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“El momentum de la luz en medios materiales”

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No te rindas
que la vida
es eso,
continuar el viaje,
perseguir tus sueños,
destrabar el tiempo,
correr los escombros
y destapar el cielo.

Mario Benedetti,
poeta uruguayo.

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Resumen

En la presente tesis estudiamos el problema de la definición adecuada y consistente del momentum de la luz en medios materiales, lo cual ha sido objeto de debate por más de 100 años y todavía no hay una respuesta definitiva y generalmente aceptada al respecto. Al realizar un completo estudio de la literatura disponible sobre el tema, nos dimos cuenta de la gran cantidad de argumentos contradictorios que se han generado durante este largo debate, que en nuestra opinión, han hecho parecer al problema más confuso y complicado de lo que en realidad es.

En 1908 y 1909, Minkowski y Abraham propusieron dos expresiones rivales para describir el tensor energía-momentum de la luz dentro de un medio material, los cuales en principio podían ponerse experimentalmente a prueba. Al pasar los años, se llevaron a cabo varios experimentos cuyos resultados parecían validar diferentes formulaciones, dando lugar a la llamada "controversia de Abraham-Minkowski". En 1966, Penfield y Haus proponen una solución formal de la controversia desde el punto de vista de la teoría clásica de campos. Ellos argumentaron que al considerar la dinámica del medio material, sólo el tensor energía-momentum total del sistema cerrado tiene significado físico y que los tensores de Abraham y Minkowski para el campo electromagnético son simplemente diferentes separaciones del mismo tensor total. Estas ideas de equivalencia entre las distintas formulaciones pasaron, sin embargo, muy desapercibidas y en los últimos 10 años la discusión ha revivido con una gran cantidad de nuevas publicaciones más orientadas a las aplicaciones ópticas. En estos trabajos actuales, los autores parecen estar completamente desinformados de los previos avances en el tema, pues continúan buscando nuevos argumentos para encontrar el momentum "correcto" de la luz en la materia. Muy recientemente, en 2010, Barnett, Loudon e independientemente Mansuripur afirman que han resuelto la controversia, pero sin ni siquiera mencionar los trabajos anteriores de Penfield y Haus, generando en nuestra opinión, más confusión que claridad en el tema.

Es por ésto, que el objetivo general de la presente tesis es contribuir en aclarar los conceptos fundamentales del problema y en encontrar un acuerdo entre los diferentes enfoques llevados a cabo por físicos de distintas áreas, que parecen estar desinformados del trabajo de los otros. Nuestra investigación nos llevó a concluir que la vieja solución de Penfield y Haus es simple, lógica y completamente consistente tanto con los experimentos como con los conceptos de la Física Clásica, por lo que adoptamos esta postura frente al tema.

En particular, hacemos uso de la electrodinámica macroscópica en una forma manifiestamente covariante para estudiar con detalle y en un contexto completamente relativista, el sistema formado por campo electromagnético y medio material macroscópico. Primero consideramos al medio material como un "escenario" fijo (sin dinámica) a través del cual se propaga la luz, para luego derivar de las ecuaciones de Maxwell macroscópicas, las ecuaciones de balance para la energía, el momentum y el momentum angular. Estudiamos la relación

entre las simetrías del medio y las cantidades electromagnéticas conservadas, para concluir que el tensor de Minkowski de la luz es el que se relaciona directamente con las simetrías del medio.

Con el fin de conectar este tratamiento con los análisis de Barnett, Loudon y otros, usamos la expresión de Penfield y Haus para el tensor total, derivada de una manera más moderna por Obukhov, para resolver con todo detalle un problema particular conocido como el experimento pensado de la "caja dieléctrica de Einstein". Esta situación particular ha suministrado el argumento más fuerte a favor de la resolución de Barnett, Loudon y Mansuripur, pues ellos la usan para seleccionar la expresión de Abraham como la única válida de describir el momentum de la luz en este caso. En nuestro anális completamente relativista, recalcamos la importancia del tensor energía-momentum total del sistema cerrado y derivamos en detalle las expresiones para los momenta de Abraham y Minkowski dentro de este medio isótropo y homogéneo en movimiento. Realizando un cálculo explícito, mostramos que el tensor de Minkowski también sirve para describir esta siuación, aunque no es tan útil como el de Abraham. Finalmente, al tomar el límite no-relativista de las expresiones finales, identificamos algunas suposiciones injustificadas que están escondidas en la descripción usual de la caja de Einstein dieléctrica y que explican el por qué los otros autores previamente sólo obtenían el momentum de Abraham para la luz en este caso.

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Chapter 1.

Introducción

Men wanted for hazardous journey.
Low wages, bitter cold,
long hours of complete darkness.
Safe return doubtful.
Honour and recognition in
event of success.[1]

Ernest Shackleton,
Irish explorer.

Si salimos a caminar en un día soleado, nos daremos cuenta tarde o temprano que la luz del Sol es capaz de calentar la materia. Luego, sea lo que sea la luz, al menos sabemos que ella transporta *energía* consigo.

Después de completar sus famosas ecuaciones en 1865, Maxwell [1] fue el primero en predecir que la luz es algún tipo de radiación electromagnética que se propaga por el espacio y que además puede ejercer presión sobre una superficie expuesta a ella. Este efecto, conocido como *presión de radiación*, es mucho más difícil de observar que la transferencia de energía. Sin embargo, éste pudo ser experimentalmente medido en 1903, por Nichols and Hull [2], lo que significa que la luz además de energía, también transporta *momentum* consigo.

El momentum de la luz describe el grado en que ella puede poner otros objetos en movimiento, cuando la absorben o reflejan. En la vida cotidiana no vemos a la luz moviendo cosas, porque la fuerza causada por ella es usualmente bastante débil, pero como dice Leonhardt en [3], evidencia visual se puede encontrar, por ejemplo, en la cola de un cometa como en la figura 1.1, donde la luz del Sol transfiere momentum a las partículas de la cola, empujando el polvo fuera del cometa.

En 1905, Einstein desarrolló su Teoría de la Relatividad Especial [4], la cual ayudó a entender que las ondas electromagnéticas son diferentes a los otros fenómenos conocidos hasta ese momento, en el sentido que ellas no necesitan de un medio material para propagarse. De hecho, las ecuaciones de Maxwell en el vacío son las únicas microscópicamente bien definidas

[1] Se buscan hombres para peligroso viaje. Salario reducido. Frío penetrante. Largos meses de completa oscuridad. Constante peligro. Dudoso regreso sano y salvo. Honor y reconocimiento en caso de éxito. *Ernest Shackleton, explorador irlandés.*

Figure 1.1.: Un padre y su hijo están disfrutando de la asombrosa vista del cometa Hale-Bopp, un poco después del paso por su perihelio, el 1 de Abril de 1997. Mientras la cola azul de iones es llevada hacia afuera por el "viento solar" de partículas cargadas desde la atmósfera solar, la cola de polvo blanquecina es empujada por la presión de radiación de la luz solar. La transferencia de momentum en el segundo caso es más débil que en el primer caso, resultando en la separación de las colas. Foto tomada por Jerry Lodriguss, 1997.

y de acuerdo con ellas, las soluciones de onda plana deben viajar a una velocidad constante c con respecto a *cualquier* observador inercial.

La teoría clásica de la radiación electromagnética tuvo mucho éxito cuando fue aplicada para describir sistemas macroscópicos, dando lugar al desarrollo de innumerables nuevas tecnologías, las cuales cambiaron la forma de vivir de las personas en el siglo XX, especialmente respecto a las comunicaciones. Sin embargo, cuando se intentó aplicar la teoría electromagnética, tal como fue formulada por Maxwell y Einstein, para describir sistemas de escala atómica y subatómica, como interacciones dentro de un átomo de hidrógeno o electrones interaccionando con luz, la teoría falló. Es otra bonita y larga historia explicar cómo la mecánica cuántica se desarrolló para describir correctamente estos sistemas microscópicos, pero podemos decir que después del trabajo por años de muchos científicos, Dirac [5] en 1927 aplicó el método de *cuantización* a la teoría de Maxwell, pudiendo exitosamente describir la radiación electromagética en términos de *fotones*, las excitaciones elementales del campo electromagnético, también interpretadas en ciertas situaciones como "partículas de luz". En forma análoga a como las ondas de luz clásicas transportan energía y momentum, en electrodinámica cuántica los fotones poseen una energía y momentum bien definidos, de tal manera que las interacciones de la luz con la material también se pueden entender en términos de transferencia de energía y momentum, sólo que esta vez en cantidades mínimas e indivisibles, llamadas "cuantos".

Lo que es realmente importante de recalcar para nuestra tesis, es que en la mayoría de los casos anteriores, sin importar si clásicos o cuánticos, para el cálculo de la radiación y de las interacciones de la luz con la materia, el campo electromagnético se asume en el vacío, donde las ecuaciones de Maxwell microscópicas están bien definidas. Pero, ¿qué pasa si consideramos un medio material macroscópico para que la luz se propague a través de él? Los medios materiales, considerados macroscópicamente neutros se caracterizan, sin embargo, por tener cargas y corrientes ligadas, las cuales se distribuyen internamente bajo la acción de un campo electromagnético local, por ejemplo una onda de luz. Como un efecto macroscópico, los medios materiales se polarizan y magnetizan, creando un campo electromagnético inducido (a pesar de ser neutros), el cual se superpone con el campo electromagnético de luz propagándose a través de él. Esta respuesta macroscópica de los medios continuos se puede describir fenomenológicamente asignando suficientes magnitudes físicas al medio, para así dar cuenta de todas las propiedades electromagnéticas que les podemos detectar en los experimentos. De esta forma, podemos encontrar medios cuya respuesta sea lineal o no-lineal en el campo electromagnético aplicado, isótropa o anisótropa, dependiendo si polarización/magnetización es paralela o no al campo aplicado, dispersiva o no-dispersiva, cuando la respuesta del medio depende de la frecuencia de la luz o no, etc.

Hay muchos tipos de medios que podremos estudiar con detalle en la tesis, pero no hemos respondido algunas preguntas importantes. Supongamos que un rayo de luz incide sobre un bloque de cristal dielétrico, como en la figura 1.2. De las ecuaciones de Maxwell en el vacío ciertamente conocemos las expresiones para la energía y el momentum de la luz en el vacío, es decir, antes y después de pasar a través del cristal, pero *¿cuál es el momentum del rayo de luz dentro del medio? y ¿cuánto momentum transfiere la luz a las cargas y corrientes ligadas del medio mientras ella se propaga?. ¿Podemos físicamente diferenciar entre el momentum del campo y del medio cuando los dos están interactuando?. ¿Tiene sentido esta pregunta?.*

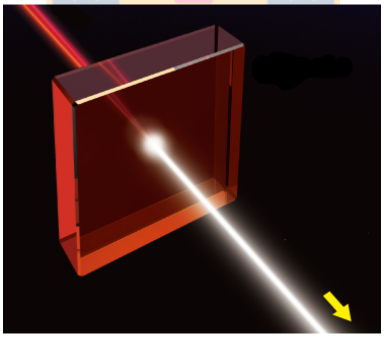

Figure 1.2.: Un rayo de luz pasando a través de un bloque de cristal dieléctrico. ¿Cuánto momentum transporta la luz dentro de él?

A primera vista uno esperaría que todas estas preguntas estuviesen contestadas en cualquier libro estándar de electrodinámica, pero *sorprendentemente* ese no es el caso, de hecho hasta en el conocido libro de Jackson [6] sólo se habla de la ecuación de balance de momentum para el campo en el vacío, haciéndole “el quite” a las preguntas que recién acabamos de plantear. En realidad no hay ningún libro que contenga respuestas definitivas y universalmente aceptadas a estas preguntas, pues constituye un problema abierto de la Física Clásica fundamental, cuyo debate a durado ya más de un siglo y a pesar de ello, todavía hay confución o al menos desacuerdo entre los autores.

En esta tesis proponemos contribuir en este estudio de la *definición adecuada del momentum de la luz dentro de un medio material*, desde el punto de vista de la electrodinámica clásica y la teoría clásica de campos. En nuestra opinión, la raíz del entendimiento del problema radica primero en la teoría clásica y por ello ahí hemos puesto todo nuestro esfuerzo. Una vez que el problema se entienda mejor desde un punto de vista clásico, puede llegar otro "Dirac" y cuantizar la teoría para encontrar el correspondiente momentum del fotón dentro de un medio macroscópico, si es que es posible, pues en la actualidad no es conocido.

En los últimos diez años, el problema de la determinación del momentum electromagnético dentro de la materia ha revivido dentro de la comunidad científica y en la literatura es conocido como la *"controversia de Abraham-Minkowski"*, en honor a los dos pioneros en el tema. En su paper original de 1908, Minkowski [7], además de desarrollar una formulación completamente covariante y cuadri-dimensional de la electrodinámica macroscópica, fue el primero en proponer un tensor energía-momentum para describir el contenido de energía y momentum del campo electromagnético en la materia. El tensor definido por Minkowski se caracterizaba por *no ser simétrico*, puede derivarse directamente de las ecuaciones de Maxwell macroscópicas y se reduce al tensor conocido tensor electromagnético simétrico en el vacío como un caso particular.

El hecho de que el tensor de Minkowski no fuera simétrico causó desconcierto en la comunidad científica, debido a la (falsa) creencia que todos los tensores energía-momentum deben ser simétricos para ser compatibles con la conservación del momentum angular del sistema. En el capítulo 4 se explica cuidadosamente que el argumento anterior es sólo verdadero en el caso de sistemas cerrados, pero para un sistema abierto, como el caso del campo electromagnético en la materia, la teoría de campos de hecho requiere que el tensor canónico no sea simétrico.

De todos modos, en 1909, un año después de la publicación de Minkowski, Abraham [8, 9] logró proponer un tensor simétrico alternativo para el campo electromagnético, pero bajo el costo de que éste no podía ser derivado a partir de primeros principios, como está detalladamente explicado en la revisión de Obukhov [10]. Abraham propuso que el tensor macroscópico debería poder obtenerse a partir de promedios sobre regiones apropiadas del espaciotiempo del tensor microscópico, que es simétrico. Como la simetría del tensor se mantiene después de calcular promedios, entonces el tensor macroscópico también debería ser simétrico. Sin embargo, Abraham no pudo dar una prueba rigurosa de su idea y definió el tensor macroscópico simétrico de una manera más o menos artificial. Hasta ahora no se tiene una derivación rigurosa de las ecuaciones de Maxwell macroscópicas como promedios de las ecuaciones microscópicas en el vacío. Además es muy probable que esto no sea posible, pues en el caso macroscópico se tienen ecuaciones *fenomenológicas* que incorporan las propiedades electromagneticas macroscópicas de los medios de una manera *efectiva* a través de las *relaciones constitutivas.* y no son directamente promedios espaciales de otras cantidades.

Cuando la luz se propaga dentro de un medio *fijo* y en reposo, por ejemplo un cristal sujeto a una mesa óptica, las correspondientes expresiones de Minkowski y Abraham para la densidad de momentum del campo dentro de la materia son definidas, en unidades SI, por

$$\boldsymbol{\pi}_M := \boldsymbol{D} \times \boldsymbol{B}, \tag{1.1}$$

$$\boldsymbol{\pi}_A := \frac{1}{c^2} \boldsymbol{E} \times \boldsymbol{H}, \tag{1.2}$$

donde $\boldsymbol{E}$ es el campo eléctrico, $\boldsymbol{B}$ el campo magnético, $\mathbf{D}$ la excitación eléctrica y $\mathbf{H}$ la excitación magnética. En el caso más simple de un pulso de onda plana propagándose dentro de un medio lineal, isótropo y homogéneo en reposo, las relaciones constitutivas son dadas por $\boldsymbol{D} = \varepsilon\varepsilon_0\boldsymbol{E}$ y $\boldsymbol{H} = \boldsymbol{B}/\mu\mu_0$, por lo que las dos expresiones rivales (1.1)-(1.2) se reducen a:

$$\boldsymbol{\pi}_M = n\frac{\mathcal{U}}{c}\,\hat{\boldsymbol{k}}, \tag{1.3}$$

$$\boldsymbol{\pi}_A = \frac{1}{n}\frac{\mathcal{U}}{c}\,\hat{\boldsymbol{k}}, \tag{1.4}$$

donde $n := \sqrt{\varepsilon\mu}$ es el índice de refracción del medio, $\mathcal{U}$ es la densidad de energía de la onda de luz plana dentro de la materia y $\hat{\boldsymbol{k}}$ es el vector unitario de propagación.

Notemos que ambas expresiones coinciden para $n = 1$, es decir, cuando el medio es el vacío, pero cuando $n \neq 1$, ellas difieren en un factor n^2. Esta clara diferencia entre las predicciones idealizadas (1.3) y (1.4) motivó el debate de determinar cuál de las dos definiciones para la densidad de momentum de la luz dentro de un medio era la correcta, y hasta el día de hoy continua, siendo una área activa de investigación.

Luego de realizar una investigación exhaustiva de la literatura, nos dimos cuenta de la cantidad de argumentos contradictorios que se han ido desarrollando en el tiempo, los cuales han generado una gran confusión en la comunidad científica y en nuestra opinión, han hecho parecer el problema más complicado de lo que en realidad es. Esto puede ser debido a que la esencia del problema es bastante transversal, por lo que ha sido estudiado por grupos de científicos de distintas áreas, los cuales por diferencias de enfonque, interés, notación o simplemente incomunicación entras las áreas, no han podido llegar a un consenso razonable o peor aún, ni siquiera se han enterado de los avances ya realizados por otros científicos. Por este motivo, *el primer objetivo en la presente tesis fue traer orden al tema, identificando los argumentos confusos y contradictorios que se han ido generando durante este largo debate y quedándonos con los que a nuestro parecer describen el problema de una manera más limpia, autoconsistente y sin necesitad de recurrir a argumentos ad-hoc para describir situaciones particulares.* En el capítulo 2 presentamos una revisión del desorrollo histórico de la controversia Abraham-Minkowski.

Dentro de los muchos trabajos que encontramos en la literatura, el que más nos influenció desde el comienzo de la investigación fue una revisión escrita por Obukhov [10] en 2008, donde se aborda el problema desde la teoría de clásica de campos. Obukhov escribe la electrodinámica en forma explícitamente covariante y usa el formalismo Lagrangeano para derivar las ecuaciones de balance y los tensores energía-momentum del sistema formado por campo electromagnético y medio material. Así, el*segundo objetivo de esta tesis fue rehacer detalladamente muchos de estos cálculos y luego extenderlos,* siendo lo expuesto en los capítulos 3, 4 y en el apéndice C, los resultados de este trabajo. Primero asumimos al medio material como un "background" sin dinámica a través del cual se propaga el campo electromagnético dinámico. Aquí resaltamos la relación entre las simetrías del medio y la obtención de cantidades electromagnéticas conservadas como energía, momentum y momentum angular, concluyendo que el tensor de Minkowski es el que está relacionado con las simetrías del medio y que la asimetría de sus componentes es completamente necesaria para describir este sistema *abierto.*

Estos simples y claros resultados nos condujeron a aceptar como válida una solución formal para la controversia postulada originalmente por Penfield y Haus [11, 12, 13] en 1966. Estos autores modelaron al medio material como un fluido dieléctrico isótropo y al considerar su dinámica derivaron mediante el formalismo Lagrangeano la forma explícita del tensor energía-momentum total para el sistema *cerrado* campo y medio material, ambos interactuando dinámicamente. Como resultado, encontraron un tensor energía-momentum total únicamente definido para el caso de medio isótropo y concluyeron que tanto el tensor de Minkowski como el de Abraham pueden describir correctamente al campo electromagnético dentro de un medio, siempre que cada uno se complemente con un tensor adecuado para el medio y así las ecuaciones de balance y el tensor del sistema total sea el mismo. En otras palabras, bajo este enfoque sólo el tensor total es el que tiene significado físico, siendo los tensores de Abraham y Minkowski distintas separaciones que no afectan las predicciones físicas del sistema. En nuestra opinión, esta forma de entender el problema es completamente satisfactoria, simple y consistente con toda la Física Clásica, pues lo único que se asume son las ecuaciones de Maxwell macroscópicas, las relaciones constitutivas para el tipo específico de material y las ecuaciones hidrodinámicas para el fluído.

Sin embargo, estos argumentos pasaron bastante desapercibidos y todavía se sigue discutiendo cuál de los dos tensores es el correcto para describir a la luz en un medio, originándose muchos argumentos teóricos y experimentos que le dan la razón a uno, a los dos a ninguno de las alternativas de momentum aquí planteadas, pero sin llegar a una respuesta definitiva. Para contribuir a la aceptación de las ideas Penfield y Haus por la comunidad científica como el correcto camino para una solución completamente y definitiva de la controversia, *el tercer y principal objetivo de esta tesis fue usar estos postulados para resolver con todo detalle un ejemplo de interacción entre la luz y un medio, donde se vea claramente que ambos tensores son igualmente correctos, mientras se apliquen bien las ecuaciones de balance totales.* Un problema particular muy simple, que ha sido estudiado por muchos autores desde que fue propuesto por primera vez en el año 1953 por Balazs [14], es el llamado *"experimento pensado de la caja de Einstein dieléctrica"*. Éste consiste en una modificación del experimento pensado de la caja de Einstein [15] de 1906, donde un pulso de luz cruza las paredes de una caja vacía inicialmente en reposo y la pone en movimiento. Einstein recurrió a este ejemplo para discutir sobre la conservación de la velocidad del centro de energía del sistema, que es una generalización relativista del centro de masa. Posteriormente Balazs toma esta idea, pero supone que la caja vacía ahora es un bloque dieléctrico con índice de refracción n y estudia el momentum que debe llevar la luz dentro de este medio. Debido a ciertas suposiciones, no justificadas a nuestro parecer, Balazs concluye que el momentum de la luz para esa situación particular al menos, debe ser dado únicamente por el momentum de Abraham. Posteriormente y hasta la actualidad, muchos autores han obtenido conclusiones erróneas, en nuestra opinión, al ser influenciados por este resultado, pues como nosotros mostramos en el capítulo 6, ésta situación no imposibilita el uso del momentum de Minkowski para el campo en el medio. Obukhov en [10], sigue la idea original de Penfield y Haus, y deriva en una notación más moderna la expresión para el tensor energía-momentum total del sistema campo más medio. Por completitud, incluímos este desarrollo en el capítulo 5, para posteriormente en el capítulo 6 poder aplicar estos resultados al problema particular de la caja dieléctrica de Einstein, donde realizamos un análisis completamente relativista detallado de la situación, calculamos explícitamente los momenta de Abraham y Minkowski para este caso y mostramos explícitamente que con ambos es posible describir la situación, pues el tensor total es el físicamente importante. Finalmente, damos razones de por qué Balazs y otros autores suelen encontrar sólamente el momentum de Abraham para describir correctamente este caso, identificando ciertas suposiciones no estrictamente justificadas y muy comunes en la literatura. Los resultados de este análisis serán publicados en [16].

Esta introducción, el resúmen y las conclusiones de la tesis están escritos en castellano, pero todos los demás capítulos del cuerpo principal están escritos en inglés. No hay que asustare por la cantidad de apéndices que incorporamos aquí, pues ellos son sólo material complementario u otros trabajos preliminares no directamente relacionados con la publicación [16]. Lo que sí es fundamental para entender el contenido del capítulo 6, son los cinco capítulos anteriores. Animo!

Chapter 2.

Motivation and historical review of the Abraham-Minkowski controversy

"La vida, para los optimistas, no es el problema, sino el remedio."

Marcel Pagnol,
poeta francés.

In order to better illustrate why the two different definitions for the momentum of light inside matter have led to so much confusion among authors, we will first shortly expose three simple theoretical arguments commonly found in the literature, which apparently do not have any contradictions, but support different momentum definitions.

After the three simple examples, we will present an historical review of the most important experiments and theoretical arguments that different authors have proposed during this hundred years of debate, from the early years until the present day. We think that this description of the most important advances in the area, will give us a global understanding of the problem and an idea of how the solution of Penfield and Haus, which we support, have gone very unnoticed in this debate. For other reviews about the subject, please see the introduction of [17], which is very good and concise and for a more detailed and historical reviews, not always free of confusion and contradictions, see [10, 13, 18, 19, 20, 21, 22].

2.1. Atom recoil and Doppler effect

Let us first consider a semi-classical example, even though we do not like it very much since it mixes concepts of classical and quantum physics, however, it is very popular in the literature. Consider that an atom of mass m with an internal transition of frequency ω_0, given by

$$\omega_0 := \omega_{\mathrm{f}} - \omega_{\mathrm{i}}, \tag{2.1}$$

is situated inside a linear, isotropic and homogeneous medium at rest, with refraction index n. Assume that the atom is initially moving with velocity v away from a monochromatic light source of frequency ω and then it absorbs a "photon", recoiling and changing its internal energy and its velocity to v', as it is shown in figure 2.1. It is important to clarify here that the photon is thought in a semi-classical sense, i.e. like a finite, localize and small electromagnetic wave packet.

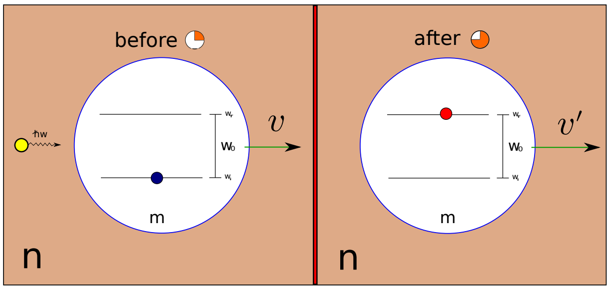

Figure 2.1.: An atom within a medium with refraction index n, absorbs a semi-classical photon and reacts by changing its internal energy and its velocity (recoil).

An atom will only successfully absorb a photon from the light source, if its velocity v is such that the (non-relativistic) Doppler shift makes the light be in resonance with the atom's transition, i.e.

$$\omega_0 = \left(1 - \frac{vn}{c}\right)\omega. \tag{2.2}$$

Additionally, we consider the conservation of energy during the process,

$$\hbar\omega + \left(\frac{1}{2}mv^2 + \hbar\omega_{\mathrm{i}}\right) = \frac{1}{2}mv'^2 + \hbar\omega_{\mathrm{f}}, \tag{2.3}$$

and the conservation of momentum,

$$mv + p = mv', \tag{2.4}$$

where p is the unknown momentum of the photon inside the medium.

If we insert (2.1) and (2.2) in (2.3), we obtain that the velocity difference of the atom, before and after the absorption, reads

$$v' - v = \frac{n\hbar\omega}{mc}\frac{2v}{v+v'}, \tag{2.5}$$

and if we replace (2.5) into (2.4), we find that p must be given by

$$p = \frac{n\hbar\omega}{c}\frac{2v}{v'+v}, \tag{2.6}$$

if the motion of the atom is non-relativistic, $v \ll c$, as it usually is.

Finally, if we consider that the energy of the photon $E = \hbar\omega$ is much smaller that the rest energy of the atom, we can assume that the velocity of the atom is eventually not affected by the recoil, i.e. $v' \approx v$ and therefore (2.6) reduces to

$$p \approx n\frac{E}{c} = p_M, \tag{2.7}$$

which indeed coincides with the Minkowski total momentum of light, integrating the result (1.3). Notice that in order to obtain the Minkowski momentum for light in this semi-classical situation, we assumed two approximations, $E = \hbar\omega \ll mc^2$ and $v \ll c$, in addition to the conservation of energy and momentum of the closed system.

2.2. De Broglie relation for a "photon"

Another semi-classical argument which is very popular in the literature is the following, related to the de Broglie relation of a semi-classical "photon". Consider a light beam of frequency ν and wavelength λ_0 striking a fixed isotropic and homogeneous crystal. From basic optics we know that when the light beam encounters the interface, it will be refracted and as a result it will change its velocity of propagation and wavelength, but not its frequency. Since the phase velocity of light inside the new medium at rest is given by $v_{(0)} = c/n$, the wavelength inside the medium λ should be related to the one in vacuum, by

$$\lambda = \frac{\lambda_0}{n}. \tag{2.8}$$

According to quantum theory, the momentum of a photon should be related to its wavelength, by the de Broglie relation,

$$p = \frac{h}{\lambda}, \tag{2.9}$$

where h is the Planck's constant. If we apply (2.9) for a photon inside a medium and we insert (2.8) in it, we obtain

$$p = n\frac{h}{\lambda_0} = n\frac{E_0}{c}, \tag{2.10}$$

where we used the well-know relation for a photon in vacuum $p_0 = E_0/c$. As it can be clearly noticed (2.10) corresponds again to the total Minkowski momentum of light, i.e. the integrated expression (1.3) and therefore we find another situation in which the Minkowski expression appears.

2.3. Dielectric Einstein box thought experiment

As a third example, we consider a modification of the famous thought experiment discussed by Einstein in 1906 [15], which we call the "dielectric Einstein box" thought experiment. Here we present the argument as it is usually found in the literature, but in chapter 6 we present our proper analysis of it.

Consider a dielectric slab of mass M with homogeneous and isotropic electromagnetic properties, floating in space. Its refraction index is n, its length is L and it occupies a finite volume V. The slab is initially at rest, but a semi-classical "photon" of total energy $\mathcal{E}_0 = \hbar\omega$ strikes it from vacuum at normal incidence putting it in motion with a *final* constant velocity $\boldsymbol{v}$. The slab is equipped with anti-reflection coatings so that the pulse can enter the slab at normal incidence without reflection and energy losses. The situation is sketched in figure 1.4. Since the photon and the slab form an isolated system, according to the first Newton law, the center of energy of the system should keep an uniform motion. Once the photon is completely inside the slab it reduces its velocity to c/n and therefore the slab will have to

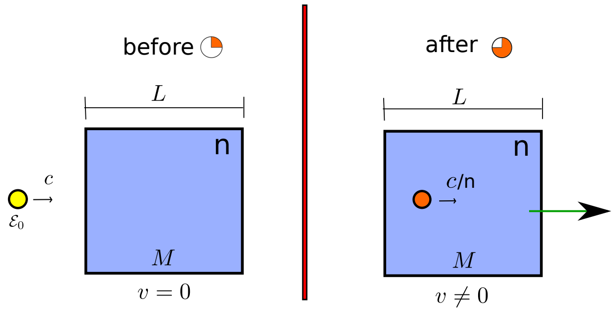

Figure 2.2.: A “photon” enters a dielectric slab initially at rest and puts it in motion.

move in the same direction of the photon in order to conserve the velocity of the center of energy of the whole system. The total energy when the photon is in vacuum is given by

$$E_{\text{tot}} = \hbar\omega + Mc^2, \tag{2.11}$$

which should be conserved during the whole process. Since in Special Relativity the mass is just a measure of the rest energy of an object, it is more convenient to talk about the *center of energy of the system* that about its center of mass. In this case, the velocity of the center of energy of the system is calculated just as if the photon and the slab were massive particles and therefore its values before and after the entrance of the photon read

$$v_{\text{ce1}} = \frac{c\hbar\omega}{\hbar\omega + Mc^2}, \tag{2.12}$$

$$v_{\text{ce2}} = \frac{(c/n)E + vE_{\text{s}}}{E + E_{\text{s}}}, \tag{2.13}$$

where E is the energy of the photon inside the slab and E_{s} is the total slab energy (rest energy plus kinetic energy) when it is moving. Using the fact that energy is a conserved quantity, we have

$$E_{\text{tot}} = E + E_{\text{s}} = \hbar\omega + Mc^2, \tag{2.14}$$

and neglecting the kinetic energy of the slab E_k with respect to its rest energy Mc^2, we obtain

$$E_{\text{s}} = Mc^2 + E_k \approx Mc^2. \tag{2.15}$$

Thus, replacing (2.15) into (2.13) and (2.14), the final velocity of the center of energy takes a simpler form,

$$v_{\text{ce2}} \approx \frac{vMc^2 + (c/n)\hbar\omega}{\hbar\omega + Mc^2}. \tag{2.16}$$

Comparing (2.12) and (2.16) we can find the final velocity v of the block, in terms of the energy of the incident photon $\mathcal{E}_0 = \hbar\omega$:

$$v \approx \frac{\mathcal{E}_0}{Mcn}(n-1) > 0. \tag{2.17}$$

As we see from (2.17), the slab will effectively move in the same direction as the photon, provided the same approximations as in subsection 2.1 are valid, i.e. $\mathcal{E}_0 = \hbar\omega \ll Mc^2$ and $v \ll c$.

Since we assume that the photon is moving at constant velocity c/n inside the medium, it will require a total time

$$\Delta t = \frac{n}{c} L, \tag{2.18}$$

in order to completely cross the slab of thickness L and therefore, the slab's net displacement Δx can be calculated as

$$\Delta x = v \Delta t \tag{2.19}$$

$$\approx (n-1) L \frac{\mathcal{E}_0}{Mc^2}. \tag{2.20}$$

Finally, imposing the conservation of linear momentum of the closed system,

$$\frac{\mathcal{E}_0}{c} = Mv + p, \tag{2.21}$$

where p is the momentum of the photon inside the slab, we can insert (2.17) in it and obtain the value of p for this case:

$$p \approx \frac{1}{n} \frac{\hbar\omega}{c} = p_A. \tag{2.22}$$

Therefore, under this analysis of the Einstein dielectric box, assuming the conservation of energy, momentum and of the center of energy velocity of the closed system, in addition to the usual approximations $\mathcal{E}_0 = \hbar\omega \ll Mc^2$ and $v \ll c$, we see that the momentum of the photon inside the slab has to be Abraham's, i.e. the integral of expression (1.4).

2.4. The early years

During the first 40 years after the formulation of the problem by Minkowski and Abraham, nobody was able to set an experiment in order to experimentally test which of the two formulations for the momentum of light in matter was the correct one. Only in 1919, Dällenbach [23] claimed to theoretically demonstrate the derivation of the Minkowski tensor from microscopic considerations, using the "electron theory", i.e. the Maxwell equations in vacuum. However, Pauli in his famous Special Relativity book of 1921 [24], argued that the argument of Dällenbach was not very cogent. Currently we know that the argument fails to justify the generalization of the fields from the electrostatic to the dynamic case, which is of course, crucial. In fact, Pauli believed in the Abraham formulation since he insisted that the energy-momentum tensor had to be symmetric.

In 1923, W. Gordon [25] wrote a very original paper, where he uses General Relativity to develop an argument in favor of the Abraham tensor. In simple terms, he considered the electromagnetic field inside a linear, isotropic and homogeneous medium as the source of gravitational field and found that the symmetric energy-momentum tensor that has to be in the right hand side of Einstein's equations is Abraham's one.

In 1939, Tamm [26] took up the discussion again concluding that the Minkowski's expression for the energy-momentum tensor is the correct one. He realized that there is no *a priori* reason for the tensor to be symmetric, as it was believed, since the electromagnetic field in matter is a non-closed system and also remarked that a *macroscopic* energy-momentum is not simply the average of the corresponding microscopic one. The macroscopic tensor must also satisfy the correct energy-momentum balance equation, as derived from macroscopic Maxwell equations. So, if the microscopic energy-momentum is symmetric, its average will be symmetric, but not necessarily the corresponding one in the macroscopic theory. Additionally, Tamm could show that in some special cases the Abraham tensor leads to wrong results, while the Minkowski's expression is in accordance with Maxwell's equations in vacuum.

Later, von Laue [27] in 1950 and similarly Møller [28] in 1952, recovered the arguments of Tamm and they found another argument in favor of the Minkowski formulation. In the simplest material medium, i.e. a linear, isotropic and homogeneous medium, we know from geometrical optics that the ray velocity defined from Huygens's principle should be identical to the velocity with which energy is propagated. Using some components of the energy-momentum tensor, the velocity of propagation of energy can be calculated, from where the authors concluded that only the Minkowski tensor satisfies this condition and not Abraham's. This argument was so influential that Pauli in the revised edition from 1958 of his book [29], inserted a supplementary note reversing his position and tending to suggest the the Minkowski tensor in "more likely to be right", but without a definite answer though.

On the other hand, Balazs in his paper of 1953 [14], was the first to propose the study of the dielectric Einstein box thought experiment, in a similar way as it was presented in 2.3, but with more general incidence directions of the light upon the slab. By assigning the momentum of a particle to the slab, he was able to show that the conservation of momentum and the conservation of the center of energy velocity of the total system can be only satisfied, if we assign the Abraham tensor to the electromagnetic field in matter, i.e. a result in agreement with the one of subsection 2.3. In chapter 6 we will present our analysis of this problem, which will be published in [16].

2.5. First experiments

In 1954, Jones and Richards [30] were the first braves to set up an experiment in order to discriminate between both definitions of the momenta of light inside matter. They basically suspended a mirror in a torsion fiber submerged in a dielectric liquid. By illuminating the mirror with light and measuring the torsion angle of the fiber, they were able to determine the torque exerted by light on the mirror and therefore its radiation pressure inside the dielectric. As a result, Jones and Richards observed that the radiation pressure was directly proportional, within about 1% to the refraction index of the liquid and therefore finding experimental evidence in favor of the Minkowski momentum. For more details about the experiment setup, see the original paper of the authors or also the review [18]. In spite of the good results in favor of the Minkowski momentum for light, the experiment of Jones and Richards went initially very unnoticed. The classical book "electrodynamics of continuous media" from Landau and Lifshitz [31] of 1960 did not even mention this experiment and uses the Abraham momentum as the correct expression.

One of the consequences of considering the Abraham momentum as the momentum of the field in matter, is that we have to assume an extra term in the momentum balance equation, which is called the Abraham force density. This term is also presented in the derivation of Landau and Lifshitz, giving for sure its existence though at that time nobody had measured it. In 1968, James in his doctoral thesis [32] performed an experiment to directly measure the Abraham force in order to determine if it was real or not. Two ferrite toroids, axially aligned with one another, were connected to a piezoelectric transducer. James applied time-varying electric and magnetic fields on the toroids and he measured the torque exerted on them by the electromagnetic field. According to James both, Minkowski's and Abraham's energy-momentum tensors, predicted a net torque on the toroids, but with differing magnitudes due to the existence of the Abraham force density. The results were consistent with the existence of the Abraham force, giving experimental evidence in favor of the Abraham momentum.

Despite the good results of James, his work went unnoticed until 1975, when G.B. Walker, Lahoz and G. Walker [33] performed a similar experiment. This team measured the torque exerted on a disc of barium titanate which was suspended on a torsion fiber in a constant axial magnetic field, and subjected to a time-varying radial electric field. A diagram of the experimental apparatus is shown in figure 2.3.

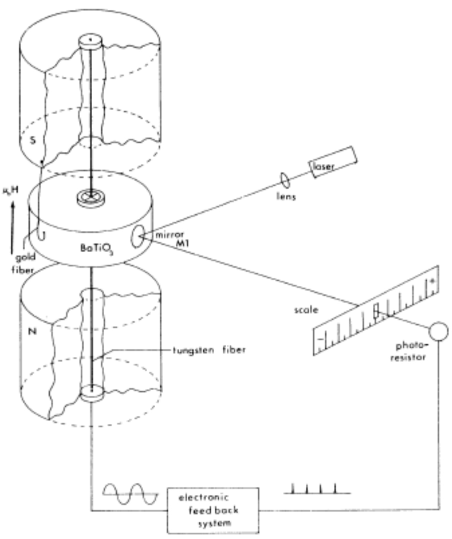

Figure 2.3.: Diagram of the experimental apparatus of Walker et al., taken from [33].

The disc behaved as a torsion pendulum and its period of oscillation was measured by reflecting a laser beam off a mirror attached to the disc. By experimentally determining the rotation period of the disc, the authors reported the existence of the Abraham force, within a 10% of error and for low rotation frequencies. In 1977, the experiment was repeated by G.B. Walker and G. Walker [34], but this time applying a time-varying magnetic field. The authors reported positive results again, supporting the momentum of Abraham for light in matter. The interpretation of the results were object of critics by Brevik [19] in 1979, who noticed various limitations in the experiment, but stressed its importance as a direct verification of the existence of the Abraham force.

On the other hand, in 1973, two years before the experiment of Walker et al. [33], Burt and Peierls [35] wrote a paper proposing a simple way to experimentally differentiate between the Minkowski and Abraham momenta. They assumed that a light pulse passes from vacuum into a medium of refractive index n, normal to the interface. By applying the rival expressions (1.3) and (1.4) to this particular simple case, together with the corresponding boundary conditions, Burt and Peierls concluded that according to Minkowski's momentum, there should be an outward force on the interface, whereas the Abraham's momentum implies an inward force or pressure on the surface. Thus, the consequences of the different momenta should be in principle observable. Besides proposing the latter test, Burt and Peierls supported the Abraham momentum and they give another argument in favor of it. However, we now know that its argument is wrong since they assume that a light pulse propagating inside a fixed medium is an isolated system, which is false. We will return to this in section 4.3.

In the same year and as a response to the paper [35], Ashkin and Dziedzic [36] performed an experiment to test the theoretical prediction of Burt and Peierls. They used a water surface as the interface between media and directed a laser beam from air into it, as shown in figure 2.4.

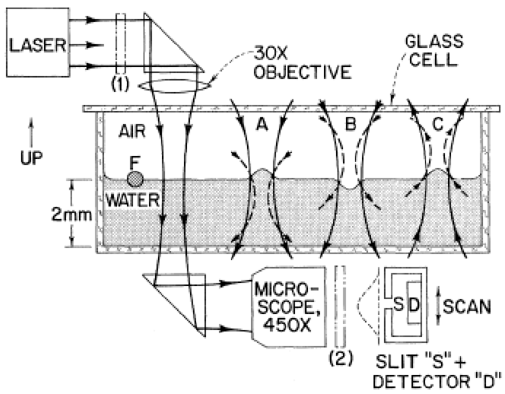

Figure 2.4.: Experimental setup of Ashkin and Dziedzic experiment. Image taken from [36].

Ashkin and Dziedzic expected that the momentum transfer from light to the fluid would cause the surface to either bulge outwards, if the Minkowski tensor was correct, or be depressed, if the Abraham tensor was correct. By studying the beam profile as it emerged from the cell, they were able to determine that the surface of the liquid was caused to bulge outwards and thus giving support to the Minkowski momentum, according to Burt and Peierls' prediction. Also in their paper, Ashkin and Dziedzic commented that they were aware of recent work by J.P. Gordon [37], at that time still unpublished, which showed that the result of their experiment did not invalidate the possibility of the Abraham tensor. Gordon modeled the material medium as a rarified gas of atoms and associated to it a mechanichal momentum in addition to the Abraham momentum for light, and therefore being able to describe the Ashkin and Dziedzic observed results also with the Abraham tensor for light.

2.6. Abraham and Minkowski equivalence ideas

In 1966, two years before the experiment of James, Penfield and Haus [11, 12, 13] made a big contribution in the understanding of the Abraham-Minkowski controversy which went very unnoticed until the mid-seventies. These authors modeled the material medium as an isotropic fluid and used the Lagrangian formalism and Hamilton's principle in order to derive an expression for the total energy-momentum of the closed system composed by electromagnetic field and material medium. In this manner, they showed that both, Minkowski and Abraham tensors, give incomplete descriptions of the total system if we use them alone, but if we complement each one with the appropriate tensor for the material medium, both give identical results. In other words, Penfield and Haus argued that *only the total energy-momentum tensor of the closed system has absolute physical meaning* and that the Minkowski, Abraham and other expressions for the electromagnetic field simply correspond to different separations of the same total tensor into different subsystems. The formalism of these authors also allowed to derive the correct energy-momentum balance equations, from where the electromagnetic forces exerted by the field on the medium could be obtained.

One year later, in 1967, de Groot and Suttorp [38]-[46] also recognized the important role of the material energy-momentum tensor in the description of the electromagnetic field in matter and that the conservation laws should only be applied to closed systems. These authors determined the total energy-momentum tensor of the system from microscopical considerations making statistical averages of the electromagnetic field in vacuum and its interaction with electrons and nuclei which constitute the medium. As a result, de Groot and Suttorp's total tensor disagreed with the one of Penfield and Haus. This difference does not bear directly upon the Abraham-Minkowski controversy, but does reflect different overall assumptions about the behaviour of matter in the presence of an electromagnetic wave, which have to be tested by experiment.

In 1975, Robinson [47] took the results of Penfield and Haus together with the ones of de Groot and Suttorp, J.P. Gordon and others, and wrote a complete review of the most important advances towards solving the controversy and stressed the importance of the arguments of Penfield and Haus, which were not generally known until that time. Robinson argued that in the past years the attention has been distracted from the significance of this work by the numerous papers in which authors tried to describe particular situations with ad-hoc arguments instead of engaging in the tedium of attempting an explicitly derived solution. Additionally, Penfield and Haus did not present they results in a concise and direct manner, but they wrote a book [13], where the conclusions relevant to this problem only emerge gradually, and somewhat indistinctly, in the course of some 250 pages of closely argued text, besides of using conventions and notations very unfamiliar to the general reader.

After Robinson's paper, the ideas of Penfield and Haus were much more spread in the scientific community and shortly appeared a large amount of publications which extended their arguments to more general cases, for instance [48, 49, 50, 51, 52, 53, 54, 55, 56]. Particularly remarkable is the work of Mikura [48], who in 1976 also using a Lagrangian method, calculated an explicit and completely covariant expression for the total energy-momentum tensor assuming the electromagnetic field interacting with a non-viscous, compressible, non-dispersive, polarizable and magnetizable fluid. Israel [49] in 1977, analysed the experiments [33, 34] of Walker et al. and proved that its results can be equally well described with either Minkowski or Abraham tensors. Therefore, it is not necessary to assume the existence of the Abraham force and the experiment of Walker et al. (as well as the one from Ashkin and Dziedzic), does not give any information to differentiate which of the tensor alternatives is the "correct" one. In fact, if the original ideas of Penfield and Haus are right, no experiment can show us the correct energy-momentum tensor for the field since we can only measure the total energy-momentum, which enters in the balance equations. At the end of his paper, Israel also emphasized that his success in casting Abraham's formulation into an unambiguous and simple form depended crucially on the simplifying assumptions he made (existence of a well-defined rest frame, absence of internal spin, simple constitutive relations), but the Minkowski formulation is independent of such details and its range of validity correspondingly much wider. In chapter 6 we will return to this observation.

Meanwhile this theoretical discussion was taken place, the experimentalists did not waste their time either. In 1978, Jones and Leslie [57] repeated the experiment of Jones and Richards [30] of 1954, but this time they confirmed with a precision of 0.05%, that the momentum associated with the radiation pressure of light inside a liquid increases directly with the refraction index of the medium into which it passes. The great improvement of precision also allowed them to discriminate substantially in favor of the phase velocity ratio and against the group velocity ratio, even though we know that energy propagates with the group one.

Later, in 1980, Gibson et al. [58] performed a new type of experiment in order to determine the radiation pressure of light inside a material medium by measuring the *photon drag effect*. This effect is characterized by the generation in a semiconductor of an electric field due to the transfer of momentum from radiation to the electrons of the valence or conduction bands of the material. By making photon drag effect measurements at sufficient long infrared wavelengths with samples of germanium and silicon, Gibson et al. concluded that the Minkowski expression for the momentum of light correctly described their experimental results. However, Brevik [59] in 1986, reanalysed the interpretation of this experiment and as it is usual by now, he stated that the experimental results do not invalidate the Abraham momentum for light. If one assigns to the light beam the Abraham momentum as the "electromagnetic momentum" and additionally a "mechanical momentum" so that their sum gives the Minkowski momentum, then the experimental results can also be interpreted with the Abraham momentum. Of course, the Abraham momentum alone does not correctly reproduce the results as Gibson et al. in [58] concluded.

Brevik, in his review of 1979 [19], predicted that the mechanical effect of light's angular momentum on a macroscopic body would be the same in a dielectric medium as in vacuum, i.e. that the total angular momentum of a light beam inside a dielectric would be independent on the refraction ondex of the medium n. No matter if we assign the Abraham or the Minkowski momentum to the light beam, its total angular momentum would be the same, contrary to the case of linear momentum, where both momenta definitions differ in a factor n^2. It was in 1994, when Kristensen and Woerdman [60] designed and experiment to test Brevik's prediction. It is surprising that in the whole literature about the Abraham-Minkowski controversy, this experiment is the only performed experiment which measures light's angular momentum inside a medium. Basically, they placed a dipole antenna within two circular waveguides, one filled with a dielectric liquid and the other empty. By emitting microwaves in the TE_{11} mode, their antenna was able to measure the total angular momentum of the electromagnetic radiation, obtaining the same results with and without the liquid and thus confirming Brevik's prediction within four standard deviations.

2.7. Revival of the discussion

As we have seen, the Abraham-Minkowski controversy has been advancing historically, but not free of confusion and contradictory arguments. Despite some authors believed that the fundamentals of Abraham-Minkowski controversy was already solved since the 1980s, in the past 10 years, the discussion of the momentum of light inside media has become relevant again. However, the revival of the controversy was not oriented to obtain important experimental confirmations or theoretical extensions of the formal solution already discussed, but there have been a large number of more practical and "optical oriented" works of Loudon, Barnett and collaborators [61, 62, 63, 64, 65, 66, 67, 68], Mansuripur [69, 70, 71, 72] and others [73, 74, 75] who seem to be completely unaware of the previous resolution of the controversy and continue trying to demonstrate which of the two tensors is the correct one. In fact, have not found any work in the literature where the ideas of Penfield and Haus are criticized. If we look at the references of all the latter papers, they do not cite any theoretical work of Penfield and Haus, de Groot and Suttorp, Israel, Mikura, etc. At most they cite the paper review of Brevik from 1979 [19] or the one of J.P. Gordon [37], so they are trying to find a completely disconnected solution of the controversy in a manner much more restrictive, in our opinion, than the one of Penfield and Haus.

This new approach is based on the assumption of the force density exerted by the field on the bound charges and currents of the medium, avoiding the use of any *a priori* expression for light's momenta. For instance, in [61, 63, 65, 66], Loudon and Barnett directly calculate the radiation pressure exerted by light, finding that the Abraham and Minkowski momenta correctly describe different situations. The ambiguity here lies in the choice of the expression for the force density. Loudon and Barnett use the usual Lorentz force density, but one could in principle choose different models to describe the medium, for instance, a medium formed by little dipoles or by individual charges, etc. Loudon and Barnett analyse the different force choices concluding that in most cases they give identical results, but not always. Additionally, Mansuripur in [69, 70, 71, 72] also applies a similar approach, but he uses a modified definition of the force density, sometimes obtaining different results as Loudon and Barnett.

In 2003, Padgett, Loudon and Barnett [62], considered a "rotational" version of the dielectric Einstein box thought experiment, i.e. a short pulse of light carrying angular momentum propagating through a transparent disc. According to their calculations, the authors claimed that the disc will or not rotate, depending if the momentum of light is given by the Abraham or Minkowski momentum. By suggesting that the disc will rotate, they gave support to the Abraham momentum.

In 2005, Loudon, Baxter and Barnett in [64], applied their Lorentz force approach to analyse the radiation pressure in the photon drag effect experiment and they agreed with experimental results of Gibson et al. in [58] in that the momentum transfer to the charge carriers alone is given by the Minkowski value. However, when the pulse is much shorter than the sample thickness, they reported that there must be a clear separation in time between surface and bulk contributions to the forces, being the total bulk momentum transfer (charges plus host) given by the Abraham expression.

Also starting from the Lorentz force, Hinds and Barnett [68] in 2009, studied the interaction between a plane wave and a unique atom with dielectric dipole moment. For this particular example, the authors found a relation between the Abraham and Minkowski expressions for the momentum of light, which allowed them to interpret the Abraham momentum as the "kinetic momentum" of the field and the Minkowski momentum as the "canonical momentum" of the field.

Particularly interesting and different from the works of Loudon, Barnett and collaborators is the work of Garrison and Chiao [76] of 2004, who assumed an isotropic, homogeneous and weakly dispersive medium at rest, and quantized the electromagnetic field inside it. They calculated three different linear momentum operators inside the medium. First, the canonical momentum density which is the generator of translations within the medium and corresponds to the total momentum operator of the system, including field and medium. Then they computed the Abraham and Minkowski momentum operators for the electromagnetic field which, as we know, are only one part of the total momentum of the system. Additionally, Garrison and Chiao took the experimental data of the experiment of Jones and Leslie [57] and compared them with the three operators expressions already derived. The authors concluded that the canonical momentum (total momentum inside matter) was the one which best fitted the experimental data, even better than Minkowski's tensor as claimed by Jones and Leslie in their experiment. The authors final conclusion was that the results obtained are consistent with assigning to each *dressed photon* inside the medium (photon coupled with matter), a unity $\hbar k$ of canonical momentum, but they argued that Abraham momentum is still necessary to describe cases when the center of mass of the medium is accelerating (they could not quantize the field for an accelerating medium).

In the recent literature one can also find some new performed experiments. For instance, in 2005 Campbell et al. [77], illuminated a Bose-Einstein condensate in order to infer the momentum of light inside this dispersive medium. The method was based on using interferometric techniques to measure the systematic shift of the *recoil frequency* of the atoms within the dilute gas, after they had absorbed a photon. The results showed that the recoil momentum of atoms caused by the absorption of a photon is $n\hbar k$, where n is the index of refraction of the gas and k the vacuum wave vector of the photon. Therefore, this experiment gave evidence in favor of the Minkowski tensor, in accordance with the first thought experiment presented in this chapter 2.1. Maybe the canonical expression of Garrison and Chiao can also better describe this results than the Minkowski momentum, but this is not reported in the literature. One year later, as a response to this experimental paper, Leonhardt [74] in 2006 studied in more detail the balance equations for energy and momentum within what he called a "quantum dielectric", such as the Bose-Einstein condensate considered in [77]. Leonhardt realized that the total momentum of the system can be expressed either using the Abraham or the Minkowski momentum for the field, but with different interpretations and he claimed that the variations of the Minkowski momentum are imprinted onto the phase of the condensate, whereas the Abraham tensor drives the flow of the dielectric medium.

In contrast to all the current works already discussed so far in this section, there are also current authors who believe in the original ideas of Penfield and Haus, and try to bring them again into the discussion. The review paper of Pfeifer et al. [18] of 2007 is particularly strong in this, where the authors, besides reviewing in detail the historical evolution of the controversy, stress the fact that the problem was solved in a formal manner in the mid-seventies. Pfeifer et al. could not understand why this new authors seem to be unaware of the resolution and continue giving arguments in favor of one or another momentum version. In another publication of the same authors [78] of 2009, they analyse the limitations of describing a system only with the Minkowski momentum, neglecting its material part. By comparing this tensor with the total tensor and the Abraham tensor alone, the authors gave an intuitive idea to explain why experiments with the medium fixed are usually well described only by Minkowski's momentum and experiments where the motion of the medium is important are better described with Abraham's one.

Other current authors who support ideas along these lines are Obukhov and Hehl. In their paper of 2003 [79], these two authors proposed a different way to calculate the energy and momentum transfer between the medium and the electromagnetic field. Since it is unimportant which tensor is assigned to each subsystem, while keeping the total balance equations unchanged, Obukhov and Hehl proposed to describe the electromagnetic field with the same form of the symmetric tensor in vacuum, but this time inside matter. If one explicitly calculate the energy-momentum balance equation for that tensor, one will see that the interactions between field and matter can be described by the Lorentz force exerted on the polarization and magnetization induced charges and currents of the medium. Finally, by calculating the corresponding forces and torques, Obukhov and Hehl could reproduce the results measured in the James experiment [32] of 1968 and also in the Walker et al. experiment [33, 34] of 1975 and thus giving positive evidence for the ideas of Penfield and Haus.

In 2007, Obukhov and Hehl [80] used a Lagrangian variational approach similar to Penfield and Haus, but in a more modern form and notation, in order to derive an explicit expression for the total energy-momentum tensor of an isotropic dielectric and magnetoelectric fluid interacting with the electromagnetic field. The authors applied their results to the particular case of the Jones et al. experiment [30, 57] and they successfully obtained the already confirmed experimental results, i.e. that the radiation pressure of light inside the medium is directly proportional to the index of refraction of the medium.

In 2008, Obukhov [10] published a very detailed review of the Abraham-Minkowski controversy, making very detailed calculations, which we found very comprehensive and understandable. The author reproduce in a modern form the original variational Lagrangian approach of Penfield and Haus and using this method, he was able to explicitly calculate the Minkowski, Abraham and total tensors for an isotropic dielectric and diamagnetic medium, but this time with much more details and including electro- and magnetostriction effects. Finally, Obukhov found a relation between the Abraham and the Minkowski tensors valid in isotropic media, which suggest that the Abraham tensor could be *only* useful in this particular case of media. In more complex media it will continue to be valid if we assign to it the correct material part, but it could lose its practical applicability. This observation inspired us to study the conditions under which the Abraham tensor may be not so useful and finally led to our publication and also to other preliminary work not presented in this thesis. We will return to this issue in chapters 5 and 6.

At the end of 2008, She et al. [81] reported the most recent experiment performed about the Abraham-Minkowski controversy. Basically, the experiment consisted in a sophistication of the Ashkin and Dziedzic [36] experiment from 1973, replacing the water surface by a nanometer silica filament (SF), fixed in one of its ends. She et al. let light travel in the SF and then emerge into air or vacuum from its free end. They thought that the free end of the SF will be pushed to move backward if Abraham momentum applies or be pulled forward if Minkowski momentum applies, when light emerges from the free end in an analogous way as the possibilities the Ashkin and Dziedzic experiment [36]. An image sequence with the visual results of SF experiment is shown in figure 2.5. The authors reported a direct observation of a push force on the end face of the SF exerted by the outgoing light and thus finding evidence in favor of the Abraham momentum, according to their very idealized considerations and calculations. As a response to this paper, Mansuripur and Zakharian [82,

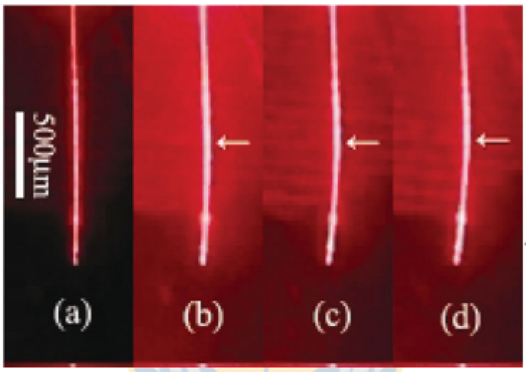

Figure 2.5.: Image sequence obtained in the experiment of She et al., taken from their paper in [81].

83] wrote a quick comment and then a detailed paper, strongly criticizing the interpretation of the results in the experiment of She et al.. The authors stressed the weak points of the analysis of She et al. and reanalysed the situation integrating the Lorentz force density exerted by the light pulse in its entire path through the nanofilament. With their numerical results, Mansuripur and Zakharian concluded that the net effect of a short pulse exiting the nanofilament should be a pull force on the end face of the filament, instead as the observed push of She et al. They explained the clear qualitative difference between their theoretical predictions and the experiment results by stating that they did not consider the possibility of mechanical momentum diffusion out of the filament, for instance in form of acoustic waves, which need a much more complicated analysis. The authors concluded by remarking that the idea of monitoring the mechanical response of a SF under the action of a light pulse can provide very important information about light momentum transfer in media, but more detailed theoretical analysis is needed and also more precise measurements to conclude something with certainty. The discussion did not finish here, because Brevik also criticized the experiment [84] and She et al. replied [85].

Very recently, in 2010, Barnett and Loudon in [86, 87] claimed that they have solved the Abraham-Minkowski controversy. This sounds strange, because according to Penfield, Haus and others, the controversy was already solved a long time ago. They used their previous results from the Lorentz force and the new observation in [68] to argue that the Abraham and the Minkowski momenta are both correct. They identify the Abraham momentum as the "kinetic" momentum of the electromagnetic field and use the Balazs idea as their strongest argument to discard the Minkowski momentum in that situation. In other types of experiments, where the medium is fixed and at rest, it is claimed that the Minkowski momentum correctly describes the situation and therefore the authors identify it as the "canonical" momentum of the field. So far the argument is very similar to the one of Penfield and Haus, but the difference is that Loudon and Barnett affirmed that using the Abraham or the Minkowski momenta is not a matter of choice, but that both can be measured in mutually exclusive situations in a kind of "complementarity postulate".

A couple of months later, in 2010, Mansuripur [88, 89] also claimed that he had solved the controversy, but in a different manner as Loudon and Barnett. One difference is that the author recognizes that the Einstein box argument does not uniquely determine the momentum of the field, but despite that he insists that this is the strongest available argument for the identification of light's momentum inside media and therefore considers the definition of the Abraham momentum for the field as an additional postulate of his theory. In [72] Mansuripur explains his theory, where he also defines a modified version of the Lorentz force and applies his postulates to solve different particular situations.

On the other hand, Saldanha in [17] also proposes his own resolution to the controversy. This time the author recognizes that the controversy is already solved for a long time and that all momenta definitions for the field are valid if we use the correct total balance equations, but despite that he proposes a particular separation, different from Abraham and Minkowski, which he argues to be more natural and convenient.

As we have seen, the Abraham-Minkowski controversy has been characterized by a large degree of disagreement among the different authors. In fact, we are not overstating if we say that each author has his own resolution. We hope that this thesis could contribute to bring clarity to this confusing debate, since we think that the answer is *simple*: The macroscopic Maxwell equations, the constitutive relations and the dynamical equations for the medium is all what is needed to derive the total energy-momentum balance equations of this coupled system and all the other choices are merely arbitrary and give equivalent physical predictions. Therefore the main objective of our publication [16] is to present this ideas in a more familiar manner and put the into the current discussion, so that authors like Barnett, Loudon, Mansuripur, etc. can understand and consider them in their analysis. Maybe this time we could find agreement between the different areas studying this problem. Meanwhile the debate continues, see Brevik [90, 91].

Chapter 3.

Covariant formulation of macroscopic electrodynamics

"Antes las distancias eran mayores, porque el espacio se mide por el tiempo."

Jorge Luis Borges,
escritor argentino.

In order to study in more depth the properties of the different energy-momentum tensors postulated for the electromagnetic field inside matter, it will be useful to express the whole theory of macroscopic electrodynamics in a manifestly covariant form, using the language of Lorentz tensors in Minkowski spacetime. This formalism automatically guarantees relativistic covariance of the theory, which will supply us an adequate framework to describe the electrodynamics of moving media in any inertial reference frame.

Additionally, the covariant formulation of electrodynamics will facilitate the application of the Lagrangian-Noether formalism in chapters 4 and 5. Another advantage of the explicitly covariant electrodynamics is that it allows a direct generalization to include gravitational effects, described by a curved spacetime in the tensorial framework of General Relativity. We will assume, however, that the gravitational effects are negligible throughout all the thesis and hence only Special Relativity will be needed.

In section 3.4 of this chapter we derive the explicit expression for the constitutive relations of an isotropic medium in motion. The expressions obtained will describe the electromagnetic properties of the medium in any inertial reference frame and they will be of great importance in order to consider the explicit expressions for the Abraham and Minkowski energy-momentum tensors in chapters 5 and 6.

3.1. Macroscopic Maxwell equations

In this thesis we consider the propagation of light inside macroscopic material media. From a classical point of view, light is an electromagnetic wave with wavelengths of the order $\sim 10^{-7}m$ and therefore it can macroscopically interact with continuum media without detecting the atomistic nature of matter, which is important in scales of the order $\sim 10^{-10}m$. Material media are characterized for having bound charges and currents, which internally redistribute under the action of a local electromagnetic field, for instance a light wave inside matter. As a macroscopic effect, the material medium gets polarized and magnetized, creating an induced electromagnetic field, which superposes with the electromagnetic field of light propagating inside it.

The dynamics of the electromagnetic field inside such a continuous material medium is described by the well-known *macroscopic Maxwell equations*, which in SI units, they are given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_{\text{ext}}, \tag{3.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{3.2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{3.3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{j}_{\text{ext}} + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{3.4}$$

where $\boldsymbol{E}$ is the *electric field strength*, $\boldsymbol{B}$ the *magnetic field strength*, $\boldsymbol{D}$ the *electric excitation* and $\boldsymbol{H}$ the *magnetic excitation*. In the case that the medium is not macroscopically neutral or if there are additional sources of *free charges and currents* moving in the material medium, they are described by the external charge and current densities ρ_{ext} and $\boldsymbol{j}_{\text{ext}}$. If the electromagnetic properties of the material medium are assumed to be fixed, without dynamics, then the electromagnetic field inside matter described by (3.1)-(3.4) constitute an open system, which interacts with the material medium and external charges and currents. Maxwell equations (3.1)-(3.4) are 8 equations for 12 unknown fields, so more information is needed to solve for the electromagnetic fields, given the sources. This extra information is supplied by the so-called *constitutive relations*, which describe all the electromagnetic properties of a given medium and are represented in general as 2 functional vector equations of $\boldsymbol{D}$ and $\boldsymbol{H}$ in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\boldsymbol{D} = \boldsymbol{D}\left[\boldsymbol{E}, \boldsymbol{B}\right], \tag{3.5}$$

$$\boldsymbol{H} = \boldsymbol{H}\left[\boldsymbol{E}, \boldsymbol{B}\right]. \tag{3.6}$$

Depending on the nature of the relations (3.5)-(3.6), the material medium can be (non)linear, (non)dispersive, (non)dissipative, (an)isotropic, (in)homogeneous, dielectric, diamagnetic, ferroelectric, ferromagnetic, magneto-electric, etc. All these cases can be phenomenologically described by the constitutive relations.

The macroscopic Maxwell equations (given the sources ρ_{ext} and $\boldsymbol{j}_{\text{ext}}$), together with the constitutive relations (given the medium properties) form a set of 14 equations for 12 unknown fields and therefore the electromagnetic field inside matter can be solved: $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{H}$.

The conservation of electric charge is a fundamental fact of nature and therefore James Clerk Maxwell himself modified the electromagnetic equations known at his time, so that the electromagnetic theory (i.e. Maxwell equations) implies the local conservation of charge in the form of a continuity equation,

$$\frac{\partial \rho_{\text{ext}}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j}_{\text{ext}} = 0. \tag{3.7}$$

The macroscopic response of the medium to the applied electromagnetic field is quantified by its polarization $\boldsymbol{P}$ and its magnetization $\boldsymbol{M}$ and, after solving for the electromagnetic field inside the macroscopic medium, we can determine $\boldsymbol{P}$ and $\boldsymbol{M}$, using the following relations:

$$\boldsymbol{P} = \boldsymbol{D} - \varepsilon_0 \boldsymbol{E}, \tag{3.8}$$

$$\boldsymbol{M} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{H}. \tag{3.9}$$

The universal constants μ_0 and ε_0 in (3.8)-(3.9) are the vacuum permeability and permittivity, respectively. In SI units, we have $\mu_0 = 4\pi \cdot 10^{-7}(N/A^2)$ and $\varepsilon_0 = 8,854188 \cdot 10^{-12}(A^2 s^2/Nm^2)$. The velocity of light in vacuum is related to these fundamental constants by $c := 1/\sqrt{\mu_0 \epsilon_0}$ and currently it is *defined* to be $c := 2,99792458 \cdot 10^8 (m/s)$. The rules to express all the electromagnetic quantities also in the gaussian system of units are summarized in table D.1.

On the other hand, the connection of the theory of macroscopic electrodynamics to the mechanics of charged bodies is established through the Lorentz force $\boldsymbol{F}_{\text{ext}}$. The total force which the electromagnetic field inside matter exerts on any given external distribution of charge and current inside a volume V is given by

$$\boldsymbol{F}_{\text{ext}} = \int_V \boldsymbol{f}_{\text{ext}} \, d^3x, \tag{3.10}$$

where $\boldsymbol{f}_{\text{ext}}$ is the Lorentz force density, defined as

$$\boldsymbol{f}_{\text{ext}} := \rho_{\text{ext}} \boldsymbol{E} + \boldsymbol{j}_{\text{ext}} \times \boldsymbol{B}. \tag{3.11}$$

We can also compute the total power of work which the electromagnetic field transfers to the external currents via the Lorentz force (3.10). The magnetic field $\boldsymbol{B}$ does no work since the magnetic contribution of the Lorentz force is perpendicular to the current density $\boldsymbol{j}_{\text{ext}} := \rho_{\text{ext}} \boldsymbol{v}$ at each point and hence to the velocity field $\boldsymbol{v}$. Using the standard definition of power of work density P_{ext}, which is valid inside any volume element of the distribution, we have

$$P_{\text{ext}} := \boldsymbol{f}_{\text{ext}} \cdot \boldsymbol{v} \tag{3.12}$$

$$= \boldsymbol{j}_{\text{ext}} \cdot \boldsymbol{E}. \tag{3.13}$$

Then, if we sum over all the volume occupied by the distribution of charge and current, we obtain the total power of work done by the electromagnetic field on the external currents $\int_V \boldsymbol{j}_{\text{ext}} \cdot \boldsymbol{E} \, d^3x$.

If we inspect Maxwell's equations (3.2) and (3.3), we see that they are homogeneous and do not depend on any given sources, so that they relate different components of the macroscopic electromagnetic field. In fact, these 2 equations can always be solved identically if we introduce the electromagnetic scalar and vector potentials ϕ and $\boldsymbol{A}$ as

$$\boldsymbol{E} =: -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{3.14}$$

$$\boldsymbol{B} =: \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{3.15}$$

Therefore, instead of using $\boldsymbol{E}$ and $\boldsymbol{B}$ we can solve for the potentials ϕ and $\boldsymbol{A}$, which reduces in two the components of independent fields. However, the fields which appear directly in the Lorentz force density (3.11) and hence which can be directly measured are $\boldsymbol{E}$ and $\boldsymbol{B}$. Because of the definitions (3.14)-(3.15), there exists a freedom for determining the potentials, given $\boldsymbol{E}$ and $\boldsymbol{B}$, which is called a *gauge transformation*:

$$\phi' = \phi - \frac{\partial \xi}{\partial t}, \tag{3.16}$$

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\xi, \tag{3.17}$$

where $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x}, t)$ is any well-behaved function of position and time. From (3.16)-(3.17) we can arbitrarily choose any set of fields ϕ and $\boldsymbol{A}$ satisfying (3.14)-(3.15), in order to do a specific calculation and the results will be independent of that choice, i.e. the gauge invariance is a symmetry intrinsically present in the theory of electrodynamics.

3.2. Four-dimensional representation of electrodynamics

Now we will exploit another symmetry intrinsically present in the theory of macroscopic electrodynamics, which is the *Lorentz covariance*. The *Lorentz transformations* are all linear transformations which keep invariant the *interval* ds^2, defined by

$$ds^2 := c^2 dt^2 - d\boldsymbol{x}^2. \tag{3.18}$$

Even though ds^2 is not always positive, we can imagine that it represent a special kind of "line element" in a four-dimensional *spacetime*. Let us denote the coordinates of any event in this 4-D spacetime by

$$x^\mu := (ct, \boldsymbol{x}), \tag{3.19}$$

where t and $\boldsymbol{x}$ are the usual time and space coordinates in the euclidean 3-D space. Greek indices label the spacetime components of *Lorentz tensors* by $\mu, \nu, \rho, ... = 0, 1, 2, 3$ and cartesian 3-D components of objects are denoted by latin indices $i, j, k, ... = 1, 2, 3$, so that the spatial components of the 4-*vector* x^μ are x^i and its temporal component is $x^0 = ct$.

Using the definition (3.19), we can express ds^2 in (3.18) as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{3.20}$$

where $\eta_{\mu\nu}$ is a second rank Lorentz tensor known as the *Minkowski metric* of the spacetime, whose components are given by

$$\eta_{\mu\nu} := \mathrm{diag}(1, -1, -1, -1). \tag{3.21}$$

The metric tensor $\eta_{\mu\nu}$ describes the local geometry of the four-dimensional spacetime, which is therefore known to have "Minkowskian geometry".

Explicitly, a Lorentz transformation can be expressed as,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{3.22}$$

where the coefficients $\Lambda^{\mu}{}_{\nu}$ must satisfy the condition,

$$\eta_{\mu\nu} = \Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\eta_{\rho\sigma}. \tag{3.23}$$

The transformation (3.22), together with the condition (3.23), can be geometrically interpreted as a kind of "rotation" of a coordinate system in the four-dimensional Minkowski spacetime, which can be decomposed in boosts (the relativistic transformation between inertial frames), 3-D spatial rotations of the coordinate axes, temporal and space inversions.

Returning to the macroscopic electromagnetic theory, we can use the 3D indicial notation to write the Maxwell equations (3.1)-(3.4), the definitions of the potentials (3.14)-(3.15), the gauge transformation (3.16)-(3.17), the continuity equation (3.7), the Lorentz force density (3.11) and the power density of electromagnetic work (3.13), in terms of the cartesian components of the fields, by

$$\begin{aligned}
\partial_i D^i &= \rho_{\text{ext}}, && (3.24)\\
\partial_i B^i &= 0, && (3.25)\\
\epsilon^{ijk}\partial_j E_k &= -\frac{\partial B^i}{\partial t}, && (3.26)\\
\epsilon^{ijk}\partial_j H_k &= j^i_{\text{ext}} + \frac{\partial D^i}{\partial t}, && (3.27)\\
E_i &= -\partial_i\phi - \frac{\partial A_i}{\partial t}, && (3.28)\\
B^i &= \epsilon^{ijk}\partial_j A_k, && (3.29)\\
\phi' &= \phi - \frac{\partial \xi}{\partial t}, && (3.30)\\
A'_i &= A_i + \partial_i \xi, && (3.31)\\
\frac{\partial \rho_{\text{ext}}}{\partial t} + \partial_i j^i_{\text{ext}} &= 0, && (3.32)\\
w^{\text{ext}} &= j^i_{\text{ext}} E_i, && (3.33)\\
f_i^{\text{ext}} &= \rho_{\text{ext}} E_i + \hat{\epsilon}_{ijk} j^j_{\text{ext}} B^k, && (3.34)
\end{aligned}$$

where the 3-D *Levi-Civita pseudo-tensor* is defined such that

$$\epsilon^{123} := \hat{\epsilon}_{123} := 1. \tag{3.35}$$

Notice that the cartesian components of $\boldsymbol{D}$, $\boldsymbol{B}$ and $\boldsymbol{j}_{\text{ext}}$ are denoted by contravariant vector components D^i, B^i and j^i_{ext}, whereas the cartesian components of $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{A}$, $\boldsymbol{f}_{\text{ext}}$ and $\boldsymbol{\nabla}$ are denoted by covariant vector components E_i, H_i, A_i, f_i^{ext} and ∂_i. Lorentz scalars are formed by contractions of contravariant with covariant objects and therefore these definitions are useful in order to better find the manifestly covariant form of Maxwell's equations.

It is an experimental fact that electric charge is locally conserved in all inertial reference frames. This is incorporated in the theory by assuming that the external charge and current densities are components of a 4-vector called the *4-current density* J^{μ}_{ext} in the form,

$$J^{\mu}_{\text{ext}} := (c\rho_{\text{ext}}, j^i_{\text{ext}}), \tag{3.36}$$

so that the continuity equation (3.32) can be cast in the Lorentz-covariant form,

$$\boxed{\partial_\mu J^\mu_{\text{ext}} = 0.} \tag{3.37}$$

The electric and magnetic excitations D^i and H_i form a second rank antisymmetric tensor, called the *excitation tensor* $H^{\mu\nu}$, whose components are identified as

$$H^{i0} := cD^i, \qquad H^{ij} := -\epsilon^{ijk} H_k, \tag{3.38}$$

or in matrix form by,

$$H^{\mu\nu} = \begin{pmatrix} 0 & -cD^1 & -cD^2 & -cD^3 \\ cD^1 & 0 & -H_3 & H_2 \\ cD^2 & H_3 & 0 & -H_1 \\ cD^3 & -H_2 & H_1 & 0 \end{pmatrix} = -H^{\nu\mu}. \tag{3.39}$$

Then, the inhomogeneous Maxwell equations (3.1) and (3.4) can be very simply written in a fully relativistically covariant form as,

$$\boxed{\partial_\mu H^{\mu\nu} = J^\nu_{\text{ext}},} \tag{3.40}$$

where $\partial_\mu := (c^{-1}\partial/\partial t, \partial_i)$ is the 4-D gradient operator. By taking the 4-divergence of (3.40), one can directly check that the Maxwell equations imply (3.37).

In the same spirit, we can assume that the scalar and vector potentials are the components of the *electromagnetic* 4*-potential* A_μ defined by,

$$A_\mu := \left(\frac{1}{c}\phi, -A_i\right), \tag{3.41}$$

then, to express the relations between E_i, B^i and ϕ, A_i in (3.28)-(3.29), we define another second rank antisymmetric tensor $F_{\mu\nu}$, called the *electromagnetic strength tensor*, as

$$\boxed{F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu,} \tag{3.42}$$

whose components explicitly read,

$$F_{i0} := -\frac{1}{c}E_i, \qquad F_{ij} := -\hat{\epsilon}_{ijk} B^k, \tag{3.43}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B^3 & B^2 \\ -E_2/c & B^3 & 0 & -B^1 \\ -E_3/c & -B^2 & B^1 & 0 \end{pmatrix} = -F_{\nu\mu}. \tag{3.44}$$

With the definition of $F_{\mu\nu}$, the homogeneous Maxwell equations can be cast in the following covariant form,

$$\boxed{\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0,} \tag{3.45}$$

which is identically satisfied by replacing the definition (3.42) into it. Furthermore, the gauge transformation (3.30)-(3.31) can be covariantly expressed as

$$\boxed{A'_\mu = A_\mu - \partial_\mu \xi.} \tag{3.46}$$

The Lorentz force density f_i and the power density of work w_{ext} are components of a more general object called the Lorentz 4-force density $\mathcal{F}_\mu^{\text{ext}}$, written in terms of $F_{\mu\nu}$ and J^μ_{ext} as,

$$\boxed{\mathcal{F}_\mu^{\text{ext}} := F_{\mu\nu} J^\nu_{\text{ext}}.} \tag{3.47}$$

Explicitly, we have the identification $\mathcal{F}_\mu^{\text{ext}} := (P_{\text{ext}}/c, -f_i^{\text{ext}})$, where P_{ext} is the power density of the work done by the electromagnetic field on the external currents (3.33) and f_i^{ext}, is the Lorentz force density exerted by the electromagnetic field on the charges and currents (3.34).

3.3. Covariant constitutive relations for linear and non-dispersive media

In the covariant expressions of section 3.2, we have not used the spacetime metric $\eta_{\mu\nu}$ explicitly, but now since we have to deal with the covariant formulation of the constitutive relations of the medium, i.e. equations which relate, for example, D^i with E_i or H_i with B^i, for a given type of medium, the metric will play a fundamental role.

Using the metric we can write the definitions (3.8)-(3.9) in indicial notation as,

$$D^i = \varepsilon_0(-\eta^{ij} E_j) + P^i, \tag{3.48}$$

$$H_i = \frac{1}{\mu_0}(-\eta_{ij} B^j) - M_i, \tag{3.49}$$

and recalling the definitions of $H^{\mu\nu}$ and $F_{\mu\nu}$ in (3.39) and (3.44), we can express (3.48) and (3.49) in covariant form:

$$H^{\mu\nu} = \frac{1}{\mu_0}\eta^{\mu\rho}\eta^{\nu\sigma} F_{\rho\sigma} + M^{\mu\nu}. \tag{3.50}$$

Here, $M^{\mu\nu}$ is the *"magnetopolarization" tensor*, which covariantly describes the polarization P^i and magnetization M_i of any medium and whose components are,

$$M^{i0} = cP^i, \qquad M^{ij} = \epsilon^{ijk} M_k, \tag{3.51}$$

$$M^{\mu\nu} = \begin{pmatrix} 0 & -cP^1 & -cP^2 & -cP^3 \\ cP^1 & 0 & M_3 & -M_2 \\ cP^2 & -M_3 & 0 & M_1 \\ cP^3 & M_2 & -M_1 & 0 \end{pmatrix} = -M^{\nu\mu}. \tag{3.52}$$

With the help of $M^{\mu\nu}$, we can express in a covariant form the induced bound charges and currents due to the polarization and magnetization $J^{\nu}_{\text{ind}} := (c\rho_{\text{ind}}, j^i_{\text{ind}})$, by

$$J^{\nu}_{\text{ind}} := -\partial_\mu M^{\mu\nu}, \tag{3.53}$$

where the induced charge and current densities are given as usual,

$$\rho_{\text{ind}} := -\partial_i P^i, \tag{3.54}$$

$$j^i_{\text{ind}} := \frac{\partial P^i}{\partial t} + \epsilon^{ijk}\partial_j M_k. \tag{3.55}$$

In the particular case of vacuum, we obviously have $M^{\mu\nu} = 0$ in (3.50) and the covariant constitutive relations for this case turn out to be,

$$H^{\mu\nu} = \frac{1}{\mu_0}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma} \tag{3.56}$$

$$= \frac{1}{2}\chi^{\mu\nu\rho\sigma}_{(\text{vac})}F_{\rho\sigma}, \tag{3.57}$$

where $\chi^{\mu\nu\rho\sigma}_{(\text{vac})}$ is a fourth rank tensor known as the *constitutive tensor of vacuum* and is explicitly defined as,

$$\boxed{\chi^{\mu\nu\rho\sigma}_{(\text{vac})} := \frac{1}{\mu_0}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}).} \tag{3.58}$$

The relation (3.50) can be cast in a more compact form, if to define a contravariant electromagnetic strength tensor $F^{\mu\nu}$ by “raising” the indices of $F_{\mu\nu}$ with the metric:

$$F^{\mu\nu} := \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}, \tag{3.59}$$

and therefore we have,

$$\boxed{H^{\mu\nu} = \frac{1}{\mu_0}F^{\mu\nu} + M^{\mu\nu}.} \tag{3.60}$$

The constitutive relations as expressed in (3.5)-(3.6) can phenomenologically describe the electrodynamics within any kind of material medium, if we add enough material functions and constants. However, to have an explicit and close form to deal with the constitutive relations in covariant formulation, we will restrict our analysis only to *linear* and *non-dispersive* media. Linear in the sense that the response of the medium, through P^i and M_i, depends only linearly on the electromagnetic field E_i and B^i, and non-dispersive in the sense that the electromagnetic properties of the medium do not depend on the frequency of the incident light which propagates through the medium. For instance, we will not be able to describe the dispersion of white light that passes through a prism or the optics of some non-linear crystals, but most of the usual cases will be included in the theory.

The most general form of the constitutive relations for this kind of medium can be covariantly written as,

$$\boxed{H^{\mu\nu} = \frac{1}{2}\chi^{\mu\nu\rho\sigma}F_{\rho\sigma},} \tag{3.61}$$

where $\chi^{\mu\nu\rho\sigma}$ is a fourth rank tensor known as the *constitutive tensor*, which describes the electromagnetic properties of a particular medium. Comparing the expressions (3.57) and (3.61), we see that $\chi^{\mu\nu\rho\sigma}_{(\mathrm{vac})}$ in (3.58) is a particular case, for vacuum, of the general relation (3.61). Since $H^{\mu\nu}$ and $F_{\mu\nu}$ in (3.61) are both antisymmetric, $\chi^{\mu\nu\rho\sigma}$ must be, by definition, also antisymmetric in the two first and the two last indices, i.e.

$$\chi^{\mu\nu\rho\sigma} = -\,\chi^{\nu\mu\rho\sigma}, \tag{3.62}$$

$$\chi^{\mu\nu\rho\sigma} = -\,\chi^{\mu\nu\sigma\rho}. \tag{3.63}$$

Taking into account these symmetries of $\chi^{\mu\nu\rho\sigma}$, we conclude that it has 36 independent components in general. Additionally, if the medium is non-dissipative, i.e. if the total energy and momentum of the electromagnetic field, the material medium, the external charges and currents is not transformed in other types of energy and momentum (for instance, heat), then the constitutive tensor $\chi^{\mu\nu\rho\sigma}$ must satisfy an extra symmetry:

$$\chi^{\mu\nu\rho\sigma} = \chi^{\rho\sigma\mu\nu}, \tag{3.64}$$

which reduces its independent components to 21.

If we decompose equation (3.61) in temporal and spatial components, we can identify the components of $\chi^{\mu\nu\rho\sigma}$ in a more familiar way by the following cartesian components:

$$D^i = \varepsilon_0\varepsilon^{ij}E_j + \beta^i{}_jB^j, \tag{3.65}$$

$$H_i = \alpha_i{}^jE_j + \mu_0^{-1}(\mu^{-1})_{ij}B^j, \tag{3.66}$$

where,

$$\varepsilon^{ij} := \mu_0\chi^{0ij0}, \tag{3.67}$$

$$(\mu^{-1})_{ij} := \frac{1}{4}\mu_0\epsilon_{ilk}\epsilon_{jmq}\chi^{lkmq}, \tag{3.68}$$

$$\alpha_i{}^j := \frac{1}{2c}\epsilon_{ilk}\chi^{lk0j}, \tag{3.69}$$

$$\beta^i{}_j := -\,\frac{1}{2c}\epsilon_{jlk}\chi^{0ilk}, \tag{3.70}$$

and inversely,

$$\chi^{0ij0} = \frac{1}{\mu_0}\varepsilon^{ij}, \tag{3.71}$$

$$\chi^{klmq} = \frac{1}{\mu_0}\epsilon^{kli}\epsilon^{mqj}(\mu^{-1})_{ij}, \tag{3.72}$$

$$\chi^{jk0i} = -\,c\,\epsilon^{ljk}\alpha_l{}^i, \tag{3.73}$$

$$\chi^{0ijk} = c\,\epsilon^{ljk}\beta^i{}_l. \tag{3.74}$$

The tensor ε^{ij} is known as the relative permittivity or the dielectric tensor of the medium, $(\mu^{-1})_{ij}$ is the inverse relative permeability tensor and $\alpha_i{}^j$, $\beta^i{}_j$ are the linear magneto-electric coupling coefficients. The quantities ε^{ij} and $(\mu^{-1})_{ij}$ are dimensionless, whereas $\alpha_i{}^j$ and $\beta^i{}_j$

have dimensions of $[\varepsilon_0^{1/2}\mu_0^{-1/2}]$. Since the relations (3.67)-(3.70) and (3.71)-(3.74) are one-on-one, it is clear that each of these 4 tensors have 9 independent components in general.

In the special case of a non-dissipative medium, the extra symmetry (3.64) implies that ε^{ij} and $(\mu^{-1})_{ij}$ must be symmetric,

$$\varepsilon^{ij} = \varepsilon^{ji}, \tag{3.75}$$

$$(\mu^{-1})_{ij} = (\mu^{-1})_{ji}, \tag{3.76}$$

and that $\alpha_i{}^j$ is the negative transpose of $\beta^i{}_j$:

$$\alpha_i{}^j = -\beta^j{}_i. \tag{3.77}$$

3.4. Isotropic medium

Since we already presented all the basic tools to work with constitutive tensors, we will now apply the formalism to find an explicit expression for the special case of an isotropic medium, whose optical properties are just described by the refraction index n.

We know that for the usual linear, non-dispersive, non-dissipative, non-magneto-electric and isotropic medium *at rest*, the constitutive relations are given by,

$$\boldsymbol{D} = \varepsilon_0\varepsilon\boldsymbol{E}, \tag{3.78}$$

$$\boldsymbol{H} = \frac{1}{\mu_0\mu}\boldsymbol{B}, \tag{3.79}$$

where $\varepsilon = \varepsilon(\boldsymbol{x}, t)$ is relative permittivity function and $\mu = \mu(\boldsymbol{x}, t)$ is the relative permeability function of the medium. The refraction index is a dimensionless quantity defined as

$$n := \sqrt{\mu\varepsilon}. \tag{3.80}$$

Writing (3.78)-(3.79) in indicial notation, we have

$$D^i = -\varepsilon_0\varepsilon\eta^{ij}E_j, \qquad H_i = -\frac{1}{\mu_0\mu}\eta_{ij}B^j, \tag{3.81}$$

and comparing them with (3.65)-(3.66), we see that the tensors $\varepsilon^{ij}_{(0)}$ and $(\mu^{-1})^{(0)}_{ij}$, in the frame where the medium is at rest, read

$$\varepsilon^{ij}_{(0)} = -\varepsilon\eta^{ij}, \qquad (\mu^{-1})^{(0)}_{ij} = -\frac{1}{\mu}\eta_{ij}, \qquad \alpha_{i\,(0)}^{\,j} = -\beta^{j\,(0)}_{\,i} = 0. \tag{3.82}$$

Finally, if we replace (3.82) in the inverse identifications (3.71)-(3.74), we find that the components of constitutive tensor of this isotropic medium in its comoving frame, are given by

$$\chi^{0ij0}_{\text{iso}(0)} = -\frac{\varepsilon}{\mu_0}\eta^{ij} = -\frac{1}{\mu_0\mu}n^2\eta^{ij}, \tag{3.83}$$

$$\chi^{0ijk}_{\text{iso}(0)} = 0, \tag{3.84}$$

$$\chi^{ijkl}_{\text{iso}(0)} = \frac{1}{\mu_0\mu}(\eta^{ik}\eta^{jl} - \eta^{il}\eta^{jk}). \tag{3.85}$$

As it is expected, if we evaluate the expressions (3.83)-(3.85) for $\varepsilon = \mu = n = 1$, we obtain the same components of the constitutive tensor for vacuum given in (3.58) in terms of the spacetime metric $\eta^{\mu\nu}$. Therefore, if we define a new diagonal second-rank tensor $\gamma^{\mu\nu}_{(0)}$ as a generalization of $\eta^{\mu\nu}$ for $n \neq 1$ in the form:

$$\gamma^{\mu\nu}_{(0)} := \mathrm{diag}(n^2, -1, -1, -1), \tag{3.86}$$

then we can express all the components of $\chi^{\mu\nu\rho\sigma}_{\mathrm{iso}(0)}$ analogously as in (3.58) by

$$\chi^{\mu\nu\rho\sigma}_{\mathrm{iso}(0)} := \frac{1}{\mu_0\mu}(\gamma^{\mu\rho}_{(0)}\gamma^{\nu\sigma}_{(0)} - \gamma^{\mu\sigma}_{(0)}\gamma^{\nu\rho}_{(0)}). \tag{3.87}$$

Since $\chi^{\mu\nu\rho\sigma}_{\mathrm{iso}(0)}$ is a fourth rank tensor, we can apply a general boost $\Lambda^{\mu}{}_{\nu}$ to it in order to obtain an expression for $\chi^{\mu\nu\rho\sigma}_{\mathrm{iso}}$ valid in a general inertial reference frame where a given volume element of the isotropic medium is moving with an arbitrary velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$. In this manner, we explicitly obtain

$$\chi^{\mu\nu\rho\sigma}_{\mathrm{iso}} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\chi^{\alpha\beta\gamma\delta}_{\mathrm{iso}(0)}, \tag{3.88}$$

which is the same as,

$$\boxed{\chi^{\mu\nu\rho\sigma}_{\mathrm{iso}} = \frac{1}{\mu_0\mu}(\gamma^{\mu\rho}\gamma^{\nu\sigma} - \gamma^{\mu\sigma}\gamma^{\nu\rho}),} \tag{3.89}$$

where $\gamma^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\gamma^{\alpha\beta}_{(0)}$ and the coefficients of the boost are given by,

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta_i \\ \gamma\beta^i & \delta^i{}_j - \frac{(\gamma-1)}{\beta^2}\beta^i\beta_j \end{pmatrix}, \tag{3.90}$$

with $\gamma := (1-\beta^2)^{-1/2}$, $\beta^2 := -\beta^i\beta_i$ and $\beta^i := v^i/c$. Finally, computing explicitly the boost, we obtain the covariant expression for $\gamma^{\mu\nu}$:

$$\boxed{\gamma^{\mu\nu} = \eta^{\mu\nu} + \frac{(n^2-1)}{c^2}u^{\mu}u^{\nu},} \tag{3.91}$$

where $u^{\mu} := (\gamma c, \gamma\boldsymbol{v})$ is the 4-velocity field of the moving isotropic medium. The result (3.91) is known in the literature as the Gordon *optical metric* and it was first found by W. Gordon in 1923 [25].

If we replace (3.91) in (3.89), we can also obtain an explicit expression for the constitutive tensor $\chi^{\mu\nu\rho\sigma}_{\mathrm{iso}}$ just in terms of the material variables of the isotropic medium: u^{μ}, μ and $n^2 = \varepsilon\mu$,

$$\boxed{\chi^{\mu\nu\rho\sigma}_{\mathrm{iso}} = \frac{1}{\mu_0\mu}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) + \frac{(n^2-1)}{\mu_0\mu c^2}\left(\eta^{\mu\rho}u^{\nu}u^{\sigma} - \eta^{\mu\sigma}u^{\nu}u^{\rho} + \eta^{\nu\sigma}u^{\mu}u^{\rho} - \eta^{\nu\rho}u^{\mu}u^{\sigma}\right).} \tag{3.92}$$

3.5. Dispersion relation and polarization condition for light in isotropic media

By studying the propagation of electromagnetic waves inside the isotropic medium, we will understand why $\gamma^{\mu\nu}$ in (3.91) is interpreted as an "optical metric". In appendix A we derive the polarization condition (A.6) for electromagnetic waves inside linear, non-dispersive and homogeneous media, which reads

$$\chi^{\mu\nu\rho\sigma} k_\nu k_\rho \tilde{A}_\sigma = 0. \tag{3.93}$$

Given any constitutive tensor $\chi^{\mu\nu\rho\sigma}$, the wave vector k_μ and the wave amplitude $\tilde{A}_\mu$ should always satisfy the condition (3.93) in order that the macroscopic Maxwell equations inside this type of media admit plane-wave solutions of the form

$$A_\sigma = \tilde{A}_\sigma e^{ik_\lambda x^\lambda}. \tag{3.94}$$

If we replace the explicit expression for the constitutive tensor of an isotropic medium (3.89) in (3.93), we obtain,

$$\gamma^{\mu\nu} k_\mu k_\sigma \tilde{A}_\nu - (\gamma^{\mu\nu} k_\mu k_\nu) \tilde{A}_\sigma = 0. \tag{3.95}$$

To solve (3.95), we need in principle to impose that the determinant of $\gamma^{\mu\sigma} k_\mu k_\lambda - \gamma^{\mu\nu} k_\mu k_\nu \delta^\sigma{}_\lambda$ be equal to zero, which is simpler to compute if we choose an appropriate gauge. Alternatively, without imposing a gauge explicitly, we notice that if the scalar $\gamma^{\mu\nu} k_\mu k_\nu \neq 0$, then from (3.95) we can solve for $\tilde{A}_\lambda$ and obtain that for this case the solution must be proportional to k_λ:

$$\tilde{A}_\lambda = \left[\frac{\gamma^{\rho\sigma} k_\rho \tilde{A}_\sigma}{\gamma^{\mu\nu} k_\mu k_\nu} \right] k_\lambda. \tag{3.96}$$

Then, if we replace (3.94) and (3.96) into the definition (3.42), we clearly see that $F_{\mu\nu}$ vanishes, for all values of $\tilde{A}_\sigma$:

$$F_{\mu\nu} = i \left(k_\mu \tilde{A}_\nu - k_\nu \tilde{A}_\mu \right) e^{k_\lambda x^\lambda} \tag{3.97}$$

$$= i \left[\frac{\gamma^{\rho\sigma} k_\rho \tilde{A}_\sigma}{\gamma^{\alpha\beta} k_\alpha k_\beta} \right] (k_\mu k_\nu - k_\nu k_\mu) \, e^{k_\lambda x^\lambda} \tag{3.98}$$

$$= 0, \tag{3.99}$$

Therefore, if $\gamma^{\mu\nu} k_\mu k_\nu \neq 0$, we obtain a "pure gauge" solution. In order to have non-zero solutions for the electromagnetic field $F_{\mu\nu}$, we conclude it is necessary that

$$\boxed{\gamma^{\mu\nu} k_\mu k_\nu = 0,} \tag{3.100}$$

and considering this in (3.95), we get a simplified condition for the plane wave,

$$\boxed{\gamma^{\mu\nu} k_\mu \tilde{A}_\nu = 0,} \tag{3.101}$$

which is gauge-independent given any allowed k_μ satisfying (3.100). The equation (3.100) defines a dispersion relation for the electromagnetic waves within this isotropic medium described by $\gamma^{\mu\nu}$ in any reference frame. For an explicit calculation of the dispersion relation (3.100), see appendix A.2.1. In the particular case of vacuum, we know that the dispersion relation is obviously,

$$\omega^2 = c^2 \boldsymbol{k}^2, \tag{3.102}$$

which, using the definition $k_\mu := (\omega/c, -k_i)$ and the metric $\eta^{\mu\nu}$, can be rewritten by,

$$\eta^{\mu\nu} k_\mu k_\nu = 0. \tag{3.103}$$

Since (3.103) is a scalar equation, we conclude that the dispersion relation (3.102) is valid in all inertial reference frames and therefore the electromagnetic waves in vacuum travel at speed c, independent of the observer, which is, of course, consistent with the principles of Special Relativity. If we compare (3.103) with (3.100), we see that the Gordon "optical metric" $\gamma^{\mu\nu}$ plays the role of the spacetime metric $\eta^{\mu\nu}$ for isotropic media with $n \neq 1$ in any state of motion.

Equation (3.101) defines a polarization condition for the plane wave in the medium, which restricts the possible values of $\tilde{A}_\mu$, given k_μ and $\gamma^{\mu\nu}$. Let us also recall that using the gauge transformation (3.46) we can change $\tilde{A}_\mu$ to

$$\tilde{A}'_\mu = \tilde{A}_\mu - i k_\mu h, \tag{3.104}$$

where h is an arbitrary constant, which together with (3.101), reduce the number of non-trivial independent components of A_μ and $\tilde{A}_\mu$ from four to two. As a result, the electromagnetic wave in the medium has two independent polarization states.

Chapter 4.

Balance equations, electromagnetic conserved quantities and medium symmetries

"Happiness is not a matter of intensity,
but of balance, order, rhythm and harmony."

Thomas Merton,
Anglo-American Trappist monk.

Other important quantities for the description of the electromagnetic field interacting with matter are the energy, momentum and angular momentum, as well as their respective balance or conservation equations. These equations describe how the energy, momentum and angular momentum of the electromagnetic field is transferred to the material medium and the external charges and currents, information which is very useful in order to gain more physical insight about the dynamics of the system and to describe different stages in its temporal evolution, without necessarily calculating the details of the interactions (i.e. computing the fields, the forces, the boundary conditions, etc.).

In this chapter we will consider the system composed by the electromagnetic field in matter as an *open* system, because only the electromagnetic field is assumed to have dynamics via the macroscopic Maxwell equations. The material medium is considered as a *fixed* background without dynamics, whose properties are specified by the non-dynamical constitutive relations, i.e. they are *given* functions of position and time. Physically this can be done by having an external agent that keeps the medium in its state of motion, for instance, a crystal fixed to an optical table by mechanical supports.

We will see that there are certain conditions, related to the *symmetries* of the fixed medium, in which the balance equations lead to *electromagnetic* conserved quantities, which will help us to better interpret the balance equations and the dynamics of light in matter, as well as to understand and solve various problems in a much easier way.

In section 4.3 we apply the Lagrange-Noether formalism to this open system, using the general results of appendix C. In fact, we will rederive the balance equations obtained directly from Maxwell's equations, but this formalism will provide us further ideas regarding the controversial asymmetry of the Minkowski tensor and the relationship between the symmetries of the medium, the Minkowski electromagnetic conserved quantities, and the interactions between field and matter.

4.1. Energy-momentum balance equation

In the covariant language of Special Relativity, the energy and momentum densities of the electromagnetic field, as well as their fluxes, are components of a more general object called the *energy-momentum tensor*, which we will study in detail in subsection 4.1.3. Therefore we will not derive separate balance equations for energy and momentum, but they will appear as different components of a general *energy-momentum balance equation*. It is important to remark that this balance equation is a direct *consequence* of the Maxwell equations for the electromagnetic field (3.40) and (3.45), together with the definition of the Lorentz 4-force density (3.47) and *not* an additional assumption of the electromagnetic theory.

4.1.1. Derivation from Maxwell's equations

In order to derive the energy-momentum balance equation, we start from the Lorentz 4-force density:

$$\begin{aligned} \mathcal{F}_\mu^{\mathrm{ext}} &:= F_{\mu\nu} J^\nu_{\mathrm{ext}} && (4.1)\\ &= \left(\frac{1}{c}P_{\mathrm{ext}}, -f_i^{\mathrm{ext}}\right) && (4.2)\\ &= \left(\frac{1}{c}j^i_{\mathrm{ext}}E_i, -\rho_{\mathrm{ext}}E_i - \hat{\epsilon}_{ijk}j^j_{\mathrm{ext}}B^k\right), && (4.3) \end{aligned}$$

which describes, in a covariant way, the rate of energy and momentum transfer from the electromagnetic field to the external charges and currents inside an infinitesimal volume element dV. According to Maxwell's equations, this conversion of energy and momentum from electromagnetic into mechanical must be balanced by a corresponding rate of decrease of energy and momentum in the electromagnetic field within the volume element dV. To exhibit this conservation law explicitly (in differential form), we take (4.1) and insert (3.40) into it in order to eliminate the external current density:

$$\mathcal{F}_\mu^{\mathrm{ext}} = F_{\mu\nu}(\partial_\lambda H^{\lambda\nu}). \tag{4.4}$$

Then, if we complete the total derivative in (4.4) and use the fact that $H^{\mu\nu}$ and $F_{\mu\nu}$ are anti-symmetric, we obtain

$$\mathcal{F}_\mu^{\mathrm{ext}} = \partial_\lambda(F_{\mu\nu}H^{\lambda\nu}) - \frac{1}{2}H^{\lambda\nu}(\partial_\lambda F_{\mu\nu}) - \frac{1}{2}H^{\nu\lambda}(\partial_\lambda F_{\nu\mu}). \tag{4.5}$$

Now we can relabel the indices $\nu \leftrightarrow \lambda$ in the last term of (4.5), use the homogeneous Maxwell equations (3.45), and write

$$\mathcal{F}_\mu^{\mathrm{ext}} = \partial_\lambda(F_{\mu\nu}H^{\lambda\nu}) - \frac{1}{4}H^{\nu\lambda}(\partial_\mu F_{\nu\lambda}) - \frac{1}{4}H^{\nu\lambda}(\partial_\mu F_{\nu\lambda}). \tag{4.6}$$

If we complete the total derivative in the second term of the right hand side of (4.6), we obtain

$$\mathcal{F}_\mu^{\text{ext}} = \partial_\lambda(H^{\lambda\nu}F_{\mu\nu}) - \frac{1}{4}\partial_\mu(F_{\nu\lambda}H^{\nu\lambda}) + \frac{1}{4}F_{\nu\lambda}(\partial_\mu H^{\nu\lambda}) - \frac{1}{4}(\partial_\mu F_{\nu\lambda})H^{\nu\lambda}, \tag{4.7}$$

and rearranging terms, the energy-momentum balance equation can be finally written as

$$\partial_\lambda\left[F_{\mu\nu}H^{\nu\lambda} + \frac{1}{4}\delta_\mu^\lambda F_{\nu\rho}H^{\nu\rho}\right] + \frac{1}{4}\left[(\partial_\mu F_{\nu\lambda})H^{\nu\lambda} - F_{\nu\lambda}(\partial_\mu H^{\nu\lambda})\right] + \mathcal{F}_\mu^{\text{ext}} = 0, \tag{4.8}$$

or, in a more compact form,

$$\boxed{\partial_\nu \Theta_\mu{}^\nu + \mathcal{F}_\mu^{\text{eff}} + \mathcal{F}_\mu^{\text{ext}} = 0,} \tag{4.9}$$

where

$$\mathcal{F}_\mu^{\text{eff}} := \frac{1}{4}\left[(\partial_\mu F_{\nu\lambda})H^{\nu\lambda} - F_{\nu\lambda}(\partial_\mu H^{\nu\lambda})\right], \tag{4.10}$$

and

$$\boxed{\Theta_\mu{}^\nu := F_{\mu\rho}H^{\rho\nu} + \frac{1}{4}\delta_\mu^\nu F_{\rho\sigma}H^{\rho\sigma}.} \tag{4.11}$$

4.1.2. Effective material 4-force density

Since $\mathcal{F}_\mu^{\text{ext}} = (P_{\text{ext}}/c, -f_i^{\text{ext}})$ is the 4-force density exerted by the electromagnetic field on the external charges and currents, then $\mathcal{F}_\mu^{\text{eff}}$ can consistently be interpreted as another 4-force density,

$$\mathcal{F}_\mu^{\text{eff}} := \left(\frac{1}{c}P_{\text{eff}}, -f_i^{\text{eff}}\right), \tag{4.12}$$

which the electromagnetic field exerts on the bound charges and currents of the material medium. We will call $\mathcal{F}_\mu^{\text{eff}}$ the *effective material* 4-force density and, as we go along the text, we will discuss its properties and its interpretation will become clearer. Notice, however, that $\mathcal{F}_\mu^{\text{eff}}$ defined in (4.10) is not the same as the Lorentz 4-force density exerted by the electromagnetic field on the induced charges and currents of the medium J^μ_{ind}, defined in (3.53). In fact, both 4-force densities differ in a 4-divergence term:

$$\mathcal{F}_\mu^{\text{eff}} = F_{\mu\nu}J^\nu_{\text{ind}} - \partial_\nu\left(F_{\mu\rho}M^{\rho\nu} + \frac{1}{4}\delta_\mu^\nu F_{\rho\sigma}M^{\rho\sigma}\right). \tag{4.13}$$

We can use (3.60) in (4.10) to express $\mathcal{F}_\mu^{\text{eff}}$ in terms of $M^{\mu\nu}$ and $F_{\mu\nu}$ as

$$\mathcal{F}_\mu^{\text{eff}} = \frac{1}{4}\left[(\partial_\mu F_{\nu\lambda})M^{\nu\lambda} - F_{\nu\lambda}(\partial_\mu M^{\nu\lambda})\right]. \tag{4.14}$$

When light is propagating in vacuum, there is no medium that can be polarized or magnetized by the electromagnetic wave and therefore we obviously have $M^{\mu\nu} = 0$. As a consequence,

$\mathcal{F}_\mu^{\text{eff}}$ in (4.14) will be identically zero for all values of $F_{\mu\nu}$. This is consistent with our physical interpretation of $\mathcal{F}_\mu^{\text{eff}}$ as a "material" 4-force density, because it can only be different from zero when there is a material medium to interact with the electromagnetic field $F_{\mu\nu}$.

There are also cases, however, in which $\mathcal{F}_\mu^{\text{eff}}$ can vanish, in spite of the interaction between light and matter and this is the reason why we interpret $\mathcal{F}_\mu^{\text{eff}}$ as an "effective" material 4-force density. To see this explicitly, we replace (3.61) into (4.10) and we find another expression for $\mathcal{F}_\mu^{\text{eff}}$ which has the advantage that the whole material medium contribution is described by the constitutive tensor $\chi^{\mu\nu\rho\sigma}$, whereas the electromagnetic field contribution is described $F_{\mu\nu}$:

$$\boxed{\mathcal{F}_\mu^{\text{eff}} = -\frac{1}{8}\left[(\partial_\mu \chi^{\alpha\beta\gamma\delta})F_{\alpha\beta}F_{\gamma\delta} + (\chi^{\alpha\beta\gamma\delta} - \chi^{\gamma\delta\alpha\beta})F_{\alpha\beta}(\partial_\mu F_{\gamma\delta})\right].} \tag{4.15}$$

Since the first term in (4.15) is proportional to the derivative of the constitutive tensor it is related to the inhomogeneities of the material medium, whereas the second term is related with the dissipation properties of the medium. For any given $F_{\mu\nu}$ field, the first term in (4.15) can only be different from zero when the electromagnetic properties of the medium are inhomogeneous in space ($\partial_i \chi^{\alpha\beta\gamma\delta} \neq 0$) and/or in time ($\partial \chi^{\alpha\beta\gamma\delta}/\partial t \neq 0$). We interpret this term as an "effective" interaction between field and matter, since light propagating through an homogeneous region of the medium will not exert any "effective" force on it even though both are interacting (the medium is changing its polarization and magnetization, while light changes its electric and magnetic fields). For instance, when light encounters itself with an interface between two different homogeneous media, only in that boundary the electromagnetic wave will exert an effective force on the medium ($f_i^{\text{eff}} \neq 0$), resulting in a decrease of the electromagnetic momentum of light. As a result, light changes its velocity of propagation (refraction), amplitude, polarization, etc. The second term in (4.15) contributes to the effective interaction between field and matter when the constitutive tensor does not present the extra symmetry $\chi^{\mu\nu\rho\sigma} = \chi^{\rho\sigma\mu\nu}$, as in (3.64). Since this contribution will be present throughout all the propagation of light, no matter the spatial inhomogeneities or changes in time of the medium, this term is interpreted as a *dissipative* energy and momentum transfer between light and matter. The corresponding decrease of energy and momentum within the electromagnetic field will be absorbed by the medium and then converted into heat or other dissipation mechanisms. Therefore, we define a *non-dissipative* medium as the one described by a constitutive tensor which satisfies the extra symmetry (3.64). Notice that this condition is more familiar than it seems, since it is just the covariant generalization of the condition $\varepsilon^{ij} = \varepsilon^{ji}$, derived in various electrodynamics books in the case of non-dissipative media, for instance see Landau and Lifshitz [31]. The condition also implies the other components to satisfy the relations (3.76)-(3.77).

4.1.3. Minkowski energy-momentum tensor

So far we have only discussed the energy and momentum transfer from the electromagnetic field to the external charges and currents through $\mathcal{F}_\mu^{\text{ext}}$, and to the bound charges and currents of the medium through the effective material 4-force density $\mathcal{F}_\mu^{\text{eff}}$, but nothing has been said about how to quantify the change of the electromagnetic energy and momentum contained in the field. For this purpose, let us assume that the medium is non-dissipative and homogeneous in space and time, so that $\mathcal{F}^{\text{eff}}_{\mu} = 0$, and additionally, that there are no external charges and currents, i.e. $J^{\mu}_{\text{ext}} = 0$ and hence $\mathcal{F}^{\text{ext}}_{\mu} = 0$. In this special case, the electromagnetic field exerts no force nor does any work on the charges and currents within any volume element and therefore, the energy and momentum of the electromagnetic field should be locally conserved. Indeed, the energy-momentum balance equation (4.9) reduces to a *continuity equation* in this case:

$$\partial_\nu \Theta_\mu{}^\nu = 0, \tag{4.16}$$

and therefore, we can interpret $\Theta_\mu{}^\nu$, defined in (4.11), as the *energy-momentum tensor* of the electromagnetic field inside matter. In the literature the tensor $\Theta_\mu{}^\nu$ is usually known as the *Minkowski tensor* of the electromagnetic field, since it was first derived by H. Minkowski in 1908 [7]. The components of any energy-momentum tensor are always identified as

$$\Theta_\mu{}^\nu = \begin{pmatrix} \mathcal{U} & S^i/c \\ -c\pi_i & -p_i{}^j \end{pmatrix}, \tag{4.17}$$

so that a *conservation equation* of the form (4.16), can be interpreted as two continuity equations, one for energy ($\mu = 0$) and another for the (three) components of momentum ($\mu = i$):

$$\frac{\partial \mathcal{U}}{\partial t} + \partial_i S^i = 0, \tag{4.18}$$

$$\frac{\partial \pi_i}{\partial t} + \partial_j p_i{}^j = 0. \tag{4.19}$$

Accordingly, $\mathcal{U}$ is the *energy density* of the field, S^i the *energy flux density* or *Poynting vector*, π_i the *momentum density* of the field and finally $p_i{}^j$ the *momentum flux density* or *stress tensor*. As usual, the meaning of the continuity equations (4.18)-(4.19) is that the time rate of change of energy (or momentum) of the field within the infinitesimal volume dV is balanced by the *net* energy (or momentum) flowing out through the boundary of the volume element per unit time, and therefore no energy (or momentum) is created or lost.

In our particular case of the electromagnetic field inside matter, we can use the identifications (3.39) and (3.44) in the definition (4.11) to explicitly compute the components of the Minkowski tensor, which read

$$\Theta_\mu{}^\nu = \begin{pmatrix} \left(E_i D^i + H_i B^i\right)/2 & \epsilon^{ijk} E_j H_k / c \\ -c\hat{\epsilon}_{ijk} D^j B^k & E_i D^j + H_i B^j - \frac{1}{2}\delta_i^j \left(E_k D^k + H_k B^k\right) \end{pmatrix}. \tag{4.20}$$

Comparing (4.20) with the identifications (4.17) we obtain that the Minkowski energy density, energy flux density, momentum density and momentum flux density for the electromagnetic field in matter are respectively given by

$$\mathcal{U} := \frac{1}{2}\left(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}\right), \tag{4.21}$$

$$\boldsymbol{S} := \boldsymbol{E}\times\boldsymbol{H}, \tag{4.22}$$

$$\boldsymbol{\pi} := \boldsymbol{D}\times\boldsymbol{B}, \tag{4.23}$$

$$p_i{}^j := -E_i D^j - H_i B^j + \frac{1}{2}\delta_i^j \left(E_k D^k + H_k B^k\right). \tag{4.24}$$

In the literature, the *Maxwell stress tensor* $T_i{}^j$ is sometimes also defined as the negative of the momentum flux density in (4.24), i.e.

$$T_i{}^j := -p_i{}^j \tag{4.25}$$

$$= E_i D^j + H_i B^j - \frac{1}{2}\delta_i^j \left(E_k D^k + H_k B^k\right). \tag{4.26}$$

In summary, when the external 4-force densities ($\mathcal{F}_\mu^{\text{eff}}$ and $\mathcal{F}_\mu^{\text{ext}}$) vanish, the Maxwell equations, via the general energy-momentum balance equation (4.9), imply that the Minkowski energy-momentum tensor of the electromagnetic field $\Theta_\mu{}^\nu$, defined as in (4.11) and with components explicitly given in (4.20) or in (4.21)-(4.24), is conserved. Notice however, that this tensor is not uniquely defined. To any given energy-momentum tensor $\Theta_\mu{}^\nu$, one can always add a "total derivative" $\partial_\lambda \xi_\mu{}^{\nu\lambda}$, with $\xi_\mu{}^{\nu\lambda} = -\xi_\mu{}^{\lambda\nu}$ an arbitrary antisymmetric function of the coordinates, so that new tensor $\Theta'_\mu{}^\nu$ has the same 4-divergence as the old tensor $\partial_\nu \Theta'_\mu{}^\nu = \partial_\nu \Theta_\mu{}^\nu$ and therefore the same balance equation. In section 4.3.2 it is argued, using the Lagrangian formalism, why the Minkowski tensor (4.11) is the simplest choice among the family of tensors $\Theta'_\mu{}^\nu = \Theta_\mu{}^\nu + \partial_\lambda \xi_\mu{}^{\nu\lambda}$ for the case of the electromagnetic field inside matter. The Abraham tensor $\Omega_\mu{}^\nu$ which we will study in depth in chapter 5 is also postulated as another choice to describe the energy-momentum content of the electromagnetic field, but it does *not* belong to the same family of tensors as the Minkowski's one.

4.1.4. Integral representation of the balance equations for energy and momentum

Since we already know how to interpret each quantity appearing in the general energy-momentum balance equation (4.9), expressed in differential form, it is time to write them explicitly and integrate them over a finite volume V, in order to obtain the integral versions of the balance equations for energy and momentum. It will be useful for our discussion to define the *total 4-momentum* of the electromagnetic field inside matter $\mathcal{P}_\mu$, by

$$\boxed{\mathcal{P}_\mu := \frac{1}{c}\int_V T_\mu{}^0 \, dV,} \tag{4.27}$$

whose components are identified like any 4-momentum as

$$\mathcal{P}_\mu := \left(\frac{E}{c}, -p_i\right), \tag{4.28}$$

where

$$p_i := \int_V \pi_i \, dV, \tag{4.29}$$

is the total linear momentum of the field inside the volume V and E is the total energy of the electromagnetic field contained in the volume V, given by

$$E := \int_V \mathcal{U} \, dV. \tag{4.30}$$

After these definitions, we can integrate the explicitly covariant energy-momentum balance equation (4.9) over a finite volume V and if we additionally use the new definition (4.27) of $\mathcal{P}_\mu$ and apply the Gauss theorem to transform the volume integral into a surface integral, the integral version of the energy-momentum balance equation, reads

$$\boxed{\frac{d\mathcal{P}_\mu}{dt} + \int_V \mathcal{F}_\mu^{\mathrm{ext}} \; dV + \int_V \mathcal{F}_\mu^{\mathrm{eff}} \; dV = -\oint_{\partial V} \Theta_\mu{}^j \hat{n}_j \; da,} \tag{4.31}$$

where $\hat{\boldsymbol{n}}$ is the outward unitary normal to the closed surface ∂V.

Then, if we evaluate for $\mu = 0$ in (4.31), we obtain the integral version of the balance equation for energy:

$$\boxed{\frac{d}{dt}\overset{\mathrm{M}}{E} + \int_V P_{\mathrm{eff}} \; dV + \int_V \boldsymbol{j}_{\mathrm{ext}} \cdot \boldsymbol{E} \; dV = -\oint_{\partial V} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \hat{\boldsymbol{n}} \; da,} \tag{4.32}$$

where the Minkowski total energy of the electromagnetic field in matter $\overset{\mathrm{M}}{E}$ within the volume V, reads

$$\overset{\mathrm{M}}{E} = \frac{1}{2}\int_V \left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}\right) \; dV, \tag{4.33}$$

and the effective material power of work density P_{eff} is in detail given by,

$$\begin{aligned} P_{\mathrm{eff}} &= \frac{1}{2}\left[E_i \frac{\partial D^i}{\partial t} - D^i \frac{\partial E_i}{\partial t} + H_i \frac{\partial B^i}{\partial t} - B^i \frac{\partial H_i}{\partial t}\right] \qquad (4.34) \\ &= \frac{1}{2}\varepsilon_0 E_i E_j \frac{\partial}{\partial t}\varepsilon^{ij} - \frac{1}{2}\mu_0^{-1} B^i B^j \frac{\partial}{\partial t}(\mu^{-1})_{ij} + \frac{1}{2} E_i B^j \frac{\partial}{\partial t}\beta^i{}_j - \frac{1}{2} B^i E_j \frac{\partial}{\partial t}\alpha_i{}^j \\ &\quad + \frac{1}{2}\varepsilon_0 \left[\varepsilon^{ij} - \varepsilon^{ji}\right] E_i \frac{\partial E_j}{\partial t} - \frac{1}{2}\mu_0^{-1}\left[(\mu^{-1})_{ij} - (\mu^{-1})_{ji}\right] B^i \frac{\partial B^j}{\partial t} \\ &\quad + \frac{1}{2}\left[\beta^i{}_j + \alpha_j{}^i\right] E_i \frac{\partial B^j}{\partial t} - \frac{1}{2}\left[\beta^j{}_i + \alpha_i{}^j\right] B^i \frac{\partial E_j}{\partial t}. \qquad (4.35) \end{aligned}$$

Paraphrasing (4.32) we can say that the time rate of change of electromagnetic energy *in matter* $\overset{\mathrm{M}}{E}$ within the finite volume V, plus the effective work per unit time (power) which the electromagnetic field does on the bound charges and currents of the medium within the volume V ($\int_V P_{\mathrm{eff}} \; dV$) plus the power transferred from the electromagnetic field to the external currents inside the volume V ($\int_V \boldsymbol{j}_{\mathrm{ext}} \cdot \boldsymbol{E} \; dV$), is equal to the negative of the *net* energy per unit time which flows through the boundary surface ∂V of the closed volume ($\oint_{\partial V} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \hat{\boldsymbol{n}} \; da$). We emphasize the word "net" in order to stress that $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ does not *only* represent the flow of electromagnetic energy, but all forms of energy flowing out the closed volume and passing through its boundaries, including also mechanical or thermal energy of the medium. This is reasonable since the fields $\boldsymbol{D}$ and $\boldsymbol{H}$ do not only depend on the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, but also on properties of the medium, through the constitutive relations, which just shows the couple nature of the system under study. Also the electromagnetic energy we have defined as $\int_V \left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}\right)/2 \; dV$ is not only "electromagnetic" in nature, but it is a *coupled* quantity which correctly describes the light propagating *inside matter*, since it is conserved in the absence of effective work transfer from the field in matter to the medium and to external currents. In particular, if the material medium has finite extent, we can choose a volume V big enough so that it encloses the material medium as a whole. The electromagnetic field evaluated at the borders of the volume will be in vacuum and therefore in that case only the "pure" electromagnetic energy will have the possibility of flowing out of the volume and we can talk of $\boldsymbol{S}$ as representing only electromagnetic energy flux.

Suppose now that we choose a volume V, where the electromagnetic properties of the medium are time independent and non-dissipative, so that $P_{\text{eff}} = 0$ in (4.35) and additionally there are no external currents so that $P_{\text{ext}} = 0$. Then, if there is no energy flux through the boundaries of the volume, the time derivative $d\overset{\text{M}}{E}/dt$ will vanish, and therefore the total electromagnetic energy in matter within V the will be a conserved quantity, in the sense that it will be time independent:

$$\overset{\text{M}}{E} = \frac{1}{2}\int_V (\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B})\ dV = constant. \tag{4.36}$$

On the other hand, if we evaluate (4.31) for $\mu = i$, the momentum balance equation in integral form, becomes

$$\boxed{\begin{aligned}\frac{d}{dt}\overset{\text{M}}{p}_i + \int_V f_i^{\text{eff}}\ dV + \int_V \left[\rho_{\text{ext}}\boldsymbol{E} + \boldsymbol{j}_{\text{ext}}\times\boldsymbol{B}\right]_i\ dV \\ = -\oint_{\partial V}\left[-E_iD^j - H_iB^j + \frac{1}{2}\delta_i^j\left(E_kD^k + H_kB^k\right)\right]\hat{n}_j\ da,\end{aligned}} \tag{4.37}$$

where the material effective force density is explicitly written as

$$\begin{aligned} f_i^{\text{eff}} &= \frac{1}{2}\left[D^j(\partial_iE_j) - E_j(\partial_iD^j) + B^j(\partial_iH_j) - H_j(\partial_iB^j)\right] && (4.38)\\ &= -\frac{1}{2}\varepsilon_0E_jE_k(\partial_i\varepsilon^{jk}) + \frac{1}{2}\mu_0^{-1}B^jB^k(\partial_i(\mu^{-1})_{jk}) - \frac{1}{2}E_jB^k(\partial_i{\beta^j}_k) + \frac{1}{2}B^jE_k(\partial_i{\alpha_j}^k)\\ &\quad -\frac{1}{2}\varepsilon_0\left[\varepsilon^{jk} - \varepsilon^{kj}\right]E_j(\partial_iE_k) + \frac{1}{2}\mu_0^{-1}\left[(\mu^{-1})_{jk} - (\mu^{-1})_{kj}\right]B^j(\partial_iB^k)\\ &\quad -\frac{1}{2}\left[{\beta^j}_k + {\alpha_k}^j\right]E_j(\partial_iB^k) + \frac{1}{2}\left[{\beta^k}_j + {\alpha_j}^k\right]B^j(\partial_iE_k), && (4.39)\end{aligned}$$

and the total electromagnetic momentum in matter inside the volume V is given by

$$\boxed{\overset{\text{M}}{\boldsymbol{p}} := \int_V (\boldsymbol{D}\times\boldsymbol{B})\ dV,} \tag{4.40}$$

i.e. the *Minkowski momentum* $\overset{\text{M}}{\boldsymbol{p}}$ of light inside matter. Notice that (4.40) is the integration of the expression (1.1) which we presented in the introduction. In section 5.3, we will compare this expression, derived from Maxwell's equations, with the corresponding Abraham expression.

In simple terms, the balance equation (4.37) states that the time rate of change of electromagnetic (Minkowski) momentum in matter contained within the finite volume V $\overset{\mathrm{M}}{p}_i$, plus the total effective force exerted by the field on the bound charges and currents of medium inside V ($\int_V f_i^{\mathrm{eff}}\ dV$), plus the total Lorentz force exerted by the field on the external charges and currents inside V ($\boldsymbol{F}^{\mathrm{ext}} = \int_V (\rho_{\mathrm{ext}}\boldsymbol{E} + \boldsymbol{j}_{\mathrm{ext}} \times \boldsymbol{B})\ dV$), is equal to the negative of the *net* momentum (electromagnetic, mechanical, thermal, etc.) which flows per unit time through the boundaries ∂V of the closed volume V and is given by the surface integral $\oint_{\partial V} \left[-E_i D^j - H_i B^j + \frac{1}{2}\delta_i^j \left(E_k D^k + H_k B^k\right)\right] \hat{n}_j\ da$.

Consequently, if we choose a finite region V of space, where there are no external charges and currents, so that $f_i^{\mathrm{ext}} = 0$, the medium is homogeneous and non-dissipative, so that $f_i^{\mathrm{eff}} = 0$ and the volume is finite and big enough, so that there is no momentum flux escaping through the borders of the volume, then the Minkowski momentum of the electromagnetic field in matter $\overset{\mathrm{M}}{p}_i$ will be a time independent conserved quantity,

$$\overset{\mathrm{M}}{p}_i = \int_V \hat{\epsilon}_{ijk} D^j B^k\ dV = constant. \tag{4.41}$$

Notice that in this case the momentum conserved quantity does not only depend on the electromagnetic field E_i and B^i, but it also depends on the medium properties through $D^i = \varepsilon_0 \varepsilon^{ij} E_j + \beta^i{}_j B^j$. This shows the coupled nature of the system and also the fact that the conserved Minkowski momentum for light in matter is an effective quantity which depends on both the electromagnetic field and the medium properties.

Moreover, is there is the case where the conditions for the conservation on energy and momentum are simultaneously satisfied, we can say that the whole Minkowski 4-momentum of the electromagnetic field, defined in (4.28), is a time independent quantity, i.e.

$$\overset{\mathrm{M}}{\mathcal{P}}_\mu = constant. \tag{4.42}$$

4.2. Angular momentum balance equation

In order to introduce the angular momentum balance equation for the open system electromagnetic field interacting with matter, we first notice that the electromagnetic field, via the Lorentz force density $\boldsymbol{f}_{\mathrm{ext}}$, can also exert a *torque density* $\boldsymbol{\tau}_{\mathrm{ext}}$ on a distribution of external charges and currents. This torque density is defined as usual by

$$\boxed{\boldsymbol{\tau}_{\mathrm{ext}} := \boldsymbol{x} \times \boldsymbol{f}_{\mathrm{ext}},} \tag{4.43}$$

and the total torque on the distribution can be computed as the volume integral of it, i.e. $\int_V (\boldsymbol{x} \times \boldsymbol{f}_{\mathrm{ext}})\ dV$.

If we want to write the torque density (4.43) in indicial notation, it is necessary to explicitly use the spacetime metric as for the case of the constitutive tensor in section 3.3. Therefore, if we denote by τ_i^{ext} the cartesian components of the torque density $\boldsymbol{\tau}_{\mathrm{ext}}$, x^i the cartesian components of the space coordinates $\boldsymbol{x}$ and f_i^{ext} the cartesian components of the external

Lorentz force density (3.34), we have

$$\begin{aligned} \tau_i^{\text{ext}} &= \hat{\epsilon}_{ijk} x^j (-\eta^{kl} f_l^{\text{ext}}) && (4.44) \\ &= \hat{\epsilon}_{ijk} x^j (-f_{\text{ext}}^k) && (4.45) \\ &= \epsilon_{ijk} x^j f_{\text{ext}}^k, && (4.46) \end{aligned}$$

where we have defined $f^i_{\text{ext}} := \eta^{ij} f_j^{\text{ext}}$ as the *contravariant* components of the external Lorentz force density $\boldsymbol{f}_{\text{ext}}$. We will use the momentum balance equation (4.37) to derive the angular momentum one. If we take (4.37), rise the free index i, relabel it as k and write it in differential form, we obtain

$$\frac{\partial \pi^k}{\partial t} + \partial_j p^{kj} + f^k_{\text{eff}} + f^k_{\text{ext}} = 0. \tag{4.47}$$

Then, if we multiply (4.47) by $\epsilon_{ijk} x^l$ and complete the total derivative ∂_j, the angular momentum balance equation in differential form reads

$$\frac{\partial}{\partial t}(\epsilon_{ijk} x^j \pi^k) + \partial_j(\epsilon_{ilk} x^l p^{kj}) + \epsilon_{ijk} x^j f^k_{\text{ext}} + \epsilon_{ijk} x^j f^k_{\text{eff}} + \epsilon_{ijk} p^{jk} = 0 \tag{4.48}$$

or, in compact form,

$$\boxed{\frac{\partial l_i}{\partial t} + \partial_j K_i{}^j + \tau_i^{\text{ext}} + \tau_i^{\text{eff}} = 0,} \tag{4.49}$$

where

$$\begin{aligned} l_i &:= \epsilon_{ijk} x^j \pi^k && (4.50) \\ &= (x^j B_j) D_i - (x^j D_j) B_i, && (4.51) \end{aligned}$$

are the cartesian components of the *orbital angular momentum density* of the electromagnetic field:

$$\boxed{\boldsymbol{l} = \boldsymbol{x} \times \boldsymbol{\pi},} \tag{4.52}$$

defined in an analogous way as (4.43), where $\boldsymbol{\pi}$ is the Minkowski momentum density of the electromagnetic field (4.23). The *net orbital angular momentum flux density* $K_i{}^j$ is defined as

$$\begin{aligned} K_i{}^j &:= \epsilon_{ilk} x^l p^{kj} && (4.53) \\ &= (\hat{\epsilon}_{ilk} x^l E^k) D^j + (\hat{\epsilon}_{ilk} x^l H^k) B^j - \frac{1}{2} \hat{\epsilon}_{ilk} \eta^{kj} x^l (E_m D^m + H_m B^m). && (4.54) \end{aligned}$$

Since f_i^{eff} is the material effective force density, it is reasonable to expect that $\epsilon_{ijk} x^j f^k_{\text{eff}}$ would be the torque density which the electromagnetic field exerts on matter, in analogy to (4.43). However, the angular momentum balance equations (4.48)-(4.49) states that this is not the case. The cartesian components of the *material effective torque density* τ_i^{eff}, which the electromagnetic field exerts on the bound charges and currents of the medium, must be defined as

$$\boxed{\tau_i^{\text{eff}} := \epsilon_{ijk}x^j f^k_{\text{eff}} + \epsilon_{ijk}p^{[jk]},} \tag{4.55}$$

where additionally to the expected first term, there is a second term proportional to the antisymmetric part of the momentum flux density ${p_i}^j$.

In an homogeneous medium, without external charges and currents, f_i^{eff} always vanishes and if $\epsilon_{ijk}x^j f^k_{\text{eff}}$ were the only contribution to τ_i^{eff}, the conservation of linear momentum would automatically imply the conservation of the angular one and it would be no sense to define the angular momentum of the field as another quantity. Therefore, it is reasonable that τ_i^{eff} in (4.55) should have another contribution besides $\epsilon_{ijk}x^j f_{\text{eff}}$, in spite of the fact that it cannot be obtained in the usual way as in (4.43). The material effective force density f_i^{eff}, as well as the material effective torque density τ_i^{eff} are not obtained as a direct sum of microscopic forces or torques between the atoms of the medium and the electromagnetic field, but they are just *effective interaction terms*, obtained from the balance equations, with which we can say in an effective way whether the energy, momentum or angular momentum of the electromagnetic field coupled with matter, is conserved or not. For instance, when the momentum of the field inside matter is conserved, light will continue propagating without change and therefore we infer that there should be no *effective* force between the material medium and the field, even though we know that they are actually microscopically interacting. It was in this sense that f_i^{eff} and P_{eff} were previously defined in (4.34) and (4.38), and exactly in the same sense τ_i^{eff} is now defined in (4.55).

If we write $\epsilon_{ijk}p^{[jk]}$ explicitly, we have

$$\epsilon_{ijk}p^{[jk]} = \hat{\epsilon}_{ijk}E^jD^k + \hat{\epsilon}_{ijk}H^jB^k, \tag{4.56}$$

and therefore we see that only in the particular case when the vector fields $\boldsymbol{E}$ and $\boldsymbol{D}$, as well as $\boldsymbol{H}$ and $\boldsymbol{B}$ are parallel to each other at each point, the contribution (4.56) will be zero. For instance, this is the case of an isotropic and non-magnetoelectric medium in its rest frame, whose constitutive relations are given in (3.78)-(3.79). By inserting (4.38) in (4.55), we can express the material effective torque density τ_i^{eff} in terms of the electromagnetic fields in matter E_i, B^i, D^i and H_i, by

$$\tau_i^{\text{eff}} = \frac{1}{2}\epsilon_{ijk}x^j\eta^{kl}\left[D^m(\partial_l E_m) - E_m(\partial_l D^m) + B^m(\partial_l H_m) - H_m(\partial_l B^m)\right] \tag{4.57}$$

$$- \epsilon_{ijk}\eta^{jl}(E_lD^k + H_lB^k), \tag{4.58}$$

which of course can also be written in terms of the medium properties ε^{ij}, $(\mu^{-1})_{ij}$, ${\alpha_i}^j$ and ${\beta^i}_j$ and the electromagnetic field E_i and B^i, if we use the constitutive relations.

4.2.1. Four-dimensional angular momentum balance equation

In subsection 4.1.3 we saw that when $J^\mu_{\text{ext}} = 0$ and the medium is homogeneous, the Minkowski tensor (4.11) satisfies a continuity equation and therefore the energy and momentum of the electromagnetic field are conserved quantities. Under these conditions, the angular momentum balance equation (4.49) reduces to

$$\frac{\partial l_i}{\partial t} + \partial_j K_i^{\ j} = -\epsilon_{ijk} p^{[jk]} \neq 0, \tag{4.59}$$

and we explicitly see that the homogeneity of the medium is not a condition for the angular momentum of the field to satisfy a continuity equation. It would be beautiful that the necessary and sufficient condition for the conservation of the orbital angular momentum of the field would be the isotropy of the electromagnetic properties of the medium. In order to answer this question and study in a simpler way the relationship between the symmetries of the medium (rotational invariance and Lorentz invariance) with the conservation of the electromagnetic angular momentum, we will first define an "angular 4-momentum tensor" analogous to the Minkowski energy-momentum tensor $\Theta_\mu^{\ \nu}$ in (4.11).

For this task, we first notice that the orbital angular momentum density of the field l_i in (4.50) can also be expressed in terms of the Minkowski energy momentum components by

$$l^i = -\frac{1}{c}\epsilon^{ijk} x_j \Theta_k^{\ 0}, \tag{4.60}$$

and multiplying (4.60) by ϵ_{lmi}, we obtain the relation

$$x_l \Theta_m^{\ 0} - x_m \Theta_l^{\ 0} = \hat{\epsilon}_{lmi}(-cl^i). \tag{4.61}$$

Therefore, we will define the covariant *angular 4-momentum tensor* $l_{\rho\sigma}^{\ \ \mu}$ as

$$\boxed{l_{\rho\sigma}^{\ \ \mu} := x_\rho \Theta_\sigma^{\ \mu} - x_\sigma \Theta_\rho^{\ \mu},} \tag{4.62}$$

so that the components $l_{lm}^{\ \ 0}$ of this antisymmetric tensor contain the electromagnetic orbital angular momentum density l_i as in (4.61). Using $l_{\rho\sigma}^{\ \ \mu}$ we can obtain the angular momentum balance equation (4.49) as components of a more general *angular 4-momentum balance equation*. If we take the 4-divergence on both sides of equation (4.62) and then insert the energy-momentum balance equation which satisfies the Minkowski tensor (4.9), we finally can write

$$\boxed{\partial_\mu l_{\rho\sigma}^{\ \ \mu} + \mathcal{T}^{\text{ext}}_{\rho\sigma} + \mathcal{T}^{\text{eff}}_{\rho\sigma} = 0,} \tag{4.63}$$

where $\mathcal{T}^{\text{ext}}_{\rho\sigma}$ is the external 4-torque density tensor defined as

$$\boxed{\mathcal{T}^{\text{ext}}_{\rho\sigma} := x_\rho \mathcal{F}^{\text{ext}}_\sigma - x_\sigma \mathcal{F}^{\text{ext}}_\rho,} \tag{4.64}$$

and the material effective 4-torque tensor density $\mathcal{T}^{\text{eff}}_{\rho\sigma}$ is defined analogously to (4.55), as

$$\boxed{\mathcal{T}^{\text{eff}}_{\rho\sigma} := x_\rho \mathcal{F}^{\text{eff}}_\sigma - x_\sigma \mathcal{F}^{\text{eff}}_\rho + 2\Theta_{[\rho\sigma]},} \tag{4.65}$$

where

$$2\Theta_{[\rho\sigma]} = \eta_{\rho\lambda} F_{\sigma\kappa} H^{\lambda\kappa} - \eta_{\sigma\lambda} F_{\rho\kappa} H^{\lambda\kappa}, \tag{4.66}$$

is the antisymmetric part of the Minkowski energy-momentum tensor.

If we evaluate the angular 4-momentum balance equation (4.62) for the spatial components $j, k = 1, 2, 3$, and multiply it by $-\epsilon^{ijk}/2$, we indeed obtain the balance equation for angular momentum (4.49), provided we do the following identifications of components:

$$l^i = -\frac{1}{2c}\epsilon^{ijk}{l_{jk}}^0, \tag{4.67}$$

$$K^{im} = -\frac{1}{2}\epsilon^{ijk}{l_{jk}}^m, \tag{4.68}$$

$$\tau^i_{\text{ext}} = -\frac{1}{2}\epsilon^{ijk}\mathcal{T}^{\text{ext}}_{jk}, \tag{4.69}$$

$$\tau^i_{\text{eff}} = -\frac{1}{2}\epsilon^{ijk}\mathcal{T}^{\text{eff}}_{jk}. \tag{4.70}$$

4.2.2. 4-torque densities and conservation of angular 4-momentum

From (4.67)-(4.68) we see that the spatial components ${l_{ij}}^0$ are related to the orbital angular momentum density l_i of the electromagnetic field and ${l_{ij}}^k$ are related to the orbital angular momentum flux density ${K_i}^j$. By inverting (4.67), we can explicitly write the antisymmetric angular 4-momentum density in terms of their components as

$${l_{ij}}^0 = c\epsilon_{ijk}l^k, \tag{4.71}$$

or in matrix form by

$${l_{\rho\sigma}}^0 = \begin{pmatrix} 0 & {l_{01}}^0 & {l_{02}}^0 & {l_{03}}^0 \\ -{l_{01}}^0 & 0 & cl_3 & -cl_2 \\ -{l_{02}}^0 & -cl_3 & 0 & cl_1 \\ -{l_{03}}^0 & cl_2 & -cl_1 & 0 \end{pmatrix} = -{l_{\sigma\rho}}^0. \tag{4.72}$$

Using the definition (4.64), we see that the 3 independent "time-space" components ${l_{0i}}^0$ are explicitly given by

$$\begin{aligned} {l_{0i}}^0 &= x_0{\Theta_i}^0 - x_i{\Theta_0}^0 & (4.73)\\ &= -c^2\left(t\pi_i + x_i\frac{\mathcal{U}}{c^2}\right), & (4.74) \end{aligned}$$

but they do not have a direct interpretation in terms of clearly identifiable properties of the system. In the next subsection 4.2.3 we will discuss the integral form of the angular 4-momentum balance equation and we will see that the volume integral of these components ${l_{0i}}^0$ are related to the motion of the center of energy of the system, a relativistic generalization of the center of mass.

Analogously to (4.71), we can also invert (4.69), and obtain

$$\mathcal{T}^{\text{ext}}_{ij} = \epsilon_{ijk}\tau^k_{\text{ext}}, \tag{4.75}$$

which allow us to express the external 4-torque density $\mathcal{T}^{\text{ext}}_{\rho\sigma}$ in matrix form as

$$\mathcal{T}^{\text{ext}}_{\rho\sigma} = \begin{pmatrix} 0 & \mathcal{T}^{\text{ext}}_{01} & \mathcal{T}^{\text{ext}}_{02} & \mathcal{T}^{\text{ext}}_{03} \\ -\mathcal{T}^{\text{ext}}_{01} & 0 & \tau^{\text{ext}}_{3} & -\tau^{\text{ext}}_{2} \\ -\mathcal{T}^{\text{ext}}_{02} & -\tau^{\text{ext}}_{3} & 0 & \tau^{\text{ext}}_{1} \\ -\mathcal{T}^{\text{ext}}_{03} & \tau^{\text{ext}}_{2} & -\tau^{\text{ext}}_{1} & 0 \end{pmatrix} = -\mathcal{T}^{\text{ext}}_{\sigma\rho}, \tag{4.76}$$

where, again, the "time-space" components $\mathcal{T}^{\text{ext}}_{0i}$ are not direct to interpret in terms of mechanical clearly identifiable quantities like power, force or torque densities. In fact, $\mathcal{T}^{\text{ext}}_{0i}$ has contributions of both the external Lorentz force density f^{ext}_i and the external power density P_{ext}, as follows:

$$\mathcal{T}^{\text{ext}}_{0i} = x_0\mathcal{F}^{\text{ext}}_i - x_i\mathcal{F}^{\text{ext}}_0 \tag{4.77}$$

$$= -\,c\left(tf^{\text{ext}}_i + x_i\frac{P_{\text{ext}}}{c^2}\right). \tag{4.78}$$

With regards to the material effective 4-torque density $\mathcal{T}^{\text{eff}}_{\rho\sigma}$, if we invert (4.70), use the definition in (4.65)-(4.66), as well as the explicit expressions for f^{eff}_i and P_{eff} in (4.34) and (4.38), we see that its components can be expressed in matrix form, in the same way as $\mathcal{T}^{\text{ext}}_{\rho\sigma}$, by

$$\mathcal{T}^{\text{eff}}_{\rho\sigma} = \begin{pmatrix} 0 & \mathcal{T}^{\text{eff}}_{01} & \mathcal{T}^{\text{eff}}_{02} & \mathcal{T}^{\text{eff}}_{03} \\ -\mathcal{T}^{\text{eff}}_{01} & 0 & \tau^{\text{eff}}_{3} & -\tau^{\text{eff}}_{2} \\ -\mathcal{T}^{\text{eff}}_{02} & -\tau^{\text{eff}}_{3} & 0 & \tau^{\text{eff}}_{1} \\ -\mathcal{T}^{\text{eff}}_{03} & \tau^{\text{eff}}_{2} & -\tau^{\text{eff}}_{1} & 0 \end{pmatrix} = -\mathcal{T}^{\text{eff}}_{\sigma\rho}. \tag{4.79}$$

where the spatial components are related to the effective torque vector field τ^{eff}_i, by

$$\mathcal{T}^{\text{eff}}_{ij} = \epsilon_{ijk}\tau^{k}_{\text{eff}} \tag{4.80}$$

$$= \left(x_i f^{\text{eff}}_j - x_j f^{\text{eff}}_i\right) - (p_{ij} - p_{ji}) \tag{4.81}$$

$$= \frac{1}{2}x_{[i}\left\{D^k(\partial_{j]}E_k) - E_k(\partial_{j]}D^k) + B^k(\partial_{j]}H_k) - H_k(\partial_{j]}B^k)\right\} + D_{[i}E_{j]} + H_{[i}B_{j]}, \tag{4.82}$$

and the time-space components $\mathcal{T}^{\text{eff}}_{0i}$ can be explicitly written as

$$\mathcal{T}^{\text{eff}}_{0i} = x_0\mathcal{F}^{\text{eff}}_i - x_i\mathcal{F}^{\text{eff}}_0 + 2\Theta_{[0i]} \tag{4.83}$$

$$= -\,c\left(tf^{\text{eff}}_i + x_i\frac{P_{\text{eff}}}{c^2}\right) + c\left(\pi_i + \frac{S_i}{c^2}\right) \tag{4.84}$$

$$\begin{aligned} = &-\frac{1}{2}ct\left[D^j(\partial_i E_j) - E_j(\partial_i D^j) + B^j(\partial_i H_j) - H_j(\partial_i B^j)\right] \\ &-\frac{1}{2}\frac{x_i}{c}\left[E_j\frac{\partial D^j}{\partial t} - D^j\frac{\partial E_j}{\partial t} + H_j\frac{\partial B^j}{\partial t} - B^j\frac{\partial H_j}{\partial t}\right] \\ &+ c\hat{\epsilon}_{ijk}\left[D^jB^k - \frac{1}{c^2}E^jH^k\right]. \end{aligned} \tag{4.85}$$

In order to interpret $\mathcal{T}^{\text{eff}}_{\rho\sigma}$ in terms of the symmetries presented by the material medium, let us insert the covariant constitutive relations (3.61) into the definition of material effective

4-torque (4.65), so that we can write a covariant expression for $\mathcal{T}^{\text{eff}}_{\rho\sigma}$ in terms of the material properties $\chi^{\mu\nu\rho\sigma}$ and the electromagnetic field $F_{\mu\nu}$:

$$\boxed{\begin{aligned}\mathcal{T}^{\text{eff}}_{\rho\sigma} = & -\frac{1}{8}F_{\alpha\beta}F_{\gamma\delta}\left[x_\rho(\partial_\sigma\chi^{\alpha\beta\gamma\delta}) - x_\sigma(\partial_\rho\chi^{\alpha\beta\gamma\delta}) + \eta_{\alpha\rho}\chi^{\sigma\beta\gamma\delta} - \eta_{\alpha\sigma}\chi^{\rho\beta\gamma\delta}\right. \\ & \left. +\eta_{\beta\rho}\chi^{\alpha\sigma\gamma\delta} - \eta_{\beta\sigma}\chi^{\alpha\rho\gamma\delta} + \eta_{\gamma\rho}\chi^{\alpha\beta\sigma\delta} - \eta_{\gamma\sigma}\chi^{\alpha\beta\rho\delta} + \eta_{\delta\rho}\chi^{\alpha\beta\gamma\sigma} - \eta_{\delta\sigma}\chi^{\alpha\beta\gamma\rho}\right] \\ & -\frac{1}{8}\left(\chi^{\alpha\beta\gamma\delta} - \chi^{\gamma\delta\alpha\beta}\right)F_{\alpha\beta}\left[x_\rho(\partial_\sigma F_{\gamma\delta}) - x_\sigma(\partial_\rho F_{\gamma\delta})\right].\end{aligned}} \tag{4.86}$$

This expression (4.86) can be understood as the angular analogous of the expression (4.15) for the material effective 4-force density $\mathcal{F}^{\text{eff}}_\mu$. By inspecting (4.86), we see that its last term is a dissipative contribution to the effective torque density between field and matter, since it is only different from zero when the constitutive tensor does not satisfy the extra symmetry (3.64) and it describes a persistent interaction throughout the propagation of light in matter, independent of the other properties of the medium, in the same way as the last term in (4.15). On the other hand, the first term in the first two lines of (4.86) has direct relation with the symmetry presented by the material medium, since if we compare it with the invariance condition (C.97) of appendix C, we see that this term is identically zero if the constitutive tensor $\chi^{\mu\nu\rho\sigma}$ is *invariant* under Lorentz transformations. If we restrict ourselves to the case of non-dissipative media, we see that there will be an effective torque interaction between the electromagnetic field and the bound charges and currents of the medium, only when the medium is *anisotropic*, so that the spatial components $\mathcal{T}^{\text{eff}}_{ij} = \epsilon_{ijk}\tau^k_{\text{eff}}$ can be different from zero. This is analogous to the spatial components of the first term in (4.15), which imply a force interaction between field and matter only then the medium is inhomogeneous.

Therefore, suppose there is a non-dissipative and isotropic medium, without external charges and currents $J^\mu_{\text{ext}} = 0$. In this case, all the components of the external 4-torque density will be zero $\mathcal{T}^{\text{ext}}_{\rho\sigma} = 0$, as well as the spatial components of the material effective 4-torque density $\mathcal{T}^{\text{eff}}_{ij} = 0$. As a consequence, the spatial components ij of the angular 4-momentum balance equation (4.63) will reduce to

$$\partial_\mu l_{ij}{}^\mu = 0, \tag{4.87}$$

which, using the identifications (4.67)-(4.68), implies the continuity equation for the orbital angular momentum of the electromagnetic field:

$$\frac{\partial l_i}{\partial t} + \partial_j K_i{}^j = 0. \tag{4.88}$$

Therefore, the absence of dissipation and the isotropy of a medium ensures the conservation of the orbital angular momentum of the electromagnetic field. However, they are not sufficient to imply a continuity equation for the components $0i$ of the angular 4-momentum:

$$\partial_\mu l_{0i}{}^\mu = \frac{\partial}{\partial t}\left(\frac{1}{c}l_{0i}{}^0\right) + \partial_j l_{0i}{}^j = -\mathcal{T}^{\text{eff}}_{0i} \neq 0, \tag{4.89}$$

since the $\mathcal{T}^{\text{eff}}_{0i}$ components of the effective 4-torque density does not vanish under these conditions. In order to obtain conserved $0i$ components of the angular 4-momentum, the constitutive tensor $\chi^{\mu\nu\rho\sigma}$ must independently be invariant under *boosts*. If this were the case, we will obtain $\mathcal{T}_{0i}^{\text{eff}} = 0$, and (4.89) will reduce to a continuity equation.

For instance, if we insert the constitutive tensor for the isotropic medium $\chi_{\text{iso}}^{\mu\nu\rho\sigma}$, given in (3.92), in the invariance condition (C.97), we can prove that in the *only* case when this equation is satisfied for the $0i$ components, is when $n = 1$, i.e. when there is no medium and we have vacuum! The only constitutive tensor invariant under boosts is the corresponding to the vacuum $\chi_{\text{vac}}^{\mu\nu\rho\sigma}$ in (3.58), which is not so surprising since when we assumed a fixed material medium in the system, there must be external forces keeping the medium in its state of motion. As a result, the different inertial frames will be not longer equivalent, since an observer could physically differentiate if he is at rest or in motion with respect to the background material medium and therefore the symmetry under boosts is lost. In vacuum, there is no material medium fixed and all inertial observers will agree that the electromagnetic field is propagating at constant speed c. In chapter 5 the material medium will be no longer fixed and therefore the invariance under boosts of the total system will be recover.

4.2.3. Integral representation of the angular 4-momentum balance equations

In order to introduce the integral version of the angular momentum balance equations, it will be very useful to define the *total orbital angular 4-momentum* of the electromagnetic field inside matter $L_{\mu\nu}$, in analogy to the total 4-momentum $\mathcal{P}_\mu$ in (4.27), by

$$L_{\rho\sigma} := \int_V \left(x_\rho \mathcal{P}_\sigma - x_\sigma \mathcal{P}_\rho\right) dV \tag{4.90}$$

$$= \frac{1}{c}\int_V l_{\rho\sigma}{}^0 \, dV, \tag{4.91}$$

which in matrix form can be expressed as,

$$L_{\mu\nu} = \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ -L_{01} & 0 & L_3 & -L_2 \\ -L_{02} & -L_3 & 0 & L_1 \\ -L_{03} & L_2 & -L_1 & 0 \end{pmatrix} = -L_{\nu\mu}, \tag{4.92}$$

where the spatial components L_{ij} can be expressed in terms of the Minkowski orbital angular momentum vector $\overset{\text{M}}{L}_i$ as usual by

$$L_{ij} = \epsilon_{ijk} \overset{\text{M}}{L}{}^k, \tag{4.93}$$

where

$$\overset{\text{M}}{L}_i := \int_V l_i \, dV \tag{4.94}$$

$$= \int_V (\boldsymbol{x} \times \boldsymbol{\pi})_i \, dV. \tag{4.95}$$

Additionally, if we insert (4.74) in the definition (4.91), the components L_{0i} of the total orbital 4-momentum are explicitly given by

$$
\begin{aligned}
L_{0i} &:= \frac{1}{c}\int_V l_{0i}{}^{0}\; dV &(4.96)\\
&= -\,c\left(t\int_V \pi_i\; dV + \frac{1}{c^2}\int_V \mathcal{U}x_i\; dV\right). &(4.97)
\end{aligned}
$$

Meanwhile, we can define the *position of the center of energy of the system* $x^i_{\mathrm{ce}}(t)$, as a relativistic generalization of the center of mass:

$$
\boxed{x^i_{\mathrm{ce}}(t) := \frac{1}{E}\int_V \mathcal{U}x^i\; dV,} \tag{4.98}
$$

where E is the total energy of the system, as defined in (4.30). Then, using (4.98) in (4.97), the components L_{0i} can be cast in a much simpler form, as

$$
\boxed{L_{0i} = -c\left(t\overset{\mathrm{M}}{p}_i + \frac{1}{c^2}\overset{\mathrm{M}}{E}x_i^{\mathrm{ce}}\right),} \tag{4.99}
$$

where we also used the definition of the total momentum of the field p_i in (4.29), which in this case coincides with Minkowski's one.

Now we are in position to integrate the covariant angular 4-momentum balance equation (4.63) over a finite volume V and use the definition (4.91) to write the balance equations for the total quantities:

$$
\frac{d}{dt}L_{\rho\sigma} + \int_V \mathcal{T}^{\mathrm{ext}}_{\rho\sigma}\; dV + \int_V \mathcal{T}^{\mathrm{eff}}_{\rho\sigma}\; dV = -\oint_{\partial V} l_{\rho\sigma}{}^{j}\hat{n}_j\; da. \tag{4.100}
$$

In the same way as we obtained (4.49) from the spatial components of (4.63), here we can evaluate (4.100) for the spatial components $k, m = 1, 2, 3$, multiply it by $-\epsilon^{ikm}/2$ and use the corresponding identifications to obtain the integral version of the balance equation for angular momentum:

$$
\boxed{\frac{d}{dt}\overset{\mathrm{M}}{L}_i + \int_V \tau_i^{\mathrm{ext}}\; dV + \int_V \tau_i^{\mathrm{eff}}\; dV = -\oint_{\partial V} K_i{}^{j}\hat{n}_j da.} \tag{4.101}
$$

Analogous to the other balance equations (4.32) and (4.37), in (4.101) the time rate of change of total Minkowski orbital angular momentum $\overset{\mathrm{M}}{L}_i$ contained in the electromagnetic field within the volume V, plus the torque exerted by the field on the external charges and currents ($\int_V \tau_i^{\mathrm{ext}}\; dV$), plus the effective torque exerted by the field on the bound charges and currents of the medium ($\int_V \tau_i^{\mathrm{eff}}\; dV$), is equal to the negative of the *net* angular momentum which flows per unit time through the boundaries ∂V of the closed volume V ($\oint_{\partial V} K_i{}^{j}\hat{n}_j\; da$).

For the specific case that we consider a volume V of the medium, where there are no external charges and currents, so that $\tau_i^{\mathrm{ext}} = 0$, the medium is isotropic and non-dissipative, so that $\tau_i^{\mathrm{eff}} = 0$, and if the volume is big enough so that there is no net angular momentum flux escaping through its boundaries, then the Minkowski total angular momentum vector inside the volume will be a conserved, i.e. a time independent quantity:

$$\overset{\mathrm{M}}{L}_i = \int_V [(x^j B_j) D_i - (x^j D_j) B_i] \; dV = constant. \tag{4.102}$$

Finally, in order to obtain the balance equation for the time-space components of the angular 4-momentum of the field L_{0i}, we just need to evaluate (4.100) for the $0i$ components:

$$\boxed{\frac{d}{dt} L_{0i} + \int_V \mathcal{T}_{0i}^{\mathrm{ext}} \; dV + \int_V \mathcal{T}_{0i}^{\mathrm{eff}} \; dV = - \oint_{\partial V} l_{0i}{}^j \hat{n}_j da.} \tag{4.103}$$

The integral balance equation (4.103) is interpreted in exactly the same way as the other ones for energy, momentum and angular momentum, which we already discussed, but here the problem is that the quantities involved are not directly recognizable. As we commented at the end of subsection 4.2.2, $\mathcal{T}_{0i}^{\mathrm{ext}}$ is only different from zero in the case of vacuum and therefore (4.103) will not lead to any conserved quantities when there is a fixed medium present.

Even though the following treatment will only lead to conserved quantities for the case of the electromagnetic field in vacuum, we will make the corresponding general derivation anyway, since we will apply the result in chapter 5 when we will deal with a closed system. Thus, let us choose a volume V, where there are no external charges and currents, so that $\mathcal{T}_{0i}^{\mathrm{ext}} = 0$, the constitutive tensor assigned to the medium is *invariant under boosts* and non-dissipative, so that $\mathcal{T}_{0i}^{\mathrm{eff}} = 0$, and the volume is big enough so that there is no flux escaping through its boundaries, then the time-space components L_{0i} will be time independent conserved quantities:

$$L_{0i} = -c \left(t \overset{\mathrm{M}}{p}_i + \frac{1}{c^2} \overset{\mathrm{M}}{E} x_i^{\mathrm{ce}} \right) = constant. \tag{4.104}$$

As a result, when the medium is invariant under boosts, a certain combination of the Minkowski momentum, the Minkowski energy and the position of the center of energy, given in (4.104), will be time independent. In fact, (4.104) can be interpreted as a constraint for the motion of the center of energy of the electromagnetic field, when the medium in which it propagates is invariant under boosts. In particular, let us define the velocity of the center of energy v_{ce}^i as

$$v_{\mathrm{ce}}^i := \frac{dx_{\mathrm{ce}}^i}{dt}, \tag{4.105}$$

and then, if we take the time derivative of equation (4.104), we obtain an explicit equation for v_{ce}^i, which reads

$$v_i^{\mathrm{ce}} + \frac{c^2}{E^M} \overset{\mathrm{M}}{p}_i + \frac{c^2 t}{E^M} \frac{d\overset{\mathrm{M}}{p}_i}{dt} + \frac{x_i^{\mathrm{ce}}}{E^M} \frac{d\overset{\mathrm{M}}{E}}{dt} = 0. \tag{4.106}$$

The equation (4.106) still looks difficult to physically interpret, but if we additionally assume the the medium is homogeneous and time independent, the Minkowski momentum and energy of the electromagnetic field will be also time independent quantities, implying a constant velocity for the center of energy of the field, simply given by

$$\boxed{v_{\mathrm{ce}}^{i} = -\frac{c^2}{E^M} \overset{\mathrm{M}}{p}{}^{i} = constant.} \tag{4.107}$$

So we conclude that the components L_{0i} of the angular 4-momentum are in some way related to the motion of the center of energy of the system. Actually, for the case of the electromagnetic field in fixed matter, the result is only valid when the medium is vacuum, in which the constant center of energy velocity of the system in (4.107), reduces to the velocity of light in vacuum c. In chapter 5, we will consider the total system formed by electromagnetic field and dynamical medium and in our analysis we will use the expression (4.107) to determine the constant center of energy velocity of the combine total system.

4.3. Lagrangian for the open system of electromagnetic field inside matter

So far we have discussed the balance equations and the conditions under which they lead to conserved quantities of the electromagnetic field, starting directly from Maxwell's equations and the constitutive relations for the non-dynamical medium. Complementary to the latter analysis, in this section we will apply the Lagrangian-Noether formalism to study in more depth this open system. The results obtained here should be the same ones derived in sections 4.1 and 4.2, but this different treatment will give us further insight into the structure of the theory, specially regarding the relation between the symmetries of the medium and the conserved quantities of the field. In chapter 5, we will also apply the Lagrangian-Noether formalism to describe the dynamics of the medium as as fluid and therefore using the results of this section, we will describe the dynamics of the total closed system, composed by electromagnetic field and material medium. The general results of Lagrangian-Noether formalism for any open or closed system are shortly summarized in appendix C.

4.3.1. Macroscopic Maxwell equations as Euler-Lagrange equations

In order to apply this general formalism to the case of the electromagnetic field in matter, we postulate the electromagnetic Lagrangian density in matter $\mathcal{L}^{\mathrm{em}}$ to be:

$$\boxed{\mathcal{L}^{\mathrm{em}} := -\frac{1}{4} F_{\mu\nu} H^{\mu\nu},} \tag{4.108}$$

since, in absence of external charges and currents, it reproduces the macroscopic Maxwell's equations as Euler-Lagrange equations (C.1) of the electromagnetic 4-potential A_μ, defined in (3.41)-(3.42). In fact, if we insert (3.61) in (4.108), the electromagnetic Lagrangian density $\mathcal{L}^{\text{em}}$ can be explicitly written as

$$\mathcal{L}^{\text{em}}(\partial_\mu A_\nu, \chi^{\mu\nu\rho\sigma}) = -\frac{1}{8}\chi^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{4.109}$$

where A_μ is the dynamical field of the system, which appears in $\mathcal{L}^{\text{em}}$ through $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\chi^{\mu\nu\rho\sigma}$ is an external field, describing the *given* electromagnetic properties of the fixed medium, in which light propagates. Here it is important to remark that only *non-dissipative* media can be analyzed in the Lagrangian formalism, because from the early beginning in definition (4.109) we have to assume that the constitutive tensor of the medium must satisfy the condition $\chi^{\mu\nu\rho\sigma} = \chi^{\rho\sigma\mu\nu}$. In order to explicitly calculate the Euler-Lagrange equations for the dynamical field A_μ, the following derivatives will be useful:

$$\frac{\partial\mathcal{L}^{\text{em}}}{\partial(\partial_\mu A_\nu)} = -\,H^{\mu\nu}, \tag{4.110}$$

$$\frac{\partial\mathcal{L}^{\text{em}}}{\partial A_\nu} = 0, \tag{4.111}$$

and then, inserting (4.110)-(4.111) in the general definition (C.1)-(C.3), we see that these Euler-Lagrange equations actually coincide with the sourceless macroscopic Maxwell equations:

$$\frac{\delta\mathcal{L}^{\text{em}}}{\delta A_\nu} = \partial_\mu H^{\mu\nu} \stackrel{!}{=} 0, \tag{4.112}$$

as given in (3.40). The homogeneous Maxwell equations (3.45) are automatically satisfied by expressing the electromagnetic field strength $F_{\mu\nu}$ in terms of the 4-potential A_μ.

If we also want to consider the interaction of electromagnetic field with external charges and currents, described in the external 4-vector field $J^\mu_{\text{ext}} := (c\rho_{\text{ext}}, j^i_{\text{ext}})$, we just have to add another term $\mathcal{L}^{\text{ext}}$ to the Lagrangian density (4.109), which must be given by,

$$\boxed{\mathcal{L}^{\text{ext}}(A_\mu, J^\mu_{\text{ext}}) := -J^\mu_{\text{ext}}A_\mu.} \tag{4.113}$$

The corresponding variational derivative of $\mathcal{L}^{\text{ext}}$ is easily calculated as

$$\frac{\delta\mathcal{L}^{\text{ext}}}{\delta A_\nu} = \frac{\partial\mathcal{L}^{\text{ext}}}{\partial A_\nu} = -J^\nu_{\text{ext}}, \tag{4.114}$$

and therefore the Euler-Lagrange equations corresponding to the composed Lagrangian density $\mathcal{L}^{\text{em}} + \mathcal{L}^{\text{ext}}$, coincide with the inhomogeneous macroscopic Maxwell equations in (3.40):

$$\frac{\delta}{\delta A_\nu}(\mathcal{L}^{\text{em}} + \mathcal{L}^{\text{ext}}) = \frac{\delta\mathcal{L}^{\text{em}}}{\delta A_\nu} + \frac{\delta\mathcal{L}^{\text{ext}}}{\delta A_\nu} = \partial_\mu H^{\mu\nu} - J^\nu_{\text{ext}} \stackrel{!}{=} 0 \quad\Leftrightarrow\quad \partial_\mu H^{\mu\nu} = J^\nu_{\text{ext}}. \tag{4.115}$$

4.3.2. Canonical energy-momentum tensor and energy-momentum balance equation

The next step in the Lagrange-Noether formalism is to calculate the canonical energy-momentum tensor of the system, which will allow us to write the canonical energy-momentum balance equation, as discussed in the appendix C.1. Therefore, using the definition (C.6) and the results (4.110) and (4.114), the canonical energy-momentum tensor for the electromagnetic field in matter is explicitly given by

$$\overset{\text{em}}{T}{}_\mu{}^\nu = \frac{\partial \mathcal{L}^{\text{em}}}{\partial(\partial_\nu A_\rho)}\partial_\mu A_\rho - \delta^\nu_\mu\left(-\frac{1}{4}F_{\rho\sigma}H^{\rho\sigma} - A_\rho J^\rho_{\text{ext}}\right) \tag{4.116}$$

$$= \left(F_{\mu\rho}H^{\rho\nu} + \frac{1}{4}\delta^\nu_\mu F_{\rho\sigma}H^{\rho\sigma}\right) + H^{\rho\nu}(\partial_\rho A_\mu) + \delta^\nu_\mu J^\rho_{\text{ext}}A_\rho \tag{4.117}$$

$$= \Theta_\mu{}^\nu + H^{\rho\nu}(\partial_\rho A_\mu) + \delta^\nu_\mu J^\rho_{\text{ext}}A_\rho, \tag{4.118}$$

where we see that the electromagnetic canonical tensor $\overset{\text{em}}{T}{}_\mu{}^\nu$ differs by two non-gauge invariant terms from the Minkowski energy-momentum tensor $\Theta_\mu{}^\nu$, previously defined in (4.11). The "on-shell" energy-momentum balance equation, which the canonical tensor satisfies, is in general given in (C.7), and in the particular case of the electromagnetic field in matter reads

$$\partial_\nu \overset{\text{em}}{T}{}_\mu{}^\nu \doteq -\frac{1}{8}\frac{\partial \mathcal{L}^{\text{em}}}{\partial \chi^{\alpha\beta\gamma\delta}}\partial_\mu \chi^{\alpha\beta\gamma\delta} - \frac{\partial \mathcal{L}^{\text{ext}}}{\partial J^\nu_{\text{ext}}}\partial_\mu J^\nu_{\text{ext}} \tag{4.119}$$

$$\doteq \frac{1}{8}F_{\alpha\beta}F_{\gamma\delta}(\partial_\mu \chi^{\alpha\beta\gamma\delta}) + A_\rho(\partial_\mu J^\rho_{\text{ext}}). \tag{4.120}$$

Notice that we put an 1/8 factor in (4.119), in order to count the terms of the sum only once, considering all the symmetries of $\chi^{\mu\nu\rho\sigma}$. We may now take the 4-divergence of the canonical energy-momentum tensor $\overset{\text{em}}{T}{}_\mu{}^\nu$ in (4.118) and using the equations of motion (4.115), we obtain the following expression on-shell:

$$\partial_\nu \overset{\text{em}}{T}{}_\mu{}^\nu \doteq \partial_\nu \Theta_\mu{}^\nu + F_{\mu\nu}J^\nu_{\text{ext}} + A_\rho(\partial_\mu J^\rho_{\text{ext}}). \tag{4.121}$$

Finally, if we replace (4.121) in the left hand side of (4.120), the non-gauge invariant terms $A_\rho(\partial_\mu J^\rho_{\text{ext}})$ cancel out on both sides and we obtain the same energy-momentum balance equation, which we already derived from Maxwell equations in (4.9),

$$\boxed{\partial_\nu \Theta_\mu{}^\nu + F_{\mu\nu}J^\nu_{\text{ext}} - \frac{1}{8}F_{\alpha\beta}F_{\gamma\delta}(\partial_\mu \chi^{\alpha\beta\gamma\delta}) \doteq 0,} \tag{4.122}$$

with the difference that now the material effective 4-force density

$$\mathcal{F}^{\text{eff}}_\mu = -\frac{1}{8}F_{\alpha\beta}F_{\gamma\delta}(\partial_\mu \chi^{\alpha\beta\gamma\delta}), \tag{4.123}$$

does not have the dissipation term, since in the Lagrange formalism we can only consider non-dissipative media with $\chi^{\mu\nu\rho\sigma} = \chi^{\rho\sigma\mu\nu}$.

4.3.3. Belinfante tensor of the electromagnetic field in matter

In order to compute the Belinfante tensor of the electromagnetic field in matter $\overset{\text{em}}{\Theta}{}_{\mu}{}^{\nu}$ and the angular 4-momentum balance equation, as defined in the appendix C.2, we first need to calculate the electromagnetic spin current density $\overset{\text{em}}{S}{}_{\rho\sigma}{}^{\mu}$. Evaluating the general definition (C.12) for our particular case, we have

$$\overset{\text{em}}{S}{}_{\rho\sigma}{}^{\mu} = \frac{\partial \mathcal{L}^{\text{em}}}{\partial(\partial_\mu A_\nu)}(s_{\rho\sigma})_\nu{}^\lambda A_\lambda \tag{4.124}$$

$$= \eta_{\sigma\nu} H^{\mu\nu} A_\rho - \eta_{\rho\nu} H^{\mu\nu} A_\sigma, \tag{4.125}$$

where we used (4.110) and the corresponding Lorentz generator derived in (C.87).

Then, if we insert (4.125) in the general definition of the Belinfante energy-momentum tensor (C.2), we obtain

$$\overset{\text{em}}{\Theta}{}_{\mu}{}^{\nu} = \overset{\text{em}}{T}{}_{\mu}{}^{\nu} - \partial_\rho (H^{\rho\nu} A_\mu) \tag{4.126}$$

$$\doteq \overset{\text{em}}{T}{}_{\mu}{}^{\nu} - H^{\rho\nu}(\partial_\rho A_\mu) - J^\nu_{\text{ext}} A_\mu, \tag{4.127}$$

and if we additionally use the explicit expression for the canonical tensor (4.118) in (4.127), we can finally see that the Belinfante energy-momentum tensor of the electromagnetic field $\overset{\text{em}}{\Theta}{}_{\mu}{}^{\nu}$ and the Minkowski tensor $\Theta_\mu{}^\nu$ are related by

$$\boxed{\overset{\text{em}}{\Theta}{}_{\mu}{}^{\nu} = \Theta_\mu{}^\nu + \delta^\nu{}_\mu J^\rho_{\text{ext}} A_\rho - A_\mu J^\nu_{\text{ext}},} \tag{4.128}$$

We know that the Belinfante tensor, by construction, have the same 4-divergence as the canonical one, since they just differ by a total derivative, but its advantage is that the non-gauge invariant term $H^{\rho\nu}(\partial_\rho A_\mu)$ in (4.118) is now absorbed. In fact, the gauge invariant Minkowski tensor $\Theta_\mu{}^\nu$ *actually is* the Belinfante energy-momentum tensor of the electromagnetic field in matter, for the important case when there are no external charges and currents $J^\mu_{\text{ext}} = 0$ and this is the reason why we denote both with very similar symbols.

4.3.4. Symmetry of the energy-momentum tensors

Let us continue with the Lagrangian-Noether analysis and use (4.128) to explicitly compute the antisymmetric part of the Belinfante tensor in terms of the external fields $\chi^{\mu\nu\rho\sigma}$ and J^μ_{ext}:

$$\overset{\text{em}}{\Theta}{}_{[\mu\nu]} = \Theta_{[\mu\nu]} - A_{[\mu} J^{\text{ext}}_{\nu]}. \tag{4.129}$$

The antisymmetric part of the Minkowski tensor can be directly calculated from its definition in (4.11). If we also use the covariant constitutive relations (3.61) and show explicitly the symmetries of $\chi^{\mu\nu\rho\sigma}$, we have

$$
\begin{aligned}
2\Theta_{[\mu\nu]} &= \Theta_{\mu\nu} - \Theta_{\nu\mu} && (4.130)\\
&= \eta_{\nu\lambda} H^{\lambda\rho} F_{\rho\mu} - \eta_{\mu\lambda} H^{\lambda\rho} F_{\rho\nu} && (4.131)\\
&= \frac{1}{2}\left(\delta^{\sigma}{}_{\mu}\eta_{\nu\lambda} - \delta^{\sigma}{}_{\nu}\eta_{\mu\lambda}\right) F_{\alpha\beta} F_{\rho\sigma} \chi^{\lambda\rho\alpha\beta} && (4.132)\\
&= -\frac{1}{8} F_{\alpha\beta} F_{\rho\sigma}\left(\eta_{\alpha\mu}\chi^{\nu\beta\rho\sigma} - \eta_{\alpha\nu}\chi^{\mu\beta\rho\sigma} + \eta_{\beta\mu}\chi^{\alpha\nu\rho\sigma} - \eta_{\beta\nu}\chi^{\alpha\mu\rho\sigma}\right.\\
&\qquad \left. + \eta_{\rho\mu}\chi^{\alpha\beta\nu\sigma} - \eta_{\rho\nu}\chi^{\alpha\beta\mu\sigma} + \eta_{\sigma\mu}\chi^{\alpha\beta\rho\nu} - \eta_{\sigma\nu}\chi^{\mu\beta\rho\mu}\right). && (4.133)
\end{aligned}
$$

By comparing (4.133) with the condition (C.97), we see that if we exclude the two first terms which have to do with the orbital angular momentum operator $(l_{\rho\sigma})^{A}{}_{B}$ defined in (C.19), the antisymmetric part of the Minkowski tensor can be written as

$$
2\Theta_{[\mu\nu]} = -\frac{1}{8} F_{\alpha\beta} F_{\rho\sigma} (s_{\mu\nu})^{\alpha\beta\rho\sigma}{}_{\gamma\delta\lambda\kappa} \chi^{\gamma\delta\lambda\kappa}, \tag{4.134}
$$

where $(s_{\mu\nu})^{\alpha\beta\rho\sigma}{}_{\gamma\delta\lambda\kappa}$ is the Lorentz generator of a fourth rank tensor and is explicitly explicitly given in (C.96). Notice also that

$$
\frac{\partial \mathcal{L}^{\text{em}}}{\partial \chi^{\alpha\beta\rho\sigma}} = -F_{\alpha\beta} F_{\rho\sigma}, \tag{4.135}
$$

and therefore we can write

$$
\boxed{2\Theta_{[\mu\nu]} = \frac{1}{8}\frac{\partial \mathcal{L}^{\text{em}}}{\partial \chi^{\alpha\beta\rho\sigma}} (s_{\mu\nu})^{\alpha\beta\rho\sigma}{}_{\gamma\delta\lambda\kappa} \chi^{\gamma\delta\lambda\kappa}.} \tag{4.136}
$$

As a result, the antisymmetric part of the Minkowski tensor must be in general different from zero, unless the constitutive tensor of the medium $\chi^{\mu\nu\rho\sigma}$ is such that all the components of the Lorentz generator applied on it vanish. Therefore, for simultaneously homogeneous and isotropic media, the spatial components Θ_{ij} will be symmetric, but the components T_{0i} can only be symmetric in the trivial case of vacuum. However, it is important to remark that this asymmetry of the Minkowski tensor is completely consistent with the Lagrange-Noether formalism and even more, it is a requirement of the theory so that we can correctly describe the external forces and torques between the field and the charges and currents of this *open* system. In fact, (4.136) corresponds to the angular momentum identity (C.27) for the Belinfante energy-momentum tensor, when $J^{\mu}_{\text{ext}} = 0$. We can obtain the more general identity for $J^{\mu}_{\text{ext}} \neq 0$, if we consider the complete Belinfante tensor $\overset{\text{em}}{\Theta}{}_{\mu}{}^{\nu}$ and replace (4.136) in (4.129). Then,

$$
\begin{aligned}
2\overset{\text{em}}{\Theta}{}_{[\mu\nu]} &= \frac{1}{8}\frac{\partial \mathcal{L}^{\text{em}}}{\partial \chi^{\alpha\beta\rho\sigma}} (s_{\mu\nu})^{\alpha\beta\rho\sigma}{}_{\gamma\delta\lambda\kappa} \chi^{\gamma\delta\lambda\kappa} + (-A_{\rho})(\eta_{\sigma\nu}\delta^{\rho}{}_{\mu} - \eta_{\sigma\nu}\delta^{\rho}{}_{\mu}) J^{\sigma}_{\text{ext}} && (4.137)\\
&= \frac{1}{8}\frac{\partial \mathcal{L}^{\text{em}}}{\partial \chi^{\alpha\beta\rho\sigma}} (s_{\mu\nu})^{\alpha\beta\rho\sigma}{}_{\gamma\delta\lambda\kappa} \chi^{\gamma\delta\lambda\kappa} + \frac{\partial \mathcal{L}^{\text{ext}}}{\partial J^{\rho}_{\text{ext}}} (s_{\mu\nu})^{\rho}{}_{\sigma} J^{\sigma}_{\text{ext}}, && (4.138)
\end{aligned}
$$

where in the right hand side of (4.138) appears the sum of all the external fields with act on the system and coincides exactly with (C.27).

The canonical energy-momentum tensor $\overset{\text{em}}{T}{}_{\mu}{}^{\nu}$ in (4.118) has even worst symmetry properties than the Belinfante and the Minkowski ones. The general Lagrange-Noether analysis in C.3 shows that it is not in general symmetric even when the system is closed. It is only directly symmetric in closed systems with trivial spin current density $S_{\rho\sigma}{}^{\mu} = 0$, i.e. when all dynamical fields are scalars.

4.3.5. Belinfante angular momentum and angular 4-momentum balance equation

To finish this section, we will rederive the angular 4-momentum balance equation of the system, obtained in (4.63), but now using the Lagrange-Noether formalism. Inserting (4.128) in the general definition (C.24), the Belinfante orbital angular 4-momentum of the electromangnetic field in matter, reads

$$\overset{\text{em}}{l}{}_{\rho\sigma}{}^{\mu} = l_{\rho\sigma}{}^{\mu} + (\delta^{\mu}{}_{\sigma} x_{\rho} - \delta^{\mu} x_{\rho} x_{\sigma}) J^{\lambda}_{\text{ext}} A_{\lambda} - (x_{\rho} A_{\sigma} - x_{\sigma} A_{\rho}) J^{\mu}_{\text{ext}}, \tag{4.139}$$

where $l_{\rho\sigma}{}^{\mu}$ is the Minkowski orbital angular 4-momentum, previously defined in (4.62). Notice that if $J^{\mu}_{\text{ext}} = 0$, both the Minkowski orbital angular 4-momentum and the Belinfante one indeed coincides. The balance equation which $\overset{\text{em}}{l}{}_{\rho\sigma}{}^{\mu}$ satisfies is given in (C.26) and in our particular open system, we have

$$\partial_{\mu} \overset{\text{em}}{l}{}_{\rho\sigma}{}^{\mu} \doteq -\frac{1}{8} \frac{\partial \mathcal{L}^{\text{em}}}{\partial \chi^{\alpha\beta\gamma\delta}} (j_{\rho\sigma})^{\alpha\beta\gamma\delta}{}_{\mu\nu\lambda\kappa} \chi^{\mu\nu\lambda\kappa} - \frac{\partial \mathcal{L}^{\text{ext}}}{\partial J^{\mu}_{\text{ext}}} (j_{\rho\sigma})^{\mu}{}_{\nu} J^{\nu}_{\text{ext}}, \tag{4.140}$$

where $(j_{\rho\sigma})^{\alpha\beta\gamma\delta}{}_{\mu\nu\lambda\kappa}$ and $(j_{\rho\sigma})^{\mu}{}_{\nu}$ are the total angular momentum operators, for a fourth rank tensor and a 4-vector, respectively, as defined in (C.18). If we take the 4-divergence on both sides of (4.139) and also consider the continuity equation for the external 4-current $\partial_{\mu} J^{\mu}_{\text{ext}} = 0$, which is a consequence of the equations of motion (4.115), we obtain an on-shell expression for the 4-divergence of the Belinfante angular 4-momentum:

$$\begin{aligned} \partial_{\mu} \overset{\text{em}}{l}{}_{\rho\sigma}{}^{\mu} &\doteq \partial_{\mu} l_{\rho\sigma}{}^{\mu} + \mathcal{T}^{\text{ext}}_{\rho\sigma} + A_{\mu} \left[(x_{\rho} \partial_{\sigma} - x_{\sigma} \partial_{\rho}) \delta^{\mu}{}_{\nu} + (\delta^{\mu}{}_{\rho} \eta_{\sigma\nu} - \delta^{\mu}{}_{\sigma} \eta_{\rho\nu}) \right] J^{\nu}_{\text{ext}} && (4.141) \\ &\doteq \partial_{\mu} l_{\rho\sigma}{}^{\mu} + \mathcal{T}^{\text{ext}}_{\rho\sigma} - \frac{\partial \mathcal{L}^{\text{ext}}}{\partial J^{\mu}_{\text{ext}}} (j_{\rho\sigma})^{\mu}{}_{\nu} J^{\nu}_{\text{ext}}, && (4.142) \end{aligned}$$

where in the last term of (4.141) we recognized the expression for $\partial \mathcal{L}^{\text{ext}} / \partial J^{\mu}_{\text{ext}}$ and $(j_{\rho\sigma})^{\mu}{}_{\nu} = (l_{\rho\sigma})^{\mu}{}_{\nu} + (s_{\rho\sigma})^{\mu}{}_{\nu}$ for the external field J^{μ}_{ext}. The tensor $\mathcal{T}^{\text{ext}}_{\rho\sigma}$ is the external 4-torque density already defined in (4.64).

Notice that if we insert (4.142) in the left hand side of the Belinfante angular 4-momentum balance equation (4.140), the terms with the total angular momentum operator acting on J^{μ}_{ext} cancel out on both sides and the balance equation reduces simply to

$$\partial_{\mu} l_{\rho\sigma}{}^{\mu} + \mathcal{T}^{\text{ext}}_{\rho\sigma} + \frac{1}{8} \frac{\partial \mathcal{L}^{\text{em}}}{\partial \chi^{\alpha\beta\gamma\delta}} (j_{\rho\sigma})^{\alpha\beta\gamma\delta}{}_{\mu\nu\lambda\kappa} \chi^{\mu\nu\lambda\kappa} \doteq 0. \tag{4.143}$$

In order to explicitly evaluate the action of $(j_{\rho\sigma})^{\alpha\beta\gamma\delta}{}_{\mu\nu\lambda\kappa}$ on the constitutive tensor, let us recall the condition for a fourth rank tensor to be invariant under Lorentz transformations of appendix C.5. If we take the left hand side of (C.97) and use the result (4.135), we obtain

$$\begin{aligned}\frac{1}{8}\frac{\partial\mathcal{L}^{\mathrm{em}}}{\partial\chi^{\alpha\beta\gamma\delta}}(j_{\rho\sigma})^{\alpha\beta\gamma\delta}{}_{\mu\nu\lambda\kappa}\chi^{\mu\nu\lambda\kappa} = & -\frac{1}{8}F_{\alpha\beta}F_{\gamma\delta}(x_\rho\partial_\sigma\chi^{\alpha\beta\gamma\delta} - x_\sigma\partial_\rho\chi^{\alpha\beta\gamma\delta} + \eta_{\alpha\rho}\chi^{\sigma\beta\gamma\delta} \\ & - \eta_{\alpha\sigma}\chi^{\rho\beta\gamma\delta} + \eta_{\beta\rho}\chi^{\alpha\sigma\gamma\delta} - \eta_{\beta\sigma}\chi^{\alpha\rho\gamma\delta} + \eta_{\gamma\rho}\chi^{\alpha\beta\sigma\delta} \\ & - \eta_{\gamma\sigma}\chi^{\alpha\beta\rho\delta} + \eta_{\delta\rho}\chi^{\alpha\beta\gamma\sigma} - \eta_{\delta\sigma}\chi^{\alpha\beta\gamma\rho}),\end{aligned} \tag{4.144}$$

which is exactly the same expression for the material effective 4-torque density $\mathcal{T}^{\mathrm{eff}}_{\rho\sigma}$ in (4.86), but in the same way as (4.123), without the dissipative term since in all our Lagrange-Noether analysis we assume $\chi^{\mu\nu\rho\sigma} = \chi^{\rho\sigma\mu\nu}$. With this result we understand clearer why the effective torque exerted by the electromagnetic field on the bound charges and currents of the medium $\mathcal{T}^{\mathrm{eff}}_{\rho\sigma}$ vanishes when the medium is isotropic or invariant under boosts: for non-dissipative media it *indeed is* the total angular operator acting on the constitutive tensor $\chi^{\mu\nu\lambda\kappa}$:

$$\boxed{\mathcal{T}^{\mathrm{eff}}_{\rho\sigma} = \frac{1}{8}\frac{\partial\mathcal{L}^{\mathrm{em}}}{\partial\chi^{\alpha\beta\gamma\delta}}(j_{\rho\sigma})^{\alpha\beta\gamma\delta}{}_{\mu\nu\lambda\kappa}\chi^{\mu\nu\lambda\kappa}.} \tag{4.145}$$

As a result, the angular 4-momentum balance equation (4.143) coincides with (4.63), derived from Maxwell equations, as it must be.

Just to see the beauty of the present theory, we can recall the definition of the 4-momentum operator $(p_\mu)^A{}_B$ in (C.70), in order to write an expression analogous to (4.145), but for $\mathcal{F}^{\mathrm{ext}}_\mu$. Therefore, if we rewrite (4.123) using the definition (C.70) and the derivative (4.135), we have

$$\boxed{\mathcal{F}^{\mathrm{eff}}_{\mu} = \frac{1}{8}\frac{\partial\mathcal{L}^{\mathrm{em}}}{\partial\chi^{\alpha\beta\gamma\delta}}(p_\mu)^{\alpha\beta\gamma\delta}{}_{\mu\nu\lambda\kappa}\chi^{\mu\nu\lambda\kappa},} \tag{4.146}$$

for the material effective 4-force density, in the case of non-dissipative media. Again, there is no surprise why $\mathcal{F}^{\mathrm{eff}}_\mu = 0$ when the constitutive tensor of the medium is homogeneous in space or time independent.

4.4. Summary table of Minkowski conserved quantities

As we have seen, the Minkowski tensor is the Belinfante tensor of the electromagnetic field in the case when there are no external charges and currents $J^{\mu}_{\text{ext}} = 0$ and therefore its conserved quantities are closely related to the symmetries of the non-dynamical medium. In qualitative terms, the conditions under which we can find Minkowski conserved quantities for the electromagnetic field inside matter are summarized in the following table:

Medium symmetries	Minkowski energy	Minkowski momentum	Minkowski angular momentum	Minkowski tensor symmetry
No symmetry				
Time independent	Conserved			
Homogeneous		Conserved		
Isotropic			Conserved	
Time independent, homogeneous & isotropic at rest	Conserved	Conserved	Conserved	Spatial components symmetric
Vacuum	Conserved	Conserved	Conserved	All components symmetric. Constant center of energy velocity: c

Table 4.1.: Minkowski conserved quantities and their relationship to the symmetries of the non-dynamical material medium, when $J^{\mu}_{\text{ext}} = 0$.

Chapter 5.

Dynamics for the medium and the Abraham tensor

Time is too slow for those who wait,
too swift for those who fear,
too long for those who grieve,
too short for those who rejoice,
but for those who love, time is eternity.

Henry Van Dyke,
American writer and cleric.

So far we have only considered the material medium as a given non-dynamical background through which the dynamical electromagnetic field propagates and therefore can be considered as an open system. This assumption is useful when the medium is not very affected by its interaction with light so that it can be approximated to be non-dynamical or when the medium is actually fixed in the laboratory by (not specified) external forces.

In this chapter we will take a step further and we will also consider the dynamics of the medium by modelling it as a relativistic ideal fluid with isotropic electromagnetic properties. Therefore, the medium will evolve in a coupled manner together with the electromagnetic field, both constituting a *closed* system, where the total energy, momentum and angular momentum will be always conserved quantities. For more details about the difference between open and closed systems, see appendix C.

Following the review [10] of Obukhov, we will apply a Lagrangian variational approach, a modernization of the original one of Penfield and Haus in [11, 12], in order to derive an explicit expression for the total energy-momentum tensor of the closed system. We will see how the Abraham tensor emerges naturally in the calculation and identify Minkowski's and Abrahams's material tensor parts to describe the dynamics of this isotropic medium interacting with light.

5.1. Lagrangian variational model for the material medium

Let us model the material medium as a relativistic ideal fluid, the elements of which are structureless particles. In the Eulerian approach this continuous medium is characterized by a *4-velocity field* $u^\mu := (c\gamma, \gamma v^i)$, a *particle number density* ν, an *entropy of each fluid element* s and the *identity (Lin) coordinate* X, whose physical meaning will be explained later. The macroscopic electromagnetic properties of the continuous medium are intrinsically isotropic and therefore we just need two quantities in order to describe them: the permittivity ε and the permeability μ of the medium, both assumed to be given functions of the particle number density ν, i.e.

$$\varepsilon = \varepsilon(\nu), \tag{5.1}$$
$$\mu = \mu(\nu), \tag{5.2}$$

in order to take into account possible electro- and magnetostriction effects. The refraction index of the medium is as usual defined by $n^2(\nu) := \varepsilon\mu$.

In section 4.3 of the previous chapter we postulated the Lagrangian density for the electromagnetic field in matter and applied the full Lagrangian-Noether formalism to this open system. Here we will also use some of that results. Recalling equations (4.109) and (4.113), the Lagrangian density for the electromagnetic field in matter $\mathcal{L}^{\text{em}}$ and the Lagrangian density of the external charges and currents $\mathcal{L}^{\text{ext}}$, are given by

$$\mathcal{L}^{\text{em}}(\partial_\mu A_\nu, \chi^{\mu\nu\rho\sigma}) = -\frac{1}{8}\chi^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{5.3}$$
$$\mathcal{L}^{\text{ext}}(A_\mu, J^\mu_{\text{ext}}) = -J^\lambda_{\text{ext}}A_\lambda. \tag{5.4}$$

Notice that $\mathcal{L}^{\text{em}}$ in (5.3) is valid for any linear material medium with a general $\chi^{\mu\nu\rho\sigma}$, but this time we know that the medium is isotropic and therefore we can use the explicit expression (3.89) derived in section 3.4 of chapter 3, which reads

$$\chi^{\mu\nu\rho\sigma}_{\text{iso}}(u^\mu, \nu) = \frac{1}{\mu_0\mu(\nu)}\left(\gamma^{\mu\rho}\gamma^{\nu\sigma} - \gamma^{\mu\sigma}\gamma^{\nu\rho}\right), \tag{5.5}$$

where the Gordon metric $\gamma^{\mu\nu}(u^\mu, \nu)$ is a given function of ν and u^μ of the form

$$\gamma^{\mu\nu}(u^\mu, \nu) = \eta^{\mu\nu} + \frac{(n^2(\nu) - 1)}{c^2}u^\mu u^\nu. \tag{5.6}$$

Finally, inserting (5.5) and (5.6) in (5.3), we can express the electromagnetic Lagrangian density $\mathcal{L}^{\text{em}}$ for the special case of an isotropic medium, just in terms of the electromagnetic field $F_{\mu\nu}$ and the material variables u^μ and ν, as

$$\boxed{\mathcal{L}^{\text{em}}(\partial_\mu A_\nu, u^\mu, \nu) = -\frac{1}{4\mu_0\mu(\nu)}F^{\mu\nu}F_{\mu\nu} - \frac{(n^2(\nu) - 1)}{2\mu_0\mu c^2}u_\nu u^\sigma F^{\mu\nu}F_{\mu\sigma}.} \tag{5.7}$$

On the other hand, the fundamental equations which govern the dynamics of the medium are the equations of the relativistic hydrodynamics for an ideal fluid, which read

$$\partial_\mu(\nu u^\mu) = 0, \tag{5.8}$$
$$u^\mu\partial_\mu s = 0, \tag{5.9}$$
$$u^\mu\partial_\mu X = 0, \tag{5.10}$$
$$u^\mu u_\mu = c^2. \tag{5.11}$$

The first equation (5.8) is the particle number continuity equation, which is analogous to the non-relativistic mass continuity equation and it describes the local conservation of the number of particles of the fluid. The second equation (5.9) takes into account the entropy conservation along the streamlines of the ideal fluid and therefore only reversible processes are allowed for this *adiabatic* or *isentropic* fluid. The entropy along each streamline is conserved, but the entropy in different streamlines do not have to be necessarily the same. In order to solve this point, Penfield and Haus in [11, 12] explain that they used the Lin coordinate X to identify the particles from different streamlines, along which the identity coordinate has also to be conserved as in (5.10). The last equation (5.11) is the usual normalization that a 4-velocity must satisfy in Special Relativity, since it actually has only 3 independent components, the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$.

In order to find the Lagrangian density $\mathcal{L}^{\mathrm{m}} = \mathcal{L}^{\mathrm{m}}(u^\mu, \nu, s, X)$, which originates the required equations (5.8)-(5.11) as Euler-Lagrange equations of it, we will use a variational approach introducing enough Lagrange multipliers Λ:

$$\boxed{\mathcal{L}^{\mathrm{m}}(u^\mu, \nu, s, X, \Lambda^{(I)}) = -\rho(\nu, s) + \Lambda^0(u^\mu u_\mu - c^2) - \nu u^\mu \partial_\mu \Lambda^1 + \Lambda^2 u^\mu \partial_\mu s + \Lambda^3 u^\mu \partial_\mu X,} \tag{5.12}$$

where $\rho(\nu, s)$ is the thermodynamic *internal energy density* of the ideal fluid of non-electromagnetic nature and $\Lambda^0, \Lambda^1, \Lambda^2, \Lambda^3$ are all the Lagrange multipliers.

Since we already know the Lagrangian densities of all parts of the system, we can write the *total Lagrangian density* $\mathcal{L}$ as the sum of the parts by

$$\mathcal{L} = \mathcal{L}^{\mathrm{m}} + \mathcal{L}^{\mathrm{em}} + \mathcal{L}^{\mathrm{ext}}, \tag{5.13}$$

where the only external field considered is the external 4-current density J^μ_{ext} and, of course, the spacetime metric. The corresponding equations of motion obtained by calculating the variational derivatives (C.2)-(C.3) of the total Lagrangian density $\mathcal{L}$ with respect to each dynamical field: A_μ, s, X, Λ^0, Λ^1, Λ^2,Λ^3, u^μ and ν, are respectively given by

$$\partial_\mu H^{\mu\nu} = J^\nu_{\mathrm{ext}}, \tag{5.14}$$

$$\partial_\mu(\Lambda^2 u^\mu) + \frac{\partial \rho}{\partial s} = 0, \tag{5.15}$$

$$\partial_\mu(\Lambda^3 u^\mu) = 0, \tag{5.16}$$

$$u^\mu u_\mu = c^2, \tag{5.17}$$

$$\partial_\mu(\nu u^\mu) = 0, \tag{5.18}$$

$$u^\mu \partial_\mu s = 0, \tag{5.19}$$

$$u^\mu \partial_\mu X = 0, \tag{5.20}$$

$$\frac{\delta \mathcal{L}^{\mathrm{em}}}{\delta u^\mu} + 2\Lambda^0 u_\mu - \nu \partial_\mu \Lambda^1 + \Lambda^2 \partial_\mu s + \Lambda^3 \partial_\mu X = 0, \tag{5.21}$$

$$\frac{\delta \mathcal{L}^{\mathrm{em}}}{\delta \nu} - \frac{\partial \rho}{\partial \nu} - u^\mu \partial_\mu \Lambda^1 = 0. \tag{5.22}$$

Notice that (5.14) and (5.17)-(5.20) are the field equations required by theory, whereas (5.15), (5.16), (5.21) and (5.22) are 4 equations that allow us to determine the 4 Lagrange multipliers and thus write the total Lagrangian density just in terms of A_μ, u^μ, ν, s and X.

Additionally, if we apply the first and second law of thermodynamics for reversible processes at each element fluid, we have

$$\boxed{Tds = d\left(\frac{\rho}{\nu}\right) + pd\left(\frac{1}{\nu}\right),} \tag{5.23}$$

where T is the *temperature* and p the *pressure* of the medium at each fluid element. From (5.23) we obtain the the following derivatives,

$$\frac{\partial\rho}{\partial s} = \nu T, \tag{5.24}$$
$$\frac{\partial\rho}{\partial\nu} = \frac{\rho+p}{\nu}. \tag{5.25}$$

which can be considered as the *definitions* of temperature T and pressure p for this medium. Then, inserting (5.24) and (5.25) into the equations of motion (5.15) and (5.22), we can express them better as

$$\partial_\mu(\Lambda^2 u^\mu) + \nu T = 0, \tag{5.26}$$
$$\frac{\delta\mathcal{L}^{\text{em}}}{\delta\nu} - \frac{\rho+p}{\nu} - u^\mu\partial_\mu\Lambda^1 = 0. \tag{5.27}$$

In order to find an explicit expression without Lagrange multipliers for the material Lagrangian density $\mathcal{L}^{\text{m}}$ in (5.12), let us multiply (5.21) by u^μ and then replace (5.25) into it, in order to obtain

$$\Lambda^0 = \frac{1}{2c^2}\left[\nu\frac{\delta\mathcal{L}^{\text{em}}}{\delta\nu} - \frac{\delta\mathcal{L}^{\text{em}}}{\delta u^\mu} - \rho - p\right]. \tag{5.28}$$

If we now insert (5.28) back into (5.21), we obtain an important result

$$-\nu\partial_\mu\Lambda^1 + \Lambda^2\partial_\mu s + \Lambda^3\partial_\mu X = \frac{u_\mu}{c^2}\left(\rho + p - \nu\frac{\delta\mathcal{L}^{\text{em}}}{\delta\nu}\right) - \left(\delta^\nu_\mu - \frac{1}{c^2}u_\mu u^\nu\right)\frac{\delta\mathcal{L}^{\text{em}}}{\delta u^\nu}. \tag{5.29}$$

Finally, if we multiply (5.29) by u^μ and then replace it into the original expression (5.12), we can write $\mathcal{L}^{\text{m}}$ "on-shell"[1], by

$$\boxed{\mathcal{L}^{\text{m}}(\partial_\mu A_\nu, u^\mu, \nu, s) \doteq p(\nu, s) - \nu\frac{\delta\mathcal{L}^{\text{em}}}{\delta\nu}(\partial_\mu A_\nu, u^\mu, \nu).} \tag{5.30}$$

5.2. Total energy-momentum tensor and balance equation for the total closed system

Now we are in conditions to derive the explicit expression for the total canonical energy-momentum of the system and the balance equation that it satisfies. Evaluating the definition

[1] See (C.7) to understand what means "on-shell"

(C.6) for the material Lagrangian density (5.12), we obtain

$$\overset{\mathrm{m}}{T}_{\mu}{}^{\nu} := \frac{\partial \mathcal{L}^{\mathrm{m}}}{\partial(\partial_\nu \Lambda^1)}\partial_\mu \Lambda^1 + \frac{\partial \mathcal{L}^{\mathrm{m}}}{\partial(\partial_\nu s)}\partial_\mu s + \frac{\partial \mathcal{L}^{\mathrm{m}}}{\partial(\partial_\nu X^\lambda)}\partial_\mu X^\lambda - \delta^\nu_\mu \mathcal{L}^{\mathrm{m}} \tag{5.31}$$

$$= u^\nu \left(-\nu \partial_\mu \Lambda^1 + \Lambda^2 \partial_\mu s + \Lambda^3 \partial_\mu X\right) - \delta^\nu_\mu p + \nu \delta^\nu_\mu \frac{\delta \mathcal{L}^{\mathrm{em}}}{\delta \nu} \tag{5.32}$$

$$= \frac{\rho}{c^2} u_\mu u^\nu + \left[\frac{1}{c^2} u_\mu u^\mu - \delta^\nu_\mu\right]\left(p - \nu \frac{\delta \mathcal{L}^{\mathrm{em}}}{\delta \nu}\right) + \left(\frac{1}{c^2} u_\mu u^\lambda - \delta^\lambda_\mu\right) u^\nu \frac{\delta \mathcal{L}^{\mathrm{em}}}{\delta u^\lambda}, \tag{5.33}$$

where we have used the relation (5.29). Since we already know the explicit expression for $\mathcal{L}^{\mathrm{em}}$ in (5.7), we can straightforwardly calculate the following derivatives:

$$\frac{\delta \mathcal{L}^{\mathrm{em}}}{\delta u^\lambda} = -\frac{(n^2-1)}{\mu_0 \mu c^2} u_\nu F^{\mu\nu} F_{\mu\lambda}, \tag{5.34}$$

$$\frac{\delta \mathcal{L}^{\mathrm{em}}}{\delta \nu} = -\frac{1}{2}\left[F^{\rho\lambda} F_{\rho\sigma} u^\sigma u_\lambda\, \varepsilon_0 \frac{\partial \varepsilon}{\partial \nu} + \left(\frac{1}{c^2} F^{\rho\lambda} F_{\rho\sigma} u^\sigma u_\lambda - \frac{1}{2} F^{\rho\sigma} F_{\rho\sigma}\right) \frac{1}{\mu_0 \mu^2} \frac{\partial \mu}{\partial \nu}\right], \tag{5.35}$$

and replacing (5.34) and (5.35) in (5.33), we see that the explicit expression for the canonical energy-momentum tensor $\overset{\mathrm{m}}{T}_{\mu}{}^{\nu}$ of the isotropic material medium, reads

$$\boxed{\overset{\mathrm{m}}{T}_{\mu}{}^{\nu} = \frac{\rho}{c^2} u_\mu u^\nu + \left(\frac{1}{c^2} u_\mu u^\nu - \delta^\nu_\mu\right) p^{\mathrm{eff}} + \frac{(n^2-1)}{\mu_0 \mu c^2}\left[F_{\mu\lambda} F^{\rho\lambda} u_\rho u^\nu - \frac{1}{c^2} F_{\rho\lambda} F^{\rho\sigma} u_\sigma u^\lambda u_\mu u^\nu\right],} \tag{5.36}$$

where the *effective pressure* takes into account the possible electro- and magnetostriction effects in the medium and is defined as

$$p^{\mathrm{eff}} := p + \frac{\nu}{2}\left[F_{\rho\sigma} F^{\rho\lambda} u^\sigma u_\lambda\, \varepsilon_0 \frac{\partial \varepsilon}{\partial \nu} + \left(\frac{1}{c^2} F_{\rho\sigma} F^{\rho\lambda} u^\sigma u_\lambda - \frac{1}{2} F_{\rho\sigma} F^{\rho\sigma}\right) \frac{1}{\mu_0 \mu^2} \frac{\partial \mu}{\partial \nu}\right]. \tag{5.37}$$

The energy-momentum balance equation for the total system composed by electromagnetic field and dynamic material medium, eventually interacting with external charges and currents J^μ_{ext}, can be obtained from the general expression (C.5). Noting that J^μ_{ext} is the only external field acting on the system, the "on-shell" total energy-momentum balance equation must be formally given by

$$\partial_\nu \overset{\mathrm{m}}{T}_{\mu}{}^{\nu} + \partial_\nu \overset{\mathrm{em}}{T}_{\mu}{}^{\nu} \doteq -\frac{\partial \mathcal{L}^{\mathrm{ext}}}{\partial J^\nu_{\mathrm{ext}}} \partial_\mu J^\nu_{\mathrm{ext}}, \tag{5.38}$$

$$\doteq A_\nu (\partial_\mu J^\nu_{\mathrm{ext}}), \tag{5.39}$$

where $\overset{\mathrm{em}}{T}_{\mu}{}^{\nu}$ is the electromagnetic energy-momentum tensor already calculated in equation (4.118) from section 4.3. The 4-divergence of $\overset{\mathrm{em}}{T}_{\mu}{}^{\nu}$ is given in (4.121) and if we insert it in (5.39), the non-gauge invariant terms cancel out, and we obtain

$$\partial_\nu \overset{\mathrm{m}}{T}_{\mu}{}^{\nu} + \partial_\nu \Theta_\mu{}^\nu + F_{\mu\nu} J^\nu_{\mathrm{ext}} \doteq 0. \tag{5.40}$$

Notice that the only non-scalar dynamical field of $\mathcal{L}^{\mathrm{m}}$ is the 4-velocity u^μ, but its derivatives do not appear in the Lagrangian. Therefore we conclude that the material spin current density $\overset{\mathrm{m}}{S}_{\rho\sigma}{}^{\mu}$ as defined in (C.12) vanishes for this ideal medium and therefore the canonical energy-momentum tensor of matter $\overset{\mathrm{m}}{T}_{\mu}{}^{\nu}$ coincides with the Belinfante's one $\overset{\mathrm{m}}{\Theta}_{\mu}{}^{\nu}$, as defined in (C.20), i.e.

$$\overset{\mathrm{m}}{\Theta}_{\mu}{}^{\nu} = \overset{\mathrm{m}}{T}_{\mu}{}^{\nu}. \tag{5.41}$$

If we define the total energy-momentum tensor $T_\mu{}^\nu$ of the system formed by electromagnetic field and dynamic material, as the sum of the Belinfante tensors of each part:

$$\boxed{T_\mu{}^\nu := \overset{\mathrm{m}}{\Theta}_{\mu}{}^{\nu} + \Theta_\mu{}^\nu,} \tag{5.42}$$

then from (5.40) we see that the total energy-momentum balance equation will be just given by

$$\boxed{\partial_\nu T_\mu{}^\nu + \mathcal{F}^{\mathrm{ext}}_\mu \doteq 0,} \tag{5.43}$$

where $\mathcal{F}^{\mathrm{ext}}_\mu = F_{\mu\nu} J^\nu_{\mathrm{ext}}$ is the external Lorentz 4-force density. As a direct consequence of (5.43), when there are no external charges and currents $J^\mu_{\mathrm{ext}} = 0$, there can still be transfer of energy and momentum between the subsystems (electromagnetic field and material medium), but the total energy and momentum of the whole system will always be conserved,

$$\partial_\nu T_\mu{}^\nu \doteq 0. \tag{5.44}$$

From now on, we will call the Belinfante energy-momentum tensor of matter $\overset{\mathrm{m}}{\Theta}_{\mu}{}^{\nu}$ as the *Minkowski material tensor* of the system, since $\overset{\mathrm{m}}{\Theta}_{\mu}{}^{\nu}$ is the tensor that we have to add to the Minkowski one of chapter 4 in order to obtain the correct total tensor of the complete system. From the explicit expression (5.36), we can straightforwardly calculate the antisymmetric part of the Minkowski material tensor, which is non-zero

$$\overset{\mathrm{m}}{\Theta}_{[\mu\nu]} = \frac{(n^2-1)}{\mu_0\mu c^2} F_{[\mu|\lambda} F^{\rho\lambda} u_\rho u_{|\nu]} \neq 0. \tag{5.45}$$

According to the Lagrange-Noether formalism of chapter C, the Belinfante total tensor of any closed system is always symmetric, but not their parts. We know that the Minkowski material and electromagnetic tensors are not symmetric, but if the derivation is correct, the sum of both tensors should produce a total *symmetric* tensor. In order to check if this is true, we will compute the explicit expression for the Minkowski tensor $\Theta_\mu{}^\nu$, for which we first need to know $H^{\mu\nu}$ in this isotropic medium. Thus,

$$H^{\mu\nu} = \frac{1}{2}\chi^{\mu\nu\rho\sigma}_{\mathrm{iso}} F_{\rho\sigma} \tag{5.46}$$

$$= \frac{1}{\mu_0\mu}\gamma^{\mu\rho}\gamma^{\nu\sigma} F_{\rho\sigma} \tag{5.47}$$

$$= \frac{1}{\mu_0\mu}\left[F^{\mu\nu} + \frac{(n^2-1)}{c^2}\left(F^{\mu\lambda}u_\lambda u^\nu - F^{\nu\lambda}u_\lambda u^\mu\right)\right]. \tag{5.48}$$

Then, inserting (5.48) in the definition of the Minkowski tensor given in (4.11), its explicit expression reads

$$\boxed{\begin{aligned}\Theta_\mu{}^\nu = \frac{1}{\mu_0\mu}\left(F_{\mu\lambda}F^{\lambda\nu} + \frac{1}{4}\delta^\nu_\mu F_{\rho\sigma}F^{\rho\sigma}\right) \\ +\frac{(n^2-1)}{\mu_0\mu c^2}\left(F_{\mu\sigma}F^{\sigma\lambda}u_\lambda u^\nu + F_{\mu\sigma}F^{\lambda\nu}u^\sigma u_\lambda + \frac{1}{2}\delta^\nu_\mu F_{\sigma\rho}F^{\sigma\lambda}u^\rho u_\lambda\right).\end{aligned}} \tag{5.49}$$

As expected, by directly calculating the antisymmetric part of $\Theta_\mu{}^\nu$, we see that it is the negative of $\overset{\mathrm{m}}{\Theta}_{[\mu\nu]}$:

$$\Theta_{[\mu\nu]} = -\frac{(n^2-1)}{\mu_0\mu c^2}F_{[\mu|\sigma}F^{\lambda\sigma}u_\lambda u_{|\nu]}, \tag{5.50}$$

and therefore the total energy-momentum tensor of the whole system is indeed *symmetric*

$$\boxed{T_{[\mu\nu]} = \overset{\mathrm{m}}{\Theta}_{[\mu\nu]} + \Theta_{[\mu\nu]} = 0.} \tag{5.51}$$

The *total angular 4-momentum tensor* of the total system is defined as the sum of the electromagnetic and material angular 4-momenta and according to (C.24), it reads

$$J_{\rho\sigma}{}^\mu := x_\rho T_\sigma{}^\mu - x_\sigma T_\rho{}^\mu \tag{5.52}$$

$$= \overset{\mathrm{m}}{l}_{\rho\sigma}{}^\mu + l_{\rho\sigma}{}^\mu. \tag{5.53}$$

Additionally, since the total energy-momentum tensor $T_\mu{}^\nu$ of the system is symmetric, taking the 4-divergence of the total energy-momentum balance equation (5.43), we can obtain the corresponding total angular 4-momentum balance equation of the whole system, which reads

$$\boxed{\partial_\mu J_{\rho\sigma}{}^\mu + \mathcal{T}^{\mathrm{ext}}_{\rho\sigma} \doteq 0,} \tag{5.54}$$

where $\mathcal{T}^{\mathrm{ext}}_{\rho\sigma}$ is the external 4-torque density, defined in (4.64). Therefore, in the same way as in (5.43), when $J^\mu_{\mathrm{ext}} = 0$, all the components of $J_{\rho\sigma}{}^\mu$ will be conserved

$$\partial_\mu J_{\rho\sigma}{}^\mu \doteq 0, \tag{5.55}$$

even though there can be still angular momentum transfer between the subsystems. This is a fundamental proof that the asymmetry of the Minkowski tensor is completely necessary for the consistency of the theory, while the total tensor is required to be symmetric.

In order to finish this section, let us add the Minkowski electromagnetic and material parts, given in (5.36) and (5.49), and find the explicit expression for the total energy-momentum of the closed system in the case of an isotropic material medium:

$$\boxed{\begin{aligned}T_\mu{}^\nu = \frac{\rho}{c^2}u_\mu u^\nu + \left(\frac{1}{c^2}u_\mu u^\nu - \delta^\nu_\mu\right)p^{\mathrm{eff}} + \frac{1}{\mu\mu_0}\left(F_{\mu\sigma}F^{\sigma\nu} + \frac{1}{4}\delta^\nu_\mu F^{\sigma\lambda}F_{\sigma\lambda}\right) \\ +\frac{(n^2-1)}{\mu\mu_0 c^2}\left(F_{\mu\sigma}F^{\lambda\nu}u^\sigma u_\lambda + \frac{1}{2}\delta^\nu_\mu F_{\sigma\rho}F^{\sigma\lambda}u^\rho u_\lambda - \frac{1}{c^2}F^{\rho\sigma}F_{\rho\lambda}u_\sigma u^\lambda u_\mu u^\nu\right).\end{aligned}} \tag{5.56}$$

5.3. The Abraham tensor

In last section we saw that the total energy-momentum of the system can be interpreted as the sum of the Minkowski momentum for the electromagnetic field in matter and the canonical energy momentum for the fluid in interaction with the field. Both energy-momentum tensors are non-symmetric, since they describe open subsystems, but their sum is symmetric.

If we inspect the explicit expression of the total energy-momentum tensor $T_\mu{}^\nu$ in (5.56), maybe we can find a more practical way to split it into different subsystems. It would be ideal if we could find a separation of $T_\mu{}^\nu$, for which the electromagnetic energy-momentum tensor would have all the terms containing the electromagnetic field $F_{\mu\nu}$, whereas the material tensor part would have all the material quantities, i.e. u^μ, ν, ρ, p, n, μ and ε. By inspecting (5.56), in fact see that the latter is impossible due to the inherently coupled nature of the combine system. Notice, however, that if we assign the third and fourth terms of (5.56) to some energy-momentum tensor $\Omega_\mu{}^\nu$ for the electromagnetic field in matter, it will indeed contain all the explicit terms with $F_{\mu\nu}$, expect the ones implicit in the effective pressure p^{eff}:

$$\boxed{\begin{aligned}\Omega_\mu{}^\nu :=& \frac{1}{\mu\mu_0}\left(F_{\mu\sigma}F^{\sigma\nu} + \frac{1}{4}\delta^\nu_\mu F^{\sigma\lambda}F_{\sigma\lambda}\right)\\ &+\frac{(n^2-1)}{\mu\mu_0 c^2}\left(F_{\mu\sigma}F^{\lambda\nu}u^\sigma u_\lambda + \frac{1}{2}\delta^\nu_\mu F_{\sigma\rho}F^{\sigma\lambda}u^\rho u_\lambda - \frac{1}{c^2}F^{\rho\sigma}F_{\rho\lambda}u_\sigma u^\lambda u_\mu u^\nu\right).\end{aligned}} \tag{5.57}$$

Actually, the tensor just defined in (5.57) is the famous *Abraham energy-momentum tensor* of the electromagnetic field in matter, first proposed by Abraham [8, 9] in 1909. It has the advantage of being *symmetric* and even more important that its material counterpart $\overset{\text{m}}{\Omega}_\mu{}^\nu$ has the same form as the tensor of an ideal fluid in isolation, but with effective pressure p^{eff}:

$$\boxed{\overset{\text{m}}{\Omega}_\mu{}^\nu := \frac{\rho}{c^2}u_\mu u^\nu + \left(\frac{1}{c^2}u_\mu u^\nu - \delta^\nu_\mu\right)p^{\text{eff}}.} \tag{5.58}$$

In contrast to the Minkowski tensor, which can be directly derived from Maxwell's equations or from the Lagrangian formalism, as the Belinfante energy-momentum tensor of light in matter, the Abraham tensor does not arise from first principles and it is not related to the symmetries of the medium when it is non-dynamical. However, due to the simple interpretation of its material part $\overset{\text{m}}{\Omega}_\mu{}^\nu$, many authors have described the momentum of light in vacuum with Abraham's expression. Thus, the total energy-momentum of the system either in the Abraham or in the Minkowski separation, reads

$$T_\mu{}^\nu = \overset{\text{m}}{\Omega}_\mu{}^\nu + \Omega_\mu{}^\nu = \overset{\text{m}}{\Theta}_\mu{}^\nu + \Theta_\mu{}^\nu. \tag{5.59}$$

There is still an open question to determine whether this appearance of the Abraham tensor within the total tensor of the system is just a coincidence or a deeper and more general feature of the theory. At the moment we have some preliminary results that in anisotropic uniaxial media the Abraham tensor will no longer be useful.

In order to obtain a more familiar explicit expression for the Abraham tensor $\Omega_\mu{}^\nu$, we can evaluate the definition (5.57) in the rest frame of the medium, i.e. in the limiting case when $u^\mu = (c, 0)$. Taking into account the previous consideration and also using the constitutive relations of an isotropic and homogeneous medium at rest, given in (3.78)-(3.79), the Abraham tensor ${\Omega_\mu}^\nu{}_{(0)}$ in matrix form, reads

$$
{\Omega_\mu}^\nu{}_{(0)} = \begin{pmatrix} \frac{1}{2}\left(D^i E_i + B^i H_i\right) & \frac{1}{c}\epsilon^{ijk} E_j H_k \\ -\frac{1}{c}\hat{\epsilon}_{ijk} E^j H^k & \frac{1}{2}\left[E_i D^j + E^j D_i + H_i B^j + H^j B_i - \delta_i^j \left(D^k E_k + B^k H_k\right)\right] \end{pmatrix}, \tag{5.60}
$$

from where we see that in the rest frame of the medium the Abraham energy density $\mathcal{U}_{A(0)}$ and the Abraham energy flux density $\boldsymbol{S}_{A(0)}$ indeed coincide with the corresponding Minkowski expressions in (4.21) and (4.22),

$$
\mathcal{U}_{A(0)} := \frac{1}{2}\left(\boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{B}\cdot\boldsymbol{H}\right) \tag{5.61}
$$

$$
\boldsymbol{S}_{A(0)} := \boldsymbol{E}\times\boldsymbol{H}. \tag{5.62}
$$

Since now the Abraham tensor is symmetric, the Abraham momentum density satisfies

$$
\boxed{\boldsymbol{\pi}_{A(0)} = \frac{1}{c^2}\boldsymbol{S}_{A(0)},} \tag{5.63}
$$

and therefore in has the famous form as given in (1.2):

$$
\boldsymbol{\pi}_{A(0)} = \frac{1}{c^2}\boldsymbol{E}\times\boldsymbol{H}. \tag{5.64}
$$

Chapter 6.

Relativistic analysis of the dielectric "Einstein box" thought experiment

"Vosotros, los que veis, ¿qué habéis hecho de la luz?"

Paul Claudel,
poeta francés.

In chapter 5, we followed Obukhov [10] and derived an explicit expression for the total energy-momentum tensor ${T_\mu}^\nu$ of the closed system composed by electromagnetic field and a dynamical isotropic material medium. We also noticed that we can express ${T_\mu}^\nu$ either in the Minkowski or in the Abraham decomposition and both should give equivalent results, since both add to the same total tensor. In this chapter, we apply those general results to analyse with great detail and in a fully relativistic manner, the particular problem of the "Einstein box" thought experiment, which we already presented as motivation in subsection 2.3.

In the past 10 years, there have been many authors, for instance [76, 86, 87, 88], who have used the Einstein box argument, first proposed by Balazs [14] in 1953, as their strongest argument to uniquely select the Abraham momentum as the momentum of the field in matter for certain cases, specially when there is uniform motion of the center of energy of the total system. Their main arguments are that the Minkowski momentum would predict a motion of the slab in the opposite direction to the incident pulse and that the Abraham momentum is the only one which simultaneously conserves the velocity of the center of energy, the total energy and the total momentum of the system.

In this chapter we make use of the general result (5.56) for the total energy-momentum tensor of the closed system and calculate in detail the relativistic expressions for the Abraham and Minkowski momenta, together with the corresponding balance equations for this isotropic and homogeneous medium. We explicitly show that using the Minkowski momentum with its adequate balance equations, one arrives at the same results as with the Abraham momentum and therefore we demonstrate that the Abraham momentum is not uniquely selected as the only "correct" momentum for light in matter for this particular case. If we take the non-relativistic approximation of the final expressions for the velocity of the slab, the velocity of the light pulse, the Minkowski and the Abraham momenta, we obtain the same solution of Barnett, Loudon, Mansuripur and others, using either tensor alternative. Additionally, we use the non-relativistic expressions to identify some unjustified assumptions tacitly made by previous authors which explain why they only obtained the Abraham momentum for this case. The results presented in this chapter are not discussed in the literature and will be publish in [16].

6.1. Relativistic model and the total energy-momentum tensor

Suppose there is a dielectric slab of mass M with homogeneous and isotropic electromagnetic properties, floating in space. In its rest frame, its index of refraction is n, its length is L and it occupies a finite volume V. The slab is initially at rest, but a light pulse of total energy $\mathcal{E}_0$ and finite volume $V_p \ll V$ strikes the slab from vacuum at normal incidence putting it in motion with a *final* constant velocity $\boldsymbol{v}$. The slab is equipped with anti-reflection coatings so that the pulse can enter the slab at normal incidence without reflection and energy losses. A sketch of the situation is show in figure 6.1. Notice that contrary to the same problem presented in 2.3, here we never speak in terms of “photons” and we assume from the beginning the light is propagating as a classical and compact wave packet.

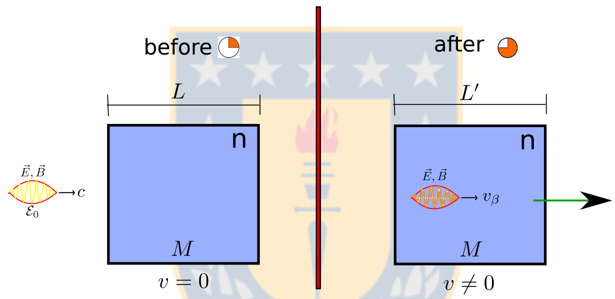

Figure 6.1.: A light pulse enters a dielectric slab initially at rest and puts it in motion.

A fully relativistic model for the total energy-momentum tensor ${T_\mu}^\nu$ for a general linear, non-dissipative, non-dispersive and isotropic dielectric fluid with proper energy density ρ, pressure p, 4-velocity field u^μ, relative permittivity ε, relative permeability μ and particle number density ν, interacting with the electromagnetic field $F_{\mu\nu}$ is given in equation (5.56) from section 5.2. Neglecting gravitational as well as possible electro- and magnetostriction effects, and assuming negligible pressure $p \approx 0$, the total tensor reduces to

$$\begin{aligned}{T_\mu}^\nu = \frac{\rho}{c^2} u_\mu u^\nu + \frac{1}{\mu\mu_0}\left(F_{\mu\sigma}F^{\sigma\nu} + \frac{1}{4}\delta_\mu^\nu F^{\sigma\lambda}F_{\sigma\lambda}\right) \\ + \frac{(n^2-1)}{\mu\mu_0 c^2}\left(F_{\mu\sigma}F^{\lambda\nu}u^\sigma u_\lambda + \frac{1}{2}\delta_\mu^\nu F_{\sigma\rho}F^{\sigma\lambda}u^\rho u_\lambda - \frac{1}{c^2}F^{\rho\sigma}F_{\rho\lambda}u_\sigma u^\lambda u_\mu u^\nu\right). \qquad (6.1)\end{aligned}$$

Due to the balance equation (5.43), when $J^\nu_{\text{ext}} = 0$, the energy-momentum tensor of the complete system is conserved and we have a closed system. If we choose a volume V' big enough so that it encloses the pulse and the slab until the pulse leaves the slab from the other side, then we can integrate the conservation equation and obtain that the total 4-momentum $\mathcal{P}_\mu := (E/c, -\boldsymbol{p})$ of the whole system, defined as

$$\mathcal{P}_\mu := \frac{1}{c}\int_V T_\mu{}^0 \, dV, \tag{6.2}$$

is a conserved, i.e. time independent, quantity. We will use this conservation of energy and momentum of the closed system to study the motion of the slab, when the pulse is propagating inside it (once the slab achieved a final constant velocity after a short deformation transient).

Because of the anti-reflection coatings, the light pulse can pass completely in the same incident direction (without reflection components), so that the problem can be treated just as a one-dimensional problem. Therefore the 4-velocity $u^\mu := (c\gamma, \gamma\boldsymbol{v})$ of the dielectric slab can be chosen to be

$$u^\mu = (c\gamma, \gamma v, 0, 0), \tag{6.3}$$

where $\gamma := (1-\beta^2)^{-1/2}$ and $\beta := v/c$ as usual. Finally, we assume that the energy density distribution in the comoving frame is homogeneous:

$$\rho = \frac{Mc^2}{V} = \text{const.} \tag{6.4}$$

6.2. Energy-momentum tensors of the electromagnetic field

As we showed in the chapter 5, the total energy-momentum tensor (6.1) can be split in different ways. For example, we can assign for the slab the energy-momentum tensor of a fluid without pressure (dust):

$$\overset{\mathrm{m}}{\Omega}_\mu{}^\nu := \frac{\rho}{c^2} u_\mu u^\nu, \tag{6.5}$$

and then light will be described by the Abraham energy-momentum tensor

$$\begin{aligned} \Omega_\mu{}^\nu &:= T_\mu{}^\nu - \overset{\mathrm{m}}{\Omega}_\mu{}^\nu & (6.6)\\ &= \frac{1}{\mu\mu_0}\left(F_{\mu\sigma}F^{\sigma\nu} + \frac{1}{4}\delta_\mu^\nu F^{\sigma\lambda}F_{\sigma\lambda}\right) \\ &+ \frac{(n^2-1)}{\mu\mu_0 c^2}\left(F_{\mu\sigma}F^{\lambda\nu}u^\sigma u_\lambda + \frac{1}{2}\delta_\mu^\nu F_{\sigma\rho}F^{\sigma\lambda}u^\rho u_\lambda - \frac{1}{c^2}F^{\rho\sigma}F_{\rho\lambda}u_\sigma u^\lambda u_\mu u^\nu\right). & (6.7) \end{aligned}$$

With this interpretation the total conserved 4-momentum of the system in (6.2) turns out to be $\mathcal{P}_\mu = \overset{\mathrm{m,A}}{\mathcal{P}}_\mu + \overset{\mathrm{A}}{\mathcal{P}}_\mu$, where

$$\overset{\mathrm{m,A}}{\mathcal{P}}_\mu := \frac{1}{c}\int_{V'} \overset{\mathrm{m}}{\Omega}_\mu{}^0 dV, \qquad \overset{\mathrm{A}}{\mathcal{P}}_\mu := \frac{1}{c}\int_{V'} \Omega_\mu{}^0 dV. \tag{6.8}$$

On the other hand, we can consider that the electromagnetic energy and momentum content is described by the Minkowski energy-momentum tensor ${\Theta_\mu}^\nu$, which is explicitly given in (5.49) by

$$\begin{aligned}{\Theta_\mu}^\nu = &\frac{1}{\mu\mu_0}\left(F_{\mu\sigma}F^{\sigma\nu} + \frac{1}{4}\delta^\nu_\mu F_{\sigma\lambda}F^{\sigma\lambda}\right)\\ &+\frac{(n^2-1)}{\mu\mu_0 c^2}\left(F_{\mu\lambda}F^{\lambda\rho}u_\rho u^\nu + F_{\mu\sigma}F^{\lambda\nu}u^\sigma u_\lambda + \frac{1}{2}\delta^\nu_\mu F_{\sigma\rho}F^{\sigma\lambda}u^\rho u_\lambda\right).\end{aligned} \tag{6.9}$$

This tensor can be obtained by adding the term

$${Q_\mu}^\nu := \frac{(n^2-1)}{\mu\mu_0 c^2}\left[F_{\mu\lambda}F^{\lambda\rho}u_\rho u^\nu + \frac{1}{c^2}F^{\rho\sigma}F_{\rho\lambda}u_\sigma u^\lambda u_\mu u^\nu\right] \tag{6.10}$$

to the Abraham tensor, so that

$${\Theta_\mu}^\nu = {\Omega_\mu}^\nu + {Q_\mu}^\nu. \tag{6.11}$$

Consequently, the total energy-momentum tensor can be written as ${T_\mu}^\nu = {\overset{\mathrm{m}}{\Theta}_\mu}{}^\nu + {\Theta_\mu}^\nu$, where

$${\overset{\mathrm{m}}{\Theta}_\mu}{}^\nu := {\overset{\mathrm{m}}{\Omega}_\mu}{}^\nu - {Q_\mu}^\nu \tag{6.12}$$

$$= \frac{\rho}{c^2}u_\mu u^\nu - \frac{(n^2-1)}{\mu\mu_0 c^2}\left[F_{\mu\lambda}F^{\lambda\rho}u_\rho u^\nu + \frac{1}{c^2}F^{\rho\sigma}F_{\rho\lambda}u_\sigma u^\lambda u_\mu u^\nu\right], \tag{6.13}$$

is the Minkowski energy-momentum tensor for matter. Finally, with this interpretation, the total conserved 4-momentum $\mathcal{P}_\mu$ can be also expressed as $\mathcal{P}_\mu = \overset{\mathrm{m,M}}{\mathcal{P}}_\mu + \overset{\mathrm{M}}{\mathcal{P}}_\mu$, where

$$\overset{\mathrm{m,M}}{\mathcal{P}}_\mu := \frac{1}{c}\int_{V'} {\overset{\mathrm{m}}{\Theta}_\mu}{}^0 dV, \qquad \overset{\mathrm{M}}{\mathcal{P}}_\mu := \frac{1}{c}\int_{V'} {\Theta_\mu}^0 dV, \tag{6.14}$$

are the Minkowski 4-momenta for matter and electromagnetic field, respectively.

6.3. Explicit calculation with the Abraham tensor

6.3.1. Abraham energy and momentum for the slab

We derive first the Abraham tensor, which is more compact in this case. If we substitute (6.3) into (6.8a) and use the identification $\overset{\mathrm{m,A}}{\mathcal{P}}_\mu = (\overset{\mathrm{m,A}}{E}/c, -\overset{\mathrm{m,A}}{\boldsymbol{p}})$, then the Abraham energy and momentum for the slab read

$$\overset{\mathrm{m,A}}{E} = \int_{V'} \rho\,\frac{u_0 u^0}{c^2} dV, \tag{6.15}$$

$$\overset{\mathrm{m,A}}{p}_i = -\frac{1}{c}\int_{V'} \rho\,\frac{u_i u^0}{c^2} dV. \tag{6.16}$$

Therefore, by explicitly calculating the integrals, we obtain

$$\overset{\text{m,A}}{E} = \rho\gamma^2 V_v, \qquad \overset{\text{m,A}}{p}_i = -\frac{1}{c^2} v_i \rho\gamma^2 V_v, \tag{6.17}$$

where V_v is the volume of the slab in the reference frame where it moves with velocity $\boldsymbol{v} = v\,\hat{\boldsymbol{x}}$. Using the relation $V_v = V/\gamma$ and expression (6.4), we have

$$\overset{\text{m,A}}{E} = \gamma M c^2, \qquad \overset{\text{m,A}}{\boldsymbol{p}} = \gamma M v\,\hat{\boldsymbol{x}}. \tag{6.18}$$

These results (6.18) are the usual expressions for the (relativistic) energy and momentum of a body of mass M moving with velocity $v\hat{\boldsymbol{x}}$, which is not surprising because of our choice (6.5) for the energy and momentum of the medium. The relation between momentum and energy is also the usual one for a relativistic massive particle:

$$\boxed{\overset{\text{m,A}}{\boldsymbol{p}} = v\,\frac{\overset{\text{m,A}}{E}}{c^2}\hat{\boldsymbol{x}}.} \tag{6.19}$$

6.3.2. Abraham energy and momentum for the light pulse

Using (6.3), (6.7), (6.8b) and the identifications of the components of $F_{\mu\nu}$ in (3.44), we can explicitly compute the energy and momentum associated to the Abraham tensor for the electromagnetic field in terms of $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{v}$:

$$\begin{aligned}\overset{\text{A}}{E} = \frac{1}{2\mu\mu_0}\int_{V'}(\boldsymbol{E}^2/c^2+\boldsymbol{B}^2)dV - \frac{(n^2-1)}{2\mu\mu_0 c^2}\int_{V'}\big\{\gamma^2(2\gamma^2-1)\big[(\boldsymbol{E}\cdot\boldsymbol{v})^2/c^2+(\boldsymbol{B}\cdot\boldsymbol{v})^2-\boldsymbol{E}^2 \\ -\,\boldsymbol{v}^2\boldsymbol{B}^2-2\boldsymbol{E}\cdot(\boldsymbol{v}\times\boldsymbol{B})\big]-2\gamma^2(\boldsymbol{E}\cdot\boldsymbol{v})^2/c^2\,\big\}\,dV,\end{aligned} \tag{6.20}$$

$$\begin{aligned}\overset{\text{A}}{\boldsymbol{p}} = \frac{1}{\mu\mu_0 c^2}\int_{V'}\boldsymbol{E}\times\boldsymbol{B}\;dV - \frac{(n^2-1)}{\mu\mu_0 c^4}\int_{V'}\big\{\gamma^2(\boldsymbol{E}\cdot\boldsymbol{v})(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) \\ +\gamma^4\boldsymbol{v}\big[(\boldsymbol{E}\cdot\boldsymbol{v})^2/c^2+(\boldsymbol{B}\cdot\boldsymbol{v})^2-\boldsymbol{E}^2-\boldsymbol{v}^2\boldsymbol{B}^2-2\boldsymbol{E}\cdot(\boldsymbol{v}\times\boldsymbol{B})\big]\big\}\;dV.\end{aligned} \tag{6.21}$$

From (6.20) and (6.21) we see that to zeroth order in $\boldsymbol{v}$ they reduce to the well known expressions for a linear, isotropic and homogeneous medium at rest,

$$\overset{\text{A}}{E}_{(0)} = \frac{1}{2}\int_{V'}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H})\;dV, \qquad \overset{\text{A}}{\boldsymbol{p}}_{(0)} = \frac{1}{c^2}\int_{V'}\boldsymbol{E}\times\boldsymbol{H}\;dV, \tag{6.22}$$

where we used the constitutive relations in the medium at rest $\boldsymbol{D} = \varepsilon\varepsilon_0\boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}/\mu\mu_0$.

We consider the special simple case in which the electromagnetic pulse can be approximated by a "cut" plane wave of finite volume $V_p \ll V$, i.e. much less than the volume of the slab, but big enough so that the continuum approximation for the slab is valid, propagating in the direction $\hat{\boldsymbol{x}}$ of the slab's motion. Solving the macroscopic Maxwell equations inside the medium for light with any polarization (see A.2.1 and A.2.2 for more details), we can write,

$$\boldsymbol{B}(x,t) = \frac{1}{v_\beta}\hat{\boldsymbol{x}}\times\boldsymbol{E}, \tag{6.23}$$

where v_β is the phase velocity of light inside the moving medium, given by

$$v_\beta := c\frac{(1+n\beta)}{(n+\beta)}. \tag{6.24}$$

Since in our one-dimensional case $\boldsymbol{v} \perp \boldsymbol{E}$ and $\boldsymbol{v} \perp \boldsymbol{B}$, the expressions (6.20) and (6.21) reduce to:

$$\overset{\mathrm{A}}{E} = \frac{1}{2\mu\mu_0 c^2}\int_{V'}(\boldsymbol{E}^2 + c^2\boldsymbol{B}^2)dV + \frac{(n^2-1)}{2\mu\mu_0 c^2}\int_{V'}\gamma^2(2\gamma^2-1)|\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}|^2\, dV, \tag{6.25}$$

$$\overset{\mathrm{A}}{\boldsymbol{p}} = \frac{1}{\mu\mu_0 c^2}\int_{V'}\boldsymbol{E}\times\boldsymbol{B}\, dV + \frac{(n^2-1)}{\mu\mu_0 c^4}\int_{V'}\gamma^4\boldsymbol{v}|\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}|^2 dV. \tag{6.26}$$

If we insert (6.23) and (6.24) into (6.25) and (6.26), we obtain more compact expressions for $\overset{\mathrm{A}}{E}$ and $\overset{\mathrm{A}}{\boldsymbol{p}}$, just in terms of $\boldsymbol{E}^2$:

$$\overset{\mathrm{A}}{E} = \frac{1}{\mu\mu_0 c^2}\int_{V'}\frac{n(n+2\beta+n\beta^2)}{(1+n\beta)^2}\boldsymbol{E}^2\, dV, \tag{6.27}$$

$$\overset{\mathrm{A}}{\boldsymbol{p}} = \frac{1}{\mu\mu_0 c^3}\hat{\boldsymbol{x}}\int_{V'}\frac{n(1+2n\beta+\beta^2)}{(1+n\beta)^2}\boldsymbol{E}^2\, dV. \tag{6.28}$$

When the pulse is fully inside the slab, we can integrate over the volume V_p of the pulse and the factors with n and β will go out the integral. Therefore, we can relate the Abraham momentum and Abraham energy of the pulse inside the medium by

$$\boxed{\overset{\mathrm{A}}{\boldsymbol{p}} = \frac{(1+2n\beta+\beta^2)}{(n+2\beta+n\beta^2)}\frac{\overset{\mathrm{A}}{E}}{c}\hat{\boldsymbol{x}},} \tag{6.29}$$

which is an important result that we will use in the next subsection. It is worthwhile to notice that (6.29) is valid for any polarization of the "cut" plane-wave pulse, but not for a general pulse form since, if we compare (6.25) with (6.26), we see that $\overset{\mathrm{A}}{\boldsymbol{p}}$ and $\overset{\mathrm{A}}{E}$ are not proportional in general. As a consistency test, it can be checked that the same result (6.29) can be obtained if we apply a boost to the well-known Abraham expression (1.4b) valid in the rest frame of the medium.

6.3.3. Conservation of the center of energy velocity

In [86], Barnett revitalized the argument of Balazs [14] which states that the conservation of the center of energy velocity, in addition to the conservation of momentum, uniquely selects the momentum of light inside the slab to be the one of Abraham. We will now examine these arguments in more detail. We can check from (6.29) that when the medium is at rest, we get the typical value of the Abraham momentum, see (1.4b), $\overset{\mathrm{A}}{\boldsymbol{p}} = (\overset{\mathrm{A}}{E}/nc)\hat{\boldsymbol{x}}$, which we can write in terms of the phase velocity of light in that reference frame $v_0 := c/n$ as

$$\overset{\mathrm{A}}{\boldsymbol{p}} = v_0\frac{\overset{\mathrm{A}}{E}}{c^2}\hat{\boldsymbol{x}}, \tag{6.30}$$

i.e. as if it was a particle of "moving mass" $\overset{\mathrm{A}}{M} := \overset{\mathrm{A}}{E}/c^2$ and velocity $\boldsymbol{v}_0 = (c/n)\hat{\boldsymbol{x}}$, in a way similar to (6.19). In [86, 87] the explicit definition of the conserved center of energy velocity of the system $\boldsymbol{v}_{\mathrm{CM}}$ is not shown, however, if we apply the result (4.107) to a closed system, $\boldsymbol{v}_{\mathrm{CM}}$ will be given as

$$\boldsymbol{v}_{\mathrm{CE}} = \frac{c^2}{E}\boldsymbol{p}, \tag{6.31}$$

where E is the total energy and $\boldsymbol{p}$ is the total momentum of the system.

Now, in order to reproduce Barnett's argument in detail, we add the momentum of light in the form (6.30) to the momentum of the slab in (6.19) and use the total energy and momentum conservation to get

$$v'_{\mathrm{CE}} = \frac{\overset{\mathrm{A}}{E}\,(c/n) + \overset{\mathrm{m,A}}{E}\,v}{\overset{\mathrm{A}}{E} + \overset{\mathrm{m,A}}{E}}, \tag{6.32}$$

which is the same expression obtained in (2.13).

Strictly speaking this argument is *incorrect*, because v'_{CE} is not a conserved quantity. The expression (6.30) for the electromagnetic field is only valid when the slab is at rest, but the final velocity of the medium is not zero and the choice of the velocity $v_0 = c/n$ for the light pulse is inappropriate. If we want to write the expression of the momentum of the pulse as if it were a particle with energy $\overset{\mathrm{A}}{E}$, then as can be seen from (6.29), the proper "particle" velocity v_{p} should be defined as

$$\boxed{v_{\mathrm{p}} := c\frac{(1 + 2n\beta + \beta^2)}{(n + 2\beta + n\beta^2)}}, \tag{6.33}$$

and hence the correct total momentum of the system would have the form $\boldsymbol{p} = (\overset{\mathrm{A}}{E}/c^2)v_{\mathrm{p}}\hat{\boldsymbol{x}} + (\overset{\mathrm{m,A}}{E}/c^2)v\hat{\boldsymbol{x}}$. Together with the total energy of the system $E = \overset{\mathrm{m,A}}{E} + \overset{\mathrm{A}}{E}$, which is also conserved, the velocity of the center of energy $\boldsymbol{v}_{CE}$ given by

$$\boldsymbol{v}_{CE} = \frac{\overset{\mathrm{A}}{E}\,v_{\mathrm{p}} + \overset{\mathrm{m,A}}{E}\,v}{\overset{\mathrm{A}}{E} + \overset{\mathrm{m,A}}{E}}\,\hat{\boldsymbol{x}}, \tag{6.34}$$

turns out to be a conserved quantity indeed. As we said in subsection 5.2, all the components of the total angular 4-momentum tensor $J_{\rho\sigma}$ defined in (5.52) are conserved in a closed system, including the components $0i$ associated to the conservation of the velocity of the center of energy. Additionally, this conservation of (6.34) holds independently of the choice of the Abraham or the Minkowski momentum to describe the electromagnetic field, because it depends on the total quantities $\boldsymbol{p}$ and E.

Although formally incorrect, in practice the naive expression $v_0 = c/n$ yields a very good approximation for the particle velocity of the pulse. As we will see in section 6.3.4, $\beta \sim 10^{-15}$

in a standard case and $\beta \sim 10^{-9}$ in an extreme case, so if we expand (6.33),

$$\boxed{v_{\mathrm{p}} = \frac{c}{n} + 2\frac{c(n^2-1)}{n^2}\beta + O(\beta^2).} \tag{6.35}$$

we see that $v_{\mathrm{p}} \approx v_0 = c/n$ is an extremely accurate approximation indeed. It is also remarkable, that the correct velocity that should enter (6.34) is also different from the relativistic phase velocity v_β in (6.24), as one could naively expect as a generalization of v_0. The non-relativistic expansion of v_β reads,

$$v_\beta = \frac{c}{n} + \frac{c(n^2-1)}{n^2}\beta + O(\beta^2), \tag{6.36}$$

differing from (6.35) by a factor 2 in the term of first order in β. This term in (6.36) has been measured with great accuracy in the Fizeau experiment. See, for instance [19], page 187.

6.3.4. Conservation equations and solution of the slab motion

Since we already know the explicit forms of the energy and momentum of the complete system in the Abraham separation, we can use the two conservation equations to solve the problem of the motion of the slab in terms of the system's parameters. We will consider two states of the system, first when the pulse is travelling with total energy $\mathcal{E}_0$ in vacuum and the slab of mass M and refraction index n is at rest, and finally when the electromagnetic pulse is completely inside the slab after it already reached a final constant velocity $\boldsymbol{v} = c\beta\hat{\boldsymbol{x}}$. Therefore, the energy conservation equation reads

$$\overset{\mathrm{m,A}}{E}{}^{(\mathrm{out})} + \overset{\mathrm{A}}{E}{}^{(\mathrm{out})} = \overset{\mathrm{m,A}}{E}{}^{(\mathrm{in})} + \overset{\mathrm{A}}{E}{}^{(\mathrm{in})}, \tag{6.37}$$

$$Mc^2 + \mathcal{E}_0 = \gamma Mc^2 + \overset{\mathrm{A}}{E}, \tag{6.38}$$

and the total momentum conservation in the propagation direction $\hat{\boldsymbol{x}}$ is given by

$$\overset{\mathrm{m,A}}{\boldsymbol{p}}{}^{(\mathrm{out})} + \overset{\mathrm{A}}{\boldsymbol{p}}{}^{(\mathrm{out})} = \overset{\mathrm{m,A}}{\boldsymbol{p}}{}^{(\mathrm{in})} + \overset{\mathrm{A}}{\boldsymbol{p}}{}^{(\mathrm{in})}, \tag{6.39}$$

$$0 \ + \frac{\mathcal{E}_0}{c} = \gamma Mc\beta + \frac{(1+2n\beta+\beta^2)}{(n+2\beta+n\beta^2)}\frac{\overset{\mathrm{A}}{E}}{c}. \tag{6.40}$$

Equations (6.38) and (6.40) constitute a system of two equations for two unknowns β and $\overset{\mathrm{A}}{E}$, which we can solve in terms of the system parameters $\mathcal{E}_0$, M and n. From (6.38) we can find an expression for $\overset{\mathrm{A}}{E}$ in terms of β, which reads

$$\overset{\mathrm{A}}{E} = \mathcal{E}_0 + Mc^2(1-\gamma). \tag{6.41}$$

This last equation already determines that the motion of the slab will be non-relativistic in most practical situations. Let us consider the extreme case when all the energy of the light pulse in vacuum is transformed in kinetic energy of the slab. Then in (6.41), $\overset{\text{A}}{E} = 0$, and we can determine γ_{max} as,

$$\gamma_{\text{max}} = 1 + q, \tag{6.42}$$

where we have defined the dimensionless parameter q by

$$q := \frac{\mathcal{E}_0}{Mc^2}. \tag{6.43}$$

The parameter q (together with n) determines the motion of the slab. In practice q is extremely small, as we shall see at the end of the section, of the order $q \sim 10^{-9}$ or less, so from (6.42) we see that γ_{max} will be very close to unity and therefore β_{max} will be at most of the order $\beta_{\text{max}} \sim \sqrt{2q} \sim 10^{-4} \ll 1$, resulting in a non-relativistic motion of the slab. Even though we know that the motion will be non-relativistic, we will first present the full equation for β, without any further approximation. Inserting (6.41) in (6.40), we get a fourth order polynomial equation for β in terms of the parameters q and n:

$$\begin{aligned}
&[(1+q-nq)^2 + n^2]\beta^4 + [4(1+q-nq)(n-q+nq) + 2n]\beta^3 \\
&+ [2(1+q-nq)^2 + 4(n-q+nq)^2 + 1 - n^2]\beta^2 \\
&+ [4(1+q-nq)(n-q+nq) - 2n]\beta + [(1+q-nq)^2 - 1] = 0.
\end{aligned} \tag{6.44}$$

This equation can be solved analytically, but the expression of the solution is large and not instructive. Since we already know that in practice $q \ll 1$ and $\beta \ll 1$, it is interesting to search for a more tractable approximated solution for β. Therefore, if we keep only the first order terms in (6.44), we get the well known solution of Balazs, Barnett, Mansuripur and many others, for the non-relativistic velocity of the dielectric slab

$$\beta \approx \frac{1}{n}(n-1)q, \tag{6.45}$$

or, in more familiar terms,

$$\boxed{v \approx \frac{(n-1)}{n}\frac{\mathcal{E}_0}{Mc} > 0,} \tag{6.46}$$

which is the same expression obtained in (2.17) and it means that the slab will move in the same direction of the electromagnetic pulse, while the pulse is propagating inside it. If we continue in the non-relativistic limit, the light pulse will spend a time interval $\Delta t \approx nL/c$ inside the slab and therefore its net displacement Δx will be as usual given by $\Delta x \approx (n-1)L\mathcal{E}_0/Mc^2 > 0$. Additionally, one can also find a solution of (6.44) which is exact to second order in q:

$$\boxed{\beta(n,q) = \frac{(n-1)}{n}q - \frac{(4n^3 - 5n^2 - 2n + 3)}{2n^3}q^2 + O(q^3),} \tag{6.47}$$

where the second term can be considered as the first "relativistic correction" for β. From (6.47) we can estimate the error that we make by using just the first order non-relativistic approximation (6.45). Suppose that the slab is made of glass with $n = 1.5$. If it has a mass of the order $M \sim 100g$ and if the light source is a good pulsed laser with energy $\mathcal{E}_0 \sim 1J$, then the parameter q will be typically of the order $q \sim 10^{-15}$ and therefore from (6.46) we see that $v \sim 0.1\ \mu m/s$. In this case, the difference between the second order solution of (6.47) and the non-relativistic solution (6.45) is of the order $\Delta\beta \sim 10^{-31}$ and the relative error of using (6.45) is $\sim 100q \sim 10^{-13}\%$. In the extreme case of the most powerful lasers available $\mathcal{E}_0 \sim 1kJ$ and a very small dielectric slab with mass $M \sim 10g$, we would in theory be able to achieve a q parameter of the order $q \sim 10^{-9}$ and a final slab velocity of order $v \sim 10\ cm/s$. In this case, $\Delta\beta \sim 10^{-19}$ and the relative error is $\sim 100q \sim 10^{-7}\%$ and hence we see that the well-known non-relativistic solution (6.46) is extremely accurate for all practical purposes.

6.4. Using the Minkowski tensor

6.4.1. Minkowski energy and momentum expressions

In [14, 19, 21, 62, 67, 75, 86] and other papers, it is argued that the Minkowski momentum fails to describe this slab experiment by predicting the motion of the slab in the opposite direction of the incident electromagnetic pulse. However, the mistake is to consider the same balance equations that are valid for the Abraham momentum, while using the Minkowski momentum for the electromagnetic pulse, thereby tacitly assigning an incorrect energy and momentum to matter in the Minkowski picture. Jones [92] noticed this deficiency and suggested that the correct momentum of matter should include the "forward bodily impulse" the nature of which he was unable to describe. The explanation is simple, though: one needs to use the *canonical momentum* of matter which, combined with the canonical momentum of the electromagnetic field, is conserved in the Minkowski picture.

As we mentioned in section 6.2, we can formally compute the correct Minkowski quantities from the Abraham expressions for the energy and momentum by adding the proper components of $Q_\mu{}^\nu$. Evaluating the expression for $Q_\mu{}^0$ in (6.10), we get

$$Q_\mu{}^0 = \frac{(n^2-1)}{\mu\mu_0 c^2}\frac{n}{(1+n\beta)^2}\boldsymbol{E}^2\,(\beta,-1,0,0)\,. \tag{6.48}$$

Then, the Minkowski energy of the pulse is of the form

$$\overset{\mathrm{M}}{E} = \overset{\mathrm{A}}{E} + \int_{V'} Q_0{}^0 dV, \tag{6.49}$$

$$= \frac{n}{\mu\mu_0 c^2}\frac{c}{v_\beta}\int_{V_p} \boldsymbol{E}^2\ dV. \tag{6.50}$$

Using (6.25) integrated over V_p, we can relate $\overset{\mathrm{M}}{E}$ to the Abraham energy of the light pulse by

$$\overset{\mathrm{A}}{E} = \frac{(n+2\beta+n\beta^2)}{(1+n\beta)(n+\beta)}\overset{\mathrm{M}}{E}. \tag{6.51}$$

These energies coincide, for a given $n \neq 1$, only in the case $\beta = 0$, i.e. in the rest frame of the medium. In the same way, starting from the definitions (6.11) and (6.12), and using the expressions (6.48) and (6.50), we can determine all the other Minkowski quantities in terms of $\overset{\mathrm{M}}{E}$:

$$\overset{\mathrm{M}}{\boldsymbol{p}} = \frac{\overset{\mathrm{M}}{E}}{v_\beta}\,\hat{\boldsymbol{x}}, \tag{6.52}$$

$$\overset{\mathrm{m,M}}{E} = \gamma M c^2 - \frac{(n^2-1)\beta}{(1+n\beta)(n+\beta)}\overset{\mathrm{M}}{E}, \tag{6.53}$$

$$\overset{\mathrm{m,M}}{\boldsymbol{p}} = \left[\gamma M c\beta - \frac{(n^2-1)}{(1+n\beta)(n+\beta)}\frac{\overset{\mathrm{M}}{E}}{c}\right]\hat{\boldsymbol{x}}. \tag{6.54}$$

The last term in the *canonical momentum* of the matter (6.54) accounts for the "forward bodily impulse" of Jones [92].

6.4.2. Defining a Minkowski velocity

With the results (6.50), (6.52), (6.53) and (6.54) we can express the center of energy velocity in the Minkowski picture as

$$v_{CE} = \frac{c^2}{E}\left[\gamma M c\beta - \frac{(n^2-1)}{(1+n\beta)(n+\beta)}\frac{\overset{\mathrm{M}}{E}}{c} + \frac{\overset{\mathrm{M}}{E}}{v_\beta}\right] \tag{6.55}$$

$$= \frac{c^2}{E}\left[v\,\frac{\gamma M c^2}{c^2} + \frac{c(1+2n\beta+\beta^2)}{(1+n\beta)(n+\beta)}\frac{\overset{\mathrm{M}}{E}}{c^2}\right] \tag{6.56}$$

$$= \frac{v\,(\gamma M c^2) + v_{\mathrm{M}}\,\overset{\mathrm{M}}{E}}{E}, \tag{6.57}$$

where we defined the Minkowski velocity v_{M} of the field as

$$v_{\mathrm{M}} := c\,\frac{(1+2n\beta+\beta^2)}{(1+n\beta)(n+\beta)}. \tag{6.58}$$

To get the velocity (6.58), we shifted terms from $\overset{\mathrm{m,M}}{p}$ to $\overset{\mathrm{M}}{p}$ and therefore it does not satisfy the prescription of a "particle velocity",

$$\boldsymbol{v}_{\mathrm{em}} = c^2\frac{\boldsymbol{p}_{\mathrm{em}}}{E_{\mathrm{em}}}, \qquad \boldsymbol{v}_{\mathrm{mat}} = c^2\frac{\boldsymbol{p}_{\mathrm{mat}}}{E_{\mathrm{mat}}}, \tag{6.59}$$

as for the case of Abraham light and matter velocities (6.19) and (6.33). In fact, v_M in (6.58) is a very artificial velocity that one would have to associate to the field with the Minkowski energy, in order to keep the velocity v for the block, and still obtain the correct (relativistic) center of energy velocity (6.34). If we expand (6.58) in powers of β we see that v_{M} is also similar to all the other previously defined velocities,

$$v_{\mathrm{p,M}} = \frac{c}{n} + \frac{c(n^2-1)}{n^2}\beta + O(\beta^2), \tag{6.60}$$

and hence equals c/n in the non-relativistic limit to zeroth order in β, as well as the Abraham particle velocity. We could also define a Minkowski "particle" velocity following the prescription (6.59). Therefore, using (6.52), (6.53) and (6.54), we have

$$v_{\mathrm{p,M}} := \frac{c^2 \overset{\mathrm{M}}{p}}{\overset{\mathrm{M}}{E}} = \frac{c^2}{v_\beta} = c\frac{(n+\beta)}{(1+n\beta)}, \tag{6.61}$$

$$v_{\mathrm{mat,M}} := \frac{c^2 \overset{\mathrm{m,M}}{p}}{\overset{\mathrm{m,M}}{E}} = c\frac{\left[\gamma M c^2(1+n\beta)(n+\beta)\beta - (n^2-1)\overset{\mathrm{M}}{E}\right]}{\left[\gamma M c^2(1+n\beta)(n+\beta) - (n^2-1)\beta\overset{\mathrm{M}}{E}\right]}. \tag{6.62}$$

With (6.61) and (6.62) the expression for the velocity of "center of energy" naturally assumes the form $v_{CE} = \left(\sum_i v_i \cdot E_i\right) / \left(\sum_i E_i\right)$, because it depends only on the total quantities, *but neither (6.61) nor (6.62) coincide with a velocity of the system which one can easily identify and interpret* (like the velocity of the block v or the phase velocity of the field v_β, for example). Indeed, $v_{\mathrm{mat,M}}$ in (6.62) depends on the energy of the pulse, which is very counter-intuitive. When there is no light pulse, i.e. $\overset{\mathrm{M}}{E} = 0$ and $v_{\mathrm{mat,M}}$ reduces to the velocity of the block v.

6.4.3. Minkowski balance equations

Since we already know all the explicit expressions for the Minkowski momentum and energy of the field and slab, we can use them to write the balance equations and correctly solve for the slab velocity also with the Minkowski formulation. From (6.52), (6.53) and (6.54), we have

$$Mc^2 + \mathcal{E}_0 = \left[\gamma M c^2 - \frac{(n^2-1)\beta}{(1+n\beta)(n+\beta)}\overset{\mathrm{M}}{E}\right] + \overset{\mathrm{M}}{E}, \tag{6.63}$$

$$\frac{\mathcal{E}_0}{c} = \left[\gamma M c\beta - \frac{(n^2-1)}{(1+n\beta)(n+\beta)}\frac{\overset{\mathrm{M}}{E}}{c}\right] + \frac{(n+\beta)}{(1+n\beta)}\frac{\overset{\mathrm{M}}{E}}{c}. \tag{6.64}$$

Taking (6.63) and dividing it by Mc^2, we get

$$q_M = \frac{(1+n\beta)(n+\beta)}{(n+2\beta+n\beta^2)}(q+1-\gamma), \tag{6.65}$$

where $q_M := \overset{\mathrm{M}}{E}/Mc^2$, following the definition (6.43). Then, if we divide (6.64) by Mc and use (6.65), we obtain after some algebra the same fourth order equation for β in (6.44), which we obtained with the Abraham formulation. Therefore in the the Minkowski picture, we obtain the same solution $\beta = \beta(n, q)$ for the motion of the slab. The authors who claim that the Minkowski momentum is unable to describe the slab plus light pulse system use the equations (6.63) and (6.64), but without the second terms inside the bracket on the r.h.s. and hence they use incorrect balance equations.

6.5. Discussion

When we use the total energy-momentum tensor, the conservation of momentum and of the velocity of center of energy is always satisfied for any specific separation, because it involves only the total quantities. However, when we use the "Abraham separation", we assign the energy-momentum tensor of a perfect fluid to the material subsystem, as if it were in isolation. As a result, the definition (6.59) of the Abraham velocity of matter coincides with the velocity of the block and it is the only separation in which this happens. This fact explains why the Abraham tensor is relevant for the case of an homogeneous and isotropic medium, and for the Einstein box theories in particular. Since in this picture we can consistently interpret the block as a particle, the remaining term can be naturally interpreted as the momentum of a "light particle" with velocity v_p given in (6.33), which in the non-relativistic approximation (very accurate for these cases, as we have demonstrated) coincides with the phase velocity of light when the medium is at rest $v_0 = c/n$. In other words, with the "Abraham separation" the property of inertia of energy is not only satisfied in the total system, as special relativity requires, but also in each subsystem separately. This was already noticed by Brevik in 1979, see [19], page 192.

In our opinion, these are the best arguments which support the usefulness of the interpretation of the Abraham momentum as the "kinetic momentum" of the field when the momentum of light is introduced in the usual non-relativistic "mv" form,

$$\overset{\mathrm{A}}{p} := \overset{\mathrm{A}}{M} v_0 = \frac{\overset{\mathrm{A}}{E}}{c^2}\frac{c}{n} = \frac{1}{n}\frac{\overset{\mathrm{A}}{E}}{c}. \tag{6.66}$$

At the same time, in spite of the fact that the Abraham choice is simpler than Minkowski's one for the case of the block, our analysis in section 6.4 clearly demonstrates that the Minkowski definition is also perfectly consistent for the Einstein box experiment, contrary to what is sometimes claimed [14, 19, 21, 62, 67, 75, 86], provided one considers the correct Minkowski expressions for the energy and momentum of matter. For a complementary discussion considering other explicit field configurations, see [91]. Again, only the total quantities are relevant for the description of the system.

In the nonrelativistic discussion of the balance equations for the slab, see for instance [14, 71, 86, 87], the Abraham momentum is selected as a consequence of treating the contribution of the light pulse to the velocity of the center of energy as if it were a particle moving with the phase velocity of the wave. *We want to stress that this choice is not justified from the point of view of field theory.* Additionally, our fully relativistic analysis shows that this assumption would only be consistent with the global conservation laws of the total system if one introduces suitable (ad hoc) "particle velocities" for the pulse, both in the Minkowski and Abraham pictures. However, these velocities do not correspond in general to any well defined velocity in the system. In particular, they do not coincide with the phase velocity of the wave in the moving medium, which is only true to zeroth order in the final slab velocity.

In any case, our explicit analysis, along with those in [10, 12, 18, 48], shows that the Abraham choice of the "correct" momentum of a light pulse is only one possibility, simple and useful for the description of isotropic media, but not at all an unique one.

Chapter 7.

Conclusiones

"A fin de cuentas, todo es un chiste."

Charles Chaplin,
cómico inglés.

Esperamos que esta tesis pueda contribuir en la clarificación de esta larga controversia sobre la definición del tensor energía-momentum de la luz dentro de un medio material. Aquí presentamos un enfoque completamente covariante y autoconsistente para estudiar este sistema, cuyos ingredientes básicos, al menos desde un punto de vista clásico, son sólamente tres: las ecuaciones de Maxwell macroscópicas para el campo electromagético, junto con las ecuaciones hidrodinámicas y las relaciones constitutivas para el tipo específico de material. Estas son las ecuaciones fundamentales que gobiernan las interacciones y el movimiento dentro del sistema total formado por campo electromagnético y medio material dinámico. Además, a partir de ellas podemos derivar directamente las ecuaciones de balance para la energía, momentum y momentum angular del sistema total. De acuerdo a este enfoque, sólo el tensor energía-momentum total tiene un significado físico claro y determina la dinámica del sistema, como mostramos explícitamente en el capítulo 6 mediante un importante ejemplo particular. De esta forma, los tensores de Abraham y Minkowski puede ser entendidos simplemente como diferentes separaciones del mismo tensor total, una elección que no modifica las predicciones físicas acerca del sistema.

En la literatura no hemos encontrado ningún trabajo criticando este enfoque, originalmente planteado por Penfield y Haus [11, 12, 13], lo que puede deberse a que en realidad estas ideas son bastante lógicas y completamente consistentes con los conceptos usuales de la Física Clásica, sin necesidad de tener que incluir suposiciones extras ad-hoc para describir casos particulares. En otros enfoques, como por ejemplo en muchas de las publicaciones actuales orientadas más a la óptica [17, 62, 75, 76, 83, 86, 87, 88], son usados conceptos extraños como la densidad de fuerza de Abraham, el momentum escondido, la masa de un pulso de luz en la materia, modificaciones de la definición de fuerza de Lorentz, etc., muchos elementos que no son necesarios de incluir en nuestro enfoque.

En el caso particular en que el medio es considerado como un escenario no dinámico donde la luz se propaga, es posible encontrar una respuesta definitiva para el tensor de la luz en un medio, siendo el tensor de *Minkowski* la respuesta. Debido a que en este caso el medio no modifica su dinámica de una manera similar al vacío, podemos consistentemente elegir no asignarle ningún tensor al medio y definir que toda la energía y el momentum del sistema está contenida en el campo electrománetico acomplado con la materia, de la misma forma que un "fotón vestido" en la teoría cuántica de campos. A partir de las ecuaciones de balance con el medio fijo, en el capítulo 4 concluímos que cuando el medio es no-disipativo y no depende del tiempo, entonces la energía de Minkowski se conserva y cuando el medio es no-disipativo y homogeńeo, entonces el momentum de Minkowski se conserva. Para el caso en que el medio es simultáneamente independiente del tiempo, no-disipativo y homogéneo, entonces el rayo de luz en la materia, incluyendo una onda de polarización y magnetización, se propagará sin cambios efectivos a una velocidad constante menor que c. Por otra parte, si el medio material es isótropo, entonces el momentum angular orbital de Minkowski es la correspondiente cantidad conservada. En el caso muy particular de un medio isótropo, homogéneo y en reposo, la energía, momentum y momentum angular de Minkowski son conservados, además de que las componentes espaciales del tensor de Minkowski son simétricas, es decir, el tensor de tensiones es simétrico en ese caso. Sin embargo, las componentes $0i$ del tensor de Minkowski no son simétricas. De hecho, en el capítulo 4 mostramos que el tensor de Minkowski puede ser completamente simétrico sólo en el caso del vacío, puesto que al fijar al medio material por fuerzas externas a un sistema de referencia en particular, la invariancia del sistema bajo boosts necesariamente se pierde. A pesar de su acusada falta de simetría, el tensor de Minkowski describe consistentemente la energía y el momentum efectivo del campo electromagnético en la materia y más aún su asimetría es un requerimiento fundamental de la teoría para describir correctamente las fuerzas efectivas ejercidas por el campo sobre el medio en estos casos.

Por otra parte, como nota Obukhov en [10], a pesar de que la definición del tensor de Abraham parezca bastante artificial a primera vista, éste aparece muy naturalmente en la expresión explícita para el tensor energía-momentum total del sistema formado por campo electromagnético y medio material con propiedas electromagnéticas isótropas, como vimos explícitamente en la sección 5.2. La descomposición de Abraham del tensor total es particularmente útil para este tipo de medio, puesto que en este caso todos los términos del tensor total en que aparece el campo electromagnético $F_{\mu\nu}$ están contenidos en el tensor de Abraham, a excepción de la presión efectiva, por lo que podemos pensar que el sistema está *casi desacomplado.* Sin embargo, estrictamente hablando, el tensor de Abraham sigue dependiendo de las propiedades del medio, como por ejemplo del índice de refracción n y de 4-velocidad u^μ del medio, por lo que en realidad el campo electromagnético siempre va a estar acoplado al medio en estos sistemas, sin importar como descompongamos el tensor total. Como resaltamos en la discusión del capítulo 6, la gran importancia de asignar al campo el tensor de Abraham, es que en el caso de un medio isótropo, el tensor energía-momentum que le corresponde al medio material es el mismo que el de un fluído ideal aislado. Como consecuencia, en el caso particular del experimento pensado de la caja de Einstein, analizado detalladamente en nuestra publicación [16], el medio se trata como una partícula de masa M y velocidad v, por lo que el pulso de luz se puede consistentemente interpretar como una partícula de luz con "masa" $\overset{\mathrm{A}}{E}/c^2$ y velocidad $v_{\mathrm{p}} = c(1 + 2n\beta + \beta^2)/(n + 2\beta + n\beta^2)$, la cual cualitativamente es diferente a la expresión relativista para la velocidad de fase v_β y también a su límite no-relativista $v_0 = c/n$. En el tratamiento usual de este problema por

Balazs, Barnett, Loudon y muchos otros, como por ejemplo en [14, 75, 86, 88], se asume a priori que el pulso de luz se debe propagar con una velocidad c/n dentro del bloque y debido a que v_{p} coincide con v_0 en el límite no-relativista, no es una sorpresa que estos autores hayan encontrado que sólo el momentum de Abraham es capaz de describir este sistema, pues tácitamente ya lo estaban asumiendo según nuestro análisis.

No hay ninguna duda de que la separación de Abraham es mucho más conveniente que la de Minkowski para sistemas como la caja dieléctrica, donde el medio es isótropo y tiene libertad de moverse, pero cualitativamente es importante de remarcar que esta situación puede ser igualmente descrita usando el tensor de Minkowski, como explícitamente mostramos en el capítulo 6 y en [16], al contrario de lo que muchos autores afirmaban, como por ejemplo en [14, 19, 21, 62, 67, 75, 86]. Desde el punto de vista de la teoría clásica de campos, sólo el tensor total es importante y hay ninguna razón fundamental para considerar que cierta separación es mejor que otra.

Obukhov en [10] comenta que el tensor de Abraham podría ser sólo una curiosa coincidencia de los medio isótropos, pero no una característica general de la teoría, válida en todos los tipos de medio. Luego, sería interesante generalizar el procedimiento del capítulo 5 con el fin de poder describir un medio con propiedades electromagnéticas anisótropas, donde el carácter acoplado del tensor total se vea más claramente. Uno podría también resolver un experimento pensado muy análogo al de la caja de Einstein para poder recalcar más aún que en todos los casos el tensor total es la cantidad física importante. El tensor constitutivo para un medio anisótropo uniaxial derivado en el apéndice B, es parte de los resultados preliminares que tenemos para llevar a cabo esta idea. También tenemos avances en determinar la densidad Lagrangeana relativista para un cristal líquido nemático, que es un modelo para estudiar un medio anisótropo uniaxial dinámico. Es muy probable que el tensor de Abraham, como fue definido en la sección 5.3 no sea útil en medios más complejos, pero quizás podramos encontrar otra tensor que cumpla su misma función en un medio anistropo y así definir un tensor de Abraham generalizado.

Otro posible trabajo futuro que es, en nuestra opinión, importante con el fin de clarificar la controversia es aplicar este formalismo para resolver explícitamente todos los experimentos que se han realizado durante el debate. Obukhov y Hehl ya han avanzado en esta línea, pues analizaron los experimentos de James y Walker et al. en [79] y el experimento de Jones et al. en [80]. Sin embargo, todavía quedan algunos experimentos pendientes, especialmente los más recientes como [77, 81].

Adicionalmente, se plantea estudiar las condiciones bajo las cuales podemos hacer la correspondencia entre el caso del medio con diámica, donde el tensor total es importante, con el caso del medio fijo, donde el tensor de Minkowski es importante. Al parecer esta correspondencia es análoga a la que se presenta, por ejemplo, en un sistema binario con una partícula mucho más masiva que la otra. En este caso, para todos los propósitos prácticos se puede asumir que una partícula está fija en el centro y la otra gira al rededor de ella, pero bajo el costo de perder la invariancia translacional del sistema. Por consiguiente el momentum total del sistema no es más conservado, pero el momentum angular y la energía total del sistema binario permanecen como cantidades conservadas.

Appendix A.

Propagation of electromagnetic waves within linear material media

"Una persona sin sentido del humor
es como un auto sin amortiguadores,
será sacudido por la más mínima
piedra en el camino."

Henry Ward Beecher,
clérigo congregacionalista estadounidense.

Now we will study the propagation of electromagnetic waves within *linear* and *non-dispersive* media. For this purpose, we will seek for plane-wave solutions of the macroscopic Maxwell equations, but in regions where no external sources are present, i.e. $J^\mu_{\text{ext}} = 0$. We will see that Maxwell's equations constrain the propagation of electromagnetic waves via a dispersion relation and a polarization condition, which will depend on the constitutive tensor $\chi^{\mu\nu\rho\sigma}$ of the specific medium. In particular, we will consider light propagation in isotropic media, using the previously derived constitutive tensors of chapter 3.

A.1. General polarization condition and extended Fresnel equation

In order to find which conditions the Maxwell equations impose to light propagation within matter in any state of motion, let us first insert (3.61) and (3.42) in (3.40) with $J^\mu_{\text{ext}} = 0$ and we obtain a second order equation for the 4-potential A_σ:

$$\chi^{\mu\nu\rho\sigma}(\partial_\mu\partial_\rho A_\sigma) + (\partial_\mu \chi^{\mu\nu\rho\sigma})(\partial_\rho A_\sigma) = 0. \tag{A.1}$$

Additionally, we will restrict our derivation to the particular case of *homogeneous* media, so that it can be assumed that

$$\partial_\mu \chi^{\mu\nu\rho\sigma} = 0, \tag{A.2}$$

and therefore the Maxwell equations in matter for the 4-potential (A.1) reduces simply to

$$\boxed{\chi^{\mu\nu\rho\sigma}(\partial_\mu\partial_\rho A_\sigma) = 0.} \tag{A.3}$$

Let us now look for plane-wave solutions for the electromagnetic field inside the medium, i.e. suppose a solution of the form,

$$A_\sigma = \tilde{A}_\sigma e^{ik_\lambda x^\lambda}, \tag{A.4}$$

where k_λ is the 4-wave vector, defined as

$$k_\lambda := \left(\frac{\omega}{c}, -\boldsymbol{k}\right), \tag{A.5}$$

and $\tilde{A}_\sigma$ is the constant complex amplitude of the wave. If we replace (A.4) into (A.3), we get the polarization condition that k_λ and $\tilde{A}_\lambda$ have to satisfy in order for the plane-wave (A.4) to be a solution of the Maxwell equations in the medium:

$$\boxed{\chi^{\mu\nu\rho\sigma}k_\nu k_\rho \tilde{A}_\sigma = 0.} \tag{A.6}$$

In order to find non-trivial solutions for (A.6), it is necessary that the determinant of $\chi^{\mu\nu\rho\sigma}k_\nu k_\rho$ be equal to zero, a condition which implies, in general, a quartic equation for k_λ, called the extended Fresnel equation. In [93] this general dispersion relation has been covariantly solved and it is explicitly given by,

$$\boxed{\mathcal{G}^{\mu\nu\rho\sigma}k_\mu k_\nu k_\rho k_\sigma = 0,} \tag{A.7}$$

where $\mathcal{G}^{\mu\nu\rho\sigma}$ is the Tamm-Rubilar tensor defined as

$$\mathcal{G}^{\mu\nu\rho\sigma} := \frac{1}{4!}\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{\theta\eta\kappa\lambda}\chi^{\alpha\beta\theta(\mu}\chi^{\nu|\gamma\eta|\rho}\chi^{\sigma)\delta\kappa\lambda}, \tag{A.8}$$

which can be analytically computed given any constitutive tensor $\chi^{\mu\nu\rho\sigma}$ of a linear, non-dispersive and quasi-homogeneous medium. Once we know the "allowed" values for the 4-vectors of light in the given medium, we can insert them back into the polarization condition (A.6) and finally solve for the amplitude $\tilde{A}_\sigma$. Since in chapter 3 we derived explicit expressions for the constitutive tensors of different kinds of media, we can use this general procedure to study the propagation of light in particular media.

A.2. Isotropic moving medium

In section 3.4 we derived in (3.89) and (3.91), the constitutive tensor for an isotropic medium as seen by any inertial reference frame, where the medium itself has a 4-velocity $u^\mu = (c\gamma, \gamma\boldsymbol{v})$. We can also assume that the velocity of the medium is a field which depends on the coordinates and time, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$.

A.2.1. Dispersion relation for light propagating parallel to medium's motion

In subsection 3.5 it is explained that with the Gordon optical metric $\gamma^{\mu\nu}$ in (3.91), we can compute the dispersion relation for light within an isotropic material medium in any state of motion, by contracting it with k^{μ}, i.e.

$$\gamma^{\mu\nu}k_{\mu}k_{\nu} = 0. \tag{A.9}$$

Therefore, in this subsection we will explicitly evaluate (A.9) for the simplest case, when light is propagating in the same direction in which the medium is moving. The results here obtained are oriented to be applied in the analysis of chapter 6, for the relativistic dielectric slab problem.

Since the problem is one-dimensional, let us assume that the direction of propagation of light and the medium is the x axis, and hence we can assume that k_{μ} and u^{μ} are of the form

$$k_{\mu} = (\omega, -k, 0, 0)\,, \tag{A.10}$$
$$u^{\mu} = (\gamma, \gamma v, 0, 0), \tag{A.11}$$

where v is the constant velocity of the medium and k the wave number of the wave. Using (A.10) and (A.11) in (3.91), the explicit expression for $\gamma^{\mu\nu}$ reads,

$$\gamma^{\mu\nu} = \begin{pmatrix} 1+(n^2-1)\gamma^2 & (n^2-1)\gamma^2\beta & 0 & 0 \\ (n^2-1)\gamma^2\beta & -1+(n^2-1)\gamma^2\beta^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.12}$$

Then, using (A.10) and (A.12) in (3.100), we obtain

$$\gamma^{00}\omega^2 - 2\gamma^{01}\omega k + \gamma^{11}k^2 = 0, \tag{A.13}$$

and finally solving for ω in terms of β, n and k, we obtain the dispersion relation to be:

$$\boxed{\omega(k) = v_{\beta}k,} \tag{A.14}$$

where v_{β} is the phase velocity of the light waves inside the moving medium, defined as

$$\boxed{v_{\beta} := \frac{c(1+n\beta)}{(n+\beta)}.} \tag{A.15}$$

It is remarkable that one can obtain the same result (A.15) by applying the relativistic velocity transformation to the well-known expression $v_0 = c/n$, valid in the comoving frame. *The relativistic transformation of the phase velocity is in general different from the usual particle velocity transformation*, but they do coincide if the phase velocity is parallel to the relative velocity. See, for instance, [94] and [95].

A.2.2. Plane-wave solution of Maxwell's equations in an isotropic moving medium

Since we already calculated the dispersion relation and therefore the phase velocity of light waves inside the moving medium, we will now evaluate the polarization condition of subsection 3.5, given by

$$\gamma^{\mu\nu} k_\mu \tilde{A}_\nu = 0, \tag{A.16}$$

in order to find the solutions for the potential $\boldsymbol{A}$ and the electromagnetic field $\boldsymbol{E}$ and $\boldsymbol{B}$. Inserting (A.10) and (A.12) into (A.16), we obtain

$$(\gamma^{00}\omega - \gamma^{01}k)\tilde{A}_0 + (\gamma^{01}\omega - \gamma^{11}k)\tilde{A}_1 = 0, \tag{A.17}$$

and recalling the gauge transformation (3.104), we can choose h so that $\tilde{A}_0 = 0$ and hence we obtain a condition only for $\tilde{A}_1$, which reads

$$(\gamma^{01}\omega - \gamma^{11}k)\tilde{A}_1 = 0. \tag{A.18}$$

Since (A.18) must be satisfied for all k, $\tilde{A}_1$ must also vanish and therefore using (A.4) and (A.14), the solution for the electromagnetic potential inside the moving medium can be written, in vector form, as

$$\boldsymbol{A}(x,t) = \Re\left\{-(\tilde{A}_2\hat{\boldsymbol{y}} + \tilde{A}_3\hat{\boldsymbol{z}})e^{i(kv_\beta t - kx)}\right\}, \tag{A.19}$$

where $\tilde{A}_2$ and $\tilde{A}_3$ are complex constants. Replacing (A.19) and $\phi := cA_0 = 0$ in the definitions (3.14)-(3.15), the electric and magnetic fields of the plane-wave turn out to be

$$\boldsymbol{E} = \Re\left\{ikv_\beta(\tilde{A}_2\hat{\boldsymbol{y}} + \tilde{A}_3\hat{\boldsymbol{z}})e^{i(kv_\beta t - kx)}\right\}, \tag{A.20}$$

$$\boldsymbol{B} = \Re\left\{ik(-\tilde{A}_3\hat{\boldsymbol{y}} + \tilde{A}_2\hat{\boldsymbol{z}})e^{i(kv_\beta t - kx)}\right\}. \tag{A.21}$$

This solution can be conveniently recast into

$$\boldsymbol{E}(x,t) = E_0\Re\left\{e^{i(kv_\beta t - kx + \varphi)}\hat{\boldsymbol{e}}\right\}, \tag{A.22}$$

$$\boldsymbol{B}(x,t) = \frac{E_0}{v_\beta}\Re\left\{e^{i(kv_\beta t - kx + \varphi)}(\hat{\boldsymbol{x}} \times \hat{\boldsymbol{e}})\right\} \tag{A.23}$$

$$= \frac{1}{v_\beta}\hat{\boldsymbol{x}} \times \boldsymbol{E}, \tag{A.24}$$

where $E_0 := |\boldsymbol{E}|$ is the amplitude of the electric field plane wave, φ is an extra phase of the field and $\hat{\boldsymbol{e}}$ is the polarization vector, a complex unitary vector that indicates the direction of the electric field $\boldsymbol{E}$ and lies in the plane perpendicular to the propagation direction $\hat{\boldsymbol{k}}$, i.e.

$$\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{k}} = 0, \qquad \hat{\boldsymbol{e}}^* \cdot \hat{\boldsymbol{e}} = 1. \tag{A.25}$$

A general way to parametrize the polarization vector $\hat{\boldsymbol{e}}$ in our case is,

$$\hat{\boldsymbol{e}} = \frac{1}{\sqrt{a^2+b^2}}\left(a\hat{\boldsymbol{y}} + be^{i\theta}\hat{\boldsymbol{z}}\right), \tag{A.26}$$

where a, b and θ are real constants, which describe a specific polarization state of light. If we insert (A.26) in (A.22)-(A.23), we obtain the final expression for the electromagnetic field of a plane-wave with any polarization, propagating within matter in the same direction as the medium:

$$\boxed{\begin{aligned} \boldsymbol{E}(x,t) &= \frac{E_0}{\sqrt{a^2+b^2}}\left[a\hat{\boldsymbol{y}}\cos(kv_\beta t - kx + \varphi) + b\hat{\boldsymbol{z}}\cos(kv_\beta t - kx + \varphi + \theta)\right], \\ \boldsymbol{B}(x,t) &= \frac{1}{v_\beta}\frac{E_0}{\sqrt{a^2+b^2}}\left[-b\hat{\boldsymbol{y}}\cos(kv_\beta t - kx + \varphi + \theta) + a\hat{\boldsymbol{z}}\cos(kv_\beta t - kx + \varphi)\right] \\ &= \frac{1}{v_\beta}\hat{\boldsymbol{x}} \times \boldsymbol{E}. \end{aligned}} \tag{A.27}$$

Appendix B.

Covariant constitutive relations for more general media

"No es bastante levantar al débil,
es necesario aún sostenerlo después..."

William Shakespeare,
escritor y dramaturgo inglés.

In section 3.4 we obtained an explicit expression for the constitutive tensor of an isotropic medium in any inertial reference frame by starting from the properties of the medium in its rest frame. Using the same method, in this chapter we will obtain explicit expressions for the constitutive tensors of an anisotropic uniaxial medium whith dielectric and diamagnetic anisotropy and of a magnetoelectric medium.

The information of this chapter is some preliminary work in order to later extend the analysis of the Abraham-Minkowski controvery to more complex media.

B.1. Uniaxial anisotropic medium

Now we will analyse a medium with anisotropic electromagnetic properties, i.e. their macroscopic response to the electromagnetic field will depend on the direction along which light propagates inside it. It is a difficult task, in general, to find an unique expression to describe any anisotropic medium and hence we will restrict us to study *uniaxial anisotropic* media, which are a bit simpler to describe mathematically due to a symmetry that they present. In any volume element of a medium of this kind, there is a preferred direction around which the electromagnetic properties of the medium are the same. This direction is usually known as the *optical axis* of the medium and it is described by a unitary vector field $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x}, t)$, i.e. such that $|\boldsymbol{n}|^2 = 1$. For instance, some crystals like calcite ($CaCO_3$), quartz (SiO_2), beta barium borate (β-BaB_2O_4), barium titanate ($BaTiO_3$) and also fluids like nematic liquid crystals [96] can present these properties.

B.1.1. Constitutive relations in the comoving frame

Analogously to the isotropic case, we will start our analysis from the knowledge of the properties of the medium in its rest frame and at the end we will try to find a covariant expression which reduces to this known comoving frame expression.

If we assume that the medium is non magneto-electric, non-dissipative and also that it has the *same optical axis for the electric and magnetic anisotropy* $\boldsymbol{n}$, we can decompose $\varepsilon^{ij}_{(0)}$ and $(\mu^{-1})^{ij}_{(0)}$ in (3.65)-(3.66) in terms of their eigenvectors (chosen as $\boldsymbol{n}$, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$) with two eigenvalues equal and one different:

$$\varepsilon^{ij}_{(0)} = \varepsilon_{\parallel} n^i n^j + \varepsilon_{\perp}(n^i_1 n^j_1 + n^i_2 n^j_2), \tag{B.1}$$
$$(\mu^{-1})^{ij}_{(0)} = \mu^{-1}_{\parallel} n^i n^j + \mu^{-1}_{\perp}(n^i_1 n^j_1 + n^i_2 n^j_2), \tag{B.2}$$

where $\varepsilon_\perp$ and $\varepsilon_\parallel$ are the relative permittivity perpendicular and parallel to the optical axis field $\boldsymbol{n}$ and $\mu_\perp$, $\mu_\parallel$ are the corresponding perpendicular and parallel relative permeability functions. Since the tensors $\varepsilon^{ij}_{(0)}$ and $(\mu^{-1})^{ij}_{(0)}$ are symmetric and real for non-dissipative and non-dispersive media, the eigenvectors satisfy the closure relation,

$$n^i n^j + n^i_1 n^j_1 + n^i_2 n^j_2 = -\eta^{ij}, \tag{B.3}$$

and therefore, replacing (B.3) into (B.1)-(B.2), we obtain a simpler expression for the dielectric and diamagnetic tensors in the comoving frame,

$$\varepsilon^{ij}_{(0)} = -\,\varepsilon_\perp \eta^{ij} + \Delta\varepsilon\; n^i n^j, \tag{B.4}$$
$$(\mu^{-1})^{(0)}_{ij} = -\,\mu^{-1}_\perp \eta_{ij} + \Delta(\mu^{-1})\; n_i n_j, \tag{B.5}$$

where,

$$\Delta\varepsilon := \varepsilon_\parallel - \varepsilon_\perp, \tag{B.6}$$
$$\Delta(\mu^{-1}) := \mu^{-1}_\parallel - \mu^{-1}_\perp, \tag{B.7}$$

which are the electric and magnetic anisotropies, respectively.

Replacing expressions (B.4)-(B.5) into (3.71)-(3.72), we find that the linearly independent components of the rest frame constitutive tensor $\chi^{\mu\nu\rho\sigma}$ must be given by,

$$\chi^{0ijk}_{\mathrm{ani}(0)} = 0, \tag{B.8}$$
$$\chi^{0ij0}_{\mathrm{ani}(0)} = -\,\mu_0^{-1}\varepsilon_\perp \eta^{ij} + \mu_0^{-1}\Delta\varepsilon\; n^i n^j, \tag{B.9}$$
$$\begin{aligned}\chi^{klmq}_{\mathrm{ani}(0)} &= \mu_0^{-1}\mu_\parallel^{-1}\left(\eta^{km}\eta^{lq} - \eta^{kq}\eta^{lm}\right)\\ &\quad + \mu_0^{-1}\Delta(\mu^{-1})\left(\eta^{km}n^l n^q - \eta^{kq}n^l n^m + \eta^{lq}n^k n^m - \eta^{lm}n^k n^q\right).\end{aligned} \tag{B.10}$$

The components of $\chi^{\mu\nu\rho\sigma}_{\mathrm{ani}(0)}$ clearly reduce to the ones of the isotropic case $\chi^{\mu\nu\rho\sigma}_{\mathrm{iso}(0)}$ in (3.83)-(3.85), in the limit $\varepsilon_\parallel \to \varepsilon_\perp = \varepsilon$ and $\mu_\parallel^{-1} \to \mu_\perp^{-1} = \mu^{-1}$.

B.1.2. Two optical metrics and factorization of the Fresnel equation

Using Maple and Grtensor, we explicitly replaced the non-vanishing components of the constitutive tensor (B.8)-(B.10) into the general dispersion relation (A.7) and its solution (A.8) of appendix A, and we obtained a factorized form of the quartic Fresnel equation in the rest frame of the medium. The two factors correspond to two different dispersion relations and therefore *two different* optical metrics can be defined in this case:

$$(\gamma^{\mu\nu}_{\mathrm{e}(0)}k_\mu k_\nu)(\gamma^{\rho\sigma}_{\mathrm{m}(0)}k_\rho k_\sigma) = 0, \tag{B.11}$$

with

$$\gamma^{\mu\nu}_{\mathrm{e}(0)}k_\mu k_\nu = n^2\,\frac{\omega^2}{c^2} - \alpha_{\mathrm{e}}\boldsymbol{k}^2 + (\alpha_{\mathrm{e}}-1)(\boldsymbol{n}\cdot\boldsymbol{k})^2, \tag{B.12}$$

$$\gamma^{\mu\nu}_{\mathrm{m}(0)}k_\mu k_\nu = n^2\,\frac{\omega^2}{c^2} - \alpha_{\mathrm{m}}\boldsymbol{k}^2 + (\alpha_{\mathrm{m}}-1)(\boldsymbol{n}\cdot\boldsymbol{k})^2, \tag{B.13}$$

where

$$n^2 := \mu_\perp\varepsilon_\perp, \tag{B.14}$$

and the parameters α_{e} and α_{m} also quantify the degree of dielectric and diamagnetic uniaxial anisotropy by,

$$\alpha_{\mathrm{e}} := \frac{\varepsilon_\perp}{\varepsilon_\parallel} \quad \text{and} \quad \alpha_{\mathrm{m}} := \frac{\mu_\perp}{\mu_\parallel}. \tag{B.15}$$

As a consequence, the reduced quadratic Fresnel equation $\gamma^{\mu\nu}_{\mathrm{e}(0)}k_\mu k_\nu = 0$ implies that if $\boldsymbol{k}$ is parallel to $\boldsymbol{n}$ then necessarily $n^2\omega^2/c^2 - \boldsymbol{k}^2 = 0$ which means that in this case light propagates with the expected effective refraction index for the ordinary ray: n. On the other hand, if $\boldsymbol{k}$ is orthogonal to $\boldsymbol{n}$ then the Fresnel equation reduces to $n^2\omega^2/c^2 - \alpha_{\mathrm{e}}\boldsymbol{k}^2 = 0$. This means that light propagates with an effective refraction index n_{e} such that $n_{\mathrm{e}}^2 := n^2/\alpha_{\mathrm{e}} = \varepsilon_\parallel\mu_\perp$. We may call this the "electric" extraordinary ray. Similarly, the second Fresnel equation $\gamma^{\mu\nu}_{\mathrm{m}(0)}k_\mu k_\nu = 0$ leads to a normal ordinary ray refraction index n for waves with wave vector parallel to the optical axis $\boldsymbol{n}$ and a refraction index n_{m} for $\boldsymbol{k} \perp \boldsymbol{n}$, with $n_{\mathrm{m}}^2 := n^2/\alpha_{\mathrm{m}} = \varepsilon_\perp\mu_\parallel$, which correspond to the "magnetic" extraordinary ray. Since light propagates inside this medium as if there were two "electric" refraction indices and two "magnetic" refraction indices (in fact they are only three, because n has degeneracy), the medium is said to present birefringence (electric and magnetic), i.e. light rays will propagate as if the medium had different refraction indices, depending on their directions of propagation and the polarization.

B.1.3. Covariant description of the constitutive tensor

Since both (B.12) and (B.13) must be covariant quantities, it is not difficult to identity the general expression of the optical metrics in a reference frame where the medium moves with 4-velocity $u^\mu = (c\gamma, \gamma\boldsymbol{v})$:

$$\boxed{\begin{aligned}\gamma^{\mu\nu}_{\mathrm{e}} &:= \alpha_{\mathrm{e}}\,\eta^{\mu\nu} + \frac{(n^2-\alpha_{\mathrm{e}})}{c^2}u^\mu u^\nu + (\alpha_{\mathrm{e}}-1)N^\mu N^\nu,\\ \gamma^{\mu\nu}_{\mathrm{m}} &:= \alpha_{\mathrm{m}}\,\eta^{\mu\nu} + \frac{(n^2-\alpha_{\mathrm{m}})}{c^2}u^\mu u^\nu + (\alpha_{\mathrm{m}}-1)N^\mu N^\nu.\end{aligned}} \tag{B.16}$$

Here N^μ is a spacelike unit 4-vector, which is also orthogonal to the timelike 4-vector u^μ, i.e.

$$N^\mu u_\mu = 0, \tag{B.17}$$
$$N^\mu N_\mu = -1, \tag{B.18}$$
$$u^\mu u_\mu = c^2, \tag{B.19}$$

and it is the covariant generalization of the optical axis direction. By definition, in the rest frame N^μ reduces to $N^\mu_{(0)} = (0, n^i)$.

Following the development of Balakin and Zimdahl in [97], where the constitutive tensor of a medium with only electric anisotropy is expressed in terms of the optical metrics, we found a natural generalization, which also includes magnetic anisotropy and which is explicitly given in terms of the two metrics (B.16) by

$$\boxed{\chi^{\mu\nu\rho\sigma}_{\text{ani}} = \frac{1}{\alpha_{\text{e}}\,\mu_0\mu_\perp}\left(\gamma^{\mu\rho}_{\text{e}}\gamma^{\nu\sigma}_{\text{e}} - \gamma^{\mu\sigma}_{\text{e}}\gamma^{\nu\rho}_{\text{e}}\right) + \frac{1}{(\alpha_{\text{m}} - \alpha_{\text{e}})\mu_0\mu_\perp}\left(\Delta\gamma^{\mu\rho}\Delta\gamma^{\nu\sigma} - \Delta\gamma^{\mu\sigma}\Delta\gamma^{\nu\rho}\right),} \tag{B.20}$$

where $\Delta\gamma^{\mu\nu}$ is the difference of the optical metrics,

$$\Delta\gamma^{\mu\nu} := \gamma^{\mu\nu}_{\text{e}} - \gamma^{\mu\nu}_{\text{m}}. \tag{B.21}$$

Replacing (B.16) into (B.20), we obtain the explicit covariant expression for $\chi^{\mu\nu\rho\sigma}$, adequate to describe the electromagnetic properties of the uniaxial anisotropic medium in any reference frame:

$$\boxed{\begin{aligned}\chi^{\mu\nu\rho\sigma}_{\text{ani}} = {} & \mu_0^{-1}\left[\mu_\perp^{-1} + \Delta(\mu^{-1})\right]\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right)\\ & + \frac{1}{\mu_0 c^2}\left[\left(n^2 - 1\right)\mu_\perp^{-1} - \Delta(\mu^{-1})\right]\left(\eta^{\mu\rho}u^\nu u^\sigma - \eta^{\mu\sigma}u^\nu u^\rho + \eta^{\nu\sigma}u^\mu u^\rho - \eta^{\nu\rho}u^\mu u^\sigma\right)\\ & + \mu_0^{-1}\Delta(\mu^{-1})\left(\eta^{\mu\rho}N^\nu N^\sigma - \eta^{\mu\sigma}N^\nu N^\rho + \eta^{\nu\sigma}N^\mu N^\rho - \eta^{\nu\rho}N^\mu N^\sigma\right)\\ & - \frac{1}{\mu_0 c^2}\left[\Delta\varepsilon + \Delta(\mu^{-1})\right]\left(u^\mu u^\rho N^\nu N^\sigma - u^\mu u^\sigma N^\nu N^\rho + u^\nu u^\sigma N^\mu N^\rho - u^\nu u^\rho N^\mu N^\sigma\right).\end{aligned}} \tag{B.22}$$

It can be easily checked that (B.22) takes into account all the necessary symmetries and reduces to (B.8), (B.9) and (B.10) in the rest frame of the medium. Additionally, it reduces to the isotropic expression (3.92), when $\varepsilon_\parallel \to \varepsilon_\perp = \varepsilon$ and $\mu_\parallel \to \mu_\perp = \mu$.

B.2. Magneto-electric medium

So far we have only discussed material media which, in its rest frame, can become polarized only by the action of an electric field and magnetized only by a magnetic field. However, there also exist materials whose electric and magnetic response are coupled. This rare type of media are known in literature as *magneto-electric* and they can be found either in natural state like Cr_2O_3 crystal (chromium sesquioxide) [98] or produced artificially with an applied external electric field as the case of $Y_3Fe_5O_12$ or with an applied magnetic field as $NiSO_4.6H_2O$. For more details, please see [99, 100].

For the description of a linear, non-dissipative, non-dispersive and magneto-electric medium we stated in (3.75), (3.76) and (3.77) that we need at most 21 independent components which are 6 $\varepsilon^{ij}_{(0)}$ and 6 $(\mu^{-1})^{(0)}_{ij}$ as usual, plus the magneto-electric coupling tensor $\beta^{i}{}^{(0)}_{j}$, which is related to the other coupling constants by $\alpha_{i}{}^{j}_{(0)} = -\beta^{j}{}^{(0)}_{i}$ and has 9 independent components. Therefore, in the rest frame of this medium, the constitutive relations in cartesian components assume the form,

$$D^i = \varepsilon_0 \varepsilon^{ij}_{(0)} E_j + \beta^{i}{}^{(0)}_{j} B^j, \tag{B.23}$$

$$H^i = \mu_0^{-1} (\mu^{-1})^{(0)}_{ij} B^j - \beta^{j}{}^{(0)}_{i} E_j. \tag{B.24}$$

In order to write (B.23)-(B.24) in the covariant form (3.61), we notice that the constitutive tensor $\chi^{\mu\nu\rho\sigma}$ can always be decomposed in the usual pure non-magnetoelectric part $\chi^{\mu\nu\rho\sigma}_{\text{nme}}$ plus a pure magneto-electric part $\chi^{\mu\nu\rho\sigma}_{\text{me}}$:

$$\chi^{\mu\nu\rho\sigma} := \chi^{\mu\nu\rho\sigma}_{\text{nme}} + \chi^{\mu\nu\rho\sigma}_{\text{me}}, \tag{B.25}$$

whose components in the rest frame are defined by

$$\chi^{0ijk}_{\text{nme}(0)} = 0, \qquad \chi^{0ij0}_{\text{nme}(0)} = \mu_0^{-1} \varepsilon^{ij}_{(0)}, \qquad \chi^{ijkl}_{\text{nme}(0)} = \mu_0^{-1} \epsilon^{ijm} \epsilon^{klq} (\mu^{-1})^{(0)}_{mq}, \tag{B.26}$$

$$\chi^{0ijk}_{\text{me}(0)} = c\, \epsilon^{ljk} \beta^{i}{}^{(0)}_{l}, \qquad \chi^{0ij0}_{\text{me}(0)} = 0, \qquad \chi^{ijkl}_{\text{me}(0)} = 0, \tag{B.27}$$

in order to be consistent with (3.71)-(3.74).

For instance, the constitutive tensors (3.87) and (B.22) which we already derived in sections 3.4 and B.1 are two examples of pure non-magnetoelectric constitutive tensors, since they satisfy $\chi^{0ijk}_{\text{iso}(0)} = \chi^{0ijk}_{\text{ani}(0)} = 0$. If we want that any of these media also present magneto-electric properties described by $\beta^{i}{}^{(0)}_{j}$, we just need to add to the known pure non-magnetoelectric constitutive tensor $\chi^{\mu\nu\rho\sigma}_{\text{nme}}$ a pure magneto-electric part $\chi^{\mu\nu\rho\sigma}_{\text{me}}$ as in (B.25). Therefore, this section is devoted to derive a covariant expression for a pure magneto-electric constitutive tensor.

In the same spirit as we defined, in section B.1.3, a covariant generalization of the optical axis vector N^μ of an uniaxial anisotropic medium, which satisfies $N^\mu_{(0)} = (0, n^i)$ and $N^\mu u_\mu = 0$, in order to find a covariant expression for $\chi^{\mu\nu\rho\sigma}_{\text{me}}$ we will need to define a magneto-electric coupling 4-tensor $\beta^\mu{}_\nu$, which in the rest frame of the medium should reduce to a completely spatial tensor:

$$\beta^{\mu}{}^{(0)}_{\nu} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \beta^1{}_1 & \beta^1{}_2 & \beta^1{}_3 \\ 0 & \beta^2{}_1 & \beta^2{}_2 & \beta^2{}_3 \\ 0 & \beta^3{}_1 & \beta^3{}_2 & \beta^3{}_3 \end{pmatrix}. \tag{B.28}$$

Here, the entries $\beta^i{}_j$ are the magneto-electric coupling coefficients which are measured in the rest frame of the medium, i.e. $\beta^{i}{}^{(0)}_{j}$, but we removed the label $^{(0)}$ to simplify the notation. By construction, we see that $\beta^\mu{}_\nu$ is orthogonal to the 4-velocity as well as N^μ, i.e.

$$\beta^\mu{}_\nu u_\mu \equiv \beta^\mu{}_\nu u^\nu \equiv 0. \tag{B.29}$$

If we recall the explicit expression for the components $\chi^{0ijk}_{\text{me}(0)}$ in (B.27a), we see that a covariant generalization $\chi^{\mu\nu\rho\sigma}_{\text{me}}$ of it must be constructed using only $\beta^{\mu}{}_{\nu}$, the completely antisymmetric 4-D Levi-Civita pseudo tensor $\epsilon^{\mu\nu\rho\sigma}$, defined such that $\epsilon_{0123} = -\epsilon^{0123} = 1$, and the 4-velocity field u^{μ} as usual. The 4-D Levi-Civita is a generalization of the usual 3-D one and both are related by

$$\epsilon^{\mu\nu\rho} := \frac{1}{c}\epsilon^{\mu\nu\rho\sigma}u_{\sigma}. \tag{B.30}$$

As a result, the spatial components of $\epsilon^{\mu\nu\rho}$ in the rest frame of the medium satisfy,

$$\epsilon^{123}_{(0)} = \frac{1}{c}\epsilon^{1230}u^{(0)}_{0} = -\epsilon^{0123} = 1, \tag{B.31}$$

in accordance with (3.35).

Using all these objects and taking care of the symmetries (3.62)-(3.64), we find a general covariant expression for the magneto-electric constitutive tensor $\chi^{\mu\nu\rho\sigma}_{\text{me}}$, valid in any inertial reference frame,

$$\boxed{\chi^{\mu\nu\rho\sigma}_{\text{me}} = \frac{1}{c}\left(\epsilon^{\lambda\alpha\rho\sigma}\beta^{\mu}{}_{\alpha}u^{\nu}u_{\lambda} - \epsilon^{\lambda\alpha\rho\sigma}\beta^{\nu}{}_{\alpha}u^{\mu}u_{\lambda} + \epsilon^{\lambda\alpha\mu\nu}\beta^{\rho}{}_{\alpha}u^{\sigma}u_{\lambda} - \epsilon^{\lambda\alpha\mu\nu}\beta^{\sigma}{}_{\alpha}u^{\rho}u_{\lambda}\right),} \tag{B.32}$$

and whose components reduce to (B.27), as can be easily verified.

Finally, if we add the pure magneto-electric part (B.32) to the pure non magneto-electric part of the uniaxial anisotropic medium (B.22), we obtain the most general constitutive tensor $\chi^{\mu\nu\rho\sigma}_{\text{ani}-\text{me}}$ that we will consider in this thesis, which describes a linear, non-dispersive, non-dissipative, magneto-electric and anisotropic uniaxial electric and magnetic medium:

$$\begin{aligned}
\chi^{\mu\nu\rho\sigma}_{\text{ani}-\text{me}} = {} & \mu_0^{-1}\left[\mu_{\perp}^{-1} + \Delta(\mu^{-1})\right]\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) \\
& + \frac{1}{\mu_0 c^2}\left[\left(n^2 - 1\right)\mu_{\perp}^{-1} - \Delta(\mu^{-1})\right]\left(\eta^{\mu\rho}u^{\nu}u^{\sigma} - \eta^{\mu\sigma}u^{\nu}u^{\rho} + \eta^{\nu\sigma}u^{\mu}u^{\rho} - \eta^{\nu\rho}u^{\mu}u^{\sigma}\right) \\
& + \mu_0^{-1}\Delta(\mu^{-1})\left(\eta^{\mu\rho}N^{\nu}N^{\sigma} - \eta^{\mu\sigma}N^{\nu}N^{\rho} + \eta^{\nu\sigma}N^{\mu}N^{\rho} - \eta^{\nu\rho}N^{\mu}N^{\sigma}\right) \\
& - \frac{1}{\mu_0 c^2}\left[\Delta\varepsilon + \Delta(\mu^{-1})\right]\left(u^{\mu}u^{\rho}N^{\nu}N^{\sigma} - u^{\mu}u^{\sigma}N^{\nu}N^{\rho} + u^{\nu}u^{\sigma}N^{\mu}N^{\rho} - u^{\nu}u^{\rho}N^{\mu}N^{\sigma}\right) \\
& + \frac{1}{c}\left(\epsilon^{\lambda\alpha\rho\sigma}\beta^{\mu}{}_{\alpha}u^{\nu}u_{\lambda} - \epsilon^{\lambda\alpha\rho\sigma}\beta^{\nu}{}_{\alpha}u^{\mu}u_{\lambda} + \epsilon^{\lambda\alpha\mu\nu}\beta^{\rho}{}_{\alpha}u^{\sigma}u_{\lambda} - \epsilon^{\lambda\alpha\mu\nu}\beta^{\sigma}{}_{\alpha}u^{\rho}u_{\lambda}\right).
\end{aligned} \tag{B.33}$$

If $\beta^{\mu}{}_{\nu} = 0$, we obtain the result (B.22) for the uniaxial anisotropic medium. Additionally, if $\Delta\varepsilon = 0$ and $\Delta(\mu^{-1}) = 0$, we get (3.87) for the isotropic medium. Furthermore, if $\varepsilon = \mu = n = 1$, we just obtain the constitutive tensor for vacuum (3.58).

Appendix C.

General Lagrange-Noether formalism for open and closed systems

“Mamá tenía una manera de explicar las cosas que yo siempre entendía...”

Forrest Gump,
personaje de película.

A very powerful way to derive the energy-momentum tensors and balance equations of non-dissipative open or closed systems is the well-known Lagrange-Noether formalism. This approach allows us to study in chapter 4 the relationship between the conserved electromagnetic quantities, i.e. energy, momentum and angular momentum of the electromagnetic field, and the symmetries of the fixed material medium in which the field propagates. In chapter 5 we will also use this approach to derive an explicit expression for the total energy-momentum tensor of the closed system formed by electromagnetic field interacting with the medium, modeled as an isotropic dielectric relativistic fluid.

Therefore, this appendix is devoted to shortly state the fundamental results and conventions that will be used in our Lagrange and symmetry analysis.

C.1. Euler-Lagrange equations, canonical energy-momentum tensor, angular momentum and balance equations

Assume we have a system of N dynamical fields, denoted *collectively* by $\Phi^A_{(I)}$, where $I = 1, ..., N$, and A is a generic way to represent their indices, depending on the tensor rank of each field. The dynamics of the system is described by field equations of the form $f\left(\Phi^A_{(I)}, \partial_\mu \Phi^A_{(I)}, \partial_\mu \partial_\nu \Phi^A_{(I)}\right) = 0$, where f is a functional which relate the time and spatial derivatives of the dynamical fields $\Phi^A_{(I)}$. Examples of these field equations are, for instance, Maxwell's equations in the case of the electromagnetic field, Einstein's equations in the case of the gravitational field, Newton's equations in the case of non-relativistic classical mechanics, Dirac's equation in the case of relativistic quantum mechanics, etc.

The condition to incorporate a system in the Lagrangian formalism is that their field equations or *"equations of motion"* must be derived as *Euler-Lagrange equations* of a function $\mathcal{L}$ called the *Lagrangian density* of the system and which are explicitly given by

$$\frac{\partial \mathcal{L}}{\partial \Phi^A_{(I)}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^A_{(I)})} \right) = 0, \qquad I = 1, ..., N. \tag{C.1}$$

The Lagrangian density $\mathcal{L}$ is a function of the dynamical fields, their first derivatives and eventually of other fields $\Psi^A_{(J)}$, in the form

$$\boxed{\mathcal{L} = \mathcal{L}\left(\Phi^A_{(I)}(x), \partial_\mu \Phi^A_{(I)}(x), \Psi^A_{(J)}(x)\right),} \tag{C.2}$$

where $\Psi^B_{(J)}$, $J = 1, ..., M$, are M *given* fields, known as *external fields*, which alter the dynamics of the system, but do not get affected by it. For instance, an external field $\Psi^A_{(J)}$ can describe an external force acting on a system of particles, an external electric field accelerating an electron, etc. If the system has one or more external fields, it is said to be an *open* system and oppositely, a system without external fields is said to be *closed*.

The left hand side of (C.1) is usually known as the *variational derivative* of $\mathcal{L}$ with respect to the field $\Phi^A_{(I)}$ and therefore the Euler-Lagrange equations of the system can be written in more compact form as

$$\frac{\delta \mathcal{L}}{\delta \Phi^A_{(I)}} = 0, \qquad I = 1, ..., N. \tag{C.3}$$

Without losing generality, we can *always* assume that the Lagrangian density $\mathcal{L}$ does not explicitly depend on the coordinates x^μ as in (C.2), provided we introduce an appropriate number of non-dynamical external fields $\Psi^A_{(J)}(x)$, which additionally can be always defined so that their derivatives do not explicitly appear in (C.2). As a result, all the Lagrangian densities considered will be invariant under infinitesimal spacetime translations of the form

$$x^\mu \to \bar{x}^\mu = x^\mu - \varepsilon^\mu + O(\varepsilon^2), \tag{C.4}$$

where ε^μ are the 4 infinitesimal parameters of the translation transformation. Assuming this invariance in $\mathcal{L}$, we can always derive the energy-momentum balance equation as an *identity* [101]:

$$\partial_\nu T_\mu{}^\nu \equiv -\frac{\delta \mathcal{L}}{\delta \Phi^A_{(I)}} \partial_\mu \Phi^A_{(I)} - \frac{\partial \mathcal{L}}{\partial \Psi^A_{(J)}} \partial_\mu \Psi^A_{(J)}, \tag{C.5}$$

where the *canonical energy-momentum tensor* $T_\mu{}^\nu$ is defined as

$$\boxed{T_\mu{}^\nu := \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi^A_{(I)})} \partial_\mu \Phi^A_{(I)} - \delta^\nu_\mu \mathcal{L}.} \tag{C.6}$$

In all practical situations, the equations of motion (C.3) will be indeed satisfied and therefore the energy-momentum balance equation can be simply written as,

$$\boxed{\partial_\nu T_\mu{}^\nu \doteq -\frac{\partial \mathcal{L}}{\partial \Psi^A_{(J)}} \partial_\mu \Psi^A_{(J)},} \tag{C.7}$$

where we used the sign "$\doteq$" instead of just "$=$" in order to remember that the equality is valid only "on-shell", i.e. when the equations of motion (C.3) are satisfied.

Additionally, in a relativistic theory the Lagrangian density $\mathcal{L}$ is invariant under infinitesimal Lorentz transformations, i.e. spatial rotations and boosts of the form

$$x^\mu \to \bar{x}^\mu = x^\mu + \lambda^\mu{}_\nu x^\nu + O(\lambda^2), \tag{C.8}$$

where $\lambda_{\mu\nu}$ are the 6 infinitesimal parameters of the Lorentz transformation, which satisfy

$$\lambda_{\mu\nu} = -\lambda_{\nu\mu}, \qquad \lambda_{\mu\nu} := \eta_{\mu\rho}\lambda^\rho{}_\nu. \tag{C.9}$$

With this assumption we can derive, after some algebra, the *angular momentum identity* [101], which reads

$$\partial_\mu S_{\rho\sigma}{}^\mu - 2T_{[\rho\sigma]} \equiv -\frac{\delta \mathcal{L}}{\delta \Phi^A_{(I)}} (s_{\rho\sigma})^A{}_B \Phi^B_{(I)} - \frac{\partial \mathcal{L}}{\partial \Psi^A_{(J)}} (s_{\rho\sigma})^A{}_B \Psi^B_{(J)}, \tag{C.10}$$

where $(s_{\rho\sigma})^A{}_B$ are the *Lorentz generators* for the fields $\Phi^A_{(I)}$, which are defined such that

$$\delta \Phi^A_{(I)} = \frac{1}{2}\lambda^{\rho\sigma}(s_{\rho\sigma})^A{}_B \Phi^B_{(I)} + O(\lambda^2), \tag{C.11}$$

where $\delta\Phi^A_{(I)}$ represent the infinitesimal change of the field $\Phi^A_{(I)}$ under the Lorentz transformation. The same definitions are valid for the Lorentz generators $(s_{\rho\sigma})^A{}_B$ for the external fields. The *spin current density* $S_{\rho\sigma}{}^\mu$, which depend only on the dynamical fields, is defined as

$$\boxed{S_{\rho\sigma}{}^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^A)} (s_{\rho\sigma})^A{}_B \Phi^B_{(I)}.} \tag{C.12}$$

Notice that the Lorentz generators and therefore the spin current density depend on the specific tensor rank of each field $\Phi^A_{(I)}$ or $\Psi^A_{(J)}$, because they transform in different ways under Lorentz transformations. In practice we will use the angular momentum relation (C.10) "on-shell", and therefore we have

$$\boxed{\partial_\mu S_{\rho\sigma}{}^\mu - 2T_{[\rho\sigma]} \doteq -\frac{\partial \mathcal{L}}{\partial \Psi^A_{(J)}} (s_{\rho\sigma})^A{}_B \Psi^B_{(J)}.} \tag{C.13}$$

It is important to stress here that in the case that the theory is non-relativistic, the Lagrangian density $\mathcal{L}$ will not be invariant under Lorentz transformations, specifically boosts, and hence the identity (C.13) will not hold for all components ρ, σ. However, a non-relativistic theory still has spatial rotation invariance and in that case the spatial components $\rho, \sigma = i, j = 1, 2, 3$ of the relation (C.13) will still be applicable.

In order to include explicitly the orbital angular momentum density of the system in (C.13), we notice that

$$T_{\rho\sigma} \equiv \partial_\mu (x_\sigma T_\rho{}^\mu) - x_\sigma \partial_\mu T_\rho{}^\mu, \tag{C.14}$$

and replace this together with (C.7) into (C.13), we get the angular momentum balance equation "on-shell":

$$\boxed{\partial_\mu J_{\rho\sigma}{}^\mu \doteq -\frac{\partial \mathcal{L}}{\partial \Psi^A_{(J)}} (j_{\rho\sigma})^A{}_B \Psi^B_{(J)},} \tag{C.15}$$

where $J_{\rho\sigma}{}^\mu$ is the canonical *total angular 4-momentum density* of the system, defined as

$$\boxed{J_{\rho\sigma}{}^\mu := S_{\rho\sigma}{}^\mu + L_{\rho\sigma}{}^\mu,} \tag{C.16}$$

with $L_{\rho\sigma}{}^\mu$ the canonical *orbital angular 4-momentum density* of the system

$$\boxed{L_{\rho\sigma}{}^\mu := x_\rho T_\sigma{}^\mu - x_\sigma T_\rho{}^\mu.} \tag{C.17}$$

The total angular 4-momentum operators $(j_{\rho\sigma})^A{}_B$ for the fields $\Psi^A_{(j)}$ are defined analogous to (C.16) as

$$(j_{\rho\sigma})^A{}_B := (s_{\rho\sigma})^A{}_B + (l_{\rho\sigma})^A{}_B, \tag{C.18}$$

where $(l_{\rho\sigma})^A{}_B$ is the usual *orbital angular 4-momentum operator*, given by

$$(l_{\rho\sigma})^A{}_B := \delta^A{}_B (x_\rho \partial_\sigma - x_\sigma \partial_\rho). \tag{C.19}$$

C.2. The Belinfante tensor

For a better understanding of the dynamics and symmetries of the system, it is usually better not to work directly with the canonical energy-momentum tensor $T_\mu{}^\nu$ in (C.6), but with another *physically equivalent* energy-momentum tensor, whose balance equations do not show the spin density $S_{\rho\sigma}{}^\mu$ explicitly. Therefore, given the canonical energy-momentum tensor $T_\mu{}^\nu$ of the system, together with the spin current density $S_{\rho\sigma}{}^\mu$, we can always define the *Belinfante energy-momentum tensor* $\Theta_\mu{}^\nu$, as follows

$$\boxed{\Theta_\mu{}^\nu := T_\mu{}^\nu + \frac{1}{2}\partial_\lambda (S^{\nu\lambda}{}_\mu + S_\mu{}^{\lambda\nu} - S_\mu{}^{\nu\lambda}),} \tag{C.20}$$

which, by construction, satisfies

$$\partial_\nu \Theta_\mu{}^\nu \equiv \partial_\nu T_\mu{}^\nu, \tag{C.21}$$

$$2\Theta_{[\mu\nu]} \equiv 2T_{[\mu\nu]} - \partial_\lambda S_{\mu\nu}{}^\lambda, \tag{C.22}$$

$$\partial_\mu l_{\rho\sigma}{}^\mu \equiv \partial_\mu (L_{\rho\sigma}{}^\mu + S_{\rho\sigma}{}^\mu). \tag{C.23}$$

We see that $T_\mu{}^\nu$ and $\Theta_\mu{}^\nu$ have equal 4-divergences and therefore they will be conserved under the same conditions. Also the total angular momentum density of the system turns out to be completely contained in the *Belinfante orbital angular momentum density* $l_{\rho\sigma}{}^\mu$, defined in the same way as (C.17), by

$$\boxed{l_{\rho\sigma}{}^\mu := x_\rho \Theta_\sigma{}^\mu - x_\sigma \Theta_\rho{}^\mu.} \tag{C.24}$$

Finally, replacing (C.21)-(C.23) in (C.7),(C.13) and (C.15), we obtain the Belinfante balance equations:

$$\partial_\nu \Theta_\mu{}^\nu \doteq -\frac{\partial \mathcal{L}}{\partial \Psi^A_{(J)}} \partial_\mu \Psi^A_{(J)}, \tag{C.25}$$

$$\partial_\mu l_{\rho\sigma}{}^\mu \doteq -\frac{\partial \mathcal{L}}{\partial \Psi^A_{(J)}} (j_{\rho\sigma})^A{}_B \Psi^B_{(J)}, \tag{C.26}$$

$$2\Theta_{[\rho\sigma]} \doteq +\frac{\partial \mathcal{L}}{\partial \Psi^A_{(J)}} (s_{\rho\sigma})^A{}_B \Psi^B_{(J)}, \tag{C.27}$$

where the spin density is explicitly absorbed.

C.3. Open and closed systems

If the system is closed, there are no external fields present in $\mathcal{L}$ and hence the right hand sides of the canonical balance equations (C.7), (C.13) and (C.15), as well as the r.h.s. of the Belinfante balance equations (C.25)-(C.27), vanish. As a consequence, the canonical energy-momentum tensor and the total angular momentum density $J_{\rho\sigma}{}^\mu = S_{\rho\sigma}{}^\mu + L_{\rho\sigma}{}^\mu$ of *any* closed system are *always* conserved:

$$\partial_\nu T_\mu{}^\nu \doteq 0, \tag{C.28}$$

$$\partial_\mu J_{\rho\sigma}{}^\mu \doteq 0, \tag{C.29}$$

but the canonical energy-momentum tensor of the closed system is not symmetric symmetric in general,

$$2T_{[\rho\sigma]} = \partial_\mu S_{\rho\sigma}{}^\mu \neq 0. \tag{C.30}$$

There are two particular cases when the cononical tensor of a closes system is symmetric. If all the dynamical fields are scalars and therefore the spin is trivial or if the canonical orbital angular momentum density $L_{\rho\sigma}{}^\mu$ also conserved in addition to $J_{\rho\sigma}{}^\mu$, i.e. separately conserved:

$$2T_{[\rho\sigma]} = \partial_\mu S_{\rho\sigma}{}^\mu \tag{C.31}$$

$$= \partial_\mu J_{\rho\sigma}{}^\mu - \partial_\mu L_{\rho\sigma}{}^\mu \tag{C.32}$$

$$\overset{!}{\doteq} \; 0. \tag{C.33}$$

The Belinfante energy-momentum tensor $\Theta_\mu{}^\nu$ and its orbital angular momentum density $l_{\rho\sigma}{}^\mu$ are also conserved,

$$\partial_\nu \Theta_\mu{}^\nu \doteq 0, \tag{C.34}$$
$$\partial_\mu l_{\rho\sigma}{}^\mu \doteq 0, \tag{C.35}$$

but in opposition to $T_\mu{}^\nu$, $\Theta_\mu{}^\nu$ is *always symmetric* when the system is closed:

$$\Theta_{[\rho\sigma]} \doteq 0, \tag{C.36}$$

and here lies its greatest advantage.

When the system is open, however, we have to keep the corresponding terms in the right hand sides of all the balance equations and therefore the energy-momentum tensors $T_\mu{}^\nu$ and $\Theta_\mu{}^\nu$ will *not be symmetric nor conserved in general*. These non-vanishing terms describe forces and torques which result from the interaction of the system with the external fields and hence the asymmetry and non-conservation of the energy-momentum tensors in open systems is completely necessarily for the correct and consistent description of the system within the Lagrange-Noether formalism.

C.4. Conserved quantities in open systems and symmetries of the external fields

Even if a system is open, there are certain cases where we can also find *conserved quantities* like energy, momentum and angular momentum. These conserved quantities are related to the *symmetries* which the specific external fields $\Psi^A_{(J)}$ [1] present. It is said that some field Ψ^A *possesses* certain symmetry, if we apply a specific transformation on it and the resulting field turns out to be geometrically the same, i.e. after the transformation we cannot physically differentiate between the transformed and the original fields. In particular, we will study transformations of the coordinate systems assigned to inertial observers, which are spacetime translations (time evolution and space translations) and Lorentz transformations (boosts between inertial observers and spatial rotations) and we will see how the invariance of the external fields under these transformations leads to conserved energy, momentum and angular momentum of the system.

C.4.1. Spacetime translation transformation

A spacetime translation is a *displacement* of the center of a coordinate system by a constant amount a^μ. If the coordinates of an arbitrary *geometrical* point P of spacetime are given by $x^\mu(P)$ in the original system, then the coordinates $\bar{x}^\mu(P)$ of the same geometrical point, but in the translated system, will be given by

$$\boxed{x^\mu(P) \to \bar{x}^\mu(P) = x^\mu(P) - a^\mu,} \tag{C.37}$$

[1] In order to simplify the notation, when there is no possibility of confusion, we will avoid to include the subindex $_{(J)}$ in the external fields, tacitly assuming that Ψ^A represent any of the M different external fields.

as schematically shown in figure C.4.1.

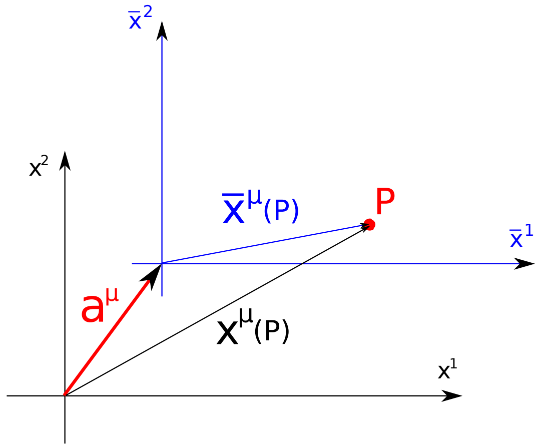

Figure C.1.: 2-D translation transformation.

The values of a field $\Psi^A(x)$ does not change under a translation transformation (C.37),

$$\Psi^A\left(x^\mu(P)\right) \to \bar{\Psi}^A\left(\bar{x}^\mu(P)\right) = \Psi(x^\mu(P)), \tag{C.38}$$

but its coordinates $\bar{x}^\mu$ do and hence

$$\boxed{\bar{\Psi}(\bar{x}^\mu) = \Psi^A\left(\bar{x}^\mu + a^\mu\right),} \tag{C.39}$$

where $\bar{\Psi}^A(\bar{x})$ is the same field $\Psi^A(x)$, but as seen in the translated coordinate system.

Suppose now that the displacement of the center of the coordinate system is infinitesimally small, i.e. the centers of the original and translated systems can be as close as we want, but not exactly in the same place. This type of transformation is called an *infinitesimal* spacetime translation and can be obtained assuming that the displacement parameters of the transformation a^μ are now infinitesimal in value

$$a^\mu = \varepsilon^\mu + (\varepsilon^2), \tag{C.40}$$

and therefore

$$\boxed{x^\mu(P) \to \bar{x}^\mu(P) = x^\mu(P) - \varepsilon^\mu + O(\varepsilon^2).} \tag{C.41}$$

Fields invariant under spacetime translations

A field $\Psi^A(x)$ is said to be *invariant under spacetime translations in the direction* ε^μ, when the field $\bar{\Psi}^A(\bar{x})$ in a translated coordinate system in the direction ε^μ, "looks" the same as in the original coordinate system. In order to mathematically express this condition, consider an arbitrary point P of spacetime with coordinates $x^\mu(P)$ in the original system and another geometrically different point Q, which in the infinitesimally translated coordinate system has the same *numerical* values of its coordinates $\bar{x}^\mu(Q)$, i.e.

$$\bar{x}^\mu(Q) = x^\mu(P). \tag{C.42}$$

Then, as shown in figure B.4.2, the field $\Psi^A(x)$ will be geometrically invariant under the a translation transformation in the direction ε^μ, when it satisfies

$$\bar{\Psi}^A(\bar{x}(Q)) = \Psi^A(x(P)). \tag{C.43}$$

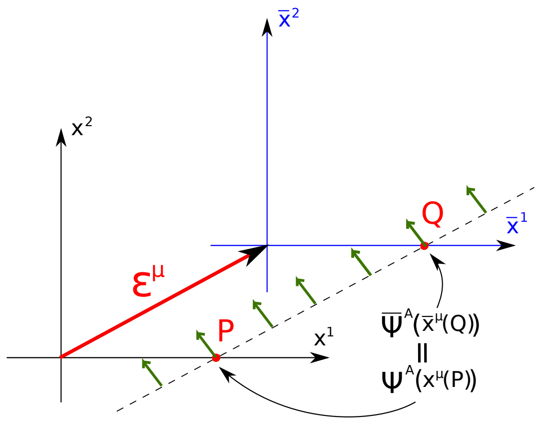

Figure C.2.: A field with translation invariance in direction ε^μ.

If we now insert (C.39) in the invariance condition (C.43) for the case of an infinitesimal transformation (C.41) and use the fact that P and Q have the same coordinates in both coordinate systems (C.42), the invariance condition simplifies to

$$\Psi^A(\bar{x}^\mu(P) + \varepsilon^\mu) = \Psi^A(x^\mu(P)). \tag{C.44}$$

Let us expand the left hand side of (C.44) in Taylor series to first order in ε^μ, since it is an infinitesimal parameter, and if we cancel the contributions of $\Psi^A(x^\mu(P))$ at both sides, we finally arrive to the *differential condition* which an invariant field under spacetime translations in the direction ε^μ must satisfy:

$$\boxed{\varepsilon^\mu \partial_\mu \Psi^A(x) = 0,} \tag{C.45}$$

i.e. the directional derivative of the field $(\hat{n}^\mu \partial_\mu)\Psi^A$ in the direction defined by $\varepsilon^\mu = \varepsilon \hat{n}^\mu$ must vanish. A field satisfying the condition (C.45) it is also said to be *homogeneous* in the direction ε^μ. When the field is "homogeneous in time", it is really a time-independent field.

C.4.2. Energy-momentum conserved quantities in an open system

The condition (C.45) of a field to be invariant under spacetime translations is exactly what we need so that the right hand side of the energy-momentum balance equations (C.7) and (C.25) can be zero. Therefore, let us multiply (C.25) by some constant parameter ε^μ, to obtain

$$\partial_\nu(\varepsilon^\mu \Theta_\mu{}^\nu) \doteq -\frac{\partial \mathcal{L}}{\partial \Psi^A_{(J)}} \partial_\mu(\varepsilon^\mu \Psi^A_{(J)}) \tag{C.46}$$

As a consequence, if *all the external fields* $\Psi^A_{(J)}$ are *invariant under spacetime translations* in the direction defined by ε^μ, i.e.

$$\boxed{\partial_\mu(\varepsilon^\mu \Psi^A_{(J)}) = 0, \qquad \forall J = 1, ..., M,} \tag{C.47}$$

then the canonical and the Belinfante energy-momentum tensors of the open system will be conserved in the direction ε^μ, even though the external fields are not trivial:

$$\boxed{\partial_\nu(\varepsilon^\mu \Theta_\mu{}^\nu) = 0.} \tag{C.48}$$

We will obtain different continuity equations and conserved quantities, depending on the direction ε^μ in which the external fields $\Psi^A_{(J)}$ are homogeneous. There can be at most 4 independent direction in which energy-momentum conserved quantities can be found, depending on the symmetries presented by the external fields.

The components of the energy-momentum tensors $T_\mu{}^\nu$ and $\Theta_\mu{}^\nu$ are always identified as

$$\Theta_\mu{}^\nu = \begin{pmatrix} \mathcal{U} & S^i/c \\ -c\pi_i & -p_i{}^j \end{pmatrix}, \tag{C.49}$$

where $\mathcal{U}$ is the *energy density* of the field, S^i the *energy flux density*, π_i the *momentum density* of the field and $p_i{}^j$ the *momentum flux density*. Therefore, in the case that the external fields are invariant under temporal translations, i.e. if they do not depend explicitly on time, then the 4-vector $\varepsilon^\mu_{(0)} := c\varepsilon\delta^\mu_0$ will lead to a continuity equation for energy:

$$\partial_\nu(\varepsilon^\mu_0 \Theta_\mu{}^\nu) = 0 \quad \Rightarrow \quad \partial_\nu \Theta_0{}^\nu = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{U}}{\partial t} + \partial_i S^i = 0. \tag{C.50}$$

On the other hand, if the external fields are spatially homogeneous in one or more directions of the coordinates axes x, y or z, the linearly independent 4-vectors $\varepsilon^\mu_{(i)} := \varepsilon\delta^\mu_i$, with $i = 1, 2, 3$, lead to the balance equation for the component i of the momentum:

$$\partial_\nu(\varepsilon^\mu_i \Theta_\mu{}^\nu) = 0 \quad \Rightarrow \quad \partial_\nu \Theta_i{}^\nu = 0 \quad \Rightarrow \quad \frac{\partial \pi_i}{\partial t} + \partial_j p_i{}^j = 0. \tag{C.51}$$

C.4.3. Lorentz transformations

Another very important group of transformations in physics, which appear in all relativistic theories are the *Lorentz transformations*. For details about the physical interpretation of a Lorentz transformation, please see the beginning of section 3.2, since here we summarize some of its geometrical properties. A Lorentz transformation can be geometrically understood as a *"spacetime rotation"* of the axes of a coordinate system by constant "angles" contained in the matrix $\Lambda^\mu{}_\nu$, where the indexes μ, ν identify the axis around which the rotation in spacetime is made. Therefore, if the coordinates of an arbitrary geometrical point P of spacetime are given by $x^\mu(P)$ in the original coordinate system, in the "rotated" system the coordinates of the same point will be given by

$$\boxed{x^\mu(P) \to \bar{x}^\mu(P) = \Lambda^\mu{}_\nu x^\nu(P).} \tag{C.52}$$

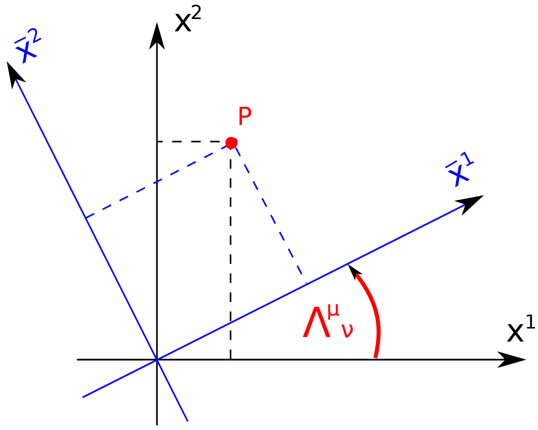

Figure C.3.: 2-D Lorentz transformation as a rotation of the coordinate systems in spacetime

A geometrical picture of this transformation is presented in figure C.4.3. In order for this spacetime rotation (C.52) to preserve the four-dimensional "length" of 4-vectors, it must satisfy the condition

$$\eta_{\mu\nu} = \Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\eta_{\rho\sigma}. \tag{C.53}$$

This is analogous to the case of a three-dimensional *rotation matrix* $R^{i}{}_{j}$, which must satisfy $\delta_{ij} = R^{k}{}_{i}R^{l}{}_{j}\delta_{kl}$, in order to preserve the length of three-dimensional vectors. In other words, a Lorentz transformation is a generalization of the usual spatial three-dimensional rotations, but now in a four-dimensional spacetime with the Minkowski metric $\eta_{\mu\nu} := \mathrm{diag}(1,-1,-1,-1)$. In fact, for a purely spatial Lorentz transformation, the components $\Lambda^{i}{}_{j}$ coincide with a usual rotation matrix $R^{i}{}_{j}$ around the axes x, y, z or combinations of them, whereas the "time-space components" $\Lambda^{0}{}_{i}$ are interpreted as a change between different inertial observers, also known as *boosts*.

The Lorentz transformations have the particularity that a field $\Psi^A(x)$ will in general change its values in different ways, depending on its *tensor rank*. For instance, a *scalar* field $\phi(x)$ is defined such that it does not change its value under a Lorentz transformation: $\bar{\phi}(\bar{x}) = \phi(x)$, whereas a *contravariant 4-vector* field $A^\mu(x)$ is defined such that their components change in the same way as the coordinates: $\bar{A}^\mu(\bar{x}) = \Lambda^{\mu}{}_{\nu}A^\nu(x)$. A tensor field of rank $(1,1)$ $\Psi^A \to A^{\mu}{}_{\nu}$, the energy-momentum tensor for example, will change as $\bar{A}^{\mu}{}_{\nu}(\bar{x}) = \Lambda^{\mu}{}_{\rho}(\Lambda^{-1})^{\sigma}{}_{\nu}A^{\rho}{}_{\sigma}$, under a Lorentz transformation, where $(\Lambda^{-1})^{\mu}{}_{\nu}$ is the inverse matrix of $\Lambda^{\mu}{}_{\nu}$. In general, for a tensor field $\Psi^A(x)$ of any rank, its values will change in the form

$$\Psi^A(x(P)) \to \bar{\Psi}^A(\bar{x}(P)) = S^{A}{}_{B}(\Lambda)\Psi^B(x(P)), \tag{C.54}$$

where $S^{A}{}_{B}(\Lambda)$ is a linear operator, function of $\Lambda^{\mu}{}_{\nu}$, whose form will depend on the field Ψ^A. Transforming also the coordinates in the right hand side of (C.54), the field in the rotated coordinate system can be finally written as

$$\boxed{\bar{\Psi}^A(\bar{x}^\mu) = S^{A}{}_{B}(\Lambda)\Psi^B\left((\Lambda^{-1})^{\mu}{}_{\nu}\bar{x}^\nu\right).} \tag{C.55}$$

Analogously as for the spacetime translation (C.41), there can also be an *infinitesimal Lorentz transformation*, if the matrix $\Lambda^\mu{}_\nu$ is supposed to be infinitesimally close to the identity, i.e.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \lambda^\mu{}_\nu + O(\lambda^2), \tag{C.56}$$

where $\lambda^\mu{}_\nu$ are the infinitesimal parameters, which relate the original and the infinitesimally transformed coordinate systems. If we replace (C.56) into the condition (C.53), we see that the parameters $\lambda^\mu{}_\nu$ must be necessarily antisymmetric in order to describe a Lorentz transformation:

$$\lambda_{\mu\nu} = -\lambda_{\nu\mu}, \qquad \text{with} \;\; \lambda_{\mu\nu} = \eta_{\mu\rho}\lambda^\rho{}_\nu. \tag{C.57}$$

Therefore, using (C.56) in (C.52) and (C.54), we obtain how the coordinates and the fields change under an infinitesimal Lorentz transformation:

$$x^\mu \to \bar{x}^\mu = x^\mu + \lambda^\mu{}_\nu x^\nu + O(\lambda^2), \tag{C.58}$$

$$\Psi^A(x^\mu) \to \bar{\Psi}^A(\bar{x}^\mu) = \Psi^A(x^\mu) + \frac{1}{2}\lambda^{\rho\sigma}(s_{\rho\sigma})^A{}_B\Psi^B(x^\mu) + O(\lambda^2), \tag{C.59}$$

where $(s_{\rho\sigma})^A{}_B$ are the so-called *Lorentz generators*, which have to be determined for each field $\Psi^A(x)$, depending on its tensor rank. For instance, the Lorentz generator of a scalar field $\phi(x)$ is zero $(s_{\rho\sigma}) = 0$ and the Lorentz generator of a contravariant 4-vector A^μ is $(s_{\rho\sigma})^\mu{}_\nu = \delta^\mu{}_\rho\eta_{\nu\sigma} - \delta^\mu{}_\sigma\eta_{\nu\rho}$.

The infinitesimal variation of a field under a transformation is defined as

$$\delta\Psi(x) := \bar{\Psi}(\bar{x}) - \Psi(x), \tag{C.60}$$

and therefore, for the case of the infinitesimal Lorentz transformation, we have

$$\delta\Psi(x^\mu) = \frac{1}{2}\lambda^{\rho\sigma}(s_{\rho\sigma})^A{}_B\Psi^B(x^\mu), \tag{C.61}$$

and $\delta x^\mu = \lambda^\mu{}_\nu x^\nu$ for the coordinates. For the case of an infinitesimal transformation, using (C.38) and (C.41), we see that the corresponding variations are $\delta_T\Psi(x) = 0$ and $\delta_T x^\mu = -\varepsilon^\mu$.

Fields invariant under Lorentz transformations

In the same way as for the spacetime translations, a field $\Psi^A(x)$ is said to be *invariant under a given Lorentz transformation* around the axis, determined by $\lambda^\mu{}_\nu$, when the field in the infinitesimally rotated coordinate system "looks" geometrically the same as in the original system, i.e. the field must satisfy

$$\bar{\Psi}(\bar{x}^\mu(Q)) = \Psi(x^\mu(P)), \tag{C.62}$$

provided P and Q are two geometrically different points in spacetime, but which have the same numerical coordinates in both systems:

$$\bar{x}^\mu(Q) = x^\mu(P). \tag{C.63}$$

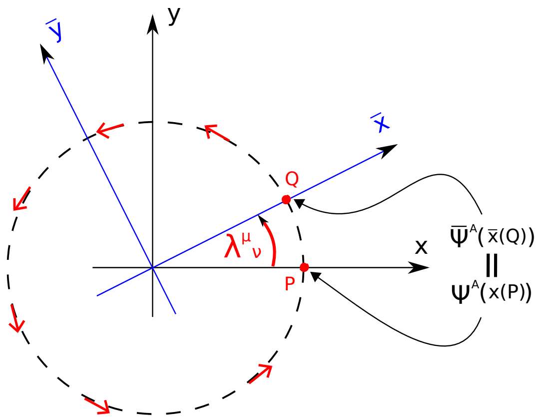

Figure C.4.: A field with rotation invariance.

A schematic picture of a field having rotation invariance is shown in figure B.4.4.

Therefore, if we insert (C.59) in the left hand side of (C.62) we obtain, to first order in $\lambda^{\rho\sigma}$:

$$\Psi^A(x^\mu(Q)) + \frac{1}{2}\lambda^{\rho\sigma}(s_{\rho\sigma})^A{}_B\Psi^B(x^\mu(Q)) + O(\lambda^2) = \Psi^A(x^\mu(P)). \tag{C.64}$$

On the other hand, if we multiply on both sides of (C.63) by an inverse Lorentz matrix $(\Lambda^{-1})^\rho{}_\mu$, and use (C.52) for the point Q, we obtain

$$x^\rho(Q) = (\Lambda^{-1})^\rho{}_\mu x^\mu(P), \tag{C.65}$$

and infinitesimally,

$$x^\mu(Q) = x^\mu(P) - \lambda^\mu{}_\nu x^\nu(P) + O(\lambda^2). \tag{C.66}$$

Then, if we insert (C.66) in the left hand side of (C.64), expand it with a Taylor series keeping terms up to first order in $\lambda^{\rho\sigma}$, and cancel the contribution of $\Psi^A(x(P))$ on both sides, the invariance condition reduces to

$$-\lambda^\mu{}_\nu x^\nu(P)\partial_\mu\Psi^A(x(P)) + \frac{1}{2}\lambda^{\rho\sigma}(s_{\rho\sigma})^A{}_B\Psi^B(x(P)) + O(\lambda^2) = 0. \tag{C.67}$$

Finally, if we antisymmetrize the first term in (C.67) and take out the label P since we can tacitly assume that the relation applies for any point P with coordinates x^μ, we obtain the *differential condition* that a field $\Psi^A(x)$ must satisfy in order to be geometrically invariant under a Lorentz transformation around the axis determined by $\lambda^{\rho\sigma}$,

$$\lambda^{\rho\sigma}\left[\delta^A{}_B(x_\rho\partial_\sigma - x_\sigma\partial_\rho) + (s_{\rho\sigma})^A{}_B\right]\Psi^B(x) = 0 \tag{C.68}$$

or, in a more compact manner,

$$\boxed{\lambda^{\rho\sigma}(j_{\rho\sigma})^A{}_B\Psi^B(x) = 0,} \tag{C.69}$$

where $j^A_{\rho\sigma B}$ is the total angular momentum operator already defined in (C.18) and (C.19).

In simpler terms, a field $\Psi^A(x)$ is invariant under a Lorentz transformation around the axis determined by $\lambda^{\rho\sigma}$, if the total angular momentum operator $(j_{\rho\sigma})^A{}_B$ applied to it turns to be zero. A field which satisfies the condition (C.69) for all purely spatial components simultaneously, i.e. $\lambda^{ij}(j_{ij})^A{}_B\Psi^B(x) = 0, \quad \forall i, j = 1, 2, 3$, is said to be *isotropic* in space, i.e. the field does not have any preferred direction.

To make an analogy of (C.69) with the spacetime translation condition (C.45), sometimes the *4-momentum operator* $(p_\mu)^A{}_B$ is defined as

$$(p_\mu)^A{}_B := \delta^A{}_B\partial_\mu, \tag{C.70}$$

so that a field $\Psi^A(x)$ which presents spacetime translational invariance in direction ε^μ, must satisfy that the 4-momentum operator $(p_\mu)^A{}_B$ applied to it turns out to be zero:

$$\varepsilon^\mu(p_\mu)^A{}_B\Psi^B(x) = 0. \tag{C.71}$$

Notice that the definition (C.70) is also consistent with the definition of the orbital angular 4-momentum in (4.90). In fact, if we replace (C.70) in the definition (C.19) of the angular momentum operator $(l_{\rho\sigma})^A{}_B$, we have

$$(l_{\rho\sigma})^A{}_B = \delta^A{}_B(x_\rho\partial_\sigma - x_\sigma\partial_\rho) \tag{C.72}$$

$$= x_\rho(p_\sigma)^A{}_B - x_\sigma(p_\rho)^A{}_B. \tag{C.73}$$

C.4.4. Four-dimensional angular momentum conserved quantities in an open system

Let us multiply the angular momentum balance equations (C.26) (the same can be done with (C.15)) by the infinitesimal parameters $\lambda^{\rho\sigma}$ of a Lorentz transformation:

$$(\lambda^{\rho\sigma}\partial_\mu)l_{\rho\sigma}{}^\mu \doteq -\frac{\partial\mathcal{L}}{\partial\Psi^A_{(J)}}\left[\lambda^{\rho\sigma}(j_{\rho\sigma})^A{}_B\Psi^B_{(J)}\right]. \tag{C.74}$$

By inspecting (C.74), we see that if *all external fields* $\Psi^A(x)$ are *invariant under a Lorentz transformation around the axis determined by* $\lambda^{\rho\sigma}$, i.e. if the total angular momentum operator $\lambda^{\rho\sigma}(j_{\rho\sigma})^A{}_B$ acting on all external fields vanish,

$$\boxed{\lambda^{\rho\sigma}(j_{\rho\sigma})^A{}_B\Psi^B_{(J)} = 0, \qquad \forall J = 1, ..., M,} \tag{C.75}$$

then the Belinfante orbital angular momentum density $l_{\rho\sigma}{}^\mu$ (as well as the total angular momentum density $J_{\rho\sigma}{}^\mu$), will be conserved around the axis $\lambda^{\rho\sigma}$:

$$\boxed{(\lambda^{\rho\sigma}\partial_\mu)l_{\rho\sigma}{}^\mu = 0.} \tag{C.76}$$

We will obtain different continuity equations and conserved quantities, depending on the axis $\lambda^{\rho\sigma}$ around which all the external fields $\Psi^A_{(J)}$ are invariant. There can be at most 6 independent directions in which angular 4-momentum conserved quantities can be found, depending on the symmetries presented by the external fields.

The spatial components ${l_{ij}}^\mu$ are identified as

$$l_i = -\frac{1}{2c}\epsilon^{mnk}\eta_{im}{l_{nk}}^0, \tag{C.77}$$

$$K_i^j = -\frac{1}{2}\epsilon^{mnk}\eta_{im}{l_{nk}}^j, \tag{C.78}$$

where $l_i := \epsilon_{ijk}\eta^{km}x^j\pi_m$ is the orbital angular momentum density of the system and $K_i^j := \epsilon_{imk}\eta^{kn}x^m {p_n}^j$ the orbital angular momentum flux density. Therefore, if the external fields are invariant under spatial rotations around one or more given spatial axes (x, y or z), one of the 3 independent parameters $\lambda^{\rho\sigma}_{(12)}$, $\lambda^{\rho\sigma}_{(13)}$ and $\lambda^{\rho\sigma}_{(23)}$, defined as $\lambda^{\rho\sigma}_{(ij)} := {\delta^\rho}_i {\delta^\sigma}_j$, will satisfy (C.75) and lead to a continuity equation for the components x, y and/or z of the orbital angular momentum. In fact, if we replace $\lambda^{\rho\sigma}_{(ij)}$ in (C.76), multiply it by $-\epsilon^{lij}\eta_{ml}/2$ and use the identifications (C.77)-(C.78), we obtain

$$\frac{\partial l_m}{\partial t} + \partial_n {K_m}^n = 0, \tag{C.79}$$

for the component l_i of the orbital angular momentum of the system. The parameters $\lambda^{\rho\sigma}_{(12)}$ correspond to infinitesimal spatial rotations around the z axis or the xy-plane, while the parameters $\lambda^{\rho\sigma}_{(13)}$ and $\lambda^{\rho\sigma}_{(23)}$ correspond to rotations around the axes y and x, respectively.

The other 3 independent Lorentz parameters $\lambda^{\rho\sigma}_{(01)}$, $\lambda^{\rho\sigma}_{(02)}$ and $\lambda^{\rho\sigma}_{(03)}$, defined as $\lambda^{\rho\sigma}_{(0i)} := {\delta^\rho}_0 {\delta^\sigma}_i$, can lead to other 3 independent conserved "time-space" components of the angular 4-momentum ${l_{0i}}^\mu$, but they are not so easy to interpret as the spatial ones, in terms of rotations around certain axes. When all the external fields $\Psi^A_{(J)}(x)$ are invariant under a boost in the direction of the axis i, with $i = 1, 2, 3$ or x, y, z, the condition (C.75) will be satisfied for the components $0i$. Then, inserting the parameters $\lambda^{\rho\sigma}_{(0i)} := {\delta^\rho}_0 {\delta^\sigma}_i$ in (C.76), we obtain the continuity equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{c}{l_{0i}}^0\right) + \partial_j {l_{0i}}^j = 0. \tag{C.80}$$

For closed systems, the integral version of (C.80) is related to the conservation of the velocity of the center of energy of the system, as it is explained in general terms in subsection 4.2.3, but for open systems this continuity equation is not clear to physically interpret. Maybe we can just say that (C.80) leads to conserved quantities related to the hypothetical invariance under boosts of the external fields acting on the open system.

C.5. Explicit expression for the Lorentz invariance condition of tensorial fields

In section 4.2.1, we studied the conditions under which the constitutive tensor $\chi^{\mu\nu\rho\sigma}$ is invariant under Lorentz transformations and spatial rotations in particular. Since the constitutive tensor is a fourth rank tensor, its explicit invariance condition (C.75) under Lorentz transformations is not trivial. Therefore, this subsection is devoted to the explicit derivation of (C.75) for different types of external fields. The main difficulty is the calculation of the Lorentz generators for each tensor rank, which can be calculated using the variation definition (C.61) and the infinitesimal Lorentz transformation (C.56), applied to tensor fields of different ranks.

Scalar field

For a scalar field ϕ, we already said in subsection C.4.3 that its Lorentz generator is zero and therefore its total angular momentum operator $(j_{\rho\sigma})^A{}_B$ coincides with the orbital angular momentum operator $(l_{\rho\sigma})^A{}_B$:

$$(j_{\rho\sigma})^A{}_B = \delta^A{}_B(x_\rho\partial_\sigma - x_\sigma\partial_\rho). \tag{C.81}$$

Now, if we replace (C.81) into (C.75), we obtain that a scalar field invariant under Lorentz transformations must satisfy

$$\boxed{\lambda^{\rho\sigma}(x_\rho\partial_\sigma\phi - x_\sigma\partial_\rho\phi) = 0.} \tag{C.82}$$

4-vector field

The variation of a contravariant 4-vector field A^μ under an infinitesimal Lorentz transformation is given by

$$\begin{aligned} \delta A^\mu(x) &= \lambda^\mu{}_\nu A^\nu && \text{(C.83)}\\ &= \delta^\mu{}_{[\rho}\eta_{\sigma]\nu}\lambda^{\rho\sigma}A^\nu && \text{(C.84)}\\ &= \frac{1}{2}\lambda^{\rho\sigma}(\delta^\mu{}_\rho\eta_{\sigma\nu} - \delta^\mu{}_\sigma\eta_{\rho\nu})A^\nu. && \text{(C.85)} \end{aligned}$$

Therefore, comparing (C.85) with (C.61), we see that the Lorentz generator for any contravariant Lorentz 4-vector field is given by

$$(s_{\rho\sigma})^\mu{}_\nu := \delta^\mu{}_\rho\eta_{\sigma\nu} - \delta^\mu{}_\sigma\eta_{\rho\nu}. \tag{C.86}$$

Notice that the Lorentz generators for a covariant vector field A_μ can be obtained from (C.86), by lowering the index μ and rising the index ν, which reads

$$(s_{\rho\sigma})_\mu{}^\nu = \delta^\nu{}_\sigma\eta_{\mu\rho} - \delta^\nu{}_\rho\eta_{\mu\sigma}. \tag{C.87}$$

If we replace (C.86) in (C.75), the condition for a 4-vector field to be invariant under Lorentz transformations is

$$\lambda^{\rho\sigma}(x_\rho\partial_\sigma A^\mu - x_\sigma\partial_\rho A^\mu + \delta^\mu{}_\rho\eta_{\sigma\nu}A^\nu - \delta^\mu{}_\sigma\eta_{\rho\nu}A^\nu) = 0. \tag{C.88}$$

Finally if we lower the free index μ in (C.88), we obtain

$$\boxed{\lambda^{\rho\sigma}(x_\rho\partial_\sigma A_\mu - x_\sigma\partial_\rho A_\mu + \eta_{\mu\rho}A_\sigma - \eta_{\mu\sigma}A_\rho) = 0.} \tag{C.89}$$

Second rank tensor field

The infinitesimal variation of a second rank tensor $A^{\mu\nu}$ is given by

$$\delta A^{\mu\nu}(x) = (\delta^{\mu}{}_{\alpha}\lambda^{\nu}{}_{\beta} + \delta^{\nu}{}_{\beta}\lambda^{\mu}{}_{\alpha})A^{\alpha\beta} \tag{C.90}$$

$$= \frac{1}{2}\lambda^{\rho\sigma}(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\rho}\eta_{\sigma\beta} - \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\sigma}\eta_{\rho\beta} + \delta^{\nu}{}_{\beta}\delta^{\mu}{}_{\rho}\eta_{\sigma\alpha} - \delta^{\nu}{}_{\beta}\delta^{\mu}{}_{\sigma}\eta_{\rho\alpha})A^{\alpha\beta}, \tag{C.91}$$

and comparing (C.91) with (C.61), we see that the Lorentz generator for a second rank tensor field $A^{\mu\nu}$ can be explicitly written by

$$(s_{\rho\sigma})^{\mu\nu}{}_{\alpha\beta} := \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\rho}\eta_{\sigma\beta} - \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\sigma}\eta_{\rho\beta} + \delta^{\nu}{}_{\beta}\delta^{\mu}{}_{\rho}\eta_{\sigma\alpha} - \delta^{\nu}{}_{\beta}\delta^{\mu}{}_{\sigma}\eta_{\rho\alpha}. \tag{C.92}$$

Then, inserting (C.92) in the general condition (C.75) and lowering the indexes μ and ν, we finally conclude that a second rank tensor with Lorentz invariance must satisfy:

$$\boxed{\lambda^{\rho\sigma}(x_{\rho}\partial_{\sigma}A_{\mu\nu} - x_{\sigma}\partial_{\rho}A_{\mu\nu} + \eta_{\nu\rho}A_{\mu\sigma} - \eta_{\nu\sigma}A_{\mu\rho} + \eta_{\mu\rho}A_{\sigma\nu} - \eta_{\mu\sigma}A_{\rho\nu}) = 0.} \tag{C.93}$$

Fourth rank tensor field

Analogously to the other cases, the infinitesimal variation of a fourth rank tensor $\chi^{\mu\nu\lambda\kappa}$ under a Lorentz transformation is given by

$$\delta\chi^{\mu\nu\lambda\kappa}(x) = (\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}\delta^{\lambda}{}_{\gamma}\lambda^{\kappa}{}_{\delta} + \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}\delta^{\kappa}{}_{\delta}\lambda^{\lambda}{}_{\gamma} + \delta^{\mu}{}_{\alpha}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\delta}\lambda^{\nu}{}_{\beta} + \delta^{\nu}{}_{\beta}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\delta}\lambda^{\mu}{}_{\alpha})\chi^{\alpha\beta\gamma\delta} \tag{C.94}$$

$$\begin{aligned} &= \frac{1}{2}\lambda^{\rho\sigma}(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\rho}\eta_{\sigma\delta} - \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\sigma}\eta_{\rho\delta} + \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}\delta^{\kappa}{}_{\delta}\delta^{\lambda}{}_{\rho}\eta_{\sigma\gamma} \\ &\quad - \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}\delta^{\kappa}{}_{\delta}\delta^{\lambda}{}_{\sigma}\eta_{\rho\gamma} + \delta^{\mu}{}_{\alpha}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\delta}\delta^{\nu}{}_{\rho}\eta_{\sigma\beta} - \delta^{\mu}{}_{\alpha}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\delta}\delta^{\nu}{}_{\sigma}\eta_{\rho\beta} \\ &\quad + \delta^{\nu}{}_{\beta}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\delta}\delta^{\mu}{}_{\rho}\eta_{\sigma\alpha} - \delta^{\nu}{}_{\beta}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\delta}\delta^{\mu}{}_{\sigma}\eta_{\rho\alpha})\chi^{\alpha\beta\gamma\delta}, \end{aligned} \tag{C.95}$$

and therefore its Lorentz generators $(s_{\rho\sigma})^{\mu\nu\lambda\kappa}{}_{\alpha\beta\gamma\delta}$ can be explicitly written as

$$\begin{aligned} (s_{\rho\sigma})^{\mu\nu\lambda\kappa}{}_{\alpha\beta\gamma\delta} &= \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\rho}\eta_{\sigma\delta} - \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\sigma}\eta_{\rho\delta} + \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}\delta^{\kappa}{}_{\delta}\delta^{\lambda}{}_{\rho}\eta_{\sigma\gamma} - \delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}\delta^{\kappa}{}_{\delta}\delta^{\lambda}{}_{\sigma}\eta_{\rho\gamma} \\ &\quad + \delta^{\mu}{}_{\alpha}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\delta}\delta^{\nu}{}_{\rho}\eta_{\sigma\beta} - \delta^{\mu}{}_{\alpha}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\delta}\delta^{\nu}{}_{\sigma}\eta_{\rho\beta} + \delta^{\nu}{}_{\beta}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\delta}\delta^{\mu}{}_{\rho}\eta_{\sigma\alpha} - \delta^{\nu}{}_{\beta}\delta^{\lambda}{}_{\gamma}\delta^{\kappa}{}_{\delta}\delta^{\mu}{}_{\sigma}\eta_{\rho\alpha}. \end{aligned} \tag{C.96}$$

Finally, if we replace (C.96) into (C.75), we see that the explicit condition for a fourth rank tensor $\chi^{\alpha\beta\gamma\delta}$ to be invariant under Lorentz transformations is

$$\boxed{\begin{aligned} &\lambda^{\rho\sigma}(x_{\rho}\partial_{\sigma}\chi_{\alpha\beta\gamma\delta} - x_{\sigma}\partial_{\rho}\chi_{\alpha\beta\gamma\delta} \\ &\quad + \eta_{\alpha\rho}\chi_{\sigma\beta\gamma\delta} - \eta_{\alpha\sigma}\chi_{\rho\beta\gamma\delta} + \eta_{\beta\rho}\chi_{\alpha\sigma\gamma\delta} - \eta_{\beta\sigma}\chi_{\alpha\rho\gamma\delta} \\ &\quad + \eta_{\gamma\rho}\chi_{\alpha\beta\sigma\delta} - \eta_{\gamma\sigma}\chi_{\alpha\beta\rho\delta} + \eta_{\delta\rho}\chi_{\alpha\beta\gamma\sigma} - \eta_{\delta\sigma}\chi_{\alpha\beta\gamma\rho}) = 0. \end{aligned}} \tag{C.97}$$

Appendix D.

Conversion table from SI to gaussian system of units

Quantity	Units		
	SI		Gaussian
Speed of Light	$c := 1/\sqrt{\varepsilon_0 \mu_0}$		
Magnetic Induction	$\vec{B}$	$=$	$\sqrt{\frac{\mu_0}{4\pi}}\vec{B}$
Magnetic Field	$\vec{H}$	$=$	$\frac{1}{\sqrt{4\pi\mu_0}}\vec{H}$
Magnetization	$\vec{M}$	$=$	$\sqrt{\frac{4\pi}{\mu_0}}\vec{M}$
Electric Field	$\vec{E}$	$=$	$\frac{1}{\sqrt{4\pi\varepsilon_0}}\vec{E}$
Electric Displacement	$\vec{D}$	$=$	$\sqrt{\frac{\varepsilon_0}{4\pi}}\vec{D}$
Polarization	$\vec{P}$	$=$	$\sqrt{4\pi\varepsilon_0}\vec{P}$
Charge Density	ρ	$=$	$\sqrt{4\pi\varepsilon_0}\rho$
Charge Current	$\vec{j}$	$=$	$\sqrt{4\pi\varepsilon_0}\vec{j}$
Relative Permittivity tensor	ε^{ij}	$=$	ε^{ij}
Relative Permeability tensor	μ_{ij}	$=$	μ_{ij}
Refraction Index	n	$=$	n

Table D.1.: Conversion table of some electromagnetic quantities from SI to gaussian units.

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