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Facultad de Ciencias Físicas y Matemática
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Maxwell Extensions of Kinematical Algebras via Semigroup Expansions and Their Chern–Simons Gravity Realizations

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Dedicado a mi familia

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Abstract

In this thesis, we present a Maxwell extension of the kinematical Lie algebras by promoting the Bacry–Lévy-Leblond cube to a semigroup expansion framework. Within this approach, we show that both non- and ultra-relativistic Maxwell algebras admitting non-degenerate invariant bilinear forms can be systematically obtained from different parent algebras through a unified expansion scheme, leading to a Maxwellian kinematical cube. We further show that both the original Bacry–Lévy-Leblond cube and its Maxwellian extension belong to an infinite hierarchy of generalized kinematical algebras generated by higher-order semigroups. The expansion method naturally provides the corresponding invariant tensors, allowing for the systematic construction of three-dimensional Chern–Simons gravity theories.

Chapter 1

Introduction

Symmetry principles provide one of the deepest organizing concepts in modern theoretical physics. At the classical level, they determine the structure of the action and the associated conservation laws. At the quantum level, they often ensure consistency, renormalizability, and predictive power. Gauge symmetry, in particular, underlies the Standard Model of particle physics and offers a unifying language for fundamental interactions.

Gravity, however, occupies a distinguished position within this framework. Although General Relativity is invariant under diffeomorphisms and local Lorentz transformations, it does not fit straightforwardly into the conventional gauge-theoretic paradigm. This tension becomes especially transparent in the first-order formalism, where the vielbein and spin connection resemble components of a gauge connection, yet no four-dimensional action invariant under the full Poincaré group can be constructed from a Lie-algebra-valued connection. This structural obstruction motivates the search for alternative gauge formulations of gravity.

In three spacetime dimensions, the situation simplifies dramatically. Gravity admits a formulation as a genuine Chern–Simons (CS) gauge theory, provided that the underlying symmetry algebra possesses a non-degenerate invariant bilinear form. This requirement is not merely technical: the non-degeneracy of the invariant tensor ensures well-defined field equations and a non-degenerate symplectic structure. Consequently, the classification and construction of Lie algebras admitting such invariant tensors becomes a central problem in the gauge formulation of gravity.

A natural arena to explore these questions is provided by kinematical Lie algebras. Under the assumptions of spacetime homogeneity, isotropy, invariance under parity and time reversal, and the requirement that transformations between inertial observers form non-compact subgroups, Bacry and Lévy-Leblond classified all possible kinematical algebras [1]. These algebras can be organized as the vertices of a cube (see Fig 1.1), where different physical regimes are connected through Inönü–Wigner contractions. Beyond the relativistic Poincaré and AdS algebras, this classification includes non-Lorentzian structures such as Galilei, Newton–Hooke, Carroll, and related algebras.

In recent years, non-Lorentzian symmetry algebras have attracted renewed interest. Non-relativistic symmetries play a central role in holography [2–13], Hořava-Lifshitz gravity [14–19], and effective descriptions of condensed matter systems such as the quantum Hall effect [20–24]. Ultra-relativistic algebras appear in contexts including tachyon condensation [25], warped conformal field theories [26], tensionless strings [27–31], holography in asymptotically flat spacetimes [32–41], asymptotic symmetries [42–44], and black hole physics [45–51].

Despite this broad applicability, non-Lorentzian algebras generically suffer from degeneracies in their invariant bilinear forms. In three-dimensional CS gravity, this degeneracy translates into dynamically undetermined gauge fields. For instance, in the non-relativistic regime a consistent CS action requires the extended Bargmann algebra [52, 53], corresponding to a double central extension of the Galilei algebra. Similarly, in ultra-relativistic settings suitable extensions are required to ensure non-degeneracy [54, 55].

At the relativistic level, the Maxwell algebra was introduced to describe a constant electromagnetic background in Minkowski spacetime [56–59]. It can be understood as an extension and deformation of the Poincaré algebra and has been extensively studied within the three-dimensional CS framework [60–64]. Non-Lorentzian limits of the Maxwell algebra have also been investigated, revealing that additional generators are typically required to avoid degeneracy in the invariant tensor [55, 65].

The standard Bacry–Lévy-Leblond cube is constructed through successive Inönü–Wigner contractions. However, contractions preserve the dimension of the algebra and therefore cannot generate the extended structures required for non-degenerate CS formulations. An alternative procedure is provided by the semigroup expansion (S-expansion) method [66], which systematically enlarges a given Lie algebra by combining it with an abelian semigroup. Remarkably, for appropriate choices of semigroup and subspace decomposition, the Inönü–Wigner contraction can be recovered as a particular case of an $S_E^{(1)}$ expansion [54]. This observation allows one to reinterpret the original kinematical cube as a diagram of expansions rather than contractions.

The main goal of this thesis is to develop a Maxwellian generalization of the Bacry–Lévy-Leblond cube by systematically applying higher-order semigroup expansions. We show that the non-degenerate non-Lorentzian Maxwell algebras introduced in [55, 65] arise naturally within this expanded framework. In particular, we construct a generalized cube in which each edge corresponds to a resonant $S_E^{(2)}$ expansion, ensuring the existence of non-degenerate invariant tensors in all sectors. This approach leads to a unified algebraic picture encompassing relativistic, non-relativistic, and ultra-relativistic Maxwell-type symmetries in three spacetime dimensions.

The thesis is organized as follows. In Chapter 2 we review the basic structure of gauge theories. Chapter 3 reformulates gravity in the language of differential forms. Chapter 4

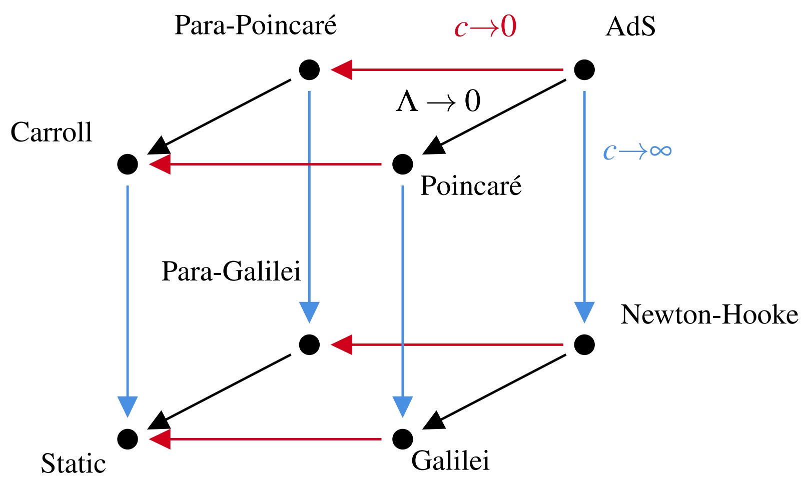

Figure 1.1: We can summarizes different limits from AdS Lie algebra through this cube.

discusses the interpretation of gravity as a gauge theory and clarifies its limitations in four dimensions. Chapter 5 introduces Chern–Simons gravity as a genuine gauge formulation in odd dimensions. Chapter 6 reviews kinematical Lie algebras and the Bacry–Lévy-Leblond cube. Chapter 7 presents the semigroup expansion method and its application to extended kinematical algebras. Chapter 8 analyzes Maxwell Chern–Simons gravity. Chapter 9 develops the Maxwellian extension of the kinematical cube. Finally, Chapter 10 constructs generalized extended kinematical algebras based on higher-order semigroup expansions.

Chapter 2

Gauge Theory

Gauge theories constitute a fundamental framework in modern theoretical physics. They are field theories whose Lagrangians remain invariant under local transformations belonging to continuous symmetry groups, known as Lie groups. This local invariance (called gauge invariance) ensures that the physical content of the theory is independent of arbitrary choices in redundant degrees of freedom, referred to as gauges. The transformations connecting different gauges form the gauge group, whose generators are associated with specific gauge fields that preserve the invariance of the Lagrangian. Upon quantization, the excitations of these fields are interpreted as gauge bosons.

The distinction between global and local symmetries is central to gauge theory. While global symmetries act identically at all points in spacetime, local symmetries allow the transformation parameters to vary with position and time, imposing a stronger constraint on the theory.

Abelian gauge theories, such as quantum electrodynamics (QED), are based on the $U(1)$ group and describe the electromagnetic interaction, where the photon emerges as the gauge boson. The Standard Model extends this principle to non-Abelian groups, $U(1) \times SU(2) \times SU(3)$, providing an unified description electromagnetic, weak, and strong interactions. Gauge principles also appear in gravitation: general relativity can be interpreted as a gauge theory under spacetime diffeomorphisms, with the graviton as the corresponding gauge boson.

Originally inspired by classical electromagnetism and later extended to quantum field theory, gauge invariance remains a cornerstone of high-energy physics and our understanding of fundamental interactions.

2.1 How General Relativity inspired Weyl

To understand how General Relativity inspired Hermann Weyl to propose his revolutionary idea of gauge invariance in 1918, it is essential to recall the fundamental principle underlying both Special and General Relativity: there are no absolute reference frames in nature. Accordingly, physical laws must be independent of the observer's choice of reference frame.

Special Relativity. Consider a particle P moving with constant velocity V with respect to an inertial frame S, while being at rest in another inertial frame S', which itself moves with velocity V relative to S. An observer in S therefore measures the velocity of the particle as V, while an observer in S' measures it as 0. The frames S and S' are related by a Lorentz transformation, which depends only on the relative velocity between the observer and the particle, and not on their position in spacetime. Thus, the Lorentz group is a typical example of a *global symmetry*.

General Relativity. In General Relativity, the motion of a particle takes place in a gravitational field. According to the equivalence principle, motion in a gravitational field is locally indistinguishable from motion in a non–inertial reference frame. Since the geometry of non–inertial frames is non–Euclidean, the motion in a gravitational field is naturally described as motion on a curved manifold.

Example 1. Consider a particle moving in a gravitational field and an observer inside a freely falling elevator. Because both the observer and the particle fall under the same gravitational acceleration, the observer describes the motion of the particle as if the gravitational field were absent.

Example 2. An observer in a global inertial frame of Special Relativity, however, will state that the elevator corresponds to an accelerated (non–inertial) frame, and therefore the particle is not truly at rest with respect to it. The elevator and the particle do not share identical acceleration at all spacetime points.

Example 3. This illustrates a key difference between Special and General Relativity: in a gravitational field, only *local* inertial frames can be defined. One may imagine a collection of nearby freely falling observers (each in their own elevator), each performing local measurements. A natural question arises: how are the measurements of different local observers related? They cannot, in general, be connected by ordinary Lorentz transformations, because that would imply uniform acceleration independent of position, in contradiction with the spatial variation of a gravitational field.

This implies that the transformations between local frames cannot be linear. Einstein introduced nonlinear coordinate transformations and the concept of a connection in physics:

$$x'^{\mu} = x^{\mu}(x^{\alpha}),$$

for example,

$$x'^{\mu} = x^{\mu} + \Gamma^{\mu}_{\alpha\beta} x^{\alpha} x^{\beta},$$

so that

$$\frac{\partial^2 x'^{\mu}}{\partial x^{\alpha} \partial x^{\beta}} = \Gamma^{\mu}_{\alpha\beta}.$$

Thus, the gravitational field varies from point to point; it is locally equivalent to a non–inertial frame, and is geometrically described by a Riemannian manifold. The equivalence principle identifies each point on the manifold with a local inertial frame, and the relation between different local observers is encoded in a connection, which measures how these frames change relative to one another. The locality of inertial frames is not just a geometric feature, but a physical one, *and it was precisely this principle of locality that led Weyl to the gauge principle*.

In General Relativity, we have seen that the direction of a vector is a relative concept: it depends on the local reference frame from which it is measured. However, the length (or norm) of the vector is the same for all observers. In this sense, the norm of a vector is an absolute quantity in General Relativity. Weyl speculated that if the gravitational interaction can be described in terms of a connection that relates the orientation of local reference frames, then it might also be possible that other fundamental interactions in nature could be described by analogous connections.

Weyl proposed a bold generalization of the relativistic principle: the magnitude of a vector representing a physical quantity should not be an absolute concept, but instead depend on its position in spacetime. This idea requires the introduction of a new connection capable of relating the norms of vectors at different spacetime points.

Let us consider a vector at spacetime position x^{μ} with norm $f(x)$. If the vector is displaced to $x^{\mu} + dx^{\mu}$, its norm becomes

$$f(x+dx) = f(x) + \partial_{\mu} f \, dx^{\mu}.$$

Now introduce a local gauge transformation through a position-dependent scale factor $S(x)$. We define $S(x) = 1$ at the point x, while at the nearby point $x + dx$ we have

$$S(x+dx) = S(x) + \partial_{\mu} S \, dx^{\mu}.$$

Under this transformation, the norm of the vector at the displaced point must be multiplied by the scale factor, since the local unit ruler used to measure lengths has changed between the two points. Thus, the transformed norm becomes

$$\begin{aligned} S(x+dx)\, f(x+dx) &= \Big(S(x) + \partial_{\mu} S \, dx^{\mu}\Big)\Big(f(x) + \partial_{\mu} f \, dx^{\mu}\Big) \\ &= f(x) + \partial_{\mu} f \, dx^{\mu} + \big(\partial_{\mu} S\big) f(x) \, dx^{\mu}, \end{aligned}$$

where we have discarded second-order terms in dx^μ. Therefore,

$$S(x+dx)f(x+dx)-f(x)=\Big(\partial_\mu+\partial_\mu S\Big)f(x)\,dx^\mu.$$

Weyl identified $\partial_\mu S$ with the electromagnetic potential A_μ, since both transform in a similar way under a gauge transformation. Indeed, a second gauge rescaling $S\to S+\Lambda$ leads to

$$\partial_\mu S \ \longrightarrow\ \partial_\mu S+\partial_\mu\Lambda,$$

which implies the transformation law

$$A_\mu \ \longrightarrow\ A_\mu+\partial_\mu\Lambda.$$

Even though Weyl's scale invariance was ultimately in conflict with well–established physical facts, the fundamental idea of a local gauge symmetry survived. At the time, it was already known that Maxwell's theory is invariant under gauge transformations, but this symmetry was regarded as merely an "accidental" redundancy associated with the arbitrariness of the electromagnetic potentials $\mathbf{A}$ and ϕ, since only the electric and magnetic fields were considered observable. Weyl's proposal was the first attempt to elevate gauge invariance to a fundamental physical principle. [67]

2.2 Quantum Mechanics and Gauge Theory

With the advent of quantum mechanics, Weyl and his followers assigned a new meaning to Weyl's original gauge theory. The guiding principle was the assumption that the phase of a wavefunction could be treated as a new local variable. Instead of a change of scale, the gauge transformation was reinterpreted as a local change in the phase of the wavefunction.

$$\psi(\mathbf{x},t)\longrightarrow\psi'(\mathbf{x},t)=\psi(\mathbf{x},t)e^{-ie\lambda}. \tag{2.2.1}$$

Indeed, the Schrödinger equations for a charged particle in a electromagnetic field is given by

$$\left[\frac{1}{2m}\Big(i\nabla-q\mathbf{A}\Big)^2+qV\right]\psi(\mathbf{x},t)=i\frac{\partial\psi}{\partial t}(\mathbf{x},t),$$

this equation can be written as

$$\left[\frac{1}{2m}\Big(i\nabla-q\mathbf{A}\Big)^2+\Big(i\frac{\partial}{\partial t}+qV\Big)\right]\psi(\mathbf{x},t)=0. \tag{2.2.2}$$

Let consider now, the Schrödinger Eq. (2.2.2) behavior under the gauge transformation

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\chi, \qquad V \longrightarrow V' = V - \frac{\partial \chi}{\partial t}. \tag{2.2.3}$$

If the Schrödinger Eq. (2.2.2) is not invariant under (2.2.3) transformations, then the Maxwell equations could enter in a contradiction with the Quantum Mechanics. Let study now, its invariance. Consider

$$\left[\frac{1}{2m}\left(i\nabla - q\mathbf{A}'\right)^2 + qV'\right]\psi'(\mathbf{x},t) = i\frac{\partial}{\partial t}\psi'(\mathbf{x},t).$$

Replacing (2.2.1) and (2.2.3), eventually we notice this is not invariant, due to we cannot recover (2.2.2). However we shall find a transformation for ψ such that together with (2.2.3) leaves Eq. (2.2.2) invariant.

Vladimir Fock, and resdicovered by Fritz London, [68] introduced the following transformation

$$\psi(\mathbf{x},t) \longrightarrow \psi'(\mathbf{x},t) = \psi(\mathbf{x},t)e^{-iq\chi(\mathbf{x},t)}, \tag{2.2.4}$$

which eventually leads invariant the Eq. (2.2.2) togheter with (2.2.3).

The previous result can also be interpreted as follows: The Schrödinger equation for a particle in an electromagnetic field is not invariant under a local phase transformation of the form given in (2.2.1). The phase of a wave function clearly satisfies the requirements of a local variable. The objections raised against Weyl's original theory are not valid, since the phase is not directly involved in the measurement of any space-time quantity analogous to the length of a vector.

In the absence of an electromagnetic field, the phase can be represented by an arbitrary constant value, as it cannot affect any observable quantity. When an electromagnetic field is present, different choices of phase at each point in space can be made naturally by interpreting the potential $A_\mu \equiv (\mathbf{A}, V)$ as a connection that relates the phases at different points in space. Choosing a particular phase $\lambda(\mathbf{x},t) \equiv iq\chi(\mathbf{x},t)$ does not affect any observable quantity, provided that the gauge transformation of A_μ is given by

$$A_\mu \longrightarrow A'_\mu = A_\mu - \partial_\mu \lambda,$$

so that the change in phase and the change in the potential exactly cancel each other. This implies that the “arbitrariness” formally attributed to the potential can be understood as the freedom to choose any value for the phase of a wave function without affecting the equations of motion.

At the end of the so-called "old period" (that is, the era when Hermann Weyl first proposed the idea of gauge invariance in 1918, at a time when the known particles were the electron and the proton, and only two fundamental forces were believed to exist) it became clear that the electromagnetic interaction of charged particles could be interpreted as a local gauge theory within the framework of quantum mechanics.

In analogy with Weyl's theory, the phase of a particle's wave function can be identified as a new physical degree of freedom that depends on the position in space-time. The value of the phase can be arbitrarily changed by applying local phase transformations to the wave function at each point. This means, as Weyl originally argued, that there must exist some kind of connection between the values of the phase at neighboring points. The role of this connection is precisely played by the electromagnetic potential.

The intimate relationship between the potential and the change of phase is clearly demonstrated by the Aharonov–Bohm effect.

Thus, by using the phase of a wave function as a local variable (rather than the norm of a vector) the electromagnetic theory can be interpreted as a gauge theory, just as Weyl had envisioned.

Finally, it should be noted that the set of all gauge transformations in (2.2.4) forms a unitary group known as the $U(1)$ group. [69]

2.3 Relativistic Generalization of the Electrodynamics Gauge Theory

Consider the simplest field theory involving spinor fields, namely the Dirac Lagrangian

$$\mathscr{L}_0 = \overline{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi(x), \tag{2.3.1}$$

where $\psi(x)$ is the electron spinor field of rest mass m, and γ^μ are the Dirac gamma matrices. Since the theory is inherently quantum mechanical, the time evolution of the spinor field (obtained from the corresponding Euler–Lagrange equations derived from the action) must remain invariant under the global phase transformation associated with the $U(1)$ symmetry group. Therefore, the Lagrangian $\mathscr{L}_0$ must be invariant under the transformation

$$\psi(x) \longrightarrow \psi'(x) = e^{i\alpha}\psi(x), \qquad \overline{\psi}(x) \longrightarrow \overline{\psi}'(x) = \overline{\psi}(x)e^{-i\alpha}, \tag{2.3.2}$$

with constant parameter $\alpha \in \mathbb{R}$. Such a transformation represents a global $U(1)$ phase rotation. It is straightforward to verify the invariance of the Lagrangian under the transformations (2.3.2). Indeed, after performing the replacement one finds

$$\mathscr{L}_0' = \overline{\psi}'(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi'(x) = \mathscr{L}_0,$$

which shows that the Lagrangian remains unchanged.

Alternatively, the result can be understood by observing how the four-gradient acting on $\psi(x)$ transforms. From (2.3.2), we obtain

$$\partial_\mu \psi(x) \longrightarrow \Big(\partial_\mu \psi(x)\Big)' = e^{i\alpha}\Big(\partial_\mu \psi(x)\Big). \tag{2.3.3}$$

On the other hand, considering the transformation from the left-hand side, one finds

$$\Big(\partial_\mu \psi(x)\Big)' = \partial'_\mu \psi'(x) = \partial'_\mu\Big(e^{i\alpha}\psi(x)\Big).$$

Thus, for an arbitrary wave function, one is naturally led to the relation

$$\partial'_\mu = e^{-i\alpha}\partial_\mu e^{i\alpha}.$$

Consequently, although the four-gradient transforms, the Lagrangian remains invariant under global $U(1)$ transformations.

However, in 1954 Yang and Mills [70] generalized the global $U(1)$ symmetry to a local one. In technical terms, they extended the $U(1)$ transformation to an arbitrary Lie group G. The essential difference lies in the fact that $U(1)$ is an abelian group with a single generator, while a non-abelian gauge group contains several generators with non-trivial commutation relations. In the general case, we consider the local transformation

$$\psi(x) \;\longrightarrow\; \psi'(x) = e^{i\alpha(x)}\psi(x), \qquad \overline{\psi}(x) \;\longrightarrow\; \overline{\psi}'(x) = \overline{\psi}(x)e^{-i\alpha(x)}. \tag{2.3.4}$$

In contrast with (2.3.2), the parameter α now depends on the spacetime position. Therefore, our goal is to generalize the invariance. That remains invariant the lagrangian not only under global phase transformations, but also under local ones, i.e. invariance under transformations where the phase varies from point to point in spacetime.

Now we reconsider the Dirac Lagrangian and analyse its behaviour under the transformation (2.3.4). After replacing into the Lagrangian, the transformed expression becomes

$$\mathscr{L}'_0 = \mathscr{L}_0 - \overline{\psi}(x)\gamma^\mu \psi(x)\partial_\mu \alpha(x),$$

which shows that $\mathscr{L}_0$ is not invariant under local $U(1)$ transformations. The origin of this non-invariance lies in the fact that the four-gradient must now transform locally as

$$\partial_\mu \;\longrightarrow\; \partial'_\mu = e^{i\alpha(x)}\Big(\partial_\mu + i\partial_\mu \alpha(x)\Big)e^{-i\alpha(x)}. \tag{2.3.5}$$

This naturally suggests introducing a suitable generalization of the operator ∂_μ. To ensure that $\mathscr{L}_0$ is locally invariant, we define a new operator, denoted by D_μ, which generalizes ∂_μ and transforms analogously to the global case (2.3.3),

$$D_\mu \psi(x) \;\longrightarrow\; \Big(D_\mu \psi(x)\Big)' = e^{i\alpha(x)}\Big(D_\mu \psi(x)\Big). \tag{2.3.6}$$

In complete analogy with the previous argument, this implies that D_μ must transform according to

$$D'_\mu = e^{-i\alpha(x)} D_\mu e^{i\alpha(x)}. \tag{2.3.7}$$

With this in mind, a Lagrangian constructed with D_μ instead of ∂_μ will be locally invariant under $U(1)$. Therefore, we define

$$\mathscr{L} = \overline{\psi}(x)\Big(i\gamma^\mu D_\mu - m\Big)\psi(x), \tag{2.3.8}$$

whose invariance follows straightforwardly from (2.3.7). By comparing (2.3.7) with (2.3.5), we are naturally led to postulate that the **covariant derivative** is defined as

$$D_\mu \equiv \partial_\mu + iA_\mu(x).$$

Thus, a new field $A_\mu(x)$ has been introduced, known as the **gauge field** or **gauge potential**. Its transformation law follows from the requirement (2.3.6) together with (2.3.5), yielding

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x). \tag{2.3.9}$$

Consequently, the Lagrangian (2.3.8) is invariant under the combined transformations (2.3.4) and (2.3.9).

Since the Lagrangian does not contain a kinetic term for the gauge field, we can recover it by computing the commutator of the covariant derivatives. In particular, we obtain

$$\Big[D_\mu, D_\nu\Big]\psi = i\Big(\partial_\mu A_\nu - \partial_\nu A_\mu\Big)\psi \equiv iF_{\mu\nu}\psi,$$

where $F_{\mu\nu}$ is an invariant gauge tensor. From this field strength, we can construct the gauge–invariant kinetic term of the theory, leading to the Lagrangian

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Hence, by combining the kinetic term of the gauge field with the minimally coupled Dirac field, we arrive at a Lagrangian that describes the interaction between a spin–1 field and a spin–1/2 field,

$$\mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(x)\big(i\gamma^\mu D_\mu - m\big)\psi(x).$$

This Lagrangian corresponds to Quantum Electrodynamics (QED), and it successfully encodes the interaction between the electromagnetic field and the electron.

In the following section, we will extend this construction to non–abelian gauge theories, introducing the Yang–Mills framework.

2.4 Yang-Mills Gauge Theory

In 1954, C. N. Yang and R. Mills proposed that the strong nuclear interaction could be described by a field theory analogous to electromagnetism, which, as we have seen, is exactly gauge invariant.[70] In direct analogy with electrodynamics, whose local symmetry group is the unitary group $U(1)$, Yang and Mills postulated that the local symmetry group associated with the strong nuclear force is the isospin group $SU(2)$.

More generally, the special unitary group $SU(N)$ is associated with the Lie algebra $\mathfrak{su}(N)$, generated by a set of matrices T^a satisfying the commutation relations

$$[T_a, T_b] = i f_{ab}{}^{c} T_c,$$

where f^{abc} are the structure constants of the algebra. An element of the group $SU(N)$ is given by

$$U = \exp(i\alpha^a T_a).$$

Therefore a wave function $\psi(x)$ transforms under this group as

$$\psi(x) \longrightarrow \psi'(x) = U\psi(x) = e^{i\alpha^a T_a}\psi(x).$$

However, we can expand the wave functions in base of the algebra, i. e.

$$\psi = \psi^a T_a.$$

We now proceed in complete analogy with the previous section, but extending the analysis to non–abelian symmetries. Let us begin by considering the Dirac Lagrangian for each spinor field ψ^a, namely

$$\mathscr{L} = i\overline{\psi}(x)\gamma^\mu \partial_\mu \psi(x). \tag{2.4.1}$$

It is straightforward to verify that this expression is invariant under global $U(1)$ transformations, as previously discussed. We now turn our attention to the local case. For a non–abelian gauge group, the spinor fields transform according to

$$\begin{aligned}
\psi(x) &\longrightarrow \psi'(x) = U(x)\psi(x) = e^{i\alpha^a(x)T_a}\psi(x),\\
\overline{\psi}(x) &\longrightarrow \overline{\psi}'(x) = \overline{\psi}(x)U^\dagger(x) = \overline{\psi}(x)e^{i\alpha^a(x)T_a}.
\end{aligned}$$

Using the same Lagrangian (2.4.1), we observe that under such local transformations one obtains

$$\partial_\mu\Big(U(x)\psi(x)\Big) = U(x)\partial_\mu\psi(x) + \Big(\partial_\mu U(x)\Big)\psi(x),$$

with

$$\partial_\mu U(x) = U(x) i \partial_\mu \Big(\alpha^a(x) T_a \Big).$$

Therefore, after transforming the Lagrangian we find

$$\mathscr{L}' = \mathscr{L} - \overline{\psi}(x) \gamma^\mu \psi(x) \partial_\mu \Big(\alpha^a(x) T_a \Big),$$

which shows that $\mathscr{L}$ is not invariant under local $SU(N)$ transformations. The source of this non–invariance is the induced transformation

$$\partial_\mu \longrightarrow \partial'_\mu = U(x) \Big(\partial_\mu + i \partial_\mu \alpha^a(x) T_a \Big) U^\dagger(x). \tag{2.4.2}$$

To restore local gauge invariance, we introduce a covariant derivative that transforms homogeneously,

$$D_\mu \psi(x) \longrightarrow \Big(D_\mu \psi(x) \Big)' = U(x) \Big(D_\mu \psi(x) \Big), \tag{2.4.3}$$

which is equivalent to requiring that

$$D'_\mu = U(x) D_\mu U^\dagger(x). \tag{2.4.4}$$

Thus, by replacing ∂_μ with D_μ, the Lagrangian becomes locally invariant under $SU(N)$. We therefore define

$$\mathscr{L} = \overline{\psi}(x) \Big(i \gamma^\mu D_\mu - m \Big) \psi(x), \tag{2.4.5}$$

whose invariance follows directly from (2.4.4). Comparing (2.4.4) with (2.4.2), we are naturally led to postulate the covariant derivative

$$D_\mu \equiv \partial_\mu + i A_\mu(x), \tag{2.4.6}$$

where

$$A_\mu(x) = A_\mu{}^a(x) T_a. \tag{2.4.7}$$

The corresponding gauge field $A_\mu(x)$, or gauge potential, transforms according to

$$A'_\mu(x) = U(x) A_\mu(x) U^\dagger(x) + i \Big(\partial_\mu U(x) \Big) U^\dagger(x), \tag{2.4.8}$$

which may equivalently be written as

$$A'_\mu(x) = U(x) A_\mu(x) U^\dagger(x) - i U(x) \Big(\partial_\mu U^\dagger(x) \Big). \tag{2.4.9}$$

We see that local invariance under $SU(N)$ comes at the price of introducing N^2-1 new fields through (2.4.7).

Since the Lagrangian does not yet contain a kinetic term for the gauge field, we compute the commutator of the covariant derivatives,

$$F_{\mu\nu} \equiv i\left[D_\mu, D_\nu\right] = i\left(D_\mu D_\nu - D_\nu D_\mu\right),$$

which transforms as

$$F'_{\mu\nu} = i\left(D'_\mu D'_\nu - D'_\nu D'_\mu\right) = U(x) F_{\mu\nu} U^\dagger(x),$$

showing that $F_{\mu\nu}$ is gauge invariant. Computing it explicitly using (2.4.6), we obtain

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu, A_\nu\right].$$

However, this term is no longer invariant because now the tensors $F_{\mu\nu}$ are matrices. The fact that they are matrices naturally leads us to consider an invariant quantity associated with matrices, namely the trace. We know that if M is a matrix, then $\mathrm{tr}\,M$ is an invariant that does not depend on the coordinate system. Indeed, if A, B and C are three matrices, we have

$$\mathrm{tr}\left(ABC\right) = \mathrm{tr}\left(BCA\right).$$

In the particular case in which $A = F$ is a matrix corresponding to a coordinate transformation, and $C = F^{-1}$, it follows that

$$\mathrm{tr}\left(FBF^{-1}\right) = \mathrm{tr}\left(BF^{-1}F\right) = \mathrm{tr}\left(B\right).$$

Taking these results into account, we can construct the following invariant Lagrangian:

$$\mathscr{L} = -\frac{1}{2g^2}\,\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right),$$

where

$$F_{\mu\nu} = {F_{\mu\nu}}^a T_a,$$

and the generators satisfy the normalization condition

$$\mathrm{tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}.$$

Here, g is the coupling constant. Thus, an invariant Lagrangian under the local symmetry group $SU(N)$ is

$$\begin{aligned}\mathscr{L}_{YM} &= -\frac{1}{2g^2}\operatorname{tr}\big(F_{\mu\nu}F^{\mu\nu}\big) + i\overline{\psi}(x)\gamma^{\mu}D_{\mu}\psi(x)\\ &= -\frac{1}{2g^2}\operatorname{tr}\big(F_{\mu\nu}F^{\mu\nu}\big) + i\overline{\psi}(x)\gamma^{\mu}\partial_{\mu}\psi(x) - \overline{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x).\end{aligned}$$

In this chapter we have established the foundations of gauge theory, showing how the requirement of local symmetry naturally introduces gauge fields and fixes their transformation properties. We derived the Yang–Mills action as the minimal dynamical realization of non-Abelian gauge invariance and recovered the electromagnetic Lagrangian as the Abelian limit. These results demonstrate that the structure of fundamental interactions is dictated not by for this assumptions, but by the internal consistency of local gauge symmetry.

Chapter 3

Gravity in the Language of Differential Forms

In this chapter we reformulate gravity using the language of differential forms, following Cartan's geometric approach. Instead of describing the gravitational field solely in terms of the metric tensor, we adopt a first-order formalism in which the fundamental variables are the vielbein and the Lorentz connection. This perspective makes the local symmetry structure manifest and provides a natural framework for extensions of gravity to higher dimensions, including torsion and coupling to fermionic matter.

The vielbein e^a is introduced as a set of orthonormal frame 1-forms that relate the spacetime manifold to a locally flat tangent space endowed with the Minkowski metric η_{ab}. Through the relation $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, the metric emerges as a derived quantity, while local Lorentz invariance acts on the tangent space indices. This separation between spacetime coordinates and internal Lorentz frames clarifies the geometric content of general covariance and local symmetry.

The spin connection ω^{ab} is introduced as an independent Lorentz-valued 1-form that defines parallel transport in the tangent bundle. Within Cartan's formalism, curvature and torsion arise naturally as 2-forms associated with this connection.

This first-order formulation highlights the gauge-theoretic structure of gravity, with the vielbein and spin connection playing roles analogous to gauge fields. Moreover, expressing the theory in terms of differential forms leads to compact and dimension-independent expressions for actions and field equations, making it particularly suitable for generalizations such as Lovelock and Chern–Simons gravities.

3.1 Einstein Field Equations from the Variational Principle

In the framework of General Relativity, the Einstein field equations (EFE) establish a direct relationship between the geometry of spacetime and the distribution of matter and energy. These equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu},$$

were introduced by A. Einstein in 1915 as a covariant tensorial expression [71], providing a unified description in which the curvature of spacetime, encoded in the Einstein tensor, is sourced by the local energy–momentum content.

A natural question concerns the derivation of the Einstein equations from a variational principle. To this end, one considers an action functional of the form

$$S_g = \int \mathscr{L}\, d^4x = \int \sqrt{-g}\, L_g\, d^4x,$$

where $\mathscr{L}$ is the generally covariant Lagrangian density. The appearance of the Jacobian factor $\sqrt{-g}$ guarantees invariance under arbitrary coordinate transformations. The task is thus to determine the appropriate scalar L_g compatible with the structure of the field equations. Since the Einstein equations involve the metric $g_{\mu\nu}$ and its first derivatives through the connection $\Gamma^{\lambda}_{\mu\nu}$, one may attempt to construct an invariant solely from these quantities; however, no nontrivial scalar invariant can be formed from $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ alone.

This difficulty was resolved by D. Hilbert in 1915. Hilbert proposed treating L_g as an invariant scalar depending not only on the metric and its first derivatives, but also linearly on its second derivatives. In this case, Gauss's theorem allows one to decompose the action into a bulk integral free of second derivatives and a boundary term,

$$S_g = \int \sqrt{-g}\, L_g\, d^4x = \int \sqrt{-g}\, L'_g\, d^4x + \oint \sqrt{-g}\, W_g{}^{\mu}\, d^4\Sigma_{\mu}.$$

After performing the variation of the action, only the bulk term contributes to the field equations, while the surface term can be handled by imposing suitable boundary conditions.

Thus, L_g must incorporate the metric, its first derivatives via $\Gamma^{\lambda}_{\mu\nu}$, and its second derivatives linearly through the Riemann tensor $R^{\lambda}{}_{\mu\nu\rho}$. In four spacetime dimensions, fourteen algebraically independent scalar invariants can be constructed from $g_{\mu\nu}$ and its derivatives. Remarkably, only one of these is linear in the second derivatives of the metric: the Ricci scalar R. Consequently, the gravitational action takes the form

$$S_g = \int \sqrt{-g}\, L_g\, d^4x = \int \sqrt{-g}\, R\, d^4x. \tag{3.1.1}$$

Varying this action with respect to the metric yields the Einstein field equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu},$$

where $G_{\mu\nu}$, the Einstein tensor, is symmetric, divergence-free, and depends on the metric and its first and second derivatives, while remaining linear in the latter.

Formulating the dynamics through an action principle provides several conceptual and practical advantages. It allows General Relativity to be seamlessly compared or unified with other classical field theories, such as electromagnetism, since they share a common variational origin. The action also specifies the natural coupling between geometry and matter, and symmetries of the action identify conserved quantities via Noether's theorem.

Typically, the Einstein–Hilbert action is constructed as a functional of the metric alone, with the Levi-Civita connection assumed *a priori*. In the Palatini formulation, however, the metric and connection are treated as independent variables, and the action is varied with respect to both. This approach is particularly useful when incorporating matter fields with non-integer spin, for which an independent connection becomes essential.

3.2 The Equivalence Principle and Local Flatness

The equivalence principle, first articulated by Einstein, asserts that within a sufficiently small region of spacetime, the effects of gravity can be eliminated by adopting a freely falling reference frame. In such a frame, all local experiments become indistinguishable from those carried out in the absence of gravity, and the laws of physics reduce to those of special relativity. Thus, any observers in a gravitational field can always identify a neighborhood in which they behave as an inertial observer. In this sense, spacetime is **locally flat**.

Mathematically, this observation leads naturally to the description of spacetime as a differentiable manifold. A manifold is a smooth geometric object that, although globally curved, admits coordinate patches that are locally isomorphic to flat Minkowski space. Formally, if $\mathscr{M}$ denotes the spacetime manifold, each point $x \in \mathscr{M}$ possesses an associated tangent space T_x with Lorentzian signature $(-,+,+,+)$, representing the locally inertial structure.

3.3 The vielbein

The map that relates the curved spacetime geometry to its locally flat tangent spaces is provided by the *vielbein* (or tetrad) field. In German, *vier* means "four", *viel* means "many", and *bein* means "leg". For this reason, in four dimensions it is often referred to as the *vierbein*, while in arbitrary dimensions it is known as the *vielbein*. The vielbein

$$e^a{}_\mu(x) = \frac{\partial z^a}{\partial x^\mu}$$

defines, at each point of spacetime, a linear isomorphism between the coordinate basis vectors x^μ on $\mathscr{M}$ and an orthonormal basis z^a on the tangent space T_x. Considering the infinitesimal coordinate transformation

$$dz^a = \frac{\partial z^a}{\partial x^\mu} dx^\mu = e^a{}_\mu(x) dx^\mu, \tag{3.3.1}$$

and using the Minkowski line element, we obtain from Eq. (3.3.1)

$$ds^2 = \eta_{ab}\, dz^a dz^b = \eta_{ab}\, e^a{}_\mu(x) e^b{}_\nu(x)\, dx^\mu dx^\nu,$$

which can be compared with the line element on the curved spacetime manifold,

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu,$$

leading to the relation

$$\eta_{ab}\, e^a{}_\mu(x) e^b{}_\nu(x) = g_{\mu\nu}. \tag{3.3.2}$$

Through the vielbein, tensors defined on the manifold can be projected onto locally inertial Minkowski frames, making the local Lorentz invariance of physical laws explicit. From now on, Greek indices refer to the curved manifold, while Latin indices label components in the locally flat tangent space.

The inverse of the vielbein is well defined by

$$e_a{}^\mu = \eta_{ab} g^{\mu\nu} e^b{}_\nu,$$

which obeys

$$e_a{}^\mu e^b{}_\mu = \delta^b_a$$
$$\eta^{ab} e_a{}^\mu e_b{}^\nu = g^{\mu\nu}.$$

Hence the vielbein with its inverse is used to interconvert Latin and Greek indices when necessary.

The Minkowski metric is preserved under Lorentz transformations,

$$\Lambda^c{}_a \Lambda^d{}_b \eta_{cd} = \eta_{ab},$$

and the vielbein transforms accordingly as

$$e^a{}_\mu(x) \longrightarrow e'^b{}_\mu(x) = \Lambda^b{}_a e^a{}_\mu(x), \tag{3.3.3}$$

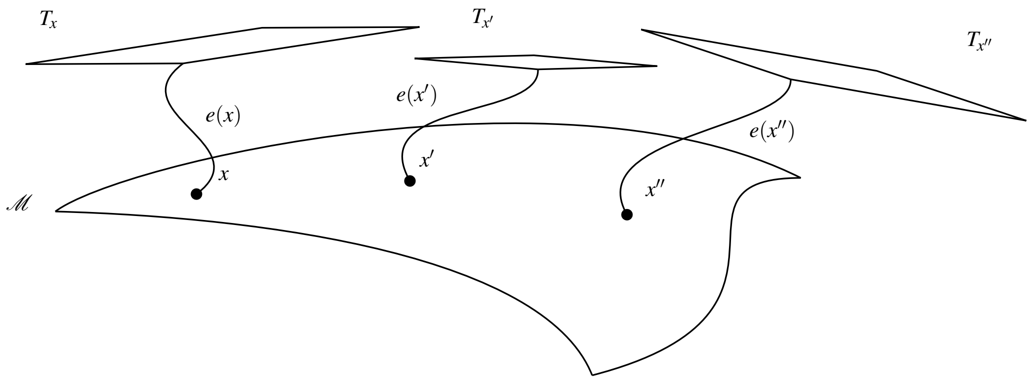

Figure 3.1: Illustration of different vielbein mappings at various points of the manifold.

leaving the metric in Eq. (3.3.2) invariant. This shows that the choice of vielbein at each point is not unique: physically, one may freely rotate the vielbein by a local Lorentz transformation without producing any observable effect on the geometry of the manifold.

In summary, the equivalence principle motivates the geometric formulation of gravity: spacetime is represented as a smooth manifold equipped with a Lorentzian metric that becomes locally flat in freely falling frames. The vielbein formalism provides the mathematical structure that relates this local flatness to the global curvature of the manifold, as illustrated in Fig. 3.1. Furthermore, the vielbein can also be expressed intrinsically on the manifold as a differential 1-form,

$$e^a(x) = e^a{}_\mu \, dx^\mu,$$

which is commonly referred to as the **vielbein 1-form**. This formulation emphasizes that the geometric and physical information encoded in the vielbein can be fully captured using differential forms, independently of any explicit coordinate basis.

Since our aim is to describe the physics on the spacetime manifold $\mathscr{M}$ using only the tangent spaces T_x, due to we are always dealing with scalars from the view of the manifold (since they do not have Greek indices), it becomes essential to understand how these tangent spaces at different points (say T_x and $T_{x'}$) are related. The appropriate mathematical framework for this task is the exterior derivate

$$d \equiv dx^\mu \frac{\partial}{\partial x^\mu}.$$

Since it contains the ordinary partial derivative ∂_μ, familiar from General Relativity, the exterior derivative plays a central role in defining how fields change across the manifold. In particular, when supplemented by a suitable connection, it will enable us to relate the dynamics over the manifold $\mathscr{M}$ through different tangent spaces via parallel transport, mirroring the procedure used in the geometric formulation of General Relativity.

3.4 The Lorentz connection

In the previous section, we concluded that the vielbein is not unique at each point x of the manifold $\mathscr{M}$ due to the transformation property in Eq. (3.3.3). After defining the vielbein as a 1-form, we may also consider the transformation

$$e^a(x) \longrightarrow e'^b(x) = \Lambda^b{}_a e^a(x),$$

which shows that, from the viewpoint of the tangent space T_x, the vielbein behaves as an ordinary Lorentz vector. For reasons discussed earlier, we now examine how the exterior derivative of the vielbein 1-form transforms. Applying the exterior derivative d to the expression above, we obtain

$$de'^b(x) = d\left(\Lambda^b{}_a e^a(x)\right) = d\Lambda^b{}_a e^a(x) + \Lambda^b{}_a de^a(x).$$

Since Λ depends on the coordinates we observe that the exterior derivative does not leave the vielbein transformation law invariant. Consequently, this operator cannot be used to construct invariant differential equations nor a derivative that transforms tangent space tensors into tangent space tensors.

In analogy with similar situations involving derivatives in gauge theories, as discussed in Sec. 2, we postulate a new derivative operator satisfying

$$De^a(x) \longrightarrow De'^b(x) = \Lambda^b{}_a De^a(x), \tag{3.4.1}$$

which leads us to define

$$De^a(x) \equiv de^a(x) + \omega^a{}_b(x)\, e^a(x),$$

where this operator is known as the **covariant exterior derivative**. Here, we introduce a new field $\omega^a{}_b(x)$, called the **Lorentz connection 1-form**. In order to satisfy Eq. (3.4.1), we conclude that this field must transform as

$$\omega'^a{}_b(x) = \Lambda^d{}_b \Lambda^a{}_c\, \omega^c{}_d(x) - \Lambda^c{}_b\, d\Lambda^a{}_c.$$

The connection $\omega^a{}_b(x)$ defines the parallel transport of Lorentz tensors within the tangent space, between infinitesimally close points where T_x is defined. In this formulation, the affine properties of spacetime are encoded in the components of the Lorentz connection, which remain, for the moment, arbitrary and independent of the metric.

In summary, the vielbein and the Lorentz connection are local 1-forms. This is not accidental. All geometric properties of $\mathscr{M}$ can in fact be encoded solely through these two 1-forms, together with their exterior products and exterior derivatives. Since both e^a and $\omega^a{}_b$ carry no coordinate indices (such as μ or ν), they behave as scalars under coordinate diffeomorphisms of $\mathscr{M}$. Like any exterior forms, they remain invariant under general coordinate transformations. Consequently, a formulation of geometry based only on these objects, their wedge products and their exterior derivatives is naturally independent of any particular coordinate system.

In the following section we will analyze the relation between the connection developed here with the Levi-Civita connection.[72]

3.5 Relation to classical tensor calculus

In General Relativity it is well known that the ordinary derivative ∂_μ acting on a tensor V^α does not transform as a tensor. For this reason one introduces the **covariant derivative** ∇_μ, which supplements the derivative with the **Levi-Civita connection** $\Gamma^\mu{}_{\alpha\beta}$ so that $\nabla_\mu V^\alpha$ transforms properly as a tensor. This leads to the standard definition of the covariant derivative acting on an arbitrary tensor,

$$\begin{aligned}\nabla_\mu A^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} \equiv& \partial_\mu A^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} + \Gamma^\alpha{}_{\lambda\mu} A^{\lambda\beta\ldots}{}_{\gamma\delta\ldots} + \Gamma^\beta{}_{\lambda\mu} A^{\alpha\lambda\ldots}{}_{\gamma\delta\ldots} + \cdots \\ & - \Gamma^\lambda{}_{\gamma\mu} A^{\alpha\beta\ldots}{}_{\lambda\delta\ldots} - \Gamma^\lambda{}_{\delta\mu} A^{\alpha\beta\ldots}{}_{\gamma\lambda\ldots} - \cdots,\end{aligned}$$

which admits a geometric interpretation as parallel transport on the manifold $\mathscr{M}$. One then imposes the **metricity condition** $\nabla_\mu g_{\alpha\beta} = 0$, meaning that parallel transport preserves the metric. This follows by considering two constant contravariant tensors (two constant vectors) v^α and u^β for which $\nabla_\mu v^\alpha = \nabla_\mu u^\beta = 0$. The covariant derivative of their inner product must vanish, $\nabla_\mu (g_{\alpha\beta} v^\alpha u^\beta) = 0$, which leads directly to metricity.

A tensor can be constructed by antisymmetrizing the Levi-Civita connection, which defines the **torsion**,

$$T^\mu{}_{\alpha\beta} \equiv \Gamma^\mu{}_{[\alpha\beta]} = \frac{1}{2}\left(\Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha}\right). \tag{3.5.1}$$

The simplest assumption that allows us to express the Levi-Civita connection in terms of known quantities such as the metric $g_{\alpha\beta}$ is to set the torsion to zero. Combined with metricity, this yields the familiar **Christoffel symbol**,

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right) \tag{3.5.2}$$

Now we have two objects, Γ and ω, whose purpose is to convert tensors into tensors on their respective spaces, the manifold $\mathscr{M}$ and the tangent space T_x. These two connections can be related through the **total covariant derivative**, which is covariant with respect to all indices of the object on which it acts. Acting on the vielbein we obtain

$$\mathscr{D}_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu - \Gamma^\lambda{}_{\nu\mu} e^a{}_\lambda + \omega^a{}_{b\mu}\, e^b{}_\nu.$$

We would like to be able to convert tangent indices into world indices and vice versa inside the total covariant derivative in the following form,

$$e^a{}_\nu \mathscr{D}_\mu \xi^\nu = D_\mu \xi^a, \tag{3.5.3}$$

that is, the projection of $\mathscr{D}$ onto the vielbein basis. Starting from

$$\mathscr{D}_\mu \xi^a = \mathscr{D}_\mu (e^a{}_\nu \xi^\nu),$$

and requiring Eq.(3.5.3), the Leibniz rule forces us to impose the **first vielbein postulate**,

$$\mathscr{D}_\mu e^a{}_\nu = 0. \tag{3.5.4}$$

From this relation we can immediately obtain the Lorentz connection ω in terms of known geometric quantities,

$$\omega^a{}_{b\mu} = \Gamma^a{}_{b\mu} - e_b{}^\nu \partial_\mu e^a{}_\nu.$$

Moreover, the Lorentz connection can be expressed completely in terms of the vielbein by using Eq. (3.3.2) together with Eq. (3.5.2).

In the next section we will focus on the geometric objects that encode the dynamics of spacetime, namely, torsion and curvature, now expressed in the tangent space.

3.6 Curvatures and Bianchi identities

Consider the torsion defined in (3.5.1). Using the first vielbein postulate, we can rewrite the Levi-Civita connection in terms of the vielbein and the Lorentz connection as

$$\Gamma^\alpha{}_{\mu\nu} = e_a{}^\alpha \partial_\mu e^a{}_\nu + e_a{}^\alpha \omega^a{}_{b\mu} e^b{}_\nu. \tag{3.6.1}$$

With this expression, the torsion takes the form

$$T^\alpha{}_{\mu\nu} = e_a{}^\alpha \left(\frac{1}{2}\partial_{[\mu} e^a{}_{\nu]} + \frac{1}{2}\omega^a{}_{b[\mu} e^b{}_{\nu]} \right) = e_a{}^\alpha T^a{}_{\mu\nu}\,,$$

where the last quantity is defined by

$$T^a{}_{\mu\nu} \equiv \left(\frac{1}{2}\partial_{[\mu} e^a{}_{\nu]} + \frac{1}{2}\omega^a{}_{b[\mu} e^b{}_{\nu]} \right).$$

In the previous section, torsion was set to zero, since otherwise we could not express the Levi-Civita connection entirely in terms of known quantities, as required in the present discussion. Hence $T^{a}{}_{\mu\nu} = 0$ and contracting the manifold indices with the dual basis dx^{μ}, we obtain the **torsion 2-form**

$$T^{a} \equiv De^{a} = 0. \tag{3.6.2}$$

Now consider the curvature tensor of General Relativity,

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}{}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\beta}.$$

Proceeding as in the torsion case, replacing (3.6.1) gives

$$R^{\alpha}{}_{\beta\mu\nu} = e_{a}{}^{\alpha}\partial_{[\mu}\omega^{a}{}_{b\nu]}e^{b}{}_{\beta} + e_{a}{}^{\alpha}\omega^{a}{}_{c[\mu}\omega^{c}{}_{b\nu]}e^{b}{}_{\beta},$$

where antisymmetrization is taken over the Greek indices. Defining

$$R^{a}{}_{b\mu\nu} \equiv \partial_{[\mu}\omega^{a}{}_{b\nu]} + \omega^{a}{}_{c[\mu}\omega^{c}{}_{b\nu]},$$

we obtain the relation between the curvatures of the two connections:

$$R^{a}{}_{b\mu\nu}(\omega) = e^{a}{}_{\alpha}R^{\alpha}{}_{\beta\mu\nu}(\Gamma)e_{b}{}^{\beta}.$$

Finally, contracting with the dual basis again yields the curvature 2-form,

$$R^{a}{}_{b} \equiv d\omega^{a}{}_{b} + \omega^{a}{}_{c}\omega^{c}{}_{b}, \tag{3.6.3}$$

where $R^{a}{}_{b}$ is the **curvature 2-form**. Together, Eqs. (3.6.2) and (3.6.3) are known as the **Cartan structure equations**, since they describe the geometric dynamics of the manifold $\mathscr{M}$.

Taking the exterior covariant derivative D of these structure equations leads to

$$DT^{a} = R^{a}{}_{b}e^{b},$$
$$DR^{a}{}_{b} = 0.$$

These are, respectively, the first and second **Bianchi identities**. There is also Dragon's theorem, which states that the second Bianchi identity can be obtained from the first by applying the exterior covariant derivative. [73, 74]

3.7 Einstein-Hilbert action in the Cartan formalism

Consider the action introduced previously in Eq. (3.1.1), whose variation yields the Einstein field equations. From Eq. (3.3.2) one can deduce that $\det e = \sqrt{-g}$. Moreover, starting from the Lagrangian appearing in Eq. (3.1.1), we write

$$R = R^{\mu\nu}{}_{\mu\nu} = \frac{1}{2}R^{ab}{}_{\mu\nu}\delta_{ab}{}^{\mu\nu}, \tag{3.7.1}$$

where we have used the identity

$$\delta_{\mu_1\cdots\mu_r}{}^{\nu_1\cdots\nu_r}\xi^{\mu_1\cdots\mu_r} = r!\,\xi^{\nu_1\cdots\nu_r}.$$

Now, from the general identity

$$\delta_{\mu_1\cdots\mu_s\alpha_{s+1}\cdots\alpha_r}{}^{\nu_1\cdots\nu_s\alpha_{s+1}\cdots\alpha_r} = \frac{(n-s)!}{(n-r)!}\delta_{\mu_1\cdots\mu_s}{}^{\nu_1\cdots\nu_s},$$

and using the fact that

$$\varepsilon_{abcd}\varepsilon^{\mu\nu cd} = \delta_{abcd}{}^{\mu\nu cd} = \frac{(4-2)!}{(4-4)!}\delta_{ab}{}^{\mu\nu} = 2\delta_{ab}{}^{\mu\nu},$$

the equality in Eq. (3.7.1) can be rewritten as

$$R = \frac{1}{2}R^{ab}{}_{\mu\nu}\delta_{ab}{}^{\mu\nu} = \frac{1}{4}\varepsilon_{abcd}R^{ab}{}_{\mu\nu}\varepsilon^{\mu\nu cd}.$$

Finally, making use of the relation

$$\varepsilon^{\mu_1\cdots\mu_n} = e^{\mu_1}{}_{\nu_1}\cdots e^{\mu_n}{}_{\nu_n}\varepsilon^{\nu_1\cdots\nu_n}(\det e)^{-1},$$

which in the present case reads

$$e^{\mu}{}_{\alpha}e^{\nu}{}_{\beta}e^{c}{}_{\rho}e^{d}{}_{\sigma}\varepsilon^{\alpha\beta\rho\sigma} = \varepsilon^{\mu\nu cd}\det e,$$

we obtain for the Lagrangian density

$$\begin{aligned}
\sqrt{-g}\,R\,d^4x &= \frac{1}{4}\varepsilon_{abcd}R^{ab}{}_{\mu\nu}\varepsilon^{\mu\nu cd}\det e\,d^4x \\
&= \frac{1}{4}\varepsilon_{abcd}R^{ab}{}_{\mu\nu}e^{\mu}{}_{\alpha}e^{\nu}{}_{\beta}e^{c}{}_{\rho}e^{d}{}_{\sigma}\varepsilon^{\alpha\beta\rho\sigma}\,d^4x \\
&= \frac{1}{4}\varepsilon_{abcd}R^{ab}{}_{\mu\nu}e^{\mu}{}_{\alpha}e^{\nu}{}_{\beta}e^{c}{}_{\rho}e^{d}{}_{\sigma}\,dx^{\alpha}dx^{\beta}dx^{\rho}dx^{\sigma} \\
&= \frac{1}{4}\varepsilon_{abcd}R^{ab}e^{c}e^{d}.
\end{aligned}$$

Therefore, we have shown that

$$\int\sqrt{-g}\,R\,d^4x = \frac{1}{4}\int\varepsilon_{abcd}R^{ab}e^{c}e^{d}.$$

The two expressions above correspond to the Einstein–Hilbert action written in the tensorial formalism and in the Cartan formalism, respectively.

3.8 Equation of motion in the Cartan formalism

To derive the equations of motion, we vary the action obtained previously,

$$S = \int \varepsilon_{abcd} R^{ab} e^c e^d, \tag{3.8.1}$$

with respect to the independent fields e and ω. The variation of the action then reads

$$\begin{aligned} \delta S &= \int \varepsilon_{abcd} \left(\delta R^{ab} e^c e^d + R^{ab} \delta e^c e^d + R^{ab} e^c \delta e^d \right) \\ &= \int \varepsilon_{abcd} \left(\delta R^{ab} e^c e^d + 2R^{ab} \delta e^c e^d \right), \end{aligned} \tag{3.8.2}$$

Next, we express the variation of the curvature 2-form δR^{ab} in terms of the variation of the Lorentz connection $\delta\omega^{ab}$. From the definition of the curvature 2-form in Eq. (3.6.3), one finds

$$\begin{aligned} \delta R^{ab} &= d(\delta\omega^{ab}) + \delta\omega^{a}{}_{c}\, \omega^{cb} + \omega^{a}{}_{c}\, \delta\omega^{cb} \\ &= d(\delta\omega^{ab}) + \omega^{b}{}_{c}\, \delta\omega^{ac} + \omega^{a}{}_{c}\, \delta\omega^{cb} \\ &= D\big(\delta\omega^{ab}\big). \end{aligned}$$

Substituting this result into Eq. (3.8.2), the variation of the action becomes

$$\begin{aligned} \delta S &= \int \varepsilon_{abcd} \left(D(\delta\omega^{ab})\, e^c e^d + 2R^{ab} e^c \delta e^d \right), \\ &= \int \varepsilon_{abcd} \left(D(\delta\omega^{ab} e^c e^d) + \delta\omega^{ab} De^c e^d - \delta\omega^{ab} e^c De^d + 2R^{ab} e^c \delta e^d \right), \\ &= \int d\left(\varepsilon_{abcd} \delta\omega^{ab} e^c e^d \right) + 2\int \varepsilon_{abcd} \delta\omega^{ab} T^c e^d + 2\int \varepsilon_{abcd} R^{ab} e^c \delta e^d. \end{aligned}$$

The first contribution is a total derivative and corresponds to a boundary term. It can be discarded by imposing that the variation of the Lorentz connection $\delta\omega^{ab}$ vanishes on the boundary of spacetime. The remaining two terms are independent variations and therefore must vanish separately in order for $\delta S = 0$. In vacuum, this requirement leads to the equations of motion

$$\begin{aligned} \varepsilon_{abcd} R^{ab} e^c &= 0, \\ \varepsilon_{abcd} T^c e^d &= 0. \end{aligned}$$

The first equation is equivalent to the Einstein field equations, while the second one enforces the vanishing of torsion, implying a torsion-free geometry.

Chapter 4

Gravity as a gauge theory

Symmetry principles play a central role in the construction of the appropriate classical action and, more importantly, they are often sufficient to guarantee the consistency of the corresponding quantum theory. In particular, gauge symmetry is essential to establish the renormalizability of the quantum field theories that successfully describe three of the four fundamental interactions in nature. Gravity, however, has persistently eluded this framework, despite being formulated as a theory invariant under general covariance, which constitutes a local symmetry closely analogous to gauge invariance.

Approximately one year after C. N. Yang and R. Mills introduced their model of non abelian gauge invariant interactions [70], R. Utiyama demonstrated that Einstein gravity can be reformulated as a gauge theory associated with the Lorentz group [75]. This fact can be directly verified from the Lagrangian given in (3.8.1), which is a Lorentz scalar and therefore manifestly invariant under local Lorentz transformations. If gravity admits a gauge theoretic interpretation, it cannot rely solely on the Lorentz group, and the inclusion of translations becomes unavoidable. This observation led to the expectation that gravity could be formulated as a gauge theory of the Poincaré group, $G = ISO(3,1)$, the symmetry group commonly employed in particle physics, which encompasses both Lorentz transformations and spacetime translations.

The introduction of translations appears natural, since a general coordinate transformation of the form

$$x_\mu \longrightarrow x_\mu + \xi_\mu$$

resembles a local translation. This transformation is indeed a local symmetry, in the sense that it leaves the action invariant and is parametrized by functions of spacetime. This analogy suggests that diffeomorphism invariance might be interpreted as a symmetry under local translations, effectively embedding the Lorentz group into the Poincaré group. Despite its appeal, constructing a local action invariant under local translations has so far proven impossible.

Within the first order formalism discussed previously, both the vielbein and the Lorentz connection are differential forms and are therefore invariant under general coordinate transformations. The Lorentz connection, commonly referred to as the spin connection, naturally arises in the Dirac equation when formulated in curved spacetime. This structure becomes necessary when coupling gravity to spinor fields, since there are no finite dimensional spinor representations of the general covariance group [76–80]. Nevertheless, this framework does not provide a clear interpretation of the vielbein as a gauge field.

Repeated attempts to identify coordinate transformations with local translations have systematically failed. The underlying issue is the absence of a known four dimensional action for general relativity that is invariant under local $ISO(3,1)$ transformations [77, 81–83]. In other words, although the fields ω^{ab} and e^a possess the appropriate index structure to be associated with the generators of the Poincaré group, there exists no Poincaré invariant four form that can be constructed from a connection valued in the Lie algebra $\mathfrak{iso}(3,1)$. This reveals that, despite its superficial plausibility, the claim that gravity is a gauge theory of translations is fundamentally limited, due to the deep differences between gauge theories defined on fiber bundles and theories with an open algebra, such as gravity.

A useful way to understand this obstruction is to assume the existence of a connection associated with local translations, analogous to the spin connection that encodes local Lorentz invariance. Since this hypothetical gauge field corresponds to translations in the tangent space, it must carry the same index structure as the generator of translations P_a. Consequently, the gauge field for local translations would necessarily have the same indices as the vielbein. A gauge theory of the Poincaré group would then be based on the connection

$$A = e^a P_a + \omega^{ab} J_a$$

However, one can show that no Poincaré invariant four form can be constructed from this connection, and therefore no Poincaré invariant gravitational action exists in four dimensions. This result will be demonstrated explicitly after we to analyze in detail the gauge transformation for A connection 1-form. Now we will introduce the standar gauge theory formalism expressed in the language of differential forms.

4.1 Gauge theory and differential forms

The standard formulation of gauge theories can be naturally expressed using the language of exterior differential forms.

Let G be a Lie group whose generators T_A satisfy the algebra

$$[T_A, T_B] = C^C{}_{AB} T_C,$$

where $C^C{}_{AB}$ are the structure constants and the uppercase indices label the generators of the Lie algebra associated with the group manifold.

In order to construct a gauge theory with symmetry group G, one introduces, for each generator T_A, a compensating field $h^A{}_\mu$. These fields can be assembled into the Lie algebra valued quantity $h_\mu = h^A{}_\mu T_A$.

The gauge potentials are defined as the one form

$$h = h^A{}_\mu T_A \, dx^\mu,$$

or equivalently as

$$h^A = h^A{}_\mu \, dx^\mu.$$

The exterior covariant derivative D is defined as $D = dx^\mu D_\mu$, with

$$D_\mu = \partial_\mu + h_\mu = \partial_\mu + h^A{}_\mu T_A.$$

From this definition, it follows that

$$D = dx^\mu D_\mu = dx^\mu (\partial_\mu + h^A{}_\mu T_A) = dx^\mu \partial_\mu + dx^\mu h^A{}_\mu T_A.$$

Since $d = dx^\mu \partial_\mu$ and $h = dx^\mu h^A{}_\mu T_A$, we obtain

$$D = d + h.$$

Following the usual construction, the curvature two form is obtained by taking the exterior product of two covariant derivatives. Let φ be a zero form. Then

$$\begin{aligned} R\varphi &= (D \wedge D)\varphi = (d+h) \wedge (d+h)\varphi \\ &= (dd\varphi + d(h\varphi) + h\,d\varphi + hh\varphi). \end{aligned}$$

Since $dd\varphi = 0$ and $d(h\varphi) = (dh)\varphi - h\,d\varphi$, it follows that

$$\begin{aligned} R\varphi &= (dh)\varphi - h\,d\varphi + h\,d\varphi + hh\varphi \\ &= (dh + hh)\varphi. \end{aligned}$$

Therefore,

$$R = dh + hh.$$

Using the decompositions $R = R^A T_A$ and $h = h^A T_A$, we find

$$\begin{aligned} R^A T_A &= dh^A T_A + h^B T_B h^C T_C \\ &= dh^A T_A + \frac{1}{2} h^B h^C T_B T_C + \frac{1}{2} h^B h^C T_B T_C. \end{aligned}$$

In the last term, the one forms can be exchanged and the indices relabeled, leading to

$$
\begin{aligned}
R^A T_A &= dh^A T_A + \frac{1}{2}h^B h^C T_B T_C - \frac{1}{2}h^B h^C T_C T_B \\
&= dh^A T_A + \frac{1}{2}h^B h^C \left(T_B T_C - T_C T_B\right) \\
&= dh^A T_A + \frac{1}{2}h^B h^C \left[T_B, T_C\right] \\
&= dh^A T_A + \frac{1}{2}h^B h^C C^A{}_{BC} T_A \\
&= \left(dh^A + \frac{1}{2}C^A{}_{BC} h^B h^C\right) T_A.
\end{aligned}
$$

Hence,

$$R^A = dh^A + \frac{1}{2}C^A{}_{BC} h^B h^C.$$

In this way, the components R^A of the curvature two form are expressed in terms of the components of the one form h^A. Writing the holonomic indices explicitly, we have

$$R^A = R^A_{\mu\nu}\, dx^\mu dx^\nu,$$

$$
\begin{aligned}
dh^A &= \partial_{[\mu} h^A_{\nu]}\, dx^\mu dx^\nu, \\
h^B h^C &= h^B_\mu h^C_\nu\, dx^\mu dx^\nu.
\end{aligned}
$$

Therefore,

$$R^A_{\mu\nu}\, dx^\mu dx^\nu = \partial_{[\mu} h^A_{\nu]}\, dx^\mu dx^\nu + \frac{1}{2}C^A{}_{BC} h^B_\mu h^C_\nu\, dx^\mu dx^\nu,$$

which implies

$$R^A_{\mu\nu} = \partial_{[\mu} h^A_{\nu]} + \frac{1}{2}C^A{}_{BC} h^B_\mu h^C_\nu.$$

The equations $R^A = 0$, with

$$R^A = dh^A + \frac{1}{2}C^A{}_{BC} h^B h^C,$$

are known as the Maurer–Cartan equations. It can be shown that the conditions

$$R^A = 0, \qquad dR^A = 0,$$

are equivalent to the Jacobi identities of the Lie algebra associated with the group G.

It is also worth noting that a gauge invariant action for the gauge field can be constructed using the components of the curvature two form R^A together with the components of the gauge potential one form h^A.

We now proceed to study the effect of gauge transformations under the Poincaré group.

4.2 Gauge transformation under Poincaré group

The Poincaré group in four dimensions consists of the following generators

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb}, \\
[J_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b, \\
[P_a, P_b] &= 0.
\end{aligned}$$

In this case, the theory has two gauge fields, the spin connection ω^{ab} and the vielbein e^a, which together form a multiplet in the adjoint representation of the Poincaré group. The associated field strength two form corresponding to the connection one form is given by

$$\begin{aligned}
F &\equiv dA + A^2, \\
F &= F^A T_A = T^a P_a + \frac{1}{2} R^{ab} J_{ab}.
\end{aligned}$$

It is important to note that, in this framework, torsion is interpreted as the field strength associated with translations, while curvature is related to the field strength corresponding to Lorentz rotations. Furthermore, the explicit expressions for torsion and curvature, written in terms of the gauge potentials, are obtained as a direct consequence of the commutation relations of the Poincaré algebra.

We now begin by studying the gauge transformation of the connection one form. We consider the group exponentiation

$$U = e^{-\lambda} = e^{-\lambda^A T_A},$$

where λ is a group parameter and T_A denote the generators. Here, the index A runs over Lorentz indices a and ab, so that

$$T_A = (P_a, J_{ab}).$$

From Section 2, we know that the theory defines the covariant derivative as

$$D \equiv d + A,$$

where the connection A transforms under the group according to

$$A \longrightarrow A' = UAU^{-1} + UdU^{-1}.$$

It can then be shown that

$$A' = A + d\lambda + [A, \lambda].$$

Therefore, the connection one form transforms as

$$\delta A = D\lambda. \tag{4.2.1}$$

Let us study the relation (4.2.1) using the infinitesimal gauge parameters ρ^a and κ^{ab}. These can be written as

$$\begin{aligned}\lambda = \lambda^A T_A =& \lambda^a P_a + \frac{1}{2}\lambda^{ab} J_{ab}\\ =& \rho^a P_a + \frac{1}{2}\kappa^{ab} J_{ab}.\end{aligned}$$

Applying the covariant derivative, we obtain

$$D\lambda = (D\rho^a + e_c \kappa^{ca}) P_a + \frac{1}{2} D\kappa^{ab} J_{ab}.$$

On the other hand, from (4.2.1) we have

$$\delta A = \delta e^a P_a + \frac{1}{2}\delta\omega^{ab} J_{ab}.$$

Therefore, we find that the connection components transform according to the following laws

$$\begin{aligned}\delta e^a &= D\rho^a + e_c \kappa^{ca},\\ \delta\omega^{ab} &= D\kappa^{ab}.\end{aligned}$$

The next step is to analyze the invariance of the Einstein–Hilbert action under the transformation laws derived for the Poincaré group. The invariance of a gravitational action under a given symmetry group would allow gravity to be formulated as a gauge theory. However, as will be shown below, the four dimensional Einstein–Hilbert action is not invariant under local Poincaré translations.

4.3 Invariance of the Einstein-Hilbert action

The four dimensional Einstein–Hilbert action

$$S = \int \varepsilon_{abcd} R^{ab} e^c e^d$$

is, by construction, invariant under general coordinate transformations as well as under local Lorentz rotations. Nevertheless, we will show below that this action fails to be invariant under local Poincaré translations. Let us therefore consider the variation of the action,

$$\begin{aligned}\delta_{plt} S =& \delta \int \varepsilon_{abcd} R^{ab} e^c e^d \\ =& \int d(\varepsilon_{abcd} \delta \omega^{ab} e^c e^d) + 2 \int \varepsilon_{abcd} \delta \omega^{ab} T^c e^d + 2 \int \varepsilon_{abcd} R^{ab} e^c \delta e^d .\end{aligned}$$

Since under local Poincaré translations the vielbein and the spin connection transform as

$$\begin{aligned}\delta e^a =& D\rho^a, \\ \delta \omega^{ab} =& 0,\end{aligned}$$

it follows that

$$\begin{aligned}\delta_{plt} S =& 2 \int \varepsilon_{abcd} R^{ab} e^c D\rho^d \\ =& -2 \int d(\varepsilon_{abcd} R^{ab} e^c \rho^d) + 2 \int \varepsilon_{abcd} R^{ab} T^c \rho^d ,\end{aligned}$$

where use has been made of the Bianchi identity $DR^{ab} = 0$. Therefore, up to boundary terms, one finds

$$\delta_{plt} S = 2 \int \varepsilon_{abcd} R^{ab} T^c \rho^d \neq 0.$$

Hence, the Einstein–Hilbert action is invariant under the Poincaré group only if the torsion is constrained to vanish. However, the condition $T^a = 0$ itself is not invariant under local Poincaré translations. Indeed, one can show that

$$\begin{aligned}\delta_{plt} T^a =& \delta(De^a) = D(\delta e^a) = DD\rho^a \\ =& R^{ab} \rho_b \neq 0.\end{aligned}$$

The lack of invariance of the four dimensional Einstein–Hilbert action may appear surprising, since translations are often identified with coordinate transformations. In fact, a coordinate transformation corresponds to a Lie derivative, and therefore gauge translations are fundamentally different from general coordinate transformations. Nevertheless, within the second order formalism, if the constraint $T^a = 0$ is imposed, gauge translations can be effectively interpreted as general coordinate transformations. In this case, the spin connection ω^{ab} is no longer an independent field, but becomes a dependent quantity.

Finally, it is worth emphasizing that the situation changes drastically in three dimensions. Under local Poincaré translations, one finds

$$\begin{aligned}\delta_{plt} S_{EH}^{(3)} =& \delta \int \varepsilon_{abc} R^{ab} e^{c} \\ =& \int \varepsilon_{abc} \delta R^{ab} e^{c} + \int \varepsilon_{abc} R^{ab} \delta e^{c} \\ =& \int \varepsilon_{abc} d(\delta \omega^{ab} e^{c}) - \int \varepsilon_{abc} R^{ab} D e^{c} + \int \varepsilon_{abc} R^{ab} \delta e^{c} \\ =& \int \varepsilon_{abc} R^{ab} D \rho^{c}.\end{aligned}$$

This expression can be rewritten as

$$\delta_{plt} S_{EH}^{(3)} = \int \varepsilon_{abc} d(R^{ab} \rho^{c}) + \int \varepsilon_{abc} D R^{ab} \rho^{c}.$$

Up to a boundary term, and making use of the Bianchi identity, one finally obtains the invariance of the three dimensional Einstein–Hilbert action under local Poincaré translations,

$$\delta_{plt} S_{EH}^{(3)} = 0.$$

Since this action is also invariant under Lorentz rotations by construction, it follows that in three dimensions it is possible to formulate a gravitational action invariant under the Poincaré group. Moreover, this invariance extends to all odd dimensions, a fact that will be discussed later in the context of Chern–Simons theories of gravity.

Chapter 5

Chern-Simons theory

Electrodynamics together with the weak and strong interactions are consistently described in the Standard Model by means of Yang–Mills gauge theories. Conventional gauge theories, much like special relativity, rely on the existence of a fixed, non-dynamical background metric structure. In order to construct an action of the Yang–Mills type,

$$S_{YM} = -\frac{1}{4g^2}\int d^4x F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4g^2}\int d^4x\, \eta^{\mu\lambda}\eta^{\nu\rho}F_{\mu\nu}F_{\lambda\rho},$$

one must assume the presence of a fixed background metric on the spacetime manifold where the gauge fields are defined. In the standard formulation this background structure is provided by the Minkowski metric $\eta = \mathrm{diag}(1,-1,-1,-1)$.

Gravity, as formulated in General Relativity, does not fit naturally into this framework despite decades of attempts to cast it as a conventional gauge theory. In General Relativity, spacetime and the gravitational field are not separate entities. The spacetime geometry itself is dynamical, carries independent degrees of freedom, and evolves according to Einstein's field equations. Geometry is therefore not prescribed a priori but determined dynamically. For this reason, any gauge formulation of gravity must avoid the introduction of a fixed background metric.

Within the first-order formalism, the vielbein e^a and the spin connection ω^{ab} are treated as independent variables. These fields can be regarded as components of a gauge connection associated with the Poincaré algebra or the (Anti-)de Sitter algebra. Nevertheless, General Relativity cannot be formulated as a genuine gauge theory in this setting, since the Einstein–Hilbert action fails to be invariant under local Poincaré translations or AdS boosts.

A metric-independent gravitational action in odd spacetime dimensions was proposed by Chamseddine [84, 85]. In the first-order formalism, the corresponding Lagrangian takes the form

$$L^{(2n+1)} = \kappa\, \varepsilon_{a_1 a_2 \cdots a_{2n+1}} \sum_{k=0}^{n} \frac{c_k}{l^{2(n-k)+1}} R^{a_1 a_2} \cdots R^{a_{2k-1} a_{2k}} e^{a_{2k+1}} \cdots e^{a_{2n+1}}, \tag{5.0.1}$$

where κ is a dimensionless constant, l is a length parameter, e^a denotes the vielbein one-form, and

$$R^{ab} = d\omega^{ab} + \omega^a{}_c\, \omega^{cb} \tag{5.0.2}$$

is the Riemann curvature two-form. For the specific choice of coefficients

$$c_k = \frac{1}{2(n+k)+1} \binom{n}{k}, \tag{5.0.3}$$

the Lagrangian (5.0.1) is identified as a Chern–Simons form for the AdS algebra.

The absence of a background metric in the definition of the Chern–Simons form is precisely what makes it suitable for constructing a genuine gauge theory of gravity. Various explicit realizations of Chern–Simons gravity theories can be found in Refs. [84–86].

5.1 Chern forms

Let T_A be a basis of the Lie algebra $\mathfrak{g}$ associated with a group G. Consider an invariant $(2n+2)$-form $P^{(2n+2)}(F)$ constructed from the curvature 2-form

$$F = dA + \frac{1}{2}[A,A] = F^A T_A,$$

where A denotes a gauge connection 1-form taking values in the Lie algebra $\mathfrak{g}$,

$$A = A^A T_A.$$

Assume now that there exists a $(2n+1)$-form $Q^{(2n+1)}$, built from A and dA, satisfying

$$dQ^{(2n+1)} = P^{(2n+2)}.$$

Under a gauge transformation, such a form does not remain strictly invariant; instead, its variation is given by a total derivative, that is, an exact form,

$$\delta Q^{(2n+1)} = d(\text{something}).$$

The $(2n+1)$-form $Q^{(2n+1)}$ is referred to as the Chern–Simons (CS) form. Because its gauge variation reduces to an exact term, it can be employed as a Lagrangian for a gauge theory formulated in terms of the connection A. In explicit form,

$$Q^{(2n+1)}_{CS} = (n+1) \int_0^1 dt \left\langle A \left(t\, dA + t^2 A^2 \right)^n \right\rangle, \tag{5.1.1}$$

where $\langle\cdots\rangle$ represents a symmetric invariant tensor of rank $n+1$ under $\mathfrak{g}$. The CS Lagrangian $Q_{CS}^{(2n+1)}$ is therefore a $(2n+1)$-form whose exterior derivative obeys

$$dQ^{(2n+1)} = \langle F\wedge\cdots\wedge F\rangle = \langle F^{n+1}\rangle.$$

It is worth emphasizing that $Q_{CS}^{(2n+1)}$ provides a nontrivial Lagrangian. Although it is not strictly invariant under gauge transformations, its variation depends exclusively on boundary contributions. In this sense, the Chern–Simons form is gauge quasi-invariant. For infinitesimal gauge transformations of the type

$$\delta A = d\lambda + [A,\lambda],$$

the CS form remains invariant up to boundary terms. This feature is sufficient for a consistent physical Lagrangian, since appropriate boundary conditions may always be imposed so that $\delta Q_{CS}^{(2n+1)} = 0$.

The equations of motion follow from varying the action with respect to the connection:

$$\delta S = \delta\int_M Q_{CS}^{(2n+1)} = \delta\int_{\partial M} Q^{(2n+1)} = n\int_{\partial M}\langle\delta F\wedge F^n\rangle.$$

Using the identity $\delta F = \nabla(\delta A)$ together with the Bianchi identity $\nabla F = 0$, one finds

$$\delta S = n\int_{\partial M}\langle\nabla(\delta A)F^n\rangle = n\int_{\partial M} d\langle\delta A F^n\rangle.$$

Applying Stokes' theorem leads to the field equations

$$\delta S = n\int_{\partial M}\delta A^A\langle T_A F^n\rangle = 0$$

$$\Rightarrow \langle F^n T_A\rangle = 0.$$

This construction is not confined to a single invariant, commonly referred to as a characteristic class. Well-known examples include the Euler, Chern, and Pontryagin classes, each of which admits an associated Chern–Simons form. In three dimensions, one obtains specific CS forms defining Lagrangians together with their corresponding topological invariants, as summarized in Table 5.1. Here R^{ab} denotes the Lorentz curvature, ω^{ab} the corresponding connection, and T^a the torsion. The quantities E_4 and P_4 represent the Euler and Pontryagin densities, while N_4 stands for the Nieh–Yan invariant [72, 86]. These Lagrangians are locally invariant under their respective gauge groups.

Finally, Chern–Simons gauge theories differ substantially from Yang–Mills theories. The essential distinction lies in the fact that CS forms are written explicitly in terms of the connection A and its exterior derivatives, whereas they cannot be expressed as local functionals depending solely on the curvature F. Moreover, unlike the Yang–Mills Lagrangian, the Chern–Simons Lagrangian exists only in odd spacetime dimensions.

Group	D = 3 Chern–Simons Lagrangians	Topological Invariant
$SO(2,2)$	$L_3^{(\text{AdS})} = \varepsilon_{abc}\left(R^{ab}e^c + \frac{1}{3l^2}e^a e^b e^c\right)$	$E_4 = \varepsilon_{abc}\left(R^{ab} \pm \frac{1}{l^2}e^a e^b\right)T^c$
$SO(2,1)$	$L_3^{\text{Lorentz}} = \omega^a{}_b d\omega^b{}_a + \frac{2}{3}\omega^a{}_b\omega^b{}_c\omega^c{}_a$	$P_4 = R^a{}_b R^b{}_a$
$SO(2,1)$	$L_3^{\text{Torsion}} = e^a T_a$	$N_4 = T^a T_a - e^a e^b R_{ab}$

Table 5.1: $D = 3$ Chern–Simons Lagrangians and associated topological invariants.

5.2 Gravity and Chern-Simons

In order to formulate a Chern–Simons theory of gravity in $D = 2n+1$ dimensions, we consider the Anti–de Sitter algebra $\mathfrak{so}(2n,2)$. Its generators P_a and J_{ab} satisfy the commutation relations

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \eta_{cb}J_{ad} - \eta_{ca}J_{bd} + \eta_{db}J_{ca} - \eta_{da}J_{cb}, \\
[J_{ab}, P_c] &= \eta_{cb}P_a - \eta_{ca}P_b, \\
[P_a, P_b] &= J_{ab}.
\end{aligned} \tag{5.2.1}$$

Here η_{ab} denotes the Minkowski metric.

A gauge connection 1-form valued in this algebra can be written as

$$A = A^A T_A = \frac{1}{l}e^a P_a + \frac{1}{2}\omega^{ab}J_{ab},$$

where the gauge fields associated with P_a and J_{ab} are identified with the vielbein e^a and the spin connection ω^{ab}, respectively. The corresponding curvature 2-form is

$$F = F^A T_A = \frac{1}{l}T^a P_a + \frac{1}{2}\left(R^{ab} + \frac{1}{l^2}e^a e^b\right)J_{ab}, \tag{5.2.2}$$

where R^{ab} and T^a denote the Lorentz curvature and torsion, given explicitly by

$$\begin{aligned}
R^{ab} &= d\omega^{ab} + \omega^a{}_c\omega^{cb}, \\
T^a &= De^a = de^a + \omega^a{}_b e^b.
\end{aligned}$$

To construct the CS Lagrangian for the AdS algebra, one needs a symmetric invariant tensor of rank $n+1$. For $\mathfrak{so}(2n,2)$, such an invariant tensor is provided by the Levi–Civita tensor,

$$\left\langle J_{a_1a_2}\cdots J_{a_{2n-1}a_{2n}}P_{a_{2n+1}}\right\rangle = \frac{2^n}{n+1}\,\varepsilon_{a_1\cdots a_{2n+1}}, \tag{5.2.3}$$

with all other components vanishing.

Substituting the algebra (5.2.1) and the invariant tensor (5.2.3) into the general Chern–Simons form (5.1.1), one obtains the Lagrangian originally proposed by Chamseddine (5.0.1), namely

$$L^{(2n+1)} = \kappa\, \varepsilon_{a_1 a_2 \cdots a_{2n+1}} \sum_{k=0}^{n} \frac{1}{l^{2(n-k)+1}} \frac{1}{2(n+k)+1} \binom{n}{k} R^{a_1 a_2} \cdots R^{a_{2k-1} a_{2k}} e^{a_{2k+1}} \cdots e^{a_{2n+1}}.$$

It is important to stress that deriving the CS form directly from (5.1.1) using the nonvanishing components of an invariant tensor is, in general, a nontrivial task. Furthermore, due to the particular structure of the invariant tensor (5.2.3), torsion does not appear in the Lagrangian. Nevertheless, the *AdS* algebra admits additional nonvanishing components of the invariant tensor that allow for torsional terms in the Lagrangian; these contributions will be analyzed later.

By definition, the CS form satisfies

$$dL_{CS}^{(2n+1)} = P^{(2n+2)},$$

where the invariant $(2n+2)$-form is given by

$$\begin{aligned} P =& \langle F \wedge \cdots \wedge F \rangle \\ =& \langle F^{n+1} \rangle. \end{aligned}$$

Using the curvature 2-form (5.2.2) together with the nonvanishing component of the invariant tensor (5.2.3), one finds

$$P = E^{2n+2} = \frac{\kappa}{l} \varepsilon_{a_1 a_2 \ldots a_{2n+1}} \left(R^{a_1 a_2} + \frac{1}{l^2} e^{a_1} e^{a_2} \right) \cdots \left(R^{a_{2n-1} a_{2n}} + \frac{1}{l^2} e^{a_{2n-1}} e^{a_{2n}} \right) T^{a_{2n+1}},$$

which corresponds to the $(2n+2)$-dimensional Euler density.

5.3 Gravity and Torsion

It is well established that the Lovelock action provides the most general metric theory of gravity in D dimensions yielding second-order field equations. It can be expressed as

$$S_d = \int \sum_{p=0}^{\left[\frac{D}{2}\right]} \alpha_p L_D^{(p)}$$

with

$$L_D^{(p)} = \varepsilon_{a_1 \cdots a_D} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_D}.$$

We have shown that varying the action with respect to the vielbein leads to the Lovelock field equations

$$\sum_{p=0}^{\left[\frac{D}{2}\right]} \alpha_p (D-2p) \varepsilon_{b_1 \cdots b_{D-1}} R^{b_1 b_2} \cdots R^{b_{2p-1} b_{2p}} e^{b_{2p+1}} \cdots e^{b_{D-1}} = 0.$$

In contrast, variation with respect to the spin connection yields

$$\sum_{p=0}^{\left[\frac{D}{2}\right]} \alpha_p p (D-2p) \varepsilon_{a b a_3 \cdots a_D} R^{a_3 a_4} \cdots R^{a_{2p-1} a_{2p}} T^{a_{2p+1}} e^{a_{2p+2}} \cdots e^{a_D} = 0. \tag{5.3.1}$$

If one imposes the vanishing of torsion identically,

$$T^a = de^a + \omega^a_{\ b} e^b = 0, \tag{5.3.2}$$

then equation (5.3.1) is automatically satisfied. However, solving (5.3.2) requires introducing additional constraints on the spin connection ω^{ab} [86]. As a result, the Lagrangian becomes a complicated functional of the vielbein e^a alone. Moreover, for $D \leq 4$, equation (5.3.1) is equivalent to (5.3.2), so that the torsion-free condition need not be imposed separately. In higher dimensions, however, equation (5.3.1) does not imply vanishing torsion, and therefore setting $T^a = 0$ amounts to an additional and generally unjustified restriction.

Furthermore, when the vielbein and the spin connection are combined into a single 1-form connection for the Lorentz group, curvature and torsion appear as different components of a unified gauge curvature 2-form. From this standpoint, imposing the condition $T^a = 0$ while allowing nonvanishing curvature seems conceptually arbitrary.

A more general framework is obtained by explicitly incorporating torsion into the Lagrangian, assuming that it is the most general D-form constructed from the vielbein and the spin connection and invariant under local Lorentz transformations. There exists a constructive algorithm that generates all possible local Lorentz invariants built from e^a, R^{ab} and T^a [87].

Chapter 6

Kinematical Algebras

Kinematical, or space-time, symmetry algebras play a central role in formulating physical theories. Based on general assumptions, Bacry and Lévy-Leblond (1968) classified all kinematical algebras that relate distinct inertial frames of reference [1]. Their generators correspond to time translations, space translations, rotations, and boosts. Beyond the relativistic AdS and Poincaré algebras, this classification includes non-Lorentzian algebras, which have attracted renewed interest due to diverse physical applications. Non-relativistic algebras arise, for instance, in holography [2–13] , Hořava-Lifshitz gravity [14–19] , and effective field theory descriptions of the quantum Hall effect [20–24] . Conversely, ultra-relativistic algebras are relevant in tachyon condensation [25], warped conformal field theories [26], tensionless strings [27–31] , holography in asymptotically flat space-times [32–41] , asymptotic symmetries [42–44] , and black hole physics [45–51].

In this section, we briefly review the kinematical Lie algebras in three spacetime dimensions, following the cube introduced by Bacry and Lévy-Leblond [1]. Beginning with the $\mathfrak{so}(2,2)$ algebra, different kinematical algebras emerge through successive Inönü-Wigner contractions, representing distinct physical regimes.

6.1 AdS and Poincaré

Given the AdS algebra in 2+1 dimensions

$$
\begin{aligned}
[J_A, J_B] &= \varepsilon_{ABC} J^C \\
[J_A, P_B] &= \varepsilon_{ABC} P^C \\
[P_A, P_B] &= \varepsilon_{ABC} J^C,
\end{aligned}
$$

we compute the equations of motion in the Chern-Simons formalism for AdS and Poincaré. We can obtain the Poincaré algebra by rescaling the translations P_A,

$$
P_A \longrightarrow \ell P_A,
$$

where ℓ is a rescaling constant with dimensions of length. Eventually, we obtain

$$[P_A, P_B] = \varepsilon_{ABC} J^C \longrightarrow [P_A, P_B] = \frac{1}{\ell^2} \varepsilon_{ABC} J^C,$$

however, $\Lambda = 1/\ell^2$, so when $\Lambda \to 0$ we recover the Poincaré algebra. This mechanism will be discussed further in the next section.

To use Chern-Simons, it is necessary to evaluate the algebra on the connection 1-form A, therefore for AdS and Poincaré we take

$$A = \omega^A J_A + e^A P_A,$$

with ω and e the Lorentz connection and the vielbein, respectively. We can obtain the curvature 2-forms $R^A(\omega)$ and $R^A(e)$, where the parenthesis indicates the translation and rotation parts respectively from the 2-form

$$F = dA + \frac{1}{2}[A, A] = R^A(\omega) J_A + R^A(e) P_A.$$

We now expand

$$\begin{aligned} dA + \frac{1}{2}[A,A] &= d(\omega^A J_A + e^A P_A) + \frac{1}{2}[\omega^A J_A + e^A P_A, \omega^B J_B + e^B P_B] \\ &= d\omega^A J_A + de^A P_A + \frac{1}{2}\omega^A \omega^B [J_A, J_B] + \omega^A e^B [J_A, P_B] + \frac{1}{2} e^A e^B [P_A, P_B] \\ &= d\omega^A J_A + de^A P_A + \frac{1}{2}\omega^A \omega^B \varepsilon_{ABC} J^C + \omega^A e^B \varepsilon_{ABC} P^C + \frac{1}{2\ell^2} e^A e^B \varepsilon_{ABC} J^C \\ &= \left(d\omega^A + \frac{1}{2}\omega^B \omega^C \varepsilon^A{}_{BC} + \frac{1}{2\ell^2} e^B e^C \varepsilon^A{}_{BC} \right) J_A + \left(de^A + \omega^B e^C \varepsilon^A{}_{BC} \right) P_A. \end{aligned}$$

In this way, we find that the curvature 2-forms for AdS are

$$\overline{R}^A \equiv d\omega^A + \frac{1}{2}\omega^B \omega^C \varepsilon^A{}_{BC} + \frac{1}{2\ell^2} e^B e^C \varepsilon^A{}_{BC} \tag{6.1.1a}$$

$$T^A \equiv de^A + \omega^B e^C \varepsilon^A{}_{BC} = De^A, \tag{6.1.1b}$$

where $\overline{R}^A$ is denoted as the AdS 2-form curvature, while T^A corresponds to the 2-form torsion. The curvatures for the Poincaré algebra are

$$R^A \equiv= d\omega^A + \frac{1}{2}\omega^B \omega^C \varepsilon^A{}_{BC} \tag{6.1.2}$$

when $\ell \to \infty$. R^A is known as the Poincaré 2-form curvature. T^A is the same as before. We now proceed to use the Chern-Simons formulation. For this we require the invariant tensors in addition to the algebra. For AdS we have

$$\begin{aligned}\langle J_A J_B\rangle &= \alpha_0\eta_{AB}\\ \langle P_A P_B\rangle &= \alpha_0\eta_{AB}\\ \langle J_A P_B\rangle &= \alpha_1\eta_{AB},\end{aligned}$$

where $\langle\cdot\rangle$ denotes the symmetrized trace. For the Poincaré algebra, we rescale the P's as before and assume α_1 rescales in the same way. Therefore, the invariant tensors for the Poincaré algebra are

$$\begin{aligned}\langle J_A J_B\rangle &= \alpha_0\eta_{AB}\\ \langle P_A P_B\rangle &= 0\\ \langle J_A P_B\rangle &= \alpha_1\eta_{AB},\end{aligned}$$

The Chern-Simons action is

$$I_{CS} = \int \langle AdA + \frac{2}{3}A^3\rangle.$$

We first compute A^3. Since the symmetrized trace is linear, we have

$$\begin{aligned}\frac{2}{3}\langle A^3\rangle &= \frac{1}{3}\langle A[A,A]\rangle = \frac{1}{3}\Big\langle A\Big(\frac{1}{2}\omega^A\omega^B\varepsilon_{ABC}J^C + \omega^A e^B\varepsilon_{ABC}P^C + \frac{1}{2\ell^2}e^A e^B\varepsilon_{ABC}J^C\Big)\Big\rangle\\
&= \frac{1}{3}\Big\langle\Big(\omega^D J_D + e^D P_D\Big)\Big(\frac{1}{2}\omega^A\omega^B\varepsilon_{ABC}J^C + \omega^A e^B\varepsilon_{ABC}P^C + \frac{1}{2\ell^2}e^A e^B\varepsilon_{ABC}J^C\Big)\Big\rangle\\
&= \frac{1}{3}\Big(\omega^D\omega^A\omega^B\varepsilon_{AB}{}^C\langle J_D J_C\rangle + 2\omega^D\omega^A e^B\varepsilon_{AB}{}^C\langle J_D P_C\rangle + \frac{1}{\ell^2}\omega^D e^A e^B\varepsilon_{AB}{}^C\langle J_D J_C\rangle\\
&\quad + e^D\omega^A\omega^B\varepsilon_{AB}{}^C\langle P_D J_C\rangle + 2e^D\omega^A e^B\varepsilon_{AB}{}^C\langle P_D P_C\rangle + \frac{1}{\ell^2}e^D e^A e^B\varepsilon_{AB}{}^C\langle P_D J_C\rangle\Big)\\
&= \frac{1}{3}\omega^D\omega^A\omega^B\varepsilon_{AB}{}^C\eta_{DC}\alpha_0 + \frac{2}{3}\omega^D\omega^A e^B\varepsilon_{AB}{}^C\eta_{DC}\alpha_1 + \frac{1}{3\ell^2}\omega^D e^A e^B\varepsilon_{AB}{}^C\eta_{DC}\alpha_0\\
&\quad + \frac{1}{3}e^D\omega^A\omega^B\varepsilon_{AB}{}^C\eta_{DC}\alpha_1 + \frac{2}{3\ell^2}e^D\omega^A e^B\varepsilon_{AB}{}^C\eta_{DC}\alpha_0 + \frac{1}{3\ell^2}e^D e^A e^B\varepsilon_{AB}{}^C\eta_{DC}\alpha_1\\
&= \frac{1}{3}\omega^D\omega^A\omega^B\varepsilon_{ABD}\,\alpha_0 + \omega^D\omega^A e^B\varepsilon_{ABD}\,\alpha_1 + \frac{1}{\ell^2}e^D\omega^A e^B\varepsilon_{ABD}\,\alpha_0 + \frac{1}{3\ell^2}e^D e^A e^B\varepsilon_{ABD}\,\alpha_1\\
&= \Big(\frac{1}{3}\omega^A\omega^B\omega^C\varepsilon_{ABC} + \frac{1}{\ell^2}e^A\omega^B e^C\varepsilon_{ABC}\Big)\alpha_0 + \Big(\omega^A\omega^B e^C\varepsilon_{ABC} + \frac{1}{3\ell^2}e^A e^B e^C\varepsilon_{ABC}\Big)\alpha_1,\end{aligned}$$

where translations P_A were rescaled. We continue with $\langle AdA\rangle$

$$\begin{aligned}\langle AdA\rangle &= \langle(\omega^A J_A + e^A P_A)(d\omega^C J_C + de^C P_C)\rangle\\
&= \omega^A d\omega^C\langle J_A J_C\rangle + \omega^A de^C\langle J_A P_C\rangle + e^A d\omega^C\langle P_A J_C\rangle + e^A de^C\langle P_A P_C\rangle\\
&= \Big(\omega^A d\omega_A + \frac{1}{\ell^2}e^A de_A\Big)\alpha_0 + \Big(\omega^A de_A + e^A d\omega_A\Big)\alpha_1.\end{aligned}$$

Notice that

$$
\begin{aligned}
& d(\omega^A e_A) = d\omega^A e_A - \omega^A de_A = e_A d\omega^A - \omega^A de_A \\
& \Rightarrow \omega^A de_A = e_A d\omega^A - d(\omega^A e_A),
\end{aligned}
\tag{6.1.3}
$$

however the last term vanishes in the integral. Finally the action for AdS is the following

$$
\begin{aligned}
I_{CS}^{AdS} = \int & \left(\omega^A d\omega_A + \frac{1}{\ell^2} e^A de_A + \frac{1}{3}\omega^A\omega^B\omega^C\varepsilon_{ABC} + \frac{1}{\ell}\omega^A e^B e^C \varepsilon_{ABC}\right)\alpha_0 \\
& + \left(2e^A d\omega^A + \omega^A\omega^B e^C \varepsilon_{ABC} + \frac{1}{3\ell^2} e^A e^B e^C \varepsilon_{ABC}\right)\alpha_1 .
\end{aligned}
$$

Recognizing the 2-form curvatures obtained in Eq. (6.1.1) and Eq. (6.1.2) we can wrote it as

$$
I_{CS}^{AdS} = \int \left(\omega^A d\omega_A + \frac{1}{3}\omega^A\omega^B\omega^C\varepsilon_{ABC} + \frac{1}{\ell^2} e^A T_A\right)\alpha_0 + \left(2e^A R_A + \frac{1}{3\ell^2} e^A e^B e^C \varepsilon_{ABC}\right)\alpha_1 .
\tag{6.1.4}
$$

Hence for Poincaré we have

$$
I_{CS}^{P} = \int \left(\omega^A d\omega_A + \frac{1}{3}\omega^A\omega^B\omega^C\varepsilon_{ABC}\right)\alpha_0 + 2e^A R_A \alpha_1 .
\tag{6.1.5}
$$

where AdS 2-form curvature $\overline{R}_A$ tends to Poincaré 2-form curvature R_A. The first term accompanied by α_0 is known as the exotic Lagrangian, in which gravity was first successfully quantized in 2+1 dimensions [88]. Now we can calculate the equation of motion for these algebras. First we vary AdS action (6.1.4) respect to the vielbein

$$
\begin{aligned}
\delta_e I_{CS}^{AdS} &= \delta_e \int \left(\omega^A d\omega_A + \frac{1}{3}\omega^A\omega^B\omega^C\varepsilon_{ABC} + \frac{1}{\ell^2} e^A T_A\right)\alpha_0 + \left(2e^A R_A + \frac{1}{3\ell^2} e^A e^B e^C \varepsilon_{ABC}\right)\alpha_1 \\
&= \int \frac{1}{\ell^2}\delta_e(e^A T_A)\alpha_0 + 2\delta_e\left(2e^A R_A + \frac{1}{3\ell^2} e^A e^B e^C \varepsilon_{ABC}\right)\alpha_1 ,
\end{aligned}
$$

from the first term, considering $T_A = De_A$ and $\delta_e De_A = D(\delta e_A)$, eventually we obtain $2\delta_e e^A T_A \alpha_0/\ell^2$. However from the sencond term we get

$$
2\delta_e\left(2e^A R_A + \frac{1}{3\ell^2} e^A e^B e^C \varepsilon_{ABC}\right) = 2\delta_e e^A \overline{R}_A .
$$

Hence the variation of AdS action respect to the vielbein is

$$
\delta_e I_{CS}^{AdS} = 2\delta_e e^A \left(\frac{1}{\ell^2} T_A \alpha_0 + \overline{R}_A \alpha_1\right),
$$

which after impose the minimum action principle respect to translation (vielbain) then we get

$$\frac{1}{\ell^2}T_A\alpha_0+\overline{R}_A\alpha_1=0,$$

since e's are arbitraries. Finally the equations of motion in Chern-Simons formalism for AdS are $T_A=0$ and $\overline{R}_A=0$ where this last one implies

$$R_A=-\frac{1}{2\ell^2}e^Be^C\varepsilon^{ABC},$$

where since $1/\ell^2$ is proportional to the cosmological constant Λ, it causes the curvature over AdS frame. Now we write the variation respect to the rotations or Lorentz connection ω obtaining the same equations of motions

$$\begin{aligned}\delta_\omega I_{CS}^{AdS}&=\delta_\omega\int\left(\omega^A d\omega_A+\frac{1}{3}\omega^A\omega^B\omega^C\varepsilon_{ABC}+\frac{1}{\ell^2}e^AT_A\right)\alpha_0+2e^A\overline{R}_A\alpha_1\\&=\int\delta_\omega\left(\omega^A d\omega_A+\frac{1}{3}\omega^A\omega^B\omega^C\varepsilon_{ABC}+\frac{1}{\ell^2}e^AT_A\right)\alpha_0+2e^A\delta_\omega\overline{R}_A\alpha_1,\end{aligned}$$

where from the first term we eventually get after impossing the minimum action principle respect to rotations (Lorentz connection)

$$\overline{R}_A\alpha_0+T_A\alpha_1=0,$$

which implies as before $T_A=0$ and $\overline{R}_A=0$.

In the following section, we shall decompose the generators, the algebra, the curvatures, and the action in order to derive new algebras via a contraction mechanism, analogous to the procedure employed to obtain Poincaré from AdS in the present section.

6.2 Decomposing space and temporal parts

In order to obtain different algebras through distinct contraction mechanisms, we first split the space and time components of the AdS (or Poincaré) generators as

$$J_A = (J_0, J_a) = (J, G_a),$$

where $A = 0,1,2$ and $a = 1,2$. Here $J \equiv J_0$ corresponds to spatial rotations, while $G_a \equiv J_a$ represent the boost generators. Similarly, the translation sector is written as

$$P_A = (P_0, P_a) = (H, P_a),$$

where $H \equiv P_0$ is the time translation (the Hamiltonian), and P_a are the spatial translations. By rewriting the AdS commutation relations, we begin with the rotational sector $[J_A, J_B]$. For $A = 0$ and $B = a$ we obtain

$$[J, G_a] = \varepsilon_{0ab} J^b = \varepsilon_{0ab} \eta^{bc} J_c = \varepsilon_{0ab} \delta^{bc} G_c = \varepsilon_{ab} G_b,$$

where we define $\varepsilon_{ab} \equiv \varepsilon_{0ab}$. Since we work in flat spacetime, indices on the spatial subspace may be raised or lowered without sign changes. Introducing $\varepsilon^{ab} \equiv \varepsilon^{0ab}$, we observe

$$\varepsilon_{ab} = \varepsilon_{0ab} = \eta_{00} \varepsilon^{0}{}_{ab} = -\varepsilon^{0ab} = -\varepsilon^{ab},$$

implying that neither $\varepsilon^{a}{}_{b}$ nor $\varepsilon_{a}{}^{b}$ are well defined. For the remaining commutator $[G_a, G_b] = \varepsilon_{abC} J^C$, since a and b only take values 1 and 2, it follows that $C = 0$, yielding

$$[G_a, G_b] = \varepsilon_{ab0} J^0 = \varepsilon_{0ab} \eta^{00} J_0 = -\varepsilon_{ab} J.$$

Proceeding with $[J_A, P_B] = \varepsilon_{ABC} P^C$, we obtain

$$[J, P_a] = \varepsilon_{ab} P_b, \quad [H, G_a] = \varepsilon_{ab} P_b,$$

while for the mixed commutator $[G_a, P_b]$,

$$[G_a, P_b] = \varepsilon_{ab0} P^0 = \varepsilon_{0ab} \eta^{00} P_0 = -\varepsilon_{ab} H.$$

Finally, for the translation sector, we find

$$[P_a, P_b] = -\varepsilon_{ab} J, \quad [H, P_a] = \varepsilon_{ab} G_b.$$

Collecting all brackets, the AdS and Poincaré algebras take the form shown in Table 6.1. The associated invariant tensors are displayed separately in Table 6.2.

Thus, the gauge connection can be written as

$$A = e^A P_A + \omega^A J_A = \tau H + e^a Pa + \omega J + \omega^a G_a,$$

Algebra	$[J,G_a]$	$[G_a,G_b]$	$[J,P_a]$	$[G_a,P_b]$	$[H,G_a]$	$[H,P_a]$	$[P_a,P_b]$
AdS	$\varepsilon_{ab}G_b$	$-\varepsilon_{ab}J$	$\varepsilon_{ab}P_b$	$-\varepsilon_{ab}H$	$\varepsilon_{ab}P_b$	$\varepsilon_{ab}G_b$	$-\varepsilon_{ab}J$
Poincaré	$\varepsilon_{ab}G_b$	$-\varepsilon_{ab}J$	$\varepsilon_{ab}P_b$	$-\varepsilon_{ab}H$	$\varepsilon_{ab}P_b$	0	0

Table 6.1: Comparison between the $(2+1)$ AdS and Poincaré algebras in kinematical basis.

Algebra	$\langle J\,J\rangle$	$\langle G_a\,G_b\rangle$	$\langle J\,H\rangle$	$\langle G_a\,P_b\rangle$	$\langle H\,H\rangle$	$\langle P_a\,P_b\rangle$
AdS	$-\alpha_0$	$\alpha_0\delta_{ab}$	$-\alpha_1$	$\alpha_1\delta_{ab}$	$-\alpha_0$	$\alpha_0\delta_{ab}$
Poincaré	$-\alpha_0$	$\alpha_0\delta_{ab}$	$-\alpha_1$	$\alpha_1\delta_{ab}$	0	0

Table 6.2: Comparison between the invariant tensors of the $(2+1)$ AdS and Poincaré algebras.

Curvature	**AdS**	**Poincaré**
$R(\omega)$	$d\omega+\frac{1}{2}\,\omega_a\omega_b\varepsilon^{ab}+\frac{1}{2}\,e_ae_b\varepsilon^{ab}$	$d\omega+\frac{1}{2}\,\omega_a\omega_b\varepsilon^{ab}$
$R^a(\omega)$	$d\omega^a+\omega\omega_b\varepsilon^{ab}+\tau e_b\varepsilon^{ab}$	$d\omega^a+\omega\omega_b\varepsilon^{ab}$
$R(\tau)$	$d\tau+\omega_ae_b\varepsilon^{ab}$	$d\tau+\omega_ae_b\varepsilon^{ab}$
$R^a(e)$	$de^a+\omega e_b\varepsilon^{ab}+\tau\omega_b\varepsilon^{ab}$	$de^a+\omega e_b\varepsilon^{ab}+\tau\omega_b\varepsilon^{ab}$

Table 6.3: Comparison between the AdS and Poincaré curvatures in $(2+1)$ dimensions.

where $\tau \equiv e^0$ and $\omega \equiv \omega^0$, such that $e^A = (\tau, e^a)$ and $\omega^A = (\omega, \omega^a)$. The corresponding curvatures are then in Table 6.3.

From this point forward, we will generate different algebras through the contraction mechanism introduced by Inönu and Wigner [89–91]. The idea is to rescale selected generators by a contraction parameter, whose choice depends on the physical limit of interest. For instance, the well-known contraction from AdS to Poincaré is achieved by rescaling the translations as $P_A \to \ell P_A$, known as a **space-time contraction**. Two additional possibilities exist: a **speed-space contraction**, where spatial generators are rescaled as $\{P_a, G_a\} \to \{\sigma P_a, \sigma G_a\}$, and a **speed-time contraction**, defined by rescaling the time-sector generators $\{G_a, H\} \to \{\kappa G_a, \kappa H\}$. These parameters are proportional or inversely proportional to the speed of light, except for ℓ, which is inversely proportional to the cosmological constant Λ, i. e., $\Lambda = 1/\ell$, and $c = \sigma = 1/\kappa$. This information is summarized in Table 6.4.

Generator	Space-time	Speed-space	Speed-time	General
J	J	J	J	J
G_a	G_a	σG_a	κG_a	$\sigma\kappa G_a$
H	ℓH	H	κH	$\ell\kappa H$
P_a	ℓP_a	σP_a	P_a	$\ell\sigma P_a$

Table 6.4: Different Inönü–Wigner contractions of the AdS Lie algebra.

We can summarize the resulting algebras in the diagram displayed in Fig. 1.1. As we will show, the algebras located at the bottom of the cube turn out to be degenerate, while those at the top admit non-degenerate invariant tensors.

6.3 Galilei algebra

We now proceed to obtain a new algebra by performing a speed-space contraction of the Poincaré algebra Table 6.1, under which the following generators are rescaled

$$\begin{aligned} P_a &\to \sigma P_a \\ G_a &\to \sigma G_a \end{aligned} \quad \Longrightarrow \quad \begin{aligned} [G_a, P_b] &= -\frac{1}{\sigma^2}\varepsilon_{ab}H \\ [G_a, G_b] &= -\frac{1}{\sigma^2}\varepsilon_{ab}J, \end{aligned}$$

from which it becomes clear that the affected commutators vanish in the limit $\sigma \to \infty$. This yields the non-relativistic limit of the Poincaré algebra, commonly known as the Galilei algebra,

$$
\begin{aligned}
&[J,G_a] = \varepsilon_{ab}G_b, \qquad [G_a,P_b] = 0, \\
&[G_a,G_b] = 0, \qquad [H,G_a] = \varepsilon_{ab}P_b, \qquad [P_a,P_b] = 0. \\
&[J,P_a] = \varepsilon_{ab}P_b, \qquad [H,P_a] = \varepsilon_{ab}G_b,
\end{aligned}
\tag{6.3.1}
$$

This algebra characterizes regimes in which relativistic effects are absent, i.e. Newtonian systems, where the original boost generators G_a reduce to standard Galilean boosts describing changes between inertial reference frames, without time dilation or spatial contraction [1].

We are now in a position to construct the Chern–Simons action associated with the Galilei algebra. For this purpose, we introduce the invariant tensors

$$
\begin{aligned}
&\langle JJ\rangle = -\alpha_0, \qquad \langle G_aP_b\rangle = 0, \\
&\langle G_aG_b\rangle = 0, \qquad \langle HH\rangle = 0, \\
&\langle JH\rangle = -\alpha_1, \qquad \langle P_aP_b\rangle = 0,
\end{aligned}
\tag{6.3.2}
$$

As will be shown later, this structure leads to a degeneracy in the resulting theory. We now compute the curvature 2-form,

$$F = dA + \frac{1}{2}[A,A] = R(\tau)H + R^a(e)P_a + R(\omega)J + R^a(\omega)G_a,$$

where the notation indicates that each curvature component multiplies its corresponding generator. From (6.3.1), the time translation generator H does not appear in any commutator contributing to $R(\tau)$, and therefore

$$R(\tau) = d\tau.$$

A similar reasoning applies to the spatial component $R^a(e)$. Since in (6.3.1) there are two contributions involving P_a, and these appear twice in F (for instance through $\omega e^b[J,P_b]$ and $e^a\omega[P_a,J]$), we obtain

$$R^a(e) = de^a + \omega e_b\varepsilon^{ab} + \tau\omega_b\varepsilon^{ab}.$$

Proceeding analogously for the remaining components of the curvature yields

$$R(\omega) = d\omega,$$
$$R^a(\omega) = d\omega^a + \omega\omega_b\varepsilon^{ab} + \tau e_b\varepsilon^{ab}.$$

Finally, the Chern–Simons action for the Galilei algebra takes the form

$$I^G_{CS} = \int \Big\langle AdA \Big\rangle,$$

since the cubic contribution vanishes, since

$$A^2 = \frac{1}{2}[A,A]$$

produces only spatial generators P_a and G_a, all of which give zero when contracted with the invariant tensor (6.3.2). Therefore, the resulting action reduces to

$$I^G_{CS} = \int \Big(\omega d\omega\Big)\alpha_0 + \Big(-\tau d\omega - \omega d\tau\Big)\alpha_1.$$

We conclude by noting that the resulting action is degenerate, as it does not produce equations of motion for the fields e^a and ω^a.

6.4 Carroll Algebra

Taking Poincaré algebra and by a speed-time contraction, eventually the modified commutators will be

$$\begin{aligned} G_a &\to \kappa G_a \\ H &\to \kappa H \end{aligned} \implies \begin{aligned} [H,G_a] &= -\frac{1}{\kappa^2}\varepsilon_{ab}H \\ [G_a,G_b] &= -\frac{1}{\kappa^2}\varepsilon_{ab}J, \end{aligned}$$

which tend to zero after $\kappa \to \infty$, deriving to the ultra relativistic limit for Poincaré

$$\begin{aligned} [J,G_a] &= \varepsilon_{ab}G_b, & [G_a,P_b] &= -\varepsilon_{ab}H, & & \\ [G_a,G_b] &= 0, & [H,G_a] &= 0, & [P_a,P_b] &= 0. \\ [J,P_a] &= \varepsilon_{ab}P_b, & [H,P_a] &= \varepsilon_{ab}G_b, & & \end{aligned} \tag{6.4.1}$$

It receives the name of Carroll in honor to the author of *Alice in Wonderland* [92], since such the algebra as the book, describe an acausal physic. The algebra consider a frame where everything is highly relativistic since $c \to 0$, this implies a physical context without causality.

We begin to construct the Chern-Simons action for this algebra, starting with the resultant invariant tensors after the speed-time contraction we get

$$\begin{aligned} \langle JJ \rangle &= -\alpha_0, & \langle G_a P_b \rangle &= \alpha_1 \delta_{ab}, \\ \langle G_a G_b \rangle &= 0, & \langle HH \rangle &= 0, \\ \langle JH \rangle &= -\alpha_1, & \langle P_a P_b \rangle &= 0, \end{aligned} \tag{6.4.2}$$

where α_1 rescales as $\kappa\alpha_1$. We continue with the calculation of the 2-form curvatures, where we will consider the Poincaré curvatures already obtained in Table 6.3 and apply the speed-time contraction to the corresponding 1-forms, namely $\{\tau, \omega^a\} \to \{\tau/\kappa, \omega^a/\kappa\}$. Thus, the 2-form Carroll are

$$\begin{aligned} R(\omega) &= d\omega, \\ R^a(\omega) &= d\omega^a + \omega\omega_b \varepsilon^{ab}, \\ R(\tau) &= d\tau + \omega_a e_b \varepsilon^{ab}, \\ R^a(e) &= de^a + \omega e_b \varepsilon^{ab}. \end{aligned} \tag{6.4.3}$$

Computing the Chern-Simons action

$$I_{CS}^C = \int \left\langle AdA + \frac{2}{3}A^3 \right\rangle,$$

we proceed with the second term

$$\begin{aligned} \left\langle \frac{2}{3}A^3 \right\rangle &= \frac{2}{3}\langle AA^2 \rangle = \frac{2}{3}\left\langle A\left(\frac{1}{2}[A,A]\right)\right\rangle \\ &= \frac{2}{3}\left\langle A\left(\varepsilon^{cb}\omega\omega^b G_c + \varepsilon^{cb}\omega^c e^b H + \varepsilon^{cb}\omega e^b P_c\right)\right\rangle \\ &= \frac{2}{3}\varepsilon^{cb} e^a \omega\omega^b \delta_{ac}\alpha_1 - \frac{2}{3}\varepsilon^{cb}\omega\omega^c e^b \alpha_1 + \frac{2}{3}\varepsilon^{cb}\omega^a \omega e^b \delta_{ac}\alpha_1 = 2\varepsilon_{ab}\omega\omega^a e^b \alpha_1. \end{aligned}$$

Thus after computing the first term eventually we obtain

$$I_{CS}^C = \int \Big(-\omega d\omega\Big)\alpha_0 + \Big(\omega^a de_a + e^a d\omega_a - (\omega d\tau + \tau d\omega) + 2\varepsilon_{ab}\omega\omega^a e^b\Big)\alpha_1,$$

where we can rewrite the lagrangian using the 2-forms curvature obtaining

$$I_{CS}^{C}=\int\Big(-\omega d\omega\Big)\alpha_0+\Big(2e^aR^a(\omega^c)-2\tau R(\omega)\Big)\alpha_1.$$

After vary this action by every field, the equation of motions obtained are every 2-form curvature (6.4.3) are 0.

6.5 Newton-Hooke

In analogy with the non-relativistic limit of the Poincaré algebra, we now carry out a speed-space contraction of the AdS algebra. Upon modifying the commutators in Table. 6.1 accordingly, we obtain the resulting algebra

$$\begin{aligned}
&[J,G_a]=\varepsilon_{ab}G_b, && [G_a,P_b]=0, && \\
&[G_a,G_b]=0, && [H,G_a]=\varepsilon_{ab}P_a, && [P_a,P_b]=0. \\
&[J,P_a]=\varepsilon_{ab}P_b, && [H,P_a]=\varepsilon_{ab}G_b, &&
\end{aligned}\tag{6.5.1}$$

And its invariant tensors are

$$\begin{aligned}
&\langle JJ\rangle=-\alpha_0, && \langle G_aP_b\rangle=0, \\
&\langle G_aG_b\rangle=0, && \langle HH\rangle=-\alpha_0, \\
&\langle JH\rangle=-\alpha_1, && \langle P_aP_b\rangle=0.
\end{aligned}\tag{6.5.2}$$

The origin of the name *Newton-Hooke* is rooted in the structure of the algebra itself. In any non-relativistic algebra where the commutator $[H,P_a]$ is proportional to the boost generator G_a, the corresponding kinematical group belongs to the Newton family. As shown in [1], such a structure leads to physical space-time transformations whose temporal evolution satisfies an equation of motion of Hooke-type. Consequently, the resulting kinematics exhibits the characteristic harmonic behavior. Therefore, the associated 2-form curvatures are

$$\begin{aligned}
R(\omega)&=d\omega,\\
R^a(\omega)&=d\omega^a+\omega\omega_b\varepsilon^{ab}+\tau e_b\varepsilon^{ab},\\
R(\tau)&=d\tau,\\
R^a(e)&=de^a+\omega e_b\varepsilon^{ab}+\tau\omega_b\varepsilon^{ab}.
\end{aligned}\tag{6.5.3}$$

The Newton–Hooke action is

$$I_{CS}^{N\text{-}H} = \int \Big(-\tau d\tau - \omega d\omega \Big)\alpha_0 + \Big(-\tau d\omega - \omega d\tau \Big)\alpha_1,$$

It is evident that this Chern–Simons action is degenerate, as the gauge field vielbein e^a and the spin connection ω^a are entirely absent from the action.

6.6 AdS-Carroll or Para-Poincaré

Now as was shown with Poincaré taking its ultra relativistic limtis, we will proceed with AdS a speed-time contraction, the resulting algebra after modified the commutators in Table 6.1 is

$$\begin{aligned}
&[J,G_a] = \varepsilon_{ab}G_b, && [G_a,P_b] = -\varepsilon_{ab}H, && \\
&[G_a,G_b] = 0, && [H,G_a] = 0, && [P_a,P_b] = -\varepsilon_{ab}J. \\
&[J,P_a] = \varepsilon_{ab}P_b, && [H,P_a] = \varepsilon_{ab}G_b, &&
\end{aligned} \tag{6.6.1}$$

And its invariant tensors are

$$\begin{aligned}
\langle JJ\rangle &= -\alpha_0, & \langle G_aP_b\rangle &= \alpha_1\delta_{ab}, \\
\langle G_aG_b\rangle &= 0, & \langle HH\rangle &= 0, \\
\langle JH\rangle &= -\alpha_1, & \langle P_aP_b\rangle &= \alpha_0\delta_{ab}.
\end{aligned}$$

The name Para-Poincaré is due to this algebra is isomorphic to Poincaré since we can exchange these generators $G_a \leftrightarrow P_b$ obtaining the other algebra and reciprocally. However, despite it is isomorphic to Poincaré, they have different physical consequences. Then a way to obtain the 2-forms curvature and the Chern-Simons action is simply by exchange every field $\omega^a \leftrightarrow e^b$ from Poincaré. Therefore the 2-form curvatures are

$$\begin{aligned}
R(\omega) &= d\omega + \frac{1}{2}e_ae_b\varepsilon^{ab}, \\
R^a(\omega) &= d\omega^a + \omega\omega_b\varepsilon^{ab} + \tau e_b\varepsilon^{ab} \\
R(\tau) &= d\tau + e_a\omega_b\varepsilon^{ab}, \\
R^a(e) &= de^a + \omega e_b\varepsilon^{ab}.
\end{aligned} \tag{6.6.2}$$

Here is important to notice that $R^a(\omega)$ and $R^a(e)$ were exchanged from Poincaré curvatures. Now the AdS-Carroll action is

$$I_{CS}^{P\text{-}P} = \int \Big(- \omega d\omega + e^a R_a(e) \Big) \alpha_0 + \Big(2\omega^a R_a(e) - 2\tau R(\omega) \Big) \alpha_1 ,$$

where the equation of motion obtained are every curvature in (6.6.2) equal to 0.

6.7 AdS-Static and Static

Finally, we will obtain the AdS–Static algebra by performing either an ultra-relativistic or non-relativistic contraction from the Newton–Hooke or AdS–Carroll algebra, respectively. We will also consider its non-cosmological limits Static. Therefore in Table 6.5, we can see the only non-vanishing commutator distinguishing the AdS-Static algebra from the Static one is $[H, P_a]$, which disappears in the flat limit.

Algebra	$[J, G_a]$	$[G_a, G_b]$	$[J, P_a]$	$[G_a, P_b]$	$[H, G_a]$	$[H, P_a]$	$[P_a, P_b]$
AdS-Static	$\varepsilon_{ab} G_b$	0	$\varepsilon_{ab} P_b$	0	0	$\varepsilon_{ab} G_b$	0
Static	$\varepsilon_{ab} G_b$	0	$\varepsilon_{ab} P_b$	0	0	0	0

Table 6.5: Comparison between the AdS-Static and Static algebras in kinematical basis.

Both algebra lead the same invariant tensors

$$\begin{aligned} \langle J J \rangle &= -\alpha_0, & \langle G_a P_b \rangle &= 0, \\ \langle G_a G_b \rangle &= 0, & \langle H H \rangle &= 0, \\ \langle J H \rangle &= -\alpha_1, & \langle P_a P_b \rangle &= 0. \end{aligned} \tag{6.7.1}$$

Bacry and Lévy showed that the Static algebra characterizes "infinitely massive" systems. In such a limit, the system is unable to undergo motion, which naturally explains the name of the group [1]. Therefore the 2-form curvatures are in Table 6.6.

Now the AdS-Static and Static action is

$$I_{CS}^{AdS\text{-}Static} = I_{CS}^{Static} = \int \Big(- \omega d\omega \Big) \alpha_0 + \Big(-2\tau R(\omega) \Big) \alpha_1 ,$$

Curvature	**AdS-Static**	**Static**
$R(\omega)$	$d\omega$	$d\omega$
$R^a(\omega)$	$d\omega^a + \omega\omega_b\varepsilon^{ab} + \tau e_b\varepsilon^{ab}$	$d\omega^a + \omega\omega_b\varepsilon^{ab}$
$R(\tau)$	$d\tau$	$d\tau$
$R^a(e)$	$de^a + \omega e_b\varepsilon^{ab}$	$de^a + \omega e_b\varepsilon^{ab}$

Table 6.6: Comparison between the AdS-Static and Static curvatures in $(2+1)$ dimensions.

It is clear that the Chern–Simons action is degenerate, since the gauge fields corresponding to the vielbein e^a and the spin connection ω^a do not appear in the action.

We have seen that the non-relativistic framework suffers from a degeneracy, as it admits a null invariant tensor for the spatial generators, as shown in (6.3.2), (6.5.2) and (6.7.1). In the following section, we will consider an alternative method to obtain different algebras, the so-called semigroup expansion (S-expansion), showing that the Inönu-Wigner contraction is nothing but a particular part of the S-expansion procedure.

Chapter 7

Extended Kinematical Algebras

The question of which Lie algebras admit a non-degenerate invariant tensor is a long-standing one, and various strategies have been proposed to address this issue. In the non-relativistic setting, it is well known that constructing a well-defined Chern–Simons action in three spacetime dimensions requires the introduction of two central extensions for both the Galilei and Newton–Hooke algebras. An alternative approach to overcome the degeneracy appearing in the original kinematical cube of [1] was presented in [54]. Since non-degeneracy demands the consideration of extended non-relativistic algebras, these structures can no longer be obtained through standard contractions of the original $\mathfrak{so}(2,2)$ algebra. Indeed, the contraction process preserves the dimension of the algebra, and therefore a suitable enlargement requires an expansion procedure [66, 93–95] .

In this section, we shall employ the semigroup expansion (S-expansion) method [66], which has recently proven useful in the non-Lorentzian context [96–103] . This framework will allow us to extend the original cube of [1] to a broader family of kinematical algebras, obtained through sequential expansions of the AdS algebra by choosing an appropriate semigroup and imposing resonant conditions. Remarkably, in contrast to [104], the method implemented here does not require central extensions of the relativistic algebras to avoid degeneracy in their non-relativistic limits. Consequently, achieving a fully non-degenerate kinematical cube in $(2+1)$-dimensions requires performing the resonant $S_E^{(2)}$-expansion in all sectors of the algebra, namely the speed–space, space–time, and speed–time decompositions.

The S-expansion procedure provides a systematic way to construct a new Lie algebra $\mathfrak{G}$ from a given one $\mathfrak{g}$ by combining its generators and structure constants with the elements of an abelian semigroup S. The resulting expanded algebra is then expressed as $\mathfrak{G} = S \times \mathfrak{g}$. Furthermore, one may extract a smaller and physically relevant subalgebra by imposing either a resonant condition or by performing a 0_S-reduction, as discussed in [66].

As an illustrative example, we shall show that the Inönü–Wigner contraction can be recovered as a particular case of the S-expansion procedure by considering the semigroup

$S_E^{(1)}$.

7.1 Inönu Wigner contraction as a semigroup expansion

In order to illustrate the expansion procedure, let us recover the Poincaré algebra starting from the decomposed AdS algebra previously introduced in Table 6.1. We consider the semigroup $S_E^{(1)} = \{\lambda_i\}_{i=0}^2$, characterized by the following Cayley Table 7.1.

λ_2	λ_2	λ_2	λ_2
λ_1	λ_1	λ_2	λ_2
λ_0	λ_0	λ_1	λ_2
$\cdot$	λ_0	λ_1	λ_2

Table 7.1: Calley table for the semigroup $S_E^{(1)}$.

We decompose the AdS algebra $\mathfrak{g}$ into two subspaces, $V_0 = \{\tilde{J}, \tilde{G}_a\}$ and $V_1 = \{\tilde{H}, \tilde{P}_a\}$, such that $\mathfrak{g} = V_0 \oplus V_1$. From Table 6.1, it is straightforward to verify that this decomposition satisfies the $\mathbb{Z}_2$-graded structure

$$[V_0, V_0] \subset V_0, \qquad [V_0, V_1] \subset V_1, \qquad [V_1, V_1] \subset V_0. \tag{7.1.1}$$

In analogy with this splitting, we introduce the resonant decomposition of $S_E^{(1)} = S_0 \cup S_1$, where $S_0 = \{\lambda_0, \lambda_2\}$ and $S_1 = \{\lambda_1, \lambda_2\}$, which satisfies

$$S_0 \cdot S_0 \subset S_0, \qquad S_0 \cdot S_1 \subset S_1, \qquad S_1 \cdot S_1 \subset S_0.$$

This compatibility condition between subsets and subspaces defines a *resonant* structure. As a consequence, the S-expanded algebra $\mathfrak{G}$ can be written as

$$\mathfrak{G} = (S_0 \times V_0) \oplus (S_1 \times V_1).$$

Explicitly, the expanded generators are given by

$$\mathfrak{G} = \{\lambda_0 \times \tilde{J}, \lambda_0 \times \tilde{G}_a, \lambda_2 \times \tilde{J}, \lambda_2 \times \tilde{G}_a\} \oplus \{\lambda_1 \times \tilde{H}, \lambda_1 \times \tilde{P}_a, \lambda_2 \times \tilde{H}, \lambda_2 \times \tilde{P}_a\},$$

where the generators are distributed according to the resonant decomposition. Applying the 0_S-reduction, which consists in setting $\lambda_2 \times T_A = 0$ for any generator $T_A \in \mathfrak{g}$, one finds

$$\mathfrak{G} = \{\lambda_0 \times \tilde{J}, \lambda_0 \times \tilde{G}_a, \lambda_2 \times \tilde{J}, \lambda_2 \times \tilde{G}_a\} \oplus \{\lambda_1 \times \tilde{H}, \lambda_1 \times \tilde{P}_a\},$$

λ_2				
λ_1			H	P_a
λ_0	J	G_a		
$\times$	$\tilde{J}$	$\tilde{G}_a$	$\tilde{H}$	$\tilde{P}_a$

Table 7.2: Generators obtained from the speed-times semigroup $S_E^{(1)}$ expansion of the AdS algebra. The generators with tilde correspond to AdS.

where the null components are omitted since we are working at the level of vector spaces. We now define the generators of the reduced expanded algebra as

$$J \equiv \lambda_0 \times \tilde{J}, \quad G_a \equiv \lambda_0 \times \tilde{G}_a, \quad P_a \equiv \lambda_1 \times \tilde{P}_a, \quad H \equiv \lambda_1 \times \tilde{H}.$$

This correspondence can be summarized in Table 7.2

The λ_2 row is empty due to the 0_S-reduction, while the remaining blank entries correspond to non-resonant combinations. Finally, we illustrate the construction of the expanded commutation relations with the example

$$[J, G_a] = [\lambda_0 \times \tilde{J}, \lambda_0 \times \tilde{G}_a] = (\lambda_0 \cdot \lambda_0) \times [\tilde{J}, \tilde{G}_a] = \lambda_0 \times (\varepsilon_{ab} \tilde{G}_b) = \varepsilon_{ab} G_b.$$

By repeating the same reasoning for the remaining brackets, one eventually reconstructs the Poincaré algebra Table 6.1. Finally, we can define the following constants arising from the invariant tensors. Denoting by $\{\tilde{\alpha}_0, \tilde{\alpha}_1\}$ the AdS constants, we obtain as shown in Table 7.3

λ_2		
λ_1		α_1
λ_0	α_0	
$\times$	$\tilde{\alpha}_0$	$\tilde{\alpha}_1$

Table 7.3: Expanded constants of invariant tensors from AdS.

Consequently, one recovers the Poincaré invariant tensors displayed in Table 6.2. Moreover, any algebra in Fig. 1.1 can be obtained through the corresponding $S_E^{(1)}$ expansion. A summary of the choice of these expansions (contractions) is presented in Table 7.4.

To overcome the degeneracy that appears in the non-relativistic sector, one may enlarge the semigroup used in the expansion procedure. In particular, we shall employ a higher-order semigroup, denoted $S_E^{(2)}$. As a starting point, let us introduce the speed–space decomposition of the AdS algebra in the following section [54].

From this point onward, every generator and invariant tensor constant denoted with a tilde (˜) will be understood as belonging to the original algebra being expanded.

Subspaces	Space-time	Speed-space	Speed-time
V_0	J, G_a	J, H	J, P_a
V_1	H, P_a	G_a, P_a	H, G_a

Table 7.4: Inönü–Wigner contractions as expansion of the AdS Lie algebra.

7.2 Extended Newton-Hooke

In what follows, we extend the non-relativistic sector of the AdS algebra by means of the $S_E^{(2)}$ semigroup expansion, which provides a non-degenerate invariant structure and therefore resolves the degeneracy problem. For this purpose, we decompose the algebra into the subspaces

$$V_0 = \{\tilde{J}, \tilde{H}\}, \qquad V_1 = \{\tilde{G}_a, \tilde{P}_a\},$$

from which the expanded generators are defined as in Table 7.5

λ_3				
λ_2	S	M		
λ_1			G_a	P_a
λ_0	J	H		
$\times$	$\tilde{J}$	$\tilde{H}$	$\tilde{G}_a$	$\tilde{P}_a$

Table 7.5: Generators obtained from the speed-space semigroup $S_E^{(2)}$ expansion of the AdS Algebra. The generators with tilde correspond to AdS.

Unlike the $S_E^{(1)}$ expansion (contraction) of Section 6.5, the current framework incorporates two central charges, S and M. These are essential for formulating a non-degenerate Chern–Simons action in three spacetime dimensions, a result that we shall demonstrate in what follows. Consequently, we obtain the following commutation relations:

$$\begin{aligned}
[J,G_a] &= \varepsilon_{ab}G_b, \qquad [G_a,P_b] = -\varepsilon_{ab}M, \\
[G_a,G_b] &= -\varepsilon_{ab}S, \qquad [H,G_a] = \varepsilon_{ab}P_b, \qquad [P_a,P_b] = -\varepsilon_{ab}S. \\
[J,P_a] &= \varepsilon_{ab}P_b, \qquad [H,P_a] = \varepsilon_{ab}G_b,
\end{aligned} \tag{7.2.1}$$

This structure corresponds to the Extended Newton–Hooke algebra [105–111] . The associated invariant tensors are given by:

$$\begin{aligned} \langle J J \rangle &= -\alpha_0, & \langle G_a P_b \rangle &= \mu_1 \delta_{ab}, & \langle J S \rangle &= -\mu_0, & & \\ \langle G_a G_b \rangle &= \mu_0 \delta_{ab}, & \langle H H \rangle &= -\alpha_0, & \langle J M \rangle &= -\mu_1, & \langle H M \rangle &= -\mu_0. \\ \langle J H \rangle &= -\alpha_1, & \langle P_a P_b \rangle &= \mu_0 \delta_{ab}, & \langle H S \rangle &= -\mu_1, & & \end{aligned} \tag{7.2.2}$$

Summarized in the following Table 7.6 the expanded constants.

λ_3		
λ_2	μ_0	μ_1
λ_1		
λ_0	α_0	α_1
$\times$	$\tilde{\alpha}_0$	$\tilde{\alpha}_1$

Table 7.6: Expanded constants of invariant tensors from AdS.

The curvatures are

$$\begin{aligned} R(\omega) &= d\omega & R^a(e) &= de^a + \omega e_b \varepsilon^{ab} + \tau \omega_b \varepsilon^{ab}, \\ R^a(\omega) &= d\omega^a + \omega \omega_b \varepsilon^{ab} + \tau e_b \varepsilon^{ab}, & R(s) &= ds + \frac{1}{2}\omega_a \omega_b \varepsilon^{ab} + \frac{1}{2} e_a e_b \varepsilon^{ab}, \\ R(\tau) &= d\tau, & R(m) &= dm + \omega_a e_b \varepsilon^{ab}. \end{aligned} \tag{7.2.3}$$

Its action is

$$\begin{aligned} I_{CS}^{eNH} = \int & \Big(-\omega R(\omega) - \tau R(\tau) \Big) \alpha_0 + \Big(-\omega R(\tau) \Big) \alpha_1 \\ & + \Big(e^a R_a(e) + \omega^a R_a(\omega) - 2sR(\omega) - 2mR(\tau) \Big) \mu_0 \\ & + \Big(e^a R_a(\omega) + \omega^a R_a(e) - 2sR(\tau) - 2mR(\omega) \Big) \mu_1 \end{aligned} \tag{7.2.4}$$

The equations of motion derived from the variation of the action are each curvature in Eq. (7.2.3) equals to zero, the variaton respecto to each gauge field is in Table A.1.

As discussed previously, the AdS–Carroll algebra leads to a degenerate invariant trace in four dimensions. However, this problem admits a natural solution via an $S_E^{(2)}$-expansion, as will be explicitly constructed in the following section.

7.3 Extended AdS-Carroll

To construct the extended AdS-Carroll (eAdS–C) algebra, we select the subspaces

$$V_0 = \{\tilde{J}, \tilde{P}_a\}, \qquad V_1 = \{\tilde{G}_a, \tilde{H}\},$$

λ_3				
λ_2	C	T_a		
λ_1			G_a	H
λ_0	J	P_a		
$\times$	$\tilde{J}$	$\tilde{P}_a$	$\tilde{G}_a$	$\tilde{H}$

Table 7.7: Generators obtained from the $S_E^{(2)}$ speed–time semigroup expansion of the AdS algebra. Tilded generators correspond to the AdS algebra.

from which we eventually define what is shown in Table 7.7

The following algebra is obtained:

$$\begin{aligned} [J,G_a] &= \varepsilon_{ab}G_b, & [G_a,P_b] &= -\varepsilon_{ab}H, & [P_a,P_b] &= -\varepsilon_{ab}J. & \\ [G_a,G_b] &= -\varepsilon_{ab}C, & [H,G_a] &= \varepsilon_{ab}T_b, & [C,P_a] &= \varepsilon_{ab}T_b, & [T_a,P_b] &= -\varepsilon_{ab}C. \\ [J,P_a] &= \varepsilon_{ab}P_b, & [H,P_a] &= \varepsilon_{ab}G_b, & [J,T_a] &= \varepsilon_{ab}T_b, & \end{aligned} \tag{7.3.1}$$

This algebra is isomorphic to the Maxwell algebra given in Eq. (8.2.1), since one can obtain either algebra by interchanging $Z \leftrightarrow C$ and $Z_a \leftrightarrow T_a$. The corresponding invariant tensors are given by:

$$\begin{aligned} \langle JJ\rangle &= -\beta_0, & \langle JH\rangle &= -\beta_1, & \langle HH\rangle &= -\beta_2, & \langle JC\rangle &= -\beta_2, \\ \langle G_a G_b\rangle &= \beta_2\delta_{ab}, & \langle G_a P_b\rangle &= \beta_1\delta_{ab}, & \langle P_a P_b\rangle &= \beta_0\delta_{ab}, & \langle T_a P_b\rangle &= \beta_2\delta_{ab}. \end{aligned} \tag{7.3.2}$$

where every constant expanded is summarized in Table 7.8.

λ_2	β_2	
λ_1		β_1
λ_0	β_0	
$\times$	$\tilde{\alpha}_0$	$\tilde{\alpha}_1$

Table 7.8: Expanded constants of invariant tensors from AdS.

The curvatures are

$$\begin{aligned} R(\omega) &= d\omega + \frac{1}{2}e_a e_b \varepsilon^{ab} & R^a(e) &= de^a + \omega e_b \varepsilon^{ab}, \\ R^a(\omega) &= d\omega^a + \omega\omega_b\varepsilon^{ab} + \tau e_b\varepsilon^{ab}, & R(c) &= dc + \frac{1}{2}\omega_a\omega_b\varepsilon^{ab} + \frac{1}{2}e_a e_b\varepsilon^{ab}, \\ R(\tau) &= d\tau + \omega_a\varepsilon_b\varepsilon^{ab}, & R^a(t) &= dt^a + \tau\omega_b\varepsilon^{ab} + ce_b\varepsilon^{ab}. \end{aligned} \tag{7.3.3}$$

Therefore its action is

$$\begin{aligned} I_{CS}^{eAdS\text{-}C} = \int \Big(-\omega d\omega + e^a R_a(e) \Big) \beta_0 + \Big(-2\tau R(\omega) + 2\omega^a R_a(e) \Big) \beta_1 \\ + \Big(-\tau R(\tau) + \omega^a R_a(\omega) + 2t^a R_a(e) - 2c R(\omega) \Big) \beta_2 . \end{aligned} \tag{7.3.4}$$

The equations of motion obtained from the variation of the action correspond to the vanishing of each curvature in Eq. (7.3.3) as result of vary (7.3.4) (for further details see Table A.1).

At this stage, we may proceed to explorem, in the following section, the non-relativistic limit of the Extended AdS–Carroll algebra, or the ultra-relativistic limit associated with the Extended Newton–Hooke algebra.

7.4 Extended AdS-Static

We are now in a position to study the non-relativistic limit of the Extended AdS–Carroll algebra 7.3, from another perspective the ultra-relativistic limit of the Extended Newton–Hooke algebra 7.2, called Extended AdS–Static (eAdS–S). The algebra is constructed by performing a resonant $S_E^{(2)}$-expansion of the starting algebra followed by a 0_S-reduction. To this end, we first consider the subspace decomposition shown in Table 7.9.

Subspaces	**Extended AdS–Carroll origin**	**Extended Newton-Hooke origin**
V_0	J, H, C	J, P_a, S
V_1	G_a, P_a, T_a	H, G_a, M

Table 7.9: Subspaces decomposition of the eAdS–C and eNH to obtain Extended AdS–Static .

From this point, from which we eventually define what is shown in Table 7.10. The following algebra is obtained:

$$\begin{aligned}
&[J, G_a] = \varepsilon_{ab} G_b, && [G_a, P_b] = -\varepsilon_{ab} M, && [P_a, P_b] = -\varepsilon_{ab} S. && \\
&[G_a, G_b] = -\varepsilon_{ab} B, && [H, G_a] = \varepsilon_{ab} T_b, && [C, P_a] = \varepsilon_{ab} T_b, && [T_a, P_b] = -\varepsilon_{ab} B. \\
&[J, P_a] = \varepsilon_{ab} P_b, && [H, P_a] = \varepsilon_{ab} G_b, && [J, T_a] = \varepsilon_{ab} T_b, &&
\end{aligned} \tag{7.4.1}$$

This algebra also recieves the name of **Para-Maxwell Extended Bargmann**, since this is isomorphic to an algebra which will be study in the next chapter. The invariants tensors

	Extended AdS–Carroll origin		**Extended Newton-Hooke origin**	
λ_3				
λ_2	$S,\ M,\ B$		$C,\ T_a,\ B$	
λ_1		$G_a,\ P_a,\ T_a$		$H,\ G_a,\ M$
λ_0	$J,\ H,\ C$		$J,\ P_a,\ S$	
$\times$	$\tilde{J},\ \tilde{H},\ \tilde{C}$	$\tilde{G}_a,\ \tilde{P}_a,\ \tilde{T}_a$	$\tilde{J},\ \tilde{P}_a,\ \tilde{S}$	$\tilde{H},\ \tilde{G}_a,\ \tilde{M}$

Table 7.10: eAdS–S generators expressed in terms of the generators of the eAdS–C and eNH generators through the $S_E^{(2)}$ semigroup elements.

are given by the expansion from either Extended AdS–Carroll invariant tensors (7.3.2), or from Extended Newton–Hooke invariant tensors (7.2.2), summarized in Table 7.11, which reads

$$\begin{array}{llll} \langle JJ\rangle = -\beta_0, & \langle HH\rangle = -\beta_2, & \langle JS\rangle = -\nu_0, & \\ \langle G_a G_b\rangle = \nu_2\delta_{ab}, & \langle P_a P_b\rangle = \nu_0\delta_{ab}, & \langle HS\rangle = -\nu_1, & \langle SC\rangle = -\nu_2, \\ \langle JH\rangle = -\beta_1, & \langle JC\rangle = -\beta_2, & \langle JM\rangle = -\nu_1, & \langle JB\rangle = -\nu_2. \\ \langle G_a P_b\rangle = \nu_1\delta_{ab}, & \langle T_a P_b\rangle = \nu_2\delta_{ab}. & \langle HM\rangle = -\nu_2, & \end{array} \tag{7.4.2}$$

	Extended AdS–Carroll origin			**Extended Newton–Hooke origin**			
λ_3							
λ_2	ν_0	ν_1	ν_2	β_2		ν_2	
λ_1					β_1		ν_1
λ_0	β_0	β_1	β_2	β_0		ν_0	
$\times$	$\tilde{\beta}_0$	$\tilde{\beta}_1$	$\tilde{\beta}_2$	$\tilde{\alpha}_0$	$\tilde{\alpha}_1$	$\tilde{\mu}_0$	$\tilde{\mu}_1$

Table 7.11: eAdS-S invariant constants expanded from eAdS-C and eNH.

The curvatures are

$$\begin{aligned} R(\omega) &= d\omega & R(m) &= dm + \omega_a e_b \varepsilon^{ab}. \\ R^a(\omega) &= d\omega^a + \omega\omega_b\varepsilon^{ab} + \tau e_b\varepsilon^{ab}, & R^a(t) &= dt^a + \tau\omega_b\varepsilon^{ab} + ce_b\varepsilon^{ab} + \omega t_a\varepsilon^{ab} \\ R(\tau) &= d\tau, & R(c) &= dc, \\ R^a(e) &= de^a + \omega e_b\varepsilon^{ab}, & R(b) &= db + \frac{1}{2}\omega_a\omega_b\varepsilon^{ab} + t_a e_b\varepsilon^{ab}. \\ R(s) &= ds + \frac{1}{2}e_a e_b\varepsilon^{ab}, \end{aligned} \tag{7.4.3}$$

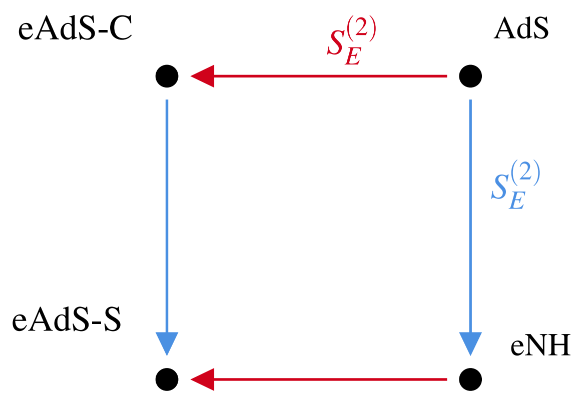

Figure 7.1: Extended kinematical algebras starting from the AdS algebra

Thus its action is

$$
\begin{aligned}
I_{CS}^{eAdS\text{-}Static} = \int & \Big(-\omega R(\omega) \Big) \beta_0 + \Big(-\omega R(\tau) \Big) \beta_1 + \Big(-\tau R(\tau) - 2\omega R(c) \Big) \beta_2 \\
& + \Big(-2sR(\omega) + e^a R_a(e) \Big) \nu_0 + \Big(e^a R_a(\omega) + \omega^a R_a(e) - 2mR(\omega) - 2sR(\tau) \Big) \nu_1 \\
& + \Big(\omega^a R_a(\omega) + t^a R_a(e) + e^a R_a(t) - 2mR(\tau) - 2bR(\omega) - 2sR(c) \Big) \nu_2
\end{aligned}
\tag{7.4.4}
$$

The equations of motion resulting from the variation of the action (7.4.4) (see Table A.3) where each curvature in Eq. (7.4.3) vanishes.

We have already constructed the back face of the cube through successive $S_E^{(2)}$ non–reativistics and ultra–relativistics expansions of the AdS algebra Fig. 7.1, together with their corresponding Chern–Simons actions. To complete the analysis, we now perform the space–time $S_E^{(2)}$ expansion, which will ultimately allow us to compute its associated Chern–Simons action. The resulting algebra is already well known in the literature; therefore, its relevance and physical implications will be discussed in the following chapter.

Chapter 8

Maxwell Chern-Simons gravity

The Maxwell algebra was originally introduced in four-dimensional spacetime as the symmetry algebra of a Minkowski background in the presence of a constant electromagnetic field [56–59]. It extends the Poincaré algebra by incorporating additional generators associated with the constant field strength, thereby modifying the commutator of translations.

Subsequently, the Maxwell algebra and its generalisations were formulated in arbitrary spacetime dimensions. In this broader setting, they provide a natural framework to relate General Relativity with Chern–Simons (CS) and Born–Infeld (BI) gravity theories [112–115]. In particular, in three-dimensional spacetime, Maxwell symmetry and its extensions, including supersymmetric and higher-spin versions, have been extensively studied within the CS gauge formalism. These analyses have led to a variety of novel insights regarding the geometric and dynamical structure of gravity theories based on enlarged kinematical algebras [60–62, 116–123].

The incorporation of a cosmological constant into Maxwell CS gravity theories can be achieved by enlarging the symmetry algebra to $\mathfrak{so}(2,2)\times\mathfrak{so}(2,1)$, commonly referred to as the AdS–Lorentz algebra [124–126]. This structure provides a natural deformation of the Maxwell algebra compatible with anti-de Sitter geometry. Applications of the AdS–Lorentz algebra and its extensions include the construction of higher-dimensional CS gravity models whose particular limits reproduce pure Lovelock gravity [127, 128]. In addition, deformations of the Maxwell algebra and their dynamical realisations through nonlinear methods have been explored in [129–131], further broadening the relevance of Maxwell-type symmetries in gravitational and effective field theory contexts.

Non-Lorentzian limits of the Maxwell algebra have also been investigated in [55, 65]. It was shown that, in both non-relativistic (NR) and ultra-relativistic (UR) regimes, additional central extensions are required in order to avoid degeneracies in the invariant bilinear form. As discussed in Section 6, a characteristic feature of certain NR limits is that the symplectic structure may become degenerate, leading to gauge fields that are not dynamically determined by the field equations.

In a three-dimensional CS formulation, the non-degeneracy of the invariant bilinear trace of the gauge generators is directly related to the non-degeneracy of the symplectic form. For instance, conditions such as $\langle G_a G_b \rangle = 0$ signal potential degeneracies that may result in dynamically indeterminate sectors of the theory. Although the Carroll contraction of the Maxwell algebra allows for the construction of a finite UR CS gravity action, it does not generically resolve the degeneracy issue. In particular, the Maxwellian Carroll algebra admits only degenerate invariant bilinear forms, in contrast with the standard Carroll and Poincaré cases.

In this chapter, we revisit the Maxwell algebra in the context of $(2+1)$-dimensional Chern–Simons gravity. We analyse its algebraic structure, derive the corresponding invariant tensors and gauge connections, and construct explicitly the associated CS action. Furthermore, we revisit its extensions and non-Lorentzian limits through expansion procedures as done in [55, 65].

8.1 Maxwell Algebra

We will compute the equations of motion obtained from the CS theory. The Maxwell algebra consists of the nine generators: Spacetime rotations J_A , spacetime translations P_A , and a new type of generators Z_A characterized and introduced in [56, 58]. The non vanishing commutators among these generators are

$$
\begin{aligned}
[J_A, J_B] &= \varepsilon_{ABC} J^C \\
[J_A, P_B] &= \varepsilon_{ABC} P^C \\
[J_A, Z_B] &= \varepsilon_{ABC} Z^C \\
[P_A, P_B] &= \varepsilon_{ABC} Z^C .
\end{aligned}
\tag{8.1.1}
$$

The implications of these symmetries have already been studied in four dimensions. However, in three dimensions they remain unexplored. Its invariant tensors are

$$
\begin{aligned}
\langle J_A J_B \rangle &= \alpha_0 \eta_{AB} \\
\langle J_A P_B \rangle &= \alpha_1 \eta_{AB} \\
\langle J_A Z_B \rangle &= \alpha_2 \eta_{AB} \\
\langle P_A P_B \rangle &= \alpha_2 \eta_{AB} .
\end{aligned}
$$

Thus, the 2-form curvatures are

$$
\begin{aligned}
R^A(\omega) &= d\omega^A + \frac{1}{2} \omega^B \omega^C \varepsilon^A{}_{BC} \\
R^A(e) &= de^A + \omega^B e^C \varepsilon^A{}_{BC} \\
R^A(z) &= dz^A + \omega^B z^C \varepsilon^A{}_{BC} + \frac{1}{2} e^B e^B \varepsilon^A{}_{BC} .
\end{aligned}
\tag{8.1.2}
$$

Hence, the CS action is

$$I_{CS}^{M} = \int \Big(\omega^A d\omega_A + \frac{1}{3}\omega^A\omega^B\omega^C\varepsilon_{ABC}\Big)\alpha_0 + 2e^A R_A(\omega)\alpha_1 + \Big(2z^A R_A(\omega) + e^A R_A(e)\Big)\alpha_2. \tag{8.1.3}$$

We can notice that this corresponds to the Poincaré action (6.1.5) with an additional term proportional to α_2. When ones vary (8.1.3) respect to the spin connection, from the α_2 we eventually get

$$\delta_e \mathscr{L}_2 = 2\delta\omega^A R_A(z).$$

While it is evident the result after vary respect to the other gauge fields. Hence the variation of this action yields field equations where each curvature to vanish.

In the next section, one could proceed by directly decomposing the Maxwell algebra into temporal and spatial components, following the strategy adopted in Chapter 6, in order to derive its non-relativistic and ultra-relativistic limits. However, instead of pursuing this direct contraction procedure, we will obtain the Maxwell algebra from a cosmological extension constructed via a semigroup expansion of the AdS algebra. As will be shown, this alternative approach leads to the same resulting structure.

8.2 (Decomposing) Maxwell Algebra from AdS

We obtain the Maxwell algebra as a space–time $S_E^{(2)}$ expansion of the AdS algebra. To this end, we choose the subspaces $V_0 = \{\tilde{J}, \tilde{G}_a\}$ and $V_1 = \{\tilde{H}, \tilde{P}_a\}$, which clearly satisfy the condition $\mathbb{Z}_2$-graded structure. Following the expansion procedure, we introduce the new set of generators summarized in Table 8.1

λ_3				
λ_2	Z	Z_a		
λ_1			H	P_a
λ_0	J	G_a		
$\times$	$\tilde{J}$	$\tilde{G}_a$	$\tilde{H}$	$\tilde{P}_a$

Table 8.1: Maxwell generators obtained from the space-time semigroup $S_E^{(2)}$ expansion of AdS.

As a result, the Maxwell algebra takes the form

$$\begin{aligned}
[J,G_a] &= \varepsilon_{ab}G_b, & [G_a,P_b] &= -\varepsilon_{ab}H, & [G_a,Z_b] &= -\varepsilon_{ab}Z, & & \\
[G_a,G_b] &= -\varepsilon_{ab}J, & [H,G_a] &= \varepsilon_{ab}P_b, & [Z,G_a] &= \varepsilon_{ab}Z_b, & [P_a,P_b] &= -\varepsilon_{ab}Z. \\
[J,P_a] &= \varepsilon_{ab}P_b, & [J,Z_a] &= \varepsilon_{ab}Z_b, & [H,P_a] &= \varepsilon_{ab}Z_b, & &
\end{aligned} \tag{8.2.1}$$

Next, we decompose the generators into temporal and spatial components as in 6.2, now including $Z_A \equiv (Z, Z_a)$, which implies $z^A \equiv (z, z^a)$. In this manner, one recovers the same algebra, confirming that (8.2.1) is its decomposition. The corresponding invariant tensor reads

$$
\begin{array}{cccc}
\langle JJ \rangle = -\alpha_0, & \langle JH \rangle = -\alpha_1, & \langle JZ \rangle = -\alpha_2, & \langle HH \rangle = -\alpha_2, \\
\langle G_a G_b \rangle = \alpha_0 \delta_{ab}, & \langle G_a P_b \rangle = \alpha_1 \delta_{ab}, & \langle G_a Z_b \rangle = \alpha_2 \delta_{ab}, & \langle P_a P_b \rangle = \alpha_2 \delta_{ab}.
\end{array}
\tag{8.2.2}
$$

In terms of the original invariant tensor, we define the constants summarized in Table 8.2

λ_2	α_2	
λ_1		α_1
λ_0	α_0	
$\times$	$\tilde{\alpha}_0$	$\tilde{\alpha}_1$

Table 8.2: Constant of invariant tensors expanded from AdS.

Using the algebraic structure, the curvature two-forms take the form

$$
\begin{array}{ll}
R(\omega) = d\omega + \frac{1}{2}\omega_a \omega_b \varepsilon^{ab}, & R^a(e) = de^a + \omega e_b \varepsilon^{ab} + \tau \omega_b \varepsilon^{ab}, \\
R^a(\omega) = d\omega^a + \omega \omega_b \varepsilon^{ab}, & R(z) = dz + \omega_a z_b \varepsilon^{ab} + \frac{1}{2} e_a e_b \varepsilon^{ab}, \\
R(\tau) = d\tau + \omega_a e_b \varepsilon^{ab}, & R^a(z) = dz^a + \omega z_b \varepsilon^{ab} + \tau e_b \varepsilon^{ab} + z \omega_b \varepsilon^{ab},
\end{array}
\tag{8.2.3}
$$

where $R(z) \equiv R^0(z)$. Finally, the corresponding Chern–Simons action is

$$
\begin{aligned}
I_{CS}^{M} = \int & \Big(-\omega d\omega + \omega^a R_a(\omega) \Big) \alpha_0 + \Big(-2\tau R(\omega) + 2e^a R_a(\omega) \Big) \alpha_1 \\
& + \Big(-2zR(\omega) + 2z^a R_a(\omega) - \tau R(\tau) + e^a R_a(e) \Big) \alpha_2 .
\end{aligned}
\tag{8.2.4}
$$

We are now ready to explore both the non-relativistic (NR) and ultra-relativistic (UR) limits of this algebra.

8.3 Chern–Simons Gravity for Non-Relativistic and Ultra-Relativistic Maxwell Algebras

We now analyze the non-relativistic (NR) and ultra-relativistic (UR) limits. In these limits, the rescaling is applied to the generators $\{G_a, P_a, Z_a\}$ for NR and $\{G_a, H, Z_a\}$ for UR, along with their corresponding fields. Applying these contractions to the Maxwell algebra (8.2.1), we obtain the resulting algebras as shown in Table 8.3. It is important to comment that the commutator $[G_a, G_b]$ has been omitted since it vanishes in both limits.

Maxwell	$[J,G_a]$	$[J,P_a]$	$[G_a,P_b]$	$[H,G_a]$	$[H,P_a]$	$[P_a,P_b]$	$[J,Z_a]$	$[Z,G_a]$
NR limit	$\varepsilon_{ab}G_b$	$\varepsilon_{ab}P_b$	0	$\varepsilon_{ab}P_b$	$\varepsilon_{ab}G_b$	0	$\varepsilon_{ab}Z_b$	$\varepsilon_{ab}Z_b$
UR limit	$\varepsilon_{ab}G_b$	$\varepsilon_{ab}P_b$	$-\varepsilon_{ab}H$	$\varepsilon_{ab}M_b$	$\varepsilon_{ab}G_b$	$-\varepsilon_{ab}Z$	$\varepsilon_{ab}Z_b$	$\varepsilon_{ab}Z_b$

Table 8.3: Comparison between the non-relativistic and ultra-relativistic limits of the algebra. The commutator $[G_a, G_b]$ is omitted since it vanishes in both cases.

Their respective invariant tensors, obtained from (8.2.2), given in Table 8.4, where α_1 for UR frame rescales.

Maxwell	$\langle JJ\rangle$	$\langle G_a G_b\rangle$	$\langle JH\rangle$	$\langle G_a P_b\rangle$	$\langle JZ\rangle$	$\langle G_a Z_b\rangle$	$\langle HH\rangle$	$\langle P_a P_b\rangle$
NR limit	$-\alpha_0$	0	$-\alpha_1$	0	$-\alpha_2$	0	$-\alpha_2$	0
UR limit	$-\alpha_0$	0	$-\alpha_1$	$\alpha_1\delta_{ab}$	$-\alpha_2$	0	0	$\alpha_2\delta_{ab}$

Table 8.4: Invariant tensor components for the non-relativistic and ultra-relativistic limits.

The curvatures (8.2.3), after the contraction, are reduced in Table 8.5.
Finally, the Chern-Simons action for each limit reads for, non-relativistic

$$I_{CS}^{M\text{-NR}} = \int (-\omega d\omega)\alpha_0 - 2\tau d\omega\alpha_1 - \Big(2zd\omega + \tau d\tau\Big)\alpha_2,$$

and ultra-relativistic

$$\begin{aligned} I_{CS}^{M\text{-UR}} = \int (-\omega d\omega)\alpha_0 + \Big(2e^a d\omega_a + 2\omega e^a\omega^b\varepsilon_{ab} - 2\tau d\omega\Big)\alpha_1 \\ + \Big(-2zd\omega + e^a de_a + \omega e^a e^b\varepsilon_{ab}\Big)\alpha_2. \end{aligned}$$

Curvature	**Maxwell NR limit**	**Maxwell UR limit**
$R(\omega)$	$d\omega$	$d\omega$
$R^a(\omega)$	$d\omega^a+\omega\omega_b\varepsilon^{ab}$	$d\omega^a+\omega\omega_b\varepsilon^{ab}$
$R(\tau)$	$d\tau$	$d\tau+\omega_a e_b\varepsilon^{ab}$
$R^a(e)$	$de^a+\omega e_b\varepsilon^{ab}+\tau\omega_b\varepsilon^{ab}$	$de^a+\omega e_b\varepsilon^{ab}$
$R(z)$	dz	$dz+\frac{1}{2}e_a e_b\varepsilon^{ab}$
$R^a(z)$	$dz^a+\omega z_b\varepsilon^{ab}+\tau e_b\varepsilon^{ab}+z\omega_b\varepsilon^{ab}$	$dz^a+\omega z_b\varepsilon^{ab}+\tau e_b\varepsilon^{ab}+z\omega_b\varepsilon^{ab}$

Table 8.5: Comparison between the non-relativistic and ultra-relativistic curvatures.

It is evident that both actions are degenerate, as the gauge field z^a does not appear in either case. In order to resolve this degeneracy, we will adopt an expansion approach instead of a contraction. Using this method, we will reconstruct the original Bacry–Lévy-Leblond kinematic cube 1.1, using only semigroup expansion, as detailed in the next section.

Chapter 9

Maxwell kinematics

Here we show that the non-degenerate non-Lorentzian Maxwell algebras introduced in [55, 65] can be naturally interpreted as part of a Maxwell-type extension of the Lévy-Leblond cube. This structure emerges once the standard Inönü-Wigner contraction is replaced by an expansion procedure. In contrast with contractions, expansion methods [66, 93–95] generally enlarge the set of generators of the initial algebra. In particular, the semigroup expansion approach developed in [66] has been extensively used to construct non-Lorentzian symmetry algebras admitting a non-degenerate invariant bilinear form and, consequently, to formulate consistent non-Lorentzian (super)gravity actions [54, 55, 96, 99, 100, 103, 132–137].

Interestingly, within this framework the dimensionality of the original algebra can be preserved. A particularly relevant example is provided by the semigroup $S_E^{(1)}$, which reproduces the Inönü-Wigner contraction through an appropriate choice of subspace decomposition, as shown in Sec. 7.1. This observation allows one to reinterpret the Lévy-Leblond cube [1] as a diagram in which the arrows correspond to $S_E^{(1)}$ expansions rather than contractions [54]. It naturally motivates the study of a generalized cube in which each arrow is replaced by an expansion associated with a higher-order semigroup.

Our starting point is the AdS algebra together with its non-Lorentzian counterparts. As shown in [54, 136], the application of an $S_E^{(2)}$ expansion to the AdS algebra, combined with suitable speed-space and speed-time subspace decompositions, reproduces the family of extended kinematical algebras developed in Chapter. 7. At the non-relativistic level, the resulting structure corresponds to the extended Newton-Hooke algebra [105–111], which reduces to the extended Bargmann algebra in the vanishing cosmological constant limit $\Lambda \to 0$ [52, 53]. In the ultra-relativistic regime, the same expansion yields the extended Para-Poincaré algebra, also known as the extended AdS-Carroll algebra [54], whose flat limit leads to an extended Carroll algebra. Moreover, two successive expansions produce an extended AdS-Static algebra. These relations are summarized in Fig. 7.1. Each of

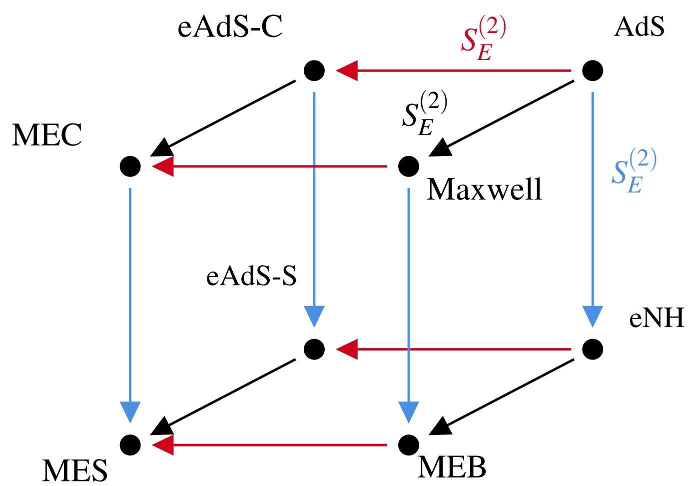

Figure 9.1: An analogous construction to that of Bacri and Lévy-Leblond, however achieved through a semigroup expansion approach.

these extended kinematical algebras, whose explicit commutation relations are listed in Chapter. 6 and Chapter. 7, admits a non-degenerate invariant tensor, which allows for the construction of well-defined Chern-Simons gravitational actions.

The Maxwellian generalization is then obtained by promoting the flat limit $\Lambda \to 0$ to an expansion procedure. In particular, the Maxwell kinematical algebras arise from a resonant $S_E^{(2)}$ expansion followed by a 0_S reduction. As illustrated by the cube in Fig. 9.1, the non-Lorentzian Maxwell algebras can also be recovered from the relativistic Maxwell algebra, inheriting the expansion relations already present in the corresponding extended kinematical algebras.

9.1 Maxwell Extended Barmann

We compute the non-relativistic extension of the Maxwell algebra, also known as the Maxwell Extended Bargmann (MEB) algebra [65]. This algebra can be obtained either from the extended Newton–Hooke algebra (see Sec. 7.2) or, alternatively, from the Maxwell algebra (8.1.1), as summarized in Fig. 9.1.

As in the previous case, the S-expansion requires a decomposition of the original algebra into subspaces (see Table 9.1) satisfying a $\mathbb{Z}_2$-graded Lie algebra structure of the form (7.1.1). Following the same procedure described in Chapter 7, we consider the resonant $S_E^{(2)}$-expansion, followed by the 0_S-reduction.

In this way, the generators of the Maxwell Extended Bargmann algebra are defined in terms of those of the extended Newton–Hooke or Maxwell algebras, as shown in Table 9.2.

It is important to note that we do not obtain new commutators here, since the only

Subspaces	Extended Newton-Hooke origin	Maxwell origin
V_0	J, G_a, S	J, H, Z
V_1	H, P_a, M	G_a, P_a, Z_a

Table 9.1: Subspaces decomposition of the extended Newton-Hooke and Maxwell algebra.

	Extended Newton–Hooke origin		Maxwell origin	
λ_3				
λ_2	$Z,\ Z_a,\ T$		$S,\ M,\ T$	
λ_1		$H,\ P_a,\ M$		$G_a,\ P_a,\ Z_a$
λ_0	$J,\ G_a,\ S$		$J,\ H,\ Z$	
$\times$	$\tilde{J},\ \tilde{G}_a,\ \tilde{S}$	$\tilde{H},\ \tilde{P}_a,\ \tilde{M}$	$\tilde{J},\ \tilde{H},\ \tilde{Z}$	$\tilde{G}_a,\ \tilde{P}_a,\ \tilde{Z}_a$

Table 9.2: MEB generators expressed in terms of the generators of the extended Newton-Hooke and Maxwell algebras through the $S_E^{(2)}$ semigroup elements.

possible new commutators would arise from the generators in the λ_2 row with those in λ_0. However, in this case, each generator corresponds to zero component, vanishing due to the antisymmetry of the ε-pseudo tensor. Therefore, the expanded algebra is

$$\begin{aligned}
&[J,G_a] = \varepsilon_{ab}G_b, && [G_a,P_b] = -\varepsilon_{ab}M, && [G_a,Z_b] = -\varepsilon_{ab}T, && \\
&[G_a,G_b] = -\varepsilon_{ab}S, && [H,G_a] = \varepsilon_{ab}P_b, && [Z,G_a] = \varepsilon_{ab}Z_b, && [P_a,P_b] = -\varepsilon_{ab}T. \\
&[J,P_a] = \varepsilon_{ab}P_b, && [J,Z_a] = \varepsilon_{ab}Z_b, && [H,P_a] = \varepsilon_{ab}Z_b, &&
\end{aligned} \tag{9.1.1}$$

By comparing with Table 8.3, it becomes evident that the commutators which previously vanished are now non-zero, reflecting the non-trivial structure generated by the expansion procedure. Furthermore we can recogonize the Extended Barman on the left column of (9.1.1).

$$\begin{aligned}
&\langle JJ\rangle = -\alpha_0, && \langle JZ\rangle = -\alpha_2, && \langle JS\rangle = -\mu_0, && \\
&\langle G_aG_b\rangle = \mu_0\delta_{ab}, && \langle G_aZ_b\rangle = \mu_2\delta_{ab}, && \langle JM\rangle = -\mu_1, && \langle ZS\rangle = -\mu_2, \\
&\langle JH\rangle = -\alpha_1, && \langle HH\rangle = -\alpha_2, && \langle HS\rangle = -\mu_1, && \langle HM\rangle = -\mu_2, \\
&\langle G_aP_b\rangle = \mu_1\delta_{ab}, && \langle P_aP_b\rangle = \mu_2\delta_{ab}, && \langle JT\rangle = -\mu_2, &&
\end{aligned} \tag{9.1.2}$$

where every constant expanded from the invariant tensors either from Extended Newton-Hooke (7.2.2) or Maxwell Algebra (8.2.2) is summarized in Table 9.3. Interestingly, the MEB algebra (9.1.1) is isomorphic to the Extended AdS-Static algebra (7.4.1) upon the following identification of the generators:

$$\tilde{G}_a \leftrightarrow P_a\,, \qquad \tilde{P}_a \leftrightarrow G_a\,, \qquad \tilde{Z} \leftrightarrow C\,, \qquad \tilde{T} \leftrightarrow Y\,, \qquad \tilde{Z}_a \leftrightarrow M_a\,,$$

where the generators with tilde ($\tilde{}$) correspond to MEB algebra. This observation suggests relabeling the extended AdS-Static algebra as the **Para-MEB** algebra.

	Extended Newton–Hooke origin				**Maxwell origin**		
λ_3							
λ_2	α_2		μ_2		μ_0	μ_1	μ_2
λ_1		α_1		α_2			
λ_0	α_0		μ_0		α_0	α_1	α_2
$\times$	$\tilde{\alpha}_0$	$\tilde{\alpha}_1$	$\tilde{\mu}_0$	$\tilde{\mu}_1$	$\tilde{\alpha}_0$	$\tilde{\alpha}_1$	$\tilde{\alpha}_2$

Table 9.3: MEB invariant constants expanded from Extended Newton-Hooke and Maxwell Algebra.

The non-degeneracy condition of the invariant tensor requires $\mu_2 \neq 0$. The curvatures are

$$\begin{aligned}
R(\omega) &= d\omega & R^a(z) &= dz^a + \omega z_b \varepsilon^{ab} + \tau e_b \varepsilon^{ab} + z\omega_b \varepsilon^{ab}, \\
R^a(\omega) &= d\omega^a + \omega\omega_b \varepsilon^{ab}, & R(s) &= ds + \frac{1}{2}\omega_a \omega_b \varepsilon^{ab}, \\
R(\tau) &= d\tau, & R(m) &= dm + \omega_a e_b \varepsilon^{ab}, \\
R^a(e) &= de^a + \omega e_b \varepsilon^{ab} + \tau\omega_b \varepsilon^{ab}, & R(t) &= dt + \omega_a z_a \varepsilon^{ab} + \frac{1}{2} e_a e_b \varepsilon^{ab}. \\
R(z) &= dz,
\end{aligned} \tag{9.1.3}$$

And its action is

$$\begin{aligned}
I_{CS}^{MEB} = \int & \Big(-\omega R(\omega)\Big)\alpha_0 + \Big(\omega^a R_a(\omega) - 2sR(\omega)\Big)\mu_0 - 2\tau R(\omega)\alpha_1 \\
&+ \Big(2e^a R_a(\omega) - 2\tau R(s) - 2mR(\omega)\Big)\mu_1 - \Big(2\omega R(z) + \tau R(\tau)\Big)\alpha_2 \\
&+ \Big(e^a R_a(e) - 2mR(\tau) - 2sR(z) - 2tR(\omega) + \omega^a R_a(z) + z^a R(\omega)\Big)\mu_2 .
\end{aligned} \tag{9.1.4}$$

The equations of motion, obtained by varying the action (9.1.4), correspond to setting each curvature in Eq. (9.1.3) to zero. The explicit variations are summarized in Table A.4. It is worth noting that the first two sectors, proportional to μ_0 and μ_1, reproduce the most general Chern–Simons action associated with the extended Bargmann algebra [97, 134]. The genuinely Maxwellian contribution arises in the sector proportional to μ_2. As in the relativistic case, a cosmological constant may be incorporated by deforming the symmetry algebra into an enlarged extended Bargmann algebra [65].

For $\mu_2 \neq 0$, the field equations are given by the vanishing of the MEB curvature two-forms introduced in (9.1.3), together with

$$R(m) = dm + \omega_a e_b \varepsilon^{ab}\,, \qquad R(t) = dt + \omega_a z_b \varepsilon^{ab} + \frac{1}{2} e_a e_b \varepsilon^{ab}\,. \tag{9.1.5}$$

9.2 Maxwell Extended Carroll

We compute the ultra-relativistic (UR) extension of the Maxwell algebra, known as the Maxwell Extended Carroll (MEC) algebra [55]. This algebra can be obtained either from the extended AdS-Carroll algebra (see Sec. 7.3) or from the Maxwell algebra (8.1.1), as illustrated in Fig. 9.1.

The MEC algebra is constructed by performing a resonant $S_E^{(2)}$-expansion of the starting algebra followed by a 0_S-reduction. To this end, we first consider the subspace decomposition shown in Table 9.4.

Subspaces	**Extended AdS-Carroll origin**	**Maxwell origin**
V_0	J, G_a, C	J, P_a, Z
V_1	H, P_a, T_a	G_a, H, Z_a

Table 9.4: Subspaces decomposition of the extended AdS-Carroll and Maxwell algebras.

The MEC generators can be obtained from the extended AdS-Carroll or Maxwell generators via the semigroup elements, as summarized in Table 9.5.

Then, the commutators of the MEC algebra can be obtained by combining either the Extended AdS–Carroll or the Maxwell commutators with the multiplication law of the

	Extended AdS–Carroll origin		**Maxwell origin**	
λ_3				
λ_2	$Z,\ Z_a,\ L$		$C,\ T_a,\ L$	
λ_1		$H,\ P_a,\ T_a$		$G_a,\ H,\ Z_a$
λ_0	$J,\ G_a,\ C$		$J,\ P_a,\ Z$	
$\times$	$\tilde{J},\ \tilde{G}_a,\ \tilde{C}$	$\tilde{H},\ \tilde{P}_a,\ \tilde{T}_a$	$\tilde{J},\ \tilde{P}_a,\ \tilde{Z}$	$\tilde{G}_a,\ \tilde{H},\ \tilde{Z}_a$

Table 9.5: MEC generators expressed in terms of the generators of the extended AdS-Carroll and Maxwell algebras through the $S_E^{(2)}$ semigroup elements.

semigroup $S_E^{(2)}$,

$$\begin{aligned}
[J,G_a] &= \epsilon_{ab}G_b, & [J,Z_a] &= \epsilon_{ab}Z_b, & & \\
[G_a,G_b] &= -\epsilon_{ab}C, & [G_a,Z_b] &= -\epsilon_{ab}L, & [C,P_a] &= \epsilon_{ab}T_b, \\
[J,P_a] &= \epsilon_{ab}P_b, & [Z,G_a] &= \epsilon_{ab}Z_b, & [J,T_a] &= \epsilon_{ab}T_b, \\
[G_a,P_b] &= -\epsilon_{ab}H, & [H,P_a] &= \epsilon_{ab}Z_b, & [T_a,P_b] &= -\epsilon_{ab}L, \\
[H,G_a] &= \epsilon_{ab}T_b, & [P_a,P_b] &= -\epsilon_{ab}Z. & &
\end{aligned} \tag{9.2.1}$$

Compared to Table 8.3, and analogously to the previous case, the commutators that previously vanished are now nonzero. It is worth noting that the MEC algebra, which contains 13 generators, is not isomorphic to the MEB algebra (9.1.1), which is spanned by 12 generators. In particular, the MEC algebra is characterized by the presence of a central charge L, whose inclusion guarantees the non-degeneracy of the invariant tensor, which reads

$$\begin{aligned}
\langle JJ\rangle &= -\beta_0, & \langle JZ\rangle &= -\beta_2, & \langle JC\rangle &= -\beta_0, \\
\langle G_a G_b\rangle &= \beta_0\delta_{ab}, & \langle G_a Z_b\rangle &= \beta_2\delta_{ab}, & \langle JL\rangle &= -\beta_2, \\
\langle JH\rangle &= -\beta_1, & \langle HH\rangle &= -\beta_2, & \langle CZ\rangle &= -\beta_2, \\
\langle G_a P_b\rangle &= \beta_1\delta_{ab}, & \langle P_a P_b\rangle &= \beta_2\delta_{ab}, & \langle T_a P_b\rangle &= \beta_2\delta_{ab}.
\end{aligned} \tag{9.2.2}$$

Every constant expanded from the invariant tensors either from Extended AdS-Carroll (7.3.2) or Maxwell Algebra (8.2.2) is summarized in Table 9.6. The non-degeneracy condition allows two of these constants to be equal.

	Extended AdS–Carroll origin			Maxwell origin		
λ_3						
λ_2	β_2		β_2	β_0		β_2
λ_1		β_1			β_1	
λ_0	β_0		β_0	β_0		β_2
$\times$	$\tilde{\beta}_0$	$\tilde{\beta}_1$	$\tilde{\beta}_2$	$\tilde{\alpha}_0$	$\tilde{\alpha}_1$	$\tilde{\alpha}_2$

Table 9.6: MEC invariant constants expanded from Extended AdS-Carroll and Maxwell Algebra.

The curvatures are

$$\begin{aligned} R(\omega) &= d\omega \\ R^a(\omega) &= d\omega^a + \omega\omega_b\varepsilon^{ab}, \\ R(\tau) &= d\tau + \omega_a e_b \varepsilon^{ab}, \\ R^a(e) &= de^a + \omega e_b \varepsilon^{ab}, \\ R(z) &= dz + \frac{1}{2} e_a e_b \varepsilon^{ab}, \end{aligned} \qquad \begin{aligned} R^a(z) &= dz^a + \omega z_b \varepsilon^{ab} + \tau e_b \varepsilon^{ab} + z\omega_b \varepsilon^{ab}, \\ R(c) &= dc + \frac{1}{2}\omega_a\omega_b\varepsilon^{ab}, \\ R^a(t) &= dt^a + \tau\omega_b\varepsilon^{ab} + ce_b\varepsilon^{ab} + \omega t_b \varepsilon^{ab}, \\ R(l) &= dl + \omega_a z_a \varepsilon^{ab} - t_a e_b \varepsilon^{ab}. \end{aligned} \tag{9.2.3}$$

Finally its action is

$$\begin{aligned} I_{CS}^{MEC} = \int & \Big(-\omega R(\omega) + \omega^a R_a(\omega) - 2cR(\omega) \Big) \beta_0 \\ & + \Big(e^a R_a(\omega) + \omega^a R_a(e) - 2\tau R(\omega) \Big) \beta_1 \\ & + \Big(-e^a R_a(e) - 2zR(\omega) + z^a R_a(\omega) - \tau R(\tau) + 2t^a R_a(e) \\ & \quad - 2cR(z) + \omega^a R_a(z) - 2lR(\omega) \Big) \beta_2. \end{aligned} \tag{9.2.4}$$

In this case, the equations of motion obtained from the variation of the action are given by the vanishing of the curvatures defined in Eq. (9.2.3) (see Table A.5).

As in the original Carroll algebra, the resulting field equations exhibit a non-vanishing temporal torsion,

$$d\tau = -\omega_a e_b \varepsilon^{ab}. \tag{9.2.5}$$

However, the Maxwell extension introduces additional gauge fields $\{c, t^a, l\}$ associated with the generators $\{C, T_a, L\}$. These fields are required to ensure the non-degeneracy of the invariant tensor and, consequently, the consistency of the field equations.

Although Maxwell gauge fields have been studied at the relativistic level in several contexts [60–64], the physical interpretation of the additional structures that arise in the Carrollian regime remains largely unexplored.

9.3 Maxwell Extended Static

For completeness, we conclude this section by introducing the cosmological extension of the eAdS–S algebra (see Sec. 7.4), which can also be interpreted as the ultra-relativistic extension of the MEB algebra (Sec. 9.1) or the non-relativistic extension of the MEC algebra (Sec. 9.2). This algebra, which we denote as the Maxwell Extended Static (MES) algebra, has not been previously discussed in the literature.

Nevertheless, we show that the S-expansion framework provides three independent constructions leading to its commutation relations. In particular, the MES algebra can be obtained from the extended AdS–static, MEB, and MEC algebras (see Table 9.7).

Subspaces	**Extended AdS-static**	**Maxwell Extended Bargmann**	**Maxwell Extended Carroll**
V_0	J,G_a,C,S,B	J,P_a,Z,S,T	J,C,Z,L,H
V_1	H,P_a,T_a,M	G_a,H,Z_a,M	G_a,P_a,T_a,Z_a

Table 9.7: Subspaces decomposition of the eAdS-S, MEB and MEC algebras.

The MES algebra is obtained through a resonant $S_E^{(2)}$-expansion of any of the parent algebras, followed by a 0_S-reduction. The resulting generators are related to those of the parent algebras through the semigroup elements, as shown in Table 9.8.

	Extended AdS-static origin		**MEB origin**		**MEC origin**	
λ_3						
λ_2	Z, Z_a, L, T, Y		C, T_a, L, B, Y		S, B, T, Y, M	
λ_1		H, P_a, T_a, M		G_a, H, Z_a, M		G_a, P_a, T_a, Z_a
λ_0	J, G_a, C, S, B		J, P_a, Z, S, T		J, C, Z, L, H	
$\times$	$\tilde{J}, \tilde{G}_a, \tilde{C}, \tilde{S}, \tilde{B}$	$\tilde{H}, \tilde{P}_a, \tilde{T}_a, \tilde{M}$	$\tilde{J}, \tilde{P}_a, \tilde{Z}, \tilde{S}, \tilde{T}$	$\tilde{G}_a, \tilde{H}, \tilde{Z}_a, \tilde{M}$	$\tilde{J}, \tilde{C}, \tilde{Z}, \tilde{L}, \tilde{H}$	$\tilde{G}_a, \tilde{P}_a, \tilde{T}_a, \tilde{Z}_a$

Table 9.8: MES generators expressed in terms of the generators of the eAdS-S, MEB and MEC algebras through the $S_E^{(2)}$ semigroup elements.

The commutation relations of the MES algebra follow from combining the commutators of the chosen starting algebra:, eAdS-S (7.4.1), MEB (9.1.1), MEC (9.2.1), with the multiplication law of the $S_E^{(2)}$ semigroup in Table 9.8, which reads:

$$
\begin{array}{lll}
[J,G_a] = \varepsilon_{ab}G_b\,, & [J,Z_a] = \varepsilon_{ab}Z_b\,, & \\
[G_a,G_b] = -\varepsilon_{ab}B\,, & [G_a,Z_b] = -\varepsilon_{ab}Y\,, & [C,P_a] = \varepsilon_{ab}T_b\,, \\
[J,P_a] = \varepsilon_{ab}P_b\,, & [Z,G_a] = \varepsilon_{ab}Z_b\,, & [J,T_a] = \varepsilon_{ab}T_b\,, \\
[G_a,P_b] = -\varepsilon_{ab}M\,, & [H,P_a] = \varepsilon_{ab}Z_b\,, & [T_a,P_b] = -\varepsilon_{ab}Y\,. \\
[H,G_a] = \varepsilon_{ab}T_b\,, & [P_a,P_b] = -\varepsilon_{ab}T\,, &
\end{array}
\tag{9.3.1}
$$

Let us note that the MES algebra contains two additional $\mathfrak{u}(1)$ central generators, S and L, satisfying

$$
[X,S] = [X,L] = 0\,, \qquad \forall X \in \mathfrak{mes}\,. \tag{9.3.2}
$$

Although S and L do not enter in the non-trivial commutators in (9.3.1), their presence is essential to ensure the non-degeneracy of the invariant bilinear form. In particular, the MES algebra admits the following non-vanishing components of the invariant tensor:

$$
\begin{array}{llll}
\langle JJ\rangle = -\beta_0\,, & \langle HH\rangle = -\beta_2\,, & \langle ZS\rangle = -\nu_2\,, & \langle JB\rangle = -\nu_0\,, \\
\langle G_aG_b\rangle = \nu_0\,, & \langle P_aP_b\rangle = \nu_2\delta_{ab}\,, & \langle HM\rangle = -\beta_2\,, & \langle SC\rangle = -\nu_0\,, \\
\langle JH\rangle = -\beta_1\,, & \langle JS\rangle = -\nu_0\,, & \langle JC\rangle = -\beta_0\,, & \langle JY\rangle = -\nu_2\,, \\
\langle G_aP_b\rangle = \nu_1\,, & \langle JM\rangle = -\nu_1\,, & \langle JL\rangle = -\beta_2\,, & \langle CT\rangle = -\nu_2\,, \\
\langle JZ\rangle = -\beta_2\,, & \langle HS\rangle = -\nu_1\,, & \langle CZ\rangle = -\beta_2\,, & \langle LS\rangle = -\nu_2\,, \\
\langle G_aZ_b\rangle = \nu_2\delta_{ab}\,, & \langle JT\rangle = -\nu_2\,, & \langle T_aP_b\rangle = \nu_2\delta_{ab}\,, & \langle ZB\rangle = -\nu_2\,.
\end{array}
\tag{9.3.3}
$$

The constants arising in each expansion are summarized in Table 9.9 for the corresponding parent algebras, namely eAdS-S (7.4.2), MEB (9.1.2), and MEC (9.2.2). The non-degeneracy condition requires $\nu_2 \neq 0$, which allows some of the constants appearing in the different expansions to be identified. Although a Maxwellian static algebra without the generators C, L, B, Y, T_a still satisfies the Jacobi identity, it does not admit a non-degenerate invariant bilinear form.

	eAdS-S origin						**MEB origin**						**MEC origin**		
λ_3															
λ_2	β_2		β_2	ν_2		ν_2	β_0		β_2	ν_0		ν_2	ν_0	ν_1	ν_2
λ_1		β_1			ν_1			β_1			ν_1				
λ_0	β_0		β_0	ν_0		ν_0	β_0		β_2	ν_0		ν_2	β_0	β_1	β_2
$\times$	$\tilde{\beta}_0$	$\tilde{\beta}_1$	$\tilde{\beta}_2$	$\tilde{\nu}_0$	$\tilde{\nu}_1$	$\tilde{\nu}_2$	$\tilde{\alpha}_0$	$\tilde{\alpha}_1$	$\tilde{\alpha}_2$	$\tilde{\mu}_0$	$\tilde{\mu}_1$	$\tilde{\mu}_2$	$\tilde{\beta}_0$	$\tilde{\beta}_1$	$\tilde{\beta}_2$

Table 9.9: Invariant constants expanded from three different algebraic origins.

The curvatures are

$$
\begin{aligned}
R(\omega) &= d\omega & R(t) &= dt + \frac{1}{2}e_a e_b \varepsilon^{ab}, \\
R^a(\omega) &= d\omega^a + \omega\omega_b \varepsilon^{ab}, & R(s) &= ds, \\
R(\tau) &= d\tau, & R(c) &= dc, \\
R^a(e) &= de^a + \omega e_b \varepsilon^{ab}, & R(l) &= dl, \\
R(z) &= dz, & R(b) &= db + \frac{1}{2}\omega_a\omega_b \varepsilon^{ab}, \\
R^a(z) &= dz^a + \omega z_b \varepsilon^{ab} + \tau e_b \varepsilon^{ab} + z\omega_b \varepsilon^{ab}, & R(y) &= dy + \omega_a z_b \varepsilon^{ab} + t_a e_b \varepsilon^{ab}, \\
R^a(t) &= dt^a + \tau\omega_b \varepsilon^{ab} + c e_b \varepsilon^{ab} + \omega t_b \varepsilon^{ab}, & R(m) &= dm + \omega_a e_b \varepsilon^{ab}.
\end{aligned}
\tag{9.3.4}
$$

Finally its action is

$$
\begin{aligned}
I_{CS}^{MES} = \int & \Big(-\omega R(\omega)\Big)\beta_0 + \Big(-2\omega R(\tau)\Big)\beta_1 + \Big(-2zR(\omega)\Big)\beta_2 \\
& + \Big(-2sR(\omega) + e^a R_a(e) - 2zR(s) - 2tR(\omega)\Big)\nu_0 \\
& + \Big(2\omega^a R_a(e) - 2mR(\omega) - 2\tau R(s)\Big)\nu_1 \\
& + \Big(\omega^a R_a(\omega) - 2cR(\omega) - 2bR(\omega) - 2sR(c) - 2cR(z) - \tau R(\tau) + 2t^a R_a(e) \\
& \quad + 2e^a R_a(t) - 2mR(\tau) + \omega^a R_a(z) + z^a R_a(\omega) - 2lR(\omega) - 2tR(c) \\
& \quad - 2yR(\omega) - 2lR(\omega) - 2bR(z)\Big)\nu_2 .
\end{aligned}
\tag{9.3.5}
$$

The equations of motion resulting from the variation of the action correspond to setting each curvature in Eq. (9.3.4) to zero. The explicit variations are summarized in Table A.6.

Therefore, the minimal content required to obtain a consistent Maxwellian extension of the static algebra, and consequently a well-defined CS action, is precisely the complete set of generators introduced here. The corresponding CS action can be constructed straightforwardly from (9.3.3) and the gauge connection associated with the MES algebra. Given its lengthy structure, we refrain from presenting it explicitly. Moreover, it may be derived directly from the extended AdS-static, MEB or MEC CS gravity actions by expressing the MES gauge field in terms of the corresponding parent algebra through the $S_E^{(2)}$ semigroup elements, according to the generator identifications displayed in Table 9.8.

Therefore, by means of the semigroup expansion method, we have derived all Non-Lorentzian (NR and UR) and zero cosmological constant extensions from AdS, as illustrated in Figure 9.1, in complete analogy with the construction originally presented by Bacri and Lévy-Leblond.

Chapter 10

Generalized extended kinematical algebras

In this section, we show that the original cube proposed by Bacry and Lévy-Leblond [1], together with its Maxwellian extension presented in the previous section (Sec. 9), can be generalized in a natural way by using an arbitrary semigroup $S_E^{(N)}$. This procedure leads to an infinite sequence of generalized kinematical cubes related to the non-Lorentzian sector of the so-called $\mathfrak{B}_k$ algebras [112–114]. The $\mathfrak{B}_k$ extension of the enlarged kinematical Lie algebras is shown schematically in Fig. 10.1. There, the vanishing cosmological constant limit of the Bacry–Lévy-Leblond cube [1] is extended through an $S_E^{(N)}$-expansion with $N = k-2$. In this setup, the $S_E^{(2)}$-expansion is kept along both the non-relativistic and ultra-relativistic directions in order to obtain a non-degenerate non-Lorentzian version of the $\mathfrak{B}_k$ algebra. If one considers higher-order semigroups along the non-relativistic or ultra-relativistic directions of the cube in Fig. 10.1, one would obtain post-Newtonian or post-Carrollian extensions of the $\mathfrak{B}_k$ algebra. These cases, however, are not studied in this work.

We start from the extended kinematical algebras summarized in Fig. 7.1. For each algebra, we introduce a subspace decomposition that satisfies a $\mathbb{Z}_2$-graded Lie algebra structure, as shown in Table 10.1.

We now consider the semigroup $S_E^{(N)} = \{\lambda_0, \lambda_1, \cdots, \lambda_{N+1}\}$, which satisfies the multiplication rule

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if } \alpha+\beta \leq N+1\,, \\ \lambda_{N+1} & \text{if } \alpha+\beta > N+1\,. \end{cases} \tag{10.0.1}$$

A subset decomposition of $S_E^{(N)}$ that is resonant with the subspace splitting in Table 10.1

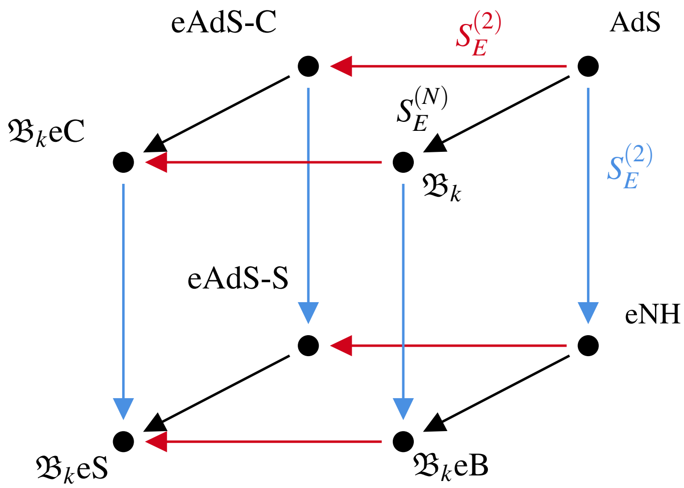

Figure 10.1: $\mathfrak{B}_k$ generalization of the extended kinematical algebras.

Subspaces	AdS origin	eNH origin	eAdS-C origin	eAdS-S origin
V_0	J, G_a	J, G_a, S	J, G_a, C	J, G_a, C, S, B
V_1	H, P_a	H, P_a, M	H, P_a, T_a	H, P_a, T_a, M

Table 10.1: Subspace decompositions of the extended kinematical algebras.

is given by

$$S_0 = \{\lambda_{2i}\} \cup \{\lambda_{N+1}\}, \qquad S_1 = \{\lambda_{2i+1}\} \cup \{\lambda_{N+1}\}, \tag{10.0.2}$$

with $i = 0, 1, 2, \cdots, [N/2]$, where $[\cdot]$ denotes the integer part. This resonant condition makes it possible to construct a resonant $S_E^{(N)}$-expanded subalgebra. The $\mathfrak{B}_k$ algebra and its non-Lorentzian versions follow from the extended kinematical algebras (see Fig. 7.1) by performing a resonant $S_E^{(N)}$-expansion and then applying the corresponding 0_S-reduction. The expanded generators are connected to those of the extended kinematical algebras through the semigroup elements, as summarized in Table 10.2.

By combining the original commutation relations of the extended kinematical algebras with the multiplication law of $S_E^{(N)}$, one finds that the $\mathfrak{B}_k$ extensions satisfy the commutation relations displayed in Table 10.3. This shows that the extended kinematical algebras [54, 136] and their Maxwellian extensions can be seen as particular cases of the $\mathfrak{B}_k$ extended kinematical algebra. When $N = 1$, the resonant 0_S-reduced $S_E^{(N)}$-expansion be-

Expanded Generators	**AdS origin**	**eNH origin**	**eAdS-C origin**	**eAdS-S origin**
$\tilde{J}^{(m)}$	$\lambda_{2m}J$	$\lambda_{2m}J$	$\lambda_{2m}J$	$\lambda_{2m}J$
$\tilde{G}_a^{(m)}$	$\lambda_{2m}G_a$	$\lambda_{2m}G_a$	$\lambda_{2m}G_a$	$\lambda_{2m}G_a$
$\tilde{S}^{(m)}$	-	$\lambda_{2m}S$	-	$\lambda_{2m}S$
$\tilde{C}^{(m)}$	-	-	$\lambda_{2m}C$	$\lambda_{2m}C$
$\tilde{B}^{(m)}$	-	-	-	$\lambda_{2m}B$
$\tilde{H}^{(m)}$	$\lambda_{2m+1}H$	$\lambda_{2m+1}H$	$\lambda_{2m+1}H$	$\lambda_{2m+1}H$
$\tilde{P}_a^{(m)}$	$\lambda_{2m+1}P_a$	$\lambda_{2m+1}P_a$	$\lambda_{2m+1}P_a$	$\lambda_{2m+1}P_a$
$\tilde{M}^{(m)}$	-	$\lambda_{2m+1}M$	-	$\lambda_{2m+1}M$
$\tilde{T}_a^{(m)}$	-	-	$\lambda_{2m+1}T_a$	$\lambda_{2m+1}T_a$

Table 10.2: Expanded generators in terms of the extended kinematical ones.

comes a contraction that corresponds to the vanishing cosmological constant limit. In that case, the expanded algebras match the $\mathfrak{B}_3$ algebra and its non-Lorentzian versions, namely the Poincaré algebra and its non-degenerate non-Lorentzian counterparts studied in [54, 136]. The expanded generators are then identified as

$$\begin{aligned}
&\tilde{J}^{(0)} \equiv J\,, \qquad &&\tilde{G}_a^{(0)} \equiv G_a\,, \qquad &&\tilde{H}^{(0)} \equiv H\,, \qquad &&\tilde{P}_a^{(0)} \equiv P_a\,,\\
&\tilde{S}^{(0)} \equiv S\,, &&\tilde{T}_a^{(0)} \equiv T_a\,, &&\tilde{C}^{(0)} \equiv C\,, &&\tilde{M}^{(0)} \equiv M\,,\\
&\tilde{B}^{(0)} \equiv B\,.
\end{aligned} \tag{10.0.3}$$

On the other hand, the case $N=2$ gives the $\mathfrak{B}_4$ extension of the kinematical algebra, which is exactly the Maxwellian extended kinematical algebra discussed before.

For $N \geq 3$, new families of kinematical symmetry algebras appear. These algebras are not post-Newtonian nor post-Carrollian, and they cannot be obtained simply as extensions of the Galilei or Carroll algebras. The special case $N=3$ gives the $\mathfrak{B}_5$ extended kinematical algebras, whose commutation relations are presented in Appendix B. At the relativistic level, the $\mathfrak{B}_5$ algebra has been useful to recover standard General Relativity

Commutators	$\mathfrak{B}_k$	$\mathfrak{B}_k$eB	$\mathfrak{B}_k$eC	$\mathfrak{B}_k$eS
$\left[J^{(m)},G_a^{(n)}\right]$	$\epsilon_{ab}G_b^{(m+n)}$	$\epsilon_{ab}G_b^{(m+n)}$	$\epsilon_{ab}G_b^{(m+n)}$	$\epsilon_{ab}G_b^{(m+n)}$
$\left[J^{(m)},P_a^{(n)}\right]$	$\epsilon_{ab}P_b^{(m+n)}$	$\epsilon_{ab}P_b^{(m+n)}$	$\epsilon_{ab}P_b^{(m+n)}$	$\epsilon_{ab}P_b^{(m+n)}$
$\left[G_a^{(m)},G_b^{(n)}\right]$	$-\epsilon_{ab}J^{(m+n)}$	$-\epsilon_{ab}S^{(m+n)}$	$-\epsilon_{ab}C^{(m+n)}$	$-\epsilon_{ab}B^{(m+n)}$
$\left[H^{(m)},G_a^{(n)}\right]$	$\epsilon_{ab}P_b^{(m+n)}$	$\epsilon_{ab}P_b^{(m+n)}$	$\epsilon_{ab}T_b^{(m+n)}$	$\epsilon_{ab}T_b^{(m+n)}$
$\left[G_a^{(m)},P_b^{(n)}\right]$	$-\epsilon_{ab}H^{(m+n)}$	$-\epsilon_{ab}M^{(m+n)}$	$-\epsilon_{ab}H^{(m+n)}$	$-\epsilon_{ab}M^{(m+n)}$
$\left[H^{(m)},P_a^{(n)}\right]$	$\epsilon_{ab}G_b^{(m+n+1)}$	$\epsilon_{ab}G_b^{(m+n+1)}$	$\epsilon_{ab}G_b^{(m+n+1)}$	$\epsilon_{ab}G_b^{(m+n+1)}$
$\left[P_a^{(m)},P_b^{(n)}\right]$	$-\epsilon_{ab}J^{(m+n+1)}$	$-\epsilon_{ab}S^{(m+n+1)}$	$-\epsilon_{ab}J^{(m+n+1)}$	$-\epsilon_{ab}S^{(m+n+1)}$
$\left[J^{(m)},T_a^{(n)}\right]$	-	-	$\epsilon_{ab}T_b^{(m+n)}$	$\epsilon_{ab}T_b^{(m+n)}$
$\left[P_a^{(m)},T_b^{(n)}\right]$	-	-	$-\epsilon_{ab}C^{(m+n+1)}$	$-\epsilon_{ab}B^{(m+n+1)}$
$\left[C^{(m)},P_a^{(n)}\right]$	-	-	$\epsilon_{ab}T_b^{(m+n)}$	$\epsilon_{ab}T_b^{(m+n)}$

Table 10.3: Commutation relations of the $\mathfrak{B}_k$ generalization of the extended kinematical algebras.

from Chern-Simons and Born-Infeld gravity actions [112–114, 138]. Its non-relativistic non-degenerate version is the $\mathfrak{B}_5$eB algebra (see Table B.1 and Table B.2), introduced in [139] as a generalized Maxwellian exotic Bargmann algebra. In that reference, it was obtained through a contraction of the algebra $\mathfrak{B}_5 \oplus \mathfrak{u}(1)^4$. The Carrollian and static cases, which had not been discussed before, correspond to the algebras $\mathfrak{B}_5$eC and $\mathfrak{B}_5$eS, shown in Table B.1 and Table B.2. From the commutation relations in Appendix B, it follows that the $\mathfrak{B}_5$ extended kinematical algebras arise as extensions of the Maxwellian extended kinematical algebras, which correspond to the first two rows of Table B.1 and Table B.2. In particular, the first row reproduces the extended kinematical algebras introduced in [54].

The $\mathfrak{B}_k$ extended kinematical algebras possess the non-vanishing components of the invariant tensor listed in Table 10.4. The associated coupling constants are related to those of the original extended kinematical algebras through the semigroup elements as

$$\alpha_{i+j} = \lambda_i\lambda_j\mu_r\,, \qquad \beta_{i+j} = \lambda_i\lambda_j\nu_r\,, \qquad \gamma_{i+j} = \lambda_i\lambda_j\sigma_s\,, \qquad \zeta_{i+j} = \lambda_i\lambda_j\rho_s\,, \qquad (10.0.4)$$

with $r = 0,1$ and $s = 0,1,2$, where the product $\lambda_i\lambda_j$ is defined by the multiplication rule (10.0.1) of $S_E^{(N)}$.

Invariant tensor	$\mathfrak{B}_k$	$\mathfrak{B}_k$eB	$\mathfrak{B}_k$eC	$\mathfrak{B}_k$eS
$\langle J^{(m)}, J^{(n)}\rangle$	$-\alpha_{2n+2m}$	0	$-\gamma_{2n+2m}$	0
$\langle J^{(m)}, H^{(n)}\rangle$	$-\alpha_{2n+2m+1}$	0	$-\gamma_{2n+2m+1}$	0
$\langle H^{(m)}, H^{(n)}\rangle$	$-\alpha_{2n+2m+2}$	0	$-\gamma_{2n+2m+2}$	0
$\langle G_a^{(m)}, G_b^{(n)}\rangle$	$\alpha_{2n+2m}\,\delta_{ab}$	$\beta_{2n+2m}\,\delta_{ab}$	$\gamma_{2n+2m}\,\delta_{ab}$	$\zeta_{2n+2m}\,\delta_{ab}$
$\langle G_a^{(m)}, P_b^{(n)}\rangle$	$\alpha_{2n+2m+1}\,\delta_{ab}$	$\beta_{2n+2m+1}\,\delta_{ab}$	$\gamma_{2n+2m+1}\,\delta_{ab}$	$\zeta_{2n+2m+1}\,\delta_{ab}$
$\langle P_a^{(m)}, P_b^{(n)}\rangle$	$\alpha_{2n+2m+2}\,\delta_{ab}$	$\beta_{2n+2m+2}\,\delta_{ab}$	$\gamma_{2n+2m+2}\,\delta_{ab}$	$\zeta_{2n+2m+2}\,\delta_{ab}$
$\langle J^{(m)}, S^{(n)}\rangle$	-	$-\beta_{2n+2m}$	-	$-\zeta_{2n+2m}$
$\langle J^{(m)}, M^{(n)}\rangle$	-	$-\beta_{2n+2m+1}$	-	$-\zeta_{2n+2m+1}$
$\langle H^{(m)}, M^{(n)}\rangle$	-	$-\beta_{2n+2m+2}$	-	$-\zeta_{2n+2m+2}$
$\langle H^{(m)}, S^{(n)}\rangle$	-	$-\beta_{2n+2m+1}$	-	$-\zeta_{2n+2m+1}$
$\langle J^{(m)}, C^{(n)}\rangle$	-	-	$-\gamma_{2n+2m}$	0
$\langle J^{(m)}, B^{(n)}\rangle$	-	-	-	$-\zeta_{2n+2m}$
$\langle P_a^{(m)}, T_b^{(n)}\rangle$	-	-	$\gamma_{2n+2m+2}\,\delta_{ab}$	$\zeta_{2n+2m+2}\,\delta_{ab}$

Table 10.4: Non-vanishing components of the invariant tensor for the $\mathfrak{B}_k$ extended kinematical algebras.

One can see that the terms proportional to α_i, with $i = 0, \dots, N-1$, reproduce the invariant tensor structure of the $\mathfrak{B}_{N+1}$ algebra. Since $N = k-2$, this recursive structure allows us to write the CS gravity action for the $\mathfrak{B}_k$ algebra as the CS action for $\mathfrak{B}_{k-1}$ plus an extra term proportional to α_N:

$$I_{\mathfrak{B}_k} = \frac{k}{4\pi}\int \left(\mathscr{L}_{\mathfrak{B}_{k-1}} + \alpha_N \mathscr{L}_N\right) . \tag{10.0.5}$$

A similar recursive structure also holds for the non-Lorentzian versions of the $\mathfrak{B}_k$ algebra.

Chapter 11

Conclusions

In this work, based on [140], we have constructed a Maxwellian extension of the kinematical Lie algebras by promoting the contraction procedure of the original Bacry and Lévy-Leblond cube [1] to an S-expansion mechanism, where the semigroup $S_E^{(2)}$ plays the fundamental role. Within this framework, we have shown that several non-degenerate non-Lorentzian Maxwell gravity theories previously introduced in the literature naturally arise as particular cases. Furthermore, both the original Bacry and Lévy-Leblond cube and its Maxwellian counterpart can be understood as members of a broader infinite hierarchy of generalized kinematical algebras generated by arbitrary $S_E^{(N)}$ semigroups. In this hierarchy, the case $N = 1$ reproduces the Bacry and Lévy-Leblond cube, while higher values of N give rise to the family of $\mathfrak{B}_k$ algebras and their associated non-Lorentzian regimes. An additional advantage of this construction is that it systematically provides the non-vanishing components of the invariant tensor required for the formulation of Chern–Simons gravity actions associated with both Lorentzian and non-Lorentzian symmetry algebras.

The results presented here open several directions for future investigation. At the gravitational level, it would be interesting to analyze the dynamical sector of the Maxwell kinematical algebras. In particular, the physical interpretation of the additional gauge fields appearing in the non-Lorentzian regimes of the Maxwell cube remains to be clarified. These fields may admit an interpretation in terms of post-Newtonian or post-Carrollian corrections, in analogy with higher-order expansions of relativistic gravity. It would also be natural to explore whether such additional non-Lorentzian gauge fields can be related to gravito-magnetic-like effects [141]. From this perspective, they may contribute to generalized notions of torsion, non-inertial forces, or effective background fluxes in non-relativistic and Carrollian geometries, providing a novel arena to investigate generalized gravitational interactions beyond the standard kinematical frameworks.

Within the generalized kinematical setting, another interesting direction concerns the analysis of classical solutions of the corresponding $\mathfrak{B}_k$ Chern–Simons gravities. In particular, it would be worthwhile to study black-hole configurations, cosmological backgrounds, and their thermodynamic properties. The enlarged gauge structure is expected to modify both the global structure of the solutions and the associated conserved charges, potentially leading to new thermodynamic contributions. The first non-trivial case beyond the Maxwellian level, corresponding to the $\mathfrak{B}_5$ algebra, deserves special attention. At the relativistic level, $\mathfrak{B}_5$ has been shown to play a central role in recovering General Relativity in suitable limits [112–114, 138]. It would therefore be interesting to investigate whether its non-Lorentzian counterparts encode analogous subleading gravitational structures and to clarify the geometric and dynamical interpretation of the associated gauge fields.

From a more conceptual perspective, it would also be interesting to explore whether the non-Lorentzian algebras obtained here can be realized as asymptotic symmetry algebras of gravitational models, possibly leading to new extensions of the $\mathfrak{bms}_3$ algebra. In a related direction, recent developments in three-dimensional AdS-Carroll gravity show that the asymptotic symmetry algebra corresponds to an infinite-dimensional extension of a generalized Maxwell algebra [142]. This suggests the possibility that infinite-dimensional extensions of the $\mathfrak{B}_k$ algebras could emerge in a similar way, potentially providing new insights into the role of non-Lorentzian limits within holography.

Another natural direction concerns the extension of the present construction to supersymmetric and higher-spin theories. In particular, the application of the S-expansion method to non-Lorentzian regimes could lead to new classes of non-degenerate supergravity models. While supersymmetric extensions of the Bacry and Lévy-Leblond cube have recently been constructed in [136], it would be interesting to generalize the present approach by considering different starting (super)algebras beyond the (super) AdS case. At the higher-spin level, it would also be worthwhile to explore whether spin-$5/2$ symmetry algebras [143–146] can be incorporated into generalized kinematical frameworks along the lines of [147].

Finally, it is worth mentioning that the original motivation of this work was to investigate Maxwell kinematical structures starting from the AdS–Lorentz algebra in $(2+1)$ dimensions. Since the Maxwell algebra can be obtained as the flat limit of AdS–Lorentz of the cosmological constant, namely through an Inönü–Wigner contraction that can be interpreted as an $S_E^{(1)}$ expansion, one could apply this contraction procedure along the different directions of the kinematical cube. However, this procedure leads to a family of algebras whose corresponding gravitational theories become degenerate, as discussed in Chapter 8. This observation naturally suggests that applying the $S_E^{(2)}$ expansion along those directions restores non-degenerate structures. Although in the present work we have focused on the AdS algebra and its contractions, due to their central role in the literature, the study of Maxwell kinematical algebras starting from the AdS–Lorentz framework remains an interesting direction for future research.

Appendix A

Results from the Variation of the Actions

In this appendix we present the results obtained from varying the actions throughout this document.

A.1 Extended Kinematical Algebras

A.1.1 Extended Newton-Hooke

Gauge field	α	μ
ω	$R(\omega)\alpha_0 + R(\tau)\alpha_1$	$R(s)\mu_0 + R(m)\mu_1$
ω_a	-	$R_a(\omega)\mu_0 + R_a(e)\mu_1$
τ	$R(\tau)\alpha_0 + R(\omega)\alpha_1$	$R(m)\mu_0 + R(s)\mu_1$
e_a	-	$R(e)\mu_0 + R_a(\omega)\mu_1$
s	-	$R(\omega)\mu_0 + R(\tau)\mu_1$
m	-	$R(\tau)\mu_0 + R(\omega)\mu_1$

Table A.1: Result after vary (7.2.4) respect to each gauge field.

A.1.2 Extended AdS-Carroll

Gauge field	β
ω	$R(\omega)\beta_0 + R(\tau)\beta_1 + R(c)\beta_2$
ω_a	$R_a(e)\beta_1 + R_a(\omega)\beta_1$
τ	$R(\omega)\beta_1 + R(\tau)\beta_2$
e_a	$R_a(e)\beta_0 + R_a(\omega)\beta_1 + R_a(t)\beta_2$
c	$R(\omega)\beta_2$
t_a	$R_a(e)\beta_2$

Table A.2: Result after vary (7.2.4) respect to each gauge field.

A.1.3 Extended AdS-Static

Gauge field	β	ν
ω	$R(\omega)\beta_0 + R(\tau)\beta_1 + R(c)\beta_2$	$R(s)\nu_0 + R(m)\nu_1 + R(b)\nu_2$
ω_a	-	$R_a(e)\nu_1 + R_a(\omega)\nu_2$
τ	$R(\tau)\beta_1 + R(\tau)\beta_2$	$R(s)\nu_1 + R(m)\nu_2$
e_a	-	$R_a(e)\nu_0 + R_a(\omega)\mu_1 + R_a(t)$
m	-	$R(\omega)\nu_1 + R(\tau)\nu_2$
t_a	-	$R_a(\omega)\nu_2$
s	-	$R(\omega)\nu_0 + R(\tau)\nu_1 + R(c)\nu_2$
c	$R(\omega)\beta_2$	$R(s)\nu_2$
b	-	$R(\omega)\nu_2$

Table A.3: Result after vary (7.4.4) respect to each gauge field.

A.2 Maxwell Kinematics

A.2.1 Maxwell Extended Bargmann Algebra

Gauge field	α	μ
ω	$R(\omega)\alpha_0 + R(\tau)\alpha_1 + R(z)\alpha_2$	$R(s)\mu_0 + R(m)\mu_1 + R(t)\mu_2$
ω_a	-	$R_a(\omega)\mu_0 + R_a(e)\mu_1 + R_a(z)\mu_2$
τ	$R(\omega)\alpha_1$	$R(s)\mu_1 + R(m)\mu_2$
e_a	-	$R_a(\omega)\mu_1 + R_a(e)\mu_2$
z	$R(\omega)\alpha_2$	$R(s)\mu_2$
z_a	-	$R_a(\omega)\mu_2$
s	-	$R(\omega)\mu_0 + R(\tau)\mu_1 + R(z)\mu_2$
m	-	$R(\omega)\mu_1 + R(\tau)\mu_2$
t	-	$R(\omega)\mu_2$

Table A.4: Result after vary (9.1.4) respect to each gauge field.

A.2.2 Maxwell Extended Carroll Algebra

Gauge field	β
ω	$(R(\omega) + R(c))\beta_0 + R(\tau)\beta_1 + (R(m) + R(z))\beta_2$
ω_a	$R_a(\omega)\beta_0 + R_a(e)\beta_1 + R_a(z)\beta_2$
τ	$R(\omega)\beta_1 + R(\tau)\beta_2$
e_a	$(R_a(e) + R_a(t))\beta_2$
z	$(R(\omega) + R(c))\beta_2$
z_a	$R_a(\omega)\beta_2$
c	$R(\omega)\beta_0 + R(z)\beta_2$
t_a	$R_a(e)\beta_2$
l	$R(\omega)\beta_2$

Table A.5: Result after vary (9.2.4) respect to each gauge field.

A.2.3 Maxwell Extended Static Algebra

Gauge field	β	ν
ω	$R(\omega)\beta_0+R(\tau)\beta_1+R(z)\beta_2$	$R(s)\nu_0+R(m)\nu_1+\Big(R(l)+R(t)+R(y)+R(c)+R(b)\Big)\nu_2$
ω_a	-	$R_a(\omega)\nu_0+R_a(e)\nu_1+R_a(z)\nu_2$
τ	$R(\omega)\beta_1$	$R(s)\nu_1+\Big(R(\tau)+R(m)\Big)\nu_2$
e_a	-	$R_a(e)\nu_0+R_a(\omega)\nu_1+\Big(R_a(e)+R_a(t)\Big)\nu_2$
z	$R(\omega)\beta_2$	$\Big(R(c)+R(s)+R(b)\Big)\nu_2$
z_a	-	$R_a(\omega)\nu_2$
t_a	-	$R_a(e)\nu_2$
t	-	$\Big(R(\omega)+R(\tau)\Big)\nu_2$
s	-	$R(\tau)\nu_1+\Big(R(\omega)+R(c)+R(z)+R(l)\Big)\nu_2$
c	-	$\Big(R(\omega)+R(s)+R(t)+R(l)\Big)\nu_2$
l	-	$\Big(R(\omega)+R(s)\Big)\nu_2$
b	-	$R(z)\nu_2$
y	-	$R(\omega)\nu_2$
m	-	$R(\omega)\nu_1+R(\tau)\nu_2$

Table A.6: Result after vary (9.3.5) respect to each gauge field.

Appendix B

Explicit commutation relations of the $\mathfrak{B}_5$ extended kinematical algebras

This appendix contains the table with the complete list of commutators of the $\mathfrak{B}_5$ algebra and its non-Lorentzian counterpart.

Here, the generators have been identified in terms of the expanded generators listed in Table 10.2 as

$$\begin{aligned}
&J^{(0)} \equiv J\,, && J^{(1)} \equiv Z\,, && G_a^{(0)} \equiv G_a\,, && G_a^{(1)} \equiv Z_a\,,\\
&H^{(0)} \equiv H\,, && H^{(1)} \equiv N\,, && P_a^{(0)} \equiv P_a\,, && P_a^{(1)} \equiv N_a\,,\\
&S^{(0)} \equiv S\,, && S^{(1)} \equiv T\,, && T_a^{(0)} \equiv T_a\,, && T_a^{(1)} \equiv L_a\,,\\
&C^{(0)} \equiv C\,, && C^{(1)} \equiv L\,, && B^{(0)} \equiv B\,, && B^{(1)} \equiv Y\,,\\
&M^{(0)} \equiv M\,, && M^{(1)} \equiv V\,. && &&
\end{aligned} \tag{B.0.1}$$

Commutators	$\mathfrak{B}_5$	$\mathfrak{B}_5$eB	$\mathfrak{B}_5$eC	$\mathfrak{B}_5$eS
$[J,G_a]$	$\varepsilon_{ab}G_b$	$\varepsilon_{ab}G_b$	$\varepsilon_{ab}G_b$	$\varepsilon_{ab}G_b$
$[J,P_a]$	$\varepsilon_{ab}P_b$	$\varepsilon_{ab}P_b$	$\varepsilon_{ab}P_b$	$\varepsilon_{ab}P_b$
$[G_a,G_b]$	$-\varepsilon_{ab}J$	$-\varepsilon_{ab}S$	$-\varepsilon_{ab}C$	$-\varepsilon_{ab}B$
$[H,G_a]$	$\varepsilon_{ab}P_b$	$\varepsilon_{ab}P_b$	$\varepsilon_{ab}T_b$	$\varepsilon_{ab}T_b$
$[G_a,P_b]$	$-\varepsilon_{ab}H$	$-\varepsilon_{ab}M$	$-\varepsilon_{ab}H$	$-\varepsilon_{ab}M$
$[J,T_a]$	-	-	$\varepsilon_{ab}T_b$	$\varepsilon_{ab}T_b$
$[C,P_a]$	-	-	$\varepsilon_{ab}T_b$	$\varepsilon_{ab}T_b$
$[H,P_a]$	$\varepsilon_{ab}Z_b$	$\varepsilon_{ab}Z_b$	$\varepsilon_{ab}Z_b$	$\varepsilon_{ab}Z_b$
$[P_a,P_b]$	$-\varepsilon_{ab}Z$	$-\varepsilon_{ab}T$	$-\varepsilon_{ab}Z$	$-\varepsilon_{ab}T$
$[J,Z_a]$	$\varepsilon_{ab}Z_b$	$\varepsilon_{ab}Z_b$	$\varepsilon_{ab}Z_b$	$\varepsilon_{ab}Z_b$
$[Z,G_a]$	$\varepsilon_{ab}Z_b$	$\varepsilon_{ab}Z_b$	$\varepsilon_{ab}Z_b$	$\varepsilon_{ab}Z_b$
$[G_a,Z_b]$	$-\varepsilon_{ab}Z$	$-\varepsilon_{ab}T$	$-\varepsilon_{ab}L$	$-\varepsilon_{ab}Y$
$[P_a,T_b]$	-	-	$-\varepsilon_{ab}L$	$-\varepsilon_{ab}Y$

Table B.1: Commutation relations of the $\mathfrak{B}_5$ extended kinematical algebras.

$[J,N_a]$	$\varepsilon_{ab}N_b$	$\varepsilon_{ab}N_b$	$\varepsilon_{ab}N_b$	$\varepsilon_{ab}N_b$
$[Z,P_a]$	$\varepsilon_{ab}N_b$	$\varepsilon_{ab}N_b$	$\varepsilon_{ab}N_b$	$\varepsilon_{ab}N_b$
$[H,Z_a]$	$\varepsilon_{ab}N_b$	$\varepsilon_{ab}N_b$	$\varepsilon_{ab}L_b$	$\varepsilon_{ab}L_b$
$[Z_a,P_b]$	$-\varepsilon_{ab}N$	$-\varepsilon_{ab}V$	$-\varepsilon_{ab}N$	$-\varepsilon_{ab}V$
$[G_a,N_b]$	$-\varepsilon_{ab}N$	$-\varepsilon_{ab}V$	$-\varepsilon_{ab}N$	$-\varepsilon_{ab}V$
$[J,L_a]$	-	-	$\varepsilon_{ab}L_b$	$\varepsilon_{ab}L_b$
$[Z,T_a]$	-	-	$\varepsilon_{ab}L_b$	$\varepsilon_{ab}L_b$
$[C,N_a]$	-	-	$\varepsilon_{ab}L_b$	$\varepsilon_{ab}L_b$
$[L,P_a]$	-	-	$\varepsilon_{ab}L_b$	$\varepsilon_{ab}L_b$

Table B.2: Commutation relations of the $\mathfrak{B}_5$ extended kinematical algebras.

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