on the simplex. These inequalities arise naturally when following the approach of [10, Proposition 3.4] to estimate the approximation error of the S_n^γ projector in Sobolev norms. We begin by introducing another weight function on the simplex and its associated orthogonal polynomials.

Definition 2.5.1. *Given $d \in \mathbb{N}_{\geq 2}$ and $\boldsymbol{\gamma} \in \mathbb{R}^{d+1}$, let the weight function $\breve{\mathrm{W}}_\gamma : \mathrm{T}^d \longrightarrow \mathbb{R}$ be defined by*

$$\breve{\mathrm{W}}_\gamma(x) := \frac{\mathrm{W}_\gamma(x)}{x_1 + x_2},$$

for all $x \in \mathrm{T}^d$.

If $\boldsymbol{\gamma} \in \{\boldsymbol{\kappa} \in \mathbb{R}^{d+1} : \breve{\mathrm{W}}_{\boldsymbol{\kappa}} \in \mathrm{L}^1(\mathrm{T}^d)\}$, we define the space $\mathrm{L}^2_\gamma(\breve{\mathrm{W}}_\gamma) := \{\breve{\mathrm{W}}_\gamma^{-1/2} f \mid f \in \mathrm{L}^2(\mathrm{T}^d)\}$, whose natural inner product is

$$\left(\forall f, g \in \mathrm{L}^2_\gamma(\breve{\mathrm{W}})\right) \quad [f, g]_\gamma := \int_{\mathrm{T}^d} f(x) g(x) \breve{\mathrm{W}}_\gamma(x) \, \mathrm{d}x.$$

We denote by $\|\!|\cdot|\!\|_\gamma$ the norm induced by the inner product $[\cdot,\cdot]_\gamma$. Given $n \in \mathbb{N}_0$, we denote by $\breve{\mathcal{V}}_n^\gamma$ the space of $\mathrm{L}^2_\gamma(\breve{\mathrm{W}})$-orthogonal polynomials of degree n; that is,

$$\breve{\mathcal{V}}_n^\gamma := \{p \in \Pi_n^2 \mid \left(\forall q \in \Pi_{n-1}^2\right) \quad [p, q]_\gamma = 0\}. \tag{2.5.1}$$

We adopt the convention $\breve{\mathcal{V}}_n^\gamma = \{0\}$ for $n < 0$. There holds (cf. [6, Sec. 3.1])

$$(\forall n \in \mathbb{N}_0) \quad \dim\left(\breve{\mathcal{V}}_n^\gamma\right) = \binom{n+d-1}{n}. \tag{2.5.2}$$

Let $\overset{\frown}{\mathrm{proj}}{}_n^\gamma$ denote the orthogonal projection from $\mathrm{L}^2_\gamma(\breve{\mathrm{W}})$ onto $\breve{\mathcal{V}}_n^\gamma$. From [6, Th. 3.2.18],

$$\Pi_n^d = \bigoplus_{k=0}^{n} \mathcal{V}_k^\gamma(\breve{\mathrm{W}}) \quad \text{and} \quad \mathrm{L}^2_\gamma(\breve{\mathrm{W}}) = \bigoplus_{k=0}^{\infty} \mathcal{V}_k^\gamma(\breve{\mathrm{W}}),$$

whence

$$(\forall u \in \mathrm{L}^2_\gamma(\breve{\mathrm{W}})) \quad u = \sum_{k=0}^{\infty} \overset{\frown}{\mathrm{proj}}{}_k^\gamma(u). \tag{2.5.3}$$

2.5.1 Schur-type inequality on the simplex

In this subsection, we will prove two lemmas with Schur-type inequalities; i.e., bounds for norms of polynomials in terms of weaker norms scaled by monotone increasing functions of their total degree. The first will help us establish a Markov-type inequality, while the second, together with said Markov-type inequality, will aid in proving our approximation result.

Lemma 2.5.2. *Let* $d \in \mathbb{N}_{\geq 2}$, $\boldsymbol{\gamma} \in (-1,\infty)^{d+1}$ *and* $n \in \mathbb{N}_0$.

(i) There exists a constant $C > 0$, *depending only on* d *and* $\boldsymbol{\gamma}$ *such that*

$$\|q\|_{\gamma}^{2} \leq C(2n + |\boldsymbol{\gamma}| + d + 1)\vert\!\vert\!\vert q \vert\!\vert\!\vert_{\gamma+e_1+e_2}^{2},$$

for all $q \in \breve{\mathcal{V}}_n^{\gamma+e_1+e_2}$.

(ii) There exists a constant $C > 0$, *depending only on* d *and* $\boldsymbol{\gamma}$ *such that*

$$\|p\|_{\gamma}^{2} \leq C(n+1)(n + |\boldsymbol{\gamma}| + d + 1)\vert\!\vert\!\vert p \vert\!\vert\!\vert_{\gamma+e_1+e_2}^{2},$$

for all $p \in \Pi_n^d$.

Lemma 2.5.3. *Let* $d \in \mathbb{N}_{\geq 2}$ *and* $\boldsymbol{\gamma} \in (-1,\infty)^{d+1}$.

(i) Let $n \in \mathbb{N}_0$ *and* $i \in [d+1]$. *Then, for all* $r \in \mathcal{V}_n^{\gamma+e_i}$,

$$\|r\|_{\gamma}^{2} = \frac{(2n + |\boldsymbol{\gamma}| + d + 1)}{\gamma_i + 1} \|r\|_{\gamma+e_i}^{2}.$$

(ii) Let $n \in \mathbb{N}_0$ *and* $i \in [d+1]$. *Then, for all* $q \in \Pi_n^d$,

$$\|q\|_{\gamma}^{2} \leq \left(\frac{(n+1)(n + |\boldsymbol{\gamma}| + d + 1)}{\gamma_i + 1}\right) \|q\|_{\gamma+e_i}^{2}.$$

(iii) Let $n \in \mathbb{N}_0$ *and* $i \in [d]$. *Then, for all* $p \in \mathcal{V}_n^{\gamma+e_i+e_{d+1}}$,

$$\|p\|_{\gamma}^{2} \leq \left(\frac{(n+1)(n + |\boldsymbol{\gamma}| + d + 1)(2n + |\boldsymbol{\gamma}| + d + 1)}{(\gamma_{d+1} + 1)(\gamma_i + 1)}\right) \|p\|_{\gamma+e_i+e_{d+1}}^{2}.$$

The main idea of the proof of Lemma 2.5.2 is to first establish the inequality in the triangular case $d = 2$, and then apply an induction argument on the dimension. To this end, it is crucial to identify relations between the norms on the d-simplex and on the $(d+1)$-simplex. This is made possible by the non-linear maps that follow.

For the case of the triangle, we define the map $\Psi : (-1, 1) \times (-1, 1) \longrightarrow \mathrm{T}^2$ as

$$\Psi(\zeta, \eta) := \left(\frac{(1-\eta)(1-\zeta)}{4}, \frac{(1-\eta)(1+\zeta)}{4} \right).$$

This map satisfies

$$\det \left[\mathrm{D}(\Psi)(\zeta, \eta) \right] = 2^{-3}(1-\eta).$$

Similarly, for the case of the d-simplex we define the map $\Psi_d : \mathrm{T}^{d-1} \times (-1, 1) \longrightarrow \mathrm{T}^d$ as

$$\Psi_d(\boldsymbol{\zeta}, \eta) := \left(\frac{(1-\eta)}{2} \boldsymbol{\zeta}, \frac{(1-\eta)(1-|\boldsymbol{\zeta}|)}{2} \right).$$

This map satisfies

$$\det \left[\mathrm{D}(\Psi_d)(\boldsymbol{\zeta}, \eta) \right] = 2^{-d}(1-\eta)^{d-1}.$$

Proposition 2.5.4. *Let* $\boldsymbol{\gamma} \in \breve{\Lambda}^d := \{\boldsymbol{\kappa} \in \mathbb{R}^{d+1} : \ \boldsymbol{\kappa} \in (-1, \infty)^{d+1} \ \wedge \ \kappa_1 + \kappa_2 > -1\}$*. Then* $\breve{\mathrm{W}}_{\boldsymbol{\gamma}} \in \mathrm{L}^1(\mathrm{T}^d)$*.*

Proof. We prove the result by induction on the dimension d.

For $d = 2$, let $\boldsymbol{\gamma} \in \breve{\Lambda}^2$. Using the change of variables $(x_1, x_2) = \Psi(\zeta, \eta)$ we have

$$\begin{aligned} \int_{\mathrm{T}^2} \breve{\mathrm{W}}_{\boldsymbol{\gamma}}(x) \, \mathrm{d}x &= 2^{-(2\gamma_1 + 2\gamma_2 + \gamma_3 + 2)} \int_{-1}^{1} \mathrm{J}(\gamma_1 + \gamma_2, \gamma_3)(\eta) \, \mathrm{d}\eta \int_{-1}^{1} \mathrm{J}(\gamma_1, \gamma_2)(\zeta) \, \mathrm{d}\zeta \\ &= \frac{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)\Gamma(\gamma_3 + 1)}{(\gamma_1 + \gamma_2 + 1)(\gamma_1 + \gamma_2 + \gamma_3 + 2)} < \infty. \end{aligned}$$

Since $\boldsymbol{\gamma} \in \breve{\Lambda}^2$, we have $\gamma_1, \gamma_2, \gamma_3 > -1$ and $\gamma_1 + \gamma_2 > -1$, which ensures the expression above is finite. Thus, $\breve{\mathrm{W}}_{\boldsymbol{\gamma}} \in \mathrm{L}^1(\mathrm{T}^2)$.

Assume for the inductive hypothesis that for some $d \in \mathbb{N}_{\geq 2}$ the statement holds for any parameter vector in $\breve{\Lambda}^d$.

Now, let $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_{d+1}, \gamma_{d+2}) \in \breve{\Lambda}^{d+1}$. Let $\tilde{\boldsymbol{\gamma}} := (\gamma_1, \dots, \gamma_{d+1})$ so this $\boldsymbol{\gamma} = (\tilde{\boldsymbol{\gamma}}, \gamma_{d+2})$. Using the change of variables $(x_1, \dots, x_{d+1}) = \Psi_{d+1}(\boldsymbol{\zeta}, \eta)$, we obtain

$$\int_{\mathrm{T}^{d+1}} \breve{\mathrm{W}}_{\boldsymbol{\gamma}}(x)\, \mathrm{d}x = \frac{\Gamma(|\tilde{\boldsymbol{\gamma}}| + d)\Gamma(\gamma_{d+2} + 1)}{\Gamma(|\boldsymbol{\gamma}| + d + 1)} \int_{\mathrm{T}^{d}} \breve{\mathrm{W}}_{\tilde{\boldsymbol{\gamma}}}(x)\, \mathrm{d}x.$$

The condition $\boldsymbol{\gamma} \in \breve{\Lambda}^{d+1}$ implies that $\tilde{\boldsymbol{\gamma}} \in \breve{\Lambda}^{d}$. By our inductive hypothesis, the integral on the right hand side is finite. Consequently, the integral on the left-hand side is also finite, which proves that $\breve{\mathrm{W}}_{\boldsymbol{\gamma}} \in \mathrm{L}^1(\mathrm{T}^{d+1})$. This completes the induction. $\square$

The map Ψ, together with a suitable basis of $\breve{\mathcal{V}}^{\boldsymbol{\gamma}}_n$, enables us to exploit properties of Jacobi polynomials to prove Lemma 2.5.2 in the special case $d = 2$. In contrast, the map Ψ_d, together with a suitable tensorization of $\breve{\mathcal{V}}^{\boldsymbol{\gamma}}_n$, enables us to prove Lemma 2.5.2 for the general case.

Definition 2.5.5. *Let $\boldsymbol{\gamma} \in \breve{\Lambda}^2$, $n \in \mathbb{N}_0$ and $j \in [n]_0$. We define the family of polynomials $\left\{\breve{\mathrm{P}}^{\boldsymbol{\gamma}}_{n,j}\right\}_{j=0}^{n}$ by*

$$\breve{\mathrm{P}}^{\boldsymbol{\gamma}}_{n,j}(x) := |x|^j\, \mathrm{P}_j^{(\gamma_1,\gamma_2)}\left(\frac{x_2 - x_1}{|x|}\right) \mathrm{P}_{n-j}^{(2j+\gamma_1+\gamma_2,\gamma_3)}(1 - 2\,|x|).$$

Remark 2.5.6. It is easy to check that $\breve{\mathrm{P}}^{\boldsymbol{\gamma}}_{n,j} \in \Pi^2_n \backslash \Pi^2_{n-1}$, for all $j \in [n]_0$ and $n \in \mathbb{N}_0$.

In the following, we present auxiliary results concerning to the polynomials described in Definition 2.5.5 and their orthogonality relations.

Proposition 2.5.7. *Let $\boldsymbol{\gamma} \in \breve{\Lambda}^2$, $m, n \in \mathbb{N}_0$, $j \in [m]_0$, and $l \in [n]_0$. Then, the following identity involving the $\breve{\mathrm{W}}_{\boldsymbol{\gamma}}$-weighted inner product $[\cdot,\cdot]_{\boldsymbol{\gamma}}$ holds:*

$$\begin{aligned}\left[\breve{\mathrm{P}}^{\boldsymbol{\gamma}}_{m,j}, \breve{\mathrm{P}}^{\boldsymbol{\gamma}}_{n,l}\right]_{\boldsymbol{\gamma}} &= 2^{-(2+l+j+2\gamma_1+2\gamma_2+\gamma_3)} \\ &\quad \times \left\langle \mathrm{P}_l^{(\gamma_1,\gamma_2)}, \mathrm{P}_j^{(\gamma_1,\gamma_2)} \right\rangle_{\mathrm{J}(\gamma_1,\gamma_2)} \left\langle \mathrm{P}_{m-j}^{(2j+\gamma_1+\gamma_2,\gamma_3)}, \mathrm{P}_{n-l}^{(2l+\gamma_1+\gamma_2,\gamma_3)} \right\rangle_{\mathrm{J}(l+j+\gamma_1+\gamma_2+1,\gamma_3)}.\end{aligned}$$

Proof. The result follows directly by applying the change of variables $(x_1, x_2) = \Psi(\zeta, \eta)$ to the left-hand side of the identity. $\square$

Remark 2.5.8. From Proposition 2.5.7 and the orthogonality properties of the Jacobi polynomials, we deduce that $\left\{\breve{\mathrm{P}}^{\boldsymbol{\gamma}}_{n,l}\right\}_{l=0}^{n}$ is a $\breve{\mathrm{W}}_{\boldsymbol{\gamma}}$-orthogonal basis of $\breve{\mathcal{V}}^{\boldsymbol{\gamma}}_n$.

Proposition 2.5.9. *Given* $\boldsymbol{\gamma} \in \breve{\Lambda}^2$*,* $m, n \in \mathbb{N}_0$*,* $j \in [m]_0$ *and* $l \in [n]_0$*. Then, the following identity involving the* $\mathrm{W}_{\boldsymbol{\gamma}}$*-weighted inner product* $\langle \cdot, \cdot \rangle_{\boldsymbol{\gamma}}$ *holds:*

$$\begin{aligned}\left\langle \breve{\mathrm{P}}^{\boldsymbol{\gamma}+e_1+e_2}_{m,j}, \breve{\mathrm{P}}^{\boldsymbol{\gamma}+e_1+e_2}_{n,l} \right\rangle_{\boldsymbol{\gamma}} &= 2^{-(3+l+j+2\gamma_1+2\gamma_2+\gamma_3)} \\ &\times \left\langle \mathrm{P}^{(\gamma_1+1,\gamma_2+1)}_{l}, \mathrm{P}^{(\gamma_1+1,\gamma_2+1)}_{j} \right\rangle_{\mathrm{J}(\gamma_1,\gamma_2)} \left\langle \mathrm{P}^{(2j+\gamma_1+\gamma_2+2,\gamma_3)}_{m-j}, \mathrm{P}^{(2l+\gamma_1+\gamma_2+2,\gamma_3)}_{n-l} \right\rangle_{\mathrm{J}(l+j+\gamma_1+\gamma_2+1,\gamma_3)}.\end{aligned}$$

Proof. The result follows directly by applying the change of variables $(x_1, x_2) = \Psi(\zeta, \eta)$ to the left-hand side of the identity. □

Remark 2.5.10. From Proposition 2.5.9 we deduce that $\left\{\breve{\mathrm{P}}^{\boldsymbol{\gamma}+e_1+e_2}_{n,l}\right\}_{l=0}^{n}$ is a $\mathrm{W}_{\boldsymbol{\gamma}}$-orthogonal set. However, in general $\breve{\mathrm{P}}^{\boldsymbol{\gamma}+e_1+e_2}_{n,l}$ is not $\mathrm{W}_{\boldsymbol{\gamma}}$-orthogonal to lower degree polynomials, so it does not need to belong to $\breve{\mathcal{V}}^{\boldsymbol{\gamma}}_n$.

The above results allow us to prove the following Schur-type inequality in the triangle.

Proposition 2.5.11. *Let* $\boldsymbol{\gamma} \in (-1, \infty)^3$ *and* $n \in \mathbb{N}_0$*. Then, there exists a constant* $C > 0$*, depending only on* $\boldsymbol{\gamma}$*, such that*

$$\|p_n\|^2_{\boldsymbol{\gamma}} \leq C\,(2n + |\boldsymbol{\gamma}| + 3)\, \| p_n \|^2_{\boldsymbol{\gamma}+e_1+e_2},$$

for all $p_n \in \breve{\mathcal{V}}^{\boldsymbol{\gamma}+e_1+e_2}_n$.

Proof. Let $n \in \mathbb{N}_0$ and $j \in [n]_0$. From Proposition 2.5.7 (with $\boldsymbol{\gamma} + e_1 + e_2$ instead of $\boldsymbol{\gamma}$) and Proposition 2.5.9 we find that

$$\frac{\left\|\breve{\mathrm{P}}^{\boldsymbol{\gamma}+e_1+e_2}_{n,j}\right\|^2_{\boldsymbol{\gamma}}}{\|\breve{\mathrm{P}}^{\boldsymbol{\gamma}+e_1+e_2}_{n,j}\|^2_{\boldsymbol{\gamma}+e_1+e_2}} = 2^3 \frac{\left\|\mathrm{P}^{(2j+\gamma_1+\gamma_2+2,\gamma_3)}_{n-j}\right\|^2_{\mathrm{J}(2j+\gamma_1+\gamma_2+1,\gamma_3)}}{\left\|\mathrm{P}^{(2j+\gamma_1+\gamma_2+2,\gamma_3)}_{n-j}\right\|^2_{\mathrm{J}(2j+\gamma_1+\gamma_2+2,\gamma_3)}} \frac{\left\|\mathrm{P}^{(\gamma_1+1,\gamma_2+1)}_{j}\right\|^2_{\mathrm{J}(\gamma_1,\gamma_2)}}{\left\|\mathrm{P}^{(\gamma_1+1,\gamma_2+1)}_{j}\right\|^2_{\mathrm{J}(\gamma_1+1,\gamma_2+1)}}. \tag{2.5.4}$$

Using Proposition 2.3.1 and Proposition 2.3.2 in (2.5.4), we obtain

$$\left\|\breve{\mathrm{P}}^{\boldsymbol{\gamma}+e_1+e_2}_{n,j}\right\|^2_{\boldsymbol{\gamma}} = \frac{(2n+\gamma_1+\gamma_2+\gamma_3+3)}{(2j+\gamma_1+\gamma_2+2)} \frac{(2j+\gamma_1+\gamma_2+3)(\gamma_1+\gamma_2+2)}{(\gamma_1+1)(\gamma_2+1)} \|\breve{\mathrm{P}}^{\boldsymbol{\gamma}+e_1+e_2}_{n,j}\|^2_{\boldsymbol{\gamma}+e_1+e_2}.$$

The first factor on the right-hand side of the above equation attains its maximum over $j \in [n]_0$

at $j = 0$, so

$$\left\|\breve{\mathrm{P}}^{\boldsymbol{\gamma}+e_1+e_2}_{n,j}\right\|^2_{\boldsymbol{\gamma}} = \frac{(2n+\gamma_1+\gamma_2+\gamma_3+3)(\gamma_1+\gamma_2+3)}{(\gamma_1+1)(\gamma_2+1)} \vert\!\vert\!\vert\breve{\mathrm{P}}^{\boldsymbol{\gamma}+e_1+e_2}_{n,j}\vert\!\vert\!\vert^2_{\boldsymbol{\gamma}+e_1+e_2}.$$

By Remark 2.5.8 and Remark 2.5.10, the above inequality generalizes to

$$\|p_n\|^2_{\boldsymbol{\gamma}} \leq \frac{(\gamma_1+\gamma_2+3)(2n+|\boldsymbol{\gamma}|+3)}{(\gamma_1+1)(\gamma_2+1)} \vert\!\vert\!\vert p_n \vert\!\vert\!\vert^2_{\boldsymbol{\gamma}+e_1+e_2}$$

for all $p_n \in \mathcal{V}^{\breve{\boldsymbol{\gamma}}}_n$. ☐

Now we define an operator whose purpose is turning $\breve{\mathrm{W}}_{\boldsymbol{\gamma}}$-orthogonal polynomials into analogues on higher dimension.

Definition 2.5.12. *Let* $d \in \mathbb{N}_{\geq 2}$, $l \in \mathbb{N}_0$, $k \in [l]_0$, *and* $\boldsymbol{\gamma} := (\tilde{\boldsymbol{\gamma}}, \gamma_{d+1}) \in \breve{\Lambda}^d$. *We define the linear operator* $\mathcal{T}^{\boldsymbol{\gamma}}_{l,k} : \Pi^{d-1}_k \longrightarrow \Pi^d_l$ *as*

$$\mathcal{T}^{\boldsymbol{\gamma}}_{l,k}(q)(x) := |x|^k\, q\left(\frac{x_1}{|x|}, \ldots, \frac{x_{d-1}}{|x|}\right) \mathrm{P}^{(2k+|\tilde{\boldsymbol{\gamma}}|+d-2,\gamma_{d+1})}_{l-k}(1-2\,|x|).$$

Proposition 2.5.13. *Let* $d \in \mathbb{N}_{\geq 2}$, $n, m \in \mathbb{N}_0$, $k \in [n]_0$, $l \in [m]_0$, *and* $\boldsymbol{\gamma} := (\tilde{\boldsymbol{\gamma}}, \gamma_{d+1}) \in \breve{\Lambda}^d$. *Then,*

$$\left[\mathcal{T}^{\boldsymbol{\gamma}}_{n,k}(p), \mathcal{T}^{\boldsymbol{\gamma}}_{m,l}(q)\right]_{\boldsymbol{\gamma}} = 2^{-(k+l+|\boldsymbol{\gamma}|+d)}\, [p,q]_{\tilde{\boldsymbol{\gamma}}} \left\langle \mathrm{P}^{(2k+|\tilde{\boldsymbol{\gamma}}|+d-2,\gamma_{d+1})}_{n-k}, \mathrm{P}^{(2l+|\tilde{\boldsymbol{\gamma}}|+d-2,\gamma_{d+1})}_{m-l} \right\rangle_{\mathrm{J}(k+l+|\tilde{\boldsymbol{\gamma}}|+d-2,\gamma_{d+1})}$$

for all $p \in \Pi^{d-1}_k$ *and* $q \in \Pi^{d-1}_l$.

Proof. The result follows directly by applying the change of variables $(x_1, \ldots, x_d) = \Psi_d(\boldsymbol{\zeta}, \eta)$ to the left-hand side of the identity. ☐

Remark 2.5.14. From Proposition 2.5.13 and the orthogonality properties of Jacobi polynomials, we deduce that if $p \in \breve{\mathcal{V}}^{\tilde{\boldsymbol{\gamma}}}_k$ and $q \in \breve{\mathcal{V}}^{\tilde{\boldsymbol{\gamma}}}_l$, then $\mathcal{T}^{\boldsymbol{\gamma}}_{n,k}(p) \perp_{\breve{\mathrm{W}}_{\boldsymbol{\gamma}}} \mathcal{T}^{\boldsymbol{\gamma}}_{m,l}(q)$ when $(n,k) \neq (m,l)$.

Remark 2.5.15. In view of Proposition 2.5.13 and (2.3.5), it follows that the operator $\mathcal{T}^{\boldsymbol{\gamma}}_{l,k}$ introduced in Definition 2.5.12 is injective.

Proposition 2.5.16. *Let* $d \in \mathbb{N}_{\geq 2}$*,* $n \in \mathbb{N}_0$*, and* $\boldsymbol{\gamma} = (\tilde{\boldsymbol{\gamma}}, \gamma_{d+1}) \in \breve{\Lambda}^d$*. Then,*

$$\breve{\mathcal{V}}_n^{\boldsymbol{\gamma}} = \bigoplus_{k=0}^{n} \mathcal{T}_{n,k}^{\boldsymbol{\gamma}}\left(\breve{\mathcal{V}}_k^{\tilde{\boldsymbol{\gamma}}}\right). \tag{2.5.5}$$

Moreover, this decomposition is $\breve{\mathrm{W}}_{\boldsymbol{\gamma}}$*-orthogonal.*

Proof. Let $i, j \in [n]_0$ be such that $i \neq j$. By Remark 2.5.14, we have that $\mathcal{T}_{n,i}^{\boldsymbol{\gamma}}(\breve{\mathcal{V}}_i^{\tilde{\boldsymbol{\gamma}}}) \perp_{\breve{\mathrm{W}}_{\boldsymbol{\gamma}}} \mathcal{T}_{n,j}^{\boldsymbol{\gamma}}(\breve{\mathcal{V}}_j^{\tilde{\boldsymbol{\gamma}}})$. Consequently, the decomposition on the right-hand side of (2.5.5) is $\breve{\mathrm{W}}_{\boldsymbol{\gamma}}$-orthogonal. This fact, together with Remark 2.5.15, leads to the following equality:

$$\dim\left(\bigoplus_{k=0}^{n} \mathcal{T}_{n,k}^{\boldsymbol{\gamma}}(\breve{\mathcal{V}}_k^{\tilde{\boldsymbol{\gamma}}})\right) = \sum_{k=0}^{n} \dim\left(\mathcal{T}_{n,k}^{\boldsymbol{\gamma}}(\breve{\mathcal{V}}_k^{\tilde{\boldsymbol{\gamma}}})\right) = \sum_{k=0}^{n} \binom{k+d-2}{k} = \binom{n+d-1}{n}. \tag{2.5.6}$$

In view of (2.5.2) and the above, it follows that $\dim\left(\bigoplus_{k=0}^{n} \mathcal{T}_{n,k}^{\boldsymbol{\gamma}}(\breve{\mathcal{V}}_k^{\tilde{\boldsymbol{\gamma}}})\right) = \dim(\breve{\mathcal{V}}_n^{\boldsymbol{\gamma}})$. We shall now show that

$$\bigoplus_{k=0}^{n} \mathcal{T}_{n,k}^{\boldsymbol{\gamma}}(\breve{\mathcal{V}}_k^{\tilde{\boldsymbol{\gamma}}}) \subset \breve{\mathcal{V}}_n^{\boldsymbol{\gamma}}, \tag{2.5.7}$$

which, combined with (2.5.6), establishes the equality in (2.5.5).

For $m \in \mathbb{N}_0$, we first observe that Remark 2.5.14 and (2.5.6) imply

$$\dim\left(\bigoplus_{l=0}^{m}\bigoplus_{r=0}^{l} \mathcal{T}_{l,r}^{\boldsymbol{\gamma}}\left(\breve{\mathcal{V}}_r^{\tilde{\boldsymbol{\gamma}}}\right)\right) = \sum_{l=0}^{m} \binom{l+d-1}{l} = \binom{m+d}{m} = \dim(\Pi_m^d).$$

Since $\mathcal{T}_{l,r}^{\boldsymbol{\gamma}}\left(\Pi_r^{d-1}\right) \subset \Pi_l^d$ (cf. Definition 2.5.12), we deduce that

$$\Pi_m^d = \bigoplus_{l=0}^{m}\bigoplus_{r=0}^{l} \mathcal{T}_{l,r}^{\boldsymbol{\gamma}}\left(\breve{\mathcal{V}}_r^{\tilde{\boldsymbol{\gamma}}}\right). \tag{2.5.8}$$

Finally, setting $m = n-1$ in (2.5.8) and applying Remark 2.5.14 yields the inclusion in (2.5.7). □

Proposition 2.5.17. *Given* $d \in \mathbb{N}_{\geq 2}$ *and* $n \in \mathbb{N}_0$*, let* $\boldsymbol{\gamma} = (\tilde{\boldsymbol{\gamma}}, \gamma_{d+1}) \in (-1, \infty)^{d+1}$*. Then, the decomposition*

$$\bigoplus_{k=0}^{n} \mathcal{T}_{n,k}^{\boldsymbol{\gamma}+e_1+e_2}\left(\breve{\mathcal{V}}_k^{\tilde{\boldsymbol{\gamma}}+e_1+e_2}\right)$$

is W_γ*-orthogonal.*

Proof. Let $l, r \in [n]$, with $p_l \in \breve{\mathcal{V}}_l^{\tilde{\gamma}+e_1+e_2}$ and $p_r \in \breve{\mathcal{V}}_r^{\tilde{\gamma}+e_1+e_2}$. Applying the change of variables $(x_1, \ldots, x_d) = \Psi_d(\boldsymbol{\zeta}, \eta)$ yields the factorization:

$$\begin{aligned}\langle \mathcal{T}_{n,l}^{\gamma+e_1+e_2}(p_l), \mathcal{T}_{n,r}^{\gamma+e_1+e_2}(p_r)\rangle_\gamma &= 2^{-(l+r+|\gamma|+d)}\langle p_l, p_r\rangle_{\tilde{\gamma}} \\ &\quad \times \langle \mathrm{P}_{n-l}^{(2l+|\tilde{\gamma}|+d,\gamma_{d+1})}, \mathrm{P}_{n-r}^{(2r+|\tilde{\gamma}|+d,\gamma_{d+1})}\rangle_{\mathrm{J}(l+r+|\tilde{\gamma}|+d-1,\gamma_{d+1})}\end{aligned}$$

The second term on the right-hand side of the above equality can be written as follows:

$$\int_{-1}^{1} \mathrm{P}_{n-l}^{(2l+|\tilde{\gamma}|+d,\gamma_{d+1})}(\eta)\, \underbrace{\mathrm{P}_{n-r}^{(2r+|\tilde{\gamma}|+d,\gamma_{d+1})}(\eta)(1-\eta)^{r-l-1}}_{Y(\eta)}(1-\eta)^{2l+|\tilde{\gamma}|+d}(1+\eta)^{\gamma_{d+1}}\,\mathrm{d}\eta \tag{2.5.9}$$

If $r \geq l+1$, then $Y \in \Pi_{n-l-1}^1$, and since $n \geq r \geq l+1$ the integral appearing in (2.5.9) vanishes, so

$$\langle \mathcal{T}_{n,l}^{\gamma+e_1+e_2}(p_l), \mathcal{T}_{n,r}^{\gamma+e_1+e_2}(p_r)\rangle_\gamma = 0.$$

If $l \geq r+1$, the approach is analogous to the above case, by swapping the roles of r and l. □

Proposition 2.5.18. *Let* $d \in \mathbb{N}_{\geq 2}$, $\boldsymbol{\gamma} = (\tilde{\boldsymbol{\gamma}}, \gamma_{d+2}) \in (-1,\infty)^{d+2}$, $n \in \mathbb{N}_0$ *and* $k \in [n]_0$. *Suppose that there exists a constant* $C > 0$, *depending only on* d *and* $\boldsymbol{\gamma}$, *such that*

$$\|q_k\|_{\tilde{\gamma}}^2 \leq C(2k + |\tilde{\boldsymbol{\gamma}}| + d + 1)\|\!|q_k|\!\|_{\tilde{\gamma}+e_1+e_2}^2 \tag{2.5.10}$$

for all $q_k \in \breve{\mathcal{V}}_k^{\tilde{\gamma}+e_1+e_2}$. *Then, there exists a constant* $C > 0$, *depending only on* d *and* $\boldsymbol{\gamma}$, *such that*

$$\|p\|_{\gamma}^2 \leq C(2n + |\boldsymbol{\gamma}| + d + 2)\|\!|p|\!\|_{\gamma+e_1+e_2}^2 \tag{2.5.11}$$

for all $p \in \breve{\mathcal{V}}_n^{\gamma+e_1+e_2}$.

Proof. Let $p \in \breve{\mathcal{V}}_n^{\gamma+e_1+e_2}$. By Proposition 2.5.16 there exist $q_k \in \breve{\mathcal{V}}_k^{\tilde{\gamma}+e_1+e_2}$ and scalars ν_k for $0 \leq k \leq n$ such that

$$p = \sum_{k=0}^{n} \nu_k \mathcal{T}_{n,k}^{\gamma+e_1+e_2}(q_k).$$

In view of Proposition 2.5.17, we have

$$\|p\|_{\gamma}^{2} = \sum_{k=0}^{n} \nu_k^2 \left\| \mathcal{T}_{n,k}^{\gamma+e_1+e_2}(q_k) \right\|_{\gamma}^{2}. \tag{2.5.12}$$

For $k \in [n]_0$, performing the change of variables $(x_1, \dots, x_{d+1}) = \Psi_{d+1}(\boldsymbol{\zeta}, \eta)$ yields

$$\left\| \mathcal{T}_{n,k}^{\gamma+e_1+e_2}(q_k) \right\|_{\gamma}^{2} = 2^{-(2k+|\gamma|+d+1)} \|q_k\|_{\tilde{\gamma}}^{2} \left\| \mathrm{P}_{n-k}^{(2k+|\tilde{\gamma}|+d+1,\gamma_{d+2})} \right\|_{\mathrm{J}(2k+|\tilde{\gamma}|+d,\gamma_{d+2})}^{2}. \tag{2.5.13}$$

By applying Proposition 2.3.1 on the last term on the right-hand side of above equality, we obtain

$$\begin{aligned} \left\| \mathcal{T}_{n,k}^{\gamma+e_1+e_2}(q_k) \right\|_{\gamma}^{2} = 2^{-(2k+|\gamma|+d+1)} \|q_k\|_{\tilde{\gamma}}^{2} & \\ \times \frac{(2n+|\boldsymbol{\gamma}|+d+2)}{2(2k+|\tilde{\boldsymbol{\gamma}}|+d+1)} & \left\| \mathrm{P}_{n-k}^{(2k+|\tilde{\gamma}|+d+1,\gamma_{d+2})} \right\|_{\mathrm{J}(2k+|\tilde{\gamma}|+d+1,\gamma_{d+2})}^{2}. \end{aligned} \tag{2.5.14}$$

Finally, by using Proposition 2.5.16 (with $\boldsymbol{\gamma} + e_1 + e_2$ instead of $\boldsymbol{\gamma}$), the hypothesis, and the identity (2.5.14), the sum in (2.5.12) reduces to the desired bound in (2.5.11). ☐

With these preliminaries in place, we proceed to prove the main results of this subsection

Proof of Lemma 2.5.2. For (i), we do strong induction on the dimension, the base case being given by Proposition 2.5.11, while the inductive step is given by Proposition 2.5.18. Let $p \in \Pi_n^d$, from the triangle inequality, (i) and the Cauchy–Schwarz inequality for $\mathbb{R}^{n+1}$,

$$\begin{aligned} \|p\|_{\gamma}^{2} \le \left[\sum_{j=0}^{n} \left\| \breve{\mathrm{proj}}_{j}^{\,\gamma+e_1+e_2}(p) \right\|_{\gamma} \right]^{2} &\le \left[\sum_{j=0}^{n} \sqrt{(2j+|\boldsymbol{\gamma}|+d+1)C} \vert\!\vert\!\vert \breve{\mathrm{proj}}_{j}^{\,\gamma+e_1+e_2}(p) \vert\!\vert\!\vert_{\gamma+e_1+e_2} \right]^{2} \\ &\le C \left[\sum_{j=0}^{n} (2j+|\boldsymbol{\gamma}|+d+1) \right] \left[\sum_{j=0}^{n} \vert\!\vert\!\vert \breve{\mathrm{proj}}_{j}^{\,\gamma+e_1+e_2}(p) \vert\!\vert\!\vert_{\gamma+e_1+e_2}^{2} \right]. \end{aligned}$$

Using that

$$\sum_{j=0}^{n} (2j+|\boldsymbol{\gamma}|+d+1) = (n+1)(n+|\boldsymbol{\gamma}|+d+1)$$

and Parseval's identity for the $\vert\!\vert\!\vert \cdot \vert\!\vert\!\vert_{\gamma+e_1+e_2}$ norm we obtain (ii). ☐

Proof of Lemma 2.5.3. For part (i), let $r \in \mathcal{V}_n^{\gamma+e_{d+1}}$. Then, there exists a homogeneous polynomial h_r of degree n such that $r - h_r \in \Pi_{n-1}^d$ and hence $x \cdot \nabla r - x \cdot \nabla h_r \in \Pi_{n-1}^d$. Thus,

$$\langle x \cdot \nabla r, r \rangle_{\gamma+e_{d+1}} = \langle x \cdot \nabla h_r, r \rangle_{\gamma+e_{d+1}} = n \langle h_r, r \rangle_{\gamma+e_{d+1}} = n \left\| r \right\|_{\gamma+e_{d+1}}^2 . \tag{2.5.15}$$

Using the fact $\operatorname{div}(x) = d$, we have

$$\begin{aligned}(2n+d) \left\| r \right\|_{\gamma+e_{d+1}}^2 &= 2\langle x \cdot \nabla r, r \rangle_{\gamma+e_{d+1}} + \operatorname{div}(x) \left\| r \right\|_{\gamma+e_{d+1}}^2 \\ &= \int_{\mathrm{T}^d} \operatorname{div}\left(r^2(x) x\right) \mathrm{W}_{\gamma+e_{d+1}}(x) \, \mathrm{d}x. \end{aligned} \tag{2.5.16}$$

Integrating by parts and using that

$$\left(\forall x \in \mathrm{T}^d\right) \;\; x \cdot \nabla \mathrm{W}_{\gamma+e_{d+1}}(x) = (\gamma_{d+1}+1)\mathrm{W}_{\gamma}(x) - (|\boldsymbol{\gamma}|+1)\mathrm{W}_{\gamma+e_{d+1}}(x)$$

we can recast (2.5.16) as

$$(2n+d) \left\| r \right\|_{\gamma+e_{d+1}}^2 = (\gamma_{d+1}+1) \left\| r \right\|_{\gamma}^2 - (|\boldsymbol{\gamma}|+1) \left\| r \right\|_{\gamma+e_{d+1}}^2$$

so

$$\left\| r \right\|_{\gamma}^2 = \frac{(2n+|\boldsymbol{\gamma}|+d+1)}{\gamma_{d+1}+1} \left\| r \right\|_{\gamma+e_{d+1}}^2 , \tag{2.5.17}$$

which is part (i) in the special case $i = d+1$. Now, given $i \in [d]$, let $\tau_i \in \mathcal{S}_{d+1}$ such that $\tau_i(d+1) = i$, $\tau_i(i) = d+1$ and $\tau_i(k) = k$ for all $k \in [d+1] \backslash \{i, d+1\}$. The associated affine map is

$$\left(\forall x \in \mathrm{T}^d\right) \;\; T_{\tau_i}(x) := \sum_{k=1}^{d} x_k v_{\tau_i(k)} + (1-|x|) v_{\tau_i(d+1)}.$$

Let $p \in \mathcal{V}_n^{\gamma+e_i}$. From (iii) of Proposition 2.2.6, we have that $T_{\tau_i}^{\star} p \in \mathcal{V}_n^{\tau_i(\gamma+e_i)}$. Since $\tau_i(i) = d+1$, we can apply the identity (2.5.17) to obtain

$$\left\| T_{\tau_i}^{\star} p \right\|_{\tau_i(\gamma)}^2 = \frac{(2n+|\boldsymbol{\gamma}|+d+1)}{\gamma_i+1} \left\| T_{\tau_i}^{\star} p \right\|_{\tau_i(\gamma+e_i)}^2 .$$

performing the change of variables $z = T_{\tau_i}(x)$ to the integrals defining each norm and using (i) of Proposition 2.2.6, this identity simplifies to

$$\|p\|_{\gamma}^{2}=\frac{(2n+|\boldsymbol{\gamma}|+d+1)}{\gamma_i+1}\|p\|_{\gamma+e_i}^{2}$$

which is the result stated in (i). On the other hand, given $q\in\Pi_n^d$. As $\Pi_n^d=\bigoplus_{k=0}^{n}\mathcal{V}_k^{\gamma+e_{d+1}}$, then

$$\begin{aligned}\|q\|_{\gamma}^{2}\le\left[\sum_{k=0}^{n}\left\|\operatorname{proj}_k^{\gamma+e_{d+1}}(q)\right\|_{\gamma}\right]^2&\overset{(i)}{=}\left[\sum_{k=0}^{n}\sqrt{\frac{2k+|\boldsymbol{\gamma}|+d+1}{\gamma_{d+1}+1}}\left\|\operatorname{proj}_k^{\gamma+e_{d+1}}(q)\right\|_{\gamma+e_{d+1}}\right]^2\\&\le\left(\sum_{k=0}^{n}\frac{2k+|\boldsymbol{\gamma}|+d+1}{\gamma_{d+1}+1}\right)\left(\sum_{k=0}^{n}\left\|\operatorname{proj}_k^{\gamma+e_{d+1}}(q)\right\|_{\gamma+e_{d+1}}^2\right)\\&=\frac{(n+1)\,(n+|\boldsymbol{\gamma}|+d+1)}{\gamma_{d+1}+1}\sum_{k=0}^{n}\left\|\operatorname{proj}_k^{\gamma+e_{d+1}}(q)\right\|_{\gamma+e_{d+1}}^2.\end{aligned}$$

Using Parseval's identity for the $\|\cdot\|_{\gamma+e_{d+1}}$ norm on the last equality we obtain (ii). Finally, let $i\in[d]$ and $q\in\mathcal{V}_n^{\gamma+e_i+e_{d+1}}$. Then,

$$\begin{aligned}\|q\|_{\gamma}&\overset{(ii)}{\le}\left(\frac{(n+1)\,(n+|\boldsymbol{\gamma}|+d+1)}{\gamma_{d+1}+1}\right)\|q\|_{\gamma+e_{d+1}}^2\\&\overset{(i)}{=}\left(\frac{(n+1)\,(n+|\boldsymbol{\gamma}|+d+1)}{\gamma_{d+1}+1}\right)\frac{(2n+|\boldsymbol{\gamma}|+d+1)}{\gamma_i+1}\|q\|_{\gamma+e_i+e_{d+1}}^2,\end{aligned}$$

concluding with the proof. ☐

2.5.2 Markov inequality

In this subsection, we establish a Markov-type inequality on the simplex associated with the weight function $\mathrm{W}_{\boldsymbol{\gamma}}$ defined in (2.1.1).

Lemma 2.5.19. *Let $d\in\mathbb{N}_{\ge 2}$, $\boldsymbol{\gamma}\in(-1,\infty)^{d+1}$ and $n\in\mathbb{N}_0$. Then, there exists a constant $C>0$, depending only on d and $\boldsymbol{\gamma}$, such that*

$$\|\nabla p\|_{\gamma}\le Cn^2\|p\|_{\gamma}$$

for all $p\in\Pi_n^d$.

This result is a consequence of the following proposition.

Proposition 2.5.20. *Let $d \in \mathbb{N}_{\geq 2}$, $\boldsymbol{\gamma} \in (-1,\infty)^{d+1}$ and $n \in \mathbb{N}_0$. Then, there exists a constant $C > 0$ such that for all $p \in \Pi_n^d$ and $x \in \mathrm{T}^d$ there holds:*

$$[\partial_{1,2}p(x)]^2 \,\frac{x_1x_2}{x_1+x_2} \leq C\left((x\cdot\nabla p(x))^2\frac{1-|x|}{|x|} + \sum_{1\leq i<j\leq d}[\partial_{i,j}p(x)]^2\,\frac{x_ix_j}{|x|}\right) \tag{2.5.18}$$

Proof. Given $x \in \mathrm{T}^d$, we define the matrices $K(x), M(x) \in \mathbb{R}^{d\times d}$ by

$$K(x) := \frac{x_1x_2}{x_1+x_2}(e_1-e_2)\otimes(e_1-e_2) = \frac{x_1x_2}{x_1+x_2}\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$M(x) := \frac{1-|x|}{|x|}x\otimes x + \sum_{1\leq i<j\leq d}\frac{x_ix_j}{|x|}(e_i-e_j)\otimes(e_i-e_j).$$

It is easy to check that $K(x)$ and $M(x)$ are symmetric matrices. We also claim that $M(x)$ is positive-definite; indeed, as $x_1,\dots,x_d > 0$ and $1-|x| > 0$, for all $z \in \mathbb{R}^d$,

$$z\cdot M(x)z = \frac{1-|x|}{|x|}(x\cdot z)^2 + \sum_{1\leq i<j\leq d}\frac{x_ix_j}{|x|}(z_i-z_j)^2 \geq 0$$

and if $z\cdot M(x)z = 0$, then $z = 0$.

Now let us note that the vector $v^{(1)}(x) := -x_2e_1 + x_1e_2$ is an eigenvector of the generalized eigenvalue problem: Find $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^d\setminus\{0\}$ such that

$$K(x)v = \lambda M(x)v. \tag{2.5.19}$$

Indeed, it is straightforward to check that

$$K(x)v^{(1)}(x) = M(x)v^{(1)}(x)$$

which goes on to show that the eigenvalue associated with the eigenvector $v^{(1)}(x)$ is $\lambda^{(1)}(x) = 1$. As $K(x)$ is symmetric and $M(x)$ is symmetric and positive-definite, we can complete the set $\{(\lambda^{(1)}(x), v^{(1)}(x))\}$ into a complete set $\{(\lambda^{(i)}(x), v^{(i)}(x))\}_{i=1}^{d}$ of eigenpairs of (2.5.19), so that $\{v^{(i)}(x)\}_{i=1}^{d}$ is an $M(x)$-orthogonal basis of $\mathbb{R}^d$. As $K(x)$ is a rank-1 matrix and $\lambda^{(1)}(x) \neq 0$, it follows that $\lambda^{(i)}(x) = 0$ for $i \in \{2, \dots, d\}$. Therefore, $\lambda^{(1)}(x) = 1$ is the maximum of the Rayleigh quotient $(v \cdot K(x)v)/(v \cdot M(x)v)$ among the $v \in \mathbb{R}^d \setminus \{0\}$, whence

$$\left(\forall\, v \in \mathbb{R}^d\right) \qquad v \cdot K(x)v \leq v \cdot M(x)v.$$

As

$$\nabla p(x) \cdot K(x)\nabla p(x) = [\partial_{1,2}p(x)]^2 \frac{x_1\, x_2}{x_1 + x_2}$$

and

$$\nabla p(x) \cdot M(x)\nabla p(x) = (x \cdot \nabla p(x))^2\, \frac{1 - |x|}{|x|} + \sum_{1 \leq i < j \leq d} [\partial_{i,j}p(x)]^2\, \frac{x_i\, x_j}{|x|},$$

we obtain (2.5.18) with $C = 1$. □

We now cite a way of expressing the weak form of the Sturm–Liouville operator $\mathcal{L}^{\gamma}$ of (2.2.8) that is an alternative to the one given in (2.2.11).

Proposition 2.5.21. *Let $d \in \mathbb{N}_{\geq 2}$, $\boldsymbol{\gamma} \in (-1, \infty)^{d+1}$ and $f, g \in \mathrm{C}^2(\mathrm{T}^d)$. Then,*

$$\int_{\mathrm{T}^d} \mathcal{L}^{\gamma} f(x) \cdot g(x) \mathrm{W}_{\gamma}(x)\; \mathrm{d}x = \int_{\mathrm{T}^d} (x \cdot \nabla f(x))(x \cdot \nabla g(x))(1 - |x|)\mathrm{W}_{\gamma}(x) \frac{\mathrm{d}x}{|x|} + \sum_{1 \leq i < j \leq d} \int_{\mathrm{T}^d} \partial_{i,j} f(x) \partial_{i,j} g(x) x_i x_j \mathrm{W}_{\gamma}(x) \frac{\mathrm{d}x}{|x|}. \quad (2.5.20)$$

Proof. This is [12, Th. 2.4]. □

Lemma 2.5.22. *Let $d \in \mathbb{N}_{\geq 2}$, $\boldsymbol{\gamma} \in (-1, \infty)^{d+1}$ and $n \in \mathbb{N}_0$. Then, there exists a constant $C > 0$, depending only on d and $\boldsymbol{\gamma}$, such that*

$$\left\| \partial_{1,2} p \right\|_{\gamma} \leq C n^2 \left\| p \right\|_{\gamma},$$

for all $p \in \Pi_n^d$.

Proof. Let $p \in \Pi_n^d$. By Proposition 2.5.20, we deduce the inequality

$$\|\!|\partial_{1,2}p|\!\|_{\gamma+e_1+e_2}^2 \leq \int_{\mathrm{T}^d} (x \cdot \nabla p)^2 \, (1-|x|) \mathrm{W}_\gamma(x) \frac{\mathrm{d}x}{|x|} + \sum_{1 \leq i < j \leq d} \int_{\mathrm{T}^d} \left[\partial_{i,j} p(x)\right]^2 x_i x_j \mathrm{W}_\gamma(x) \frac{\mathrm{d}x}{|x|}.$$

Furthermore, applying Proposition 2.5.21 to the right hand side yields the bound

$$\|\!|\partial_{1,2}p|\!\|_{\gamma+e_1+e_2}^2 \leq |\langle \mathcal{L}^\gamma p, p\rangle_\gamma| \leq \|\mathcal{L}^\gamma p\|_\gamma \, \|p\|_\gamma \, .$$

On the other hand,

$$\|\mathcal{L}^\gamma p\|_\gamma = \left\| \sum_{k=0}^n \mathcal{L}^\gamma \operatorname{proj}_k^\gamma(p) \right\|_\gamma \overset{(2.2.8)}{=} \left\| \sum_{k=0}^n k(k+|\boldsymbol{\gamma}|+d) \operatorname{proj}_k^\gamma(p) \right\|_\gamma \overset{\text{Parseval}}{\leq} n(n+|\boldsymbol{\gamma}|+d) \, \|p\|_\gamma \, .$$

From the last two inequalities and part (ii) of Lemma 2.5.2, we deduce that

$$\begin{aligned} \|\partial_{1,2}p\|_\gamma^2 &\leq \hat{C} n(n+|\boldsymbol{\gamma}|+d) \|\!|\partial_{1,2}p|\!\|_{\gamma+e_1+e_2}^2 \\ &\leq \hat{C} n^2 (n+|\boldsymbol{\gamma}|+d)^2 \, \|p\|_\gamma^2 \, . \end{aligned}$$

The inequality then follows after realizing that there exists a positive constant C depending on $\boldsymbol{\gamma}$ and d only such that $\hat{C} n^2 (n+|\boldsymbol{\gamma}|+d)^2 \leq C n^4$ for all $n \in \mathbb{N}$. $\square$

Proof of Lemma 2.5.19. Let $p \in \Pi_n^d$. Since $T_{\sigma_1}^\star p \in \Pi_n^d$, it follows from Lemma 2.5.22 that

$$\left\| \partial_{1,2} T_{\sigma_1}^\star p \right\|_{\sigma_1(\gamma)}^2 \leq C n^4 \left\| T_{\sigma_1}^\star p \right\|_{\sigma_1(\gamma)}^2 .$$

Performing the change of variables $z = T_{\sigma_1}(x)$ and applying (i) in Proposition 2.2.9, we deduce that the inequality above is equivalent to

$$\|\partial_2 p\|_\gamma^2 \leq C n^4 \, \|p\|_\gamma^2 \, .$$

By analogous arguments, we obtain the same inequality for $\partial_1 p$.

Now let $i \in \{3, \ldots, d\}$. From (iii) in Proposition 2.2.9, and after the change of variables

$z = T_{\sigma_i}(x)$, we observe that

$$\left\|\partial_{1,2} T^{\star}_{\sigma_i} p\right\|^2_{\sigma_i(\gamma)} = \left\|T^{\star}_{\sigma_i} \partial_i p\right\|^2_{\sigma_i(\gamma)} = \|\partial_i p\|^2_{\gamma} \quad \text{and} \quad \left\|T^{\star}_{\sigma_i} p\right\|^2_{\sigma_i(\gamma)} = \|p\|^2_{\gamma}.$$

Since $T^{\star}_{\sigma_i} p \in \Pi^d_n$, applying again Lemma 2.5.22 yields

$$\|\partial_i p\|^2_{\gamma} \leq C n^4 \|p\|^2_{\gamma}.$$

This completes the proof. □

Corollary 2.5.23. *Given $d \in \mathbb{N}_{\geq 2}$, let $\boldsymbol{\gamma} \in (-1, \infty)^{d+1}$, $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then, there exists a constant $C > 0$, depending only on d, r and $\boldsymbol{\gamma}$, such that*

$$\|\nabla_r p\|_{\gamma} \leq C n^{2r} \|p\|_{\gamma}$$

for all $p \in \Pi^d_n$.

Proof. We proceed by induction on the order of differentiation, observing that the base case corresponds to Lemma 2.5.19. □

2.6 Approximation results

In this section, we present two approximation results. First, we establish an estimate for the projector S^{γ}_n in the Lebesgue norm $\|\cdot\|_{\gamma}$. Although similar results were established in [7] and [3] on the triangle and the simplex, respectively. We provide an alternative proof that exploits the properties of the Sturm-Liouville operator $\mathcal{L}^{\gamma}$. Finally, we present the proof of the main result of this chapter, Theorem 2.1.1.

Definition 2.6.1. *Given $d \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\boldsymbol{\gamma} \in (-1, \infty)^{d+1}$, we define H^m_{γ} as the topological completion of $(\mathrm{C}^m(\overline{\mathrm{T}^d}),\ \|\cdot\|_{\gamma;m})$.*

That is, up to isometry, H^m_{γ} is the space of the equivalence classes of Cauchy sequences of

$(\mathrm{C}^m(\overline{\mathrm{T}^d}),\ \|\cdot\|_{\gamma;m})$ with respect to the equivalence relation $\sim$ defined by

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \iff \lim_{n\to\infty} \|x_n - y_n\|_{\gamma;m} = 0,$$

equipped with the metric

$$(x, y) \mapsto \lim_{n\to\infty} \|x_n - y_n\|_{\gamma;m},$$

where $(x_n)_{n\in\mathbb{N}}$ y $(y_n)_{n\in\mathbb{N}}$ are any representatives of the equivalence classes x and y, respectively, which makes it a complete metric space. Identifying each $f \in \mathrm{C}^m(\overline{\mathrm{T}^d})$ with the equivalence class of the constant sequence $(f)_{n\in\mathbb{N}}$, $\mathrm{C}^m(\overline{\mathrm{T}^d})$ is a dense subset of H^m_γ [15, Th. III.33.VII].

It is easy checked that the map $(x, y) \mapsto \lim_{n\to\infty}\langle x_n, y_n\rangle_{\gamma;m}$, where again $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are any representatives of the equivalence classes x and y, respectively, is a well defined inner product that induces the above metric, whence H^m_γ is a Hilbert space. We denote that inner product by $\langle\cdot,\cdot\rangle_{\gamma;m}$ as well.

Proposition 2.6.2. *Let $d \in \mathbb{N}$ and $\boldsymbol{\gamma} \in (-1,\infty)^{d+1}$. Then, the bilinear form $B^\gamma : \mathrm{H}^1_\gamma \times \mathrm{H}^1_\gamma \longrightarrow \mathbb{R}$ defined in (2.2.10) satisfies*

$$\left(\forall u, v \in \mathrm{H}^1_\gamma\right) \quad |B^\gamma(u, v)| \le \|\nabla u\|_\gamma \|\nabla v\|_\gamma. \tag{2.6.1}$$

Proof. Let $u \in \mathrm{C}^1(\overline{\mathrm{T}^d})$. First we note that

$$\sum_{i=1}^d |\partial_i u|^2 \mathrm{W}_{e_i+e_{d+1}} = \sum_{i=1}^d |\partial_i u|^2 \mathrm{W}_{e_i}\left[1 - \sum_{j=1}^d \mathrm{W}_{e_j}\right] = \sum_{i=1}^d |\partial_i u|^2 \mathrm{W}_{e_i} - \sum_{i,j=1}^d |\partial_i u|^2 \mathrm{W}_{e_i+e_j}$$

and

$$\begin{aligned}\sum_{1\le i<j\le d} |\partial_{i,j} u|^2 \mathrm{W}_{e_i+e_j} &= \sum_{1\le i<j\le d}\left(|\partial_j u|^2 - 2\partial_i u\ \partial_j u + |\partial_i u|^2\right)\mathrm{W}_{e_i+e_j}\\ &= \sum_{i,j=1}^d |\partial_i u|^2 \mathrm{W}_{e_i+e_j} - \sum_{1\le i<j\le d} 2\partial_i u \partial_j u \mathrm{W}_{e_i+e_j}.\end{aligned}$$

Thus,

$$\sum_{i=1}^{d} |\partial_i u|^2 \mathrm{W}_{e_i+e_{d+1}} + \sum_{1\le i<j\le d} |\partial_{i,j} u|^2 \mathrm{W}_{e_i+e_j} = \sum_{i=1}^{d} |\partial_i u|^2 \mathrm{W}_{e_i} - 2 \sum_{1\le i<j\le d} \partial_i u \partial_j u \mathrm{W}_{e_i+e_j}.$$

Since $\mathrm{W}_{e_i}(x) \le 1$ for all $x \in \mathrm{T}^d$, we deduce that

$$\sum_{i=1}^{d} |\partial_i u|^2 \mathrm{W}_{e_i+e_{d+1}} + \sum_{1\le i<j\le d} |\partial_{i,j} u|^2 \mathrm{W}_{e_i+e_j} \le \sum_{i=1}^{d} |\partial_i u|^2 .$$

Multiplying both sides by $\mathrm{W}_{\boldsymbol{\gamma}}(x)$ and integrating over $x \in \mathrm{T}^d$, we obtain

$$|B^{\boldsymbol{\gamma}}(u,u)| \le \|\nabla u\|_{\boldsymbol{\gamma}}^2 .$$

By the Cauchy–Schwarz inequality for the positive semi-definite bilinear form $B^{\boldsymbol{\gamma}}$ it follows that

$$\left(\forall u, v \in \mathrm{C}^1(\overline{\mathrm{T}^d})\right) \quad |B^{\boldsymbol{\gamma}}(u,v)| \le \|\nabla u\|_{\boldsymbol{\gamma}} \|\nabla v\|_{\boldsymbol{\gamma}} .$$

The general result then follows from Definition 2.6.1. □

Proposition 2.6.3. *Let $d \in \mathbb{N}$ and $\boldsymbol{\gamma} \in (-1,\infty)^{d+1}$. Then,*

$$\left(\forall u \in \mathrm{H}^1_{\boldsymbol{\gamma}}\right) (\forall n \in \mathbb{N}_0) \quad \|u - S_n^{\boldsymbol{\gamma}}(u)\|_{\boldsymbol{\gamma}} \le (\lambda_{n+1}^{\boldsymbol{\gamma}})^{-1/2} \inf_{p_n \in \Pi_n^d} \|\nabla u - \nabla p_n\|_{\boldsymbol{\gamma}} \tag{2.6.2}$$

and

$$\left(\forall u \in \mathrm{H}^2_{\boldsymbol{\gamma}}\right) (\forall n \in \mathbb{N}_0) \quad \|u - S_n^{\boldsymbol{\gamma}}(u)\|_{\boldsymbol{\gamma}} \le (\lambda_{n+1}^{\boldsymbol{\gamma}})^{-1} \|\mathcal{L}^{\boldsymbol{\gamma}}(u) - S_n^{\boldsymbol{\gamma}}(\mathcal{L}^{\boldsymbol{\gamma}}(u))\|_{\boldsymbol{\gamma}} \tag{2.6.3}$$

Proof. Let $v \in \mathrm{H}^1_{\boldsymbol{\gamma}}$. By Definition 2.6.1, it is the limit in $\mathrm{H}^1_{\boldsymbol{\gamma}}$ of a sequence of functions in $\mathrm{C}^1(\overline{\mathrm{T}^d})$. Hence, exploiting the bound (2.6.1) and the structure of the Sobolev inner product, we can extend (2.2.11) to:

$$(\forall p_n \in \mathcal{V}_n^{\boldsymbol{\gamma}}) \quad B^{\boldsymbol{\gamma}}(p_n, v) = \lambda_n^{\boldsymbol{\gamma}} \langle p_n, v\rangle.$$

Then, given $n \in \mathbb{N}_0$ and $q \in \Pi_n^d$,

$$B^\gamma(q,v) = \sum_{k=0}^{n} B^\gamma(\mathrm{proj}_k^\gamma(q), v) = \sum_{k=0}^{n} \lambda_k^\gamma \langle \mathrm{proj}_k^\gamma(q), v\rangle_\gamma$$

so

$$B^\gamma(v-q, v-q) = B^\gamma(v,v) + \sum_{k=0}^{n} \lambda_k^\gamma \left[\left\| \mathrm{proj}_k^\gamma(q) \right\|^2 - 2\langle \mathrm{proj}_k^\gamma(q), v\rangle_\gamma \right]. \tag{2.6.4}$$

As the eigenvalues λ_n^γ are nonnegative (2.2.9), a minimizer of the left-hand side of (2.6.4) among all $q \in \Pi_n^d$ is obtained by minimizing what lies inside the square brackets of the right-hand side for each n independently, and that is attained by choosing q so that $\mathrm{proj}_k^\gamma(q) = \mathrm{proj}_k^\gamma(v)$ for $0 \le k \le n$; hence,

$$\left(\forall v \in \mathrm{H}_\gamma^1\right) \ (\forall n \in \mathbb{N}_0) \quad S_n^\gamma(v) \in \operatorname*{arg\,min}_{q \in \Pi_n^d} B^\gamma(v-q, v-q). \tag{2.6.5}$$

Now, for all $n \in \mathbb{N}_0$,

$$0 \le B^\gamma(v - S_n^\gamma(v), v - S_n^\gamma(v)) \overset{(2.6.4)}{=} B^\gamma(v,v) - \sum_{k=0}^{n} \lambda_k^\gamma \left\| \mathrm{proj}_k^\gamma(q) \right\|^2.$$

Thus, every partial sum of the series of non-negative terms $\sum_{k=0}^{n} \lambda_k^\gamma \left\| \mathrm{proj}_k^\gamma(v) \right\|_\gamma^2$ is bounded by the finite quantity $B^\gamma(v,v)$, so the series converges and we obtain the Bessel-type bound

$$\left(\forall v \in \mathrm{H}_\gamma^1\right) \ (\forall n \in \mathbb{N}_0) \quad \sum_{k=0}^{\infty} \lambda_k^\gamma \left\| \mathrm{proj}_k^\gamma(v) \right\| \le B^\gamma(v,v). \tag{2.6.6}$$

Therefore, given $u \in \mathrm{H}_\gamma^1$ and $n \in \mathbb{N}_0$,

$$\begin{aligned} \|u - S_n^\gamma(u)\|_\gamma^2 &= \sum_{k=n+1}^{\infty} \left\| \mathrm{proj}_k^\gamma(u) \right\|_\gamma^2 \\ &\le (\lambda_{n+1}^\gamma)^{-1} \sum_{k=n+1}^{\infty} \lambda_k^\gamma \left\| \mathrm{proj}_k^\gamma(u) \right\|_\gamma^2 \overset{(2.6.6)}{\le} (\lambda_{n+1}^\gamma)^{-1} B^\gamma(u - S_n^\gamma(u), u - S_n^\gamma(u)) \\ &\overset{(2.6.5)}{=} (\lambda_{n+1}^\gamma)^{-1} \min_{p_n \in \Pi_n^d} B^\gamma(u - p_n, u - p_n) \overset{(2.6.1)}{\le} (\lambda_{n+1}^\gamma)^{-1} \inf_{p_n \in \Pi_n^d} \|\nabla u - \nabla p_n\|_\gamma^2. \end{aligned}$$

The desired bound (2.6.2) then follows by taking square roots on both sides of the above inequality, utilizing the fact that the function $x \mapsto \sqrt{x}$ is monotonically increasing.

Let $w \in \mathrm{C}^2(\overline{\mathrm{T}^d})$. From (2.2.8), (2.2.9), (2.2.10) and the self-adjointness of B^γ made evident there, for all $k \in \mathbb{N}_0$ and $q_k \in \mathcal{V}_k^\gamma$,

$$\langle \mathcal{L}^\gamma w, q_k \rangle_\gamma = \lambda_k^\gamma \langle w, q_k \rangle_\gamma = \langle \lambda_k^\gamma \operatorname{proj}_k^\gamma(w), q_k \rangle_\gamma.$$

So, let $u \in \mathrm{H}_\gamma^2$ and $n \in \mathbb{N}_0$. The above inequality, the fact $\mathcal{L}^\gamma$ is a continuous map from H_γ^2 to L_γ^2 and Definition 2.6.1 give

$$(\forall k \in \mathbb{N}_0) \quad \operatorname{proj}_k^\gamma(\mathcal{L}^\gamma u) = \lambda_k^\gamma \operatorname{proj}_k^\gamma(u). \tag{2.6.7}$$

Then, the bound (2.6.3) follows from

$$\begin{aligned}\|u - S_n^\gamma(u)\|_\gamma^2 &\le (\lambda_{n+1}^\gamma)^{-2} \sum_{k=n+1}^{\infty} \|\lambda_k^\gamma \operatorname{proj}_k^\gamma(u)\|_\gamma^2 \\ &\overset{(2.6.7)}{=} (\lambda_{n+1}^\gamma)^{-2} \sum_{k=n+1}^{\infty} \|\operatorname{proj}_k^\gamma(\mathcal{L}^\gamma u)\|_\gamma^2 = (\lambda_{n+1}^\gamma)^{-2} \left\|\mathcal{L}^\gamma u - S_n^\gamma(\mathcal{L}^\gamma u)\right\|_\gamma^2.\end{aligned}$$

Finally, taking square roots and noting the monotonicity of the function $x \mapsto \sqrt{x}$ completes the proof of the bound (2.6.3). □

Proposition 2.6.4. *Let $d \in \mathbb{N}$, $m \in \mathbb{N}$ and $\boldsymbol{\gamma} \in (-1, \infty)^{d+1}$. Then, for all $u \in \mathrm{H}_\gamma^{2m-1}$ and $n \in \mathbb{N}_0$ we have*

$$\|u - S_n^\gamma(u)\|_\gamma \le (\lambda_{n+1}^\gamma)^{-\frac{2m-1}{2}} \left\|\nabla [\mathcal{L}^\gamma]^{2m-2}(u)\right\|_\gamma. \tag{2.6.8}$$

Also, for all $v \in \mathrm{H}_\gamma^{2m}$ and $n \in \mathbb{N}_0$ we have

$$\|v - S_n^\gamma(v)\|_\gamma \le (\lambda_{n+1}^\gamma)^{-m} \left\|[\mathcal{L}^\gamma]^m(v) - S_n^\gamma([\mathcal{L}^\gamma]^m(v))\right\|_\gamma. \tag{2.6.9}$$

Proof. We proceed by induction on the regularity parameter m. For $m = 1$, the statement is proven by the inequalities in (2.6.2) and (2.6.3).

Now, assume the proposition holds for some $m \in \mathbb{N}$. Let $u \in \mathrm{H}_\gamma^{2(m+1)-1}$ and $n \in \mathbb{N}_0$. We note that $u \in \mathrm{H}_\gamma^{2(m+1)-2}$ and that $[\mathcal{L}^\gamma]^{2(m+1)-2}(u) \in \mathrm{H}_\gamma^1$. Applying the inductive hypothesis for the even case (with regularity $2m$), yields

$$\|u - S_n^\gamma(u)\|_\gamma \leq (\lambda_{n+1}^\gamma)^{-\frac{2m}{2}} \left\|[\mathcal{L}^\gamma]^{2m}(u) - S_n^\gamma\left([\mathcal{L}^\gamma(u)]^{2m}\right)\right\|_\gamma \overset{(2.6.2)}{\leq} (\lambda_{n+1}^\gamma)^{-\frac{2m+1}{2}} \left\|\nabla[\mathcal{L}^\gamma]^{2m}(u)\right\|_\gamma .$$

Similarly, let $v \in \mathrm{H}_\gamma^{2(m+1)}$. We note that $v \in \mathrm{H}_\gamma^{2(m+1)-2}$ and that $[\mathcal{L}^\gamma]^{2(m+1)-2}(u) \in \mathrm{H}_\gamma^2$. Applying the inductive hypothesis for the even case (with regularity $2m$), yields

$$\begin{aligned}\|u - S_n^\gamma(u)\|_\gamma &\leq (\lambda_{n+1}^\gamma)^{-\frac{2(m+1)-2}{2}} \left\|[\mathcal{L}^\gamma]^{2(m+1)-2}(u) - S_n^\gamma\left([\mathcal{L}^\gamma]^{2(m+1)-2}(u)\right)\right\|_\gamma \\ &\overset{(2.6.3)}{\leq} (\lambda_{n+1}^\gamma)^{-(m+1)} \left\|[\mathcal{L}^\gamma]^{m+1}(u) - S_n^\gamma\left([\mathcal{L}^\gamma]^{m+1}(u)\right)\right\|_\gamma .\end{aligned}$$

This completes the inductive step and concludes the proof. □

Lemma 2.6.5. *Let $d \in \mathbb{N}$, $m \in \mathbb{N}$ and $\boldsymbol{\gamma} \in (-1,\infty)^{d+1}$. Then, there exists a constant $C > 0$, depending only on d and $\boldsymbol{\gamma}$ such that*

$$\left(\forall u \in \mathrm{H}_\gamma^m\right) (\forall n \in \mathbb{N}_0) \quad \|u - S_n^\gamma(u)\|_\gamma \leq C(\lambda_{n+1}^\gamma)^{-\frac{m}{2}} \|u\|_{\gamma;m} .$$

Proof. Let $u \in \mathrm{H}_\gamma^{2m-1}$. Since the operator $\nabla[\mathcal{L}^\gamma]^{2m-1}$ is bounded from H_γ^{2m-1} to L_γ^2, we deduce that there exists a constant $\tilde{C}_{\text{odd}} := \tilde{C}_{\text{odd}}(\boldsymbol{\gamma}, m) > 0$ such that

$$\left\|\nabla[\mathcal{L}^\gamma]^{2m-2}u\right\|_\gamma \leq \tilde{C}_{\text{odd}}(\lambda_{n+1}^\gamma)^{-\frac{2m-1}{2}} \|u\|_{\gamma,2m-1} . \tag{2.6.10}$$

Combining (2.6.8) of Proposition 2.6.4 and the inequality (2.6.10), we obtain the desired result for all $u \in \mathrm{H}_\gamma^{2m-1}$.

On the other hand, let $v \in \mathrm{H}_\gamma^{2m}$. Since the operator is bounded from H_γ^{2m} to L_γ^2, we deduce that there exists a constant $\tilde{C}_{\text{even}} := \tilde{C}_{\text{even}}(\boldsymbol{\gamma}, m) > 0$ such that

$$\left\|[\mathcal{L}^\gamma]^{2m}v\right\|_\gamma \leq \tilde{C}_{\text{even}}(\lambda_{n+1}^\gamma)^{-\frac{m}{2}} \|v\|_{\gamma,2m} \tag{2.6.11}$$

Combining (2.6.9) of Proposition 2.6.4 and the inequality (2.6.11), we obtain the desired result for all $v \in \mathrm{H}_\gamma^{2m}$. □

Proposition 2.6.6. *Let* $d \in \mathbb{N}$*,* $\boldsymbol{\gamma} \in (-1,\infty)^{d+1}$ *and* $l \in \mathbb{N}$*. Then, there exists a constant* $C > 0$*, depending only on* d *and* $\boldsymbol{\gamma}$ *such that for all* $u \in \mathrm{H}^l_{\boldsymbol{\gamma}}$*,* $n \in \mathbb{N}$ *and* $j \in [d]$*,*

$$\left\|\partial_j S^{\boldsymbol{\gamma}}_n(u) - S^{\boldsymbol{\gamma}}_{n-1}(\partial_j u)\right\|_{\boldsymbol{\gamma}} \leq C n^{\frac{5}{2}-l} \left\|\partial_j u\right\|_{\boldsymbol{\gamma};l-1}.$$

Proof. Let us first assume that $u \in \mathrm{C}^l(\overline{\mathrm{T}^d})$. Combining part (v) of Proposition 2.2.2 and part (iv) of Proposition 2.2.1, we obtain

$$\begin{aligned}\partial_j \operatorname{proj}^{\boldsymbol{\gamma}}_{k+1}(u) - \operatorname{proj}^{\boldsymbol{\gamma}}_k(\partial_j u) &= \overbrace{\operatorname{proj}^{\boldsymbol{\gamma}+e_j+e_{d+1}}_k \circ \operatorname{proj}^{\boldsymbol{\gamma}}_{k+1}(\partial_j u) - \operatorname{proj}^{\boldsymbol{\gamma}+e_j+e_{d+1}}_{k-1} \circ \operatorname{proj}^{\boldsymbol{\gamma}}_k(\partial_j u)}^{\mathrm{FDiff}_k} \\ &\quad + \underbrace{\operatorname{proj}^{\boldsymbol{\gamma}+e_j+e_{d+1}}_k \circ \operatorname{proj}^{\boldsymbol{\gamma}}_{k+2}(\partial_j u) - \operatorname{proj}^{\boldsymbol{\gamma}+e_j+e_{d+1}}_{k-2} \circ \operatorname{proj}^{\boldsymbol{\gamma}}_k(\partial_j u)}_{\mathrm{SDiff}_k}.\end{aligned}$$

Summing this expression from 0 to $n-1$ yields

$$\sum_{k=0}^{n-1} \left[\partial_j \operatorname{proj}^{\boldsymbol{\gamma}}_{k+1}(u) - \operatorname{proj}^{\boldsymbol{\gamma}}_k(\partial_j u)\right] = \sum_{k=0}^{n-1} \mathrm{FDiff}_k + \sum_{k=0}^{n-1} \mathrm{SDiff}_k \tag{2.6.12}$$

Since both sums on the right-hand side are telescoping, they can be evaluated directly. The first sum results in

$$\sum_{k=0}^{n-1} \mathrm{FDiff}_k = \operatorname{proj}^{\boldsymbol{\gamma}+e_j+e_{d+1}}_{n-1} \circ \operatorname{proj}^{\boldsymbol{\gamma}}_n(\partial_j u) \tag{2.6.13}$$

and the second sum evaluates to

$$\sum_{k=0}^{n-1} \mathrm{SDiff}_k = \operatorname{proj}^{\boldsymbol{\gamma}+e_j+e_{d+1}}_{n-1} \circ \operatorname{proj}^{\boldsymbol{\gamma}}_{n+1}(\partial_j u) + \operatorname{proj}^{\boldsymbol{\gamma}+e_j+e_{d+1}}_{n-2} \circ \operatorname{proj}^{\boldsymbol{\gamma}}_n(\partial_j u). \tag{2.6.14}$$

Using (2.2.3) to express $S^{\boldsymbol{\gamma}}_n$ in terms of the $\operatorname{proj}^{\boldsymbol{\gamma}}_k$, and combining this with the identities in (2.6.12), (2.6.13), and (2.6.14), we have

$$\begin{aligned}\partial_j S^{\boldsymbol{\gamma}}_n(u) - S^{\boldsymbol{\gamma}}_{n-1}(\partial_j u) = \sum_{k=0}^{n-1} \left[\partial_j \operatorname{proj}^{\boldsymbol{\gamma}}_{k+1}(u) - \operatorname{proj}^{\boldsymbol{\gamma}}_k(\partial_j u)\right] &= \operatorname{proj}^{\boldsymbol{\gamma}+e_j+e_{d+1}}_{n-2} \circ \operatorname{proj}^{\boldsymbol{\gamma}}_n(\partial_j u) \\ + \operatorname{proj}^{\boldsymbol{\gamma}+e_j+e_{d+1}}_{n-1} \circ \operatorname{proj}^{\boldsymbol{\gamma}}_n(\partial_j u) + \operatorname{proj}^{\boldsymbol{\gamma}+e_j+e_{d+1}}_{n-1} &\circ \operatorname{proj}^{\boldsymbol{\gamma}}_{n+1}(\partial_j u).\end{aligned} \tag{2.6.15}$$

Now, by part (iii) of Lemma 2.5.3, the fact that $\left\|\mathrm{proj}_n^{\gamma+e_j+e_{d+1}}\right\|_{\mathcal{L}(\mathrm{L}^2_{\gamma+e_j+e_{d+1}})} \leq 1$ and the fact that $\|\cdot\|_{\gamma+e_j+e_{d+1}} \leq \|\cdot\|_{\gamma}$ in L^2_{γ}, we have that for all $n \in \mathbb{N}$

$$\left\|\mathrm{proj}_{n-1}^{\gamma+e_j+e_{d+1}} \circ \mathrm{proj}_{n+1}^{\gamma}(\partial_j u)\right\|_{\gamma}^2 \leq C_{\gamma,j;1}(n) \left\|\mathrm{proj}_{n+1}^{\gamma}(\partial_j u)\right\|_{\gamma}^2 \tag{2.6.16}$$

and

$$\left\|\mathrm{proj}_{n-1}^{\gamma+e_j+e_{d+1}} \circ \mathrm{proj}_{n}^{\gamma}(\partial_j u)\right\|_{\gamma}^2 \leq C_{\gamma,j;1}(n) \left\|\mathrm{proj}_{n}^{\gamma}(\partial_j u)\right\|_{\gamma}^2, \tag{2.6.17}$$

where

$$C_{\gamma,j;1}(n) := \frac{n(n+|\boldsymbol{\gamma}|+d)(2n+|\boldsymbol{\gamma}|+d-1)}{(\gamma_{d+1}+1)(\gamma_j+1)}.$$

Analogous arguments show that for all $n \in \mathbb{N}$,

$$\left\|\mathrm{proj}_{n-2}^{\gamma+e_j+e_{d+1}} \circ \mathrm{proj}_{n}^{\gamma}(\partial_j u)\right\|_{\gamma}^2 \leq C_{\gamma,j;2}(n) \left\|\mathrm{proj}_{n}^{\gamma}(\partial_j u)\right\|_{\gamma}^2, \tag{2.6.18}$$

where

$$C_{\gamma,j;2}(n) := \frac{(n-1)(n+|\boldsymbol{\gamma}|+d-1)(2n+|\boldsymbol{\gamma}|+d-1)}{(\gamma_{d+1}+1)(\gamma_j+1)}.$$

Taking the squared L^2_{γ} norm of both ends of (2.6.15), using the triangle inequality and the bounds (2.6.16),(2.6.17) and (2.6.18), we observe that

$$\left\|\partial_j S_n^{\gamma}(u) - S_{n-1}^{\gamma}(\partial_j u)\right\|_{\gamma}^2 \leq C_{\gamma,j;1}(n) \left\|\partial_j u - S_{n-1}^{\gamma}(\partial_j u)\right\|_{\gamma}^2.$$

As $\partial_j u \in \mathrm{H}_{\gamma}^{l-1}$, we can appeal to Lemma 2.6.5 to obtain the desired result for $u \in \mathrm{C}^l(\overline{\mathrm{T}^d})$ after realizing that there exists a constant $C > 0$, depending only on d and $\boldsymbol{\gamma}$ such that $(C_{\gamma,j;1}(n))^{1/2}(\lambda_n^{\gamma})^{-\frac{l-1}{2}} \leq Cn^{\frac{5-2l}{2}}$. The general result then follows via the density result in Definition 2.6.1. □

Remark 2.6.7. Numerical experiments suggest that the case $l = 1$ of Proposition 2.6.6 is sharp if one relies on the $\|\partial_j u\|_{\gamma}$ seminorm. However, switching to the *directionally* stronger $\|\nabla u\|_{\gamma}$ seminorm reduces the observed power of n from 3/2 to 1/2. This reduction, if true, would impact Corollary 2.6.8 and the main result of this section, Theorem 2.1.1, making the results comparable to those in [9].

Corollary 2.6.8. *Given* $d \in \mathbb{N}_{\geq 2}$, $\boldsymbol{\gamma} \in (-1,\infty)^{d+1}$ *and* $r, l \in \mathbb{N}$ *with* $r \leq l$. *Then, there exists a constant* $C > 0$, *depending only on* d, l, r *and* $\boldsymbol{\gamma}$ *such that for all* $u \in \mathrm{H}^l_{\boldsymbol{\gamma}}$ *and* $n \in \mathbb{N}$,

$$\left\| \nabla_r S^{\boldsymbol{\gamma}}_n(u) - S^{\boldsymbol{\gamma}}_n\left(\nabla_r u\right) \right\|_{\boldsymbol{\gamma}} \leq C n^{2r+1/2-l} \left\| u \right\|_{\boldsymbol{\gamma};l}.$$

Proof. We will now operate by induction on r. Taking the square root of the sum with respect to j of the square of both sides of the inequality in Proposition 2.6.6 the case $r = 1$ follows at most immediately. Let us suppose now that our desired result holds for some $r \in [l]$ and that $r + 1 \leq l$. Then, for all $j \in [d]$, by triangle inequality,

$$\left\| \nabla_r \partial_j S^{\boldsymbol{\gamma}}_n(u) - S^{\boldsymbol{\gamma}}_n(\nabla_r \partial_j u) \right\|_{\boldsymbol{\gamma}} \leq \left| \partial_j S^{\boldsymbol{\gamma}}_n(u) - S^{\boldsymbol{\gamma}}_n(\partial_j u) \right|_{\boldsymbol{\gamma};r} + \left\| \nabla_r S^{\boldsymbol{\gamma}}_n(\partial_j u) - S^{\boldsymbol{\gamma}}_n(\nabla_r \partial_j u) \right\|_{\boldsymbol{\gamma}}.$$

By Corollary 2.5.23 and Proposition 2.6.6, the first term is bounded by an appropriate constant times $n^{2r} n^{5/2-l} \left\| \partial_j u \right\|_{\boldsymbol{\gamma};l-1}$. By the induction hypothesis and the fact that $\partial_j u \in \mathrm{H}^{l-1}_{\boldsymbol{\gamma}}$, the second term is bounded by a appropriate constant times $n^{2r+1/2-(l-1)} \left\| \partial_j u \right\|_{\boldsymbol{\gamma};l-1}$. Then, the desired result in the $r + 1$ case follows from summing up with respect to j and standard inequalities connecting vector 1- and 2-norm. □

Proof of Theorem 2.1.1. For every $k \in \{1, \ldots, r\}$,

$$\begin{aligned} \left| u - S^{\boldsymbol{\gamma}}_n(u) \right|^2_{\boldsymbol{\gamma};k} &\leq 2 \left\| \nabla_k u - S^{\boldsymbol{\gamma}}_n(\nabla_k u) \right\|^2_{\boldsymbol{\gamma}} + 2 \left\| S^{\boldsymbol{\gamma}}_n(\nabla_k u) - \nabla_k S^{\boldsymbol{\gamma}}_n(u) \right\|^2_{\boldsymbol{\gamma}} \\ &\leq C_1 (\lambda^{\boldsymbol{\gamma}}_{n+1})^{-(l-k)/2} \sum_{|\alpha|=k} \binom{k}{\alpha} \left\| \partial_\alpha u \right\|^2_{\boldsymbol{\gamma};l-k} + C_2 n^{4k+1-2l} \left\| u \right\|^2_{\boldsymbol{\gamma};l} \leq C_3 n^{4r+1-2l} \left\| u \right\|^2_{\boldsymbol{\gamma};l} \end{aligned}$$

where we have used Lemma 2.6.5, Corollary 2.6.8 and C_1 and C_2 depend on $\boldsymbol{\gamma}$, d, l and k only and C_3 depends on $\boldsymbol{\gamma}$, d, l and r only. Thus,

$$\left\| u - S^{\boldsymbol{\gamma}}_n(u) \right\|^2_{\boldsymbol{\gamma};r} \leq \left(C_4 (\lambda^{\boldsymbol{\gamma}}_{n+1})^{-l} + r C_3 n^{4r+1-2l} \right) \left\| u \right\|^2_{\boldsymbol{\gamma};l} \leq C_5 n^{4r+1-2l} \left\| u \right\|^2_{\boldsymbol{\gamma};l},$$

where we have again used Lemma 2.6.5, C_4 depends on $\boldsymbol{\gamma}$, d and only l and C_5 depends on $\boldsymbol{\gamma}$, d, l and r only. □

Chapter 3

Characterization of Sobolev orthogonal polynomials on the triangle

3.1 Introduction

Given $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \in (-1, \infty)^3$ and persisting with the notation of Section 2.1, we define, inspired by the terms involving derivatives of order two in the expansion of the inner product in [21, Eq. (7.1)], the bilinear form

$$\mathfrak{B}^{\boldsymbol{\gamma}}(u, v) := \langle \partial_3 \partial_1 u, \partial_3 \partial_1 v \rangle_{\boldsymbol{\gamma}+e_1} + \langle \partial_3 \partial_2 u, \partial_3 \partial_2 v \rangle_{\boldsymbol{\gamma}+e_2} + \langle \partial_1 \partial_2 u, \partial_1 \partial_2 v \rangle_{\boldsymbol{\gamma}+e_3}, \tag{3.1.1}$$

where $\partial_3 := \partial_2 - \partial_1$. Let us recall that $S_0^{\boldsymbol{\gamma}}$ is the orthogonal projection from $\mathrm{L}^2_{\boldsymbol{\gamma}}$ onto constant polynomials. We define the Sobolev space $\mathbb{H}^2_{\boldsymbol{\gamma}}$ as the topological completion of $\mathrm{C}^2(\overline{\mathrm{T}^2})$ with respect to the inner product

$$\langle u, v \rangle_{\mathbb{H}^2_{\boldsymbol{\gamma}}} := \mathfrak{B}^{\boldsymbol{\gamma}}(u, v) + \sum_{i=1}^{3} \langle S_0^{\boldsymbol{\gamma}}(\partial_i u), S_0^{\boldsymbol{\gamma}}(\partial_i v) \rangle_{\boldsymbol{\gamma}} + \langle S_0^{\boldsymbol{\gamma}}(u), S_0^{\boldsymbol{\gamma}}(v) \rangle_{\boldsymbol{\gamma}}. \tag{3.1.2}$$

In [21, Sec. 14] orthogonal projectors with respect to an analogue of $\mathbb{H}^2_{\boldsymbol{\gamma}}$ for the case $\boldsymbol{\gamma} = (0, 0, 0)$ were shown to provide quasioptimal polynomial approximants in the full unweighted Sobolev norm $\mathrm{W}^{1,2}(\mathrm{T}^2) = \mathrm{H}^1(\mathrm{T}^2)$. However, the role that the projectors $S_0^{\boldsymbol{\gamma}}$ play in the lower-order terms of (3.1.2) is in [21] played by trace operators, which are not necessarily well defined in weighted Sobolev spaces. Another observation is that in [21] explicit *bases* of orthogonal polynomials lie in the center of the arguments, whereas we strive to reserve that place for *spaces* of orthogonal polynomials. Given $n \in \mathbb{N}_0$ we denote by $\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}}$ the space of orthogonal polynomials of degree n with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}^2_{\boldsymbol{\gamma}}}$ of (3.1.2); that is,

$$\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}} := \left\{ p \in \Pi_n^2 \mid (\forall\, q \in \Pi_{n-1}^2)\ \langle p, q \rangle_{\mathbb{H}^2_{\boldsymbol{\gamma}}} = 0 \right\}. \tag{3.1.3}$$

In this chapter, we focus on the study and characterization of each space of Sobolev orthogonal polynomials $\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}}$. For lower degrees ($n \in [3]$) we found an explicit basis of $\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}}$ (see Theorem 3.3.8 and Theorem 3.3.10), while for higher degrees ($n \in \mathbb{N}_{\geq 4}$) we characterize $\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}}$ as a direct sum of the image of (mostly) weighted Lebesgue orthogonal polynomial spaces through compositions of instances of the first-order differential operators of (2.2.5) and (2.2.6) with the right parameters, namely,

$$m^{\boldsymbol{\gamma}} := d_1^{\boldsymbol{\gamma}-e_1-e_2-e_3} d_2^{\boldsymbol{\gamma}-e_2} \quad \text{and} \quad \mathrm{M}^{\boldsymbol{\gamma}} := m^{\boldsymbol{\gamma}} \left(d_{1,2}^{\boldsymbol{\gamma}+e_3} \right). \tag{3.1.4}$$

The characterization which we obtain is summarized in the following theorem.

Theorem 3.1.1. *Let $\boldsymbol{\gamma} \in (-1, \infty)^3$ be such that $\boldsymbol{\gamma} \neq (0,0,0)$, and $n \in \mathbb{N}_{\geq 4}$. Then, there exists a subspace $\mathcal{J}_n^{\boldsymbol{\gamma}} \subset \mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}}$ such that*

$$\partial_1 \partial_3 \mathcal{J}_n^{\boldsymbol{\gamma}} \subset \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_1}, \quad \partial_2 \partial_3 \mathcal{J}_n^{\boldsymbol{\gamma}} \subset \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_2} \quad \text{and} \quad \partial_1 \partial_2 \mathcal{J}_n^{\boldsymbol{\gamma}} \subset \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_3}.$$

Moreover, we have the direct sum decomposition

$$\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}} = \mathcal{J}_n^{\boldsymbol{\gamma}} \oplus \mathrm{M}^{\boldsymbol{\gamma}} \left(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+e_1+e_2+e_3} \right).$$

The subspace $\mathcal{J}_n^{\boldsymbol{\gamma}}$, defined in Theorem 3.1.1, will be presented later and is constructed from the polynomials in Section 3.5. An important detail of this subspace is that it depends on the nature of the parameter $\boldsymbol{\gamma}$; specifically, on whether one, several, or none of its elements are zero.

The second main contribution of this chapter is the characterization of the elements of $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$ as eigenfunctions of the Sturm–Liouville-type operator $\tilde{\mathcal{L}}^{\gamma} := \mathcal{L}^{\gamma-e_1-e_2-e_3} + \mathrm{LOT}^{\gamma}$, where $\mathcal{L}^{\gamma-e_1-e_2-e_3}$ is the operator defined in (2.2.7), and LOT^{γ} denotes a suitable lower-order finite-rank operator defined in Definition 3.6.11. This characterization is summarized in the following theorem.

Theorem 3.1.2. *Let $\boldsymbol{\gamma} \in (-1, \infty)^3$ be such that $\boldsymbol{\gamma} \neq (0,0,0)$, $n \in \mathbb{N}_0$ and $p_n \in \mathcal{V}_n^{\gamma;\mathrm{Sob}}$. Then,*

$$\tilde{\mathcal{L}}^{\gamma}(p_n) = \tilde{\lambda}_n^{\gamma} p_n, \quad \text{where} \quad \tilde{\lambda}_n^{\gamma} = \begin{cases} 0 & \text{if } 0 \leq n \leq 1, \\ n(n + |\boldsymbol{\gamma}| - 1) & \text{if } 2 \leq n. \end{cases}$$

The case $\boldsymbol{\gamma} = (0,0,0)$ was studied in [1], where a Sturm–Liouville operator for Sobolev orthogonal polynomials on the triangle was derived. Furthermore, in that work, this result was extended to Sobolev orthogonal polynomials on the simplex when $\boldsymbol{\gamma} = (0, \ldots, 0)$.

In the previous chapter (see Theorem 2.1.1) we proved that for all $u \in \mathrm{H}^l_{\gamma}$, there exists $C > 0$ such that

$$\|u - S_n^{\gamma}(u)\|_{\gamma;1} \leq C n^{5/2-l} \|u\|_{\gamma;l} .$$

As mentioned in Section 2.1, this result appears to be suboptimal with respect to the power on n. However, even the optimal power on n attainable while sticking with the projector S_n^{γ} as above may well be strictly worse than the power attained by a better projector, as it is the case in the ball (e.g., compare the case $r = 1$ of [9, Th. 1.1] against [11, Cor. 4.7]). This situation is consistent with the fact that in the above equation there is a mismatch between the norm in which the error is measured and the norm that corresponds to the projector operator.

Motivated by the work of [21], we define a Sobolev-type projector based on the space $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$, with the expectation of proving in future work that this projector achieves an optimal approximation rate in H^1_{γ}.

Moreover, the characterization of $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$ in terms of L^2_{γ}-orthogonal polynomial spaces allows us to exploit well-known properties of the spaces $\mathcal{V}_n^{\gamma}$ to analyze the structure of $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$. For example, the existence of a Sturm–Liouville operator associated with $\mathcal{V}_n^{\gamma}$ (cf. (2.2.8)) provides a natural pathway for constructing an analogous operator for $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$.

Following the ideas in [11] and the proof of Proposition 2.6.3, this approach could allow us to obtain first-order Sobolev-type approximation bounds for the associated projector.

Throughout this chapter we abbreviate

$$\mathrm{T} := \mathrm{T}^2, \quad \partial_3 := \partial_{1,2} \quad \text{and} \quad d_3^{\gamma} := d_{1,2}^{\gamma}.$$

3.2 Sobolev orthogonal polynomials on the interval

In this section we study univariate Sobolev orthogonal polynomials with respect to an inner product inspired by (3.1.2). The primary motivation for this is characterizing $\mathcal{J}_n^{\gamma}$ for the various regimes of $\boldsymbol{\gamma}$.

Let $\alpha, \beta > -1$ and $I := (-1, 1)$. Let $\mathcal{V}_n^{\mathrm{J}(\alpha,\beta)}$ be the space of the orthogonal polynomials of degree n with respect to the Jacobi weight $\mathrm{J}(\alpha, \beta)$ defined in Section 2.3; i.e.,

$$\mathcal{V}_n^{\mathrm{J}(\alpha,\beta)} := \left\{p \in \Pi_n^1 \middle| \ \left(\forall q \in \Pi_{n-1}^1\right) \ \langle p, q\rangle_{\mathrm{J}(\alpha,\beta)} = 0\right\}.$$

Let $\mathrm{proj}_n^{\mathrm{J}(\alpha,\beta)}$ denote the orthogonal projector from $\mathrm{L}^2_{\alpha,\beta}(I)$ onto $\mathcal{V}_n^{\mathrm{J}(\alpha,\beta)}$. From [6, Th. 3.2.18], $\Pi_n^1 = \bigoplus_{k=0}^{n} \mathcal{V}_k^{\mathrm{J}(\alpha,\beta)}$ and $\mathrm{L}^2_{\alpha,\beta} = \bigoplus_{k=0}^{\infty} \mathcal{V}_k^{\mathrm{J}(\alpha,\beta)}$, whence

$$(\forall n \in \mathbb{N}_0) \quad S_n^{\mathrm{J}(\alpha,\beta)} = \sum_{k=0}^{n} \mathrm{proj}_k^{\mathrm{J}(\alpha,\beta)} \quad \text{and} \quad (\forall u \in \mathrm{L}^2_{\alpha,\beta}) \quad u = \sum_{k=0}^{\infty} \mathrm{proj}_k^{\mathrm{J}(\alpha,\beta)}(u).$$

For $\alpha > -2$ and $\beta > -1$, we define the Sobolev space

$$\mathbb{H}^{\circ}_{\alpha,\beta} := \left\{f \in \mathrm{L}^2_{\alpha+1,\beta} : f' \in \mathrm{L}^2_{\alpha+1,\beta} \ \wedge \ f'' \in \mathrm{L}^2_{\alpha+1,\beta+1}\right\}. \tag{3.2.1}$$

By standard arguments (see, e.g., [14, Th. 1.1]), $\mathbb{H}^{\circ}_{\alpha,\beta}$ equipped with the inner product

$$\langle f, g\rangle^{\circ}_{\alpha,\beta} = \langle f'', g''\rangle_{\mathrm{J}(\alpha+1,\beta+1)} + \sum_{k=0}^{1} \left\langle S_0^{\mathrm{J}(\alpha+1,\beta)}(f^{(k)}), S_0^{\mathrm{J}(\alpha+1,\beta)}(g^{(k)})\right\rangle_{\mathrm{J}(\alpha+1,\beta)}, \tag{3.2.2}$$

is a Hilbert space. Given $n \in \mathbb{N}_0$ we denote by $\mathcal{V}_n^{\alpha,\beta;\circ}$ the space of orthogonal polynomials of degree n with respect to this inner product $\langle \cdot, \cdot \rangle_{\alpha,\beta}^{\circ}$ of (3.2.1); that is,

$$\mathcal{V}_n^{\alpha,\beta;\circ} = \left\{ p \in \Pi_n^1 \middle| \left(\forall q \in \Pi_{n-1}^1 \right) \ \langle p, q \rangle_{\alpha,\beta}^{\circ} = 0 \right\}. \tag{3.2.3}$$

A first characterization of $\mathcal{V}_n^{\alpha,\beta;\circ}$ is given by the following proposition.

Proposition 3.2.1. *Let $\alpha > -2$, $\beta > -1$ and $n \in \mathbb{N}_{\geq 2}$. Then,*

$$\mathcal{V}_n^{\alpha,\beta;\circ} = \left\{ p \in \Pi_n^1 \middle| \ (\forall q \in \Pi_{n-1}^1) \ \langle p'', q'' \rangle_{\mathrm{J}(\alpha+1,\beta+1)} = 0 \ \wedge \ (\forall k \in \{0,1\}) \ S_0^{\mathrm{J}(\alpha+1,\beta)}(p^{(k)}) = 0 \right\}. \tag{3.2.4}$$

Proof. Let $p \in \mathcal{V}_n^{\alpha,\beta;\circ}$. It is straightforward to check that

$$S_0^{\mathrm{J}(\alpha+1,\beta)}(p) \, \|1\|_{\mathrm{J}(\alpha+1,\beta)}^2 = \langle p, 1 \rangle_{\alpha,\beta}^{\circ} = 0 \quad \text{and} \quad S_0^{\mathrm{J}(\alpha+1,\beta)}(p') \, \|1\|_{\mathrm{J}(\alpha+1,\beta)}^2 = \langle p, \mathrm{J}(0,1) \rangle_{\alpha,\beta}^{\circ} = 0.$$

Since $\|1\|_{\mathrm{J}(\alpha+1,\beta)} \neq 0$, we must have $S_0^{\mathrm{J}(\alpha+1,\beta)}(p) = 0$ and $S_0^{\mathrm{J}(\alpha+1,\beta)}(p') = 0$. These two identities, together with the fact that $p \in \mathcal{V}_n^{\alpha,\beta;\circ}$, imply the left-to-right inclusion. The reverse inclusion is immediate. □

In what follows, we introduce and study some differential operators that will be useful in the characterization of $\mathcal{V}_n^{\alpha,\beta;\circ}$. We begin by introducing the following first-order differential operator, which is a univariate variant of the operator d_j^{γ} of (2.2.6)

$$\begin{aligned} d^{\alpha,\beta} p &:= -\mathrm{J}(\alpha,\beta)^{-1} \left[\mathrm{J}(\alpha+1,\beta+1) p \right]' \\ &= \left[(\alpha+1)\mathrm{J}(0,1) - (\beta+1)\mathrm{J}(1,0) \right] p - \mathrm{J}(1,1) p'. \end{aligned} \tag{3.2.5}$$

Proposition 3.2.2. *Let $\alpha, \beta > -1$ and $k \in \mathbb{N}_0$. Then, the following properties hold:*

(i) $d^{\alpha,\beta}$ *maps* Π_k^1 *into* Π_{k+1}^1.

(ii) *Given* $p, q \in \mathrm{C}^1(\bar{I})$, *then* $\langle p', q \rangle_{\mathrm{J}(\alpha+1,\beta+1)} = \langle p, d^{\alpha,\beta} q \rangle_{\mathrm{J}(\alpha,\beta)}$.

(iii) *Let* $r \in \mathcal{V}_k^{\mathrm{J}(\alpha+1,\beta+1)}$, *then* $d^{\alpha,\beta} r \in \mathcal{V}_{k+1}^{\mathrm{J}(\alpha,\beta)}$.

(iv) *Let* $h \in \mathcal{V}_k^{\mathrm{J}(\alpha,\beta)}$, *then* $h' \in \mathcal{V}_{k-1}^{\mathrm{J}(\alpha+1,\beta+1)}$.

Proof. Analogous to the proof of Proposition 2.2.2. ☐

Proposition 3.2.3. *Let* $\alpha, \beta \in \mathbb{R}$. *The following identity holds:*

$$\left(d^{\alpha-1,\beta-1} p\right)' = d^{\alpha,\beta} p' + (\alpha+\beta) p,$$

for all $p \in \mathrm{C}^2(\bar{I})$.

Proof. The result follows directly from (3.2.5) and by application of the calculus rules of differentiation. ☐

Definition 3.2.4. *Let* $\alpha, \beta \in \mathbb{R}$. *We define the second-order differential operator (cf. [11, eq. (38)] and* (3.1.4) *below)*

$$\mathrm{M}^{\alpha,\beta} := d^{\alpha-1,\beta-1} d^{\alpha,\beta} = \left[\mathrm{J}(\alpha-1,\beta-1)\right]^{-1} \left[\mathrm{J}(\alpha+1,\beta+1)p\right]''.$$

From (3.2.5), Definition 3.2.4 and by applying the calculus rules of differentiation, it follows that

$$\begin{aligned}\mathrm{M}^{\alpha,\beta} p = {} & \alpha(\alpha+1)\mathrm{J}(0,2)p + \beta(\beta+1)\mathrm{J}(2,0)p - 2(\alpha+1)(\beta+1)\mathrm{J}(1,1)p \\ & - 2(\alpha+1)\mathrm{J}(1,2)p' + 2(\beta+1)\mathrm{J}(2,1)p' + \mathrm{J}(2,2)p''. \end{aligned} \tag{3.2.6}$$

Then, $\mathrm{M}^{\alpha,\beta}$ maps Π_k^1 into Π_{k+2}^1. Moreover, from the above expression, we can deduce that

$$\mathrm{M}^{\alpha,\beta} p(1) = 4\alpha(\alpha+1)p(1) \quad \text{and} \quad \mathrm{M}^{\alpha,\beta} p(-1) = 4\beta(\beta+1)p(-1). \tag{3.2.7}$$

Proposition 3.2.5. *Let* $\alpha, \beta \in \mathbb{R}$. *The following identities hold for all* $p \in \mathrm{C}^2(\bar{I})$:

(i) $\left(\mathrm{M}^{\alpha,\beta} p\right)' = \mathrm{M}^{\alpha+1,\beta+1} p' + 2(\alpha+\beta+1) d^{\alpha,\beta} p,$

(ii) $\left(\mathrm{M}^{\alpha,\beta}p\right)'' = \mathrm{M}^{\alpha+2,\beta+2}p'' + 4(\alpha+\beta+2)d^{\alpha+1,\beta+1}p' + 2(\alpha+\beta+2)(\alpha+\beta+1)p.$

Proof. To prove (i), we use Definition 3.2.4 and Proposition 3.2.3 to obtain

$$\begin{aligned}\left(\mathrm{M}^{\alpha,\beta}p\right)' &= \left(d^{\alpha-1,\beta-1}d^{\alpha,\beta}p\right)' \\ &= d^{\alpha,\beta}(d^{\alpha,\beta}p)' + (\alpha+\beta)d^{\alpha,\beta}p \\ &= d^{\alpha,\beta}d^{\alpha+1,\beta+1}p' + (\alpha+\beta+2)d^{\alpha,\beta}p + (\alpha+\beta)d^{\alpha,\beta}p \\ &= \mathrm{M}^{\alpha+1,\beta+1}p' + 2(\alpha+\beta+1)d^{\alpha,\beta}p.\end{aligned}$$

Thus, (i) is established. For (ii), we apply (i) and Proposition 3.2.3 again:

$$\begin{aligned}\left(\mathrm{M}^{\alpha,\beta}p\right)'' &= \left(\mathrm{M}^{\alpha+1,\beta+1}p' + 2(\alpha+\beta+1)d^{\alpha,\beta}p\right)' \\ &= \left(\mathrm{M}^{\alpha+1,\beta+1}p'\right)' + 2(\alpha+\beta+1)\left(d^{\alpha,\beta}p\right)' \\ &= \mathrm{M}^{\alpha+2,\beta+2}p'' + 2(\alpha+\beta+3)d^{\alpha+1,\beta+1}p' + 2(\alpha+\beta+1)\left[d^{\alpha+1,\beta+1}p' + (\alpha+\beta+2)p\right] \\ &= \mathrm{M}^{\alpha+2,\beta+2}p'' + 4(\alpha+\beta+3)d^{\alpha+1,\beta+1}p' + 2(\alpha+\beta+1)(\alpha+\beta+2)p.\end{aligned}$$

Hence, (ii) holds. □

Proposition 3.2.6. *Let* $\alpha > -2$, $\beta > -1$ *and* $n \in \mathbb{N}_{\geq 2}$. *The following property holds:*

$$\left(\mathrm{M}^{\alpha,\beta}\right)''\left(\mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}\right) \subseteq \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}.$$

Proof. Let $p \in \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}$. First, we note from (ii) of Proposition 3.2.5 that

$$\left(\mathrm{M}^{\alpha,\beta}p\right)'' = \mathrm{M}^{\alpha+2,\beta+2}p'' + 4(\alpha+\beta+2)d^{\alpha+1,\beta+1}p' + 2(\alpha+\beta+2)(\alpha+\beta+1)p. \tag{3.2.8}$$

Next, from Definition 3.2.4, we deduce that

$$\mathrm{M}^{\alpha+2,\beta+2}p'' = d^{\alpha+1,\beta+1}d^{\alpha+2,\beta+2}p''.$$

From (iv) of Proposition 3.2.2 and (iii) of Proposition 3.2.2, we observe the following mapping for the first term on the right-hand side of (3.2.8):

$$\mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)} \xrightarrow{()''} \mathcal{V}_{n-4}^{\mathrm{J}(\alpha+3,\beta+3)} \xrightarrow{\mathrm{M}^{\alpha+2,\beta+2}} \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}.$$

Similarly, from (iv) of Proposition 3.2.2 and (iii) of Proposition 3.2.2, we establish the mapping for the second term on the right-hand side of (3.2.8):

$$\mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)} \xrightarrow{()'} \mathcal{V}_{n-3}^{\mathrm{J}(\alpha+2,\beta+2)} \xrightarrow{d^{\alpha+1,\beta+1}} \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}.$$

Since $p \in \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}$, all terms on the right-hand side of (3.2.8) belong to $\mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}$. Therefore, we conclude that $\left(\mathrm{M}^{\alpha,\beta}p\right)'' \in \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}$. □

Proposition 3.2.7. *Let $\alpha > -2$, $\beta > -1$ and $n \in \mathbb{N}_{\geq 4}$. The following property holds:*

$$\mathrm{M}^{\alpha,\beta}\left(\mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}\right) \subset \mathcal{V}_{n-3}^{\mathrm{J}(\alpha+1,\beta)} \oplus \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta)} \oplus \mathcal{V}_{n-1}^{\mathrm{J}(\alpha+1,\beta)} \oplus \mathcal{V}_{n}^{\mathrm{J}(\alpha+1,\beta)}.$$

Proof. Let $p \in \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}$ and $q \in \Pi_{n-4}^{1}$. Applying Definition 3.2.4 and integrating by parts twice, we find that the inner product $\langle \mathrm{M}^{\alpha,\beta}p, q\rangle_{\mathrm{J}(\alpha+1,\beta)}$ can be simplified as follows:

$$\begin{aligned}
\langle \mathrm{M}^{\alpha,\beta}p, q\rangle_{\mathrm{J}(\alpha+1,\beta)} &= \int_{[-1,1]} \mathrm{J}(\alpha-1,\beta-1)^{-1}\left(\mathrm{J}(\alpha+1,\beta+1)p\right)'' q\mathrm{J}(\alpha+1,\beta) \\
&= \int_{[-1,1]} \left(\mathrm{J}(\alpha+1,\beta+1)p\right)'' q\mathrm{J}(2,1) \\
&= -\int_{[-1,1]} \left(\mathrm{J}(\alpha+1,\beta+1)p\right)' \left[q\mathrm{J}(2,1)\right]' \\
&= \langle p, (q\mathrm{J}(2,1))''\rangle_{\mathrm{J}(\alpha+1,\beta+1)}.
\end{aligned}$$

Since $p \in \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}$ and $(q\mathrm{J}(2,1))'' \in \Pi_{n-3}^{1}$, we conclude that $\langle \mathrm{M}^{\alpha,\beta}p, q\rangle_{\mathrm{J}(\alpha+1,\beta)} = 0$. This establishes the desired result. □

Proposition 3.2.8. *Let $\alpha > -2$, $\beta > -1$ and $n \in \mathbb{N}_{\geq 2}$. The following property holds:*

$$\left(\mathrm{M}^{\alpha,\beta}\right)'\left(\mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}\right) \subset \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta)} \oplus \mathcal{V}_{n-1}^{\mathrm{J}(\alpha+1,\beta)}.$$

Proof. Let $p \in \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}$ and $q \in \Pi_{n-3}^1$. From (i) of Proposition 3.2.5 we have

$$\left(\mathrm{M}^{\alpha,\beta} p\right)' = \mathrm{M}^{\alpha+1,\beta+1} p' + 2(\alpha+\beta+1) d^{\alpha,\beta} p = d^{\alpha,\beta} \left(d^{\alpha+1,\beta+1} p' + 2(\alpha+\beta+1) p\right).$$

Letting $h := d^{\alpha+1,\beta+1} p' + 2(\alpha+\beta+1)p$ and using (3.2.5) we have that

$$\left(\mathrm{M}^{\alpha,\beta} p\right)' = d^{\alpha+1,\beta} h - \mathrm{J}(0,1) h. \tag{3.2.9}$$

From (iii) and (iv) of Proposition 3.2.2, and the fact that $p \in \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}$, we deduce that $d^{\alpha+1,\beta+1} p' \in \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}$, so $h \in \mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}$. This and (3.2.9) yield

$$\left\langle \left(\mathrm{M}^{\alpha,\beta} p\right)', q \right\rangle_{\mathrm{J}(\alpha+1,\beta)} = \langle d^{\alpha+1,\beta} h, q \rangle_{\mathrm{J}(\alpha+1,\beta)} - \langle h, q \rangle_{\mathrm{J}(\alpha+1,\beta+1)} = \langle d^{\alpha+1,\beta} h, q \rangle_{\mathrm{J}(\alpha+1,\beta)}.$$

By part (ii) of Proposition 3.2.2, we have

$$\langle d^{\alpha+1,\beta} h, q \rangle_{\mathrm{J}(\alpha+1,\beta)} = \langle h, q' \rangle_{\mathrm{J}(\alpha+2,\beta+1)} = \langle h, \mathrm{J}(1,0) q' \rangle_{\mathrm{J}(\alpha+1,\beta+1)} = 0.$$

This establishes the desired result. □

Proposition 3.2.9. *Let $\alpha > -2$, $\beta > -1$ and $n \in \mathbb{N}_0$. Then, the operator $\mathrm{M}^{\alpha,\beta}|_{\Pi_n^1}$ is injective.*

Proof. Let $p \in \Pi_n^1$ be such that $\mathrm{M}^{\alpha,\beta} p = 0$. From Definition 3.2.4, we have that

$$\mathrm{J}(\alpha-1,\beta-1)^{-1} \left(\mathrm{J}(\alpha+1,\beta+1) p\right)'' = 0.$$

Since $\mathrm{J}(\alpha-1,\beta-1)^{-1} \neq 0$ on I, we deduce that $(\mathrm{J}(\alpha+1,\beta+1)p)'' = 0$ on I. The above implies that $\mathrm{J}(\alpha+1,\beta+1)p \in \Pi_1^1$. If p is a non-null polynomial, then $\mathrm{J}(\alpha+1,\beta+1)$ is polynomial; consequently $\alpha+1 \in \mathbb{N}_0$ and $\beta+1 \in \mathbb{N}_0$. Since $\alpha > -2$ and $\beta > -1$, the polynomial $\mathrm{J}(\alpha+1,\beta+1)$ has degree at least one. Thus the product, $\mathrm{J}(\alpha+1,\beta+1)p$ has degree at least $n+1$, which is a contradiction. Therefore, we conclude that $p = 0$. □

Lemma 3.2.10. *Let $\alpha > -2$, $\beta > -1$ and $n \in \mathbb{N}_{\geq 4}$. Then, the following property holds:*

$$\mathcal{V}_n^{\alpha,\beta;\circ} = \mathrm{M}^{\alpha,\beta}\left(\mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}\right)$$

Proof. The inclusion

$$\mathrm{M}^{\alpha,\beta}\left(\mathcal{V}_{n-2}^{\mathrm{J}(\alpha+1,\beta+1)}\right) \subseteq \mathcal{V}_n^{\alpha,\beta;\circ}$$

follows from the structure of the inner product (3.2.2), Proposition 3.2.6, Proposition 3.2.8 and Proposition 3.2.7. Furthermore, from Proposition 3.2.9 we deduce that the dimensions of both spaces are the same. Therefore, the desired result holds. ☐

Definition 3.2.11. *We define $\eta : \mathrm{T} \longrightarrow I$ as the map given by $\eta(x) = 1 - 2x_1 - 2x_2$ and $\eta^\star f := f \circ \eta$.*

Proposition 3.2.12. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$, $n \in \mathbb{N}_0$ and $p \in \Pi_n^1$. Then, the polynomial $\eta^\star p$ satisfies*

(i) $S_0^{\boldsymbol{\gamma}}(\eta^\star p) = S_0^{\mathrm{J}(\gamma_1+\gamma_2+1,\gamma_3)}(p)$,

(ii) $\partial_1 \eta^\star p = -2\eta^\star p'$,

(iii) $\partial_2 \eta^\star p = -2\eta^\star p'$,

(iv) $\partial_3 \eta^\star p = 0$.

Proof. Part (i) follows from the fact that

$$(\forall q \in \Pi_n^1) \quad \langle \eta^\star q, 1\rangle_{\boldsymbol{\gamma}} = 2^{-3-2\gamma_1-2\gamma_2-\gamma_3}\langle 1,1\rangle_{\mathrm{J}(\gamma_1,\gamma_2)}\langle q,1\rangle_{\mathrm{J}(\gamma_1+\gamma_2+1,\gamma_3)}.$$

On the other hand (ii), (iii) and (iv), follow from the rules of calculus. ☐

We close this section with the following lemma, stating that the univariate Sobolev orthogonal polynomials given in (3.2.4) become bivariate Sobolev orthogonal polynomials defined in (3.1.3) under the pullback $\eta^\star$, provided the degree is sufficiently large. Together with the results from the upcoming section, this will allow for the definition of $\mathcal{J}_n^{\boldsymbol{\gamma}}$ in specific cases of $\boldsymbol{\gamma}$.

Lemma 3.2.13. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$. *Then,*

$$\eta^\star \mathcal{V}_n^{\gamma_1+\gamma_2,\gamma_3;\circ} \subset \mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}}.$$

Proof. Let $p \in \mathcal{V}_n^{\gamma_1+\gamma_2,\gamma_3;\circ}$. Using Proposition 3.2.12, we have

$$S_0^{\boldsymbol{\gamma}}(\partial_1\eta^\star p) = S_0^{\boldsymbol{\gamma}}(\partial_2\eta^\star p) = -2S_0^{\mathrm{J}(\gamma_1+\gamma_2+1,\gamma_3)}(p') \quad \text{and} \quad S_0^{\boldsymbol{\gamma}}(\partial_3\eta^\star p) = 0.$$

From Proposition 3.2.1, we deduce that $S_0^{\mathrm{J}(\gamma_1+\gamma_2+1,\gamma_3)}(p') = 0$ and $S_0^{\mathrm{J}(\gamma_1+\gamma_2+1,\gamma_3)}(p) = 0$. Therefore, $S_0^{\boldsymbol{\gamma}}(\partial_1\eta^\star p) = 0$, $S_0^{\boldsymbol{\gamma}}(\partial_2\eta^\star p) = 0$, $S_0^{\boldsymbol{\gamma}}(\partial_3\eta^\star p) = 0$ and $S_0^{\boldsymbol{\gamma}}(\eta^\star p) = 0$.

Now, let $k \in [n-1]_0$ and $l \in [k]_0$. Since $\partial_3\eta^\star p = 0$, we deduce that

$$\mathfrak{B}^{\boldsymbol{\gamma}}(\eta^\star p, J_{k,l}^{\boldsymbol{\gamma}}) = \langle \partial_1\partial_2\eta^\star p, \partial_1\partial_2 J_{k,l}\rangle_{\boldsymbol{\gamma}+e_3} \overset{\text{Proposition 3.2.1}}{=} 4\langle \eta^\star p'', \partial_1\partial_2 J_{k,l}^{\boldsymbol{\gamma}}\rangle_{\boldsymbol{\gamma}+e_3}$$

Since $\partial_1\partial_2 J_{k,l}^{\boldsymbol{\gamma}} \in \mathcal{V}_{k-2}^{\boldsymbol{\gamma}+e_1+e_2+2e_3}$, there exist $c_0^{\boldsymbol{\gamma}}, \ldots, c_{k-2}^{\boldsymbol{\gamma}} \in \mathbb{R}$ such that

$$\mathfrak{B}^{\boldsymbol{\gamma}}(\eta^\star p, J_{k,l}^{\boldsymbol{\gamma}}) = \sum_{i=0}^{k-2} 4c_i^{\boldsymbol{\gamma}} \langle \eta^\star p'', J_{k-2,i}^{\boldsymbol{\gamma}+e_1+e_2+2e_3}\rangle_{\boldsymbol{\gamma}+e_3}.$$

By performing the change of variables $x = \Psi(\zeta,\eta)$, we find that (cf. (2.4.1))

$$\begin{aligned}\langle \eta^\star p'', J_{k-2,i}^{\boldsymbol{\gamma}+e_1+e_2+2e_3}\rangle_{\boldsymbol{\gamma}+e_3} = 2^{-4-2\gamma_1-2\gamma_2-\gamma_3-i}&\langle \mathrm{P}_i^{(\gamma_1+1,\gamma_2+1)}, 1\rangle_{\mathrm{J}(\gamma_1,\gamma_2)}\\ &\times \langle p'', \mathrm{J}(i,0)\mathrm{P}_{k-2-i}^{(2i+\gamma_1+\gamma_2+3,\gamma_3+2)}\rangle_{\mathrm{J}(\gamma_1+\gamma_2+1,\gamma_3+1)}.\end{aligned}$$

Since $\mathrm{J}(i,0)\mathrm{P}_{k-2-i}^{(2i+\gamma_1+\gamma_2+3,\gamma_3+2)} \in \Pi_{k-2}^1$, there exists a polynomial $h_i \in \Pi_k^1$ such that

$$h_i'' = \mathrm{J}(i,0)\mathrm{P}_{k-2-i}^{(2i+\gamma_1+\gamma_2+3,\gamma_3+2)}.$$

Then, from Proposition 3.2.1, we get that

$$\langle p'', \mathrm{J}(i,0)\mathrm{P}_{k-2-i}^{(2i+\gamma_1+\gamma_2+3,\gamma_3+2)}\rangle_{\mathrm{J}(\gamma_1+\gamma_2+1,\gamma_3+1)} = \langle p'', h_i''\rangle_{\mathrm{J}(\gamma_1+\gamma_2+1,\gamma_3+1)} = 0.$$

Consequently, $\mathfrak{B}^{\gamma}(\eta^{\star}p, J^{\gamma}_{k,l}) = 0$. Since this holds for every basis member $J^{\gamma}_{k,l}$, we deduce that $\eta^{\star}p \in \mathcal{V}_n^{\gamma;\mathrm{Sob}}$. $\square$

3.3 Sobolev orthogonal polynomials on the triangle

In this section, we study some properties of the space of Sobolev orthogonal polynomials $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$ and construct explicit bases for low degrees. Given $\boldsymbol{\gamma} \in (-1,\infty)^3$, we define the number b_{γ} as follows:

$$b_{\gamma} := \int_{\mathrm{T}} \mathrm{W}_{\gamma}(x)\ \mathrm{d}x = \frac{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)\Gamma(\gamma_3+1)}{\Gamma(\gamma_1+\gamma_2+\gamma_3+3)}.$$

The following result yields a preliminary characterization of the orthogonal polynomials $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}^2_{\gamma}}$.

Lemma 3.3.1. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$ and $n \in \mathbb{N}_{\geq 2}$. Then,*

$$\mathcal{V}_n^{\gamma;\mathrm{Sob}} = \left\{p \in \Pi^2_n \middle|\ (\forall q \in \Pi^2_{n-1})\ \mathfrak{B}^{\gamma}(p,q) = 0\ \wedge\ S^{\gamma}_0(\bar{\nabla}p) = 0\ \wedge\ S^{\gamma}_0(p) = 0\right\},$$

where $\bar{\nabla} = (\partial_1, \partial_2, \partial_3)^t$.

Proof. Let $p \in \mathcal{V}_n^{\gamma;\mathrm{Sob}}$. It is straightforward to check that

$$\langle p, 1\rangle_{\mathrm{H}^2_{\gamma}} = S^{\gamma}_0(p)b_{\gamma}, \quad \langle p, \mathrm{W}_{e_3}\rangle_{\mathrm{H}^2_{\gamma}} = -S^{\gamma}_0(\partial_1 p)b_{\gamma} - S^{\gamma}_0(\partial_2 p)b_{\gamma}, \quad \text{and} \quad \langle p, \mathrm{W}_{e_1}\rangle_{\mathrm{H}^2_{\gamma}} = -S^{\gamma}_0(\partial_3 p)b_{\gamma}.$$

Since $b_{\gamma} \neq 0$, $\mathrm{W}_{e_3}, \mathrm{W}_{e_1} \in \Pi^2_1$, $\partial_3 = \partial_2 - \partial_2$ and $p \in \mathcal{V}_n^{\gamma;\mathrm{Sob}}$, we deduce that $S^{\gamma}_0(p) = 0$ and $S^{\gamma}_0(\bar{\nabla}p) = 0$. Now, let $q \in \Pi^2_{n-1}$. By definition of $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$, we have $\langle p, q\rangle_{\mathrm{H}^2_{\gamma}} = 0$, which implies that $\mathfrak{B}^{\gamma}(p,q) = 0$. Therefore, the left-to-right inclusion is established. The other inclusion is immediate. $\square$

We observe that $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$ is a subspace of

$$V^{\gamma}_n(\mathfrak{B}) := \{p \in \Pi^2_n |\ (\forall q \in \Pi^2_{n-1})\mathfrak{B}^{\gamma}(p,q) = 0\}.$$

This motivates us to investigate the relationship between these spaces in the results that follow.

Definition 3.3.2. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}$. *We define the operator* $\mathrm{R}^{\boldsymbol{\gamma}} : \Pi_n^2 \longrightarrow \Pi_1^2$ *as follows:*

$$\mathrm{R}^{\boldsymbol{\gamma}}(p) := -\frac{S_0^{\boldsymbol{\gamma}}(\partial_1 p)}{|\boldsymbol{\gamma}|+3} T_{\sigma_2}^{\star} J_{1,0}^{\sigma_1(\boldsymbol{\gamma})} - \frac{S_0^{\boldsymbol{\gamma}}(\partial_2 p)}{|\boldsymbol{\gamma}|+3} T_{\sigma_1}^{\star} J_{1,0}^{\sigma_2(\boldsymbol{\gamma})} + S_0^{\boldsymbol{\gamma}}(p).$$

Proposition 3.3.3. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$, $n \in \mathbb{N}_0$ *and* $p \in \Pi_n^2$. *Then,* $S_0^{\boldsymbol{\gamma}}(\bar{\nabla}(\mathrm{I}-\mathrm{R}^{\boldsymbol{\gamma}})p) = 0$ *and* $S_0^{\boldsymbol{\gamma}}((\mathrm{I}-\mathrm{R}^{\boldsymbol{\gamma}})p) = 0$.

Proof. By (iii) in Proposition 2.2.6, we have that $T_{\sigma_1}^{\star} J_{1,0}^{\sigma_2(\boldsymbol{\gamma})}, T_{\sigma_2}^{\star} J_{1,0}^{\sigma_1(\boldsymbol{\gamma})} \in \mathcal{V}_1^{\boldsymbol{\gamma}}$. Thus,

$$S_0^{\boldsymbol{\gamma}}((\mathrm{I}-\mathrm{R}^{\boldsymbol{\gamma}})p) = S_0^{\boldsymbol{\gamma}}(p) - S_0^{\boldsymbol{\gamma}}(\mathrm{R}^{\boldsymbol{\gamma}}(p)) = S_0^{\boldsymbol{\gamma}}(p) - S_0^{\boldsymbol{\gamma}}(p) = 0.$$

On the other hand, from (2.4.6) and (2.4.7) we deduce that

$$\partial_1 T_{\sigma_1}^{\star} J_{1,0}^{\sigma_2(\boldsymbol{\gamma})} = \partial_2 T_{\sigma_2}^{\star} J_{1,0}^{\sigma_1(\boldsymbol{\gamma})} = 0 \quad \text{and} \quad \partial_2 T_{\sigma_1}^{\star} J_{1,0}^{\sigma_2(\boldsymbol{\gamma})} = \partial_1 T_{\sigma_2}^{\star} J_{1,0}^{\sigma_1(\boldsymbol{\gamma})} = -(|\boldsymbol{\gamma}|+3).$$

Thus,

$$S_0^{\boldsymbol{\gamma}}(\partial_1(\mathrm{I}-\mathrm{R}^{\boldsymbol{\gamma}})p) = 0 \quad \text{and} \quad S_0^{\boldsymbol{\gamma}}(\partial_2(\mathrm{I}-\mathrm{R}^{\boldsymbol{\gamma}})p) = 0.$$

Therefore, $S_0^{\boldsymbol{\gamma}}((\mathrm{I}-\mathrm{R}^{\boldsymbol{\gamma}})p) = 0$ and $S_0^{\boldsymbol{\gamma}}(\bar{\nabla}(\mathrm{I}-\mathrm{R}^{\boldsymbol{\gamma}})p) = 0$. □

Remark 3.3.4. Let $p \in \Pi_1^2$. From Proposition 3.3.3 we deduce that $\|(\mathrm{I}-\mathrm{R}^{\boldsymbol{\gamma}})p\|_{\mathbb{H}_{\boldsymbol{\gamma}}^2} = 0$. Thus, $\mathrm{R}^{\boldsymbol{\gamma}}(p) = p$. Therefore, it is straightforward to check that $\ker(\mathrm{I}-\mathrm{R}^{\boldsymbol{\gamma}}) = \Pi_1^1$.

The next theorem connects the orthogonal polynomial spaces $\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}}$ with the spaces $V_n^{\boldsymbol{\gamma}}(\mathfrak{B})$. The latter are earlier to analyze because the bilinear form $\mathfrak{B}^{\boldsymbol{\gamma}}$ does not involve lower-order terms. A similar phenomenon was observed in [8] and [5] regarding the construction of bases of Sobolev orthogonal polynomials on product domains.

Theorem 3.3.5. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 2}$. *Then,*

$$\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}} = (\mathrm{I}-\mathrm{R}^{\boldsymbol{\gamma}}) V_n^{\boldsymbol{\gamma}}(\mathfrak{B}).$$

Proof. Let $p \in \mathcal{V}_n^{\gamma;\mathrm{Sob}}$. From Lemma 3.3.1, we deduce that $p \in V_n^\gamma(\mathfrak{B})$, with $S_0^\gamma(\bar{\nabla} p) = 0$ and $S_0^\gamma(p) = 0$. This implies that $\mathrm{R}^\gamma(p) = 0$. Consequently, we can write $p = p - \mathrm{R}^\gamma(p)$, which shows that $p \in (\mathrm{I} - \mathrm{R}^\gamma)V_n^\gamma(\mathfrak{B})$. Thus, the first inclusion holds.

Conversely, let $q \in (\mathrm{I} - \mathrm{R}^\gamma)V_n^\gamma(\mathfrak{B})$. Then, there exists $h \in V_n^\gamma(\mathfrak{B})$ such that

$$q = h - \mathrm{R}^\gamma(h).$$

Since $h \in \Pi_n^2$, we deduce from Proposition 3.3.3 that $S_0^\gamma(\bar{\nabla} h) = 0$ and $S_0^\gamma(h) = 0$. Thus, $q \in \mathcal{V}_n^{\gamma;\mathrm{Sob}}$. Therefore, the second inclusion holds. $\square$

Proposition 3.3.6. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $\sigma \in \Sigma_2$. *Then, for all* $f, g \in \mathrm{C}^2(\overline{\mathrm{T}})$, *we have* $\langle T_\sigma^\star f, T_\sigma^\star g\rangle_{\mathbb{H}^2_{\sigma(\gamma)}} = \langle f, g\rangle_{\mathbb{H}^2_\gamma}$.

Proof. We will prove the result for σ_1; the case for σ_2 is analogous. Let $f, g, h \in \mathrm{C}^2(\overline{\mathrm{T}})$. From Proposition 2.2.9 we have the following identities:

$$\begin{aligned}
\partial_1\partial_2 T_{\sigma_1}^\star h &= -\partial_1 T_{\sigma_1}^\star \partial_1 h = -T_{\sigma_1}^\star \partial_3\partial_1 h,\\
\partial_1\partial_3 T_{\sigma_1}^\star h &= -\partial_1 T_{\sigma_1}^\star \partial_2 h = -T_{\sigma_1}^\star \partial_3\partial_2 h,\\
\partial_3\partial_2 T_{\sigma_1}^\star h &= -\partial_3 T_{\sigma_1}^\star \partial_1 h = T_{\sigma_1}^\star \partial_2\partial_1 h,
\end{aligned}$$

which allows us to prove that

$$\begin{aligned}
\mathfrak{B}^{\sigma_1(\gamma)}(T_{\sigma_1}^\star f, T_{\sigma_1}^\star g) &= \langle \partial_1\partial_3 T_{\sigma_1}^\star f, \partial_1\partial_3 T_{\sigma_1}^\star g\rangle_{\sigma_1(\gamma)+e_1} + \langle \partial_2\partial_3 T_{\sigma_1}^\star f, \partial_2\partial_3 T_{\sigma_1}^\star g\rangle_{\sigma_1(\gamma)+e_2}\\
&+ \langle \partial_1\partial_2 T_{\sigma_1}^\star f, \partial_1\partial_2 T_{\sigma_1}^\star g\rangle_{\sigma_1(\gamma)+e_3}\\
&= \langle T_{\sigma_1}^\star \partial_2\partial_3 f, T_{\sigma_1}^\star \partial_2\partial_3 g\rangle_{\sigma_1(\gamma)+e_1} + \langle T_{\sigma_1}^\star \partial_2\partial_1 f, T_{\sigma_1}^\star \partial_2\partial_1 g\rangle_{\sigma_1(\gamma)+e_2}\\
&+ \langle T_{\sigma_1}^\star \partial_1\partial_3 f, T_{\sigma_1}^\star \partial_1\partial_3 g\rangle_{\sigma_1(\gamma)+e_3}.
\end{aligned}$$

Now, from (ii) of Proposition 2.2.6, we deduce the following:

$$\mathfrak{B}^{\sigma_1(\gamma)}(T_{\sigma_1}^\star f, T_{\sigma_1}^\star g) = \mathfrak{B}^\gamma(f, g).$$

On the other hand, from (i) of Proposition 2.2.9 and (ii) of Proposition 2.2.6 we have

$$\begin{aligned} S_0^{\sigma_1(\gamma)}(\partial_1 T^\star_{\sigma_1} h) &= \frac{\langle T^\star_{\sigma_1}\partial_3 h, 1\rangle_{\sigma_1(\gamma)}}{\|1\|^2_{\sigma_1(\gamma)}} = S_0^\gamma(\partial_3 h),\\ S_0^{\sigma_1(\gamma)}(\partial_2 T^\star_{\sigma_1} h) &= -\frac{\langle T^\star_{\sigma_1}\partial_1 h, 1\rangle_{\sigma_1(\gamma)}}{\|1\|^2_{\sigma_1(\gamma)}} = -S_0^\gamma(\partial_1 h),\\ S_0^{\sigma_1(\gamma)}(\partial_3 T^\star_{\sigma_1} h) &= -\frac{\langle T^\star_{\sigma_1}\partial_2 h, 1\rangle_{\sigma_1(\gamma)}}{\|1\|^2_{\sigma_1(\gamma)}} = -S_0^\gamma(\partial_2 h). \end{aligned}$$

Similarly, using again (ii) of Proposition 2.2.6

$$S_0^\gamma(T^\star_{\sigma_1} h) = \frac{\langle T^\star_{\sigma_1} h, 1\rangle_{\sigma_1(\gamma)}}{\|1\|^2_{\sigma_1(\gamma)}} = \frac{\langle h, 1\rangle_\gamma}{\|1\|^2_\gamma} = S_0^\gamma(h).$$

Combining these results, the desired result follows. □

The following lemma shows that the Sobolev orthogonal polynomials $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$ are covariant under permutations of $(\gamma_1, \gamma_2, \gamma_3)$ and their corresponding vertices in $\mathbb{V}_2$. While this property is standard for L^2_γ-orthogonal polynomials, it does not hold for the Sobolev orthogonal polynomials introduced in [21], owing to the asymmetric structure of the inner product. In contrast, in [17] certain Sobolev orthogonal polynomials on the triangle that preserve covariance are studied, and, similarly to our approach, this property is exploited in order to construct explicit bases. This property is crucial for the construction of $\mathcal{J}_n^\gamma$ in Theorem 3.1.1.

Lemma 3.3.7. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$, $n \in \mathbb{N}_0$ and $\sigma \in \Sigma_2$. Then, $T^\star_\sigma \mathcal{V}_n^{\gamma;\mathrm{Sob}} = \mathcal{V}_n^{\sigma(\gamma);\mathrm{Sob}}$.*

Proof. Let $p \in \mathcal{V}_n^{\gamma;\mathrm{Sob}}$ and $q \in \Pi^2_{n-1}$. It follows from Proposition 3.3.6 that

$$\langle T^\star_\sigma p, q\rangle_{\mathbb{H}^2_{\sigma(\gamma)}} = \langle p, T^\star_{\sigma^{-1}} q\rangle_{\mathbb{H}^2_\gamma} = 0,$$

since $T^\star_{\sigma^{-1}} q \in \Pi^2_{n-1}$. Consequently, $T^\star_\sigma p \in \mathcal{V}_n^{\sigma(\gamma);\mathrm{Sob}}$. This establishes the inclusion $T^\star_\sigma \mathcal{V}_n^{\gamma;\mathrm{Sob}} \subset \mathcal{V}_n^{\sigma(\gamma);\mathrm{Sob}}$. Finally, since $T^\star_\sigma$ is injective, we deduce that $\dim(T^\star_\sigma \mathcal{V}_n^{\gamma;\mathrm{Sob}}) = \dim(\mathcal{V}_n^{\sigma(\gamma);\mathrm{Sob}})$. The desired equality then follows. □

In the next two theorems, we construct explicit bases for $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$ for $n \leq 3$.

Theorem 3.3.8. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \{0,1\}$. *Then,* $\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}} = \mathcal{V}_n^{\boldsymbol{\gamma}}$.

Proof. The case $n = 0$ is straightforward. If $n = 1$, let $p_1 \in \mathcal{V}_1^{\boldsymbol{\gamma}}$ and $q_0 \in \Pi_0^2$. It is easy to check that $S_0^{\boldsymbol{\gamma}}(p_1) = 0$ and $\mathfrak{B}^{\boldsymbol{\gamma}}(p_1, q_0) = 0$. Since $q_0 \in \Pi_0^2$, we deduce that $\partial_i q_0 = 0$ for all $i \in \{1,2,3\}$, so $\langle p_1, q_0\rangle_{\mathbb{H}^2_{\boldsymbol{\gamma}}} = 0$. Thus, we have $\mathcal{V}_1^{\boldsymbol{\gamma}} \subset \mathcal{V}_1^{\boldsymbol{\gamma};\mathrm{Sob}}$, and since $\dim\left(\mathcal{V}_1^{\boldsymbol{\gamma}}\right) = \dim\left(\mathcal{V}_1^{\boldsymbol{\gamma};\mathrm{Sob}}\right)$, the desired equality for the case $n = 1$ follows. □

Definition 3.3.9. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$, *we define the polynomials*

$$\begin{aligned}
&\rho_{2,0}^{\boldsymbol{\gamma}} = (\mathrm{I} - \mathrm{R}^{\boldsymbol{\gamma}}) J_{2,0}^{\boldsymbol{\gamma}}, \quad \rho_{2,1}^{\boldsymbol{\gamma}} = T_{\sigma_1}^{\star} \rho_{2,0}^{\sigma_2(\boldsymbol{\gamma})}, \quad \rho_{2,2}^{\boldsymbol{\gamma}} = T_{\sigma_2}^{\star} \rho_{2,0}^{\sigma_1(\boldsymbol{\gamma})},\\
&\rho_{3,0}^{\boldsymbol{\gamma}} = (\mathrm{I} - \mathrm{R}^{\boldsymbol{\gamma}}) \left(J_{3,0}^{\boldsymbol{\gamma}} - a_1^{\boldsymbol{\gamma}} J_{2,0}^{\boldsymbol{\gamma}}\right), \quad \rho_{3,1}^{\boldsymbol{\gamma}} = T_{\sigma_1}^{\star} \rho_{3,0}^{\sigma_2(\boldsymbol{\gamma})}, \quad \rho_{3,2}^{\boldsymbol{\gamma}} = T_{\sigma_2}^{\star} \rho_{3,0}^{\sigma_1(\boldsymbol{\gamma})}\\
&\text{and} \quad \rho_{3,3}^{\boldsymbol{\gamma}} = (\mathrm{I} - \mathrm{R}^{\boldsymbol{\gamma}}) \left(J_{3,1}^{\boldsymbol{\gamma}} - a_2^{\boldsymbol{\gamma}} J_{2,0}^{\boldsymbol{\gamma}} - a_3^{\boldsymbol{\gamma}} T_{\sigma_1}^{\star} J_{2,0}^{\sigma_2(\boldsymbol{\gamma})} - a_4^{\boldsymbol{\gamma}} T_{\sigma_2}^{\star} J_{2,0}^{\sigma_1(\boldsymbol{\gamma})}\right)
\end{aligned}$$

where,

$$a_1^{\boldsymbol{\gamma}} := \frac{S_0^{\boldsymbol{\gamma}+e_3}(\partial_1\partial_2 J_{3,0}^{\boldsymbol{\gamma}})}{(5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|)}, \quad a_2^{\boldsymbol{\gamma}} := \frac{S_0^{\boldsymbol{\gamma}+e_3}(\partial_1\partial_2 J_{3,1}^{\boldsymbol{\gamma}})}{(5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|)}, \quad a_3^{\boldsymbol{\gamma}} := -\frac{S_0^{\boldsymbol{\gamma}+e_1}(\partial_1\partial_3 J_{3,1}^{\boldsymbol{\gamma}})}{(5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|)}$$

$$\text{and} \quad a_4^{\boldsymbol{\gamma}} := \frac{S_0^{\boldsymbol{\gamma}+e_2}(\partial_2\partial_3 J_{3,1}^{\boldsymbol{\gamma}})}{(5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|)}.$$

Theorem 3.3.10. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \{2,3\}$. *Then,* $\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}} = \mathrm{span}\left(\{\rho_{n,0}^{\boldsymbol{\gamma}}, \ldots, \rho_{n,n}^{\boldsymbol{\gamma}}\}\right)$.

Proof. If $n = 2$, since $\rho_{2,0}^{\boldsymbol{\gamma}} \in \Pi_2^2$, we deduce that $J_{2,0}^{\boldsymbol{\gamma}} \in V_2^{\boldsymbol{\gamma}}(\mathfrak{B})$. Thus, by Theorem 3.3.5, we have that $\rho_{2,0}^{\boldsymbol{\gamma}} \in \mathcal{V}_2^{\boldsymbol{\gamma};\mathrm{Sob}}$. Invoking Lemma 3.3.7, and since $\rho_{2,0}^{\sigma_1(\boldsymbol{\gamma})} \in \mathcal{V}_2^{\sigma_1(\boldsymbol{\gamma});\mathrm{Sob}}$ and $\rho_{2,0}^{\sigma_2(\boldsymbol{\gamma})} \in \mathcal{V}_2^{\sigma_2(\boldsymbol{\gamma});\mathrm{Sob}}$, it follows that $\rho_{2,1}^{\boldsymbol{\gamma}} \in \mathcal{V}_2^{\boldsymbol{\gamma};\mathrm{Sob}}$ and $\rho_{2,2}^{\boldsymbol{\gamma}} \in \mathcal{V}_2^{\boldsymbol{\gamma};\mathrm{Sob}}$.

Now, we need to prove that $\dim(\mathcal{V}_2^{\boldsymbol{\gamma};\mathrm{Sob}}) = \dim(\mathrm{span}(\{\rho_{2,0}^{\boldsymbol{\gamma}}, \rho_{2,1}^{\boldsymbol{\gamma}}, \rho_{2,2}^{\boldsymbol{\gamma}}\}))$. Let, $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ be scalars such that

$$\mu_1 \rho_{2,0}^{\boldsymbol{\gamma}} + \mu_2 \rho_{2,1}^{\boldsymbol{\gamma}} + \mu_3 \rho_{2,2}^{\boldsymbol{\gamma}} = 0. \tag{3.3.1}$$

Combining (2.4.5) and (2.4.6) with the inclusion $\mathrm{R}^{\boldsymbol{\gamma}}(\Pi_2^2) \subset \Pi_1^2$, we conclude that $\partial_3 \rho_{2,0}^{\boldsymbol{\gamma}} \in \Pi_0^2$, $\partial_1 \rho_{2,1}^{\boldsymbol{\gamma}} \in \Pi_0^2$ and $\partial_2 \rho_{2,2}^{\boldsymbol{\gamma}} \in \Pi_0^2$. Consequently, applying mixed partial derivatives to (3.3.1) yields the following system:

$$\begin{bmatrix} \partial_1\partial_2\rho_{2,0}^{\gamma} & 0 & 0 \\ 0 & \partial_2\partial_3\rho_{2,1}^{\gamma} & 0 \\ 0 & 0 & \partial_1\partial_3\rho_{2,2}^{\gamma} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using (2.4.7), it is easy to check that $\partial_1\partial_2\rho_{2,0}^{\gamma} = (5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|)$. Furthermore, from Proposition 2.2.9 we have that $\partial_1\partial_3\rho_{2,1}^{\gamma} = \partial_2\partial_3\rho_{2,2}^{\gamma} = (5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|)$. Hence, the system has a unique solution given by $\mu_1 = 0$, $\mu_2 = 0$ and $\mu_3 = 0$. Therefore, $\dim(\operatorname{span}(\{\rho_{2,0}^{\gamma}, \rho_{2,1}^{\gamma}, \rho_{2,2}^{\gamma}\})) = 3$. Since $\dim(\mathcal{V}_2^{\gamma;\mathrm{Sob}}) = 3$, we conclude with the desired result.

If $n = 3$, let $q_2 \in \Pi_2^2$. From (2.4.5), we have that $\partial_3(J_{3,0}^{\gamma} - a_1^{\gamma}J_{2,0}^{\gamma}) = 0$. Noticing that $\partial_1\partial_2 q_2 \in \Pi_0^2$, we obtain

$$\begin{aligned} \mathfrak{B}^{\gamma}((J_{3,0}^{\gamma} - a_1^{\gamma}J_{2,0}^{\gamma}), q_2) &= \langle \partial_1\partial_2(J_{3,0}^{\gamma} - a_1^{\gamma}J_{2,0}^{\gamma}), \partial_1\partial_2 q_2\rangle_{\gamma+e_3} \\ &= \partial_1\partial_2 q_2 b_{\gamma+e_3}\left(S_0^{\gamma+e_3}(\partial_1\partial_2 J_{3,0}^{\gamma}) - \frac{S_0^{\gamma+e_3}(\partial_1\partial_2 J_{3,0}^{\gamma})}{(5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|)} S_0^{\gamma+e_3}(\partial_1\partial_2 J_{2,0}^{\gamma})\right). \end{aligned} \tag{3.3.2}$$

Using (2.4.7), we deduce that $\partial_1\partial_2 J_{2,0}^{\gamma} = (5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|)$. Then, the expression above vanishes. Consequently, $(J_{3,0}^{\gamma} - a_1^{\gamma}J_{2,0}^{\gamma}) \in V_3^{\gamma}(\mathfrak{B})$. Therefore, this result and Theorem 3.3.5 imply that $\rho_{3,0}^{\gamma} \in \mathcal{V}_3^{\gamma;\mathrm{Sob}}$. From Lemma 3.3.7, and since $\rho_{3,0}^{\sigma_1(\gamma)} \in \mathcal{V}_3^{\sigma_1(\gamma);\mathrm{Sob}}$ and $\rho_{3,0}^{\sigma_2(\gamma)} \in \mathcal{V}_3^{\sigma_2(\gamma);\mathrm{Sob}}$, we deduce that $\rho_{3,1}^{\gamma} \in \mathcal{V}_3^{\gamma;\mathrm{Sob}}$ and $\rho_{3,2}^{\gamma} \in \mathcal{V}_3^{\gamma;\mathrm{Sob}}$.

On the other hand, since $\partial_1\partial_2 q_2, \partial_1\partial_3 q_2, \partial_2\partial_3 q_2 \in \Pi_0^2$, letting

$$\chi^{\gamma} := J_{3,1}^{\gamma} - a_2^{\gamma}J_{2,0}^{\gamma} - a_3^{\gamma}T_{\sigma_1}^{\star}J_{2,0}^{\sigma_2(\gamma)} - a_4^{\gamma}T_{\sigma_2}^{\star}J_{2,0}^{\sigma_1(\gamma)},$$

we have that

$$\begin{aligned} \mathfrak{B}^{\gamma}(\chi^{\gamma}, q_2) = S_0^{\gamma+e_3}(\partial_1\partial_2\chi^{\gamma})\partial_1\partial_2 q_2 b_{\gamma+e_3} \\ + S_0^{\gamma+e_1}(\partial_1\partial_3\chi^{\gamma})\partial_1\partial_3 q_2 b_{\gamma+e_1} \\ + S_0^{\gamma+e_2}(\partial_2\partial_3\chi^{\gamma})\partial_2\partial_3 q_2 b_{\gamma+e_2}. \end{aligned}$$

Using (2.4.7), we deduce that $\partial_1\partial_2 J_{2,0}^{\gamma} = \partial_1\partial_3 T_{\sigma_1}^{\star}J_{2,0}^{\sigma_2(\gamma)} = \partial_2\partial_3 T_{\sigma_2}^{\star}J_{2,0}^{\sigma_1(\gamma)} = (5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|)$.

Then, we have the following identities:

$$\begin{aligned} S_0^{\gamma+e_3}(\partial_1\partial_2\chi^\gamma) &= S_0^{\gamma+e_3}(\partial_1\partial_2 J_{3,1}^\gamma) - a_2^\gamma(5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|) \overset{\text{Definition 3.3.9}}{=} 0,\\ S_0^{\gamma+e_1}(\partial_1\partial_3\chi^\gamma) &= S_0^{\gamma+e_1}(\partial_1\partial_3 J_{3,1}^\gamma) + a_3^\gamma(5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|) \overset{\text{Definition 3.3.9}}{=} 0,\\ S_0^{\gamma+e_2}(\partial_2\partial_3\chi^\gamma) &= S_0^{\gamma+e_2}(\partial_2\partial_3 J_{3,1}^\gamma) - a_4^\gamma(5+|\boldsymbol{\gamma}|)(4+|\boldsymbol{\gamma}|) \overset{\text{Definition 3.3.9}}{=} 0. \end{aligned} \tag{3.3.3}$$

Consequently, $\chi^\gamma \in V_3^\gamma(\mathfrak{B})$. Therefore, from Theorem 3.3.5 we have that $\rho_{3,3}^\gamma \in \mathcal{V}_3^{\gamma;\mathrm{Sob}}$. Now, we need to prove that $\dim(\mathcal{V}_3^{\gamma;\mathrm{Sob}}) = \dim(\mathrm{span}(\{\rho_{3,0}^\gamma,\ldots,\rho_{3,3}^\gamma\}))$. Let $\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\mu}_3,\tilde{\mu}_4 \in \mathbb{R}$ be scalars such that

$$\tilde{\mu}_1\rho_{3,0}^\gamma + \tilde{\mu}_2\rho_{3,1}^\gamma + \tilde{\mu}_3\rho_{3,2}^\gamma + \tilde{\mu}_4\rho_{3,3}^\gamma = 0. \tag{3.3.4}$$

Using (2.4.5), (2.4.6) and $\mathrm{R}^\gamma(\Pi_3^2) \subset \Pi_1^2$, we have that $\partial_3\rho_{3,0}^\gamma, \partial_1\rho_{3,1}^\gamma, \partial_2\rho_{3,2}^\gamma \in \Pi_1^2$. Then, $\partial_1\partial_2\partial_3\rho_{3,i}^\gamma = 0$ for all $i \in \{0,1,2\}$. Furthermore, using (2.4.1) and (2.3.3) we deduce that

$$\begin{aligned} \partial_1\partial_2\partial_3 J_{3,1}^\gamma(x) = {}& \frac{(7+|\boldsymbol{\gamma}|)(6+|\boldsymbol{\gamma}|)}{2}\\ &\times \partial_1\partial_2\partial_3\left((\gamma_1+1)x_2^3 - (\gamma_2+1)x_1^3 + (\gamma_1-2\gamma_2-1)x_1^2x_2 + (2\gamma_1-\gamma_2+1)x_1x_2^2\right)\\ &= (7+|\boldsymbol{\gamma}|)(6+|\boldsymbol{\gamma}|)(\gamma_1+\gamma_2+2), \end{aligned}$$

and since $\partial_1\partial_2\partial_3\rho_{3,3}^\gamma = \partial_1\partial_2\partial_3 J_{3,1}^\gamma$, we deduce that $\partial_1\partial_2\partial_3\rho_{3,3}^\gamma \neq 0$. Thus, applying the operator $\partial_1\partial_2\partial_3$ to (3.3.4), we find that $\tilde{\mu}_4 = 0$. Now, applying the operators $\partial_1\partial_1\partial_2$, $\partial_2\partial_2\partial_3$ and $\partial_1\partial_1\partial_3$ to (3.3.4), we obtain the following system:

$$\begin{bmatrix} \partial_1\partial_1\partial_2\rho_{3,0}^\gamma & 0 & 0\\ 0 & \partial_2\partial_2\partial_3\rho_{3,1}^\gamma & 0\\ 0 & 0 & \partial_3\partial_3\partial_1\rho_{3,2}^\gamma \end{bmatrix} \begin{bmatrix} \tilde{\mu}_1\\ \tilde{\mu}_2\\ \tilde{\mu}_3 \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}.$$

Using (2.4.7), we observe that $\partial_1\partial_1\partial_2\rho_{3,0}^\gamma = -(7+|\boldsymbol{\gamma}|)(6+|\boldsymbol{\gamma}|)(5+|\boldsymbol{\gamma}|)$. Furthermore, from Proposition 2.2.9 we have that $\partial_2\partial_2\partial_3\rho_{3,1}^\gamma = -(7+|\boldsymbol{\gamma}|)(6+|\boldsymbol{\gamma}|)(5+|\boldsymbol{\gamma}|) = \partial_3\partial_3\partial_1\rho_{3,2}^\gamma$. Hence, the system has a unique solution given by $\tilde{\mu}_1 = 0$, $\tilde{\mu}_2 = 0$ and $\tilde{\mu}_3 = 0$. Therefore,

$$\dim\left(\mathrm{span}\left(\{\rho_{3,0}^\gamma,\ldots,\rho_{3,3}^\gamma\}\right)\right) = 4.$$

Since $\dim(\mathcal{V}_3^{\gamma;\mathrm{Sob}}) = 4$, we conclude with the desired result. □

3.4 Sobolev orthogonal polynomial in the range of M^{γ}

In this section, we focus on studying the properties of the differential operators m^{γ} and M^{γ} defined in (3.1.4). Our main results in this section aim to prove that the restriction $\mathrm{M}^{\gamma}|_{\Pi_n^2}$ is injective for any non-zero $\boldsymbol{\gamma}$ in $(-1, \infty)^3$ (see Lemma 3.4.2), and that M^{γ} maps $\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}$ into $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$ for all $\boldsymbol{\gamma} \in (-1, \infty)^3$ (see Lemma 3.4.20). The proofs of these results follow the same spirit as those in Section 3.2 (see Proposition 3.2.9 and Lemma 3.2.10).

We begin by rewriting the operators m^{γ} and M^{γ} in alternative forms. From (2.2.5), we can express m^{γ} as:

$$m^{\gamma}(u) = \mathrm{W}_{\gamma-e_1-e_2-e_3}^{-1} \partial_1 \partial_2 \left(\mathrm{W}_{\gamma+e_3} u \right), \tag{3.4.1}$$

and can also be expressed as:

$$m^{\gamma}(u) = d_1^{\gamma-e_1-e_2-e_3} d_2^{\gamma-e_2} u = d_2^{\gamma-e_1-e_2-e_3} d_1^{\gamma-e_1}, \tag{3.4.2}$$

where the first form comes straight from (3.1.4), and the other can be deduced from (3.4.1). Similarly, using (3.4.1) and (2.2.6), the operator M^{γ} is given by:

$$\mathrm{M}^{\gamma}(u) = -\mathrm{W}_{\gamma-e_1-e_2-e_3}^{-1} \partial_1 \partial_2 \partial_3 (\mathrm{W}_{\gamma+e_1+e_2+e_3} u), \tag{3.4.3}$$

and can also be expressed as:

$$\mathrm{M}^{\gamma}(u) = d_1^{\gamma-e_1-e_2-e_3} d_2^{\gamma-e_2} d_3^{\gamma+e_3} u = d_2^{\gamma-e_1-e_2-e_3} d_3^{\gamma-e_1} d_1^{\gamma+e_2} u = d_3^{\gamma-e_1-e_2-e_3} d_1^{\gamma-e_3} d_2^{\gamma+e_1} u,$$

where the first form comes straight from (3.1.4), and the other two can be deduced from (3.4.3).

Let $\sigma \in \Sigma_2$. From the expressions above, together with (i) in Proposition 2.2.6 and Proposition 2.2.10, we deduce the following symmetry property:

$$\mathrm{M}^{\gamma}(u) = T_{\sigma^{-1}}^{\star} \mathrm{M}^{\sigma(\gamma)} (T_{\sigma}^{\star} u). \tag{3.4.4}$$

On the other hand, expanding (3.4.1) using the product rule yields the following expression for m^{γ}:

$$\begin{aligned} m^{\gamma}(u) = {} & \left[\gamma_1\gamma_2 \mathrm{W}_{2e_3} - \gamma_1(\gamma_3+1)\mathrm{W}_{e_2+e_3} - (\gamma_3+1)\gamma_2\mathrm{W}_{e_1+e_3} + (\gamma_3+1)\gamma_3\mathrm{W}_{e_1+e_2}\right] u \\ & + \left[\gamma_2\mathrm{W}_{e_1+2e_3} - (\gamma_3+1)\mathrm{W}_{e_1+e_2+e_3}\right]\partial_1 u + \left[\gamma_1\mathrm{W}_{e_2+2e_3} - (\gamma_3+1)\mathrm{W}_{e_1+e_2+e_3}\right]\partial_2 u \\ & + \mathrm{W}_{e_1+e_2+2e_3}\partial_1\partial_2 u. \end{aligned} \tag{3.4.5}$$

This expansion allows us to evaluate $m^{\gamma}(u)$ at the vertices of the triangle T directly:

$$(\forall u \in \mathrm{C}^2(\overline{\mathrm{T}}))\ m^{\gamma}(u)(1,0) = 0, \quad m^{\gamma}(u)(0,1) = 0 \quad \text{and} \quad m^{\gamma}(u)(0,0) = \gamma_1\gamma_2 u(0,0). \tag{3.4.6}$$

In particular, since $\mathrm{M}^{\gamma} = m^{\gamma} d_3^{\gamma+e_3}$ and (cf. (2.2.6)) $d_3^{\gamma+e_3} u(0,0) = 0$, we deduce that $\mathrm{M}^{\gamma}u$ vanishes on $\mathbb{V}_2$. This property shall play a crucial role in the proof of Theorem 3.1.1, as it will allow us to show that:

$$\dim\left(\mathcal{J}_n^{\gamma} \cap \mathrm{M}^{\gamma}(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3})\right) = 0.$$

We now aim to prove that $\mathrm{M}^{\gamma}|_{\Pi_n^2}$ is injective. To do so, we begin by proving the injectivity of $m^{\gamma}|_{\Pi_n^2}$ under the assumption that $\gamma_1 \neq 0$ or $\gamma_3 \neq 0$. Then, we use this result, together with the symmetry property of M^{γ} given in (3.4.4), to attain this aim.

Proposition 3.4.1. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$ be such that $\gamma_1 \neq 0$ or $\gamma_3 \neq 0$, and $n \in \mathbb{N}$. Then the operator $m^{\gamma} : \Pi_n^2 \longrightarrow \Pi_{n+2}^2$ described in (3.1.4), is injective.*

Proof. Let $p_1 \in \Pi_{n+1}^2$ such that

$$(\forall x \in \mathrm{T})\ d_1^{\gamma-e_1-e_2-e_3} p_1(x) \overset{(2.2.5)}{=} \left[\gamma_3 x_1 - \gamma_1(1-x_1-x_2)\right] p_1(x) - x_1(1-x_1-x_2)\partial_1 p_1(x) = 0.$$

Fixing $x_2 \in (0,1)$, we observe that $\hat{p}_1 := p_1(\cdot, x_2) \in \Pi_{n+1}^1$ and satisfies

$$(\forall x_1 \in (0, 1-x_2))\ \left[\gamma_3 x_1 - \gamma_1(1-x_1-x_2)\right]\hat{p}_1(x_1) - x_1(1-x_1-x_2)\hat{p}_1'(x_1) = 0. \tag{3.4.7}$$

This is a first-order linear ordinary differential equation, whose general solution is given by

$$(\forall x_1 \in (0, 1-x_2)) \quad \hat{p}_1(x_1) = c(x_2) x_1^{-\gamma_1}(1-x_1-x_2)^{-\gamma_3}$$

for some constant $c(x_2) \in \mathbb{R}$. However, since $\hat{p}_1 \in \Pi^1_{n+1}$ and $\gamma_1 \neq 0$ or $\gamma_3 \neq 0$, we deduce that $c(x_2) = 0$. Therefore, $\hat{p}_1 = 0$. Since x_2 is arbitrary, we conclude that $p_1 = 0$ on T. Now, let $p_2 \in \Pi^2_n$ such that

$$(\forall x \in \mathrm{T})\ d_2^{\gamma - e_2} p_2(x) \overset{(2.2.5)}{=} \left[(\gamma_3 + 1)x_2 - \gamma_2(1-x_1-x_2)\right] p_2(x) - x_2(1-x_1-x_2)\partial_2 p_2(x) = 0.$$

Fixing $x_1 \in (0,1)$, we observe that $\hat{p}_2 := p_2(x_1, \cdot) \in \Pi^1_n$ and satisfies

$$(\forall x_2 \in (0, 1-x_1))\ \left[(\gamma_3 + 1)x_2 - \gamma_2(1-x_1-x_2)\right] \hat{p}_2(x_2) - x_2(1-x_1-x_2)\hat{p}_2'(x_2) = 0. \quad (3.4.8)$$

This is a first-order linear ordinary differential equation, whose general solution is given by

$$(\forall x_2 \in (0, 1-x_1)) \quad \hat{p}_2(x_2) = \tilde{c}(x_1) x_2^{-\gamma_2}(1-x_1-x_2)^{-1-\gamma_3}$$

for some constant $\tilde{c}(x_1) \in \mathbb{R}$. However, since $\hat{p}_2 \in \Pi^1_n$ and $\gamma_3 > -1$, we deduce that $\tilde{c}(x_1) = 0$. Therefore, $\hat{p}_2 = 0$. Since x_1 is arbitrary, we conclude that $p_2 = 0$ on T. Finally, since the composition of injective operators is an injective operator, we conclude that m^{γ} is injective. $\square$

Lemma 3.4.2. *Let $\boldsymbol{\gamma} \in (-1, \infty)^3$, be such that $\boldsymbol{\gamma} \neq (0,0,0)$, and $n \in \mathbb{N}$. Then the operator $\mathrm{M}^{\gamma} : \Pi^2_n \longrightarrow \Pi^2_{n+3}$ described in (3.1.4), is injective.*

Proof. First we observe that $d_3^{\gamma + e_3}$ is injective. Indeed, let $p \in \Pi^2_n$ be such that

$$d_3^{\gamma + e_3} p \overset{(2.2.6)}{=} -\mathrm{W}^{-1}_{\gamma + e_3} \partial_3 (\mathrm{W}_{\gamma + e_1 + e_2 + e_3} p) = 0 \quad \text{on } \mathrm{T}.$$

Since $\mathrm{W}^{-1}_{\gamma + e_3} > 0$ on T, we have that $\mathrm{W}_{\gamma + e_1 + e_2 + e_3} p$, satisfies the boundary value problem

$$b \cdot \nabla \omega = 0 \quad \text{on } \mathrm{T},$$

$$\omega = 0 \quad \text{on } \partial\mathrm{T},$$

where $b = (-1,1)^t$. This problem has the unique solution $\omega = 0$. Then, $\mathrm{W}_{\gamma+e_1+e_2+e_3} p = 0$ on T. Since $\mathrm{W}_{\gamma+e_1+e_2+e_3} > 0$ on T, we deduce that $p = 0$.

If $\gamma_1 \neq 0$ or $\gamma_3 \neq 0$, the injectivity of m^γ follows from Proposition 3.4.1, since $\mathrm{M}^\gamma = m^\gamma d_3^{\gamma+e_3}$ and the composition of injective operators is an injective operator.

If $\gamma_1 = 0$ and $\gamma_3 = 0$, we observe first that from the above case, $\mathrm{M}^{\sigma_1(\gamma)}$ is injective. From (3.4.4), we have that $\mathrm{M}^\gamma = T^\star_{\sigma_2} \mathrm{M}^{\sigma_1(\gamma)} T^\star_{\sigma_1}$. Since $T^\star_{\sigma_1}$ and $T^\star_{\sigma_2}$ are injective, we conclude that M^γ is injective. ☐

Remark 3.4.3. In the proof of Lemma 3.4.2, we showed that the operator

$$d_3^{\gamma+e_3} : \Pi^2_n \longrightarrow \Pi^2_{n+1}$$

is injective. In contrast to m^γ, this result holds for all $\boldsymbol{\gamma} \in (-1,\infty)^3$. Moreover, by applying the same argument, we can prove that the operators

$$d_1^{\gamma+e_2} : \Pi^2_n \longrightarrow \Pi^2_{n+1} \quad \text{and} \quad d_2^{\gamma+e_1} : \Pi^2_n \longrightarrow \Pi^2_{n+1}$$

are also injective.

In the remainder of this section, we will focus on proving that M^γ maps $\mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3}$ into $\mathcal{V}^{\gamma,\mathrm{Sob}}_n$. To this end, we first define some auxiliary differential operators that will enable us to establish commutation relations between m^γ and its derivatives.

Definition 3.4.4. *Given $\boldsymbol{\gamma} \in \mathbb{R}^3$ and $j \in \{1,2\}$, we define the first-order differential operators $\mathcal{D}^\gamma_j := (\gamma_3+1)\mathrm{I} - \mathrm{W}_{e_3}\partial_j$.*

The mapping sequence $\Pi^2_n \xrightarrow{\partial_j} \Pi^2_{n-1} \xrightarrow{\mathrm{W}_{e_3}} \Pi^2_n$ implies the inclusion $\mathcal{D}^\gamma_j(\Pi^2_n) \subset \Pi^2_n$, for all $n \in \mathbb{N}_0$. Furthermore, if $\boldsymbol{\gamma} \in (-1,\infty)^3$, we have the following result

Proposition 3.4.5. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$, $n \in \mathbb{N}_0$ and $j \in \{1,2\}$. Then, $\mathcal{D}^\gamma_j\left(\mathcal{V}^{\gamma+e_3}_n\right) \subset \mathcal{V}^{\gamma+e_j}_n$.*

Proof. Let $p \in \mathcal{V}_n^{\gamma+e_3}$ and $q \in \Pi_{n-1}^2$. From Definition 3.4.4 and (ii) in Proposition 2.2.2 we have

$$\begin{aligned}\langle \mathcal{D}_j^{\gamma} p, q\rangle_{\gamma+e_j} &= (\gamma_3+1)\langle p, q\rangle_{\gamma+e_j} - \langle \partial_j p, q\rangle_{\gamma+e_j+e_3} \\ &= (\gamma_3+1)\langle p, q\rangle_{\gamma+e_j} - \langle p, d_j^{\gamma} q\rangle_{\gamma} \\ &= (\gamma_j+1)\langle p, q\rangle_{\gamma+e_3} + \langle p, \mathrm{W}_{e_j}\partial_j q\rangle_{\gamma+e_3}.\end{aligned}$$

The last term on the right-hand side is zero since $q \in \Pi_{n-1}^2$, $\mathrm{W}_{e_j}\partial_j q \in \Pi_{n-1}^2$ and $p \in \mathcal{V}_n^{\gamma+e_3}$. Therefore, $\mathcal{D}_j^{\gamma} p \in \mathcal{V}_n^{\gamma+e_j}$. ☐

Definition 3.4.6. *Given $\boldsymbol{\gamma} \in \mathbb{R}^3$, $i \in \{1,2,3\}$ and $j \in \{1,2\}$, we introduce the second-order differential operators $\Lambda_i^{\gamma} := d_i^{\gamma}\partial_i$ and $\Psi_j^{\gamma} := d_j^{\gamma}\partial_3$.*

The mapping sequence $\Pi_n^2 \xrightarrow{\partial_i} \Pi_{n-1}^2 \xrightarrow{d_j^{\gamma}} \Pi_n^2$ implies the inclusions $\Lambda_i^{\gamma}(\Pi_n^2) \subset \Pi_n^2$ and $\Psi_i^{\gamma}(\Pi_n^2) \subset \Pi_n^2$ for all $n \in \mathbb{N}_0$. Furthermore, provided that $\boldsymbol{\gamma} \in (-1,\infty)^3$, the following result holds:

Proposition 3.4.7. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$, $n \in \mathbb{N}_0$ and $i \in \{1,2,3\}$. Then, the following inclusions hold:*

(i) $\Lambda_i^{\gamma}\left(\mathcal{V}_n^{\gamma}\right) \subset \mathcal{V}_n^{\gamma}$.

(ii) $\Psi_1^{\gamma+e_2}\left(\mathcal{V}_n^{\gamma+e_3}\right) \subset \mathcal{V}_n^{\gamma+e_2}$.

(iii) $\Psi_2^{\gamma+e_1}\left(\mathcal{V}_n^{\gamma+e_3}\right) \subset \mathcal{V}_n^{\gamma+e_1}$.

Proof. For (i), it suffices to note that, by virtue of Proposition 2.2.2 and Proposition 2.2.3, we have the mapping sequence $\mathcal{V}_n^{\gamma} \xrightarrow{\partial_i} \mathcal{V}_{n-1}^{\gamma+e_i+e_3} \xrightarrow{d_i^{\gamma}} \mathcal{V}_n^{\gamma}$. Similarly, for (ii) and (iii), we apply the same reasoning, observing that $\mathcal{V}_n^{\gamma+e_3} \xrightarrow{\partial_3} \mathcal{V}_{n-1}^{\gamma+e_1+e_2+e_3} \xrightarrow{d_1^{\gamma+e_2}} \mathcal{V}_n^{\gamma+e_2}$ and $\mathcal{V}_n^{\gamma+e_3} \xrightarrow{\partial_3} \mathcal{V}_{n-1}^{\gamma+e_1+e_2+e_3} \xrightarrow{d_2^{\gamma+e_1}} \mathcal{V}_n^{\gamma+e_1}$ ☐

Definition 3.4.8. *Given $\boldsymbol{\gamma} \in \mathbb{R}^3$, $i \in \{1,2,3\}$ and $j \in \{1,2\}$, we introduce the third-order differential operators $\Theta_{i,j}^{\gamma} := \Lambda_i^{\gamma}\mathcal{D}_j^{\gamma}$.*

By combining Definition 3.4.4 and Definition 3.4.6, we deduce the identity:

$$\Theta_{i,j}^{\gamma}(p) = (\gamma_3+1)\Lambda_i^{\gamma}(p) + (1-\delta_{i,3})d_i^{\gamma}\partial_j p - d_i^{\gamma}\left(\mathrm{W}_{e_3}\partial_i\partial_j p\right), \tag{3.4.9}$$

where $\delta_{i,3}$ denotes the Kronecker delta.

Based on the mapping sequence $\Pi_n^2 \xrightarrow{\mathcal{D}_j^{\gamma}} \Pi_n^2 \xrightarrow{\Lambda_i^{\gamma}} \Pi_n^2$, we infer the inclusions $\Theta_{i,j}^{\gamma}(\Pi_n^2) \subset \Pi_n^2$, for all $n \in \mathbb{N}_0$. Furthermore, if $\boldsymbol{\gamma} \in (-1,\infty)^3$ we have the following result

Proposition 3.4.9. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$, $n \in \mathbb{N}_0$, $i \in \{1,2,3\}$ *and* $j \in \{1,2\}$. *Then, the inclusion* $\Theta_{i,j}^{\gamma+e_j}(\mathcal{V}_n^{\gamma+e_3}) \subset \mathcal{V}_n^{\gamma+e_j}$ *holds.*

Proof. It is direct consequence of Proposition 3.4.5 and Proposition 3.4.7. ☐

Definition 3.4.10. *Given* $\boldsymbol{\gamma} \in \mathbb{R}^3$, $i \in \{1,2,3\}$ *and* $j \in \{1,2\}$, *we introduce the fourth-order differential operators* $\Phi_{i,j}^{\gamma} := \Lambda_i^{\gamma}\Psi_j^{\gamma}$.

Based on the mapping sequence $\Pi_n^2 \xrightarrow{\Psi_j^{\gamma}} \Pi_n^2 \xrightarrow{\Lambda_i^{\gamma}} \Pi_n^2$, we infer the inclusions $\Phi_{i,j}^{\gamma}(\Pi_n^2) \subset \Pi_n^2$, for all $n \in \mathbb{N}_0$. Furthermore, if $\boldsymbol{\gamma} \in (-1,\infty)^3$ we have the following result

Proposition 3.4.11. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$, $n \in \mathbb{N}_0$, $i \in \{1,2,3\}$ *and* $j \in \{1,2\}$. *Then, the following inclusions hold:*

(i) $\Phi_{i,1}^{\gamma+e_2}(\mathcal{V}_n^{\gamma+e_3}) \subset \mathcal{V}_n^{\gamma+e_2}$.

(ii) $\Phi_{i,2}^{\gamma+e_1}(\mathcal{V}_n^{\gamma+e_3}) \subset \mathcal{V}_n^{\gamma+e_1}$.

Proof. This follows directly from Definition 3.4.10 and Proposition 3.4.7. ☐

Proposition 3.4.12. *Let* $\boldsymbol{\gamma} \in \mathbb{R}^3$. *Then, the following commutation relations hold:*

(i) $\partial_1 d_1^{\gamma-e_1-e_3} = d_1^{\gamma}\partial_1 + (\gamma_1+\gamma_3)\mathrm{I}$.

(ii) $\partial_2 d_2^{\gamma-e_2-e_3} = d_2^{\gamma}\partial_2 + (\gamma_2+\gamma_3)\mathrm{I}$.

(iii) $\partial_3 d_3^{\gamma-e_1-e_2} = d_3^{\gamma}\partial_3 + (\gamma_1+\gamma_2)\,\mathrm{I}$.

where I *is the identity operator.*

Proof. (i), (ii) and (iii) are direct consequences of the definition of the operators (2.2.5) and (2.2.6), and the rules of calculus. ☐

Proposition 3.4.13. *Let* $\boldsymbol{\gamma} \in \mathbb{R}^3$*. Then, the following commutation relations hold:*

(i) $\partial_2 d_1^{\gamma-e_1-e_2-e_3} = d_1^{\gamma-e_1}\partial_2 - \mathrm{W}_{e_1}\partial_3 + \gamma_1 \mathrm{I}$.

(ii) $\partial_1 d_2^{\gamma-e_1-e_2-e_3} = d_2^{\gamma-e_2}\partial_1 + \mathrm{W}_{e_2}\partial_3 + \gamma_2 \mathrm{I}$.

(iii) $\partial_3 d_1^{\gamma+e_3} = d_1^{\gamma+e_1+e_2+e_3}\partial_3 + \mathrm{W}_{e_3}\partial_2 - (\gamma_3+2)\mathrm{I}$.

(iv) $\partial_3 d_2^{\gamma+e_3} = d_2^{\gamma+e_1+e_2+e_3}\partial_3 - \mathrm{W}_{e_3}\partial_1 + (\gamma_3+2)\mathrm{I}$.

Proof. (i), (ii), (iii) and (iv) are direct consequences of the definition of the operators (2.2.5) and (2.2.6), and the rules of calculus. ☐

From (2.2.5) and (2.2.6), we observe that the operators d_1^{γ}, d_2^{γ}, and d_3^{γ} are independent of the parameters γ_2, γ_1, and γ_3, respectively. We will use this observation in the following results.

Proposition 3.4.14. *Let* $\boldsymbol{\gamma} \in \mathbb{R}^3$*. Then, the following commutation relation holds:*

$$\partial_1 m^{\gamma} = m^{\gamma+e_1+e_2+e_3}\partial_1 + d_1^{\gamma}\left[\mathrm{W}_{e_3}\partial_1 + \mathrm{W}_{e_2}\partial_3 + \gamma_2 \mathrm{I}\right] + (\gamma_1+\gamma_3) d_2^{\gamma-e_2}.$$

Proof. Let $p \in \mathrm{C}^3(\overline{\mathrm{T}})$. From (3.1.4) and part (i) of Proposition 3.4.12 we have

$$\partial_1 m^{\gamma} p = \left[d_1^{\gamma}\partial_1 + (\gamma_1+\gamma_3)\mathrm{I}\right] d_2^{\gamma-e_2} p = d_1^{\gamma}\partial_1 d_2^{\gamma-e_2} p + (\gamma_1+\gamma_3) d_2^{\gamma-e_2} p. \tag{3.4.10}$$

Now, from (2.2.5), it is straightforward to verify that

$$\partial_1 d_2^{\gamma-e_2} p = d_2^{\gamma+e_1+e_3}\partial_1 p + \mathrm{W}_{e_3}\partial_1 p + \mathrm{W}_{e_2}\partial_3 p + \gamma_2 p. \tag{3.4.11}$$

So, putting together (3.4.10) and (3.4.11), we conclude the desired equality. ☐

In what follows, the proofs of Proposition 3.4.15 and Proposition 3.4.16 are lengthy but elementary in terms of previous results and the rules of calculus. For this reason they have been thoroughly verified with the help of a computer algebra system.

Proposition 3.4.15. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$. Then, the following commutation relation holds:*

$$\partial_1\partial_2 m^{\gamma} = m^{\gamma+e_1+e_2+2e_3}\partial_1\partial_2 + \sum_{i=1}^{3} c_i\Lambda_i^{\gamma+e_3} + c_4 I,$$

where $c_1 = (2\gamma_2+\gamma_3+1)$, $c_2 = (2\gamma_1+\gamma_3+1)$, $c_3 = -(\gamma_3+1)$ *and* $c_4 = [\gamma_1 c_1 - c_3(\gamma_2+\gamma_3)]$.

Proof. Let $p \in \mathrm{C}^4(\overline{\mathrm{T}})$. Combining (3.1.4), Proposition 3.4.12, and Proposition 3.4.13, we obtain

$$\partial_1\partial_2 m^{\gamma} p = m^{\gamma+e_1+e_2+2e_3}\partial_1\partial_2 p + \mathcal{A}^{\gamma}(p),$$

where $\mathcal{A}^{\gamma}$ denotes a third-order differential operator defined by

$$\begin{aligned}\mathcal{A}^{\gamma}(p) := (\gamma_2+\gamma_3+1)\Lambda_1^{\gamma+e_3}p + (\gamma_1+\gamma_3)\Lambda_2^{\gamma+e_3}p + (\gamma_1+\gamma_3)(\gamma_2+\gamma_3+1)p + (\gamma_2+1)d_1^{\gamma+e_3}\partial_2 p \\ + (\gamma_1+1)\partial_1 d_2^{\gamma-e_2}p + \mathrm{W}_{e_2}d_1^{\gamma+e_3}\partial_3\partial_2 p - \mathrm{W}_{e_1}\partial_1\partial_3 d_2^{\gamma-e_2}p.\end{aligned}$$

It remains to show that $\mathcal{A}^{\gamma}$ can be written in the form:

$$\mathcal{A}^{\gamma}(p) = c_1\Lambda_1^{\gamma+e_3}p + c_2\Lambda_2^{\gamma+e_3}p + c_3\Lambda_3^{\gamma+e_3}p + c_4 p.$$

By adding and subtracting appropriate terms, we can rearrange $\mathcal{A}^{\gamma}$ as follows:

$$\begin{aligned}\mathcal{A}^{\gamma}(p) &= c_1\Lambda_1^{\gamma+e_3}p + c_2\Lambda_2^{\gamma+e_3}p \\ &\quad + (\gamma_1+\gamma_3)(\gamma_2+\gamma_3+1)p - \gamma_2\Lambda_1^{\gamma+e_3}p - (\gamma_1+1)\Lambda_2^{\gamma+e_3}p \\ &\quad + (\gamma_2+1)d_1^{\gamma+e_3}\partial_2 p + (\gamma_1+1)\partial_1 d_2^{\gamma-e_2}p + \mathrm{W}_{e_2}d_1^{\gamma+e_3}\partial_3\partial_2 p - \mathrm{W}_{e_1}\partial_1\partial_3 d_2^{\gamma-e_2}p \\ &= c_1\Lambda_1^{\gamma+e_3}p + c_2\Lambda_2^{\gamma+e_3}p + c_4 p \\ &\quad - \gamma_2(1+\gamma_1)p - \gamma_2\Lambda_1^{\gamma+e_3}p - (\gamma_1+1)\Lambda_2^{\gamma+e_3}p \\ &\quad + (\gamma_2+1)d_1^{\gamma+e_3}\partial_2 p + (\gamma_1+1)\partial_1 d_2^{\gamma-e_2}p + \mathrm{W}_{e_2}d_1^{\gamma+e_3}\partial_3\partial_2 p - \mathrm{W}_{e_1}\partial_1\partial_3 d_2^{\gamma-e_2}p.\end{aligned}$$

Expanding the differential terms in the expression above yields

$$\mathcal{A}^{\gamma}(p) = c_1\Lambda_1^{\gamma+e_3}p + c_2\Lambda_2^{\gamma+e_3}p$$

$$\begin{aligned}&-\Big[\left((1+\gamma_1)\mathrm{W}_{e_2}-(1+\gamma_2)\mathrm{W}_{e_1}\right)\partial_2 p-\mathrm{W}_{e_1+e_2}\partial_2^2 p+\left(\mathrm{W}_{e_1}-\mathrm{W}_{e_2}\right)\partial_1 p\\&-\gamma_1\mathrm{W}_{e_2}\partial_1 p+\gamma_2\mathrm{W}_{e_1}\partial_1 p+2\mathrm{W}_{e_1+e_2}\partial_1\partial_2 p-\mathrm{W}_{e_1+e_2}\partial_1^2 p\Big]+c_4 p.\end{aligned}$$

Finally, observing that the term in brackets corresponds precisely to $(\gamma_3+1)\Lambda_3^{\gamma+e_3}$, we conclude the desired equality. □

Proposition 3.4.16. *Let $\boldsymbol{\gamma}\in(-1,\infty)^3$. Then, the following commutation relation holds:*

$$\partial_1\partial_3 m^{\gamma}=m^{\gamma+2e_1+e_2+e_3}\partial_1\partial_3+\sum_{i=1}^{3}c_i\Theta_{i,1}^{\gamma+e_1}+c_4\Psi_2^{\gamma+e_1}+c_5\mathcal{D}_1^{\gamma},$$

where $c_1=1$, $c_2=-1$, $c_3=-1$, $c_4=2(1+\gamma_1+\gamma_3)$, and $c_5=(\gamma_1-\gamma_2+\gamma_3)$.

Proof. Let $p\in\mathrm{C}^4(\overline{\mathrm{T}})$. Combining (3.1.4), Proposition 3.4.12 and Proposition 3.4.13 we have

$$\partial_1\partial_3 m^{\gamma}(p)=m^{\gamma+2e_1+e_2+e_3}\partial_1\partial_3(p)+\mathcal{B}^{\gamma}(p),$$

where $\mathcal{B}^{\gamma}$ denotes a third-order differential operator defined by

$$\begin{aligned}\mathcal{B}^{\gamma}(p)=&-d_1^{\gamma+e_1}\left(\mathrm{W}_{e_3}\partial_1^2 p\right)+(\gamma_3+2)\Lambda_1^{\gamma+e_1}(p)+d_1^{\gamma+e_1}\partial_3\left(\mathrm{W}_{e_2}\partial_3 p\right)+\gamma_2 d_1^{\gamma+e_1}\partial_3 p\\&+\mathrm{W}_{e_3}\partial_1\partial_2 d_2^{\gamma-e_2}p-(\gamma_3+1)\partial_1 d_2^{\gamma-e_2}p+(\gamma_1+\gamma_3)\partial_3 d_2^{\gamma-e_2}p.\end{aligned}$$

Invoking property (iv) of Proposition 3.4.12, together with (3.4.9), Definition 3.4.4, and Definition 3.4.6, we can rewrite $\mathcal{B}^{\gamma}$ as follows:

$$\begin{aligned}\mathcal{B}^{\gamma}(p)=&\,\Theta_{1,1}^{\gamma+e_1}(p)+(\gamma_1+\gamma_3)\Psi_2^{\gamma+e_1}(p)+(\gamma_1+\gamma_3)\mathcal{D}_1^{\gamma}(p)+d_1^{\gamma+e_1}\partial_3\left(\mathrm{W}_{e_2}\partial_3 p\right)+\gamma_2 d_1^{\gamma+e_1}\partial_3 p\\&+\mathrm{W}_{e_3}\partial_1\partial_2 d_2^{\gamma-e_2}p-(\gamma_3+1)\partial_1 d_2^{\gamma-e_2}p.\end{aligned}$$

Next, utilizing (ii) from Proposition 3.4.12 and Proposition 3.4.13, we obtain

$$\begin{aligned}\mathcal{B}^{\gamma}(p)=&\,\Theta_{1,1}^{\gamma+e_1}(p)+(\gamma_1+\gamma_3)\Psi_2^{\gamma+e_1}(p)+(\gamma_1+\gamma_3)\mathcal{D}_1^{\gamma}(p)+d_1^{\gamma+e_1}\partial_3\left(\mathrm{W}_{e_2}\partial_3 p\right)+\gamma_2 d_1^{\gamma+e_1}\partial_3 p\\&+\mathrm{W}_{e_3}d_2^{\gamma+e_1+2e_3}\partial_1\partial_2 p+\mathrm{W}_{e_2+e_3}\partial_3\partial_2 p+(\gamma_2+1)\mathrm{W}_{e_3}\partial_2 p+(\gamma_2+\gamma_3+1)\mathrm{W}_{e_3}\partial_1 p\end{aligned}$$

$$
\begin{aligned}
&- (\gamma_3+1) d_2^{\gamma+e_1-e_2+e_3}\partial_1 p - (\gamma_3+1)\mathrm{W}_{e_2}\partial_3 p - (\gamma_3+1)\gamma_2 p \\
&= \Theta_{1,1}^{\gamma+e_1}(p) + (\gamma_1+\gamma_3)\Psi_2^{\gamma+e_1}(p) + (\gamma_1-\gamma_2+\gamma_3)\mathcal{D}_1^{\gamma}(p) + d_1^{\gamma+e_1}\partial_3\left(\mathrm{W}_{e_2}\partial_3 p\right) + \gamma_2 d_1^{\gamma+e_1}\partial_3 p \\
&+ \mathrm{W}_{e_3} d_2^{\gamma+e_1+2e_3}\partial_1\partial_2 p + \mathrm{W}_{e_2+e_3}\partial_3\partial_2 p + (\gamma_2+1)\mathrm{W}_{e_3}\partial_2 p + (\gamma_3+1)\mathrm{W}_{e_3}\partial_1 p \\
&- (\gamma_3+1) d_2^{\gamma+e_1-e_2+e_3}\partial_1 p - (\gamma_3+1)\mathrm{W}_{e_2}\partial_3 p.
\end{aligned}
$$

Expanding the terms using (2.2.5) yields

$$
\begin{aligned}
\mathcal{B}^\gamma(p) &= \Theta_{1,1}^{\gamma+e_1}(p) + (\gamma_1+\gamma_3)\Psi_2^{\gamma+e_1}(p) + (\gamma_1-\gamma_2+\gamma_3)\mathcal{D}_1^{\gamma}(p) \\
&+ (\gamma_3+1)\mathrm{W}_{e_1}\partial_3\left(\mathrm{W}_{e_2}\partial_3 p\right) - (\gamma_1+2)\mathrm{W}_{e_3}\partial_3\left(\mathrm{W}_{e_2}\partial_3 p\right) - \mathrm{W}_{e_1+e_3}\partial_1\partial_3\left(\mathrm{W}_{e_2}\partial_3 p\right) \\
&+ \gamma_2(\gamma_3+1)\mathrm{W}_{e_1}\partial_3 p - \gamma_2(\gamma_1+2)\mathrm{W}_{e_3}\partial_3 p - \gamma_2\mathrm{W}_{e_1+e_3}\partial_1\partial_3 p \\
&+ (\gamma_3+3)\mathrm{W}_{e_2+e_3}\partial_1\partial_2 p - (\gamma_2+1)\mathrm{W}_{2e_3}\partial_1\partial_2 p - \mathrm{W}_{e_2+2e_3}\partial_2^2\partial_1 p \\
&+ \mathrm{W}_{e_2+e_3}\partial_3\partial_2 p + (\gamma_2+1)\mathrm{W}_{e_3}\partial_2 p + (\gamma_3+1)\mathrm{W}_{e_3}\partial_1 p \\
&- (\gamma_3+1)(\gamma_3+2)\mathrm{W}_{e_2}\partial_1 p + \gamma_2(\gamma_3+1)\mathrm{W}_{e_3}\partial_1 p + (\gamma_3+1)\mathrm{W}_{e_2+e_3}\partial_2\partial_1 p \\
&- (\gamma_3+1)\mathrm{W}_{e_2}\partial_3 p \\
&= \Theta_{1,1}^{\gamma+e_1}(p) + (\gamma_1+\gamma_3)\Psi_2^{\gamma+e_1}(p) + (\gamma_1-\gamma_2+\gamma_3)\mathcal{D}_1^{\gamma}(p) \\
&+ (\gamma_3+1)\mathrm{W}_{e_1+e_2}\partial_3^2 p + (\gamma_3+1)\mathrm{W}_{e_1}\partial_3 p - (\gamma_1+2)\mathrm{W}_{e_2+e_3}\partial_3^2 p - (\gamma_1+2)\mathrm{W}_{e_3}\partial_3 p \\
&- \mathrm{W}_{e_1+e_3}\partial_1\partial_3 p - \mathrm{W}_{e_1+e_2+e_3}\partial_1\partial_3^2 p \\
&+ \gamma_2(\gamma_3+1)\mathrm{W}_{e_1}\partial_3 p - \gamma_2(\gamma_1+2)\mathrm{W}_{e_3}\partial_3 p - \gamma_2\mathrm{W}_{e_1+e_3}\partial_1\partial_3 p \\
&+ (\gamma_3+3)\mathrm{W}_{e_2+e_3}\partial_1\partial_2 p - (\gamma_2+1)\mathrm{W}_{2e_3}\partial_1\partial_2 p - \mathrm{W}_{e_2+2e_3}\partial_2^2\partial_1 p \\
&+ \mathrm{W}_{e_2+e_3}\partial_3\partial_2 p + (\gamma_2+1)\mathrm{W}_{e_3}\partial_2 p + (\gamma_3+1)\mathrm{W}_{e_3}\partial_1 p \\
&- (\gamma_3+1)(\gamma_3+2)\mathrm{W}_{e_2}\partial_1 p + \gamma_2(\gamma_3+1)\mathrm{W}_{e_3}\partial_1 p + (\gamma_3+1)\mathrm{W}_{e_2+e_3}\partial_2\partial_1 p \\
&- (\gamma_3+1)\mathrm{W}_{e_2}\partial_3 p.
\end{aligned}
$$

Finally, by regrouping the terms, we identify the resulting expression as

$$
\mathcal{B}^\gamma(p) = \sum_{i=1}^{3} c_i\Theta_{i,1}^{\gamma+e_1}(p) + c_4\Psi_2^{\gamma+e_1}(p) + c_5\mathcal{D}_1^{\gamma}(p),
$$

which completes the proof. □

Let $\boldsymbol{\gamma} \in (-1, \infty)^3$ and $n \in \mathbb{N}_{\geq 2}$. From Lemma 3.3.1 and Theorem 3.3.5, we observe that $p_n \in \mathcal{V}_n^{\boldsymbol{\gamma},\mathrm{Sob}}$ if and only if $p_n \in V_n^{\boldsymbol{\gamma}}(\mathfrak{B})$ and $\mathrm{R}^{\boldsymbol{\gamma}}(p_n) = 0$. The following results enable us to show that polynomials in $\mathrm{M}^{\boldsymbol{\gamma}}(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+e_1+e_2+e_3})$ satisfy both conditions.

Proposition 3.4.17. *Let* $\boldsymbol{\gamma} \in (-1, \infty)^3$ *and* $n \in \mathbb{N}$. *Then, the following inclusions hold:*

(i) $\partial_1 \partial_2 m^{\boldsymbol{\gamma}} \left(\mathcal{V}_n^{\boldsymbol{\gamma}+e_3}\right) \subset \mathcal{V}_n^{\boldsymbol{\gamma}+e_3}$.

(ii) $\partial_1 \partial_3 m^{\boldsymbol{\gamma}} \left(\mathcal{V}_n^{\boldsymbol{\gamma}+e_3}\right) \subset \mathcal{V}_n^{\boldsymbol{\gamma}+e_1}$.

(iii) $\partial_2 \partial_3 m^{\boldsymbol{\gamma}} \left(\mathcal{V}_n^{\boldsymbol{\gamma}+e_3}\right) \subset \mathcal{V}_n^{\boldsymbol{\gamma}+e_2}$.

Proof. Let $p \in \mathcal{V}_n^{\boldsymbol{\gamma}+e_3}$. According to Proposition 3.4.15, we have

$$\partial_1 \partial_2 m^{\boldsymbol{\gamma}}(p) = m^{\boldsymbol{\gamma}+e_1+e_2+2e_3} \partial_1 \partial_2 p + \sum_{i=1}^{3} c_i \Lambda_i^{\boldsymbol{\gamma}+e_3}(p) + c_4 p.$$

Using parts (iii) and (iv) of Proposition 2.2.2, we deduce the following mapping sequence:

$$\mathcal{V}_n^{\boldsymbol{\gamma}+e_3} \xrightarrow{\partial_2} \mathcal{V}_{n-1}^{\boldsymbol{\gamma}+e_2+2e_3} \xrightarrow{\partial_1} \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_1+e_2+3e_3} \xrightarrow{d_2^{\boldsymbol{\gamma}+e_1+2e_3}} \mathcal{V}_{n-1}^{\boldsymbol{\gamma}+e_1+2e_3} \xrightarrow{d_1^{\boldsymbol{\gamma}+e_3}} \mathcal{V}_n^{\boldsymbol{\gamma}+e_3}.$$

Consequently, we have that $m^{\boldsymbol{\gamma}+e_1+e_2+2e_3} \partial_1 \partial_2(p) \in \mathcal{V}_n^{\boldsymbol{\gamma}+e_3}$, while it follows from Proposition 3.4.7 that $\Lambda_i^{\boldsymbol{\gamma}+e_3}(p) \in \mathcal{V}_n^{\boldsymbol{\gamma}+e_3}$. Thus, we conclude that $\partial_1 \partial_2 m^{\boldsymbol{\gamma}}(p) \in \mathcal{V}_n^{\boldsymbol{\gamma}+e_3}$, which completes the proof of (i).

Similarly, applying Proposition 3.4.16 yields

$$\partial_1 \partial_3 m^{\boldsymbol{\gamma}}(p) = m^{\boldsymbol{\gamma}+2e_1+e_2+e_3} \partial_1 \partial_3(p) + \sum_{i=1}^{3} c_i \Theta_{i,1}^{\boldsymbol{\gamma}+e_1}(p) + c_4 \Psi_2^{\boldsymbol{\gamma}+e_1}(p) + c_5 \mathcal{D}_1^{\boldsymbol{\gamma}}(p).$$

Invoking part (iii) of Proposition 2.2.2 and part (iv) of Proposition 2.2.3, we observe the sequence:

$$\mathcal{V}_n^{\boldsymbol{\gamma}+e_3} \xrightarrow{\partial_3} \mathcal{V}_{n-1}^{\boldsymbol{\gamma}+e_1+e_2+e_3} \xrightarrow{\partial_1} \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+2e_1+e_2+2e_3} \xrightarrow{d_2^{\boldsymbol{\gamma}+2e_1+e_3}} \mathcal{V}_{n-1}^{\boldsymbol{\gamma}+2e_1+e_3} \xrightarrow{d_1^{\boldsymbol{\gamma}+e_1}} \mathcal{V}_n^{\boldsymbol{\gamma}+e_1}.$$

Therefore, $m^{\boldsymbol{\gamma}+2e_1+e_2+e_3} \partial_1 \partial_3(p) \in \mathcal{V}_n^{\boldsymbol{\gamma}+e_1}$. Moreover, it follows from Proposition 3.4.7 and Proposition 3.4.9 that $\Psi_2^{\boldsymbol{\gamma}+e_1}(p) \in \mathcal{V}_n^{\boldsymbol{\gamma}+e_1}$ and $\Theta_{i,1}^{\boldsymbol{\gamma}+e_1}(p) \in \mathcal{V}_n^{\boldsymbol{\gamma}+e_1}$, and from Proposition 3.4.5 that

$\mathcal{D}_1^\gamma(p) \in \mathcal{V}_n^{\gamma+e_1}$. Hence, $\partial_1\partial_3 m^\gamma(p) \in \mathcal{V}_n^{\gamma+e_1}$, concluding the proof of (ii).

Finally, let $\tau \in \mathcal{S}_3$ be the permutation defined by $\tau(1) = 2$, $\tau(2) = 1$ and $\tau(3) = 3$. From Proposition 2.2.7 we have that $T_\tau^\star \partial_1 = \partial_2 T_\tau^\star$, $T_\tau^\star \partial_2 = \partial_1 T_\tau^\star$ and $T_\tau^\star \partial_3 = -\partial_3 T_\tau^\star$. Combining Proposition 2.2.6 with the identities above and the fact that $\tau^{-1} = \tau$, we obtain the following symmetry relation:

$$\partial_2\partial_3 m^\gamma = -T_\tau^\star \partial_1\partial_3 T_\tau^\star m^\gamma \overset{(3.4.1)}{=} -T_\tau^\star \partial_1\partial_3 T_\tau^\star \mathrm{W}_{\gamma-e_1-e_2-e_3}^{-1} \partial_1\partial_2 \left(\mathrm{W}_{\gamma+e_3}\right) = -T_\tau^\star \partial_1\partial_3 m^{\tau(\gamma)} T_\tau^\star. \tag{3.4.12}$$

Moreover, the previous case implies that

$$\partial_1\partial_3 m^{\tau(\gamma)} \left(\mathcal{V}_n^{\tau(\gamma)+e_3}\right) \subset \mathcal{V}_n^{\tau(\gamma)+e_1}. \tag{3.4.13}$$

Using the identity $\mathcal{V}_n^{\tau(\gamma)+e_3} = T_\tau^\star \mathcal{V}_n^{\gamma+e_3}$ (cf. (iii) of Proposition 2.2.6), we apply $T_\tau^\star$ to (3.4.13). Recalling (3.4.12), we find:

$$\partial_2\partial_3 m^\gamma \left(\mathcal{V}_n^{\gamma+e_3}\right) = -T_\tau^\star \partial_1\partial_3 m^{\tau(\gamma)} T_\tau^\star \left(\mathcal{V}_n^{\gamma+e_3}\right) \subset T_\tau^\star \mathcal{V}_n^{\tau(\gamma)+e_1}.$$

Since $T_\tau^\star \mathcal{V}_n^{\tau(\gamma)+e_1} = \mathcal{V}_n^{\gamma+e_2}$ from (iii) of Proposition 2.2.6, the proof of (iii) is complete. □

Proposition 3.4.18. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}$. *Then, the following inclusion holds:*

$$m^\gamma \left(\mathcal{V}_{n+1}^{\gamma+e_3}\right) \subset \mathcal{V}_n^\gamma \oplus \mathcal{V}_{n+1}^\gamma \oplus \mathcal{V}_{n+2}^\gamma \oplus \mathcal{V}_{n+3}^\gamma.$$

Proof. Let $p_n \in \mathcal{V}_{n+1}^{\gamma+e_3}$ and $q \in \Pi_{n-1}^2$. Using (3.4.1) and integrating by parts twice, we obtain

$$\begin{aligned}\langle m^\gamma(p), q\rangle_\gamma &= \langle \partial_1\partial_2 \left(\mathrm{W}_{\gamma+e_3} p\right), q\mathrm{W}_{e_1+e_2+e_3}\rangle_0 \\ &= \langle p, \partial_2\partial_1 \left(q\mathrm{W}_{e_1+e_2+e_3}\right)\rangle_{\gamma+e_3} - \oint_{\partial\mathrm{T}} \partial_1 \left(q\mathrm{W}_{e_1+e_2+e_3}\right) \mathrm{W}_{\gamma+e_3} p \nu_2 \, \mathrm{d}\ell \\ &\qquad + \oint_{\partial\mathrm{T}} \partial_2 \left(\mathrm{W}_{\gamma+e_3} p\right) \mathrm{W}_{e_1+e_2+e_3} q \nu_1 \, \mathrm{d}\ell.\end{aligned} \tag{3.4.14}$$

The first term vanishes because $p \in \mathcal{V}_{n+1}^{\gamma+e_3}$ and $\partial_2\partial_1(\mathrm{W}_{e_1+e_2+e_3} q) \in \Pi_n^2$. Since $\partial_1 \mathrm{W}_{e_2} = 0$, the second term can be written as follows:

$$\oint_{\partial\mathrm{T}} \partial_1 \left(q\mathrm{W}_{e_1+e_2+e_3}\right) \mathrm{W}_{\gamma+e_3} p\nu_2 \, \mathrm{d}\ell = \oint_{\partial\mathrm{T}} \partial_1 \left(q\mathrm{W}_{e_1+e_3}\right) \mathrm{W}_{\gamma+e_2+e_3} p\nu_2 \, \mathrm{d}\ell.$$

In this expression, the Integral is zero because $\mathrm{W}_{\gamma+e_2+e_3} = 0$ on $\{x_2 = 0\} \cup \{1 - x_1 - x_2 = 0\}$, and $\nu_2 = 0$ on $\{x_1 = 0\}$.

The third term can be expanded as follows:

$$\oint_{\partial\mathrm{T}} \partial_2 \left(\mathrm{W}_{\gamma+e_3} p\right) \mathrm{W}_{e_1+e_2+e_3} q\nu_1 \, \mathrm{d}\ell = \gamma_2 \oint_{\partial\mathrm{T}} pq\mathrm{W}_{\gamma+e_1+2e_3}\nu_1 \, \mathrm{d}\ell - (\gamma_3 + 1) \oint_{\partial\mathrm{T}} pq\mathrm{W}_{\gamma+e_1+e_2+e_3}\nu_1 \, \mathrm{d}\ell \\ + \oint_{\partial\mathrm{T}} \partial_2 pq\mathrm{W}_{\gamma+e_1+e_2+2e_3}\nu_1 \, \mathrm{d}\ell. \quad (3.4.15)$$

In this expression, the first integral is zero because $\mathrm{W}_{\gamma+e_1+e_3} = 0$ on $\{x_1 = 0\} \cup \{1 - x_1 - x_2 = 0\}$ and $\nu_1 = 0$ on $\{x_2 = 0\}$. Finally, the second and third integrals vanish because $\mathrm{W}_{\gamma+e_1+e_2+e_3} = 0$ on $\partial\mathrm{T}$. □

Proposition 3.4.19. *Let $\boldsymbol{\gamma} \in (-1, \infty)^3$, $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$. Then, the following inclusion holds:*

$$\partial_i m^\gamma \left(\mathcal{V}_{n+1}^{\gamma+e_3}\right) \subset \mathcal{V}_{n+1}^\gamma \oplus \mathcal{V}_{n+2}^\gamma.$$

Proof. In the case $i = 1$, let $p \in \mathcal{V}_{n+1}^{\gamma+e_3}$ and $q \in \Pi_n^2$. Invoking Proposition 3.4.14 yields

$$\langle \partial_1 m^\gamma p, q \rangle_\gamma = \langle m^{\gamma+e_1+e_2+e_3} \partial_1 p, q \rangle_\gamma + \langle d_1^\gamma \left[\mathrm{W}_{e_3}\partial_1 + \mathrm{W}_{e_2}\partial_3 + \gamma_2 \mathrm{I}\right] p, q \rangle_\gamma + (\gamma_1 + \gamma_3) \langle d_2^{\gamma-e_2} p, q \rangle_\gamma.$$

Using part (ii) of Proposition 2.2.2 and the identity $m^{\gamma+e_1+e_2+e_3} = d_1^\gamma d_2^{\gamma+e_1+e_3}$, we obtain

$$\langle \partial_1 m^\gamma p, q \rangle_\gamma = \langle \partial_1 p, \mathrm{W}_{e_2}\partial_2\partial_1 q \rangle_{\gamma+e_1+2e_3} + \langle \partial_1 p, \partial_1 q \rangle_{\gamma+e_1+2e_3} + \langle \partial_3 p, \partial_1 q \rangle_{\gamma+e_1+e_2+e_3} \\ + \gamma_2 \langle p, \mathrm{W}_{e_1}\partial_1 q \rangle_{\gamma+e_3} + (\gamma_1 + \gamma_3) \langle d_2^{\gamma-e_2} p, q \rangle_\gamma. \quad (3.4.16)$$

From part (iv) of Proposition 2.2.2 and part (iv) of Proposition 2.2.3, we deduce that $\partial_1 p \in \mathcal{V}_n^{\gamma+e_1+2e_3}$ and $\partial_3 p \in \mathcal{V}_n^{\gamma+e_1+e_2+e_3}$. Since $\mathrm{W}_{e_1}\partial_1 q \in \Pi_n^2$ and $\mathrm{W}_{e_2}\partial_2\partial_1 q, \partial_1 q \in \Pi_{n-1}^2$, equation

(3.4.16) simplifies to:

$$\langle \partial_1 m^\gamma p, q\rangle_\gamma = (\gamma_1 + \gamma_3)\langle d_2^{\gamma-e_2} p, q\rangle_\gamma .$$

Next, using (2.2.5) and applying integration by parts, we find that

$$\langle d_2^{\gamma-e_2} p, q\rangle_\gamma = \langle p, \partial_2 \left(q\mathrm{W}_{e_2}\right)\rangle_{\gamma+e_3} - \oint_{\partial \mathrm{T}} pq\mathrm{W}_{\gamma+e_2+e_3}\nu_2 \mathrm{d}\ell. \tag{3.4.17}$$

The first term of the right-hand side of (3.4.17) vanishes because $p \in \mathcal{V}_{n+1}^{\gamma+e_3}$ and $\partial_2\left(\mathrm{W}_{e_2} q\right) \in \Pi_n^2$. The second term also vanishes, because $\mathrm{W}_{\gamma+e_2+e_3} = 0$ on $\{x_2 = 0 \vee 1 - x_1 - x_2 = 0\}$ and $\nu_2 = 0$ on $\{x_1 = 0\}$. Thus, we conclude that $\langle \partial_1 m^\gamma p, q\rangle_\gamma = 0$ and so $\partial_1 m^\gamma p \in \mathcal{V}_{n+1}^\gamma \oplus \mathcal{V}_{n+2}^\gamma$.

For the case $i = 2$, let $\tau \in \mathcal{S}_2$ be the permutation defined by $\tau(1) = 2$, $\tau(2) = 1$ and $\tau(3) = 3$. From Proposition 2.2.7 we have that $T_\tau^\star \partial_1 = \partial_2 T_\tau^\star$, $T_\tau^\star \partial_2 = \partial_1 T_\tau^\star$ and $T_\tau^\star \partial_3 = -\partial_3 T_\tau^\star$. Combining Proposition 2.2.6, with the identities above and the fact that $\tau^{-1} = \tau$, we obtain the following symmetry relation:

$$\partial_2 m^\gamma = T_\tau^\star \partial_1 T_\tau^\star m^\gamma \overset{(3.4.1)}{=} T_\tau^\star \partial_1 T_\tau^\star \mathrm{W}_{\gamma-e_1-e_2-e_3}^{-1} \partial_1 \partial_2 \left(\mathrm{W}_{\gamma+e_3}\right) = T_\tau^\star \partial_1 m^{\tau(\gamma)} T_\tau^\star . \tag{3.4.18}$$

Moreover, the previous case implies that

$$\partial_1 m^{\tau(\gamma)} \left(\mathcal{V}_{n+1}^{\tau(\gamma)+e_3}\right) \subset \mathcal{V}_{n+1}^{\tau(\gamma)} \oplus \mathcal{V}_{n+2}^{\tau(\gamma)} . \tag{3.4.19}$$

Thus, using the identity $\mathcal{V}_{n+1}^{\tau(\gamma)+e_3} = T_\tau^\star \mathcal{V}_{n+1}^{\gamma+e_3}$ (cf.(iii) of Proposition 2.2.6), applying $T_\tau^\star$ to (3.4.19) and recalling (3.4.18), we find that

$$\partial_2 m^\gamma \left(\mathcal{V}_{n+1}^{\gamma+e_3}\right) = T_\tau^\star \partial_1 m^{\tau(\gamma)} T_\tau^\star \left(\mathcal{V}_{n+1}^{\gamma+e_3}\right) \subset T_\tau^\star \mathcal{V}_{n+1}^{\tau(\gamma)} \oplus T_\tau^\star \mathcal{V}_{n+2}^{\tau(\gamma)} .$$

Furthermore, from (iii) of Proposition 2.2.6 we deduce the desired inclusion for the case $i = 2$.

The case $i = 3$ follows directly from the identity $\partial_3 = \partial_2 - \partial_1$ and the results established above. □

Lemma 3.4.20. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$. *Then, the following inclusion holds:*

$$\mathrm{M}^{\gamma}(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}) \subset \mathcal{V}_n^{\gamma;\mathrm{Sob}}.$$

Proof. Let $p \in \mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}$. From (3.1.4), part (iii) of Proposition 2.2.3, and Proposition 3.4.17, it follows that $\mathrm{M}^{\gamma}(p) \in V_n^{\gamma}(\mathfrak{B}^{\gamma})$. Furthermore, as n is high enough, invoking Proposition 3.4.18 and Proposition 3.4.19, we find that $\mathrm{M}^{\gamma}(p) \in \ker(\mathrm{R}^{\gamma})$.

Finally, combining these facts with Theorem 3.3.5, we conclude that $\mathrm{M}^{\gamma}(p) \in \mathcal{V}_n^{\gamma;\mathrm{Sob}}$, thereby establishing the inclusion. □

We finish this section with a digression on tools which facilitated the discovery and verification of inclusions such as Proposition 3.4.7, Proposition 3.4.9, Proposition 3.4.11, and Proposition 3.4.17. With the help of a computer algebra system (*Mathematica*).

Proposition 3.4.21. *Let* $d, n \in \mathbb{N}$, $\boldsymbol{\gamma} \in (-1,\infty)^{d+1}$ *and* $\mathcal{O} : \Pi^d \longrightarrow \Pi^d$ *be a linear operator which does not raise the degree of its arguments. Then* $\mathcal{O}$ *commutes with* $\mathcal{L}^{\gamma}$ *in* $\mathcal{V}_n^{\gamma}$ *if and only if* $(\forall n \in \mathbb{N}_0)$ $\mathcal{O}(\mathcal{V}_n^{\gamma}) \subseteq \mathcal{V}_n^{\gamma}$.

Proof. Let $p \in \mathcal{V}_n^{\gamma}$. Suppose that $\mathcal{O}$ commutes with $\mathcal{L}^{\gamma}$. Then, using (2.2.8), we have

$$(\mathcal{L}^{\gamma} \circ \mathcal{O})(p) = (\mathcal{O} \circ \mathcal{L}^{\gamma})(p) = \mathcal{O}(\mathcal{L}^{\gamma}(p)) = \lambda_n^{\gamma}\mathcal{O}(p).$$

Therefore, $\mathcal{O}(\mathcal{V}_n^{\gamma}) \subset \mathcal{V}_n^{\gamma}$.

On the other hand, suppose that $(\forall n \in \mathbb{N}_0)$ $\mathcal{O}(\mathcal{V}_n^{\gamma}) \subset \mathcal{V}_n^{\gamma}$. Then, using (2.2.8), we have

$$(\forall p \in \mathcal{V}_n^{\gamma}) \quad (\mathcal{L}^{\gamma} \circ \mathcal{O})(p) = \lambda_n^{\gamma}\mathcal{O}(p) = \mathcal{O}(\lambda_n^{\gamma}p) = \mathcal{O}(\mathcal{L}^{\gamma}(p)) = (\mathcal{O} \circ \mathcal{L}^{\gamma})(p).$$

Therefore, $\mathcal{O}$ commutes with $\mathcal{L}^{\gamma}$ in $\mathcal{V}_n^{\gamma}$. □

Proposition 3.4.22. *Let* $d, n \in \mathbb{N}$, $\boldsymbol{\gamma}, \hat{\boldsymbol{\gamma}} \in (-1,\infty)^{d+1}$ *with* $|\boldsymbol{\gamma}| = |\hat{\boldsymbol{\gamma}}|$ *and* $\mathcal{O} : \Pi^d \longrightarrow \Pi^d$ *be a linear operator which does not raise the degree of its arguments. Then* $\mathcal{O}\mathcal{L}^{\gamma} = \mathcal{L}^{\hat{\gamma}}\mathcal{O}$ *in* $\mathcal{V}_n^{\gamma}$ *if and only if* $\mathcal{O}(\mathcal{V}_n^{\gamma}) \subset \mathcal{V}_n^{\hat{\gamma}}$.

Proof. It follows from repeating the arguments used in the proof of Proposition 3.4.21 and using the fact that $\lambda_n^{\gamma} = \lambda_n^{\hat{\gamma}}$. $\square$

3.5 Complementary Sobolev orthogonal polynomials

In view of Lemma 3.4.2 and Lemma 3.4.20, for n high enough and $\boldsymbol{\gamma} \neq (0,0,0)$ we have a characterization of a subspace of $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$ of dimension $n-2$. Since $\dim\left(\mathcal{V}_n^{\gamma;\mathrm{Sob}}\right) = n+1$, we need to find three linearly independent polynomials to complete the space. These polynomials shall for a basis of the space $\mathcal{J}_n^{\gamma}$ described in Theorem 3.1.1. The aim of this section is to explicitly construct such polynomials; to this end, we define three families of polynomials that will enable us to prove our decomposition for different $\boldsymbol{\gamma}$ profiles.

We begin by defining a first family of polynomials.

Definition 3.5.1. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$ and $n \geq 3$. We define the polynomials*

$$h_{n,1}^{\gamma} := m^{\gamma}\left(J_{n-2,0}^{\gamma+e_3}\right), \quad h_{n,2}^{\gamma} := T_{\sigma_2}^{\star} h_{n,1}^{\sigma_1(\gamma)} \quad \text{and} \quad h_{n,3}^{\gamma} := T_{\sigma_1}^{\star} h_{n,1}^{\sigma_2(\gamma)}.$$

We denote by $\mathcal{H}_n^{\gamma}$ the space spanned by the polynomials $h_{n,1}^{\gamma}$, $h_{n,2}^{\gamma}$ and $h_{n,3}^{\gamma}$.

In what follows, we present some results concerning the subspace $\mathcal{H}_n^{\gamma}$, some of which require additional comments on the parameter $\boldsymbol{\gamma}$.

Proposition 3.5.2. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$ be such that $\gamma_i \neq 0$ for all $i \in \{1,2,3\}$ and $n \geq 3$. Then,* $\dim(\mathcal{H}_n^{\gamma}) = 3$.

Proof. From Definition 3.5.1, (3.4.6), (2.4.1) and (2.3.2), it follows that

$$h_{n,1}^{\gamma}(1,0) = 0, \quad h_{n,1}^{\gamma}(0,1) = 0 \quad \text{and} \quad h_{n,1}^{\gamma}(0,0) = \frac{\gamma_1\gamma_2\Gamma(\gamma_1+\gamma_2+n)}{\Gamma(n-1)\Gamma(\gamma_1+\gamma_2+2)}.$$

From the above, the fact that $h_{n,2}^{\gamma} = T_{\sigma_2}^{\star} h_{n,1}^{\sigma_1(\gamma)}$ and Remark 2.2.11,

$$h_{n,2}^{\gamma}(1,0) = \frac{\gamma_2\gamma_3\Gamma(\gamma_2+\gamma_3+n)}{\Gamma(n-1)\Gamma(\gamma_2+\gamma_3+2)}, \quad h_{n,2}^{\gamma}(0,1) = 0 \quad \text{and} \quad h_{n,2}^{\gamma}(0,0) = 0.$$

Similarly, we obtain

$$h^{\gamma}_{n,3}(1,0) = 0, \quad \text{and} \quad h^{\gamma}_{n,3}(0,1) = \frac{\gamma_1\gamma_3\Gamma(\gamma_1+\gamma_3+n)}{\Gamma(n-1)\Gamma(\gamma_1+\gamma_3+2)} \quad \text{and} \quad h^{\gamma}_{n,3}(0,0) = 0.$$

Now, let $n \geq 3$ and let $\{\alpha_i\}_{i=1}^{3} \subset \mathbb{R}$ be such that

$$(\forall x \in \overline{\mathrm{T}}) \quad \sum_{i=1}^{3} \alpha_i h^{\gamma}_{n,i}(x_1,x_2) = 0. \tag{3.5.1}$$

Evaluating (3.5.1) at the vertices of $\overline{\mathrm{T}}$, we obtain the system

$$\begin{bmatrix} h^{\gamma}_{n,1}(0,0) & 0 & 0 \\ 0 & h^{\gamma}_{n,2}(1,0) & 0 \\ 0 & 0 & h^{\gamma}_{n,3}(0,1) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.5.2}$$

Since γ_1, γ_2, γ_3 and all the involved evaluations of Γ are non-zero, it follows that the diagonal matrix in (3.5.2) is invertible; hence the system admits only the trivial solution. Thus, $\dim(\mathcal{H}^{\gamma}_n) = 3$. □

Proposition 3.5.3. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 3}$*. Then,*

$$\mathcal{V}^{\gamma+e_3}_{n-2} = \operatorname{span}\left(\left\{J^{\gamma+e_3}_{n-2,0}\right\}\right) \oplus_{\perp_{\gamma+e_3}} d_3^{\gamma+e_3}\left(\mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3}\right).$$

Proof. First, we observe that from Section 2.4 and (iii) in Proposition 2.2.3, we have that

$$\operatorname{span}\left(\left\{J^{\gamma+e_3}_{n-2,0}\right\}\right) + d_3^{\gamma+e_3}\left(\mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3}\right) \subseteq \mathcal{V}^{\gamma+e_3}_{n-2}.$$

Now, let $p \in \mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3}$. From (ii) in Proposition 2.2.2, (iii) in Proposition 2.2.3 and since $\partial_3 J^{\gamma+e_3}_{n-2,0} = 0$, we get

$$\langle J^{\gamma+e_3}_{n-2,0}, d_3^{\gamma+e_3} p\rangle_{\gamma+e_3} = \langle \partial_3 J^{\gamma+e_3}_{n-2,0}, p\rangle_{\gamma+e_1+e_2+e_3} = 0.$$

Thus, $\operatorname{span}\left(\left\{J^{\gamma+e_3}_{n-2,0}\right\}\right) \perp_{\gamma+e_3} d_3^{\gamma+e_3}\left(\mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3}\right)$. Furthermore, using Remark 3.4.3 and (2.2.2), we have

$$\begin{aligned}\dim\left(\operatorname{span}\left(\left\{J_{n-2,0}^{\gamma+e_3}\right\}\right)\oplus d_3^{\gamma+e_3}\left(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}\right)\right) &= \dim\left(\operatorname{span}\left(\left\{J_{n-2,0}^{\gamma+e_3}\right\}\right)\right)+\dim\left(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}\right)\\ &= 1+(n-3)+1=\dim\left(\mathcal{V}_{n-2}^{\gamma+e_3}\right).\end{aligned}$$

Therefore, the desired result follows. □

Proposition 3.5.4. *Let $\boldsymbol{\gamma}\in(-1,\infty)^3$, be such that $\gamma_i\neq 0$ for all $i\in\{1,2,3\}$, and $n\in\mathbb{N}_{\geq 3}$. Then,*

$$\dim\left(\mathcal{H}_n^{\gamma}\oplus \mathrm{M}^{\gamma}(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3})\right)=\dim\left(\mathcal{H}_n^{\gamma}\right)+\dim\left(\mathrm{M}^{\gamma}(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3})\right)$$

Proof. Let $p\in\mathcal{H}_n^{\gamma}\cap\mathrm{M}^{\gamma}(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3})$. Then, there exist $q\in\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}$ and constants $\kappa_1,\kappa_2,\kappa_3\in\mathbb{R}$ such that

$$p=\sum_{i=1}^{3}\kappa_i h_{n,i}^{\gamma}\quad\text{and}\quad p=\mathrm{M}^{\gamma}(q). \tag{3.5.3}$$

Evaluating the second equality in (3.5.3) at the vertices $(1,0)$ and $(0,1)$ with the help of (3.4.6), we deduce that $p(1,0)=p(0,1)=0$. Since $h_{n,1}^{\gamma}(1,0)=0$, $h_{n,2}^{\gamma}(1,0)\neq 0$ and $h_{n,3}^{\gamma}(1,0)=0$, it follows that $\kappa_2=0$. Likewise, since $h_{n,1}^{\gamma}(0,1)=0$, $h_{n,2}^{\gamma}(0,1)=0$ and $h_{n,3}^{\gamma}(0,1)\neq 0$, we obtain $\kappa_3=0$. Consequently,

$$\mathrm{M}^{\gamma}(q)=\kappa_1 h_{n,1}^{\gamma}.$$

This is equivalent to writing

$$m^{\gamma}d_3^{\gamma+e_3}(q)=\kappa_1 m^{\gamma}\left(J_{n-2,0}^{\gamma+e_3}\right). \tag{3.5.4}$$

Since $\gamma_i\neq 0$ for all $i\in\{1,2,3\}$, from Proposition 3.4.1 it follows that

$$d_3^{\gamma+e_3}(q)=\kappa_1 J_{n-2,0}^{\gamma+e_3}.$$

However, by Proposition 3.5.3, we know that $J_{n-2,0}^{\gamma+e_3}\notin d_3^{\gamma+e_3}\left(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}\right)$. This implies, $\kappa_1=0$.

Thus, $p = 0$ and hence

$$\dim\left(\mathcal{H}^\gamma_n \cap \mathrm{M}^\gamma(\mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3})\right) = 0.$$

Therefore, the desired result follows. $\square$

Proposition 3.5.5. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$ and let $n \in \mathbb{N}_{\geq 3}$. Then,*

(i) $\partial_1\partial_2 h^\gamma_{n,1} \in \mathcal{V}^{\gamma+e_3}_{n-2}$, $\partial_1\partial_3 h^\gamma_{n,1} \in \mathcal{V}^{\gamma+e_1}_{n-2}$ *and* $\partial_2\partial_3 h^\gamma_{n,1} \in \mathcal{V}^{\gamma+e_2}_{n-2}$.

(ii) $\partial_1\partial_2 h^\gamma_{n,2} \in \mathcal{V}^{\gamma+e_3}_{n-2}$, $\partial_1\partial_3 h^\gamma_{n,2} \in \mathcal{V}^{\gamma+e_1}_{n-2}$ *and* $\partial_2\partial_3 h^\gamma_{n,2} \in \mathcal{V}^{\gamma+e_2}_{n-2}$.

(iii) $\partial_1\partial_2 h^\gamma_{n,3} \in \mathcal{V}^{\gamma+e_3}_{n-2}$, $\partial_1\partial_3 h^\gamma_{n,3} \in \mathcal{V}^{\gamma+e_1}_{n-2}$ *and* $\partial_2\partial_3 h^\gamma_{n,3} \in \mathcal{V}^{\gamma+e_2}_{n-2}$.

Proof. Part (i) is a direct consequence of Proposition 3.4.17. For part (ii), we have

$$\partial_1\partial_2 h^\gamma_{n,2} \overset{\text{Def.3.5.1}}{=} \partial_1\partial_2 T^\star_{\sigma_2} h^{\sigma_1(\gamma)}_{n,1} \overset{\text{Prop.2.2.9(ii)}}{=} T^\star_{\sigma_2}\partial_2\partial_3 h^{\sigma_1(\gamma)}_{n,1} \overset{\text{(i)}}{\in} T^\star_{\sigma_2}(\mathcal{V}^{\sigma_1(\gamma)+e_2}_{n-2}) \overset{\text{Prop.2.2.6(iii)}}{=} \mathcal{V}^{\gamma+e_3}_{n-2}.$$

The rest of part (ii) and all of part (iii), is analogous. $\square$

Proposition 3.5.6. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$ and $n \in \mathbb{N}_{\geq 4}$. Then, $\mathcal{H}^\gamma_n \subset \mathcal{V}^\gamma_{n-3} \oplus \mathcal{V}^\gamma_{n-2} \oplus \mathcal{V}^\gamma_{n-1} \oplus \mathcal{V}^\gamma_n$.*

Proof. Thanks to Proposition 3.4.18, we obtain

$$h^\gamma_{n,1} = m^\gamma(J^{\gamma+e_3}_{n-2}) \in \mathcal{V}^\gamma_{n-3} \oplus \mathcal{V}^\gamma_{n-2} \oplus \mathcal{V}^\gamma_{n-1} \oplus \mathcal{V}^\gamma_n. \tag{3.5.5}$$

Now,

$$h^\gamma_{n,2} = T^\star_{\sigma_2} h^{\sigma_1(\gamma)}_{n,1} \overset{(3.5.5)}{\in} T^\star_{\sigma_2}\left(\mathcal{V}^{\sigma_1(\gamma)}_{n-3} \oplus \mathcal{V}^{\sigma_1(\gamma)}_{n-2} \oplus \mathcal{V}^{\sigma_1(\gamma)}_{n-1} \oplus \mathcal{V}^{\sigma_1(\gamma)}_{n}\right) \overset{\text{Prop.2.2.6(iii)}}{=} \mathcal{V}^\gamma_{n-3} \oplus \mathcal{V}^\gamma_{n-2} \oplus \mathcal{V}^\gamma_{n-1} \oplus \mathcal{V}^\gamma_n.$$

A completely analogous argument shows that $h^\gamma_{n,3} \in \mathcal{V}^\gamma_{n-3} \oplus \mathcal{V}^\gamma_{n-2} \oplus \mathcal{V}^\gamma_{n-1} \oplus \mathcal{V}^\gamma_n$. $\square$

Proposition 3.5.7. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$, $i \in \{1,2,3\}$ and $n \in \mathbb{N}_{\geq 4}$. Then, $\partial_i \mathcal{H}^\gamma_n \subset \mathcal{V}^\gamma_{n-2} \oplus \mathcal{V}^\gamma_{n-1}$.*

Proof. First, we observe that proving the result is equivalent to verifying the corresponding property for each $h^\gamma_{n,i}$, for $i \in \{1,2,3\}$. We begin with the case of $h^\gamma_{n,1}$ which is a direct consequence of Proposition 3.4.19. Now,

$$\partial_1 h_{n,2}^{\gamma} = \partial_1 T_{\sigma_2}^{\star} h_{n,1}^{\sigma_1(\gamma)} \overset{\text{Prop.2.2.9(ii)}}{=} -T_{\sigma_2}^{\star}\partial_2 h_{n,1}^{\sigma_1(\gamma)} \overset{\text{Above}}{\in} T_{\sigma_2}^{\star}\left(\mathcal{V}_{n-2}^{\sigma_1(\gamma)} \oplus \mathcal{V}_{n-1}^{\sigma_1(\gamma)}\right) \overset{\text{Prop.2.2.6(iii)}}{=} \mathcal{V}_{n-2}^{\gamma} \oplus \mathcal{V}_{n-1}^{\gamma}.$$

The rest of the cases follow analogously. □

Lemma 3.5.8. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$*. Then,* $\mathcal{H}_n^{\gamma} \subset \mathcal{V}_n^{\gamma;\mathrm{Sob}}$*.*

Proof. Let $h \in \mathcal{H}_n^{\gamma}$ and $q \in \Pi_{n-1}^2$. From Proposition 3.5.5, we deduce that $\mathfrak{B}^{\gamma}(h,q) = 0$. Furthermore, Proposition 3.5.6 implies $S_0^{\gamma}(h) = 0$ and Proposition 3.5.7 implies $S_0^{\gamma}(\bar{\nabla} h) = 0$. Therefore, by the characterization provided in Lemma 3.3.1, $h \in \mathcal{V}_n^{\gamma;\mathrm{Sob}}$. □

A second family of auxiliary polynomials, inspired by Section 3.2, is defined in a fashion involving a Jacobi polynomial, an instance of the $\mathrm{M}^{\alpha,\beta}$ operator of Definition 3.2.4 and the operator $\eta^{\star}$ of Definition 3.2.11 as follows.

Definition 3.5.9. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$*. We define the polynomials*

$$\varrho_{n,1}^{\gamma} := \eta^{\star}\mathrm{M}^{\gamma_1+\gamma_2,\gamma_3}\mathrm{P}_{n-2}^{(\gamma_1+\gamma_2+1,\gamma_3+1)}, \quad \varrho_{n,2}^{\gamma} := T_{\sigma_1}^{\star}\varrho_{n,1}^{\sigma_2(\gamma)} \quad \text{and} \quad \varrho_{n,3}^{\gamma} := T_{\sigma_2}^{\star}\varrho_{n,1}^{\sigma_1(\gamma)}.$$

Proposition 3.5.10. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$*. Then,*

$$\partial_1\partial_2\varrho_{n,1}^{\gamma} \in \mathcal{V}_{n-2}^{\gamma+e_3}, \quad \partial_1\partial_3\varrho_{n,1}^{\gamma} \in \mathcal{V}_{n-2}^{\gamma+e_1} \quad \text{and} \quad \partial_2\partial_3\varrho_{n,1}^{\gamma} \in \mathcal{V}_{n-2}^{\gamma+e_2}.$$

Proof. First we note that by (iv) of Proposition 3.2.12 $\partial_3\varrho_{n,1}^{\gamma} = 0$, then $\partial_1\partial_3\varrho_{n,1}^{\gamma} \in \mathcal{V}_{n-2}^{\gamma+e_1}$ and $\partial_2\partial_3\varrho_{n,1}^{\gamma} \in \mathcal{V}_{n-2}^{\gamma+e_2}$.

Now, let $k \in [n-3]_0$ and $l \in [k]_0$. By applying (ii) and (iii) of Proposition 3.2.12, performing the change of variables $x = \Psi(\zeta,\eta)$ and noting that

$$J_{k,l}^{\gamma}(\Psi(\zeta,\eta)) = \frac{(1-\eta)^l}{2^l}\mathrm{P}_l^{(\gamma_1,\gamma_2)}(\zeta)\mathrm{P}_{k-l}^{(2l+\gamma_1+\gamma_2+1,\gamma_3)}(\eta),$$

it follows that:

$$\begin{aligned}\langle \partial_1\partial_2 \varrho^{\boldsymbol{\gamma}}_{n,1}, J^{\boldsymbol{\gamma}}_{k,l}\rangle_{\boldsymbol{\gamma}+e_3} &= 4\langle \eta^\star(\mathrm{M}^{\gamma_1+\gamma_2,\gamma_3}\mathrm{P}^{(\gamma_1+\gamma_2+1,\gamma_3+1)}_{n-2})'', J^{\boldsymbol{\gamma}}_{k,l}\rangle_{\boldsymbol{\gamma}+e_3}\\ &= 2^{-2-2\gamma_1-2\gamma_2-\gamma_3-l}\langle \mathrm{P}^{(\gamma_1,\gamma_2)}_l, 1\rangle_{\mathrm{J}(\gamma_1,\gamma_2)}\\ &\quad\times \langle (\mathrm{M}^{\gamma_1+\gamma_2,\gamma_3}\mathrm{P}^{(\gamma_1+\gamma_2+1,\gamma_3+1)}_{n-2})'', \mathrm{J}(l,0)\mathrm{P}^{(2l+\gamma_1+\gamma_2+1,\gamma_3)}_{k-l}\rangle_{\mathrm{J}(\gamma_1+\gamma_2+1,\gamma_3+1)}.\end{aligned}$$

Since $\mathrm{J}(l,0)\mathrm{P}^{(2l+\gamma_1+\gamma_2+1,\gamma_3)}_{k-l} \in \Pi^1_k$, there exists a polynomial h of degree at most $k+2$ such that $h'' = \mathrm{J}(l,0)\mathrm{P}^{(2l+\gamma_1+\gamma_2+1,\gamma_3)}_{k-l}$. Thus, combining the fact that $k+2 \le n-1$ with Lemma 3.2.10 and Proposition 3.2.1, we deduce that

$$\begin{aligned}&\langle (\mathrm{M}^{\gamma_1+\gamma_2,\gamma_3}\mathrm{P}^{(\gamma_1+\gamma_2+1,\gamma_3+1)}_{n-2})'', \mathrm{J}(l,0)\mathrm{P}^{(2l+\gamma_1+\gamma_2+1,\gamma_3)}_{k-l}\rangle_{\mathrm{J}(\gamma_1+\gamma_2+1,\gamma_3+1)}\\ &\qquad\qquad = \langle \mathrm{M}^{\gamma_1+\gamma_2,\gamma_3}\mathrm{P}^{(\gamma_1+\gamma_2+1,\gamma_3+1)}_{n-2}, h\rangle^{\circ}_{\gamma_1+\gamma_2,\gamma_3} = 0.\end{aligned}$$

Therefore, $\partial_1\partial_2\varrho^{\boldsymbol{\gamma}}_{n,1} \in \mathcal{V}^{\boldsymbol{\gamma}+e_3}_{n-2}$. □

Remark 3.5.11. From Proposition 3.5.10, Proposition 2.2.9 and (iii) in Proposition 2.2.6, it follows that the same result holds for $\varrho^{\boldsymbol{\gamma}}_{n,2}$ and $\varrho^{\boldsymbol{\gamma}}_{n,3}$.

Proposition 3.5.12. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\ge 4}$. *Then,* $\operatorname{span}\left(\left\{\varrho^{\boldsymbol{\gamma}}_{n,1}, \varrho^{\boldsymbol{\gamma}}_{n,2}, \varrho^{\boldsymbol{\gamma}}_{n,3}\right\}\right) \subset \mathcal{V}^{\boldsymbol{\gamma};\mathrm{Sob}}_n$.

Proof. From Lemma 3.2.10 and Lemma 3.2.13 we deduce that $\varrho^{\boldsymbol{\gamma}}_{n,1} \in \mathcal{V}^{\boldsymbol{\gamma};\mathrm{Sob}}_n$. Furthermore, $\varrho^{\boldsymbol{\gamma}}_{n,2} \in \mathcal{V}^{\boldsymbol{\gamma};\mathrm{Sob}}_n$ and $\varrho^{\boldsymbol{\gamma}}_{n,3} \in \mathcal{V}^{\boldsymbol{\gamma};\mathrm{Sob}}_n$ follow from Definition 3.5.9 and Lemma 3.3.7. □

Remark 3.5.13. There is some overlap between Lemma 3.2.13, which leads to Proposition 3.5.10, and Proposition 3.5.12, but this arrangement befits arguments that appear in the sequel.

Finally, we introduce a third family of auxiliary polynomials as follows.

Definition 3.5.14. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\ge 3}$. *We define the polynomials*

$$\phi^{\boldsymbol{\gamma}}_{n,1} := d_3^{\boldsymbol{\gamma}-e_1-e_2} J^{\boldsymbol{\gamma}}_{n-1,n-1}, \quad \phi^{\boldsymbol{\gamma}}_{n,2} := T^\star_{\sigma_1}\phi^{\sigma_2(\boldsymbol{\gamma})}_{n,1} \quad \text{and} \quad \phi^{\boldsymbol{\gamma}}_{n,2} := T^\star_{\sigma_2}\phi^{\sigma_1(\boldsymbol{\gamma})}_{n,1}.$$

Proposition 3.5.15. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\ge 4}$. *Then,* $S^{\boldsymbol{\gamma}}_0(\phi^{\boldsymbol{\gamma}}_{n,1}) = 0$.

Proof. From (2.2.6) and by integration by parts, we deduce that

$$\langle \phi^{\boldsymbol{\gamma}}_{n,1}, 1\rangle_{\boldsymbol{\gamma}} = \int_{\mathrm{T}} J^{\boldsymbol{\gamma}}_{n-1,n-1}\partial_3(\mathrm{W}_{e_1+e_2})\mathrm{W}_{\boldsymbol{\gamma}} - \int_{\partial\mathrm{T}} J^{\boldsymbol{\gamma}}_{n-1,n-1}\mathrm{W}_{\boldsymbol{\gamma}+e_1+e_2}(\nu_2-\nu_1). \tag{3.5.6}$$

The first term on the right-hand side of (3.5.6) is vanishes because $n-1>1$, $J^{\gamma}_{n-1,n-1}\in\mathcal{V}^{\gamma}_{n-1}$ and $\partial_3(\mathrm{W}_{e_1+e_2})\in\Pi^2_1$. Furthermore, the second term is vanishes because $\mathrm{W}_{\gamma+e_1+e_2}=0$ on $\{x_1=0\vee x_2=0\}$ and $\nu_2-\nu_1=0$ on $\{1-x_1-x_2=0\}$. This completes the proof. □

Proposition 3.5.16. *Let* $\boldsymbol{\gamma}\in(-1,\infty)^3$ *and* $n\in\mathbb{N}_{\geq 4}$. *Then,* $S^{\gamma}_0(\bar{\nabla}\phi^{\gamma}_{n,1})=0$.

Proof. From part (iii) of Proposition 3.4.12, we obtain

$$\partial_3\phi^{\gamma}_{n,1}=\Lambda^{\gamma}_3 J^{\gamma}_{n-1,n-1}+(\gamma_1+\gamma_2)J^{\gamma}_{n-1,n-1}.$$

Since $J^{\gamma}_{n-1,n-1}\in\mathcal{V}^{\gamma}_{n-1}$ and $\Lambda^{\gamma}_3 J^{\gamma}_{n-1,n-1}\in\mathcal{V}^{\gamma}_{n-1}$ (cf. part (i) of Proposition 3.4.7), we deduce that $S^{\gamma}_0(\partial_3\phi^{\gamma}_{n,1})=0$. As $\partial_3=\partial_2-\partial_1$, this implies that $S^{\gamma}_0(\partial_1\phi^{\gamma}_{n,1})=S^{\gamma}_0(\partial_2\phi^{\gamma}_{n,1})$. Therefore, it now suffices to prove that $S^{\gamma}_0(\partial_1\phi^{\gamma}_{n,1})=0$. By the rules of calculus it is straightforward to check that

$$\partial_1 d_3^{\gamma-e_1-e_2}=-\gamma_2\mathrm{I}-\gamma_2\mathrm{W}_{e_1}\partial_1-\mathrm{W}_{e_1+e_2}\partial_1\partial_2-\mathrm{W}_{e_2}\partial_2+(\gamma_1+1)\mathrm{W}_{e_2}\partial_1+\mathrm{W}_{e_1+e_2}\partial_1^2.$$

Then, we have

$$\begin{aligned}S^{\gamma}_0(\partial_1\phi^{\gamma}_{n,1})&=-\gamma_2 S^{\gamma}_0(J^{\gamma}_{n-1,n-1})-\gamma_2 S^{\gamma}_0(\mathrm{W}_{e_1}\partial_1 J^{\gamma}_{n-1,n-1})\\&\quad-S^{\gamma}_0(\mathrm{W}_{e_1+e_2}\partial_1\partial_2 J^{\gamma}_{n-1,n-1})-S^{\gamma}_0(\mathrm{W}_{e_2}\partial_2 J^{\gamma}_{n-1,n-1})\\&\quad+(\gamma_1+1)S^{\gamma}_0(\mathrm{W}_{e_2}\partial_1 J^{\gamma}_{n-1,n-1})+S^{\gamma}_0(\mathrm{W}_{e_1+e_2}\partial_1^2 J^{\gamma}_{n-1,n-1}).\end{aligned}\tag{3.5.7}$$

The first term on the right-hand side of (3.5.7) vanishes because $J^{\gamma}_{n-1,n-1}\in\mathcal{V}^{\gamma}_{n-1}$ and $n-1>1$. Furthermore, from (2.4.10), we deduce that

$$\partial_1 J^{\gamma}_{n-1,n-1}=-(n-1+\gamma_2)J^{\gamma+e_1+e_3}_{n-2,n-2}\quad\text{and}\quad\partial_2 J^{\gamma}_{n-1,n-1}=(n-1+\gamma_1)J^{\gamma+e_2+e_3}_{n-2,n-2}.\tag{3.5.8}$$

From (2.4.1), we observe that $J^{\gamma}_{k,k}(x)$ does not depend on γ_3. Thus, from (3.5.8), we obtain

$$\partial_1 J^{\gamma}_{n-1,n-1}\in\mathcal{V}^{\gamma+e_1}_{n-2},\quad\partial_2 J^{\gamma}_{n-1,n-1}\in\mathcal{V}^{\gamma+e_2}_{n-2}\quad\text{and}\quad\partial_1\partial_2 J^{\gamma}_{n-1,n-1}\in\mathcal{V}^{\gamma+e_1+e_2}_{n-3}.\tag{3.5.9}$$

Consequently, (3.5.9) implies that the second, third, and fourth terms on the right-hand side of (3.5.7) vanish. For the fifth term, we note that, with the help of the changue of variables $x = \Psi(\zeta, \eta)$ and the fact that $\mathrm{J}(0,1)(t) = 2 - \mathrm{J}(1,0)(t)$,

$$\begin{aligned}\langle \mathrm{W}_{e_2}\partial_1 J^{\gamma}_{n-1,n-1}, 1\rangle_\gamma &= -(n-1+\gamma_2)\langle J^{\gamma+e_1}_{n-2,n-2}, 1\rangle_{\gamma+e_2}\\ &= -(n-1+\gamma_2)2^{-3-2\gamma_1-2\gamma_2-\gamma_3-n}\,\|1\|^2_{\mathrm{J}(\gamma_1+\gamma_2+n,\gamma_3)}\,\langle \mathrm{P}^{(\gamma_1+1,\gamma_2)}_{n-2}, \mathrm{J}(0,1)\rangle_{\mathrm{J}(\gamma_1,\gamma_2)}\\ &= -(n-1+\gamma_2)2^{-2-2\gamma_1-2\gamma_2-\gamma_3-n}\,\|1\|^2_{\mathrm{J}(\gamma_1+\gamma_2+n,\gamma_3)}\,\langle \mathrm{P}^{(\gamma_1+1,\gamma_2)}_{n-2}, 1\rangle_{\mathrm{J}(\gamma_1,\gamma_2)}.\end{aligned}$$

On the other hand, we can write the sixth term as follows:

$$\langle \mathrm{W}_{e_1+e_2}\partial_1^2 J^{\gamma}_{n-1,n-1}, 1\rangle_\gamma = (n-1+\gamma_2)(n-2+\gamma_2)\langle J^{\gamma+2e_1}_{n-3,n-3}, 1\rangle_{\gamma+e_1+e_2}.$$

By performing the change of variables $x = \Psi(\zeta, \eta)$ and using the fact that $\mathrm{J}(0,1)(t) = 2 - \mathrm{J}(1,0)(t)$, we have

$$\begin{aligned}\langle J^{\gamma+2e_1}_{n-3,n-3}, 1\rangle_{\gamma+e_1+e_2} &= 2^{-4-2\gamma_1-2\gamma_2-\gamma_3-n}\,\|1\|^2_{\mathrm{J}(\gamma_1+\gamma_2+n,\gamma_3)}\,\langle \mathrm{P}^{(\gamma_1+2,\gamma_2)}_{n-3}, \mathrm{J}(0,1)\rangle_{\mathrm{J}(\gamma_1+1,\gamma_2)}\\ &= 2^{-3-2\gamma_1-2\gamma_2-\gamma_3-n}\,\|1\|^2_{\mathrm{J}(\gamma_1+\gamma_2+n,\gamma_3)}\,\langle \mathrm{P}^{(\gamma_1+2,\gamma_2)}_{n-3}, 1\rangle_{\mathrm{J}(\gamma_1+1,\gamma_2)}\\ &\overset{\text{Prop.2.3.4}}{=} 2^{-2-2\gamma_1-2\gamma_2-\gamma_3-n}\left(\frac{\gamma_1+1}{n-2+\gamma_2}\right)\|1\|^2_{\mathrm{J}(\gamma_1+\gamma_2+n,\gamma_3)}\,\langle \mathrm{P}^{(\gamma_1+1,\gamma_2)}_{n-2}, 1\rangle_{\mathrm{J}(\gamma_1,\gamma_2)}.\end{aligned}$$

Thus, we obtain

$$\langle \mathrm{W}_{e_1+e_2}\partial_1^2 J^{\gamma}_{n-1,n-1}, 1\rangle_\gamma = -(\gamma_1+1)\langle \mathrm{W}_{e_2}\partial_1 J^{\gamma}_{n-1,n-1}, 1\rangle_\gamma. \tag{3.5.10}$$

Finally, the projection $S^\gamma_0(\partial_1\phi^\gamma_{n,1})$ simplifies to

$$S^\gamma_0(\partial_1\phi^\gamma_{n,1}) = \frac{1}{\|1\|^2_\gamma}\Big((\gamma_1+1)\langle \mathrm{W}_{e_2}\partial_1 J^{\gamma}_{n-1,n-1}, 1\rangle_\gamma + \langle \mathrm{W}_{e_1+e_2}\partial_1^2 J^{\gamma}_{n-1,n-1}, 1\rangle_\gamma\Big) \overset{(3.5.10)}{=} 0.$$

Therefore, the proof is complete. □

Let us recall the map $\Psi\colon (-1,1)^2 \to \mathrm{T}$ and introduce its inverse $\Phi\colon \mathrm{T} \to (-1,1)^2$ through the formulae

$$\Psi(\zeta,\eta) := \left(\frac{(1-\eta)(1-\zeta)}{4}, \frac{(1-\eta)(1+\zeta)}{4}\right) \quad \text{and} \quad \Phi(x,y) = \left(\frac{y-x}{x+y}, 1-2x-2y\right).$$

Given a differential operator A defined on T, the operator $\Phi_\star A$ defined for all regular enough functions $g\colon (-1,1)^2 \to \mathbb{R}$ as

$$\Phi_\star A[g] := A[g\circ\Phi]\circ\Psi \tag{3.5.11}$$

is the corresponding differential operator defined on $(-1,1)^2$. In particular, for $i \in \{1,2\}$,

$$\Phi_\star\partial_i[g] = \partial_i[g\circ\Phi]\circ\Psi = ((\partial_1 g\circ\Phi)\partial_i\Phi_1 + (\partial_2 g\circ\Phi_2)\partial_i\Phi_2)\circ\Psi = (\partial_1 g)(\partial_i\Phi_1\circ\Psi)+(\partial_2 g)(\partial_i\Phi_2\circ\Psi).$$

As

$$\begin{aligned} \partial_1\Phi_1\circ\Psi(\zeta,\eta) &= -2\frac{1+\zeta}{1-\eta}, & \partial_2\Phi_1\circ\Psi(\zeta,\eta) &= 2\frac{1-\zeta}{1-\eta},\\ \partial_1\Phi_2\circ\Psi(\zeta,\eta) &= -2, & \partial_2\Phi_2\circ\Psi(\zeta,\eta) &= -2, \end{aligned}$$

it follows that

$$\begin{aligned} \Phi_\star\partial_1[g](\zeta,\eta) &= -2\frac{1+\zeta}{1-\eta}\frac{\partial g}{\partial\zeta}(\zeta,\eta) - 2\frac{\partial g}{\partial\eta}(\zeta,\eta),\\ \Phi_\star\partial_2[g](\zeta,\eta) &= 2\frac{1-\zeta}{1-\eta}\frac{\partial g}{\partial\zeta}(\zeta,\eta) - 2\frac{\partial g}{\partial\eta}(\zeta,\eta),\\ \Phi_\star\partial_3[g](\zeta,\eta) &= \Phi_\star\partial_2[g](\zeta,\eta) - \Phi_\star\partial_1[g](\zeta,\eta) = \frac{4}{1-\eta}\frac{\partial g}{\partial\zeta}(\zeta,\eta). \end{aligned} \tag{3.5.12}$$

Given two differential operators A and B defined on T, for all regular enough functions $g\colon (-1,1)^2 \to \mathbb{R}$, there holds the composition rule

$$\Phi_\star(AB)[g] = \Phi_\star A\,\Phi_\star B[g]. \tag{3.5.13}$$

Proposition 3.5.17. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$ and $i \in \{1,2\}$, we introduce the second-order differen-*

tial operators $\Xi_i^{\gamma} := \mathrm{W}_{e_i}\partial_i^2 + (\gamma_i + 1)\partial_i$. *Then, the following identity holds:*

$$\partial_1\partial_2 d_3^{\gamma-e_1-e_2} = d_3^{\gamma}\partial_1\partial_2 + \Xi_1^{\gamma} - \Xi_2^{\gamma}.$$

Proof. This follows directly from (2.2.6) and by application of the calculus rules of differentiation. □

Proposition 3.5.18. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$. *Then,*

$$\Xi_1^{\gamma} J_{n-1,n-1}^{\gamma} \in \mathcal{V}_{n-2}^{\gamma+e_3} \quad \text{and} \quad \Xi_2^{\gamma} J_{n-1,n-1}^{\gamma} \in \mathcal{V}_{n-2}^{\gamma+e_3}.$$

Proof. Based on (2.4.10), we deduce that

$$\partial_1 J_{n-1,n-1}^{\gamma} = -(n-1+\gamma_2) J_{n-2,n-2}^{\gamma+e_1+e_3} \quad \text{and} \quad \partial_1^2 J_{n-1,n-1}^{\gamma} = (n-1+\gamma_2)(n-2+\gamma_2) J_{n-3,n-3}^{\gamma+e_2+e_3}.$$

Let $k \in [n-3]_0$ and $l \in [k]_0$. Then,

$$\begin{aligned}\left\langle \Xi_1^{\gamma} J_{n-1,n-1}^{\gamma}, J_{k,l}^{\gamma} \right\rangle_{\gamma+e_3} = (n+\gamma_2-1)(n+\gamma_2-2)\langle J_{n-3,n-3}^{\gamma+2e_1}, J_{k,l}^{\gamma}\rangle_{\gamma+e_1+e_3} \\ - (\gamma_1+1)(n+\gamma_2-1)\langle J_{n-2,n-2}^{\gamma+e_1}, J_{k,l}^{\gamma}\rangle_{\gamma+e_3}. \quad (3.5.14)\end{aligned}$$

Performing the change of variables $x = \Psi(\zeta,\eta)$ to the first integral on the right-hand side of (3.5.14) yields

$$\begin{aligned}\langle J_{n-3,n-3}^{\gamma+2e_1}, J_{k,l}^{\gamma}\rangle_{\gamma+e_1+e_3} = 2^{-3-2\gamma_1-2\gamma_2-\gamma_3-l-n} \\ \times \left\langle \mathrm{P}_{n-3}^{(\gamma_1+2,\gamma_2)}, \mathrm{P}_{l}^{(\gamma_1,\gamma_2)} \right\rangle_{\mathrm{J}(\gamma_1+1,\gamma_2)} \\ \times \left\langle \mathrm{P}_{k-l}^{(2l+\gamma_1+\gamma_2+1,\gamma_3)}, 1 \right\rangle_{\mathrm{J}(\gamma_1+\gamma_2+n+l-1,\gamma_3+1)}.\end{aligned}$$

From Proposition 2.3.4, we can rewrite the expression above as follows:

$$\langle J_{n-3,n-3}^{\gamma+2e_1}, J_{k,l}^{\gamma}\rangle_{\gamma+e_1+e_3} = 2^{-2-2\gamma_1-2\gamma_2-\gamma_3-l-n}$$

$$\times \left(\frac{\gamma_1+1}{n+\gamma_2-2}\right)\left\langle \mathrm{P}_{n-2}^{(\gamma_1+1,\gamma_2)}, \mathrm{P}_{l}^{(\gamma_1,\gamma_2)}\right\rangle_{\mathrm{J}(\gamma_1,\gamma_2)} \times \left\langle \mathrm{P}_{k-l}^{(2l+\gamma_1+\gamma_2+1,\gamma_3)}, 1\right\rangle_{\mathrm{J}(\gamma_1+\gamma_2+n+l-1,\gamma_3+1)}. \tag{3.5.15}$$

Similarly, performing the change of variables $x=\Psi(\zeta,\eta)$ to the second term on the right-hand side of (3.5.14), we find

$$\langle J_{n-2,n-2}^{\gamma+e_1}, J_{k,l}^{\gamma}\rangle_{\gamma+e_3} = 2^{-2-2\gamma_1-2\gamma_2-\gamma_3-l-n}\left\langle \mathrm{P}_{n-2}^{(\gamma_1+1,\gamma_2)}, \mathrm{P}_{l}^{(\gamma_1,\gamma_2)}\right\rangle_{\mathrm{J}(\gamma_1,\gamma_2)} \times \left\langle \mathrm{P}_{k-l}^{(2l+\gamma_1+\gamma_2+1,\gamma_3)}, 1\right\rangle_{\mathrm{J}(\gamma_1+\gamma_2+n+l-1,\gamma_3+1)}. \tag{3.5.16}$$

Substituting (3.5.15) and (3.5.16) into (3.5.14), we conclude that $\langle \Xi_1^{\gamma} J_{n-1,n-1}^{\gamma}, J_{k,l}^{\gamma}\rangle_{\gamma+e_3}=0$. Consequently, we deduce that

$$\Xi_1^{\gamma} J_{n-1,n-1}^{\gamma} \in \mathcal{V}_{n-2}^{\gamma+e_3}. \tag{3.5.17}$$

Now, let $\tau\in\mathcal{S}_3$ be the permutation defined by $\tau(1)=2$ and $\tau(2)=1$. Using part (ii) of Proposition 2.2.7 and part (i) of Proposition 2.2.6, we can express Ξ_2^{γ} as:

$$\Xi_2^{\gamma} = T_{\tau}^{\star}\Xi_1^{\tau(\gamma)}T_{\tau}^{\star}. \tag{3.5.18}$$

Furthermore, from (2.4.1) and (2.3.6), it follows that

$$T_{\tau}^{\star} J_{n-1,n-1}^{\gamma} = (-1)^{n-1} J_{n-1,n-1}^{\tau(\gamma)}. \tag{3.5.19}$$

Finally, combining (3.5.17) with (3.5.18), (3.5.19) and (iii) of Proposition 2.2.6, we conclude that $\Xi_2^{\gamma} J_{n-1,n-1}^{\gamma}\in\mathcal{V}_{n-2}^{\gamma+e_3}$. □

Proposition 3.5.19. *Let* $\boldsymbol{\gamma}\in(-1,\infty)^3$ *and* $n\in\mathbb{N}_{\geq 2}$. *Then,* $\Lambda_3^{\gamma} J_{n,n}^{\gamma} = n(n+\gamma_1+\gamma_2+1)J_{n,n}^{\gamma}$.

Proof. From Definition 3.4.6 and (2.2.6), we obtain

$$\Lambda_3^{\gamma} J_{n,n}^{\gamma} = -\mathrm{W}_{\gamma}^{-1}\partial_3\left(\mathrm{W}_{\gamma+e_1+e_2}\partial_3 J_{n,n}^{\gamma}\right)$$

The transformed function $J_{n,n}^{\gamma}\circ\Psi$ has a simpler formula than $J_{n,n}^{\gamma}$ itself (cf. (3.5.20)). Thus, it suits us to transform $\Lambda_3^{\boldsymbol{\gamma}}$ into an operator on the square $(-1,1)^2$ according to (3.5.11). By (3.5.13),

$$\Phi_\star \Lambda_3^{\boldsymbol{\gamma}}[g] = -\Phi_\star \mathrm{W}_{\boldsymbol{\gamma}}^{-1} \Phi_\star \partial_3 \Phi_\star \mathrm{W}_{\boldsymbol{\gamma}+e_1+e_2} \Phi_\star \partial_3[g],$$

where $\mathrm{W}_{\boldsymbol{\gamma}}$ and $\mathrm{W}_{\boldsymbol{\gamma}+e_1+e_2}$ must be understood as multiplication operators. We compute

$$\Phi_\star \mathrm{W}_{\boldsymbol{\gamma}}[g](\zeta,\eta) = \frac{\mathrm{J}(\gamma_1+\gamma_2,\gamma_3)(\eta)\mathrm{J}(\gamma_1,\gamma_2)(\zeta)}{2^{2\gamma_1+2\gamma_2+\gamma_3}} g(\zeta,\eta)$$

and

$$(J_{n,n}^{\boldsymbol{\gamma}} \circ \Psi)(\zeta,\eta) = \frac{(1-\eta)^n}{2^n} \mathrm{P}_n^{(\gamma_1,\gamma_2)}(\zeta) \tag{3.5.20}$$

and recall that $\partial_3 \Phi_\star$ was computed in (3.5.12) to obtain

$$\begin{aligned}
\Phi_\star \Lambda_3^{\boldsymbol{\gamma}}[J_{n,n}^{\boldsymbol{\gamma}} \circ \Psi](\zeta,\eta) &= -\frac{\mathrm{J}(\gamma_1+\gamma_2,\gamma_3)^{-1}(\eta)\mathrm{J}(\gamma_1,\gamma_2)^{-1}(\zeta)}{2^{-2\gamma_1-2\gamma_2-\gamma_3}} 4\mathrm{J}(1,0)^{-1}(\eta) \\
\times \partial_\zeta &\left[\frac{\mathrm{J}(\gamma_1+\gamma_2+2,\gamma_3)(\eta)\mathrm{J}(\gamma_1+1,\gamma_2+1)(\zeta)}{2^{2\gamma_1+2\gamma_2+\gamma_3+4}} 4\mathrm{J}(1,0)^{-1}(\eta)\partial_\zeta \left(\frac{\mathrm{J}(n,0)}{2^n} \mathrm{P}_n^{(\gamma_1,\gamma_2)}(\zeta) \right) \right] \\
&= -\frac{\mathrm{J}(n,0)(\eta)}{2^n} \mathrm{J}(\gamma_1,\gamma_2)^{-1}(\zeta) \partial_\zeta \left[\mathrm{J}(\gamma_1+1,\gamma_2+1)(\zeta) \partial_\zeta \mathrm{P}_n^{(\gamma_1,\gamma_2)}(\zeta) \right] \\
&\overset{(2.3.4)}{=} n(n+\gamma_1+\gamma_2+1) \frac{\mathrm{J}(n,0)(\eta)}{2^n} \mathrm{P}_n^{(\gamma_1,\gamma_2)}(\zeta) = n(n+\gamma_1+\gamma_2+1) \Phi_\star [J_{n,n}^{\boldsymbol{\gamma}} \circ \Psi](\zeta,\eta).
\end{aligned}$$

Then,

$$\Lambda_3^{\boldsymbol{\gamma}}[J_{n,n}^{\boldsymbol{\gamma}}] \circ \Psi \overset{(3.5.11)}{=} \Phi_\star \Lambda_3^{\boldsymbol{\gamma}}[J_{n,n}^{\boldsymbol{\gamma}} \circ \Psi] = n(n+\gamma_1+\gamma_2+1) \Phi_\star [J_{n,n}^{\boldsymbol{\gamma}} \circ \Psi] \overset{(3.5.11)}{=} n(n+\gamma_1+\gamma_2+1)(J_{n,n}^{\boldsymbol{\gamma}} \circ \Psi);$$

as Ψ is invertible we obtain the desired result. □

Proposition 3.5.20. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$. *Then,*

$$\partial_1 \partial_3 \phi_{n,1}^{\boldsymbol{\gamma}} \in \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_1}, \quad \partial_2 \partial_3 \phi_{n,1}^{\boldsymbol{\gamma}} \in \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_2} \quad \text{and} \quad \partial_1 \partial_2 \phi_{n,1}^{\boldsymbol{\gamma}} \in \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_3}.$$

Proof. From Proposition 3.5.17 we get

$$\partial_1 \partial_2 \phi_{n,1}^{\boldsymbol{\gamma}} = d_3^{\boldsymbol{\gamma}} \partial_1 \partial_2 J_{n-1,n-1}^{\boldsymbol{\gamma}} + \Xi_1^{\boldsymbol{\gamma}} J_{n-1,n-1}^{\boldsymbol{\gamma}} - \Xi_2^{\boldsymbol{\gamma}} J_{n-1,n-1}^{\boldsymbol{\gamma}} \tag{3.5.21}$$

Thanks to (3.5.8) we have that $\partial_1\partial_2 J^{\gamma}_{n-1,n-1} \in \mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3}$. Then, since $d_3^{\gamma+e_3}\mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3} \subset \mathcal{V}^{\gamma+e_3}_{n-2}$, we deduce that $d_3^{\gamma}\partial_1\partial_2 J^{\gamma}_{n-1,n-1} \in \mathcal{V}^{\gamma+e_3}_{n-2}$ (for $d_3^{\gamma} = d_3^{\gamma+e_3}$). On the other hand, from Proposition 3.5.18, the second and third terms on the right-hand side of (3.5.21) lies in $\mathcal{V}^{\gamma+e_3}_{n-2}$. Therefore, $\partial_1\partial_2\phi^{\gamma}_{n,1}$ also lie in $\mathcal{V}^{\gamma+e_3}_{n-2}$.

Now, using (iii) on Proposition 3.4.12 and Proposition 3.5.19, we get

$$\partial_1\partial_3\phi^{\gamma}_{n,1} = \partial_1 d_3^{\gamma}\partial_3 J^{\gamma}_{n-1,n-1} + (\gamma_1+\gamma_2)\partial_1 J^{\gamma}_{n-1,n-1} = n(n-1+\gamma_1+\gamma_2)\partial_1 J^{\gamma}_{n-1,n-1}.$$

Since,

$$\partial_1 J^{\gamma}_{n-1,n-1} \overset{(3.5.8)}{=} -(n-1+\gamma_2)J^{\gamma+e_1+e_3}_{n-2,n-2} \overset{(2.4.1)}{=} -(n-1+\gamma_2)J^{\gamma+e_1}_{n-2,n-2} \in \mathcal{V}^{\gamma+e_1}_{n-2},$$

we deduce that $\partial_1\partial_3\phi^{\gamma}_{n,1} \in \mathcal{V}^{\gamma+e_1}_{n-2}$. Similarly, using (iii) on Proposition 3.4.12 and Proposition 3.5.19

$$\partial_2\partial_3\phi^{\gamma}_{n,1} = \partial_2 d_3^{\gamma}\partial_3 J^{\gamma}_{n-1,n-1} + (\gamma_1+\gamma_2)\partial_2 J^{\gamma}_{n-1,n-1} = n(n-1+\gamma_1+\gamma_2)\partial_2 J^{\gamma}_{n-1,n-1}.$$

Since $\partial_2 J^{\gamma}_{n-1,n-1} \in \mathcal{V}^{\gamma+e_2}_{n-2}$, we deduce that $\partial_2\partial_3\phi^{\gamma}_{n,1} \in \mathcal{V}^{\gamma+e_2}_{n-2}$. ☐

Remark 3.5.21. From Proposition 3.5.20, Proposition 2.2.9 and (iii) in Proposition 2.2.6, it follows that the same result holds for $\phi^{\gamma}_{n,2}$ and $\phi^{\gamma}_{n,3}$.

Proposition 3.5.22. *Let* $\gamma \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$. *Then,* $\operatorname{span}\left(\left\{\phi^{\gamma}_{n,1}, \phi^{\gamma}_{n,2}, \phi^{\gamma}_{n,3}\right\}\right) \subset \mathcal{V}^{\gamma;\mathrm{Sob}}_{n}$.

Proof. From Proposition 3.5.15, Proposition 3.5.16, Proposition 3.5.20 and Lemma 3.3.1, we deduce that $\phi^{\gamma}_{n,1} \in \mathcal{V}^{\gamma;\mathrm{Sob}}_{n}$. Furthermore, using this result and Lemma 3.3.7, we conclude that $\phi^{\gamma}_{n,2}, \phi^{\gamma}_{n,3} \in \mathcal{V}^{\gamma;\mathrm{Sob}}_{n}$ as well. ☐

3.6 Proof of the main results

In this section, we exploit results from earlier sections, together with some new results, to prove the main results of this chapter, Theorem 3.1.1 and Theorem 3.1.2.

Lemma 3.6.1. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *be such that* $\gamma_i \neq 0$ *for all* $i \in [3]$*, and* $n \in \mathbb{N}_{\geq 4}$*. Then,*

$$\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}} = \mathcal{H}_n^{\boldsymbol{\gamma}} \oplus \mathrm{M}^{\boldsymbol{\gamma}}(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+e_1+e_2+e_3}).$$

Proof. Let $p_n \in \mathcal{H}_n^{\boldsymbol{\gamma}} \oplus \mathrm{M}^{\boldsymbol{\gamma}}(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+e_1+e_2+e_3})$. Then there exist $r_{n-3} \in \mathcal{V}_{n-3}^{\boldsymbol{\gamma}+e_1+e_2+e_3}$ and constants $\{\vartheta_i\}_{i=1}^4 \subset \mathbb{R}$ such that

$$p_n = \sum_{i=1}^{3} \vartheta_i h_{3,i}^{\boldsymbol{\gamma}} + \vartheta_4 \mathrm{M}^{\boldsymbol{\gamma}}(r_{n-3}).$$

Since, by part (iii) of Proposition 2.2.3, $d_3^{\boldsymbol{\gamma}+e_3} r_{n-3} \in \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_3}$, using the factorized form of $\mathrm{M}^{\boldsymbol{\gamma}}$ given in (3.1.4), Proposition 3.4.18 and Proposition 3.4.19 we deduce that

$$S_0^{\boldsymbol{\gamma}}(\mathrm{M}^{\boldsymbol{\gamma}}(r_{n-3})) = S_0^{\boldsymbol{\gamma}}(m^{\boldsymbol{\gamma}}(d_3^{\boldsymbol{\gamma}+e_3} r_{n-3})) = 0 \quad \text{and} \quad S_0^{\boldsymbol{\gamma}}(\partial_i \mathrm{M}^{\boldsymbol{\gamma}}(r_{n-3})) = S_0^{\boldsymbol{\gamma}}(\partial_i m^{\boldsymbol{\gamma}}(d_3^{\boldsymbol{\gamma}+e_3} r_{n-3})) = 0$$

for all $i \in \{1,2,3\}$. Similarly by Proposition 3.5.6 and Proposition 3.5.7 we deduce that $S_0^{\boldsymbol{\gamma}}(h_{n,j}^{\boldsymbol{\gamma}}) = 0$ and $S_0^{\boldsymbol{\gamma}}(\partial_i h_{n,j}^{\boldsymbol{\gamma}}) = 0$ with $i,j \in \{1,2,3\}$. Then for all $q \in \Pi_{n-1}^2$ we have

$$\langle p_n, q \rangle_{\mathbb{H}_{\boldsymbol{\gamma}}^2} = \sum_{i=1}^{3} \vartheta_i \mathfrak{B}^{\boldsymbol{\gamma}}(h_{n,i}^{\boldsymbol{\gamma}}, q) + \vartheta_4 \mathfrak{B}^{\boldsymbol{\gamma}}(m^{\boldsymbol{\gamma}}(d_3^{\boldsymbol{\gamma}+e_3} r_{n-3}), q). \tag{3.6.1}$$

Now from Proposition 3.4.17, we deduce that

$$\partial_1\partial_2 m^{\boldsymbol{\gamma}}(d_3^{\boldsymbol{\gamma}+e_3} r_{n-3}) \in \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_3}, \quad \partial_1\partial_3 m^{\boldsymbol{\gamma}}(d_3^{\boldsymbol{\gamma}+e_3} r_{n-3}) \in \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_1} \quad \text{and} \quad \partial_2\partial_3 m^{\boldsymbol{\gamma}}(d_3^{\boldsymbol{\gamma}+e_3} r_{n-3}) \in \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_3}.$$

As $\partial_1\partial_2 q, \partial_1\partial_3 q, \partial_2\partial_3 q \in \Pi_{n-3}^2$, we deduce that the second term of (3.6.1) vanishes. Similarly, using Proposition 3.5.5 we deduce that the first term of (3.6.1) vanishes. Thus, $p_n \in \mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}}$. Finally, from Proposition 3.5.4, Lemma 3.4.2, Proposition 3.5.2 and (2.2.2) we deduce that

$$\dim\left(\mathcal{H}_n^{\boldsymbol{\gamma}} \oplus \mathrm{M}^{\boldsymbol{\gamma}}\left(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+e_1+e_2+e_3}\right)\right) = \dim\left(\mathcal{H}_n^{\boldsymbol{\gamma}}\right) + \dim\left(\mathrm{M}^{\boldsymbol{\gamma}}\left(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+e_1+e_2+e_3}\right)\right) = n+1 = \dim\left(\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}}\right)$$

which proves the desired equality. □

Definition 3.6.2. *Let* $\boldsymbol{\gamma} \in (-1, \infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$*. We define the polynomial space*

$$\mathcal{N}_n^{\boldsymbol{\gamma}} := \operatorname{span}\left(\{\varrho_{n,1}^{\boldsymbol{\gamma}}, \varrho_{n,2}^{\boldsymbol{\gamma}}, h_{n,1}^{\boldsymbol{\gamma}}\}\right).$$

Proposition 3.6.3. *Let* $\boldsymbol{\gamma} \in (-1, \infty)^3$ *be such that* $\gamma_1 \neq 0$ *and* $\gamma_2 = \gamma_3 = 0$*, and let* $n \in \mathbb{N}_{\geq 4}$*. Then,* $\dim\left(\mathcal{N}_n^{\boldsymbol{\gamma}}\right) = 3$*.*

Proof. Let $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ be such that

$$\mu_1 \varrho_{n,1}^{\boldsymbol{\gamma}} + \mu_2 \varrho_{n,2}^{\boldsymbol{\gamma}} + \mu_3 h_{n,1}^{\boldsymbol{\gamma}} = 0. \tag{3.6.2}$$

Given that $\gamma_2 = 0$, it follows from (3.4.6) that $h_{n,1}^{\boldsymbol{\gamma}}(v) = 0$ for all $v \in \mathbb{V}_2$. By evaluating (3.6.2) at the vertices $(0,0)$ and $(0,1)$, and applying Definition 3.5.9 of $\varrho_{n,1}^{\boldsymbol{\gamma}}$ and (3.2.7), we obtain the following system:

$$\begin{bmatrix} 4\gamma_1(\gamma_1+1)\mathrm{P}_{n-2}^{(\gamma_1+1,1)}(1) & 0 \\ 0 & 4\gamma_1(\gamma_1+1)\mathrm{P}_{n-2}^{(\gamma_1+1,1)}(1) \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\gamma_1 \neq 0$ and $\mathrm{P}_{n-2}^{(\gamma_1+1,1)}(1) \neq 0$, we must have $\mu_1 = \mu_2 = 0$. Consequently, (3.6.2) reduces to $\mu_3 h_{n,1}^{\boldsymbol{\gamma}} = 0$. Since $h_{n,1}^{\boldsymbol{\gamma}} \neq 0$, we conclude that $\mu_3 = 0$. Therefore, the desired result holds. □

Lemma 3.6.4. *Let* $\boldsymbol{\gamma} \in (-1, \infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$*. Then, if* $\gamma_i \neq 0$ *for exactly one* $i \in \{1, 2, 3\}$*, there exists a permutation* $\sigma \in \mathcal{S}_3$ *be such that*

$$\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}} = T_{\sigma^{-1}}^{\star} \mathcal{N}_n^{\sigma(\boldsymbol{\gamma})} \oplus \mathrm{M}^{\boldsymbol{\gamma}}(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+e_1+e_2+e_3}).$$

Proof. Let us suppose first that $\gamma_1 \neq 0$ and $\gamma_2 = \gamma_3 = 0$. From Proposition 3.5.12, Lemma 3.5.8 and Lemma 3.4.20 we get that $\mathcal{N}_n^{\boldsymbol{\gamma}} \oplus \mathrm{M}^{\boldsymbol{\gamma}}(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+e_1+e_2+e_3}) \subset \mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}}$. Let $q \in \mathcal{N}_n^{\boldsymbol{\gamma}} \cap \mathrm{M}^{\boldsymbol{\gamma}}(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+e_1+e_2+e_3})$. Then, there exist $p \in \mathcal{V}_{n-3}^{\boldsymbol{\gamma}+e_1+e_2+e_3}$ and $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ be such that

$$q = \mathrm{M}^{\boldsymbol{\gamma}}(p) \quad \text{and} \quad q = \mu_1 \varrho_{n,1}^{\boldsymbol{\gamma}} + \mu_2 \varrho_{n,2}^{\boldsymbol{\gamma}} + \mu_3 h_{n,1}^{\boldsymbol{\gamma}}. \tag{3.6.3}$$

As $\gamma_2 = \gamma_3 = 0$, it follows from (3.4.6) that $\mathrm{M}^{\boldsymbol{\gamma}}(p)(v)$ and $h_{n,1}^{\boldsymbol{\gamma}}(v) = 0$ for all $v \in \mathbb{V}_2$. Then,

$q(v) = 0$ for all $v \in \mathbb{V}_2$. By evaluating (3.6.3) at the vertices $(0,0)$ and $(0,1)$, from the Definition 3.5.9 of $\varrho^{\gamma}_{n,1}$ and (3.2.7), we obtain the following system:

$$\begin{bmatrix} 4\gamma_1(\gamma_1+1)\mathrm{P}^{(\gamma_1+1,1)}_{n-2}(1) & 0 \\ 0 & 4\gamma_1(\gamma_1+1)\mathrm{P}^{(\gamma_1+1,1)}_{n-2}(1) \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\gamma_1 \neq 0$ and $\mathrm{P}^{(\gamma_1+1,1)}_{n-2}(1) \neq 0$, we must have $\mu_1 = \mu_2 = 0$. From (3.6.3) we have $\mu_3 h^{\gamma}_{n,1} = \mathrm{M}^{\gamma}(p)$. This is equivalent to $m^{\gamma}\mu_3 J^{\gamma+e_3}_{n-2,0} = m^{\gamma} d_3^{\gamma+e_3} p$. Furthermore, since $\gamma_1 \neq 0$, Proposition 3.4.1 implies that $\mu_3 J^{\gamma+e_3}_{n-2,0} = d_3^{\gamma+e_3} p$. However, by Proposition 3.5.3, we know that $J^{\gamma+e_3}_{n-2,0} \notin d_3^{\gamma+e_3}\mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-2}$. Consequently, we must have $\mu_3 = 0$. Therefore, $q = 0$. Combining this result with Proposition 3.6.3, Lemma 3.4.2 and (2.2.2), we conclude that

$$\dim\left(\mathcal{N}^{\gamma}_n \oplus \mathrm{M}^{\gamma}\left(\mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3}\right)\right) = \dim\left(\mathcal{N}^{\gamma}_n\right) + \dim\left(\mathrm{M}^{\gamma}\left(\mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3}\right)\right) = n+1 = \dim\left(\mathcal{V}^{\gamma;\mathrm{Sob}}_n\right).$$

Hence, the result follows with $\sigma = id$.

If $\gamma_2 \neq 0$ and $\gamma_1 = \gamma_3 = 0$. From Lemma 3.3.7 we have that

$$\mathcal{V}^{\gamma;\mathrm{Sob}}_n = T^{\star}_{\sigma_2}\mathcal{V}^{\sigma_1(\gamma);\mathrm{Sob}}_n.$$

We, observe that $\sigma_1(\boldsymbol{\gamma}) = (\gamma_{\sigma_1(1)}, \gamma_{\sigma_1(2)}, \gamma_{\sigma_1(3)}) = (\gamma_2, 0, 0)$. Then, from the previous case,

$$\mathcal{V}^{\sigma_1(\gamma);\mathrm{Sob}}_n = \mathcal{N}^{\sigma_1(\gamma)}_n \oplus \mathrm{M}^{\sigma_1(\gamma)}(\mathcal{V}^{\sigma_1(\gamma)+e_1+e_2+e_3}_{n-3}).$$

Thus, since $T^{\star}_{\sigma_2}$ is injective and using (3.4.4), we deduce that

$$\mathcal{V}^{\gamma;\mathrm{Sob}}_n = T^{\star}_{\sigma_2}\mathcal{N}^{\sigma_1(\gamma)}_n \oplus T^{\star}_{\sigma_2}\mathrm{M}^{\sigma_1(\gamma)}(\mathcal{V}^{\sigma_1(\gamma)+e_1+e_2+e_3}_{n-3}) = T^{\star}_{\sigma_2}\mathcal{N}^{\sigma_1(\gamma)}_n \oplus \mathrm{M}^{\gamma}(\mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3}).$$

Similarly, if $\gamma_3 \neq 0$ and $\gamma_1 = \gamma_2 = 0$, then

$$\mathcal{V}^{\gamma;\mathrm{Sob}}_n = T^{\star}_{\sigma_1}\mathcal{N}^{\sigma_2(\gamma)}_n \oplus \mathrm{M}^{\gamma}(\mathcal{V}^{\gamma+e_1+e_2+e_3}_{n-3}).$$

Finally, the desired result holds. □

Definition 3.6.5. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$. *We define the polynomial space*

$$\mathcal{R}_n^{\boldsymbol{\gamma}} := \operatorname{span}\left(\{\phi_{n,1}^{\boldsymbol{\gamma}}, \phi_{n,2}^{\boldsymbol{\gamma}}, h_{n,1}^{\boldsymbol{\gamma}}\}\right).$$

Proposition 3.6.6. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *be such that* $\gamma_1 \neq 0$, $\gamma_2 = 0$ *and* $\gamma_3 \neq 0$, *and let* $n \in \mathbb{N}_{\geq 4}$. *Then,* $\dim(\mathcal{R}_n^{\boldsymbol{\gamma}}) = 3$.

Proof. Let $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$ be such that

$$\kappa_1 \phi_{n,1}^{\boldsymbol{\gamma}} + \kappa_2 \phi_{n,2}^{\boldsymbol{\gamma}} + \kappa_3 h_{n,1}^{\boldsymbol{\gamma}} = 0. \tag{3.6.4}$$

Given that $\gamma_2 = 0$, it follows from Definition 3.5.1 and (3.4.6) that $h_{n,1}^{\boldsymbol{\gamma}}(v) = 0$ for all $v \in \mathbb{V}_2$. By evaluating (3.6.4) at the vertices $(0,1)$ and $(0,0)$, and applying Definition 3.5.14 of $\phi_{n,1}^{\boldsymbol{\gamma}}$ and $\phi_{n,2}^{\boldsymbol{\gamma}}$, (2.2.6), (2.3.6) and (2.4.1), we obtain the following system:

$$\begin{bmatrix} \gamma_1 \mathrm{P}_{n-1}^{(\gamma_1,\gamma_2)}(1) & 0 \\ 0 & (-1)^n \gamma_1 \mathrm{P}_{n-1}^{(\gamma_1,\gamma_3)}(1) \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\gamma_1 \neq 0$, $\mathrm{P}_{n-1}^{(\gamma_1,\gamma_2)}(1) \neq 0$ and $\mathrm{P}_{n-1}^{(\gamma_1,\gamma_3)}(1) \neq 0$, we must have $\kappa_1 = \kappa_2 = 0$. Consequently, (3.6.4) reduces to $\kappa_3 h_{n,1}^{\boldsymbol{\gamma}} = 0$. Since $h_{n,1}^{\boldsymbol{\gamma}} \neq 0$, we conclude that $\kappa_3 = 0$. Therefore, the desired result holds. □

Lemma 3.6.7. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *and* $n \in \mathbb{N}_{\geq 4}$. *Then, if* $\gamma_i = 0$ *for exactly one* $i \in \{1,2,3\}$, *there exists a permutation* $\sigma \in \mathcal{S}_3$ *be such that*

$$\mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}} = T_{\sigma^{-1}}^{\star} \mathcal{R}_n^{\sigma(\boldsymbol{\gamma})} \oplus \mathrm{M}^{\boldsymbol{\gamma}}(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3}).$$

Proof. First, we suppose that $\gamma_2 = 0$. From Proposition 3.5.22, Lemma 3.5.8 and Lemma 3.4.20 we get that $\mathcal{R}_n^{\boldsymbol{\gamma}} \oplus \mathrm{M}^{\boldsymbol{\gamma}}(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3}) \subset \mathcal{V}_n^{\boldsymbol{\gamma};\mathrm{Sob}}$. Let $q \in \mathcal{R}_n^{\boldsymbol{\gamma}} \cap \mathrm{M}^{\boldsymbol{\gamma}}(\mathcal{V}_{n-3}^{\boldsymbol{\gamma}+\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3})$. Then, there exist

$p \in \mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}$ and $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$ be such that

$$q = \mathrm{M}^{\gamma}(p) \quad \text{and} \quad q = \kappa_1 \phi_{n,1}^{\gamma} + \kappa_2 \phi_{n,2}^{\gamma} + \kappa_3 h_{n,1}^{\gamma}. \tag{3.6.5}$$

Given that $\gamma_2 = 0$, it follows from Definition 3.5.1 and (3.4.6) that $\mathrm{M}^{\gamma}(p)(v) = 0$ and $h_{n,1}^{\gamma}(v) = 0$ for all $v \in \mathbb{V}_2$. Then, $q(v) = 0$ for all $v \in \mathbb{V}_2$. By evaluating (3.6.5) at the vertices $(0,1)$ and $(0,0)$, and applying Definition 3.5.14 of $\phi_{n,1}^{\gamma}$ and $\phi_{n,2}^{\gamma}$, (2.2.6), (2.3.6) and (2.4.1), we obtain the following system:

$$\begin{bmatrix} \gamma_1 \mathrm{P}_{n-1}^{(\gamma_1,\gamma_2)}(1) & 0 \\ 0 & (-1)^n \gamma_1 \mathrm{P}_{n-1}^{(\gamma_1,\gamma_3)}(1) \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\gamma_1 \neq 0$, $\mathrm{P}_{n-1}^{(\gamma_1,\gamma_2)}(1) \neq 0$ and $\mathrm{P}_{n-1}^{(\gamma_1,\gamma_3)}(1) \neq 0$, we must have $\kappa_1 = \kappa_2 = 0$. From (3.6.5) we have $\kappa_3 h_{n,3}^{\gamma} = \mathrm{M}^{\gamma}(p)$. This is equivalent to $m^{\gamma}\kappa_3 J_{n-2,0}^{\gamma+e_3} = m^{\gamma} d_3^{\gamma+e_3} p$. Furthermore, since $\gamma_1 \neq 0$, Proposition 3.4.1 implies that $\kappa_3 J_{n-2,0}^{\gamma+e_3} = d_3^{\gamma+e_3} p$. However, by Proposition 3.5.3, we know that $J_{n-2,0}^{\gamma+e_3} \notin d_3^{\gamma+e_3} \mathcal{V}_{n-2}^{\gamma+e_1+e_2+e_3}$. Consequently, we must have $\kappa_3 = 0$. Therefore, $q = 0$. Combining this result with Proposition 3.6.6, Lemma 3.4.2 and (2.2.2), we obtain

$$\dim\left(\mathcal{R}_n^{\gamma} \oplus \mathrm{M}^{\gamma}\left(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}\right)\right) = \dim\left(\mathcal{R}_n^{\gamma}\right) + \dim\left(\mathrm{M}^{\gamma}\left(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}\right)\right) = n+1 = \dim\left(\mathcal{V}_n^{\gamma;\mathrm{Sob}}\right).$$

Hence, the result follows with $\sigma = id$.

Let us now consider the case in which $\gamma_1 = 0$. We observe that $\sigma_2(\boldsymbol{\gamma}) = (\gamma_3, 0, \gamma_2)$. Then, using the previous case, Lemma 3.3.7, the injectivity of $T_{\sigma_1}^{\star}$ and (3.4.4) we have

$$\mathcal{V}_n^{\gamma;\mathrm{Sob}} = T_{\sigma_1}^{\star} \mathcal{R}_n^{\sigma_2(\gamma)} \oplus \mathrm{M}^{\gamma}(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}).$$

Similarly, if $\gamma_3 = 0$, then

$$\mathcal{V}_n^{\gamma;\mathrm{Sob}} = T_{\sigma_2}^{\star} \mathcal{R}_n^{\sigma_1(\gamma)} \oplus \mathrm{M}^{\gamma}(\mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}).$$

Then, the desired result holds. □

Proof of Theorem 3.1.1. The proof is divided into three cases. First, assume that for all $i \in$

$\{1,2,3\}$, $\gamma_i \neq 0$. In this case, the result follows from Proposition 3.5.5 and Lemma 3.6.1. Thus, in this case, we can take $\mathcal{J}_n^\gamma = \mathcal{H}_n^\gamma$. Second, suppose there exists a unique $i \in \{1,2,3\}$ such that $\gamma_i = 0$. Here, the result follows from Proposition 3.5.5, Proposition 3.5.10, Remark 3.5.11, and Lemma 3.6.4. Thus, there exists a permutation $\sigma \in \mathcal{S}_3$ such that we can take $\mathcal{J}_n^\gamma = T^\star_{\sigma^{-1}}\mathcal{N}_n^{\sigma(\gamma)}$. Finally, assume there exists a unique $i \in \{1,2,3\}$ such that $\gamma_i \neq 0$. The result follows from Proposition 3.5.5, Proposition 3.5.20, Remark 3.5.21, and Lemma 3.6.7. Thus, there exists a permutation $\sigma \in \mathcal{S}_3$ such that we can take $\mathcal{J}_n^\gamma = T^\star_{\sigma^{-1}}\mathcal{R}_n^{\sigma(\gamma)}$. □

Corollary 3.6.8. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$ be such that $\boldsymbol{\gamma} \neq (0,0,0)$ and $n \in \mathbb{N}_0$. Then,*

$$\partial_1\partial_3\mathcal{V}_n^{\gamma;\mathrm{Sob}} \subset \mathcal{V}_{n-2}^{\gamma+e_1}, \quad \partial_2\partial_3\mathcal{V}_n^{\gamma;\mathrm{Sob}} \subset \mathcal{V}_{n-2}^{\gamma+e_2} \quad \text{and} \quad \partial_1\partial_2\mathcal{V}_n^{\gamma;\mathrm{Sob}} \subset \mathcal{V}_{n-2}^{\gamma+e_3}.$$

Proof. If $0 \leq n \leq 1$, the properties hold trivially. If $n = 2$, let $p_2 \in \mathcal{V}_2^{\gamma;\mathrm{Sob}}$. Then, $\partial_1\partial_3 p_2, \partial_2\partial_3 p_2, \partial_1\partial_2 p_2 \in \Pi_0^2$. Since $\mathcal{V}_0^{\gamma+e_1} = \mathcal{V}_0^{\gamma+e_2} = \mathcal{V}_0^{\gamma+e_3} = \Pi_0^2$, the desired result holds. If $n = 3$, let $p_3 \in \mathcal{V}_3^{\gamma;\mathrm{Sob}}$. By Theorem 3.3.10, there exist constants $\upsilon_1, \ldots, \upsilon_4 \in \mathbb{R}$ such that

$$p_3 = \upsilon_1\rho_{3,0}^\gamma + \upsilon_2\rho_{3,1}^\gamma + \upsilon_3\rho_{3,2}^\gamma + \upsilon_4\rho_{3,3}^\gamma.$$

From the definition of $\rho_{3,0}^\gamma$ in Definition 3.3.9 and the fact that R^γ maps Π^2 to Π_1^2, we learn that

$$\partial_1\partial_2\rho_{3,0}^\gamma = \partial_1\partial_2(J_{3,0}^\gamma - a_1^\gamma J_{2,0}^\gamma) \overset{(2.4.7)}{=} \partial_1\partial_2 J_{3,0}^\gamma - a_1^\gamma(4+|\boldsymbol{\gamma}|)(5+|\boldsymbol{\gamma}|). \tag{3.6.6}$$

Taking the $S_0^{\gamma+e_3}$ projection of the above equation and using the particular form of a_1^γ given in Definition 3.3.9 we find that $S_0^{\gamma+e_3}(\partial_1\partial_2\rho_{3,0}^\gamma) = 0$. Using again the fact that R^γ maps Π^2 to Π_1^2, but now combined with (2.4.5), it readily follows that

$$\partial_1\partial_3\rho_{3,0}^\gamma = 0 \quad \text{and} \quad \partial_2\partial_3\rho_{3,0}^\gamma = 0. \tag{3.6.7}$$

Thus,

$$S_0^{\gamma+e_1}(\partial_1\partial_3\rho_{3,0}^\gamma) = 0, \quad S_0^{\gamma+e_2}(\partial_2\partial_3\rho_{3,0}^\gamma) = 0 \quad \text{and} \quad S_0^{\gamma+e_3}(\partial_1\partial_2\rho_{3,0}^\gamma) = 0. \tag{3.6.8}$$

Now, from definition of $\rho^{\gamma}_{3,1}$ in Definition 3.3.9 and the fact that R^{γ} maps Π^2 to Π^2_1, we deduce that

$$\partial_2\partial_3\rho^{\gamma}_{3,1} = \partial_2\partial_3 T^{\star}_{\sigma_1}\rho^{\sigma_2(\gamma)}_{3,0} \overset{\text{Prop.2.2.9(i)}}{=} T^{\star}_{\sigma_1}\partial_1\partial_2\rho^{\sigma_2(\gamma)}_{3,0} \overset{(3.6.6)}{=} T^{\star}_{\sigma_1}\partial_1\partial_2 J^{\sigma_2(\gamma)}_{3,0} - a_1^{\sigma_2(\gamma)}(4+|\boldsymbol{\gamma}|)(5+|\boldsymbol{\gamma}|).$$

Taking the $S_0^{\gamma+e_2}$ projection of the above equation and using the particular form of $a_1^{\sigma_2(\gamma)}$ given in Definition 3.3.9 we find that

$$\begin{aligned} S_0^{\gamma+e_2}(\partial_2\partial_3\rho^{\gamma}_{3,1}) &= S_0^{\gamma+e_2}(T^{\star}_{\sigma_1}\partial_1\partial_2 J^{\sigma_2(\gamma)}_{3,0}) - a_1^{\sigma_2(\gamma)}(4+|\boldsymbol{\gamma}|)(5+|\boldsymbol{\gamma}|) \\ &= b^{-1}_{\gamma+e_2}\langle T^{\star}_{\sigma_1}\partial_1\partial_2 J^{\sigma_2(\gamma)}_{3,0}, 1\rangle_{\gamma+e_2} - a_1^{\sigma_2(\gamma)}(4+|\boldsymbol{\gamma}|)(5+|\boldsymbol{\gamma}|) \\ &\overset{\text{Prop.2.2.6(ii)}}{=} b^{-1}_{\sigma_2(\gamma)+e_3}\langle \partial_1\partial_2 J^{\sigma_2(\gamma)}_{3,0}, 1\rangle_{\sigma_2(\gamma)+e_3} - a_1^{\sigma_2(\gamma)}(4+|\boldsymbol{\gamma}|)(5+|\boldsymbol{\gamma}|) \\ &= S_0^{\sigma_2(\gamma)+e_3}(\partial_1\partial_2 J^{\sigma_2(\gamma)}_{3,0}) - a_1^{\sigma_2(\gamma)}(4+|\boldsymbol{\gamma}|)(5+|\boldsymbol{\gamma}|) = 0. \end{aligned}$$

Using again the fact that R^{γ} maps Π^2 to Π^2_1, but now combined with (3.6.7), it readily follows that

$$\begin{aligned} \partial_1\partial_2\rho^{\gamma}_{3,1} &= \partial_1\partial_2 T^{\star}_{\sigma_1}\rho^{\sigma_2(\gamma)}_{3,0} \overset{\text{Prop.2.2.9(i)}}{=} -T^{\star}_{\sigma_1}\partial_1\partial_3\rho^{\sigma_2(\gamma)}_{3,0} = 0. \\ \partial_1\partial_3\rho^{\gamma}_{3,1} &= \partial_1\partial_3 T^{\star}_{\sigma_1}\rho^{\sigma_2(\gamma)}_{3,0} \overset{\text{Prop.2.2.9(i)}}{=} -T^{\star}_{\sigma_1}\partial_2\partial_3\rho^{\sigma_2(\gamma)}_{3,0} = 0. \end{aligned}$$

Thus,

$$S_0^{\gamma+e_1}(\partial_1\partial_3\rho^{\gamma}_{3,1}) = 0, \quad S_0^{\gamma+e_2}(\partial_2\partial_3\rho^{\gamma}_{3,1}) = 0 \quad \text{and} \quad S_0^{\gamma+e_3}(\partial_1\partial_2\rho^{\gamma}_{3,1}) = 0. \tag{3.6.9}$$

Using the same arguments, we also find that

$$S_0^{\gamma+e_1}(\partial_1\partial_3\rho^{\gamma}_{3,2}) = 0, \quad S_0^{\gamma+e_2}(\partial_2\partial_3\rho^{\gamma}_{3,2}) = 0 \quad \text{and} \quad S_0^{\gamma+e_3}(\partial_1\partial_2\rho^{\gamma}_{3,2}) = 0. \tag{3.6.10}$$

Furthermore, from the definition of $\rho^{\gamma}_{3,3}$ in Definition 3.3.9, the fact that R^{γ} maps Π^2 to Π^2_1 and (3.3.3), we deduce that

$$S_0^{\gamma+e_1}(\partial_1\partial_3\rho^{\gamma}_{3,3}) = 0, \quad S_0^{\gamma+e_2}(\partial_2\partial_3\rho^{\gamma}_{3,3}) = 0 \quad \text{and} \quad S_0^{\gamma+e_3}(\partial_1\partial_2\rho^{\gamma}_{3,3}) = 0. \tag{3.6.11}$$

Then, by comibining (3.6.8), (3.6.9), (3.6.10) and (3.6.11) we deduce that

$$S_0^{\gamma+e_1}(\partial_1\partial_3 p_3) = 0, \quad S_0^{\gamma+e_2}(\partial_2\partial_3 p_3) = 0 \quad \text{and} \quad S_0^{\gamma+e_3}(\partial_1\partial_2 p_3) = 0.$$

This, combined with the fact that for all $q \in \Pi_0^2$,

$$\langle \partial_1\partial_3 p_3, q\rangle_{\gamma+e_1} = b_{\gamma+e_1} S_0^{\gamma+e_1}(\partial_1\partial_3 p_3)q, \quad \langle \partial_2\partial_3 p_3, q\rangle_{\gamma+e_2} = b_{\gamma+e_2} S_0^{\gamma+e_2}(\partial_2\partial_3 p_3)q$$
$$\text{and} \quad \langle \partial_1\partial_2 p_3, q\rangle_{\gamma+e_3} = b_{\gamma+e_3} S_0^{\gamma+e_3}(\partial_1\partial_2 p_3)q,$$

establishes the result for $n = 3$. Finally, suppose $n \geq 4$. Let $p_n \in \mathcal{V}_n^{\gamma;\mathrm{Sob}}$. According to Theorem 3.1.1, there exist $v_{n-3} \in \mathcal{V}_{n-3}^{\gamma+e_1+e_2+e_3}$ and $w_n \in \mathcal{J}_n^{\gamma}$ such that

$$p_n = w_n + \mathrm{M}^{\gamma}(v_{n-3}).$$

From the same theorem, it follows that $\partial_1\partial_3 w_n \in \mathcal{V}_{n-2}^{\gamma+e_1}$, $\partial_2\partial_3 w_n \in \mathcal{V}_{n-2}^{\gamma+e_2}$ and $\partial_1\partial_2 w_n \in \mathcal{V}_{n-2}^{\gamma+e_3}$. Furthermore, Proposition 3.4.17 and (3.1.4) imply that

$$\partial_1\partial_3 \mathrm{M}^{\gamma}(v_{n-3}) \in \mathcal{V}_{n-2}^{\gamma+e_1}, \quad \partial_2\partial_3 \mathrm{M}^{\gamma}(v_{n-3}) \in \mathcal{V}_{n-2}^{\gamma+e_2} \quad \text{and} \quad \partial_1\partial_2 \mathrm{M}^{\gamma}(v_{n-3}) \in \mathcal{V}_{n-2}^{\gamma+e_3}.$$

Consequently, the desired result holds for $n \geq 4$. □

Definition 3.6.9. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$. We define the bilinear form $B^{\gamma;\mathrm{Sob}} : \mathrm{C}^3(\overline{\mathrm{T}}) \times \mathrm{C}^3(\overline{\mathrm{T}}) \longrightarrow \mathbb{R}$ in terms of the bilinear form B^{γ} of* (2.2.10) *as follows:*

$$B^{\gamma;\mathrm{Sob}}(p,q) := B^{\gamma+e_1}(\partial_1\partial_3 p, \partial_1\partial_3 q) + B^{\gamma+e_2}(\partial_2\partial_3 p, \partial_2\partial_3 q) + B^{\gamma+e_3}(\partial_1\partial_2 p, \partial_1\partial_2 q).$$

Theorem 3.6.10. *Let $\boldsymbol{\gamma} \in (-1,\infty)^3$ be such that $\boldsymbol{\gamma} \neq (0,0,0)$, $n \in \mathbb{N}_0$ and $p_n \in \mathcal{V}_n^{\gamma;\mathrm{Sob}}$. Then, for all $q \in \mathrm{C}^3(\overline{\mathrm{T}})$,*

$$B^{\gamma;\mathrm{Sob}}(p_n,q) = \lambda_n^{\gamma;\mathrm{Sob}} \langle p_n, q\rangle_{\mathbb{H}^2_{\gamma}}, \quad \text{where} \quad \lambda_n^{\gamma;\mathrm{Sob}} = \begin{cases} 0 & \text{if } n \leq 1, \\ (n-2)(n+|\boldsymbol{\gamma}|+1) & \text{if } n \geq 2. \end{cases}$$

Proof. By Corollary 3.6.8, it follows that $\partial_1\partial_3 p_n \in \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_1}$, $\partial_2\partial_3 p_n \in \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_2}$ and $\partial_1\partial_2 p_n \in \mathcal{V}_{n-2}^{\boldsymbol{\gamma}+e_3}$. Consequently, using (2.2.11), we obtain

$$\begin{aligned}
B^{\boldsymbol{\gamma}+e_1}(\partial_1\partial_3 p_n, \partial_1\partial_3 q) &= \lambda_{n-2}^{\boldsymbol{\gamma}+e_1}\langle \partial_1\partial_3 p_n, \partial_1\partial_3 q\rangle_{\boldsymbol{\gamma}+e_1},\\
B^{\boldsymbol{\gamma}+e_2}(\partial_2\partial_3 p_n, \partial_2\partial_3 q) &= \lambda_{n-2}^{\boldsymbol{\gamma}+e_2}\langle \partial_2\partial_3 p_n, \partial_2\partial_3 q\rangle_{\boldsymbol{\gamma}+e_2},\\
B^{\boldsymbol{\gamma}+e_3}(\partial_1\partial_2 p_n, \partial_1\partial_2 q) &= \lambda_{n-2}^{\boldsymbol{\gamma}+e_3}\langle \partial_1\partial_2 p_n, \partial_1\partial_2 q\rangle_{\boldsymbol{\gamma}+e_3}.
\end{aligned}$$

From (2.2.9), we observe that $\lambda_n^{\boldsymbol{\gamma};\mathrm{Sob}} = \lambda_{n-2}^{\boldsymbol{\gamma}+e_1} = \lambda_{n-2}^{\boldsymbol{\gamma}+e_2} = \lambda_{n-2}^{\boldsymbol{\gamma}+e_3}$. Therefore,

$$B^{\boldsymbol{\gamma};\mathrm{Sob}}(p_n, q) = \lambda_n^{\boldsymbol{\gamma};\mathrm{Sob}}\mathfrak{B}^{\boldsymbol{\gamma}}(p_n, q). \tag{3.6.12}$$

In the case $n \le 1$, we note that $B^{\boldsymbol{\gamma};\mathrm{Sob}}(p_n, q) = 0$. Hence, by setting $\lambda_n^{\boldsymbol{\gamma};\mathrm{Sob}} = 0$, the desired result holds. Finally, for $n \geq 2$, Lemma 3.3.1 implies that $S_0^{\boldsymbol{\gamma}}(\bar{\nabla} p_n) = 0$ and $S_0^{\boldsymbol{\gamma}}(p_n) = 0$. This, combined with (3.1.2) and (3.6.12), establishes the desired result. ☐

Definition 3.6.11. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$. *We define the differential operator* $\tilde{\mathcal{L}}^{\boldsymbol{\gamma}} : \mathrm{C}^2(\overline{\mathrm{T}}) \longrightarrow \mathrm{C}(\overline{\mathrm{T}})$ *as follows:*

$$\tilde{\mathcal{L}}^{\boldsymbol{\gamma}}(p) = \mathcal{L}^{\boldsymbol{\gamma}-e_1-e_2-e_3}(p) + \frac{\mathrm{proj}_0^{\boldsymbol{\gamma}}(\varepsilon_1^{\boldsymbol{\gamma}}(p))}{|\boldsymbol{\gamma}|+3} T_{\sigma_2}^{\star} J_{1,0}^{\sigma_1(\boldsymbol{\gamma})} + \frac{\mathrm{proj}_0^{\boldsymbol{\gamma}}(\varepsilon_2^{\boldsymbol{\gamma}}(p))}{|\boldsymbol{\gamma}|+3} T_{\sigma_1}^{\star} J_{1,0}^{\sigma_2(\boldsymbol{\gamma})} - \mathrm{proj}_0^{\boldsymbol{\gamma}}(\boldsymbol{\delta}\cdot\nabla p),$$

where $\varepsilon_1^{\boldsymbol{\gamma}}, \varepsilon_2^{\boldsymbol{\gamma}} : \mathrm{C}^2(\overline{\mathrm{T}}) \longrightarrow \mathrm{C}(\overline{\mathrm{T}})$ *and* $\boldsymbol{\delta} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *are given by*

$$\varepsilon_1^{\boldsymbol{\gamma}}(p) := |\boldsymbol{\gamma}|\,\partial_1 p + \mathrm{W}_{e_3}\partial_1\partial_2 p + \mathrm{W}_{e_1}\partial_1\partial_3 p, \quad \varepsilon_2^{\boldsymbol{\gamma}}(p) := |\boldsymbol{\gamma}|\,\partial_2 p + \mathrm{W}_{e_3}\partial_1\partial_2 p - \mathrm{W}_{e_2}\partial_2\partial_3 p$$
$$\text{and} \quad \boldsymbol{\delta}(x) := (1-3x_1, 1-3x_2)^t.$$

Proposition 3.6.12. *Let* $\boldsymbol{\gamma} \in \mathbb{R}^3$. *Then,* $\mathcal{L}^{\boldsymbol{\gamma}} = \mathcal{L}^{\boldsymbol{\gamma}-e_1-e_2-e_3} - \boldsymbol{\delta}\cdot\nabla$.

Proof. Let $j \in \{1,2\}$. From (2.2.5), we can express $d_j^{\boldsymbol{\gamma}}$ as

$$d_j^{\boldsymbol{\gamma}} = \Big[(\gamma_3+1)\mathrm{W}_{e_j} - (\gamma_j+1)\mathrm{W}_{e_3}\Big]\,\mathrm{I} - \mathrm{W}_{e_j+e_3}\partial_j = d_j^{\boldsymbol{\gamma}-e_1-e_2-e_3} + \mathrm{W}_{e_j}\mathrm{I} - \mathrm{W}_{e_3}\mathrm{I}. \tag{3.6.13}$$

Likewise, using (2.2.6), we find that

$$d_3^{\gamma} = [(\gamma_1+1)\mathrm{W}_{e_2} - (\gamma_2+1)\mathrm{W}_{e_1}]\,\mathrm{I} - \mathrm{W}_{e_1+e_2}\partial_3 = d_3^{\gamma-e_1-e_2-e_3} + \mathrm{W}_{e_2}\mathrm{I} - \mathrm{W}_{e_1}\mathrm{I}. \tag{3.6.14}$$

Substituting (3.6.13) and (3.6.14) into (2.2.7) and simplifying, we arrive at the desired result.

□

Proposition 3.6.13. *Let* $\boldsymbol{\gamma} \in \mathbb{R}^3$*. Then,* $\nabla\mathcal{L}^{\gamma-e_1-e_2-e_3} = \mathcal{L}^{\gamma}\nabla + \boldsymbol{\varepsilon}^{\gamma}$*, where* $\boldsymbol{\varepsilon}^{\gamma} = (\varepsilon_1^{\gamma}, \varepsilon_2^{\gamma})^t$*.*

Proof. From (2.2.7), we have

$$\partial_1\mathcal{L}^{\gamma-e_1-e_2-e_3} = \partial_1 d_1^{\gamma-e_1-e_2-e_3}\partial_1 + \partial_1 d_2^{\gamma-e_1-e_2-e_3}\partial_2 + \partial_1 d_3^{\gamma-e_1-e_2-e_3}\partial_3 \tag{3.6.15}$$

Applying (i) of Proposition 3.4.12, the first term on the right-hand side of (3.6.15) can be written as:

$$\partial_1 d_1^{\gamma-e_1-e_2-e_3}\partial_1 = d_1^{\gamma}\partial_1^2 + (\gamma_1+\gamma_3)\partial_1. \tag{3.6.16}$$

Next, using (2.2.5) and standard differentiation rules, the second term on the right-hand side of (3.6.15) becomes:

$$\partial_1 d_2^{\gamma-e_1-e_2-e_3}\partial_2 = d_2^{\gamma}\partial_1\partial_2 + \mathrm{W}_{e_2}\partial_3\partial_2 + \mathrm{W}_{e_3}\partial_1\partial_2 + \gamma_2\partial_2. \tag{3.6.17}$$

Similarly, for the last term on the right-hand side of (3.6.15), we obtain:

$$\partial_1 d_3^{\gamma-e_1-e_2-e_3}\partial_3 = d_3^{\gamma}\partial_1\partial_3 - \mathrm{W}_{e_2}\partial_2\partial_3 + \mathrm{W}_{e_1}\partial_1\partial_3 - \gamma_2\partial_3. \tag{3.6.18}$$

Substituting (3.6.16), (3.6.17) and (3.6.18) into (3.6.15) and simplifying, we obtain:

$$\begin{aligned}\partial_1\mathcal{L}^{\gamma-e_1-e_2-e_3} &= \mathcal{L}^{\gamma}\partial_1 + (\gamma_1+\gamma_3)\partial_1 + \mathrm{W}_{e_2}\partial_3\partial_2 + \mathrm{W}_{e_3}\partial_1\partial_2 + \gamma_2\partial_2 - \mathrm{W}_{e_2}\partial_2\partial_3 + \mathrm{W}_{e_1}\partial_1\partial_3 - \gamma_2\partial_3\\ &= \mathcal{L}^{\gamma}\partial_1 + (\gamma_1+\gamma_2+\gamma_3)\partial_1 + \mathrm{W}_{e_3}\partial_1\partial_2 + \mathrm{W}_{e_1}\partial_1\partial_3. \end{aligned} \tag{3.6.19}$$

Similarly, from (2.2.7), we have

$$\partial_2\mathcal{L}^{\gamma-e_1-e_2-e_3} = \partial_2 d_1^{\gamma-e_1-e_2-e_3}\partial_1 + \partial_2 d_2^{\gamma-e_1-e_2-e_3}\partial_2 + \partial_2 d_3^{\gamma-e_1-e_2-e_3}\partial_3 \tag{3.6.20}$$

Using (2.2.5) and differentiation rules, the first term on the right-hand side of (3.6.20) can be expressed as:

$$\partial_2 d_1^{\gamma-e_1-e_2-e_3}\partial_1 = d_1^{\gamma}\partial_2\partial_1 - \mathrm{W}_{e_1}\partial_3\partial_1 + \mathrm{W}_{e_3}\partial_2\partial_1 + \gamma_1\partial_1. \tag{3.6.21}$$

Now, by applying (ii) of Proposition 3.4.12, the second term on the right-hand side of (3.6.20) becomes:

$$\partial_2 d_2^{\gamma-e_1-e_2-e_3}\partial_2 = d_2^{\gamma}\partial_2^2 + (\gamma_2+\gamma_3)\partial_2. \tag{3.6.22}$$

Likewise, for the last term on the right-hand side of (3.6.20), we obtain:

$$\partial_2 d_3^{\gamma-e_1-e_2-e_3}\partial_3 = d_3^{\gamma}\partial_2\partial_3 + \mathrm{W}_{e_1}\partial_1\partial_3 - \mathrm{W}_{e_2}\partial_2\partial_3 + \gamma_1\partial_3. \tag{3.6.23}$$

Substituting (3.6.21), (3.6.22) and (3.6.23) into (3.6.20) and simplifying, we obtain:

$$\begin{aligned}\partial_2\mathcal{L}^{\gamma-e_1-e_2-e_3} &= \mathcal{L}^{\gamma}\partial_2 + (\gamma_2+\gamma_3)\partial_2 - \mathrm{W}_{e_1}\partial_3\partial_1 + \mathrm{W}_{e_3}\partial_2\partial_1 + \gamma_1\partial_1 + \mathrm{W}_{e_1}\partial_1\partial_3 - \mathrm{W}_{e_2}\partial_2\partial_3 + \gamma_1\partial_3\\ &= \mathcal{L}^{\gamma}\partial_2 + (\gamma_1+\gamma_2+\gamma_3)\partial_2 + \mathrm{W}_{e_3}\partial_1\partial_2 - \mathrm{W}_{e_2}\partial_2\partial_3. \end{aligned}\tag{3.6.24}$$

Combining (3.6.19) and (3.6.24), the desired result follows. □

Proposition 3.6.14. *Let $\boldsymbol{\gamma} \in \mathbb{R}^3$. Then, the following hold:*

(i) $\mathcal{L}^{\gamma}\partial_1 = \partial_1\mathcal{L}^{\gamma-e_1-e_3} - (1+|\boldsymbol{\gamma}|)\partial_1$.

(ii) $\mathcal{L}^{\gamma}\partial_2 = \partial_2\mathcal{L}^{\gamma-e_2-e_3} - (1+|\boldsymbol{\gamma}|)\partial_2$.

(iii) $\mathcal{L}^{\gamma}\partial_3 = \partial_3\mathcal{L}^{\gamma-e_1-e_2} - (1+|\boldsymbol{\gamma}|)\partial_3$.

Proof. From (2.2.7), we have

$$\partial_1\mathcal{L}^{\gamma-e_1-e_3} = \partial_1 d_1^{\gamma-e_1-e_3}\partial_1 + \partial_1 d_2^{\gamma-e_1-e_3}\partial_2 + \partial_1 d_3^{\gamma-e_1-e_3}\partial_3 \tag{3.6.25}$$

Using (i) of Proposition 3.4.12, the first term on the right-hand side of (3.6.25) can be written as follows:

$$\partial_1 d_1^{\gamma-e_1-e_3}\partial_1 = d_1^{\gamma}\partial_1^2 + (\gamma_1+\gamma_3)\partial_1. \tag{3.6.26}$$

Next, by applying (2.2.5) and standard calculus rules, the second term on the right-hand side of (3.6.25) becomes:

$$\partial_1 d_2^{\gamma-e_1-e_3}\partial_2 = d_2^{\gamma}\partial_1\partial_2 + \mathrm{W}_{e_2}\partial_2\partial_3 + (\gamma_2+1)\partial_2. \tag{3.6.27}$$

Similarly, for the last term on the right-hand side of (3.6.25), we obtain

$$\partial_1 d_3^{\gamma-e_1-e_3}\partial_3 = d_3^{\gamma}\partial_1\partial_3 - \mathrm{W}_{e_2}\partial_2\partial_3 - (\gamma_2+1)\partial_3. \tag{3.6.28}$$

Substituting (3.6.26), (3.6.27) and (3.6.28) into (3.6.25) and performing some algebraic manipulations, we obtain part (i). Finally, using similar arguments, we can prove (ii) and (iii). □

Proposition 3.6.15. *Let* $\boldsymbol{\gamma} \in \mathbb{R}^3$*. Then, the following hold:*

(i) $\mathcal{L}^{\gamma}\partial_1\partial_2 = \partial_1\partial_2\mathcal{L}^{\gamma-e_1-e_2-2e_3} - 2\,|\boldsymbol{\gamma}|\,\partial_1\partial_2$.

(ii) $\mathcal{L}^{\gamma}\partial_1\partial_3 = \partial_1\partial_3\mathcal{L}^{\gamma-2e_1-e_2-e_3} - 2\,|\boldsymbol{\gamma}|\,\partial_1\partial_3$.

(iii) $\mathcal{L}^{\gamma}\partial_2\partial_3 = \partial_2\partial_3\mathcal{L}^{\gamma-e_1-2e_2-e_3} - 2\,|\boldsymbol{\gamma}|\,\partial_2\partial_3$.

Proof. Applying parts (i) and (ii) of Proposition 3.6.14 we obtain

$$\begin{aligned}\mathcal{L}^{\gamma}\partial_1\partial_2 = \partial_1\mathcal{L}^{\gamma-e_1-e_3}\partial_2 - (1+|\boldsymbol{\gamma}|)\partial_1\partial_2 &= \partial_1\partial_2\mathcal{L}^{\gamma-e_1-e_2-2e_3} - (|\boldsymbol{\gamma}|-1)\partial_1\partial_2 - (1+|\boldsymbol{\gamma}|)\partial_1\partial_2\\ &= \partial_1\partial_2\mathcal{L}^{\gamma-e_1-e_2-2e_3} - 2\,|\boldsymbol{\gamma}|\,\partial_1\partial_2.\end{aligned}$$

Similarly, by invoking (i) and (iii) of Proposition 3.6.14, it follows that

$$\begin{aligned}\mathcal{L}^{\gamma}\partial_1\partial_3 = \partial_1\mathcal{L}^{\gamma-e_1-e_3}\partial_3 - (1+|\boldsymbol{\gamma}|)\partial_1\partial_3 &= \partial_1\partial_3\mathcal{L}^{\gamma-2e_1-e_2-e_3} - (|\boldsymbol{\gamma}|-1)\partial_1\partial_3 - (1+|\boldsymbol{\gamma}|)\partial_1\partial_3\\ &= \partial_1\partial_3\mathcal{L}^{\gamma-2e_1-e_2-e_3} - 2\,|\boldsymbol{\gamma}|\,\partial_1\partial_3.\end{aligned}$$

Finally, using (ii) and (iii) of Proposition 3.6.14 we have

$$\begin{aligned}\mathcal{L}^{\boldsymbol{\gamma}}\partial_2\partial_3 = \partial_2\mathcal{L}^{\boldsymbol{\gamma}-e_2-e_3}\partial_3 - (1+|\boldsymbol{\gamma}|)\partial_2\partial_3 = \partial_2\partial_3\mathcal{L}^{\boldsymbol{\gamma}-e_1-2e_2-e_3} - (|\boldsymbol{\gamma}|-1)\partial_2\partial_3 - (1+|\boldsymbol{\gamma}|)\partial_2\partial_3 \\ = \partial_2\partial_3\mathcal{L}^{\boldsymbol{\gamma}-e_1-2e_2-e_3} - 2\,|\boldsymbol{\gamma}|\,\partial_2\partial_3.\end{aligned}$$

□

In what follows, we will employ the previous results to study some properties of the operator $\tilde{\mathcal{L}}^{\boldsymbol{\gamma}}$, which in turn will allow us to prove Theorem 3.1.2.

Proposition 3.6.16. *Let* $\boldsymbol{\gamma} \in (-1,\infty)^3$ *be such that* $\boldsymbol{\gamma} \neq (0,0,0)$. *Then, the operator* $\tilde{\mathcal{L}}^{\boldsymbol{\gamma}}$ *is self-adjoint in* $\mathbb{H}^2_{\boldsymbol{\gamma}}$ *with respect to the* $\langle\cdot,\cdot\rangle_{\mathbb{H}^2_{\boldsymbol{\gamma}}}$*-inner product; indeed, for all* $p \in \mathrm{C}^4(\overline{\mathrm{T}})$ *and* $q \in \mathrm{C}^2(\overline{\mathrm{T}})$,

$$\langle\tilde{\mathcal{L}}^{\boldsymbol{\gamma}}p, q\rangle_{\mathbb{H}^2_{\boldsymbol{\gamma}}} = B^{\boldsymbol{\gamma};\mathrm{Sob}}(p,q) + 2(|\boldsymbol{\gamma}|+1)\mathfrak{B}^{\boldsymbol{\gamma}}(p,q).$$

Proof. From Proposition 3.6.15, we obtain

$$\mathfrak{B}^{\boldsymbol{\gamma}}(\mathcal{L}^{\boldsymbol{\gamma}-e_1-e_2-e_3}p, q) = B^{\boldsymbol{\gamma};\mathrm{Sob}}(p,q) + 2(|\boldsymbol{\gamma}|+1)\mathfrak{B}^{\boldsymbol{\gamma}}(p,q).$$

Next, combining Definition 3.6.11, Proposition 3.6.13, (2.4.6), Proposition 2.2.9 and (2.4.7), we find that

$$\nabla\tilde{\mathcal{L}}^{\boldsymbol{\gamma}}p = \nabla\mathcal{L}^{\boldsymbol{\gamma}-e_1-e_2-e_3}p - \mathrm{proj}^{\boldsymbol{\gamma}}_0(\boldsymbol{\varepsilon}^{\boldsymbol{\gamma}}(p)) = \mathcal{L}^{\boldsymbol{\gamma}}\nabla p + \boldsymbol{\varepsilon}^{\boldsymbol{\gamma}}(p) - \mathrm{proj}^{\boldsymbol{\gamma}}_0(\boldsymbol{\varepsilon}^{\boldsymbol{\gamma}}(p)).$$

Consequently, using the relation above and the self-adjointness of $\mathcal{L}^{\boldsymbol{\gamma}}$, we deduce that

$$S^{\boldsymbol{\gamma}}_0(\partial_i\tilde{\mathcal{L}}^{\boldsymbol{\gamma}}p) = S^{\boldsymbol{\gamma}}_0(\mathcal{L}^{\boldsymbol{\gamma}}\partial_i p) = \frac{\langle\mathcal{L}^{\boldsymbol{\gamma}}\partial_i p, 1\rangle_{\boldsymbol{\gamma}}}{\|1\|^2_{\boldsymbol{\gamma}}} = \frac{\langle\partial_i p, \mathcal{L}^{\boldsymbol{\gamma}}(1)\rangle_{\boldsymbol{\gamma}}}{\|1\|^2_{\boldsymbol{\gamma}}} = 0,$$

for all $i \in \{1,2\}$. Moreover, since $\partial_3 = \partial_2 - \partial_1$, it follows that $S^{\boldsymbol{\gamma}}_0(\bar{\nabla}\tilde{\mathcal{L}}^{\boldsymbol{\gamma}}p) = 0$.

On the other hand, part (iii) of Proposition 2.2.6 implies that $T^{\star}_{\sigma_1}J^{\sigma_2(\boldsymbol{\gamma})}_{1,0}, T^{\star}_{\sigma_2}J^{\sigma_1(\boldsymbol{\gamma})}_{1,0} \in \mathcal{V}^{\boldsymbol{\gamma}}_1$.

Therefore, applying Definition 3.6.11 and Proposition 3.6.12, we compute:

$$S_0^{\gamma}(\tilde{\mathcal{L}}^{\gamma}p) = S_0^{\gamma}(\mathcal{L}^{\gamma-e_1-e_2-e_3}(p)) - S_0^{\gamma}(\boldsymbol{\delta}\cdot\nabla p) = S_0^{\gamma}(\mathcal{L}^{\gamma}(p)) = \frac{\langle \mathcal{L}^{\gamma}(p), 1\rangle_{\gamma}}{\|1\|_{\gamma}^2} = \frac{\langle p, \mathcal{L}^{\gamma}(1)\rangle_{\gamma}}{\|1\|_{\gamma}^2} = 0,$$

and the desired result follows from the structure (3.1.2) of the $\langle\cdot,\cdot\rangle_{\mathbb{H}^2_{\gamma}}$-inner product. ☐

Proof of Theorem 3.1.2. From Theorem 3.6.10 and Proposition 3.6.16, for any polynomial q

$$\langle \tilde{\mathcal{L}}^{\gamma}(p_n), q\rangle_{\mathbb{H}^2_{\gamma}} = B^{\gamma;\mathrm{Sob}}(p_n,q) + 2(|\boldsymbol{\gamma}|+1)\mathfrak{B}^{\gamma}(p_n,q) = \lambda_n^{\gamma;\mathrm{Sob}}\langle p_n, q\rangle_{\mathbb{H}^2_{\gamma}} + 2(|\boldsymbol{\gamma}|+1)\mathfrak{B}^{\gamma}(p_n,q).$$

If $n\le 1$, it is straightforward to see that $\tilde{\lambda}_n^{\gamma}=0$, while if $n\ge 2$, from Lemma 3.3.1 we have

$$\langle \tilde{\mathcal{L}}^{\gamma}(p_n), q\rangle_{\mathbb{H}^2_{\gamma}} = (\lambda_n^{\gamma;\mathrm{Sob}} + 2(|\boldsymbol{\gamma}|+1))\langle p_n, q\rangle_{\mathbb{H}^2_{\gamma}} = n(n+|\boldsymbol{\gamma}|-1)\langle p_n, q\rangle_{\mathbb{H}^2_{\gamma}}.$$

As, $\tilde{\mathcal{L}}^{\gamma}(p_n)$ itself a polynomial (cf. Definition 3.6.11), the desired result follows. ☐

We conclude this section by providing some comparisons between the results obtained in this chapter and those presented in [11].

Regarding Theorem 3.1.1, we observe that a direct sum decomposition is obtained in the triangle, in contrast to the case in the ball [11, Lem. 4.2], where an orthogonal decomposition is achieved. Furthermore, it is worth noting that M^{γ} (cf. (3.4.18)) is a third-order differential operator, whereas its counterpart M^{α} (cf. [11, Eq. (38)]) is a second-order differential operator.

On the other hand, concerning the space $\mathcal{J}_n^{\gamma}$ in Theorem 3.1.1, we obtained a characterization that depends on whether the parameter $\boldsymbol{\gamma}$ has any vanishing component. In contrast, this distinction does not occur in the space $\mathcal{H}_n^d$ (cf. [11, Sec. 2]) from [11, Lem. 4.2].

When comparing Theorem 3.6.10 with its counterpart in the ball [11, Thm. 3.11], it is observed that the eigenvalues associated with the weak problem in the triangle are non-zero for $n\ge 3$, while in the ball they are non-zero for $n\ge 2$. This discrepancy mainly stems from the fact that a second-order Sobolev inner product is employed in the triangle (cf. (3.1.2)), whereas a first-order Sobolev inner product was used in the ball (cf. [11, Eq. (30)]).

Finally, regarding Theorem 3.1.2 and its counterpart in the ball [11, Thm. 3.14], we observe that both Sturm-Liouville operators arise from a classical operator with shifted parameters, minus a finite-rank perturbation. Specifically, in the triangle, (2.2.7) is used with the shift $\boldsymbol{\gamma} \to \boldsymbol{\gamma} - e_1 - e_2 - e_3$, while in the ball [11, Eq. (11)] is employed with the shift $\alpha \to \alpha - 1$. Nevertheless, the perturbation in the triangle is of rank 2 (cf. Definition 3.6.11), unlike in the ball, where it is of rank 1 (cf. [11, Eq. (45)]). This difference is due to the fact that in the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}^2_{\boldsymbol{\gamma}}}$ (cf. (3.1.2)), the low-order terms involve first-order derivatives, whereas in the inner product in the ball $\langle \cdot, \cdot \rangle_{\gamma,1}$ (cf. [11, Eq. (30)]), these terms do not involve derivatives.

Chapter 4

Conclusions and future work

4.1 Conclusions

In Chapter 2, we define the affine maps T_σ, which allow us to exploit the covariance of the polynomial spaces $\mathcal{V}_n^\gamma$ and use this property to prove Lemma 2.5.19. This leads to a family of integral inequalities on the simplex, which in turn enables us to derive an upper bound for the approximation error of the orthogonal projector S_n^γ in Sobolev norms. Unlike its analogue in the unit ball, our bound exhibits a slower approximation rate with respect to the power on n. The reason is that our Schur-type inequality in part (iii) of Lemma 2.5.3 depends cubically on the total polynomial degree, whereas in the case of the ball the dependence is linear [9, Prop. 3.4(i)].

In Chapter 3, we introduce the operators m^γ and M^γ, study their properties, and use them to characterize the space $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$ in terms of the images of these operators restricted to the orthogonal polynomial spaces defined in (2.2.1). Finally, we define a second-order differential operator $\tilde{\mathcal{L}}^\gamma$ and prove that the polynomials in $\mathcal{V}_n^{\gamma;\mathrm{Sob}}$ are eigenfunctions of this operator.

4.2 Future work

Possible future research directions derived from this thesis include:

- Provide an intrinsic characterization of the weighted Sobolev space H^m_γ of Definition 2.6.1.
- Establishing the Markov inequality for a broader class of weights.
- Improving the decay rate established in Theorem 2.1.1.
- Estimating the approximation error, in the $\|\cdot\|_{\gamma;1}$ norm, of the orthogonal projector associated with $\langle\cdot,\cdot\rangle_{\mathbb{H}^2_\gamma}$.

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