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# ASYMPTOTIC STRUCTURE OF MAXWELL CHERN-SIMONS GRAVITY THEORY COUPLED WITH SPIN-3 FIELDS

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# Resumen

En esta tesis analizamos las simetrías asintóticas de la teoría de gravedad Chern-Simons (CS) en tres dimensiones para una extensión de espín más alto de la llamada álgebra de Maxwell. Proponemos un conjunto de condiciones de borde para la teoría de gravedad antes mencionada y mostramos que la correspondiente álgebra de carga define una extensión de espín más alto del álgebra $\mathfrak{max}\text{-}\mathfrak{bms}_3$, la cual, a su vez, corresponde a la simetría asintótica de la gravedad CS de Maxwell. También mostramos que el álgebra $\mathfrak{hs}_3\mathfrak{max}\text{-}\mathfrak{bms}_3$ se puede obtener alternativamente como un límite plano desde tres copias del álgebra $\mathcal{W}_3$, con tres cargas centrales independientes.

***Keywords*** – Teorías clásicas de gravedad, simetrías espacio-tiempo, teorías Chern-Simons, simetría de gauge

# Abstract

In this thesis we analyze the asymptotic symmetries of the three-dimensional Chern-Simons (CS) gravity theory for a higher spin extension of the so-called Maxwell algebra. We propose a generalized set of asymptotic boundary conditions for the aforementioned flat gravity theory and we show that the corresponding charge algebra defines a higher-spin extension of the $\mathfrak{max}\text{-}\mathfrak{bms}_3$ algebra, which in turn corresponds the asymptotic symmetries of the Maxwell CS gravity. We also show that the $\mathfrak{hs}_3\mathfrak{max}\text{-}\mathfrak{bms}_3$ algebra can alternatively be obtained as a vanishing cosmological constant limit of three copies of the $\mathcal{W}_3$ algebra, with three independent central charges.

***Keywords*** – Classical theories of gravity, space-time symmetries, Chern-Simons theories, gauge symmetry

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# Capítulo 1

# Introduction

There is now widespread agreement that unifying the four fundamental forces of nature requires going beyond Einstein's General Relativity (GR). While GR has successfully explained all known gravitational phenomena and has profound astrophysical implications, it ultimately proves incompatible with the quantum mechanical description of the microscopic world. In three-dimensional settings, Chern-Simons (CS) gravity provides a distinctive approach, presenting gravity as a gauge theory independent of a metric [1–3], which is particularly useful when investigating quantum gravity principles. This framework not only surpasses traditional gauge theories but also offers a fertile ground for testing ideas like the AdS/CFT correspondence [4], uncovering deep links between gravitational theories and their boundary representations.

Three-dimensional gravity, with or without a cosmological constant, can be formulated through a Chern-Simons action based on the AdS algebra and the Poincaré algebra, respectively . At present, there is growing interest in developing CS (super)gravity models based on extensions and deformations of the Poincaré algebra, suggesting intriguing generalizations of GR. However, many aspects, such as their solutions, thermodynamics, asymptotic symmetries, and higher-spin extensions, are still not fully understood. Similarly, progress has been made in the study of the Maxwell algebra in three dimensions.

The Maxwell symmetry, as well as its extensions with a non-vanishing cosmological constant, have received a growing interest in the context of (super)gravity during the last years [5–13]. The Maxwell algebra can be defined in any spacetime dimension and can be obtained by considering an extension and deformation of the Poincaré symmetry. It was first introduced in the literature to describe the presence of a constant classical electromagnetic background in Minkowski spacetime [14–16]. In three spacetime dimensions, a gravity theory invariant under this symmetry can be constructed using the CS formulation of gravity [17, 18]. The CS action turns out to be given by three independent sectors, one of them is given by the usual Einstein-Hilbert (EH) term without a cosmological constant, while the other sectors are given by the exotic Lagrangian [2] and a term which involves the gravitational Maxwell field. The corresponding field equations are those of Poincaré gravity, describing a torsionless and flat spacetime, plus a third one involving the gravitational Maxwell gauge field. This theory has been deeply studied in different contexts such as (super)gravity theories [5, 6, 12, 19–31], higher-spin extensions [32–36], non- and ultra-relativistic gravity theories [37–42] and asymptotic symmetries [43–45].

Recently, significant attention has been given to the infinite-dimensional symmetries of asymptotically flat spacetimes at null infinity, which were initially proposed to be governed by the $\mathfrak{bms}$ algebra, discovered over half a century ago [46, 47]. In three dimensions, it has been demonstrated in [48, 49] that these asymptotic symmetries are described by the $\mathfrak{bms}_3$ algebra. One of the recent results showed that the gravitational Maxwell field not only modifies the vacuum of the CS theory but also its asymptotic structure [43]. The asymptotic symmetry algebra in this case the asymptotic symmetry algebra is described by a deformed $\mathfrak{bms}_3$ algebra, denoted in this work as $\mathfrak{max}$-$\mathfrak{bms}_3$, and which was first introduced in [50] as an $S$-expansion of the Virasoro algebra. Asymptotic symmetries are key physical symmetries in the theory, significantly influencing the system's state. Its relevance becomes even more important in topological theories like three-dimensional gravity, where the dynamic is entirely captured by boundary degrees of freedom and holonomies. Therefore, grasping the asymptotic dynamics of extended (super)gravity theories is crucial, for instance, for developing dual theories for three-dimensional extended supergravities. This topic is especially compelling, as it could provide valuable insights into holography in non-AdS or Poincaré contexts.

Within the context of higher-spin (HS) gravity in three-dimensional spacetime, it was found in [33] the extension with spin-3 gauge fields of the Maxwell CS gravity. The underlying symmetry corresponds to a spin-3 extension of the Maxwell algebra, denoted here as $\mathfrak{hs}_3\mathfrak{max}$, allowing the inclusion of a new gauge field being the spin-3 analogue of the gravitational Maxwell field. Interestingly, the $\mathfrak{hs}_3\mathfrak{max}$ algebra can also be obtained as the vanishing cosmological constant limit of three copies of the $\mathfrak{sl}(3,\mathbb{R})$ algebra. Higher-spin fields have been of great interest due to their appearance in the spectrum of string theory and simplified models of the AdS/CFT conjecture [51–60]. In particular, in three-dimensions the coupling of massless HS fields to anti-de-Sitter (AdS) gravity is consistently described by a CS action whose gauge group is given, in the simplest case, by two copies of $SL(3,\mathbb{R})$ [61–63], which describes non-propagating spin-3 fields coupled to AdS gravity. Despite the lack of local degrees of freedom, CS theories exhibit a rich structure that merits further investigation. In fact, akin to the situation in pure gravity, the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ CS theory features interesting solutions, including HS black holes [64–71] and conical singularities [72, 73]. As explained in [74], it is possible to perform the vanishing cosmological constant limit in a straightforward way when an appropriate gauge is chosen, so that the HS black hole solution reduces to a HS extension of locally flat cosmological spacetimes [75, 76]. Furthermore, the asymptotic symmetry of the HS AdS theory realizes two copies of the centrally-extended $\mathcal{W}_3$ algebra [63, 77], whose flat limit was studied in [74, 78–80].

It is the purpose of this thesis to extend the asymptotic conditions considered for the Maxwell CS gravity in [43], to include spin-3 gauge fields non minimally coupled to the theory. Thus, our results can be seen as a novel set of asymptotic boundary conditions for higher spin gravity without cosmological constant in three dimensions.

## 1.1. Outline

This thesis is organized as follows:

- In Chapter 2 we review the theory of GR, the Einstein-Hilbert action and introduce Cartan's formalism for gravity.
- In Chapter 3 we introduce the Chern-Simons form and the Chern-Simons action. In particular, we briefly review the construction of the AdS and

Maxwell CS gravity theories.

- In Chapter 4 we review the CS higher-spin gravity theory in three dimensions. Specifically, the construction of the three-dimensional Maxwell and AdS-Lorentz gravities coupled to spin-3 fields is presented.
- In Chapter 5 we review the asymptotic structure of three-dimensional gravity and we show the main results of the previous work [43].
- In Chapter 6 we provide the asymptotic symmetry algebra for the minimal HS Maxwell CS gravity, which turns out to be given by a HS extension of the deformed $\mathfrak{bms}_3$ algebra, denoted as $\mathfrak{hs}_3\mathfrak{maxbms}_3$, with three central charges. We propose suitable fall-off conditions for the gauge fields at infinity and the gauge transformations preserving the boundary conditions. We explicitly show that the $\mathfrak{hs}_3\mathfrak{maxbms}_3$ algebra can be found as the vanishing cosmological constant limit ($\ell \to \infty$) of three copies of the $W_3$ algebra. The results presented in this chapter can be found in the published article [81]. Finally, in the last chapter we provide some discussion and possible future developments.

# Capítulo 2

# First order formulation for gravity

## 2.1. General Relativity in the Einstein Formalism

Geometry can be understood as the collection of statements that describe the relationships among points, lines, and higher-dimensional submanifolds embedded within a given manifold [3]. This general concept is often encapsulated in the metric tensor $g_{\mu\nu}(x)$, which defines the notion of distance between two infinitesimally close points with coordinates $x^\mu$ and $x^\mu + dx^\mu$,

$$ds^2 = g_{\mu\nu} dx^{\mu\nu} \,. \tag{2.1.1}$$

This is the case in Riemannian geometry, where all relevant objects defined on the manifold (distance, area, angles, parallel transport operations, curvature, etc.) can be constructed from the metric. The coordinate components of this tensor correspond to the dot product between the $\partial_\mu$ vectors of the coordinate basis

$$g_{\mu\nu} \equiv \partial_\mu \cdot \partial_\nu \,. \tag{2.1.2}$$

The equation (2.1.2) introduce the dot product between arbitrary vectors $A$ and $B$, through

$$\begin{aligned} A \cdot B &= (A^\mu \partial_\mu) \cdot (B^\nu \partial_\nu) \,, \\ &= A^\mu B^\nu (\partial_\mu \cdot \partial_\nu) \,, \\ &= g_{\mu\nu} A^\mu B^\nu \,. \end{aligned} \tag{2.1.3}$$

However, a distinction should be made between metric and affine features of spacetime. Metricity refers to measurements of lengths, angles, volumes of objects which are locally defined in spacetime. Affinity refers to properties which remain invariant under translations, such as parallelism.

In differential geometry, parallelism is encoded in the affine connection $\Gamma^{\alpha}_{\beta\gamma}(x)$: a vector $u_{\parallel}$ at the point of coordinates $x$ is said to be parallel to the vector $u$ at a point with coordinates $x + dx$, if their components are related by parallel transport

$$\begin{aligned} u^{\alpha}_{\parallel}(x) &= u^{\alpha}(x+dx) + dx^{\mu}\Gamma^{\alpha}{}_{\mu\beta}u^{\beta}(x)\,, \\ &= u^{\alpha}(x) + dx^{\mu}\left[\partial_{\mu}u^{\alpha} + \Gamma^{\alpha}{}_{\mu\beta}u^{\beta}(x)\right]. \end{aligned} \tag{2.1.4}$$

The expression between parentheses corresponds to the *covariant derivative* of $u^{\alpha}$ with respect to the connection $\Gamma^{\alpha}{}_{\mu\beta}$, which we will denote as

$$D_{\mu} \equiv \partial_{\mu}u^{\alpha} + \Gamma^{\alpha}_{\mu\beta}u^{\beta}\,. \tag{2.1.5}$$

As we have mentioned, the affine connection $\Gamma^{\alpha}_{\mu\beta}(x)$ needs to be functionally related to the metric tensor $g_{\mu\nu}(x)$. However, Einstein formulated GR adopting the point of view that the spacetime metric should be the only dynamically independent field, while the affine connection should be a function of the metric given by the Christoffel symbol,

$$\Gamma^{\alpha}{}_{\mu\beta} = \frac{1}{2}g^{\alpha\lambda}(\partial_{\mu}g_{\lambda\beta} + \partial_{\beta}g_{\lambda\mu} - \partial_{\lambda}g_{\mu\beta})\,. \tag{2.1.6}$$

Thus, in the original formulation of GR, Einstein considered that the spacetime metric should be the only dynamically independent field, while the affine connection should be a function of the metric, as shown by (2.1.6). However, it is important to note that if we consider that these properties are not independent, it is necessary to introduce a constraint: the torsion tensor is assumed to be zero on the entire variety.

Using the definition (2.1.5) it is possible to calculate how the commutator of two covariant derivatives acts on a vector $u^{\alpha}$

$$[D_{\mu}, D_{\nu}]u^{\alpha} = R^{\alpha}{}_{\beta\mu\nu}u^{\beta} - T^{\lambda}{}_{\mu\nu}D_{\lambda}u^{\alpha}\,, \tag{2.1.7}$$

where $T^{\lambda}_{\mu\nu}$ corresponds to the torsion tensor and $R^{\alpha}_{\beta\mu\nu}$ is known as the *Riemann curvature tensor*, which is given by

$$R^{\alpha}{}_{\beta\mu\nu} \equiv \partial_\mu \Gamma^{\alpha}{}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\beta} \,. \tag{2.1.8}$$

We also define

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \,, \tag{2.1.9}$$

$$R = g^{\mu\nu} R_{\mu\nu} \,, \tag{2.1.10}$$

which are known as the *Ricci tensor* and the *Ricci scalar curvature*, respectively.

## 2.2. Einstein-Hilbert Action

It is a well-known fact that Einstein's field equations (in vacuum)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \,, \tag{2.2.1}$$

can be obtained from a variational principle

$$I_g = \int d^4x \mathcal{L}_g = \int d^4x \sqrt{-g} L_g \,, \tag{2.2.2}$$

where $g = \det(g_{\mu\nu}) < 0$ is the determinant of the metric tensor. Since the field equations contain derivatives of the metric up to second order, the scalar $L_g$ must contain only the components of the metric tensor $g_{\mu\nu}$ and its first derivatives through the affine connection $\Gamma^{\alpha}_{\mu\nu}$. However, it is not possible to construct an invariant scalar from only $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$. In 1915, this problem was solved by the mathematician David Hilbert. Suppose that $L_g$ is an invariant scalar that, in addition to containing $g_{\mu\nu}$ and its first derivatives, also contains second derivatives. Of all the curvature scalars that can be formed in four dimensions, Hilbert chose the Ricci scalar curvature $R$, since it is uniquely linear in the second derivative of $g_{\mu\nu}$ and provides us with second-order equations for the metric. Thus we have that

$$I^{(4)}_{EH} = \int d^4x \sqrt{-g} R \,. \tag{2.2.3}$$

The variation of the action leads us to the Einstein field equations (2.2.1).

So far, we have reviewed the formulation of GR considering that the metric and affine properties are not independent. For this, it was necessary to introduce a constraint: the torsion tensor was assumed to be zero on the entire manifold. However, these properties can be considered as independent notions.

In the following sections, the formulation of the theory of GR will be briefly reviewed, considering the independence of metricity and parallelism. This formalism is known as Cartan gravity (when working with differential forms) or Palatini formalism (in the tensor formulation).

## 2.3. Vielbein, Lorentz connection, Curvature, and Torsion

### 2.3.1. Vielbein

As is shown in [82] the equivalence principle allows us to choose a coordinate system in which the spacetime looks locally as Minkowski. This choice can be implemented through the vielbein $e^a{}_\mu(x)$ as

$$g_{\mu\nu}(x) = e^a{}_\mu(x)\eta_{ab}e^b{}_\nu(x)\,, \tag{2.3.1}$$

where $\eta_{ab} = \mathrm{diag}(-,+,\ldots,+)$ is the Minkowski metric. From hereon, latin and greek characters denote Lorentz and spacetime indices, respectively.

Looking at the left-hand side of Eq. (2.3.1), one observes that there are $D(D+1)/2$ independent components of the metric, since it represents a symmetric rank-2 tensor in $D-$ dimension. The vielbein, on the other hand, has no symmetries on their indices whatsoever, and it has therefore $D^2$ independent component. This means that, given a metric, the choice of the vielbein is not unique. Nevertheless, the difference in their independent components exactly $D^2 - D(D+1)/2 = D(D-1)/2$, which matches precisely the number of generators of the local Lorentz group in $D-$dimensions. The vielbein, on the other hand, has no symmetries on their indices whatsoever and it has therefore $D^2$ independent components. this means that, given a metric, the choice of the vielbein is unique. Nevertheless, the difference in their independent components is exactly $D^2 - D(D+1)/2 = D(D-1)/2$, which matches precisely the number of generators of the local Lorentz group in $D-$dimensions. Therefore, all the equivalence choices of the vielbein are related by $\Lambda^a{}_b(x) \in SO(1, D-1)$ as

$$e'^a{}_\mu(x) = \Lambda^a{}_b e^b{}_\mu(x)\,. \tag{2.3.2}$$

Additionally, Eq. (2.3.1) requires that the vielbein field transform as a 1-form under general coordinate transformations, that is

$$e'^a{}_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} e^a{}_\nu(x)\,. \tag{2.3.3}$$

If the vielbein is a non-singular matrix, i.e, det $e^a{}_\mu \neq 0$, then there exist and inverse vielbein field $E^\mu{}_a$ such that $E^\mu{}_a e^b{}_\mu = \delta^a_b$ and $E^\mu{}_a e^a{}_\nu = \delta^\mu_\nu$. The inverse vielbein transforms according to

$$E'^\mu_a(x) = \Lambda^b{}_a E^\mu_b \quad \text{and} \quad E'^\mu_a(x') = \frac{\partial x'^\mu}{\partial x^\nu} E^\nu{}_a(x)\,, \tag{2.3.4}$$

under local Lorentz transformations and general coordinate transformations, respectively. Using the inverse vielbein, it is possible to rewrite the Eq. (2.3.1) in terms of

$$E^\mu{}_a(x) g_{\mu\nu}(x) E^\nu{}_b(x) = \eta_{ab}\,. \tag{2.3.5}$$

In fact, it is possible to change the coordinate basis of a type-$(p, q)$ spacetime tensor to a Lorentz one, and vice versa, via

$$T^{a_1 \dots a_p}{}_{b_1 \dots b_q} = e^{a_1}{}_{\mu_1} \dots e^{a_p}{}_{\mu_p} \dots E^{\nu_1}{}_{b_q} T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_q}\,, \tag{2.3.6}$$

$$T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} = E^{\mu_1}_{a_1} \dots E^{\mu_p}_{a_p} \dots e^{b_1}{}_{\mu_1} T^{a_1 \dots a_p}{}_{b_1 \dots b_q}\,. \tag{2.3.7}$$

The components of the inverse vielbein field at some point $p \in \mathcal{M}$ with local coordinates $x^\mu$, can be used to construct the vector basis on the tangent space as $E_a = E^\mu{}_a \partial_\mu$. Analogously, the vielbein 1-form defined as

$$e^a = e^a{}_\mu dx^\mu\,, \tag{2.3.8}$$

can be used to construct a new basis for the vector space $\Omega^p(\mathcal{M})$ of differential forms. Thus, the inverse vielbein spans an orthonormal basis for vector fields, in the same way as the vielbein 1-form spans an orthonormal basis for differential forms. For instance, it is possible to define a differential $p-$form using

$$\alpha = \frac{1}{p!}\alpha_{\mu_1\dots\mu_p} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} = \frac{1}{p!}\alpha_{a_1\dots a_p} e^{a_1}\wedge\cdots\wedge e^{a_p}\,. \tag{2.3.9}$$

### 2.3.2. The Lorentz connection

In order to define covariant derivatives under local Lorentz transformations, we need to introduce a gauge connection for such a group. This is called the Lorentz connection 1-form denoted by $\omega^a{}_b(x) = \omega^a{}_{b\mu}(x)dx^\mu$, whose transformation law under $SO(1,D-1)$ is given by

$$\omega^a{}_b \to \omega'^a{}_b = \Lambda^a{}_c\omega^c{}_d\Lambda^d{}_b + \Lambda^a{}_b d\Lambda^c{}_d\,. \tag{2.3.10}$$

In this way, we can define the exterior covariant derivative with respect to $\omega^a{}_b$ denoted by $D$, as a map $D:\Omega^p(\mathcal{M})\to\Omega^{p+1}(\mathcal{M})$ acting on Lorentz tensor $T^{a_1\dots a_p}{}_{b_1\dots b_q}\in\Omega^p(\mathcal{M})$ as

$$\begin{aligned} DT^{a_1\dots a_p}{}_{b_1\dots b_q} \;=\;& dT^{a_1\dots a_p}{}_{b_1\dots b_q} + \omega^{a_1}_{\ c_1}\wedge T^{c_1\dots a_p}{}_{b_1\dots b_q} + \cdots + \omega^{a_p}_{\ c_p}\wedge T^{a_1\dots c_p}{}_{b_1\dots b_q} \\ & -\omega^{c_1}_{\ b_1}\wedge T^{a_1\dots a_p}{}_{c_1\dots b_q} - \cdots - \omega^{c_q}_{\ b_q}\wedge T^{a_1\dots a_p}{}_{b_1\dots c_q}\,. \end{aligned} \tag{2.3.11}$$

This object transforms covariantly under local Lorentz transformations, provided that the Lorentz connection transform as (2.3.10).

### 2.3.3. Volume element

The Levi-Civita symbol $\epsilon_{a_1\dots a_D}$ is an invariant tensor under $SO(1,D-1)$, where

$$\epsilon_{a_1\dots a_D} = \begin{cases} +1, & \text{for even permutation of } a_1\dots a_D\,, \\ -1, & \text{for odd permutation of } a_d\dots a_D\,, \\ \ 0, & \text{if any of the indices appears repeated at least once.} \end{cases} \tag{2.3.12}$$

The volume element $D-$form, denoted by $\epsilon$, can be defined by means of the Levi-Civita symbol according to

$$\begin{aligned}\epsilon &= \frac{1}{D!}\epsilon_{a_1\dots a_D}e^{a_1}\wedge\dots e^{a_D}\,,\\ &= \frac{1}{D!}\epsilon_{a_1\dots a_D}e^{a_1}e^{a_1}_{\ \ \mu_1}\dots e^{a_D}_{\ \ \mu_D}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_D} = ed^Dx\,,\end{aligned} \tag{2.3.13}$$

where $e = \det e^a_{\ \mu}$, $d^Dx = \frac{1}{D!}\epsilon_{\mu_1\dots\mu_D}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_D}$, and we have used the formula of the determinant of a $D\times D$ matrix $M^a_{\ b}$

$$\det\, M_{\epsilon_{a_1}\dots a_D} = \epsilon_{m_1\dots m_D}M^{m_1}_{\ \ a_1}\dots M^{m_D}_{\ \ a_D}\,. \tag{2.3.14}$$

Moreover, the volume element satisfies $i_{a_D\dots a_pc_{p+1}\dots c_D}\epsilon_{b_1\dots b_pc_{p+1}\dots c_D}$. Using the identity

$$\epsilon^{a_1\dots a_pc_{p+1}\dots c_D}\epsilon_{b_1\dots b_pc_{p+1}\dots c_D} = -p!(D-p)!\delta^{a_1}_{[b_1}\dots\delta^{a_p}_{b_p]}\,, \tag{2.3.15}$$

for Lorentzian manifolds, we arrive to

$$e^{a_1}\wedge\dots\wedge e^{a_D} = -\epsilon^{a_1\dots a_D}ed^Dx\,, \tag{2.3.16}$$

which can be regarded as the covariant volume element upon the identification $\sqrt{-g} = e$ from Eq. (2.3.1)

### 2.3.4. Cartan's structure equations

The curvature and torsion 2-forms are defined through the Cartan's structure equations, which are given by

$$R^a_{\ b} = d\omega^a_{\ b} + \omega^a_{\ c}\wedge\omega^c_{\ b} = \frac{1}{2}\mathcal{R}^{ab}_{\ \ cd}e^c\wedge e^d\,, \tag{2.3.17}$$

$$T^a = de^a + \omega^a_{\ b}\wedge e^b = \frac{1}{2}\mathcal{T}^a_{\ bc}e^b\wedge e^c\,. \tag{2.3.18}$$

Moreover, they satisfy the Bianchi identities

$$DR^a_{\ b} = 0 \qquad \text{and} \qquad DT^a = R^a_{\ b}\wedge e^b. \tag{2.3.19}$$

The curvature and torsion $2-$forms will play an essential role in gravitational theories written in Cartan's language. This stems from the fact these objects transform as tensors under $SO(1, D-1)$ and they can be used to construct invariant action principles. Even more, they encode the dynamics of the fundamental fields $e^a$ and $\omega^a{}_b$, since they are constructed out of their derivatives.

## 2.4. Einstein-Hilbert action in differential forms

In the context of the Cartan formalism, the Einstein-Hilbert action in 4 dimensions is given by

$$I_{EH}^{(4)} = \int \varepsilon_{abcd} R^{ab} e^c e^d \,, \tag{2.4.1}$$

where $R^{ab} = d\omega^{ab} + \omega^a{}_c \omega^{cb}$ is the 2-form curvature and $e^a$ is the vielbein.

To check that this action corresponds effectively to the Einstein-Hilbert action written in tensor language, it is necessary to explicitly write the basis of differential forms in the vierbein. Thus, first, expanding the 2-form curvature $R^{ab}$ in the basis of 2-forms $\{e^i e^j\}$, we obtain

$$\varepsilon_{abcd} R^{ab} e^c e^d = \varepsilon_{abcd} R^{ab}{}_{ij} e^i e^j e^c e^d \,. \tag{2.4.2}$$

Now we expand the equation in the basis $\{dx^\mu\}$

$$\varepsilon_{abcd} R^{ab} e^c e^d = \varepsilon_{abcd} R^{ab}{}_{ij} e^i{}_\mu e^j{}_\nu e^c{}_\rho e^d{}_\sigma dx^\mu dx^\nu dx^\rho dx^\sigma \,, \tag{2.4.3}$$

$$= \varepsilon_{abcd} R^{ab}{}_{ij} e^i{}_\mu e^j{}_\nu e^c{}_\rho e^d{}_\sigma \varepsilon^{\mu\nu\rho\sigma} d^4x \,, \tag{2.4.4}$$

where we used the fact that

$$dx^\mu dx^\nu dx^\rho dx^\sigma = \varepsilon^{\mu\nu\rho\sigma} dx^0 dx^1 dx^2 dx^3 = \varepsilon^{\mu\nu\rho\sigma} d^4x \,. \tag{2.4.5}$$

Using the well-known result

$$\varepsilon^{i_1 \ldots i_n} = e^{i_1}{}_{\mu_1} \ldots e^{\mu_1 \ldots \mu_n}{}_{\mu_n} (\det e)^{-1} \,, \tag{2.4.6}$$

we can write

$$e^i{}_\mu e^j{}_\nu e^c{}_\rho e^d{}_\sigma \varepsilon^{\mu\nu\rho\sigma} = \varepsilon^{ijcd} (\det e) \,. \tag{2.4.7}$$

Therefore, making use of the identities

$$\delta = \frac{(n-s)!}{(n-r)!}\delta_{h_1 \ldots h_s}{}^{j_1 \ldots j_s}\,, \tag{2.4.8}$$

$$= r! B^{j_1 \ldots j_r}\,. \tag{2.4.9}$$

We have to

$$\varepsilon_{abcd} R^{ab} e^c e^d = \varepsilon_{abcd} R^{ab}{}_{ij} \varepsilon^{ijcd} (\det e) d^4x\,, \tag{2.4.10}$$

$$= \delta_{abcd}{}^{ijcd} R^{ab}{}_{ij} (\det e) d^4x\,, \tag{2.4.11}$$

$$= 2\delta_{ab}{}^{ij} R^{ab}{}_{ij} (\det e) d^4x\,, \tag{2.4.12}$$

$$= 4R^{ij}_{ij} (\det e) d^4x\,. \tag{2.4.13}$$

Finally, since $R^{ij}_{ij} = R$ and that $\det e = \sqrt{-g}$,

$$\varepsilon_{abcd} R^{ab} e^c e^d = 4\sqrt{-g} R d^4x\,. \tag{2.4.14}$$

Thus,

$$\int \varepsilon_{abcd} R^{ab} e^c e^d = 4 \int \sqrt{-g} R d^4x\,. \tag{2.4.15}$$

This shows us the equivalence between the Einstein-Hilbert action written in differential forms and that written in tensor language.

## 2.5. Equations of motion in Cartan formalism

To obtain the equations of motion, we must perform the variation of the action assuming that $\delta e^a$ and $\delta\omega^{ab}$ are infinitesimal variations, so we have that

$$\delta I = \int (\delta R^{ab} e^c e^d + R^{ab} \delta e^c e^d + R^{ab} e^c \delta e^d)\,, \tag{2.5.1}$$

$$= \int \varepsilon_{abcd} (\delta R^{ab} e^c e^d + 2R^{ab} e^c \delta e^d)\,. \tag{2.5.2}$$

where we have made use of the antisymmetric property of the levi-civita. From the definition of the 2-form curvature we can write

$$\begin{aligned} \delta R^{ab} &= \delta\omega^{ab} + \delta\omega^{a}{}_{c}\omega^{cd} + \omega^{a}{}_{c}\delta\omega^{cb} \,, && (2.5.3)\\ &= d\delta\omega^{ab} + \omega^{b}{}_{c}\delta\omega^{ac} + \omega^{a}{}_{c}\delta\omega^{cb} \,, && (2.5.4)\\ &= D(\delta\omega^{ab}) \,. && (2.5.5) \end{aligned}$$

Then replacing above we found,

$$\begin{aligned} \delta I &= \int \varepsilon_{abcd}(D(\delta\omega^{ab})e^{c}e^{d} + 2R^{ab}e^{c}\delta e^{d}) \,, && (2.5.6)\\ &= \int \varepsilon_{abcd}(D(\delta\omega^{ab}e^{c}e^{d}) + \delta\omega^{ab}De^{c}e^{d} - \delta\omega^{ab}e^{c}De^{d} + 2R^{ab}e^{c}\delta e^{d}) \,, && (2.5.7)\\ &= \int d(\varepsilon_{abcd}\delta\omega^{ab}e^{c}e^{d}) + 2\int \varepsilon_{abcd}\delta\omega^{ab}T^{c}e^{d} + 2\int \varepsilon_{abcd}R^{ab}e^{c}\delta e^{d} \,. && (2.5.8) \end{aligned}$$

The first term corresponds to a boundary term and can be depreciated by requiring that the variation of the spin connection $\omega^{ab}$ vanishes at the boundary of spacetime. The other two terms are independent and give the necessary conditions for the vanishing of $\delta S$. That is, $\delta S = 0$ requires that the following equations of motion be satisfied in vacuum:

$$\begin{aligned} \varepsilon_{abcd}R^{ab}e^{c} &= 0 \,, && (2.5.9)\\ \varepsilon_{abcd}T^{c}e^{d} &= 0 \,. && (2.5.10) \end{aligned}$$

The first equation is actually equivalent to Einstein's field equations, while the second expresses the cancellation of torsion.

## 2.6. Invariance of Einstein-Hilbert action

### 2.6.1. Poincaré group

One of the simplest examples of a gauge theory for gravity is obtained considering the Poincaré group. The generators of this group are given by

$$T_A = (P_a, J_{ab}) \,, \qquad (2.6.1)$$

where $P_a$ corresponds to the translations generators and $J_{ab} = -J_{ba}$ are the Lorentz rotations. The generators of the Poincaré group satisfy the Lie algebra

$$[J_{ab}, J_{cd}] = \eta_{cb}J_{ad} - \eta_{ca}J_{bd} + \eta_{db}J_{ca} - \eta_{da}J_{cb}\,, \tag{2.6.2}$$

$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b\,, \tag{2.6.3}$$

$$[P_a, P_b] = 0\,. \tag{2.6.4}$$

In this case, the theory has two gauge fields, the spin connection $\omega^{ab}$ and the vielbein $e^a$. The fundamental observation is that $\left\{e^a, \omega^{ab}\right\}$, consider as, a multiplet in the adjunt representation of the Poincaré group. This observation is the key, since it allow us to write the 1-form connection as

$$A = A^A T_A = \frac{1}{\ell}e^a P_a + \frac{1}{2}\omega^{ab}J_{ab}\,. \tag{2.6.5}$$

The introduction of the length parameter $\ell$ is necessary in order to interpret the vielbein as the gauge field associated to the translation generator $P_a$. Indeed, one can always choose the generators of a dimensionless Lie algebra $T_A$ so that the 1-form connection A must also be dimensionless. However, the vielbein $e^a = e^a{}_i dx^i$ must have dimension of length by being related to the spacetime metric $g_{ik} = e^a{}_i e^b{}_k \eta_{ab}$ through the equation ab. So the gauge field must be of the form $e^a/\ell$, where $\ell$ is a length parameter.

Similarly it is possible to write the 2-form field intensity associated with the 1-form connection $A$ as

$$F \equiv dA + A^2\,, \tag{2.6.6}$$

$$F \equiv F^A T_A = \frac{1}{\ell}T^a P_a + \frac{1}{2}R^{ab}J_{ab}\,. \tag{2.6.7}$$

It is important to note that in this context, torsion is interpreted as the field strength related to translations and curvature is related to the field strength of Lorentz rotations. Note further that the explicit expressions for torsion and curvature given in terms of gauge potentials are obtained as a direct consequence of the commutation relations of Poincaré algebra. This formalism explicitly shows the close relationship between the geometrical structure of the manifold and the algebraic structure of the fundamental symmetry group.

Next, if we want to know how the 1-form connection A transforms under the Poincaré group, it is necessary to remember that the transformation law depends on how the group is exponentiated. Let the exponentiation

$$U = e^{-\lambda} = e^{-\lambda^A T_A} \,. \tag{2.6.8}$$

We know that the invariance of the theory must be define through a covariant derivative

$$D = d + A \,, \tag{2.6.9}$$

where $A$ transform under the group as

$$A \to A' = UAU^{-a} + UdU^{-1} \,. \tag{2.6.10}$$

Then making use of the equation (2.6.8) it is possibly to show that

$$A' = A + d\lambda + [A, \lambda] \,, \tag{2.6.11}$$

transforms as

$$\delta A = D\lambda \,.$$

Let us consider now the parameter of the transformation, which can be built as

$$\lambda = \lambda^A T_A = \frac{1}{l}\lambda^a P_a + \frac{1}{2}\lambda^{ab} J_{ab} \,, \tag{2.6.12}$$

$$\equiv \frac{1}{l}\rho^a P_a + \frac{1}{2}\kappa^{ab} J_{ab} \,. \tag{2.6.13}$$

Then, if we introduce (2.6.13) in (2.6.1) we found

$$\delta A = \frac{1}{l}(d\rho^a + \omega^a{}_b \rho^b + e_c \kappa^{ca}) P_a + \frac{1}{2}(d\kappa^{ab} + \omega^{ac}\kappa_c{}^b + \omega^{bc}\kappa^a{}_c) J_{ab} \,, \tag{2.6.14}$$

$$\equiv \frac{1}{l}(D\rho^a + e_c \kappa^{ca}) P_a + \frac{1}{2} D\kappa^{ab} J_{ab} \,. \tag{2.6.15}$$

since

$$\delta A = \frac{1}{l}\delta e^a P_a + \frac{1}{2}\delta\omega^{ab} J_{ab} \,, \tag{2.6.16}$$

we have that the components $e^a$ y $\omega^{ab}$ of the connection have the following transformation law

$$\delta e^a = D\rho^a + e_c \kappa^{ca} \,, \tag{2.6.17}$$

$$\delta\omega^{ab} = d\kappa^{ab} + \omega^{ac}\kappa_c{}^b + \omega^{bc}\kappa^a{}_c \,, \tag{2.6.18}$$

so that under local Poincare translations we have that

$$\delta e^a = D\rho^a \,, \tag{2.6.19}$$

$$\delta\omega^{ab} = 0 \,, \tag{2.6.20}$$

and under Lorentz transformations

$$\delta e^a = e^c \kappa_c{}^a \,, \tag{2.6.21}$$

$$\delta\omega^{ab} = D\kappa^{ab} \,. \tag{2.6.22}$$

The next step is to analyze the invariance of the Einstein-Hilbert action under the transformation laws derived for the Poincaré group. Establishing the invariance of a gravitational action under a given symmetry group is essential for formulating gravity as a gauge theory. However, as we will demonstrate below, the four-dimensional Einstein-Hilbert action is not invariant under local Poincaré translations.

## 2.7. Invariance of the EH action under the Poincaré group

The EH action in $D = 4$ dimensions

$$I = \int \varepsilon_{abcd} R^{ab} e^e e^d \,, \tag{2.7.1}$$

is, by construction, invariant under general coordinate transformations and under Lorentz rotations. However, we will show below that this action is not invariant under local Poincaré translations. Let us consider in fact the variation of the action,

$$\delta_{tlp} I = \delta \int \varepsilon_{abcd} R^{ab} e^c e^d \,, \tag{2.7.2}$$

$$= \int d(\varepsilon_{abcd} \delta\omega^{ab} e^c e^d) + 2\int \varepsilon_{abcd} \delta\omega^{ab} T^c e^d + 2\int \varepsilon_{abcd} R^{ab} e^c \delta e^d \,. \tag{2.7.3}$$

Then, since under local Poincaré translations, the Vierbein and the spin connection transform as

$$\delta e^a = D\rho^a \,, \tag{2.7.4}$$

$$\delta\omega^{ab} = 0 \,, \tag{2.7.5}$$

we have that

$$\delta_{tlp} I = 2\varepsilon_{abcd} R^{ab} e^c D\rho^d \,, \tag{2.7.6}$$

$$= -2\int d(\varepsilon_{abcd} R^{ab} e^c \rho^d) + 2\int \varepsilon_{abcd} R^{ab} T^c \rho^d \,, \tag{2.7.7}$$

where we have made use of the Bianchi identity $DR^{ab} = 0$. Then, except for border terms, we have that

$$\delta_{tlp} I = 2\int \varepsilon_{abcd} R^{ab} T^c \rho^d \neq 0 \,. \tag{2.7.8}$$

Thus, the Einstein-Hilbert action remains invariant under the Poincaré group only if the torsion is constrained to vanish. However, the condition $T^a = 0$ is not preserved under local Poincaré translations. In fact, it can be shown that

$$\delta T^a = \delta(De^a) = D(\delta e^a) = DD\rho^a \,, \tag{2.7.9}$$

$$= R^{ab}\rho_b \neq 0 \,. \tag{2.7.10}$$

The lack of invariance of the four-dimensional Einstein-Hilbert action may seem unexpected, as translations are often regarded as coordinate transformations. However, a coordinate transformation corresponds to a Lie derivative, meaning that gauge translations are fundamentally different from general coordinate transformations.

Nevertheless, within the second-order formalism, imposing the condition $T^a = 0$ allows gauge translations to be treated as general coordinate transformations. In this case, the component $\omega^{ab}$ of the connection becomes a dependent field.

Finally, it is worth noting that in three-dimensional spacetime ($D = 3$), the situation is fundamentally different. Indeed, under local Poincaré translations, it can be shown that

$$\delta_{tlp} I_{EH}^{(3)} = \delta \int \varepsilon_{abc} R^{ab} e^c \,, \tag{2.7.11}$$

$$= \int \varepsilon_{abc} (\delta R^{ab}) e^c + \int \varepsilon_{abc} R^{ab} \delta e^c \,, \tag{2.7.12}$$

$$= \int \varepsilon_{abc} d(\delta\omega^{ab} e^c) - \int \varepsilon_{abc} \delta\omega^{ab} D e^c + \int \varepsilon_{abc} R^{ab} \delta e^c \,, \tag{2.7.13}$$

$$= \int \varepsilon_{abc} R^{ab} D\rho^c \,, \tag{2.7.14}$$

which can be rewritten as

$$\delta_{tlp} I_{EH}^{(3)} = \int \varepsilon_{abc} d(R^{ab} \rho^c) + \int \varepsilon_{abc} D R^{ab} \rho^c \,. \tag{2.7.15}$$

Thus, apart from a boundary term and utilizing the Bianchi identity, the invariance of the three-dimensional action under local Poincaré translations is ultimately established.

$$\delta_{tlp} I_{EH}^{(3)} = 0 \,. \tag{2.7.16}$$

Since this action is inherently invariant under Lorentz rotations, it follows that in three dimensions, one can formulate a gravitational action that remains invariant under the Poincaré group. Moreover, this invariance extends to all odd dimensions, a property that will be further examined in the context of Chern-Simons gravity theories.

# Capítulo 3

# Chern-Simons gravity theory

The Standard Model of high-energy physics is an exceptionally successful theory, providing precise and predictive descriptions of particle interactions [3]. This model accounts for three of the four fundamental forces of nature: electromagnetism, the weak interaction, and the strong interaction. At its core, the Standard Model is governed by a Yang-Mills action, predicated on the principle that nature exhibits invariance under a set of transformations that act independently at each point in spacetime—referred to as local or gauge symmetry. Crucially, the Yang-Mills theory relies on the presence of a non-dynamical background metric structure. Specifically, the Minkowski metric, $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, is indispensable for the formulation of the gauge theory, as becomes evident in the structure of the action

$$I = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int d^4x \eta_{\mu\rho}\eta_{\nu\lambda}F^{\rho\lambda}F^{\mu\nu} \,. \tag{3.0.1}$$

This implies that, in Yang-Mills theory, the space-time metric represents a fixed, non-dynamical background.

On the other hand, gravity described by GR is invariant under general coordinate transformations. This invariance is a local symmetry, analogous to the gauge invariance of the other three interactions, however, GR does not qualify as a gauge theory, except in three dimensions. However, there is a major difference: in GR, the metric is a dynamical object, which has independent degrees of freedom and obeys dynamical equations of motion given by Einstein's equations. This tells us that in the general theory of relativity, the geometry is dynamically determinate. Thus, the construction of a gauge theory for gravity requires an action that does not consider a fixed space-time background, or in other words, does not consider a fixed background metric.

We will see that the only way to formulate an action for gravity is to construct it in terms of a connection, without assuming a fixed space-time background.

In odd dimensions $D = 2n+1$, an action satisfying these conditions was proposed by Chamseddine in refs [83–85] . In the first-order formalism, the action can be written as

$$L_g^{(2n+1)} = \kappa\varepsilon_{a_1\ldots a_{2n+1}} \sum_{k=0}^{n} \frac{c_k}{l^{2(n-k)+1}} R^{a_1a_2}\ldots R^{a_{2k-1}}e^{2_{2k+1}}\ldots e^{a_{2n+1}}\,, \tag{3.0.2}$$

where

$$c_k = \frac{1}{2(n-k)+1}\binom{n}{k}\,, \tag{3.0.3}$$

the constants and $c_k$ are dimensionless and $l$ is a length parameter.

## 3.1. Chern-Simons Forms

The CS forms have been studied from different points of view, see for example refs [3, 86–89]. Let us see a brief construction of these CS forms: Let $\{T_a\}$ be a basis for the Lie algebra $\mathfrak{g}$ of a group $G$. Let $A$ be the 1-form gauge-valued connection in the Lie algebra $\mathfrak{g}$

$$A = A^A T_A\,, \tag{3.1.1}$$

whose 2-form curvature is given by

$$F = dA + \frac{1}{2}[A, A]\,. \tag{3.1.2}$$

We can define the following characteristic class as the product of $n+1$ curvatures

$$P^{(2n+2)} = \left\langle F^{n+1}\right\rangle\,, \tag{3.1.3}$$

where $\langle\ldots\rangle$ denotes a symmetric invariant tensor of rank $n+1$ for $\mathfrak{g}$ and $P^{(2n+2)}$

is a $2n+2$ invariant form. Furthermore, it can be shown that

$$d\omega_{2n+1} = \left\langle F^{n+1} \right\rangle \,, \tag{3.1.4}$$

where $\omega_{2n+1}$ is the CS form associated with the characteristic class , and is a local polynomial function of the 1-form $A$ valued in the Lie algebra $\mathfrak{g}$. Explicitly this is given by

$$\omega_{2n+1} = (n+1) \int_0^1 dt \left\langle A(tdA + t^2 A^2)^n \right\rangle \,. \tag{3.1.5}$$

Under infinitesimal gauge transformations of the form

$$\delta_\lambda A = d\lambda + [A, \lambda] \,, \tag{3.1.6}$$

the CS form is gauge invariant modulo boundary terms. Now, if we perform this gauge transformation on both sides of , we find

$$d\delta\omega_{2n+1} = 0 \,, \tag{3.1.7}$$

and by Poincaré's lemma, $\delta\omega_{2n+1}$ is an exact form.

Under a non-infinitesimal gauge transformation

$$A \to A^g = g^{-1} A g + g^{-1} dg \,, \tag{3.1.8}$$

the CS form transforms as

$$\omega^g_{2n+1} = \omega_{2n+1} + d\beta + (-1)^n \frac{n!(n+1)!}{(2n+1)!} \left\langle (g^{-1}dg)^{2n+1} \right\rangle \,, \tag{3.1.9}$$

where $\beta$ is a 2-form, which is a function of $A$ and depends on $g$ through the combination $g^{-1}dg$.

From the above, we can state the following lemma [3]

> **Lemma:** Let $\mathcal{P}(F)$ be an invariant $2n-$form constructed with the field strength $F = dA + A^2$, where $A$ is the connection for some gauge group $G$. If there exists a $2n-1$ form, $\mathcal{C}$, depending on $A$ and $dA$, such that $dC = \mathcal{P}$, then under a gauge transformation, $\mathcal{C}$ changes by a total derivative (exact form) $\delta\mathcal{C} = \mathrm{d(something)}$

Since the CS form changes to an exact form, it can be used as a Lagrangian for a gauge theory for the gauge connection A. It should be emphasized that $\omega_{2n+1}$ denotes a non-trivial Lagrangian which is not invariant under gauge transformations, but which changes to a function which depends only on the fields at the boundary: that is, quasi-invariant. This is sufficient to define a physical Lagrangian such that the principle of least action considers variations of the fields subject to appropriate boundary conditions. In this way it is always possible to select the boundary condition on the fields such that $\delta\omega_{2n+1} = 0$.

## 3.2. Chern-Simons Action

As we have already seen, CS forms can be used to construct gauge-invariant actions. A CS action is completely characterized if the Lie algebra $\mathfrak{g}$ and the invariant tensor are known. In a $2n+1$-dimensional spacetime, the CS action is given by

$$I = (n+1)k \int_M \int_0^1 dt \left\langle A(tdA + t^2A^2)^n \right\rangle . \tag{3.2.1}$$

By varying the action with respect to the connection we obtain the corresponding equations of motion

$$\langle F^n T_a \rangle = 0 . \tag{3.2.2}$$

However, despite the presence of powers of curvature greater than two in the action, the equations of motion are of first order in $A$. In the following, three-dimensional CS gravity will be described using the AdS group. In addition, in the following chapters of the thesis gravity will be described using the Maxwell group and its extension with spin-3 generators.

## 3.3. AdS Chern-Simons gravity theory

Let us briefly review three-dimensional AdS CS gravity theory, whose underlying symmetry corresponds to the AdS algebra. The generators of the AdS algebra

$(J_a, P_a)$ satisfy the following commutation relations:

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc} J^c \,, && (3.3.1) \\ [J_a, P_b] &= \epsilon_{abc} P^c \,, && (3.3.2) \\ [P_a, P_b] &= \frac{1}{\ell^2} \epsilon_{abc} J^c \,, && (3.3.3) \end{aligned}$$

where $a, b, \ldots = 0, 1, 2$ are Lorentz indices raised and lowered with the Minkowski metric $\eta_{ab}$, $\epsilon_{abc}$ is the Levi-Civita tensor in three dimensions and $\ell$ is the AdS radius. The 1-form gauge connection valued in the AdS algebra takes the form[1]

$$A = e^a P_a + \omega^a J_a \,, \tag{3.3.4}$$

where $e^a$ and $\omega^a$ are interpreted as the Vielbein and the spin connection, corresponding to the gauge fields associated to $P_a$ and $J_a$, respectively. The corresponding curvature two-form is

$$F = T^a P_a + \left( R^a + \frac{1}{2\ell^2} \epsilon^{abc} e_b e_c \right) J_a \,, \tag{3.3.5}$$

where

$$R^a = d\omega^a + \frac{1}{2} \epsilon^{abc} \omega_b \omega_c \,, \tag{3.3.6}$$

$$T^a = De^a = de^a + \epsilon^{abc} \omega_b e_c \,, \tag{3.3.7}$$

correspond to the Lorentz curvature and torsion two-forms, respectively. The Lorentz covariant derivative is defined by $Dv^a = dv^a + \epsilon^{abc} \omega_b v_c$.

Since this algebra admits an invariant bilinear form, whose only non-vanishing components are given by $\langle J_a P_b \rangle = \eta_{ab}$[2], the three-dimensional CS action

$$I_{\text{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \left\langle A\text{d}A + \frac{2}{3} A^3 \right\rangle \,, \tag{3.3.8}$$

[1] Here, $\omega^a = \frac{1}{2} \epsilon_{abc} \omega^{bc}$ is the dual spin-connection one-form.

[2] For simplicity, we have omitted here the component $\langle J_a J_b \rangle \sim \eta_{ab}$ which allows for the exotic term in the AdS CS action

with $k = \frac{1}{4G}$, reduces, up to boundary terms, to

$$I[e,\omega] = \frac{k}{4\pi}\int 2R^a e_a + \frac{1}{3\ell^2}\epsilon_{abc}e^a e^b e^c\,. \tag{3.3.9}$$

Note that this is precise form of the action of three-dimensional GR with negative constant $-1/\ell^2$. The field equations $F = 0$ imply that

$$R^a = -\frac{1}{2\ell^2}\epsilon^{abc}e_b e_c\,, \qquad T^a = De^a = 0\,, \tag{3.3.10}$$

which means that the spacetime curvature is constant and it has vanishing torsion. Here $D$ denotes the Lorentz covariant derivative. By construction, the action changes by a boundary term under the following infinitesimal local gauge transformations spanned by the parameter $\lambda = \xi^a P_a + \Lambda^a J_a$

$$\delta e^a = D\xi^a - \epsilon^{abc}\Lambda_b e_c\,, \qquad \delta\omega^a = D\Lambda^a - \frac{1}{\ell^2}\epsilon^{abc}\xi_b e_c\,. \tag{3.3.11}$$

Note that after performing the flat space limit $\ell \to \infty$ the AdS CS gravity reduces to the Poincaré CS gravity theory. Indeed, the flat limit can be applied at the level of the AdS algebra, the invariant tensor, the field equations, and the gauge transformations.

It is worth noting that the AdS algebra in three dimensions $so(2,2)$ is isomorphic to two copies of the spacial linear algebra in two dimensions, $sl(2,R)$. Then $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, where $\mathfrak{g}_\pm$ stands for two copies of $sl(2,R)$. The $sl(2,R)$ algebra reads

$$[L_i, L_j] = (i-j)L_{i+j}\,, \tag{3.3.12}$$

where the generators $L_i$, with $i = -1, 0, 1$, are assumed to be the same for both copies and are chosen to be

$$L_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_0 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{3.3.13}$$

Splitting the connection in two independent $sl(2,R)-$ valued gauge fields, according to $A = A^+ + A^-$, the Chern-Simons action reduces to

$$I_{CS} = I_{CS}[A^+] - I_{CS}[A^-]\,. \tag{3.3.14}$$

## 3.4. Maxwell Chern-Simons gravity theory

It is known since 1970, see [15], that the presence of a constant classical electromagnetic field background in Minkowski space-time leads to the modification of Poincaré symmetries. One obtains the enlargement of Poincaré algebra, called Maxwell algebra [14, 90] which is obtained by the replacement of the commutative momentum generators $P_a, (a = 0, 1, \ldots, d)$ by

$$[P_a\,, P_b] = ieZ_{ab}\,, \quad Z_{ba} = -Z_{ab}\,, \tag{3.4.1}$$

where $e$ is the electromagnetic coupling constant.

In this section, using the CS formalism, we review the three-dimensional gravity theory based on the Maxwell algebra. In three-dimensions, the Maxwell algebra is generated by translations $P_a$, Lorentz transformations $J_a$ and an Abelian ideal of generators $Z_a$ satisfying the commutation relations

$$\begin{aligned} &[J_a, J_b] = \epsilon_{abc} J^c\,, && [P_a, P_b] = \epsilon_{abc} Z^c\,, \\ &[J_a, Z_b] = \epsilon_{abc} Z^c\,, && [Z_a, Z_b] = 0\,, \\ &[J_a, P_b] = \epsilon_{abc} P^c\,, && [Z_a, P_b] = 0\,. \end{aligned} \tag{3.4.2}$$

All generators are anti-Hermitean. Note that the generators $(P_a, J_a)$ do not form a Poincaré subalgebra of (3.4.2) because the translations $P_a$ do not commute. At first sight, it looks like $Z_a$ should be interpreted as the Poincaré translations because the set $(Z_a, J_a)$ is closed, forming a Poincaré subalgebra. However, as we shall see below, identifying $P_a$ with the translational generator gives a good gravitational dynamics, where GR is reproduced in a particular limit. For the present algebra (3.4.2), the relevant tensor in three-dimensional has rank 2, and its components are given by

$$\begin{aligned} &\langle J_a J_b \rangle = \alpha_0\, \eta_{ab}\,, && \langle P_a P_b \rangle = \alpha_2\, \eta_{ab}\,, \\ &\langle J_a P_b \rangle = \alpha_1\, \eta_{ab}\,, && \langle Z_a Z_b \rangle = 0\,, \\ &\langle J_a Z_b \rangle = \alpha_2\, \eta_{ab}\,, && \langle Z_a P_b \rangle = 0\,, \end{aligned} \tag{3.4.3}$$

where $\langle \ldots \rangle$ stands for a non-degenerate invariant symmetric bilinear form and $\alpha_0$,

$\alpha_1$ and $\alpha_2$ are dimensionless constants. The fundamental field associated to the Maxwell algebra is the one-form potential

$$A = e^a P_a + \omega^a J_a + \sigma^a Z_a \,, \tag{3.4.4}$$

whose components are the vielbein $e^a(x)$, the spin connection $\omega^a(x)$ and the gravitational Maxwell gauge field $\sigma^a(x)$. The dynamics of the field $A$ in three dimensions is described by the CS action (3.3.8). The Maxwell symmetry is guaranteed, up to a total derivative, by the use of the Maxwell gauge field (3.4.4), the invariant tensor (3.4.3) and the algebra (3.4.2). The CS action with Maxwell symmetry reads

$$\begin{aligned} I[A] &= \frac{k}{4\pi}\int\limits_{\mathcal{M}} \Big[\alpha_0 \left(\omega^a d\omega_a + \frac{1}{3}\,\epsilon^{abc}\omega_a\omega_b\omega_c\right) + 2\alpha_1 R_a e^a \\ &\quad + \alpha_2 \left(T^a e_a + 2R^a\sigma_a\right) - d\left(\alpha_1\omega^a e_a + \alpha_2\omega^a\sigma_a\right)\Big] \,. \end{aligned} \tag{3.4.5}$$

The first term in the action is the gravitational CS term with the coupling constant $\alpha_0$. Next term is the EH one, so that its coupling can be normalized to $\alpha_1 = 1$ . The last term, with the coupling constant $\alpha_2$, gives the dynamics to the gravitational Maxwell field and also contributes of the other fields.

The action (3.4.5) is invariant, up to boundary terms, under the action of the infinitesimal gauge transformations, $\delta A = d\lambda + [A, \lambda]$. In terms of components, the local gauge parameter reads $\lambda = \xi^a(x)P_a + \Lambda^a(x)J_a + \chi^a(x)Z_a$, and the gauge field change as

$$\delta_\lambda e^a = D\xi^a - \epsilon^{abc}\Lambda_b e_c \,, \tag{3.4.6}$$

$$\delta_\lambda \omega^a = D\Lambda^a \,, \tag{3.4.7}$$

$$\delta_\lambda \sigma^a = D\chi^a + \epsilon^{abc}(e_a\xi_c - \Lambda_b\sigma_c) \,. \tag{3.4.8}$$

Extremization of the action (3.4.5) gives rise to the following equations of motion,

$$\delta e^a \; : \qquad 0 = \alpha_1 R_a + \alpha_2 T_a \,, \tag{3.4.9}$$

$$\delta\omega^a \; : \qquad 0 = \alpha_0 R_a + \alpha_1 T_a + \alpha_2 \left(D\sigma_a + \frac{1}{2}\,\epsilon_{abc}e^b e^c\right) \,, \tag{3.4.10}$$

$$\delta\sigma^a \; : \qquad 0 = \alpha_2 R_a \,, \tag{3.4.11}$$

where the curvature and torsion two-forms are $R^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c$, and $T^a = De^a$, respectively. In the limit $\alpha_2 = 0$, the equations of motion of GR are recovered without ($\alpha_0 = 0$) and with ($\alpha_0 \neq 0$) the gravitational CS term. When $\alpha_2 \neq 0$, the above equations can be equivalently written as

$$T^a = 0\,, \tag{3.4.12}$$

$$R^a = 0\,, \tag{3.4.13}$$

$$D\sigma^a + \frac{1}{2}\epsilon^{abc}e_b e_c = 0\,. \tag{3.4.14}$$

Similarly to GR, the geometries described by the equations of motion (3.4.13) are Riemannian (torsionless) and locally flat. A difference with respect to GR is that the gravitational Maxwell field, $\sigma^a$, does not vanish on-shell when $\alpha_2 \neq 0$. This can be seen by multiplying the last equation in (3.4.13) by $e_a$, which on-shell means that $D(\sigma^a e_a)$ is proportional to the volume form and, therefore, cannot vanish. Furthermore, $\sigma^a$ couples to the geometry through the interaction with other fields that backreact on it on-shell, that leads to new effects compared to GR. For example, it modifies the asymptotic sector of the spacetime and the asymptotic charges of the solutions (See section 5.3).

# Capítulo 4

# Chern-Simons higher-spin gravity theories in three dimensions

Higher-spin (HS) fields have received great interest due to their appearance in the spectrum of string theory and simplified models of the AdS/CFT conjecture [51–60]. In particular, in three dimensions the coupling of massless HS fields to AdS gravity is consistently described by a CS action whose gauge group is given, in the simplest case, by two copies of $SL(3,\mathbb{R})$ [61–63], which describes non-propagating spin-3 fields coupled to AdS gravity. The extension of the previously described Maxwell CS gravity with spin-3 gauge fields was also found in [33]. The underlying symmetry corresponds to a spin-3 extension of the Maxwell algebra, denoted here as $\mathfrak{hs}_3\mathfrak{max}$, allowing the inclusion of a new gauge field, which is the spin-3 analogue of the gravitational Maxwell field. This chapter is devoted to reviewing these HS gravity theories.

## 4.1. Review of $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons gravity

The spin-3 extension of three-dimensional AdS gravity can be formulated as a CS theory for the group $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ [77]. As pure gravity corresponds to the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ sector of the theory, the field content of the full theory is determined by the embedding of the $\mathfrak{sl}(2,\mathbb{R})$ algebra in $\mathfrak{sl}(3,\mathbb{R})$. The $\mathfrak{sl}(3,\mathbb{R})$

algebra is defined by the commutation relations

$$\begin{aligned}
[L_i, L_j] &= (i-j)L_{i+j}\,,\\
[L_i, W_m] &= (2i-m)W_{i+m}\,,\\
[W_m, W_n] &= \frac{\sigma}{3}(m-n)(2m^2+2n^2-mn-8)L_{m+n}\,,
\end{aligned} \tag{4.1.1}$$

where $i,j=-1,0,1$ and $m,n=-2,-1,0,1,2$. Here, we consider $\sigma<0$ [1]and the corresponding Killing form in the fundamental representation is normalized such that

$$\begin{aligned}
\langle L_0 L_0\rangle &= \frac{1}{2}\,, \quad \langle W_0 W_0\rangle = -\frac{2}{3}\sigma\,,\\
\langle L_1 L_{-1}\rangle &= -1\,, \quad \langle W_2 W_{-2}\rangle = -4\sigma\,,\\
\langle W_1 W_{-1}\rangle &= \sigma\,.
\end{aligned} \tag{4.1.2}$$

The algebra $\mathfrak{sl}(2,\mathbb{R})$ can be non-trivially embedded in $\mathfrak{sl}(3,\mathbb{R})$ in two inequivalent ways: the principal embedding $\{L_0, L_{\pm1}\}$, which gives rise to an interacting theory of massless spin−2 and spin−3 field; and the diagonal embedding $\left\{\frac{1}{2}L_0, \frac{1}{4}W_{\pm2}\right\}$, leading to a theory for a spin−2 field, two spin−3/2 fields and a spin−1 current [67, 68]. As we want to describe gravity coupled to spin−3 matter fields, the principal embedding of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{sl}(3,\mathbb{R})$ will be considered throughout this thesis. For our purposes it will be convenient to write $\mathfrak{sl}(3,\mathbb{R})$ in the form

$$\begin{aligned}
[J_a, J_b] &= \epsilon_{abc}J^c\,,\\
[J_a, J_{bc}] &= \epsilon^m_{a(b}T_{c)m}\,,\\
[T_{ab}, T_{cd}] &= \sigma\big(\eta_{a(c}\epsilon_{d)bm}+\eta_{b(c}\epsilon_{d)am}\big)J^m\,,
\end{aligned} \tag{4.1.3}$$

where the generators $\{J_a, T_{ab}\}$ are related to those of (4.1.1) by

$$\begin{aligned}
J_0 &= \tfrac{1}{2}(L_{-1}+L_1)\,,\\
J_1 &= \tfrac{1}{2}(L_{-1}-L_1)\,,\\
J_2 &= L_0\,,
\end{aligned} \tag{4.1.4}$$

[1]As discussed in [77], the case $\mathfrak{sl}(3,\mathbb{R})$ arises for $\sigma<0$, whereas $\sigma>0$ corresponds to the $\mathfrak{su}(1,2)$ algebra. To ensure generality, we will consider an arbitrary $\sigma$, while emphasizing that our primary focus is on the negative values of $\sigma$.

$$\begin{aligned} T_{00} &= \tfrac{1}{4}(W_2 + W_{-2} + 2W_0)\,, \\ T_{11} &= \tfrac{1}{4}(W_2 + W_{-2} - 2W_0)\,, \\ T_{22} &= W_0\,, \\ T_{01} &= \tfrac{1}{4}(W_2 - W_{-2})\,, \\ T_{02} &= \tfrac{1}{2}(W_1 + W_{-1})\,, \\ T_{12} &= \tfrac{1}{2}(W_1 - W_{-1})\,. \end{aligned} \tag{4.1.5}$$

In this case, instead of 8 generators of the fundamental representation, there are 9 generators $\{J_a, T_{ab} = T_{ba}\}; a, b = 1, 2, 3$, plus the constraint $T^a_a = 0$, where indices lowered and raised with the metric $\eta_{ab} = diag(-1, 1, 1)$.

The action of the system is given by

$$I = I_{CS}[A] - I_{CS}[\bar{A}]\,, \tag{4.1.6}$$

where $I_{CS}[A]$ corresponds to the CS action (3.3.8). The components of the invariant tensor are given by

$$\begin{aligned} \langle J_a J_b \rangle &= \frac{1}{2}\eta_{ab}\,, \\ \langle J_a T_{bc} \rangle &= 0\,, \\ \langle T_{ab} T_{cd} \rangle &= -\frac{\sigma}{2}\left(\eta_{a(c}\eta_{d)b} - \frac{2}{3}\eta_{ab}\eta cd\right)\,, \end{aligned} \tag{4.1.7}$$

and the $\mathfrak{sl}(3,\mathbb{R})$ valued connection one-forms $A$ and $\bar{A}$ have the form

$$A = \left(\omega^a + \frac{1}{\ell}e^a\right)J_a + \left(\omega^{ab} + \frac{1}{\ell}e^{ab}\right)T_{ab} \tag{4.1.8}$$

$$\bar{A} = \left(\omega^a - \frac{1}{\ell}e^a\right)\bar{J}_a + \left(\omega^{ab} - \frac{1}{\ell}e^{ab}\right)\bar{T}_{ab}\,. \tag{4.1.9}$$

The field equations are naturally given by the vanishing of the curvatures associated to $A$ and $\bar{A}$

$$dA + A \wedge A = 0\,, \qquad d\bar{A} + \bar{A} \wedge \bar{A} = 0. \tag{4.1.10}$$

As each subset of $\mathfrak{sl}(3,\mathbb{R})$ generators satisfies (4.1.3) and (4.1.7), the action (4.1.6) takes the form

$$\begin{aligned} I = \frac{k}{2\pi} \int \Big[ & e^a \Big( d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b\omega^c + 2\sigma\epsilon_{aec}\omega^{cd}\omega^e{}_d \Big) \\ & - 2\sigma e^{ab} \Big( d\omega_{ab} + 2\epsilon_{acd}\omega^c\omega^d{}_b + \frac{1}{6\ell^2}\epsilon_{abc}e^a e^b e^c + \frac{2\sigma}{\ell^2}\epsilon_{aec}e^a e^{cd} e^e{}_d \Big) \Big], \end{aligned} \tag{4.1.11}$$

The field equations coming from this action can be expanded around a vacuum solution and, after using the torsion constraints to express $\omega^a$ and $\omega^{ab}$ in terms of $e^a$ and $e^{ab}$, they reduce to the Fronsdal equations [91] for the space-time metric and a spin-3 field.

## 4.2. Three-dimensional Maxwell gravity coupled to spin-3 fields

In this section, we briefly review the three-dimensional Maxwell CS gravity coupled to spin-3 fields first presented in [33]. The CS gravity action is constructed from the spin-3 extension of the Maxwell algebra, referred to in this work as $\mathfrak{hs}_3\mathfrak{max}$, whose generators satisfy the following non-vanishing commutators:

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc}J^c, & [J_a, P_b] &= \epsilon_{abc}P^c\,, \\ [P_a, P_b] &= \epsilon_{abc}Z^c, & [J_a, Z_b] &= \epsilon_{abc}Z^c\,, \\ [J_a, J_{bc}] &= \epsilon^m_{a(b}J_{c)m}, & [J_a, P_{bc}] &= \epsilon^m_{a(b}P_{c)m}\,, \\ [P_a, J_{bc}] &= \epsilon^m_{a(b}P_{c)m}, & [P_a, P_{bc}] &= \epsilon^m_{a(b}Z_{c)m}\,, \\ [Z_a, J_{bc}] &= \epsilon^m_{a(b}Z_{c)m}, & [J_a, Z_{bc}] &= \epsilon^m_{a(b}Z_{c)m}\,, \\ [J_{ab}, J_{cd}] &= -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right)J^m\,, \\ [J_{ab}, P_{cd}] &= -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right)P^m\,, \\ [J_{ab}, Z_{cd}] &= -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right)Z^m\,, \\ [P_{ab}, P_{cd}] &= -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\epsilon_{d)am}\right)Z^m\,, \\ \text{others} &= 0\,, \end{aligned} \tag{4.2.1}$$

where $a, b, \cdots = 0, 1, 2$ are Lorentz indices raised and lowered with the Minkowski metric $\eta_{ab}$ and $\epsilon_{abc}$ is the Levi-Civita tensor. This algebra turns out to be a generalization of the Maxwell algebra and describes the coupling of spin-3 generators $\{J_{ab}, P_{ab}, Z_{ab}\}$ to the Maxwell ones $\{J_a, P_a, Z_a\}$. It was first derived as an expansion [92] of the $\mathfrak{sl}(3,\mathbb{R})$ algebra with a particular semigroup, and is naturally recovered through an Inönü-Wigner contraction of three copies of $\mathfrak{sl}(3,\mathbb{R})$. It is important to recall that the spin-3 generators are assumed to be symmetric and traceless.

In order to write down a CS action for this algebra, we define the one-form gauge connection

$$A = \omega^a J_a + e^a P_a + \sigma^a Z_a + \omega^{ab} J_{ab} + e^{ab} P_{ab} + \sigma^{ab} Z_{ab} \,, \tag{4.2.2}$$

where $e^{ab}, \omega^{ab}$ and $\sigma^{ab}$ correspond to the spin-3 analogues of the vielbein, spin connection and Maxwell field, respectively. The non-vanishing components of the invariant tensor are given by:

$$\begin{aligned} \langle J_a J_b \rangle &= \alpha_0 \eta_{ab} \,, & \langle P_a P_b \rangle &= \alpha_2 \eta_{ab} \,, \\ \langle J_a P_b \rangle &= \alpha_1 \eta_{ab} \,, & \langle J_a Z_b \rangle &= \alpha_2 \eta_{ab} \,, \end{aligned} \tag{4.2.3}$$

$$\begin{aligned} \langle J_{ab} J_{cd} \rangle &= \alpha_0 \left( \eta_{a(c}\eta_{d)b} - \tfrac{2}{3}\eta_{ab}\eta_{dc} \right) \,, & \langle P_{ab} P_{cd} \rangle &= \alpha_2 \left( \eta_{a(c}\eta_{d)b} - \tfrac{2}{3}\eta_{ab}\eta_{dc} \right) \,, \\ \langle J_{ab} P_{cd} \rangle &= \alpha_1 \left( \eta_{a(c}\eta_{d)b} - \tfrac{2}{3}\eta_{ab}\eta_{dc} \right) \,, & \langle J_{ab} Z_{cd} \rangle &= \alpha_2 \left( \eta_{a(c}\eta_{d)b} - \tfrac{2}{3}\eta_{ab}\eta_{dc} \right) \,. \end{aligned} \tag{4.2.4}$$

Then, considering the previous invariant tensor and the one-form gauge connection (4.2.2) in the CS action

$$I[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \left\langle AdA + \frac{2}{3} A^3 \right\rangle \,, \tag{4.2.5}$$

defined on a three-dimensional manifold $\mathcal{M}$, and where $k = \frac{1}{4G}$ is the level of the theory related to the gravitational constant $G$, we obtain

$$
\begin{aligned}
I_{\mathfrak{hs}_3\mathfrak{Max}} = & \frac{k}{4\pi}\int \alpha_0 \left[\left(\omega^a d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a\omega^b\omega^c\right) + 2\left(\omega^a_{\ b} d\omega^b_{\ a} + 2\epsilon_{abc}\omega^a\omega^{bd}\omega^c_{\ d}\right)\right] \\
& +2\alpha_1\left[e^a\left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b\omega^c + 2\epsilon_{abc}\omega^{bd}\omega^c_{\ d}\right) + 2e^{ab}\left(d\omega_{ab} + 2\epsilon_{acd}\omega^c\omega^d_{\ b}\right)\right] \\
& +\alpha_2\left[e^a\left(de_a + \epsilon_{abc}\omega^b e^c\right) + 2\sigma^a\left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b\omega^c\right)\right. \\
& +2e^{ab}\left(de_{ab} + 2\epsilon_{acd}\omega^c e^d_{\ b} + 4\epsilon_{acd}e^c\omega^d_{\ b}\right) \\
& \left. +4\left(\omega^{ab}d\sigma_{ab} + \epsilon_{abc}\sigma^a\omega^{be}\omega^c_{\ e} + 2\epsilon_{abc}\omega^a\sigma^{be}\omega^c_{\ e}\right)\right] . \qquad (4.2.6)
\end{aligned}
$$

This CS action describes the coupling of spin-3 gauge fields to three-dimensional Maxwell gravity and corresponds to a novel extension of higher-spin three-dimensional gravity in flat space including topological HS matter. It has three different independent sectors proportional to $\alpha_0, \alpha_1$ and $\alpha_2$. The term proportional to $\alpha_1$ corresponds to an Euler type CS form while the term proportional to $\alpha_0$ and $\alpha_2$ are Pontryagin type CS forms. As it was mentioned in [33], similarly to the spin-3 extension of the Poincaré gravity, the action (4.2.6) does not contain the cosmological constant term. Extremization of the action gives rise to the following field equations for the spin-2 fields

$$
\begin{aligned}
\mathcal{T}^a &\equiv de^a + \epsilon^{abc}\omega_b e_c + 4\epsilon^{abc}e^{bd}\omega_c^{\ d} = 0\,, && (4.2.7)\\
\mathcal{R}^a &\equiv d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c + 2\epsilon^{abc}\omega_{bd}\omega_c^{\ d} = 0\,, && (4.2.8)\\
\mathcal{F}^a &\equiv d\sigma^a + \epsilon^{abc}\omega_b\sigma_c + \frac{1}{2}\epsilon^{abc}e_b e_c + 2\epsilon^{abc}\left(2\omega_{bd}\sigma_c^{\ d} + e_{bd}e_c^{\ d}\right) = 0\,, && (4.2.9)
\end{aligned}
$$

while the corresponding field equations for the spin-3 fields are

$$
\begin{aligned}
\mathcal{T}^{ab} &\equiv de^{ab} + \epsilon^{cd(a|}\omega_c e_d^{\ |b)} + \epsilon^{cd(a|}e_c\omega_d^{\ |b)} = 0\,, && (4.2.10)\\
\mathcal{R}^{ab} &\equiv d\omega^{ab} + \epsilon^{cd(a|}\omega_c\omega_d^{\ |b)} = 0\,, && (4.2.11)\\
\mathcal{F}^{ab} &\equiv d\sigma^{ab} + \epsilon^{cd(a|}\omega_c\sigma_d^{\ |b)} + \epsilon^{cd(a|}\sigma_c\omega_d^{\ |b)} + \epsilon^{cd(a|}e_c e_d^{\ |b)} = 0\,. && (4.2.12)
\end{aligned}
$$

The CS action (4.2.6) is invariant, up to boundary terms, under the action of the infinitesimal gauge transformations $\delta A = D\lambda = d\lambda + [A,\lambda]$, where the local gauge parameter is given by

$$
\lambda = \Lambda^a J_a + \xi^a P_a + \chi^a Z_a + \Lambda^{ab}J_{ab} + \xi^{ab}P_{ab} + \chi^{ab}Z_{ab}\,. \qquad (4.2.13)
$$

For the spin-2 gauge fields we get

$$
\begin{aligned}
\delta\omega^a &= D_\omega\Lambda^a + 4\epsilon^{abc}\omega_{bd}\Lambda_c^{\ d}\,, && (4.2.14)\\
\delta e^a &= D_\omega\xi^a - \epsilon^{abc}\Lambda_b e_c + 4\epsilon^{abc}\omega_{bd}\xi_c^{\ d} + 4\epsilon^{abc}e_{bd}\Lambda_c^{\ d}\,, && (4.2.15)\\
\delta\sigma^a &= D_\omega\chi^a - \epsilon^{abc}\xi_b e_c - \epsilon^{abc}\Lambda_b\sigma_c + 4\epsilon^{abc}e_{bd}\xi_c^{\ d} \\
&\quad +4\epsilon^{abc}\omega_{bd}\chi_c^{\ d} + 4\epsilon^{abc}\sigma_{bd}\Lambda_c^{\ d}\,, && (4.2.16)
\end{aligned}
$$

where, besides the usual gauge transformations of CS Maxwell gravity, there are new terms involving the spin-3 gauge parameters $\xi^{ab}$, $\Lambda^{ab}$ and $\chi^{ab}$. On the other hand, the spin-3 gauge fields transform as follows

$$
\begin{aligned}
\delta\omega^{ab} &= d\Lambda^{ab} + \epsilon^{cd(a|}\omega_c\Lambda_d^{\ |b)} + \epsilon^{cd(a}\omega^{b)}_{\ c}\Lambda_d\,, && (4.2.17)\\
\delta e^{ab} &= d\xi^{ab} + \epsilon^{cd(a|}\omega_c\xi_d^{\ |b)} + \epsilon^{cd(a|}e_c\Lambda_d^{\ |b)} + \epsilon^{cd(a}e^{b)}_{\ c}\Lambda_d + \epsilon^{cd(a}\omega^{b)}_{\ c}\xi_d\,, && (4.2.18)\\
\delta\sigma^{ab} &= d\chi^{ab} + \epsilon^{cd(a|}\sigma_c\Lambda_d^{\ |b)} + \epsilon^{cd(a|}\omega_c\chi_d^{\ |b)} + \epsilon^{cd(a|}e_c\xi_d^{\ |b)} \\
&\quad + \epsilon^{cd(a}\omega^{b)}_{\ c}\chi_d + \epsilon^{cd(a}\sigma^{b)}_{\ c}\Lambda_d + \epsilon^{cd(a}e^{b)}_{\ c}\xi_d\,. && (4.2.19)
\end{aligned}
$$

A consistent set of boundary conditions for the pure Maxwell gravity theory was proposed in [43], whose asymptotic symmetry algebra was shown to be given by a deformation of the $\mathfrak{bms}_3$ algebra, here referred to as $\mathfrak{max}\text{-}\mathfrak{bms}_3$, with three independent central charges. Subsequently, in [93] it was shown that the $\mathfrak{max}\text{-}\mathfrak{bms}_3$ can be recovered from three copies of the Virasoro algebra, which in turn were shown to correspond to the asymptotic symmetry algebra of the AdS-Lorentz algebra. In Chapter 5 we shall extend these results to the previously discussed Maxwell CS gravity coupled with spin-3 gauge fields. We will include chemical potentials without spoiling the original deformed $\mathfrak{bms}_3$ symmetry. As we will see, the corresponding asymptotic symmetry algebra will correspond to a spin-3 extension of $\mathfrak{max}\text{-}\mathfrak{bms}_3$. The charge algebra has three central charges defined in terms of the coupling constants appearing in the CS action (4.2.6). We will also show that this asymptotic symmetry can alternatively be obtained through a well-defined flat limit of three copies of the $\mathcal{W}_3$ algebra. Before considering the analysis of the asymptotic structure of the Maxwell gravity theory and its corresponding higher-spin extension, in the next section we will review the construction of the AdS-Lorentz gravity coupled to spin-3 fields, which allows for the introduction of a non-vanishing cosmological constant.

## 4.3. AdS-Lorentz gravity coupled to spin-3 fields

In this section, we are review the coupling of spin-3 gauge field to three-dimensional AdS-Lorentz gravity.

The generators of the expanded algebra satisfy

$$
\begin{aligned}
&[J_a, J_b] = \epsilon_{abc} J^c\,, && [J_a, P_b] = \epsilon_{abc} P^c\,,\\
&[P_a, P_b] = \epsilon_{abc} Z^c\,, && [J_a, Z_b] = \epsilon_{abc} Z^c\,,\\
&[P_a, Z_b] = \frac{1}{\ell^2}\epsilon_{abc} P^c\,, && [Z_a, Z_b] = \frac{1}{\ell^2}\epsilon_{abc} Z^c\,,\\
&[J_a, J_{bc}] = \epsilon^m{}_{a(b} J_{c)m}\,, && [J_a, P_{bc}] = \epsilon^m{}_{a(b} P_{c)m}\,,\\
&[P_a, J_{bc}] = \epsilon^m{}_{a(b} P_{c)m}\,, && [P_a, P_{bc}] = \epsilon^m{}_{a(b} Z_{c)m}\,,\\
&[Z_a, J_{bc}] = \epsilon^m{}_{a(b} Z_{c)m}\,, && [Z_a, P_{bc}] = \frac{1}{\ell^2}\epsilon^m{}_{a(b} P_{c)m}\,,\\
&[J_a, Z_{bc}] = \epsilon^m{}_{a(b} Z_{c)m}\,, && [P_a, Z_{bc}] = \frac{1}{\ell^2}\epsilon^m{}_{a(b} P_{c)m}\,,\\
&[Z_a, Z_{bc}] = \frac{1}{\ell^2}\epsilon^m{}_{a(b} Z_{c)m}\,,\\
&[J_{ab}, J_{cd}] = -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\eta_{d)am}\right) J^m\,,\\
&[J_{ab}, P_{cd}] = -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\eta_{d)am}\right) P^m\,,\\
&[J_{ab}, Z_{cd}] = -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\eta_{d)am}\right) Z^m\,,\\
&[P_{ab}, P_{cd}] = -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\eta_{d)am}\right) Z^m\,,\\
&[P_{ab}, Z_{cd}] = -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\eta_{d)am}\right) P^m\,,\\
&[Z_{ab}, Z_{cd}] = -\left(\eta_{a(c}\epsilon_{d)bm} + \eta_{b(c}\eta_{d)am}\right) Z^m\,.
\end{aligned}
\tag{4.3.1}
$$

Let us consider now the one-form gauge connection

$$
A = \omega^a J_a + e^a P_a + \sigma^a Z_a + \omega^{ab} J_{ab} + e^{ab} P_{ab} + \sigma^{ab} Z_{ab}\;, \tag{4.3.2}
$$

and the corresponding invariant tensor

$$
\begin{aligned}
\langle J_a J_b \rangle &= \alpha_0 \eta_{ab} \,, & \langle J_a P_b \rangle &= \alpha_1 \eta_{ab} \,, \\
\langle P_a P_b \rangle &= \alpha_2 \eta_{ab} \,, & \langle J_a Z_b \rangle &= \alpha_2 \eta_{ab} \,, \\
\langle P_a Z_b \rangle &= \frac{\alpha_1}{\ell^2} \eta_{ab} \,, & \langle Z_a Z_b \rangle &= \frac{\alpha_2}{\ell^2} \eta_{ab} \,, \\
\langle J_{ab} J_{bc} \rangle &= \alpha_0 \left( \eta_{a(c} \eta_{d)b} - \tfrac{2}{3} \eta_{ab} \eta_{dc} \right) \,, \\
\langle P_{ab} P_{bc} \rangle &= \alpha_2 \left( \eta_{a(c} \eta_{d)b} - \tfrac{2}{3} \eta_{ab} \eta_{dc} \right) \,, \\
\langle J_{ab} P_{bc} \rangle &= \alpha_1 \left( \eta_{a(c} \eta_{d)b} - \tfrac{2}{3} \eta_{ab} \eta_{dc} \right) \,, \\
\langle J_{ab} Z_{bc} \rangle &= \alpha_2 \left( \eta_{a(c} \eta_{d)b} - \tfrac{2}{3} \eta_{ab} \eta_{dc} \right) \,, \\
\langle P_{ab} Z_{bc} \rangle &= \frac{\alpha_1}{\ell} \left( \eta_{a(c} \eta_{d)b} - \tfrac{2}{3} \eta_{ab} \eta_{dc} \right) \,, \\
\langle Z_{ab} Z_{bc} \rangle &= \frac{\alpha_2}{\ell^2} \left( \eta_{a(c} \eta_{d)b} - \tfrac{2}{3} \eta_{ab} \eta_{dc} \right) \,.
\end{aligned}
\tag{4.3.3}
$$

The CS action (4.2.5) in this case takes the form

$$
\begin{aligned}
I \;=\; \kappa \int & \alpha_0 \left[ \left( \omega^a d\omega_a + \frac{1}{3} \epsilon_{abc} \omega^a \omega^b \omega^c \right) + 2 \left( \omega^a{}_b d\omega^b{}_a + 2 \epsilon_{abc} \omega^a \omega^{bd} \omega^c{}_d \right) \right] \\
& + 2\alpha_1 \left[ e^a \left( d\omega_a + \frac{1}{2} \epsilon_{abc} \left\{ \omega^b \omega^c + \frac{1}{2\ell^4} \sigma^b \sigma^c \right\} + \frac{1}{\ell^2} d\sigma_a + \frac{1}{\ell^2} \epsilon_{abc} \left\{ \omega^b \sigma^c + \frac{1}{\ell^4} \sigma^b \sigma^c \right\} \right) \right. \\
& + \frac{1}{6\ell^2} \epsilon_{abc} \left( e^a e^b e^c + 12 \epsilon_{abc} e^a e^{bd} e^c{}_d \right) + 2 e^a \left( \epsilon_{abc} \omega^{bd} \omega^c{}_d + \frac{2}{\ell^2} \epsilon_{abc} \omega^{bd} \sigma^c{}_d \right) \\
& + 2 e^{ab} \left( d\omega_{ab} + 2 \epsilon_{acd} \left\{ \omega^c \omega^d{}_b + \frac{1}{\ell^2} \sigma^c \omega^d{}_b + \frac{1}{\ell^4} \sigma^c \sigma^d{}_b \right\} + \frac{1}{\ell^2} d\sigma_{ab} \right. \\
& \left. \left. + \frac{2}{\ell^2} \epsilon_{acd} \left\{ \sigma^c \omega^d{}_b + 2 \omega^c \sigma^d{}_b + \frac{2}{\ell^2} \sigma^c \sigma^d{}_b \right\} \right) \right] \\
& + 2\alpha_2 \left[ \frac{1}{2} e^a \left( de_a + \epsilon_{abc} \omega^b e^c + \frac{1}{\ell^2} \epsilon_{abc} \sigma^b e^c \right) \right. \\
& + \sigma^a \left( d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \omega^c + \frac{1}{\ell^2} \left\{ d\sigma_a + \frac{1}{2} \epsilon_{abc} \omega^b \sigma^c + \frac{1}{3\ell^2} \epsilon_{abc} \sigma^b \sigma^c \right\} \right) \\
& + 2 \left( \omega^{ab} d\sigma_{ab} + \frac{1}{2\ell^2} \sigma^{ab} d\sigma_{ab} + \epsilon_{abc} \sigma^a \omega^{be} \omega^c{}_e + \frac{2}{\ell^2} \epsilon_{abc} \sigma^a \omega^{be} \sigma^c{}_e \right. \\
& \left. + \frac{1}{\ell^4} \epsilon_{abc} \sigma^a \sigma^{be} \sigma^c{}_e + 2 \epsilon_{abc} \omega^a \sigma^{be} \omega^c{}_e + \frac{1}{\ell^2} \epsilon_{abc} \omega^a \sigma^{be} \sigma^c{}_e \right) \\
& \left. + e^{ab} \left( de_{ab} + 2 \epsilon_{acd} \omega^c e^d{}_b + \frac{2}{\ell^2} \epsilon_{acd} \sigma^c e^d{}_b + 4 \epsilon_{acd} e^c \omega^d{}_b + \frac{4}{\ell^2} \epsilon_{acd} e^c \sigma^d{}_b \right) \right] \,.
\end{aligned}
\tag{4.3.4}
$$

and it describes the coupling of spin-3 fields to the *AdS*-Lorentz gravity. Note that the absence of abelian generators in this new HS symmetry gives terms proportional to $\alpha_1$ and $\alpha_2$ that are different to the ones that appear in the spin-3 Maxwell case.

The field equations in this case are given by

$$\mathcal{T}^a \equiv de^a + \epsilon^{abc}\left(\omega_b e_c + \frac{1}{\ell^2}\sigma_b e_c\right) + 4\epsilon^{abc}\left(e^{bd}\omega_c{}^d + \frac{1}{\ell^2}e^{bd}\sigma_c{}^d\right) = 0\,, \tag{4.3.5}$$

$$\mathcal{R}^a \equiv d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c + 2\epsilon^{abc}\omega_{bd}\omega_c{}^d = 0\,, \tag{4.3.6}$$

$$\begin{aligned}\mathcal{F}^a \equiv\, & d\sigma a + \epsilon^{abc}\left(\omega_b\sigma_c + \frac{1}{\ell^2}\sigma_b\sigma_c + \frac{1}{2}e_b e_c\right) \\ & + 2\epsilon^{abc}\left(2\omega_{bd}\sigma_c{}^d + \frac{1}{\ell^2}\sigma_{bd}\sigma_c{}^d + e_{bd}e_c{}^d\right) = 0\,,\end{aligned} \tag{4.3.7}$$

$$\mathcal{T}^{ab} \equiv de^{ab} + \epsilon^{cd(a|}\left(\omega_c e_d{}^{|b)} + \frac{1}{\ell^2}\sigma_c e_d{}^{|b)} + e_c\omega_d{}^{|b)} + \frac{1}{\ell^2}e_c\sigma_d{}^{|b)}\right) = 0\,, \tag{4.3.8}$$

$$\mathcal{R}^{ab} \equiv d\omega^{ab} + \epsilon^{cd(a|}\omega_c\omega_d{}^{|b)} = 0\,, \tag{4.3.9}$$

$$\begin{aligned}\mathcal{F}^{ab} \equiv\, & d\sigma^{ab} \\ & + \epsilon^{cd(a|}\left(\omega_c\sigma_d{}^{|b)} + \frac{1}{\ell^2}\sigma_c\sigma_d{}^{|b)} + \sigma_c\omega_d{}^{|b)} + \frac{1}{\ell^2}\sigma_c\sigma_d{}^{|b)} + e_c e_d{}^{|b)}\right) = 0\,,\end{aligned} \tag{4.3.10}$$

where the presence of non-abelian generators also modifies the gauge transformations with respect to the spin-3 Maxwell case. Specifically, the spin-2 gauge transformations take the form

$$\delta\omega^a = D_\omega\Lambda^a - 4\sigma\epsilon^{abc}\omega_{bd}\Lambda_c{}^d\,, \tag{4.3.11}$$

$$\begin{aligned}\delta e^a =\, & D_\omega\xi^a + \frac{1}{\ell^2}\epsilon^{abc}\sigma_b\xi_c - \epsilon^{abc}\Lambda_b e_c - \frac{1}{\ell^2}\epsilon^{abc}\chi_b e_c \\ & -4\sigma\epsilon^{abc}\left(\omega_{bd}\xi_c{}^d + \frac{1}{\ell^2}\sigma_{bd}\xi_c{}^d + e_{bd}\Lambda_c{}^d + \frac{1}{\ell^2}e_{bd}\chi_c{}^d\right),\end{aligned} \tag{4.3.12}$$

$$\delta\sigma^a = D_\omega\chi^a + \frac{1}{\ell^2}\epsilon^{abc}\sigma_b\chi_c - \epsilon^{abc}\xi_b e_c - 4\sigma\epsilon^{abc}\left(e_{bd}\xi_c{}^d + \sigma_{bd}\chi_c{}^d\right)\,, \tag{4.3.13}$$

while the spin-3 gauge transformations are given by

$$\delta\omega^{ab} = d\Lambda^{ab} + \epsilon^{cd(a|}\omega_c\Lambda_d^{\ |b)} + \epsilon^{cd(a}\omega^{b)}_{\ c}\Lambda_d\,, \tag{4.3.14}$$

$$\begin{aligned}\delta e^{ab} &= d\xi^{ab} + \epsilon^{cd(a|}\omega_c\xi_d^{\ |b)} + \epsilon^{cd(a}\omega^{b)}_{\ c}\xi_d + \frac{1}{\ell^2}\epsilon^{cd(a|}\sigma_c\xi_d^{\ |b)} + \frac{1}{\ell^2}\epsilon^{cd(a}\sigma^{b)}_{\ c}\xi_d \\ &\quad + \epsilon^{cd(a|}e_c\Lambda_d^{\ |b)} + \frac{1}{\ell^2}\epsilon^{cd(a|}e_c\chi_d^{\ |b)} + \epsilon^{cd(a}e^{b)}_{\ c}\Lambda_d + \frac{1}{\ell^2}\epsilon^{cd(a}e^{b)}_{\ c}\chi_d\,,\end{aligned} \tag{4.3.15}$$

$$\begin{aligned}\delta\sigma_{ab} &= d\chi^{ab} + \epsilon^{cd(a|}\omega_c\chi_d^{\ |b)} + \epsilon^{cd(a}\omega^{b)}_{\ c}\chi_d + \frac{1}{\ell^2}\epsilon^{cd(a|}\sigma_c\chi_d^{\ |b)} + \frac{1}{\ell^2}\epsilon^{cd(a}\sigma^{b)}_{\ c}\chi_d \\ &\quad + \epsilon^{cd(a|}e_c\xi_d^{\ |b)} + \epsilon^{cd(a}e^{b)}_{\ c}\xi_d\,.\end{aligned} \tag{4.3.16}$$

Note that the limit $\ell \to \infty$ properly reproduces the spin-3 Maxwell field equations and gauge transformations.

# Capítulo 5

# Asymptotic structure of three-dimensional gravity

## 5.1. Introduction and state of the art

The notion of the *principle of least action* of classical mechanics is the cornerstone of modern theoretical physics. This fundamental concept states that the action remains stationary under arbitrary variations of the dynamical variables while keeping the initial and final conditions fixed.

The concept of asymptotic symmetries in GR, which corresponds to those gauge transformations that map the field configurations into themselves, plays a fundamental role. Indeed, they are essential in order to have a suitable definition of the canonical generators that define the global charges of any physical theory. A deep understanding of a physical theory requires having control of its asymptotic structure. This is described by the asymptotic behaviour of the physical fields, far away from any physical process. The case of three-dimensional gravity with negative cosmological constant has been extensively studied due to the existence of the Bañados-Teitelboim-Zanelli (BTZ) black hole and its extraordinarily asymptotic structure [94]. Regarding the asymptotic structure, the groundbreaking work by D. Brown and M. Henneaux is regarded as a precursor to the AdS/CFT correspondence. In particular, they demonstrated that the asymptotic symmetry algebra at spacelike infinity of anti-de Sitter spaces is represented by two copies of the infinite-dimensional Virasoro algebra, which is associated with the conformal algebra in two dimensions. The Brown-Henneaux results were important precursors of the AdS/CFT correspondence. In the case of vanishing cosmological constant, the $\mathfrak{bms}_3$ algebra is found as the asymptotic symmetry of Einstein gravity at null infinity:

$$\begin{aligned} [\mathcal{J}_m, \mathcal{J}_n] &= (m-n)\mathcal{J}_{m+n} + \frac{c_1}{12}(m^3 - m)\delta_{m+n,0}\,, \\ [\mathcal{J}_m, \mathcal{P}_n] &= (m-n)\mathcal{P}_{m+n} + \frac{c_2}{12}(m^3 - m)\delta_{m+n,0}\,, \\ [\mathcal{P}_m, \mathcal{P}_n] &= 0\,. \end{aligned} \tag{5.1.1}$$

On the other hand, the **Virasoro algebra** is generated by $\ell_n$ for $n \in \mathbb{Z}$ and the central charge $c$. It corresponds to the central extension of the Witt algebra

$$\begin{aligned} [\ell_n, \ell_n] &= (m-n)\ell_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}\,, \\ [\ell_m, c] &= 0\,. \end{aligned} \tag{5.1.2}$$

The Virasoro symmetry appears in any physical system with conformal invariance defined on a two-dimensional space.

This algebra is given by the semi-direct sum of the infinitesimal diffeomorphism on the circle with an Abelian ideal of super translations. The Poincaré algebra is a finite subalgebra of the $\mathfrak{bms}_3$ one, formed by the generators $\{\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_{-1}, \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_{-1}\}$. This can be made explicit in terms of generators $\{J_a, P_a\}$ obtained through the following change of basis

$$\begin{aligned} &\mathcal{J}_{-1} = -2J_0\,, \quad \mathcal{J}_0 = J_2\,, \quad \mathcal{J}_1 = J_1\,, \\ &\mathcal{P}_{-1} = -2P_0\,, \quad \mathcal{P}_0 = P_2\,, \quad \mathcal{P}_1 = P_1\,. \end{aligned} \tag{5.1.3}$$

As will be explained in more detail in the next section, the $\mathfrak{bms}_3$ algebra can be obtained from two copies of the Virasoro algebra through an Inönü-Wigner contraction.

## 5.2. Asymptotic structure of General Relativity in 3D

Three-dimensional gravity possesses an extraordinarily rich asymptotic structure. The three-dimensional result has been very relevant and indeed it can be seen as the precursor of the so called AdS/CFT correspondence [95] and also has served to recover the Bekenstein-Hawking entropy for the BTZ black hole [96,97] through a microscopical derivation [98].

The Brown-Henneaux boundary conditions for three-dimensional gravity with a negative cosmological constant, formulated in terms of gauge fields [99], are defined as

$$A^{\pm}_{\phi} = L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^{\pm} L_{\mp 1} + \mathcal{O}\left(\frac{1}{r}\right), \qquad A^{\pm}_{r} = \mathcal{O}\left(\frac{1}{r}\right), \tag{5.2.1}$$

where $\mathcal{L}^{\pm}$ depends on the time $t$ and the angular coordinate $\phi$. Notably, the radial coordinate is fully determined by a gauge transformation of the connection.

$$A^{\pm} = g_{\pm}^{-1} a^{\pm} g_{\pm} + g_{\pm}^{-1} d g_{\pm}\,, \tag{5.2.2}$$

where the group element is defined as $g_{\pm} = e^{\pm r L_0}$. In this context, the dynamical fields correspond to the leading terms of the asymptotic behavior (5.2.1), specifically:

$$a^{\pm}_{\phi} = L_{\pm 1} - \frac{2\pi}{k}\mathcal{L}^{\mp}_{\pm}, \qquad a^{\pm}_{r} = 0\,. \tag{5.2.3}$$

Consequently, the relevant components of the dynamical field are confined to the angular components of the gauge connection $a$. The asymptotic structure of the dynamical fields must remain invariant under gauge transformations of the form

$$\delta a^{\pm}_{\phi} = \partial_{\phi}\lambda^{\pm} + \left[a^{\pm}_{\phi}, \lambda^{\pm}\right]. \tag{5.2.4}$$

Thus, the Lie-algebra valued parameters have to be given by

$$\lambda^{\pm}[\epsilon_{\pm}] = \epsilon_{\pm} L_{\pm 1} \mp \epsilon'_{\pm} L_0 + \frac{1}{2}\left(\epsilon''_{\pm} - \frac{4\pi}{k}\epsilon_{\pm}\mathcal{L}^{\pm}\right) L_{\mp 1}\,, \tag{5.2.5}$$

provided the functions $\mathcal{L}^{\pm}$ transform as

$$\delta\mathcal{L}_{\pm} = \epsilon_{\pm}\mathcal{L}^{\pm\prime} + 2\epsilon_{\pm}'\mathcal{L}^{\pm} - \frac{k}{4\pi}\epsilon_{\pm}''' \,, \tag{5.2.6}$$

where $\epsilon_{\pm} = \epsilon_{\pm}(t, \phi)$ are arbitrary functions and primes denote derivatives with respect to $\phi$.

At spatial infinity, the time evolution of the dynamical fields of the connection $a_{\phi}^{\pm}$ corresponds to a gauge transformation with gauge parameters given by the Lagrange multipliers $a_t^{\pm}$. The most general Lagrange multipliers that preserve the asymptotic conditions (5.2.1) are given by [100].

$$a_t^{\pm} = \lambda^{\pm}[\xi_{\pm}] \,. \tag{5.2.7}$$

Here, $\xi_{\pm}$ are also arbitrary functions of $t$ and $\phi$, which are assumed to be fixed at the boundary. To ensure the consistency of the Lagrange multipliers under gauge transformations, the fields $\mathcal{L}^{\pm}$ must satisfy the following field equations at the asymptotic region

$$\dot{\mathcal{L}}_{\pm} = \xi_{\pm}\mathcal{L}^{\pm\prime} + 2\chi_{\pm}'\mathcal{L}^{\pm} - \frac{k}{4\pi}\xi_{\pm}''' \,, \tag{5.2.8}$$

while the parameters of the asymptotic symmetries fulfill

$$\dot{\epsilon}_{\pm} = \epsilon_{\pm}'\xi_{\pm} - \epsilon_{\pm}\xi_{\pm}' \,. \tag{5.2.9}$$

These conditions are necessary in order to ensure the conservation of the global charges.

Replacing the gauge parameters (5.2.5) and the asymptotic conditions (5.2.1) in the variation of the canonical generators $\delta Q(\Lambda) = -\frac{k}{2\pi}\int_{\Sigma} \langle \Lambda \delta A_{\phi} \rangle \, d\phi$ , we get that

$$\delta Q[\lambda] = \delta Q[\lambda^{+}] - \delta Q[\lambda^{-}] \,, \tag{5.2.10}$$

which can be readily integrated as

$$Q[\lambda^{\pm}] = -\int \epsilon_{\pm}\mathcal{L}_{\pm} d\phi \,. \tag{5.2.11}$$

Since the Poisson brackets fulfill

$$\left\{Q[\lambda_1^{\pm}],Q[\lambda_2^{\pm}]\right\} = \delta_{\lambda_2^{\pm}} Q[\lambda_1^{\pm}]\,, \tag{5.2.12}$$

The algebra of the canonical generators can be directly derived from the transformation properties of the fields (5.2.6). By expanding in Fourier modes as $X = \frac{1}{2\pi}\sum_n X_n e^{in\phi}$ , the Poisson brackets simplify to two copies of the Virasoro algebra, both sharing the same central charge $c = 6k = 3\ell/2G$. Explicitly, the algebras take the following form.

$$i\left\{\mathcal{L}_m^{\pm},\mathcal{L}_n^{\pm}\right\} = (m-n)\mathcal{L}_{m+n}^{\pm} + \frac{k}{2}m^3\delta_{m+n,0}\,, \tag{5.2.13}$$

which coincide with the asymptotic symmetry algebra found in the metric formulation.

Hereon we are going to review the asymptotic structure in the case of three-dimensional asymptotically flat spacetimes. The asymptotic conditions in terms of gauge fields were first proposed in the context of flat higher spin gravity in [78, 79], and in the context of flat supergravity in [101]. For this purpose we are going to relabel the Poincaré generators according to

$$\begin{aligned} \hat{J}_{-1} &= -2J_0\,, \quad \hat{J}_1 = J_1\,, \quad \hat{J}_0 = J_2\,, \\ \hat{P}_1 &= -2P_0\,, \quad \hat{P}_1 = P_1\,, \quad \hat{P}_0 = P_2\,, \end{aligned}$$

such that the non-vanishing commutation relations of the Poincaré algebra can be written as

$$\begin{aligned} \left[\hat{J}_m,\hat{J}_n\right] &= (m-n)\hat{J}_{m+n}\,, \\ \left[\hat{J}_m,\hat{P}_n\right] &= (m-n)\hat{P}_{m+n}\,. \end{aligned} \tag{5.2.14}$$

Then, with this relabel the asymptotic behavior in this case is given by

$$a_\phi = \hat{J}_1 - \frac{\pi}{k}\left(\mathcal{J}\hat{P}_{-1} + \mathcal{P}\hat{P}_{-1}\right)\,, \tag{5.2.15}$$

where the radial coordinate can be switched on by gauge transformation with group element $g = e^{\frac{r}{2}\hat{P}_{-1}}$. The functions $\mathcal{J}$ and $\mathcal{P}$ depend on the null coordinate $u$ and the angular coordinate $\phi$. The asymptotic form of the dynamical fields, in this case is preserved under the action of gauge transformations spanned by the following parameter

$$\begin{aligned}\lambda[T,Y] &= T\hat{P}_1 + Y\hat{J}_1 - T'\hat{P}_0 - Y'\hat{J}_0 \\ &\quad - \frac{1}{2}\left(\frac{2\pi}{k}Y\mathcal{P} - Y''\right)\hat{J}_{-1} - \frac{\pi}{k}\left(T\mathcal{P} + Y\mathcal{J} - \frac{k}{2\pi}T''\right)\hat{P}_{-1}\,, \end{aligned} \tag{5.2.16}$$

provided the transformation laws of the fields are given by

$$\delta\mathcal{P} = 2\mathcal{P}Y' + \mathcal{P}'Y - \frac{k}{2\pi}Y'''\,, \tag{5.2.17}$$

$$\delta\mathcal{J} = 2\mathcal{J}Y' + \mathcal{J}'Y + 2\mathcal{P}T' + \mathcal{P}'T - \frac{k}{2\pi}T'''\,, \tag{5.2.18}$$

with $T(u,\phi)$ and $Y(u,\phi)$ arbitrary functions. The Lagrange multiplier reads [100]

$$a_u = \lambda[\mu_\mathcal{P}, \mu_\mathcal{J}]\,, \tag{5.2.19}$$

where $\mu_\mathcal{P}, \mu_\mathcal{J}$ also stand for arbitrary functions of $u, \phi$ and they are assumed to be fixed at the boundary. Consistency of preserving the asymptotic form of the Lagrange multiplier now leads to the following field equations

$$\dot{\mathcal{P}} = 2\mathcal{P}\mu_\mathcal{J}' + \mathcal{P}'\mu_\mathcal{J} - \frac{k}{2\pi}\mu_\mathcal{J}'''\,, \tag{5.2.20}$$

$$\dot{\mathcal{J}} = 2\mathcal{J}\mu_\mathcal{J}' + \mathcal{J}'\mu_\mathcal{J} + 2\mathcal{P}\mu_\mathcal{P}' + \mathcal{P}'\mu_\mathcal{P} - \frac{k}{2\pi}\mu_\mathcal{P}'''\,, \tag{5.2.21}$$

which have to be fulfilled in the asymptotic region. In turn, the parameters of the transformation satisfy the following conditions

$$\dot{Y} = \mu_\mathcal{J}Y' - \mu_\mathcal{J}'Y\,, \tag{5.2.22}$$

$$\dot{T} = \mu_\mathcal{J}T' - \mu_\mathcal{J}'T + \mu_\mathcal{P}Y' - \mu_\mathcal{P}'Y\,. \tag{5.2.23}$$

The canonical generator in this case is also easily integrated, which reads

$$Q[T,Y] = -\int (T\mathcal{P} + Y\mathcal{P})d\phi\,. \tag{5.2.24}$$

Expanding in Fourier modes, the nonvanishing components of the Poisson brackets are given by

$$
\begin{aligned}
i\{\mathcal{J}_m, \mathcal{J}_n\} &= (m-n)\mathcal{J}_{m+n}\,, &\quad (5.2.25)\\
i\{\mathcal{J}_m, \mathcal{P}_n\} &= (m-n)\mathcal{P}_{m+n} + k m^3 \delta_{m+n,0}\,, &\quad (5.2.26)
\end{aligned}
$$

which corresponds to the infinite-dimensional $\mathfrak{bms}_3$ algebra with a central charge $c = 3/G$ found in [48, 49].

Let us point out that by making the following change of basis on the generators of the Virasoro algebras [102](5.2.13)

$$
\mathcal{P}_n = \frac{1}{\ell}(\mathcal{L}^+_n + \mathcal{L}^-_{-n})\,, \qquad \mathcal{J}_n = \mathcal{L}^+_n - \mathcal{L}^-_{-n}\,, \tag{5.2.27}
$$

and rescaling the AdS level according to $k \to k\ell$, in the flat limit $\ell \to \infty$, the $\mathfrak{bms}_3$ algebra is recovered.

## 5.3. Asymptotic symmetries of 3D Maxwell Chern-Simons gravity

In this section we briefly review the results of [43], where the authors found the asymptotic symmetry algebra of the three-dimensional Maxwell CS gravity theory. To start with, we propose the following behavior of the gauge fields at the boundary

$$
\begin{aligned}
A \;=\; & \left(-dr + \frac{1}{2}\mathcal{M}du + \frac{1}{2}\mathcal{N}d\phi\right) P_0 + du\, P_1 + r d\phi\, P_2 + \frac{1}{2}\mathcal{M}d\phi\, J_0 + d\phi\, J_1 \\
& + \frac{1}{2}\left(\mathcal{N}du + \mathcal{F}d\phi - r^2 d\phi\right) Z_0 + r du\, Z_2\,.
\end{aligned} \tag{5.3.1}
$$

For the moment, let us assume that the functions $\mathcal{M}$, and $\mathcal{N}$ and $\mathcal{F}$ depend on all boundary coordinates $x^i = (u, \phi)$, and we shall set them on-shell later. The radial dependence of the gauge field $A$ can be eliminated by gauge transformation.

$$
A = h^{-1}dh + h^{-1}ah\,. \tag{5.3.2}
$$

Using the identity $h^{-1}dh = -drP_0$ and the Baker-Campbell-Hausdorff formula, we obtain

$$h^{-1}ah = a + rdu\, Z_2 + rd\phi\, P_2 - r^2 d\phi\, Z_0\,. \tag{5.3.3}$$

The final effect is that the radial dependence from the gauge field $A$ is dropped out and the new gauge field $a$ becomes the asymptotic field,

$$a = \frac{1}{2}\left(\mathcal{M}du + \mathcal{N}d\phi\right) P_0 + du\, P_1 + \frac{1}{2}\mathcal{M}d\phi\, J_0 + d\phi\, J_1 + \frac{1}{2}\left(\mathcal{N}du + \mathcal{F}d\phi\right) Z_0\,. \tag{5.3.4}$$

with the angular component given by

$$a_\phi = \frac{1}{2}\mathcal{N}\, P_0 + \frac{1}{2}\mathcal{M}\, J_0 + J_1 + \frac{1}{2}\mathcal{F}\, Z_0\,. \tag{5.3.5}$$

The asymptotic symmetry is a residual symmetry, that leaves the asymptotic conditions (5.3.1) invariant. In order to find it, we consider gauge parameters of the form

$$\Lambda = h^{-1}\lambda h\,, \qquad \lambda = \xi^a(u,\phi)\, P_a + \Lambda^a(u,\phi)\, J_a + \chi^a(u,\phi)\, Z_a\,. \tag{5.3.6}$$

Gauge transformations of the full connection $A$ with gauge parameters $\Lambda$ lead to $r$-independent gauge transformations of $a$ with gauge parameter $\lambda$. Now, we require that the transformed field, $a + D\lambda$, and the original one, $a$, have the same form (5.3.4). A change of the boundary field (5.3.4) is given by

$$\begin{aligned}\delta_\lambda a \;&=\; \frac{1}{2}\left[\delta_\lambda\mathcal{M}(u,\phi)\, du + \frac{1}{2}\delta_\lambda\mathcal{N}(u,\phi)\, d\phi\right] P_0 + \frac{1}{2}\delta_\lambda\mathcal{M}(u,\phi)\, d\phi\, J_0 \\ &\quad + \frac{1}{2}\left(\delta_\lambda\mathcal{N}(u,\phi)\, du + \delta_\lambda\mathcal{F}(u,\phi)\, d\phi\right) Z_0\,.\end{aligned} \tag{5.3.7}$$

On the other hand, this change must to be equal to the gauge transformation $\delta_\lambda a = D\lambda$. The angular component $a_\phi$ is left invariant for the Lie-algebra-valued parameter $\lambda = \lambda(y, f, h)$ of the form

$$\begin{aligned}\lambda = &\left(\frac{\mathcal{M}}{2}f + \frac{\mathcal{N}}{2}y - f''\right)P_0 + fP_1 - f'P_2 + \left(\frac{\mathcal{M}}{2}y - y''\right)J_0 + yJ_1 - y'J_2 \\ &+ \left(\frac{1}{2}\mathcal{M}h + \frac{1}{2}\mathcal{F}y + \frac{1}{2}\mathcal{N}f - h''\right)Z_0 + hZ_1 - h'Z_2\,,\end{aligned} \tag{5.3.8}$$

provided that the functions $\mathcal{M}, \mathcal{N}$ and $\mathcal{F}$ transform as follows:

$$\begin{aligned} \delta\mathcal{M} &= \mathcal{M}'y + 2\mathcal{M}y' - 2y''' \,, \\ \delta\mathcal{N} &= \mathcal{M}'f + 2\mathcal{M}f' - 2f''' + \mathcal{N}'y + 2\mathcal{N}y' \,, \\ \delta\mathcal{F} &= \mathcal{M}'h + 2\mathcal{M}h' - 2h''' + \mathcal{N}'f + 2\mathcal{N}f' + \mathcal{F}'y + 2\mathcal{F}y' \,, \end{aligned} \tag{5.3.9}$$

where $y = y(\phi, u)$, $f = f(\phi, u)$, and $h = h(\phi, u)$ are arbitrary functions defined on $\partial\Sigma$ and the prime denotes the derivative with respect to the $\phi$ coordinate.

The asymptotic symmetries along time will be preserved whenever the Lagrange multiplier is $A_u = h^{-1} a_u h$, with

$$a_u = \lambda[\mu, \xi, \vartheta] \,, \tag{5.3.10}$$

where $\mu, \xi$, and $\vartheta$ are arbitrary functions of $(u, \phi)$ which are assumed to be fixed at the boundary [68, 100]. The time evolution of the gauge fields in the asymptotic region is given by the following conditions

$$\begin{aligned} \dot{\mathcal{M}} &= \mathcal{M}'\mu + 2\mathcal{M}\mu' - 2\mu''' \,, \\ \dot{\mathcal{N}} &= \mathcal{M}'\xi + 2\mathcal{M}\xi' + \mathcal{N}'\mu + 2\mathcal{N}\mu' - 2\xi''' \,, \\ \dot{\mathcal{F}} &= \mathcal{M}'\vartheta + 2\mathcal{M}\vartheta' + \mathcal{N}'\xi + 2\mathcal{N}\xi' + \mathcal{F}'\mu + 2\mathcal{F}\mu' - 2\vartheta''' \,, \end{aligned} \tag{5.3.11}$$

where dot corresponds to the derivative with respect to $u$.

In summary, the asymptotic behavior is described by gauge fields of the form given in (5.3.2), where the components $a_\phi$ and $a_u$ of the asymptotic gauge field $a$ are given by (5.3.5) and (5.3.10), respectively.

Let us now compute the charge algebra of the theory. As discussed in Ref.[97], the algebra is spanned by the conserved charges $Q[\Lambda]$, which, as mentioned before, are on-shell equivalent to diffeomorphism charges of the form $Q[\xi] = \frac{k}{2\pi}\int_{\partial\Sigma} \langle A \iota_\xi A \rangle$ with $\Lambda = \iota_\xi A$. Furthermore, the charge algebra in representation of Poisson brackets can be obtained using the Regge-Teitelboim method [103] directly from the transformation law

$$\delta_{\Lambda_2} Q[\Lambda_1] = \{Q[\Lambda_1], Q[\Lambda_2]\} \,. \tag{5.3.12}$$

On the other hand, the variation of the charge in CS theory is given by [97]

$$\delta Q[\Lambda] = \frac{k}{2\pi} \int\limits_{\partial\Sigma} \langle \Lambda \delta A \rangle \ . \tag{5.3.13}$$

After applying the gauge transformation (5.3.2) which introduces the asymptotic field (5.3.5), and using (5.3.6), we get

$$\delta Q[\lambda] = \frac{k}{2\pi} \int d\phi \, \langle \lambda \delta a_\phi \rangle \ . \tag{5.3.14}$$

Since the non-vanishing components of the invariant tensor for the Maxwell algebra is known, as well as the gauge field $a$, after a straightforward calculation one arrives to

$$\delta Q[y,f,h] = \frac{k}{4\pi} \int d\phi \left[ y \left( \alpha_2 \delta\mathcal{F} + \alpha_0 \delta\mathcal{M} + \delta\mathcal{N} \right) + f \left( \alpha_2 \delta\mathcal{N} + \delta\mathcal{M} \right) + \alpha_2 h \delta\mathcal{M} \right] \ . \tag{5.3.15}$$

The functions $y$, $f$ and $h$ do not depend on the fields, thus it is trivial to integrate the variation out, finding

$$Q[y,f,h] = \frac{k}{4\pi} \int d\phi \left[ y \left( \alpha_2 \mathcal{F} + \alpha_0 \mathcal{M} + \mathcal{N} \right) + f \left( \alpha_2 \mathcal{N} + \mathcal{M} \right) + \alpha_2 h \mathcal{M} \right] \ . \tag{5.3.16}$$

Now we define the asymptotic charges which correspond to the independent terms in (5.3.16),

$$\begin{aligned} j[y] &= \frac{k}{4\pi} \int d\phi \, y \left( \alpha_2 \mathcal{F} + \mathcal{N} + \alpha_0 \mathcal{M} \right) \ , \\ p[f] &= \frac{k}{4\pi} \int d\phi \, f \left( \alpha_2 \mathcal{N} + \mathcal{M} \right) \ , \\ z[h] &= \frac{k}{4\pi} \int d\phi \, \alpha_2 h \mathcal{M} \, . \end{aligned} \tag{5.3.17}$$

Using (5.3.12), they give rise to the centrally extended Poisson algebra

$$
\begin{aligned}
\{j[y_1], j[y_2]\} &= j\,[[y_1, y_2]] - \frac{k\alpha_0}{2\pi}\int d\phi\, y_1 y_2''' \,, \\
\{j[y], p[f]\} &= p\,[[y, f]] - \frac{k}{2\pi}\int d\phi\, y f''' \,, \\
\{j[y], z[h]\} &= z\,[[y, h]] - \frac{k\alpha_2}{2\pi}\int d\phi\, y h''' \,, \\
\{p[f_1], p[f_2]\} &= z\,[[f_1, f_2]] - \frac{k\alpha_2}{2\pi}\int d\phi\, f_1 f_2''' \,, \\
\{p[f], z[h]\} &= 0\,, \\
\{z[h_1], z[h_2]\} &= 0\,,
\end{aligned}
\tag{5.3.18}
$$

where here $[x, y] = xy' - yx'$, stands for the Lie bracket of the vector field components $x(\phi)$ and $y\,(\phi)$ on $\partial\Sigma$. The result describes a deformed $\mathfrak{bms}_3$ algebra, as expected, which is an infinite-dimensional enhancement of the Maxwell algebra [50], with three central charges. It is common to write down the algebra in Fourier modes,

$$
\mathcal{J}_m = j[e^{im\phi}]\,, \qquad \mathcal{P}_m = p[e^{im\phi}]\,, \qquad \mathcal{Z}_m = z[e^{im\phi}]\,, \qquad m \in \mathbb{Z}\,, \tag{5.3.19}
$$

with all $\phi$-dependent functions expanded on the circle $\partial\Sigma$. The corresponding deformed $\mathfrak{bms}_3$ symmetry reads

$$
\begin{aligned}
i\,\{\mathcal{J}_m, \mathcal{J}_n\} &= (m-n)\,\mathcal{J}_{m+n} + \frac{c_1}{12}\, m^3 \delta_{m+n,0}\,, \\
i\,\{\mathcal{J}_m, \mathcal{P}_n\} &= (m-n)\,\mathcal{P}_{m+n} + \frac{c_2}{12}\, m^3 \delta_{m+n,0}\,, \\
i\,\{\mathcal{P}_m, \mathcal{P}_n\} &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\, m^3 \delta_{m+n,0}\,, \\
i\,\{\mathcal{J}_m, \mathcal{Z}_n\} &= (m-n)\,\mathcal{Z}_{m+n} + \frac{c_3}{12}\, m^3 \delta_{m+n,0}\,, \\
i\,\{\mathcal{P}_m, \mathcal{Z}_n\} &= 0\,, \\
i\,\{\mathcal{Z}_m, \mathcal{Z}_n\} &= 0\,,
\end{aligned}
\tag{5.3.20}
$$

where we have used the integral representation of the Kronecker delta $\delta_{mn} = \frac{1}{2\pi}\int d\phi\, e^{i(m-n)\phi}$. In the above classical algebra, the three central charges are associated to three terms in the gravitational action: the standard one $c_2 = 12k$ along the EH term, the gravitational CS one $c_1 = 12k\alpha_0$ and, finally, a new ingredient, a charge along the gravitational Maxwell term with torsion, $c_3 = 12k\alpha_2$.

Note that the Maxwell algebra is a finite subalgebra of (5.3.20) formed by the generators $\{\mathcal{J}_{-1},\mathcal{J}_0,\mathcal{J}_1,\mathcal{P}_{-1},\mathcal{P}_0,\mathcal{P}_1,\mathcal{Z}_{-1},\mathcal{Z}_0,\mathcal{Z}_1\}$ (see [50]).

# Capítulo 6

# Asymptotic structure of gravity coupled to spin-3 fields

In this Chapter, following the methodology used in [74] we derive the asymptotic symmetry algebra for the spin-3 Maxwell CS gravity theory. First, inspired by the boundary conditions we have to impose in the pure Maxwell gravity, we provide the suitable fall-off conditions for the gauge fields at infinity and the gauge transformations preserving our proposed boundary conditions. Then, using the Regge-Teitelboim method [103] we find the charge algebra which, as it is expected, will be a higher spin extension of the $\mathfrak{max}\text{-}\mathfrak{bms}_3$ algebra [43, 50]. The Poisson algebra structure we find is then obtained as a flat limit from a classical centrally extended $\mathcal{W}_3 \oplus \mathcal{W}_3 \oplus \mathcal{W}_3$ algebra.

## 6.1. On the boundary conditions

In our analysis we will consider the BMS gauge where the spacetime manifold is parametrized by the local coordinates $x^\mu = (u, r, \phi)$. Here, $u$ corresponds to the retarded time coordinate, while $r \to \infty$ indicates the boundary region.

To start with, we propose the following behavior of the gauge fields at the boundary

$$A = h^{-1}dh + h^{-1}ah \,, \tag{6.1.1}$$

where the radial dependence is entirely captured by the group element $h = e^{-rP_0}$.

The auxiliary gauge field $a$ has the form

$$a = a_\phi d\phi + a_u du\,, \tag{6.1.2}$$

with the angular component given by

$$a_\phi \;=\; J_1 + \frac{1}{2}\mathcal{N}P_0 + \frac{1}{2}\mathcal{M}J_0 + \frac{1}{2}\mathcal{F}Z_0 + \mathcal{V}P_{00} + \mathcal{X}Z_{00}\,, \tag{6.1.3}$$

where the functions $\mathcal{M}, \mathcal{N}, \mathcal{F}, \mathcal{V}, \mathcal{W}$ and $\mathcal{X}$ are assumed to depend on all boundary coordinates $x^i = (u, \phi)$. Let us note that the form of the angular component generalizes the asymptotic behavior of the gauge field considered in [74, 79] to the presence of the gravitational Maxwell field along its spin-3 counterpart.

As is well-known, the asymptotic symmetries correspond to the set of gauge transformations $\delta A = d\lambda + [A, \lambda]$ that preserve the asymptotic conditions (6.1.1), with

$$\begin{aligned}\lambda = {}& \Lambda^a(u,\phi)J_a + \xi^a(u,\phi)P_a + \chi^a(u,\phi)Z_a \\ & + \Lambda^{ab}(u,\phi)J_{ab} + \xi^{ab}(u,\phi)P_{ab} + \chi^{ab}(u,\phi)Z_{ab}\,.\end{aligned} \tag{6.1.4}$$

Thus, the angular component $a_\phi$ is left invariant for the Lie-algebra-valued parameter $\lambda = \lambda(y, f, h, v, w, g)$ of the form

$$
\begin{aligned}
\lambda = & \left(\frac{\mathcal{M}}{2} f + \frac{\mathcal{N}}{2} y + 4\mathcal{V} v + 4\mathcal{W} w - f''\right) P_0 + f P_1 - f' P_2 \\
& + \left(\frac{\mathcal{M}}{2} y + 4\mathcal{W} v - y''\right) J_0 \\
& + y J_1 - y' J_2 \\
& + \left(\frac{1}{2}\mathcal{M} h + \frac{1}{2}\mathcal{F} y + \frac{1}{2}\mathcal{N} f + 4\mathcal{W} g + 4\mathcal{X} v + 4\mathcal{V} w - h''\right) Z_0 + h Z_1 - h' Z_2 \\
& + \left[\frac{1}{6} w^{(4)} - \frac{2}{3}\mathcal{M} w'' - \frac{7}{12}\mathcal{M}' w' + \frac{1}{12}(3\mathcal{M}^2 - 2\mathcal{M}'') w \right. \\
& \quad \left. - \frac{2}{3}\mathcal{N} v'' - \frac{7}{12}\mathcal{N}' v' + \frac{1}{12}(6\mathcal{M}\mathcal{N} - 2\mathcal{N}'') v + \mathcal{V} y + \mathcal{W} f \right] P_{00} \\
& + (\mathcal{M} w + \mathcal{N} v - w'') P_{01} + \left(\frac{1}{3} w''' - \frac{5}{6}\mathcal{M} w' - \frac{1}{3}\mathcal{M}' w - \frac{5}{6}\mathcal{N} v' - \frac{1}{3}\mathcal{N}' v\right) P_{02} \\
& + w P_{11} - w'' P_{12} \\
& + \left[\frac{v^{(4)}}{6} - \frac{2}{3}\mathcal{M} v'' - \frac{7}{12}\mathcal{M}' v' + \frac{1}{12}(3\mathcal{M}^2 - 2\mathcal{M}'') v + \mathcal{W} y\right] J_{00} \\
& + (\mathcal{M} v - v'') J_{01} + \left(\frac{v'''}{3} - \frac{5}{6}\mathcal{M} v' - \frac{1}{3}\mathcal{M}' v\right) J_{02} + v J_{11} - v' J_{12} \\
& + \left[\frac{1}{6} g^{(4)} - \frac{2}{3}\mathcal{M} g'' - \frac{7}{12}\mathcal{M}' g' + \frac{1}{12}(3\mathcal{M}^2 - 2\mathcal{M}'') g + \mathcal{V} f + \mathcal{W} h + \mathcal{X} y \right. \\
& - \frac{2}{3}\mathcal{F} v'' - \frac{7}{12}\mathcal{F}' v' \\
& \left. + \frac{1}{12}(6\mathcal{F}\mathcal{M} + 3\mathcal{N}^2 - 2\mathcal{F}'') v - \frac{2}{3}\mathcal{N} w'' - \frac{7}{12}\mathcal{N}' w' + \frac{1}{12}(6\mathcal{M}\mathcal{N} - 2\mathcal{N}'') w \right] Z_{00} \\
& + (\mathcal{M} g + \mathcal{F} v + \mathcal{N} w - g'') Z_{01} \\
& + \left(\frac{1}{3} g''' - \frac{5}{6}\mathcal{M} g' - \frac{1}{3}\mathcal{M}' g - \frac{5}{6}\mathcal{F} v' - \frac{1}{3}\mathcal{F}' v - \frac{5}{6}\mathcal{N} w' - \frac{1}{3}\mathcal{N}' w\right) Z_{02} \\
& + g Z_{11} - g' Z_{12} ,
\end{aligned}
\tag{6.1.5}
$$

provided the functions $\mathcal{M}, \mathcal{N}, \mathcal{F}, \mathcal{V}, \mathcal{W}, \mathcal{X}$ transform as follows

$$
\begin{aligned}
\delta\mathcal{M} &= \mathcal{M}'y + 2\mathcal{M}y' - 2y''' + 4(2\mathcal{W}'v + 3\mathcal{W}v')\,,\\
\delta\mathcal{N} &= \mathcal{M}'f + 2\mathcal{M}f' + \mathcal{N}'y + 2\mathcal{N}y' - 2f''' + 4(2\mathcal{W}'w + 3\mathcal{W}w')\\
&\quad +4(2\mathcal{V}'v + 3\mathcal{V}v')\,,\\
\delta\mathcal{F} &= \mathcal{M}'h + 2\mathcal{M}h' + \mathcal{N}'f + 2\mathcal{N}f' + \mathcal{F}'y + 2\mathcal{F}y' - 2h'''\\
&\quad +4(2\mathcal{W}'g + 3\mathcal{W}g') + 4(2\mathcal{V}'w + 3\mathcal{V}w') + 4(2\mathcal{X}'v + 3\mathcal{X}v')\,,\\
\delta\mathcal{W} &= \mathcal{W}'y + 3\mathcal{W}y' + \frac{1}{12}(8\mathcal{M}\mathcal{M}' - 2\mathcal{M}''')v + \frac{1}{12}\big(8\mathcal{M}^2 - 9\mathcal{M}''\big)v'\\
&\quad -\frac{5}{4}\mathcal{M}'v'' - \frac{5}{6}\mathcal{M}v''' + \frac{1}{6}v^{(5)}\,,\\
\delta\mathcal{V} &= \mathcal{W}'f + 3\mathcal{W}f' + 3\mathcal{V}y' + \frac{1}{12}(8\mathcal{N}\mathcal{M}' + 8\mathcal{M}\mathcal{N}' - 2\mathcal{N}''')v\\
&\quad +\frac{1}{12}(16\mathcal{M}\mathcal{N} - 9\mathcal{N}'')v' - \frac{5}{4}\mathcal{N}'v'' - \frac{5}{6}\mathcal{N}v'''\\
&\quad +\frac{1}{12}(8\mathcal{M}\mathcal{M}' - 2\mathcal{M}''')w + \frac{1}{12}(8\mathcal{M}^2 - 9\mathcal{M}'')w' - \frac{5}{4}\mathcal{M}'w''\\
&\quad -\frac{5}{6}\mathcal{M}w''' + \frac{1}{6}w^{(5)}\\
\delta\mathcal{X} &= \mathcal{W}'h + 3\mathcal{W}h' + \mathcal{V}'f + 3\mathcal{V}f' + \mathcal{X}'y + 3\mathcal{X}y'\\
&\quad +\frac{1}{12}(8\mathcal{M}\mathcal{F}' + 8\mathcal{F}\mathcal{M}' + 8\mathcal{N}\mathcal{N}' - 2\mathcal{F}''')v\\
&\quad +\frac{1}{12}\big(16\mathcal{F}\mathcal{M} + 8\mathcal{N}^2 - 9\mathcal{F}''\big)v' - \frac{5}{4}\mathcal{F}'v'' - \frac{5}{6}\mathcal{F}v'''\\
&\quad +\frac{1}{12}(8\mathcal{N}\mathcal{M}' + 8\mathcal{M}\mathcal{N}' - 2\mathcal{N}''')w\\
&\quad +\frac{1}{12}(16\mathcal{M}\mathcal{N} - 9\mathcal{N}'')w' - \frac{5}{4}\mathcal{N}'w'' - \frac{5}{6}\mathcal{N}w'''\\
&\quad +\frac{1}{12}(8\mathcal{M}\mathcal{M}' - 2\mathcal{M}''')g + \frac{1}{12}\big(8\mathcal{M}^2 - 9\mathcal{M}''\big)g'\\
&\quad -\frac{5}{4}\mathcal{M}'g'' - \frac{5}{6}\mathcal{M}g''' + \frac{1}{6}g^{(5)}\,,
\end{aligned}
\tag{6.1.6}
$$

where prime denotes the derivative with respect to the $\phi$ coordinate. The most general time component of the gauge field $a_u$ which preserves the boundary conditions above is given by

$$
a_u = \lambda[\mu, \xi, \vartheta, \varrho, \varepsilon, \varphi]\,, \tag{6.1.7}
$$

where $\mu, \xi, \vartheta, \varrho, \varepsilon$ and $\varphi$ are arbitrary functions of $(u, \phi)$ which are assumed to be fixed at the boundary [68, 100].The time evolution of the gauge fields in the asymptotic region is given by the following conditions

$$\begin{aligned}
\dot{\mathcal{M}} &= \mathcal{M}'\mu + 2\mathcal{M}\mu' - 2\mu''' + 4(2\mathcal{W}'\varrho + 3\mathcal{W}\varrho')\,, \\
\dot{\mathcal{N}} &= \mathcal{M}'\xi + 2\mathcal{M}\xi' + \mathcal{N}'\mu + 2\mathcal{N}\mu' - 2\xi''' \\
&\quad + 4(2\mathcal{W}'w + 3\mathcal{W}w') + 4(2\mathcal{V}'\varrho + 3\mathcal{V}\varrho')\,, \\
\dot{\mathcal{F}} &= \mathcal{M}'\vartheta + 2\mathcal{M}\vartheta' + \mathcal{N}'\xi + 2\mathcal{N}\xi' + \mathcal{F}'\mu + 2\mathcal{F}\mu' - 2\vartheta''' + 4(2\mathcal{W}'\varphi + 3\mathcal{W}\varphi') \\
&\quad + 4(2\mathcal{V}'\varepsilon + 3\mathcal{V}\varepsilon') + 4(2\mathcal{X}'\varrho + 3\mathcal{X}\varrho')\,, \\
\dot{\mathcal{W}} &= \mathcal{W}'\mu + 3\mathcal{W}\mu' + \frac{1}{12}(8\mathcal{M}\mathcal{M}' - 2\mathcal{M}''')\varrho + \frac{1}{12}\left(8\mathcal{M}^2 - 9\mathcal{M}''\right)\varrho' - \frac{5}{4}\mathcal{M}'\varrho'' \\
&\quad - \frac{5}{6}\mathcal{M}\varrho''' + \frac{1}{6}\varrho^{(5)}\,, \\
\dot{\mathcal{V}} &= \mathcal{W}'\xi + 3\mathcal{W}\xi' + 3\mathcal{V}\mu' + \frac{1}{12}(8\mathcal{N}\mathcal{M}' + 8\mathcal{M}\mathcal{N}' - 2\mathcal{N}''')\varrho \\
&\quad + \frac{1}{12}(16\mathcal{M}\mathcal{N} - 9\mathcal{N}'')\varrho' - \frac{5}{4}\mathcal{N}'\varrho'' - \frac{5}{6}\mathcal{N}\varrho''' \\
&\quad + \frac{1}{12}(8\mathcal{M}\mathcal{M}' - 2\mathcal{M}''')\varepsilon + \frac{1}{12}(8\mathcal{M}^2 - 9\mathcal{M}'')\varepsilon' \\
&\quad - \frac{5}{4}\mathcal{M}'\varepsilon'' - \frac{5}{6}\mathcal{M}\varepsilon''' + \frac{1}{6}\varepsilon^{(5)} \\
\dot{\mathcal{X}} &= \mathcal{W}'\vartheta + 3\mathcal{W}\vartheta' + \mathcal{V}'\xi + 3\mathcal{V}\xi' + \mathcal{X}'\mu + 3\mathcal{X}\mu' \\
&\quad + \frac{1}{12}(8\mathcal{M}\mathcal{F}' + 8\mathcal{F}\mathcal{M}' + 8\mathcal{N}\mathcal{N}' - 2\mathcal{F}''') + \frac{1}{12}\left(16\mathcal{F}\mathcal{M} + 8\mathcal{N}^2 - 9\mathcal{F}''\right)\varrho'\varrho \\
&\quad - \frac{5}{4}\mathcal{F}'\varrho'' - \frac{5}{6}\mathcal{F}\varrho''' + \frac{1}{12}(8\mathcal{N}\mathcal{M}' + 8\mathcal{M}\mathcal{N}' - 2\mathcal{N}''')\varepsilon \\
&\quad + \frac{1}{12}(16\mathcal{M}\mathcal{N} - 9\mathcal{N}'')\varepsilon' - \frac{5}{4}\mathcal{N}'\varepsilon'' - \frac{5}{6}\mathcal{N}\varepsilon''' \\
&\quad + \frac{1}{12}(8\mathcal{M}\mathcal{M}' - 2\mathcal{M}''')\varphi + \frac{1}{12}\left(8\mathcal{M}^2 - 9\mathcal{M}''\right)\varphi' - \frac{5}{4}\mathcal{M}'\varphi'' - \frac{5}{6}\mathcal{M}\varphi''' \\
&\quad + \frac{1}{6}\varphi^{(5)}\,,
\end{aligned} \tag{6.1.8}$$

where dot corresponds to the derivative with respect to $u$.

In summary, the asymptotic behaviour is described by gauge fields of the form given in (6.1.1), where the components $a_\phi$ and $a_u$ of the asymptotic gauge field $a$ are given by (6.1.3) and (6.1.7), respectively.

The structure outlined above provides insights into the asymptotic symmetries of the higher spin Maxwell Chern-Simons gravity and its associated algebra. In fact, the charge algebra of this theory can be derived using the Regge-Teitelboim approach [103]. In the following sections, we will explore this construction in greater detail.

## 6.2. Charge algebra: spin-3 extension of the deformed $\mathfrak{bms}_3$ algebra

The charge algebra of the three-dimensional HS Maxwell CS gravity theory in representation of Poisson brackets can be obtained using the Regge-Teitelboim method [103] directly from the transformation law

$$\delta_{\Lambda_2} Q[\Lambda_1] = \{Q[\Lambda_1], Q[\Lambda_2]\} \ , \tag{6.2.1}$$

where $Q[\Lambda]$ are the conserved charges spanning the algebra [97]. On the other hand, the variation of the charge in a CS theory is given by

$$\delta Q[\Lambda] = \frac{k}{2\pi} \int_{\partial\Sigma} \langle \Lambda \delta A \rangle \, . \tag{6.2.2}$$

After applying the gauge transformation (6.1.1) which introduces the asymptotic gauge field (6.1.3) we get [96]

$$\delta Q[\lambda] = \frac{k}{2\pi} \int d\phi \, \langle \lambda \delta a_\phi \rangle \ , \tag{6.2.3}$$

where $\Lambda = h^{-1}\lambda h$. Considering the invariant tensor (4.2.3)-(4.2.4) and the gauge field $a$ defined in (6.1.3) in the previous expression, we get

$$\delta Q[y,f,h,v,w,g] = \int d\phi \left( y\delta\mathbf{J} + f\delta\mathbf{P} + h\delta\mathbf{Z} + v\delta\mathbf{V} + w\delta\mathbf{W} + g\delta\mathbf{X} \right) \ , \tag{6.2.4}$$

where we have defined

$$\begin{aligned}
\mathbf{J} &= \frac{k}{4\pi}\left(\alpha_0\mathcal{M} + \alpha_1\mathcal{N} + \alpha_2\mathcal{F}\right) \ , \\
\mathbf{P} &= \frac{k}{4\pi}\left(\alpha_1\mathcal{M} + \alpha_2\mathcal{N}\right) \ , \\
\mathbf{Z} &= \frac{k}{4\pi}\alpha_2\mathcal{M} \, , \\
\mathbf{V} &= \frac{k}{\pi}\left(\alpha_0\mathcal{W} + \alpha_1\mathcal{V} + \alpha_2\mathcal{X}\right) \ , \\
\mathbf{W} &= \frac{k}{\pi}\left(\alpha_1\mathcal{W} + \alpha_2\mathcal{V}\right) \ , \\
\mathbf{X} &= \frac{k}{\pi}\alpha_2\mathcal{W} .
\end{aligned} \tag{6.2.5}$$

Assuming that the functions $y$, $f$, $h$, $v$, $w$ and $g$ do not depend on the fields we can directly integrate the variation on the phase space, and we find

$$Q[y,f,h,v,w,g]=\int d\phi\,(y\mathbf{J}+f\mathbf{P}+h\mathbf{Z}+v\mathbf{V}+w\mathbf{W}+g\mathbf{X})\ . \tag{6.2.6}$$

$$\delta_{\lambda_2}Q[\lambda_1]=\{Q[\lambda_1],Q[\lambda_2]\}\ . \tag{6.2.7}$$

As it is expected, the asymptotic symmetries associated to $y$, $f$ and $h$ span the $\mathfrak{max}$-$\mathfrak{bms}_3$ algebra [43]. Indeed, expanding in Fourier modes according to

$$X=\frac{1}{2\pi}\sum X_m e^{im\phi}\,, \tag{6.2.8}$$

one obtain the following algebra:

$$\begin{aligned}
i\{J_m,J_n\} &= (m-n)\,J_{m+n}+\alpha_0 k m^3\delta_{m+n,0}\,,\\
i\{J_m,P_n\} &= (m-n)\,P_{m+n}+\alpha_1 k m^3\delta_{m+n,0}\,,\\
i\{P_m,P_n\} &= (m-n)\,Z_{m+n}+\alpha_2 k m^3\delta_{m+n,0}\,,\\
i\{J_m,Z_n\} &= (m-n)\,Z_{m+n}+\alpha_2 k m^3\delta_{m+n,0}\,,\\
i\{P_m,Z_n\} &= 0\,,\\
i\{Z_m,Z_n\} &= 0\,,
\end{aligned} \tag{6.2.9}$$

which corresponds to the asymptotic symmetry algebra for the three-dimensional Maxwell CS gravity derived in [50] and subsequently in [43]. The brackets of the previous Fourier modes with the other charges associated to $v$, $w$ and $g$ read

$$\begin{aligned}
i\{J_m,V_n\} &= (2m-n)\,V_{m+n}\,,\\
i\{J_m,W_n\} &= (2m-n)\,W_{m+n}\,,\\
i\{P_m,V_n\} &= (2m-n)\,W_{m+n}\,,\\
i\{P_m,W_n\} &= (2m-n)\,X_{m+n}\,,\\
i\{J_m,X_n\} &= (2m-n)\,X_{m+n}\,,\\
i\{Z_m,V_n\} &= (2m-n)\,X_{m+n}\,,\\
i\{P_m\,,X_n\} &= 0\,,\\
i\{Z_m\,,W_n\} &= 0\,,\\
i\{Z_m\,,X_n\} &= 0\,,
\end{aligned} \tag{6.2.10}$$

where we have defined the Fourier modes:

$$V_m = Q[v = e^{im\phi}]\,, \qquad W_m = Q[w = e^{im\phi}]\,, \qquad X_m = Q[g = e^{im\phi}]\,, \tag{6.2.11}$$

which correspond to spin-3 generators. Finally, these generators satisfy the following brackets:

$$\begin{aligned}
i\{W_m, W_n\} &= \frac{8}{3\alpha_2 k}(m-n)\Omega^{ZZ}_{m+n} + \frac{1}{3}(m-n)(2m^2+2n^2-nm)Z_{m+n} \\
&\quad + \frac{\alpha_2 k}{3} m^5 \delta_{m+n,0}\,, \\
i\{W_m, V_n\} &= \frac{8}{3\alpha_2 k}(m-n)\left(2\Omega^{PZ}_{m+n} - \frac{\alpha_1}{\alpha_2}\Omega^{ZZ}_{m+n}\right) \\
&\quad + \frac{1}{3}(m-n)(2m^2+2n^2-nm)P_{m+n} + \frac{\alpha_1 k}{3} m^5 \delta_{m+n,0}\,, \\
i\{V_m, X_n\} &= \frac{8}{3\alpha_2 k}(m-n)\Omega^{ZZ}_{m+n} + \frac{1}{3}(m-n)(2m^2+2n^2-nm)Z_{m+n} \\
&\quad + \frac{\alpha_2 k}{3} m^5 \delta_{m+n,0}]\,, \\
i\{V_m, V_n\} &= \frac{8}{3\alpha_2 k}(m-n) \\
&\quad \times \left[\frac{1}{\alpha_2}\left(\frac{\alpha_1^2}{\alpha_2} - \alpha_0\right)\Omega^{ZZ}_{m+n} + \Omega^{PP}_{m+n} - 2\frac{\alpha_1}{\alpha_2}\Omega^{PZ}_{m+n} + 2\Omega^{ZJ}_{m+n}\right] \\
&\quad + \frac{1}{3}(m-n)(2m^2+2n^2-nm)J_{m+n} + \frac{\alpha_0 k}{3} m^5 \delta_{m+n,0}\,, \\
i\{W_m, X_n\} &= 0\,, \\
i\{X_m, X_n\} &= 0\,,
\end{aligned} \tag{6.2.12}$$

where we have defined the following terms

$$\Omega^{T\bar{T}}_m = \sum_{j=-\infty}^{\infty} T_j \bar{T}_{m-j}\,. \tag{6.2.13}$$

The previous algebra corresponds to a higher spin-extension of the $\mathfrak{max}$-$\mathfrak{bms}_3$ found in [43]. Note that the nonlinearity of the algebra is not trivial and does not simply correspond to a Maxwell generalization of the $\mathfrak{bms}_3$ algebra. Furthermore, in this case we have three central terms switched on, as can be seen by defining

$$c_i = 12k\alpha_{i-1}\,, \qquad \text{with} \qquad i = 1, 2, 3\,. \tag{6.2.14}$$

With this definition of the central charges, the algebra can be written as

$$\begin{aligned}
i\left\{J_m,J_n\right\} &= (m-n)\,J_{m+n}+\frac{c_1}{12}\,m^3\delta_{m+n,0}\,,\\
i\left\{J_m,P_n\right\} &= (m-n)\,P_{m+n}+\frac{c_2}{12}\,m^3\delta_{m+n,0}\,,\\
i\left\{P_m,P_n\right\} &= (m-n)\,Z_{m+n}+\frac{c_3}{12}\,m^3\delta_{m+n,0}\,,\\
i\left\{J_m,Z_n\right\} &= (m-n)\,Z_{m+n}+\frac{c_3}{12}\,m^3\delta_{m+n,0}\,,
\end{aligned} \tag{6.2.15}$$

$$\begin{aligned}
i\left\{J_m,V_n\right\} &= (2m-n)\,V_{m+n}\,, & i\left\{J_m,W_n\right\} &= (2m-n)\,W_{m+n}\,,\\
i\left\{P_m,V_n\right\} &= (2m-n)\,W_{m+n}\,, & i\left\{P_m,W_n\right\} &= (2m-n)\,X_{m+n}\,,\\
i\left\{J_m,X_n\right\} &= (2m-n)\,X_{m+n}\,, & i\left\{Z_m,V_n\right\} &= (2m-n)\,X_{m+n}\,,
\end{aligned} \tag{6.2.16}$$

$$\begin{aligned}
i\left\{W_m,W_n\right\} &= \frac{32}{c_3}(m-n)\Omega^{ZZ}_{m+n}+\frac{1}{3}(m-n)(2m^2+2n^2-nm)Z_{m+n}\\
&\quad +\frac{c_3}{36}m^5\delta_{m+n,0}\,,\\
i\left\{W_m,V_n\right\} &= \frac{32}{c_3}(m-n)\left(2\Omega^{PZ}_{m+n}-\frac{c_2}{c_3}\Omega^{ZZ}_{m+n}\right)\\
&\quad +\frac{1}{3}(m-n)(2m^2+2n^2-nm)P_{m+n}+\frac{c_2}{36}m^5\delta_{m+n,0}\,,\\
i\left\{V_m,X_n\right\} &= \frac{32}{c_3}(m-n)\Omega^{ZZ}_{m+n}+\frac{1}{3}(m-n)(2m^2+2n^2-nm)Z_{m+n}\\
&\quad +\frac{c_3}{36}m^5\delta_{m+n,0}\,,\\
i\left\{V_m,V_n\right\} &= \frac{32}{c_3}(m-n)\left[\frac{1}{c_3}\left(\frac{c_2^2}{c_3}-c_1\right)\Omega^{ZZ}_{m+n}+\Omega^{PP}_{m+n}-2\frac{c_2}{c_3}\Omega^{PZ}_{m+n}+2\Omega^{ZJ}_{m+n}\right]\\
&\quad +\frac{1}{3}(m-n)(2m^2+2n^2-nm)J_{m+n}+\frac{c_1}{36}m^5\delta_{m+n,0}\,.
\end{aligned} \tag{6.2.17}$$

In sum, the commutation relations (6.2.15)-(6.2.17) provides the asymptotic symmetries of spin-3 fields coupled to Maxwell CS gravity. In the next section, we show that this algebra can alternatively be recovered as the flat limit of three copies of the $\mathcal{W}_3$ algebra, mimicking the relation between HS Maxwell gravity and three copies of $\mathfrak{sl}(3,\mathbb{R})$.

## 6.3. Spin-3 extension of the deformed $\mathfrak{bms}_3$ algebra as a vanishing cosmological constant limit

It is possible to check that the asymptotic symmetries, described by three copies of the $W_3$ algebra, lead to the spin-3 extension of the $\mathfrak{max}$-$\mathfrak{bms}_3$ algebra. Indeed, by redefining the generators as

$$
\begin{aligned}
J_m &= \mathcal{L}^+_m - \mathcal{L}^-_{-m} - \hat{\mathcal{L}}_{-m} \,, \\
P_m &= \frac{1}{\ell}\left(\mathcal{L}^+_m + \mathcal{L}^-_{-m}\right) \,, \\
Z_m &= \frac{1}{\ell^2}\left(\mathcal{L}^+_m - \mathcal{L}^-_{-m}\right) \,, \\
V_m &= \mathcal{W}^+_m - \mathcal{W}^-_{-m} - \hat{\mathcal{W}}_{-m} \,, \\
W_m &= \frac{1}{\ell}\left(\mathcal{W}^+_m + \mathcal{W}^-_{-m}\right) \,, \\
X_m &= \frac{1}{\ell^2}\left(\mathcal{W}^+_m - \mathcal{W}^-_{-m}\right) \,,
\end{aligned}
\tag{6.3.1}
$$

along the redefinition of the central charges as follows,

$$
c_1 = c^+ - c^- - \hat{c} \,, \qquad c_2 = \frac{1}{\ell}(c^+ + c^-) \,, \qquad c_3 = \frac{1}{\ell^2}(c^+ - c^-) \,, \tag{6.3.2}
$$

the three copies of the $W_3$ algebra given by

$$
\begin{aligned}
i\left\{\mathcal{L}^\pm_m, \mathcal{L}^\pm_n\right\} &= (m-n)\,\mathcal{L}^\pm_{m+n} + \frac{c^\pm}{12} m^3 \delta_{m+n,0} \,, \\
i\left\{\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n\right\} &= (m-n)\,\hat{\mathcal{L}}_{m+n} + \frac{\hat{c}}{12} m^3 \delta_{m+n,0} \,, \\
i\left\{\mathcal{L}^\pm_m, \mathcal{W}^\pm_n\right\} &= (2m-n)\,\mathcal{W}^\pm_{m+n} \,, \\
i\left\{\hat{\mathcal{L}}_m, \hat{\mathcal{W}}_n\right\} &= (2m-n)\,\hat{\mathcal{W}}_{m+n} \,, \\
i\left\{\mathcal{W}^\pm_m, \mathcal{W}^\pm_n\right\} &= \frac{32}{c^\pm}(m-n)\Omega^\pm_{m+n} + \frac{1}{3}(m-n)(2m^2 - 2n^2 - mn)\mathcal{L}^\pm_{m+n} \\
&\quad + \frac{c^\pm}{36} m^5 \delta_{m+n,0} \,, \\
i\left\{\hat{\mathcal{W}}_m, \hat{\mathcal{W}}_n\right\} &= \frac{32}{\hat{c}}(m-n)\hat{\Omega}_{m+n} + \frac{1}{3}(m-n)(2m^2 - 2n^2 - mn)\hat{\mathcal{L}}_{m+n} \\
&\quad + \frac{\hat{c}}{36} m^5 \delta_{m+n,0} \,,
\end{aligned}
\tag{6.3.3}
$$

with

$$\Omega^{\pm}_{m} \equiv \sum_{j\in\mathbb{Z}} \mathcal{L}^{\pm}_{m+j}\mathcal{L}^{\pm}_{-j}\,, \qquad \hat{\Omega}_{m} \equiv \sum_{j\in\mathbb{Z}} \hat{\mathcal{L}}_{m+j}\hat{\mathcal{L}}_{-j}\,, \tag{6.3.4}$$

can be written as

$$\begin{aligned}
i\{J_m,J_n\} &= (m-n)\,J_{m+n}+\frac{c_1}{12}m^3\delta_{m+n,0}\,,\\
i\{J_m,P_n\} &= (m-n)\,P_{m+n}+\frac{c_2}{12}m^3\delta_{m+n,0}\,,\\
i\{P_m,P_n\} &= (m-n)\,Z_{m+n}+\frac{c_3}{12}m^3\delta_{m+n,0}\,,\\
i\{J_m,Z_n\} &= (m-n)\,Z_{m+n}+\frac{c_3}{12}m^3\delta_{m+n,0}\,,\\
i\{P_m,Z_n\} &= \frac{1}{\ell^2}(m-n)\,P_{m+n}+\frac{c_2}{12\ell^2}m^3\delta_{m+n,0}\,,\\
i\{Z_m,Z_n\} &= \frac{1}{\ell^2}(m-n)\,Z_{m+n}+\frac{c_3}{12\ell^2}m^3\delta_{m+n,0}\,,
\end{aligned} \tag{6.3.5}$$

$$\begin{aligned}
i\{J_m,V_n\} &= (2m-n)\,V_{m+n}\,,\\
i\{J_m,W_n\} &= (2m-n)\,W_{m+n}\,,\\
i\{P_m,V_n\} &= (2m-n)\,W_{m+n}\,,\\
i\{P_m,W_n\} &= (2m-n)\,X_{m+n}\,,\\
i\{J_m,X_n\} &= (2m-n)\,X_{m+n}\,,\\
i\{Z_m,V_n\} &= (2m-n)\,X_{m+n}\,,\\
i\{P_m,X_n\} &= \frac{1}{\ell^2}(2m-n)\,W_{m+n}\,,\\
i\{Z_m,W_n\} &= \frac{1}{\ell^2}(2m-n)\,W_{m+n}\,,\\
i\{Z_m,X_n\} &= \frac{1}{\ell^2}(2m-n)\,X_{m+n}\,,
\end{aligned} \tag{6.3.6}$$

$$
\begin{aligned}
i\{W_m, W_n\} &= \frac{32}{(c_2^2/\ell^2 - c_3^2)}(m-n)\left[2\frac{c_2}{\ell^2}\Omega^{PZ}_{m+n} - c_3\left(\Omega^{ZZ}_{m+n} + \frac{\Omega^{PP}_{m+n}}{\ell^2}\right)\right] \\
&\quad + \frac{1}{3}(m-n)(2m^2+2n^2-nm)Z_{m+n} + \frac{c_3}{36}m^5\delta_{m+n,0}, \\
i\{W_m, V_n\} &= \frac{32}{c_2^2/\ell^2 - c_3^2}(m-n)\left[-2c_3\Omega^{PZ}_{m+n} + c_2\left(\Omega^{ZZ}_{m+n} + \frac{\Omega^{PP}_{m+n}}{\ell^2}\right)\right] \\
&\quad + \frac{1}{3}(m-n)(2m^2+2n^2-nm)P_{m+n} + \frac{c_2}{36}m^5\delta_{m+n,0}, \\
i\{V_m, X_n\} &= \frac{32}{(c_2^2/\ell^2 - c_3^2)}(m-n)\left[2\frac{c_2}{\ell^2}\Omega^{PZ}_{m+n} - c_3\left(\Omega^{ZZ}_{m+n} + \frac{\Omega^{PP}_{m+n}}{\ell^2}\right)\right] \\
&\quad + \frac{1}{3}(m-n)(2m^2+2n^2-nm)Z_{m+n} + \frac{c_3}{36}m^5\delta_{m+n,0}, \\
i\{W_m, X_n\} &= \frac{32}{(c_2^2/\ell^2 - c_3^2)\ell^2}(m-n)\left[-2c_3\Omega^{PZ}_{m+n} + c_2\left(\Omega^{ZZ}_{m+n} + \frac{\Omega^{PP}_{m+n}}{\ell^2}\right)\right] \\
&\quad + \frac{1}{3\ell^2}(m-n)(2m^2+2n^2-nm)P_{m+n} + \frac{c_2}{36\ell^2}m^5\delta_{m+n,0}, \\
i\{X_m, X_n\} &= \frac{32}{(c_2^2/\ell^2 - c_3^2)\ell^2}(m-n)\left[2\frac{c_2}{\ell^2}\Omega^{PZ}_{m+n} - c_3\left(\Omega^{ZZ}_{m+n} + \frac{\Omega^{PP}_{m+n}}{\ell^2}\right)\right] \\
&\quad + \frac{1}{3\ell^2}(m-n)(2m^2+2n^2-nm)Z_{m+n} + \frac{c_3}{36\ell^2}m^5\delta_{m+n,0}, \\
i\{V_m, V_n\} &= \frac{32}{(-c_1c_2^2/\ell^4 + c_2c_3/\ell^2 + c_1c_3^2/\ell^2 - c_3^3)}(m-n)\left[-2\frac{c_1c_2}{\ell^2}\Omega^{PZ}_{m+n}\right. \\
&\quad + c_3^2\left(-\Omega^{PP}_{m+n} + \frac{\Omega^{JJ}_{m+n}}{\ell^2} - 2\Omega^{JZ}_{m+n}\right) + c_1c_3\left(\Omega^{ZZ}_{m+n} + \frac{\Omega^{PP}_{m+n}}{\ell^2}\right) \\
&\quad \left. - c_2^2\left(\frac{\Omega^{JJ}_{m+n}}{\ell^4} - 2\frac{\Omega^{JZ}_{m+n}}{\ell^2} + \Omega^{ZZ}_{m+n}\right)\right] \\
&\quad + \frac{1}{3}(m-n)(2m^2+2n^2-nm)J_{m+n} + \frac{c_1}{36}m^5\delta_{m+n,0}. \qquad (6.3.7)
\end{aligned}
$$

Let us note that the $\ell$ parameter corresponds to the AdS radius which is related to the cosmological constant through $\Lambda = -1/\ell^2$ in three-dimensional spacetime. Then, taking the flat limit $\ell \to \infty$, the asymptotic symmetry algebra given by three copies of the $\mathcal{W}_3$ algebra leads to the $\mathfrak{hs}_3\mathfrak{max}$-$\mathfrak{bms}_3$ obtained in the previous section. The algebra (6.3.5)-(6.3.7) is a spin-3 extension of the enlarged $\mathfrak{bms}_3$ algebra introduced in [93], and corresponds to the asymptotic symmetry algebra of AdS-Lorentz CS gravity coupled to a spin$-3$ field.

# Capítulo 7

# Conclusion

In this thesis, we have studied the asymptotic structure of the Chern-Simons gravity theory invariant under a higher spin extension of the so-called Maxwell symmetry. In particular, we proposed some consistent asymptotic conditions that allow us to canonically realized a nonlinear higher-spin extension of the Maxwell generalization of the $\mathfrak{bms}_3$ symmetry that we call $\mathfrak{hs}_3\mathfrak{max}\text{-}\mathfrak{bms}_3$. Moreover, we have also shown that this algebra can be obtained as a vanishing cosmological constant limit of three copies of the $\mathcal{W}_3$ algebra, each with an independent central charge.

In Chapter 2, we reviewed the theory of General Relativity, the Einstein-Hilbert action and introduced the Cartan's formalism for gravity.

In Chapter 3, we introduced the Chern-Simons form and the Chern-Simons action. In particular, we briefly reviewed the construction of the AdS and Maxwell Chern-Simons gravity theories.

In Chapter 4, we reviewed the Chern-Simons higher-spin gravity theory in three dimensions. Specifically, the construction of the three-dimensional Maxwell and AdS-Lorentz gravities coupled to spin-3 fields was presented.

In Chapter 5, we reviewed the asymptotic structure of three-dimensional gravity and we showed the results of the previous work [43].

In conclusion, we can say that one of the most fascinating aspects of three-dimensional higher-spin gravities is the presence of exact solutions that carry higher-spin charges. This thesis serves as a foundational step in exploring the thermodynamics of such configurations through topological arguments [74, 87].

In the present case we expect to find a novel higher-spin extension of locally flat cosmological spacetimes. Since we have incorporated the chemical potential conjugated to the higher spin charges, the thermodynamics properties of the solution can be analyzed along the lines of [74]. An interesting direction now available for exploration involves the potential application of the theoretical description of the theory in terms of three copies of $SL(3, R)$. This framework may provide insights in the context of thermodynamic properties of either black hole solutions of (higher-spin extension of) AdS-Lorentz gravity or cosmological solutions of (higher-spin extension of) Maxwell gravity.

A possible extension of our results can be carried out by incorporating a negative cosmological constant to the higher-spin Maxwell CS theory. This can be done by considering the higher-spin extension of the so-called AdS-Lorentz gravity constructed in [33]. In this case, we expect to find higher-spin black holes generalizing the BTZ-type solution of the AdS-Lorentz gravity studied in [93] including spin-3 charges. Then, the higher spin black hole solution should lead to the higher spin extension of locally flat cosmological spacetimes, after performing the vanishing cosmological constant limit. In this scenario, we could also study the solutions and the thermodynamics of the higher-spin extension of the Maxwell teleparallel gravity first presented in [45] and analyze how the presence of a non-vanishing torsion modifies the solutions of the spin-3 extension of the Maxwell gravity theory. As solutions we expect to find a spin-3 extension of a Maxwell teleparallel black hole [work in progress]. We guess that both spin-3 black holes will reduce to the Maxwellian generalization of the spin-3 extension of flat cosmologies in a flat limit.

Another interesting aspect worth exploring is the study of the asymptotic symmetry of the three-dimensional Maxwell CS gravity theory coupled to spin-5/2 gauge field [34, 104]. In such case, the HS field theory contains fields of spins 2, 4 and 5/2 whose invariance is extended to HS fermionic symmetry transformations, referred to as hypersymmetry [105–110]. Following the results obtained here, one could expect to find, after imposing suitable boundary conditions, a deformation of the hyper-$\mathfrak{bms}_3$ which should be alternatively recovered as a vanishing cosmological constant limit of a precise combination of the $\mathcal{W}_{(2,\frac{5}{2},4)}$ and $\mathcal{W}_{(2,4)}$ algebra [111, 112]. The asymptotic algebra could serve to derive hypersymmetry bounds which could imply interesting properties for solutions as in [107, 109].

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