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FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS

LORENTZ AND DIFFEOMORPHISM VIOLATION IN EFFECTIVE THEORIES OF GRAVITY AND MATTER

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A la memoria de mi querida abuela y segunda madre,
Josefina Del Carmen Riquelme Muñoz (1943-2017);
del mejor hombre que he conocido,
José Heriberto Bravo Reyes (1941-2025),
y de nuestras queridas mascotas y familiares,
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Resumen

En la presente tesis se estudió en profundidad el sector minimal de la extensión del Modelo Estándar Gravitacional (gSME) en un escenario cosmológico, con el objetivo de desvelar las condiciones de compatibilidad entre los campos *background* y la dinámica de la teoría. Para ello, se analizó el rompimiento explícito y espontáneo de la invariancia bajo difeomorfismos de la Relatividad General dentro del marco de las teorías efectivas. En ambos casos se derivaron las ecuaciones de Friedmann modificadas, así como las configuraciones requeridas por los campos de *background* para preservar la isotropía y la homogeneidad. En este contexto, se identificaron configuraciones que permiten multiples fases de expansión del universo.

Este estudio condujo al desarrollo de un método para tratar las incompatibilidades entre las violaciones explícitas de difeomorfismos y la geometría Riemanniana, basado en el análisis de las isometrías de un sistema gravitacional. Se concluye que las configuraciones de los campos de *background* son más propensas a determinarse cuanto mayor sea el número de difeomorfismos que actúan como isometrías del sistema en el caso de rompimiento explícito; en cambio, en el caso de rompimiento espontáneo, el campo de *background* no requiere respetar dichas isometrías para garantizar la compatibilidad. Prueba de la efectividad del método, se logra estudiar violaciones de causalidad en presencia de campos de *background* de forma consistente.

Keywords – Modified gravity theories, Accelerated cosmic expansion, Diffeomorphism symmetry breaking, Lorentz symmetry breaking

Abstract

This thesis conducted an in-depth study of the minimal sector of the gravitational Standard-Model Extension (gSME) in a cosmological scenario, with the aim of uncovering the compatibility conditions between the background fields and the dynamics of the theory. To this end, the explicit and spontaneous breaking of diffeomorphism invariance in General Relativity was analyzed within the framework of effective field theories. In both cases, the modified Friedmann equations were derived, as well as the configurations required by the background fields to preserve isotropy and homogeneity. In this context, configurations that allow for multiple phases of universe expansion were identified.

This study led to the development of a method for addressing the incompatibilities between explicit diffeomorphism violations and Riemannian geometry, based on the analysis of the isometries of a gravitational system. It is concluded that the configurations of the background fields are more likely to be determined the greater the number of diffeomorphisms that act as isometries of the system in the case of explicit breaking; in contrast, in the case of spontaneous breaking, the background field does not need to respect these isometries to guarantee compatibility. As proof of the method's effectiveness, causality violations in the presence of background fields were consistently studied.

Keywords – Modified gravity theories, Accelerated cosmic expansion, Diffeomorphism symmetry breaking, Lorentz symmetry breaking

Notation and Conventions

In this thesis, we adopt the sign conventions of C. W. Misner, K. S. Thorne, and J. A. Wheeler as used in their seminal text on gravitation, *Gravitation* [1]. In particular, we use metric signature $(-+++)$, or *mostly plus*. Through the thesis we will use geometrized units, where the speed of light is set equal to one.

The index notation used in this work is as follows: Greek indices $(\mu, \nu, \dots)$ denote spacetime components of tensors on the manifold $\mathcal{M}$, while Latin indices $(a, b, \dots)$ denote spatial components of tensors on the hypersurface σ, which is embedded via the ADM foliation.

For the reader's convenience, we list below frequently used symbols, including those defined in this work and standard notation from gravitation:

Symbol	**Description**
$\mathcal{M}$	Spacetime manifold
$g_{\mu\nu}$	Metric tensor of $\mathcal{M}$
$\Gamma^{\lambda}{}_{\mu\nu}$	Affine connection (Christoffel symbols) of $g_{\mu\nu}$
∇_λ	Covariant derivative on $\mathcal{M}$
∂_μ	Partial derivative with respect to x^μ on $\mathcal{M}$
${}^{(4)}R^{\lambda}{}_{\mu\rho\nu}$	Riemann curvature tensor on $\mathcal{M}$
${}^{(4)}R_{\mu\nu}$	Ricci tensor on $\mathcal{M}$
${}^{(4)}R$	Ricci scalar on $\mathcal{M}$
$\mathcal{L}_\xi$	Lie derivative along the ξ direction
Σ_t	Hypersurface in $\mathcal{M}$ at time t
σ	Three-dimensional spatial manifold
q_{ab}	Induced metric on σ
K_{ab}	Extrinsic curvature of σ
N	Lapse function (temporal displacement)
N^a	Shift function (spatial displacement)
$X^\mu_t(y^a)$	Embedding of σ with coordinates y^a into $\mathcal{M}$ at time t
$e^\mu{}_a$	Projector vectors tangential to Σ_t
n^μ	Normal vector to Σ_t
D_a	Covariant derivative on σ
∂_a	Partial derivative with respect to y^a on σ
$\Gamma^{c}{}_{ab}$	Affine connection (Christoffel) of q_{ab}
${}^{(3)}R^{d}{}_{acb}$	Riemann curvature tensor on σ
${}^{(3)}R_{ab}$	Ricci tensor on σ
${}^{(3)}R$	Ricci scalar on σ

$\mathbf{e}_{\mu}{}^{a}$	Vierbein (tetrad) relating $g_{\mu\nu}$ to η_{ab}
$\omega_{\mu}{}^{ab}$	Spin connection
D_{μ}	Covariant derivative with Gauge and Spin connection
A_{μ}	Gauge field
$F_{\mu\nu}$	Gauge curvature tensor ($F = dA$)
$SO^{+}(3,1)$	Proper orthochronous Lorentz group
$E(3)$, $ISO(3)$	Euclidean group (group of isometries of Euclidean space)
$\Lambda^{a}{}_{b}$	Matrix components for a Lorentz transformation
$T^{\mu\nu}_{\mathbf{e}}$	Energy-momentum tensor of the field $\mathbf{e}$
$S_{\omega}{}^{\lambda\mu\nu}$	Spin density tensor
$(T_m)^{\mu\nu}$	Matter source (energy-momentum tensor)
$T^{\lambda\mu\nu}$	Torsion tensor
$\mathcal{L}_g^{(d)}$	Lagrange density of mass dimension d
Λ	Cosmological constant
H	Hubble parameter
w	Ratio between pressure and density of matter content ($w = P/\rho$)
k	Curvature number for FLRW metric ($k = 0, \pm 1$)
ρ	Density of a perfect fluid
P	Pressure of a perfect fluid
B^{μ}	Bumblebee field
b^{μ}	Vacuum expectation value (VEV) of the Bumblebee field
ω	Angular velocity in Gödel universe
r_c	Critical radius in Gödel universe

The Latin index in the vierbein $\mathbf{e}_{\mu}{}^{a}$ is a flat spacetime index and should not be interpreted as a σ manifold index. Auxiliary quantities such as $\xi^{\rho}, \chi^{\sigma}, T^{\mu\nu}$ have a different meaning in each chapter unless otherwise stated.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

It is well known that Quantum Field Theory (QFT) suffers from ultraviolet (UV) divergences, and it has been a fundamental problem in high energy physics for decades. Quantum field theory, our mathematical framework for describing nature, inherently limits the influence of smaller distance scales on the description of larger objects [2]. The UV divergences appear because it is assumed that QFT is valid to arbitrarily short distance scales, corresponding to arbitrarily high energies.

These divergences can be addressed using standard techniques such as momentum cutoffs, dimensional regularization, and renormalization. These approaches often introduce unphysical parameters, rely on arbitrary prescriptions, or break fundamental symmetries, making them mathematically effective but conceptually unsatisfactory [3]. For example, lattice discretization explicitly breaks the continuum nature of space-time [4], while dimensional regularization introduces a formally consistent but physically unintuitive analytic continuation of spacetime dimensions [5]. Similarly, Pauli-Villars regularization modifies propagators by introducing unphysical ghost fields [6], which may not emerge naturally from fundamental principles.

However, the existence of these divergences seems incompatible with the predictions of General Relativity (GR) predictions. They originate from integrals where internal momenta can grow arbitrarily large. Virtual particles with high energy E and momentum P have a Compton wavelength $\lambda = \hbar/P$, and general relativity predicts that such energy densities collapse into black holes when λ approaches the Schwarzschild radius, $r = 2GE/c^4$. This transition occurs at the Planck scale

(10^{19} [GeV]), characterized by the Planck energy $E_P = \sqrt{8\pi\hbar c/\kappa}$ and the Planck length $\ell_P = \sqrt{\hbar\kappa c/8\pi}$, where quantum gravity effects become significant. At these scales, the nature of virtual particles changes dramatically, becoming black holes potentially radiating all particle species via Hawking radiation.

These qualitative arguments suggest that gravity may provide a natural UV cut-off, as Planck-scale black holes would decay within a Planck time $t_P = \ell_P/c$. Moreover, this implies that spacetime may possess a minimal length scale, beyond which further physical resolution becomes impossible, possibly due to the inaccessibility of regions behind event horizons. While speculative, these ideas point toward the need for a consistent theory of quantum gravity to may resolve the UV divergences in QFT.

On the other hand, classical general relativity exhibits fundamental limitations. The prediction of spacetime singularities such as those associated with black holes and the Big Bang indicates that the theory is already being extrapolated beyond its regime of validity. In scenarios involving gravitational collapse, matter reaches energy densities so extreme that quantum effects are expected to become significant. A consistent quantum theory of gravity is not only expected to resolve these singularities, but may also offer insights into other open problems in fundamental physics, including the cosmological constant value problem, the nature of dark matter, and the dynamics of the pre-inflationary universe.

The preceding discussion highlight the limitations of both QFT and GR when approaching the Planck scale. Various approaches to a quantum theory of gravity suggest that novel phenomenological effects may arise from the breakdown of fundamental symmetries underlying conventional QFT and general relativity. Most notably, we are looking for deviations from Lorentz invariance.

Lorentz symmetry is the invariance of physical laws under Lorentz transformations namely spatial rotations and boosts, corresponding to the proper orthochronous Lorentz group $SO^+(3,1)$. As a global symmetry of Minkowski spacetime, it underlies the theory of Special Relativity (SR) and the Standard Model of particle physics (SM), where it is intimately connected to Charge-Parity-Time reversal (CPT) symmetry invariance. As a local symmetry of the laws of physics in freely falling frames, Lorentz invariance is a cornerstone of General Relativity.

Nevertheless, it remains possible that nature exhibits small violations of Lorentz symmetry in a quantum gravity regime [7].

A plausible origin for such violations appears in string field theory [8], where certain interaction terms can induce spontaneous Lorentz symmetry breaking due to nonzero vacuum expectation values of tensor fields coupled to scalars. More recently, a variety of alternative mechanisms for Lorentz violation at the fundamental level have been proposed, including scenarios involving noncommutative field theories [9], spacetime-varying fields [10], quantum gravity effects [11], brane-world scenarios [12], supersymmetry breaking [13], and massive gravity [14], among others.

For any consistent treatment of Lorentz violation, one must uphold the principle of observer independence: physical predictions should not depend on the choice of coordinates used to describe them. This requirement holds regardless of whether the coordinate transformation corresponds to a Lorentz transformation or not.

To date, no compelling experimental evidence for Lorentz violation has been found; consequently, current efforts in physics focus on searching for potential violations rather than modeling an already observed phenomenon. In this exploratory phase, it is desirable to adopt the most general possible theoretical framework, so that no region of parameter space remains unexplored.

A systematic way to study potential violations is through effective field theory. Starting from the Standard Model coupled to General Relativity, the action can be augmented by incorporating all possible scalar terms formed by contracting Lorentz-violating operators, each multiplied by coefficients that control the magnitude of the effects. The resulting framework is known as the Standard-Model Extension (SME) [15], [16]. Since CPT violation is intrinsically associated with Lorentz violation in realistic field theories, the SME also includes general CPT violation. Coordinate invariance ensures that the physics remains unaffected by changes in the observer frame, including Lorentz and other coordinate transformations, while allowing observable effects of Lorentz violation to arise from particle transformations. Under these considerations, the general SME formulation offers a guide for experimental searches [17].

This thesis deals with the exploration in various sectors of high-energy physics of phenomenology associated with violations of Lorentz symmetry, focusing particularly on modified gravity theories and cosmology. The research was carried out using effective field theories within the framework of the Standard-Model Extension (SME). The primary objective of this research is to determine whether the introduction of background fields in theories that break diffeomorphism and Lorentz symmetry is compatible with the cosmological framework.

To achieve this, we initially worked within the fermionic sector of the Standard-Model Extension (SME), specifically with the Myers-Pospelov model. Despite the indefinite-metric states in the model, microcausality and unitarity are proven for tree-level and one-loop processes using the Lee-Wick prescription and a modified Cutkosky rule that restricts the state space to stable, positive-norm particles. This investigation was complemented by formal studies of Quantum Field Theory with Lorentz violation, which provided essential conceptual background and familiarity with the theoretical foundations. This research culminated in the publication of two articles [18], [19], which serve as an introduction to SME phenomenology. The current thesis will focus on the interaction between gravity and background fields; therefore, the results concerning fermionic matter in flat space-time will not be explored here.

In Chapter 2 we will begin by discussing how the symmetries and principles of General Relativity shape the theory of gravity, in particular the role of Lorentz symmetry. Subsequently, in Chapter 3 the motivation for Lorentz symmetry breaking will be presented, as well as the introduction of a framework that categorizes and incorporates the breaking in all sectors of the Standard Model and Gravity, called the Standard-Model Extension (SME). In particular, we will focus on the so-called minimal gravitational SME sector.

In Chapter 4, making use of the ADM decomposition detailed in Chapter 2 and the minimal extension of gravitational SME from Chapter 3, we will project the entire model into ADM variables to recover the usual cosmological setup in the Gaussian normal coordinates limit for the FLRW metric. Subsequently, we will seek to evaluate the compatibility between symmetry violation and the notion of isotropy and homogeneity, through different breaking mechanisms. The mechanisms will be treated separately, dedicating Chapter 4 to explicit symmetry breaking and Chapter 5 to spontaneous breaking. In both chapters we will study accelerated expansion of the Universe with influence of external background fields.

In Chapter 6, the role played by the isometries of spacetimes in the compatibility of admitting Lorentz symmetry violations without causing discrepancies between geometry and dynamics will be discussed, as a result of the research carried out in Chapters 4 and 5.

As an example of how isometries constrain the degrees of freedom in gravitational setups to yield more compatible configurations in gravitational models with explicit violation of diffeomorphism symmetry, we present our study of causality violations in the presence of background fields in a Gödel universe in 7.

This work includes a appendix (Appendix A) where we present a mechanism developed for the optimal decomposition of tensorial quantities in terms of ADM variables, which makes it possible to identify the geometric contributions made by the breaking mechanism in the dynamics of spacetime.

Chapter 2

Theoretical Framework I: General Relativity

On November 25th 1915, Albert Einstein presented the gravitational field equations to the Prussian Academy of Natural Sciences [20]. Around the same time, David Hilbert had proposed the correct action principle for gravity, which yielded the Einstein equations, based on a communication in which Einstein had outlined the general idea of General Relativity [21]. The only action in four dimensions that is compatible with the postulates of General Relativity and that recovers Special Relativity as its local limit is the Einstein–Hilbert action. This marks the beginning of the modern study of spacetime dynamics.

2.1 Principles of General Relativity

The framework of General Relativity requires that physical laws remain invariant under general coordinate transformations. As we shall see, this emerges as a direct consequence of the diffeomorphism invariant realization of the underlying physical principles [22].

The Equivalence Principle of Gravitation and Inertia describes how physical systems respond to external gravitational fields. It rests on the equivalence between inertial mass, m_i, and gravitational mass, m_g. Einstein realized that this implies that no static, homogeneous gravitational field can be detected within a free-falling reference frame. Consider two observers: one in an inertial frame and another in free fall. The exact cancellation between inertial forces and gravitational forces due to the $m_i = m_g$ equivalence means the freely falling observer perceives no gravitational field. However, this cancellation must be modified when inhomogeneities in the gravitational field are present. For instance, while Earth is in free fall toward the Sun, we don't perceive the Sun's gravitational field in our daily experience, but the slight inhomogeneity in this field will be enough to produce significant waves in the sea.

In reality, inertial forces do not perfectly cancel gravitational forces in free-falling systems when field inhomogeneities or time dependencies exist, but a crucial feature emerges when we examine sufficiently small spacetime regions. This leads us to the formal statement of the Equivalence Principle:

At every point in spacetime within an arbitrary gravitational field, there exists a locally inertial coordinate system in which, within a sufficiently small neighborhood, the laws of nature coincide with those of an unaccelerated coordinate system in the absence of gravitation.

An alternative perspective comes from the Principle of General Covariance, which imposes two requirements for any valid physical equation:

- The equation must hold in the absence of gravitation: it must reduce to the laws of Special Relativity when the metric tensor $g_{\mu\nu}$ equals the Minkowski metric $\eta_{\mu\nu}$, causing the affine connection ${\Gamma^\lambda}_{\mu\nu}$ to vanish.
- The equation must be generally covariant, maintaining its form under arbitrary coordinate transformations.

In essence, the Principle of General Covariance extends the Equivalence Principle. Constructing generally covariant equations naturally introduces new objects that will describe the geometry and the gravitational phenomena: the metric tensor and affine connection.

These principles define the framework of General Relativity: General Covariance requires the field equations to remain invariant under coordinate transformations, expressed through tensorial quantities on a manifold. Meanwhile, the Equivalence Principle requires that gravity be encoded in the metric tensor and connection, which must locally reduce to Special Relativity and the Minkowski spacetime in appropriate coordinates.

With this formulation, we now formally define our spacetime framework equipped to describe gravity.

A **spacetime** is defined as the pair $(\mathcal{M}, \mathbf{g})$ where $\mathcal{M}$ is a four-dimensional real smooth manifold equipped with a metric following a spacetime signature $(\mp, \pm, \pm, \pm)$, also called Lorentzian manifolds. Spacetime signatures will represent the causal structure, connecting with the respective flat spacetime solution $\eta_{\mu\nu} =$ Diag.$(\mp, \pm, \pm, \pm)$, also being a geometrical invariant under diffeomorphism.

In the tangent space of $\mathcal{M}$, we define objects that transform under general coordinate transformations, also referred to as general tensors.. Tensors are defined with respect to a group of transformations, for example the Poincaré and Lorentz tensors in SR. In the case of GR we have Diff$(\mathcal{M})$, the group of automorphisms of $\mathcal{M}$, acting as the group of coordinate transformations. If we define around a point $P \in \mathcal{M}$ a coordinate patch with coordinates $\{x^\lambda, \lambda = 0, 1, 2, 3\}$, under a change of coordinates $x^\mu \to \bar{x}^\mu(x^\lambda)$ a tensor of rank (p, q) will transform as follows

$$\bar{T}^{\mu_1' \cdots \mu_p'}{}_{\nu_1' \cdots \nu_q'}(\bar{x}^\lambda) = \frac{\partial \bar{x}^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial \bar{x}^{\mu_p'}}{\partial x^{\mu_p}} \frac{\partial x^{\nu_1}}{\partial \bar{x}^{\nu_1'}} \cdots \frac{\partial x^{\nu_q}}{\partial \bar{x}^{\nu_q'}} T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q}(x^\lambda) \,. \tag{2.1.1}$$

The covariant derivative provides the correct differentiation operator to preserve the tensorial character of equations. The covariant derivative as we know corresponds to

$$\begin{aligned} \nabla_\lambda T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} \;=\; & \partial_\lambda T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} + \Gamma^{\mu_1}{}_{\kappa\lambda} T^{\kappa \cdots \mu_p}{}_{\nu_1 \cdots \nu_q} + \cdots + \Gamma^{\mu_p}{}_{\kappa\lambda} T^{\mu_1 \cdots \kappa}{}_{\nu_1 \cdots \nu_q} \\ & - \Gamma^{\kappa}{}_{\nu_1\lambda} T^{\mu_1 \cdots \mu_p}{}_{\kappa \cdots \nu_q} - \cdots - \Gamma^{\kappa}{}_{\nu_q\lambda} T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \kappa} \,, \end{aligned} \tag{2.1.2}$$

where

$$\Gamma^\lambda{}_{\mu\nu} \;=\; \frac{1}{2} g^{\lambda\kappa} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu} \right) , \tag{2.1.3}$$

corresponds to the Christoffel connection. The fact that the Christoffel connection is not a tensor allows us to identify coordinate systems in which the connection vanishes, recovering the Minkowskian flat space.

The equations of motion for fields in GR are covariant under general coordinate transformations, and the complete theory is invariant under diffeomorphisms as a fundamental symmetry. This is true if we regard the diffeomorphism as a passive one, i.e., such that all it does is permute the points of the manifold. In other words, giving different names to points should not affect the physics [23]. Active diffeomorphisms are quite different, however, and not every theory is invariant under their action. When people say that GR is a *diffeomorphism invariant* theory, what they really mean is that GR is invariant under *active* diffeomorphisms. This happens to have profound implications in the interpretation of spacetime, as opposed to the 'triviality' of general covariance, which is physically meaningless.

In a spacetime $(\mathcal{M}, \mathbf{g})$ the metric tensor $\mathbf{g}$ allow us to define the infinitesimal distance between two points in the manifold, or spacetime interval. Taking this in consideration we can introduce a notion of distance $d_{\mathbf{g}}(p, p')$ between any two points p and p' on the manifold via integration:

$$d_{\mathbf{g}}(p, p') \equiv \int_p^{p'} \mathrm{d}s \,. \tag{2.1.4}$$

In this sense, we can say that the metric defines a map from the Cartesian product of $\mathcal{M}$ with itself to the real set:

$$\begin{aligned} d_{\mathbf{g}} : \mathcal{M} \times \mathcal{M} \quad &\rightarrow \quad \mathbb{R} \\ (p, p') \quad &\longmapsto \quad d_{\mathbf{g}}(p, p') \,. \end{aligned} \tag{2.1.5}$$

It is clear that a passive diffeomorphism gives the same physical situation, because the way we label the points of $\mathcal{M}$ is not associated to any physical observable.

Now, consider a diffeomorphism Φ from $\mathcal{M}$ to itself. In general, we have

$$d_{\mathbf{g}}(p, p') \neq d_{\mathbf{g}}(\Phi^{-1}(p), \Phi^{-1}(p')) \,. \tag{2.1.6}$$

However, since Φ^{-1} is a smooth map, we can define a *new metric* $\tilde{\mathbf{g}}$ on $\mathcal{M}$ that its associated distance function is given by

$$d_{\tilde{\mathbf{g}}}(p,p') \equiv d_{\mathbf{g}}(\Phi^{-1}(p),\Phi^{-1}(p'))\,. \tag{2.1.7}$$

The pair $(\mathcal{M},\tilde{\mathbf{g}})$ is still a valid spacetime and the claim is that it is physically indistinguishable from $(\mathcal{M},\mathbf{g})$. This is equivalent to saying that $\tilde{\mathbf{g}}$ solves the *same* equations as $\mathbf{g}$ does. This is what the sentence *GR is a diffeomorphism invariant theory* really means and it is clearly not a trivial fact as passive diffeomorphism invariance is. Invariance under passive diffeomorphisms talks about the *form* of equations. Invariance under active diffeomorphisms tells us something about the *mathematical structure* of the theory and inevitably hints towards a *relational* interpretation:

(...) as long as the relations between entities remain the same, it does not matter 'where' or 'how' they are distributed.

Perhaps the source of confusion stems from the fact that active and passive diffeomorphisms are simply two interpretations of the same mathematical transformation: a diffeomorphism. There is a trivial relation between transforming the tensorial components of a tensor while keeping the basis fixed, and keeping the components fixed while transforming the basis. The relevance of the distinction between transformations appears when we can no longer act on fixed fields; if we can't move field components but can only change the basis, we lose the relation between active and passive transformations.

In this way, we have discussed how diffeomorphism invariance implements the principles of GR, and made an introduction to the idea of *active* and *passive*. The discussion about active and passive diffeomorphism transformations will be detailed in Chapter 3.

2.2 The ADM formalism

Another relevant aspect of General Relativity is its ADM formalism. The Arnowitt–Deser–Misner (ADM) formalism is the Hamiltonian formulation of general relativity [24]. To perform the Hamiltonian formulation, it is necessary to separate space and time. This is achieved by foliating spacetime into hypersurfaces that are perpendicular to a congruence of curves. The spacetime foliation in this formalism helps provide a clearer understanding of the dynamics of gravitational fields and their symmetries, but also is strongly compatible (in the Gaussian normal coordinates limit) with the cosmological setup giving a better understanding about the nature of some geometrical quantities, that will be our strategy to study cosmology in modified gravitational models (see Chapter 4).

The space and time separation is required for the canonical formulation because, otherwise, velocities cannot be defined, and therefore the conjugate momentum, which seems to break invariance under diffeomorphisms. However, this is not the case since we do not fix the separation but keep it arbitrary, so we do not fix a coordinate system. The arbitrariness in fact covers the entire group of diffeomorphisms. Since the action is invariant under diffeomorphisms, it will not depend on this auxiliary separation.

We start from the Einstein-Hilbert action principle for metric tensor fields $g_{\mu\nu}$ of Lorentzian signature propagating on a four-dimensional manifold $\mathcal{M}$

$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,{}^{(4)}R\,. \tag{2.2.1}$$

Here we will use the Lorentzian convention $(-,+,+,+)$ so that timelike vectors have negative norm. The coordinates $x^\mu,\ \mu = 0,1,2,3$ are the coordinates of $\mathcal{M}$ in a local trivialization. ${}^{(4)}R$ is the curvature scalar associated with $g_{\mu\nu}$, and $\kappa = 8\pi G$. To make the action principle well-defined in general, boundary terms must be added (unless one assumes that $\mathcal{M}$ is spatially compact without boundary).

To formulate the action principle in its canonical form, one must assume that $\mathcal{M}$ has a special topology $\mathcal{M} \cong \mathbb{R}\times\sigma$, where σ is a fixed three-dimensional manifold of arbitrary topology [25]. Having made this assumption, $\mathcal{M}$ is foliated into hypersurfaces $\Sigma_t := x_t(\sigma)$; for each fixed $t\in\mathbb{R}$, there exists an embedding $x_t : \sigma \to \mathcal{M}$ defined by $x_t(y) := x(t,y)$, where $y^a,\ a = 1,2,3$ are the local coordinates of σ. Likewise, we have a diffeomorphism

$$X : \mathbb{R}\times\sigma \longmapsto \mathcal{M} \tag{2.2.2}$$

$$(t,y) \longmapsto X(t,y) := X_t(y)\,. \tag{2.2.3}$$

Any diffeomorphism $\varphi \in \mathrm{Diff}(\mathcal{M})$ of $\mathcal{M}$ is of the form $\varphi = X' \circ X^{-1}$, where X, X' are two different foliations related by a diffeomorphism $X' = \varphi \circ X$. It follows then that at this point the freedom to choose the foliation is equivalent to $\mathrm{Diff}(\mathcal{M})$. In fact, since the action is invariant under all diffeomorphisms of $\mathcal{M}$, the foliations X have not been specified and we must allow them to be completely arbitrary.
Now we will use these foliations to provide a decomposition of $\mathcal{M}$. A useful parametrization of the embedding and its arbitrariness can be given by its deformation vector field

$$t^\mu(x) := \left(\frac{\partial x^\mu(t,y)}{\partial t}\right)\bigg|_{x=x(t,y)} =: N(x)n^\mu + N^a(x)e^\mu{}_a\,, \tag{2.2.4}$$

$$e^\mu{}_a(x) := \frac{\partial x^\mu}{\partial y^a}\bigg|_{x=x(t,y)}\,. \tag{2.2.5}$$

Here n^μ is a time-like unit vector normal to the hypersurface Σ_t, and $e^\mu{}_a$ is tangential to Σ_t, i.e. it satisfies $g_{\mu\nu}n^\mu e^\nu{}_a = 0$. Clearly, the vector field n^μ is completely determined as a function of (g, x) by these two requirements. The proportionality coefficients N and N^a are called the Lapse and Shift, respectively.

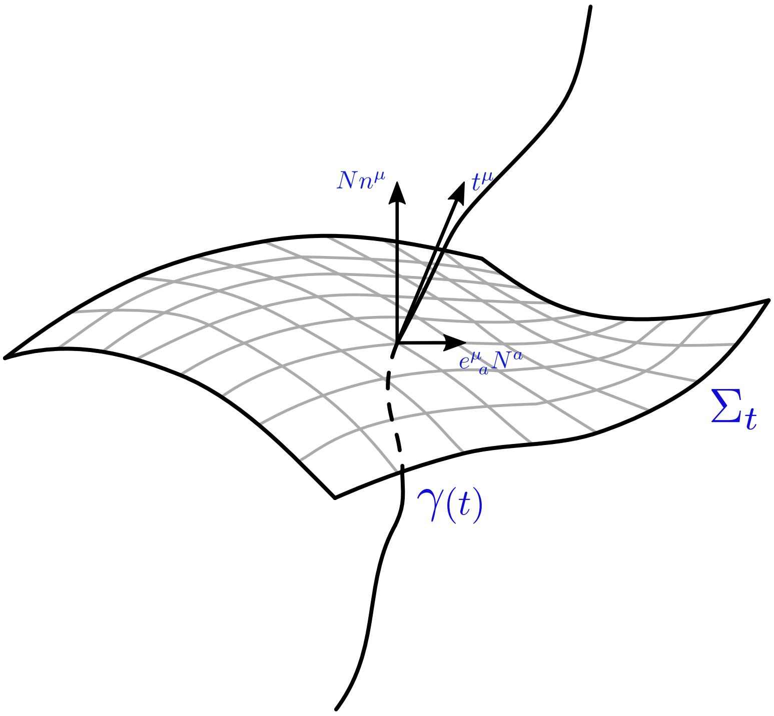


Figure 2.2.1: A region of the hypersurface Σ_t, which is normal to the congruence of curves $\gamma(t)$ at a point in $\mathcal{M}$. The time direction vector t^μ is decomposed into its normal and tangential components relative to Σ_t.

Note that information about the metric $g_{\mu\nu}$ has been implicitly involved since we are only dealing with spacelike embeddings and metrics with a given signature. Thus the foliation t must be timelike everywhere, which leads to the constraint

$$-N^2 + g_{\mu\nu} e^{\mu}{}_{a} e^{\nu}{}_{b} N^a N^b < 0 \,, \tag{2.2.6}$$

which in particular implies that the Lapse does not vanish anywhere. Furthermore, we take N as positive everywhere since we want a future-oriented foliation. At this point we are dealing with a proper subset of all embeddings, and this subset is dynamically constrained since it depends on the metric tensor.

Consider instead a single hypersurface σ embedded in $\mathcal{M}$ through the embedding X. Let n be its unit normal vector and $\Sigma = X(\sigma)$ its image. At this point we have the choice of working either in σ or in Σ when developing the tensor calculus of spatial tensor fields. Working in Σ has the advantage that we can compare spatial tensor fields with arbitrary tensors restricted to Σ, since both are tensors on a subset of $\mathcal{M}$. Moreover, once we develop the tensor calculus in Σ, we will immediately have that of σ by using the pullback of tensor fields in Σ to σ through the embedding.

The previous decomposition gives us the elements to express tensorial quantities in $\mathcal{M}$ in terms of objects living in Σ or σ. Thus now we will construct all the relevant quantities and identities to decompose spacetime quantities into their tangential (spatial) and normal (temporal) components.

2.2.1 Projective technology

We recall the embedding (2.2.5) of σ in $\mathcal{M}$. If x^μ are the coordinates on $\mathcal{M}$ and y^a are the coordinates on the hypersurface, the embedding corresponds to

$$e^{\mu}{}_{a} := \frac{\partial x^\mu}{\partial y^a} \,. \tag{2.2.7}$$

Given a metric $g_{\mu\nu}$ on $\mathcal{M}$, the induced metric q_{ab} through the embedding of the hypersurface into the larger manifold is given by

$$q_{ab} = e^{\mu}{}_{a} e^{\nu}{}_{b} g_{\mu\nu} \,, \tag{2.2.8}$$

that corresponds to the first fundamental form in Gaussian terminology. The vectors $e^{\mu}{}_{a}$ are tangent to curves contained in Σ, so $e^{\mu}{}_{a}n_{\mu} = 0$. Given this orthogonality, we are in a position to write the completeness relation for the inverse metric

$$g^{\alpha\beta} = q^{ab}e^{\alpha}{}_{a}e^{\beta}{}_{b} - n^{\alpha}n^{\beta}\,, \tag{2.2.9}$$

which in turn leads to the expressions

$$\begin{aligned} \delta^{\alpha}{}_{\beta} &= e^{\alpha}{}_{a}(g_{\beta\gamma}e^{\gamma}{}_{b}q^{ba}) - n^{\alpha}n_{\beta}\,, && (2.2.10)\\ g_{\alpha\beta} &= (g_{\alpha\rho}e^{\rho}{}_{a}q^{ac})q_{cd}(q^{db}e^{\sigma}{}_{b}g_{\sigma\beta}) - n_{\alpha}n_{\beta}\,. && (2.2.11)\end{aligned}$$

This motivates the definition of the object

$$e_{\alpha}{}^{a} := g_{\alpha\beta}q^{ab}e^{\beta}{}_{b}\,, \tag{2.2.12}$$

which possesses the following properties:

$$\begin{aligned} n^{\alpha}e_{\alpha}{}^{a} &= 0\,, && (2.2.13)\\ e^{\alpha}{}_{b}e_{\alpha}{}^{a} &= \delta^{a}{}_{b}\,, && (2.2.14)\\ e^{\alpha}{}_{c}e_{\beta}{}^{c} &= \delta^{\alpha}{}_{\beta} + n^{\alpha}n_{\beta}\,. && (2.2.15)\end{aligned}$$

Therefore the completeness relations can be written as

$$\begin{aligned} g^{\alpha\beta} &= e^{\alpha}{}_{a}q^{ab}e^{\beta}{}_{b} - n^{\alpha}n^{\beta}\,, && (2.2.16)\\ \delta^{\alpha}{}_{\beta} &= e^{\alpha}{}_{a}e_{\beta}{}^{a} - n^{\alpha}n_{\beta}\,, && (2.2.17)\\ g_{\alpha\beta} &= e_{\alpha}{}^{a}q_{ab}e_{\beta}{}^{b} - n_{\alpha}n_{\beta}\,. && (2.2.18)\end{aligned}$$

These relations allow us to write any tensor in terms of its components tangent to the hypersurface and orthogonal to it:

$$\begin{aligned} V^{\alpha} &= e^{\alpha}{}_{a}V^{a} - n^{\alpha}V^{\mathbf{n}}\,, && (2.2.19)\\ V_{\alpha} &= e_{\alpha}{}^{a}V_{a} - n_{\alpha}V_{\mathbf{n}}\,. && (2.2.20)\end{aligned}$$

Further generalizations follow the same index pattern.

2.2.1.1 Line element

Recalling (2.2.4) and (2.2.5), and considering that n^μ is a normal unitary 4-vector we can obtain how the vector is written in terms of the Shift and Lapse parameters as follows.
Considering that $x^\mu = (t, y^a)$ we describe the infinitesimal element as

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + 2g_{ta}\mathrm{d}t\mathrm{d}y^a + g_{ab}\mathrm{d}y^a\mathrm{d}y^b\,. \tag{2.2.21}$$

Considering that

$$g_{tt} := t^\mu t^\nu g_{\mu\nu}\,, \tag{2.2.22}$$
$$g_{ta} := t^\mu e^\nu{}_a g_{\mu\nu}\,, \tag{2.2.23}$$
$$g_{ab} := e^\mu{}_a e^\nu{}_b g_{\mu\nu}\,, \tag{2.2.24}$$

we replace the definitions (2.2.4) and (2.2.5) to obtain

$$g_{tt} = N^2 n^2 + N^a N^b q_{ab}\,, \tag{2.2.25}$$
$$g_{ta} = N^a q_{ab}\,, \tag{2.2.26}$$
$$g_{ab} = q_{ab}\,. \tag{2.2.27}$$

Considering $n^\mu n_\mu = -1$ (a purely timelike unit vector) it allow us to write the line element as

$$\mathrm{d}s^2 = \big(-N^2 + N^a N_a\big)\mathrm{d}t^2 + 2N_a\mathrm{d}t\mathrm{d}y^a + q_{ab}\mathrm{d}y^a\mathrm{d}y^b\,, \tag{2.2.28}$$

but also gives the tools to write n^μ exclusively in terms of the ADM parameters.

Considering the result (2.2.28), the normalization, and the orthogonality condition we obtain two equations:

$$g_{\mu\nu}n^\mu n^\nu = \big(-N^2 + N^a N_a\big)(n^0)^2 + 2N_a(n^0)(n^a) + q_{ab}(n^a)(n^b) \tag{2.2.29}$$
$$g_{\mu\nu}e^\mu{}_c n^\nu = \big(N_a(n^0) + q_{ab}(n^b)\big)e^a{}_c \tag{2.2.30}$$

We solve (2.2.30) obtaining

$$(n^a) = -N^a(n^0)\,, \tag{2.2.31}$$

and replacing in (2.2.29) we obtain

$$(n^0)^2 = \frac{1}{N^2}\,. \tag{2.2.32}$$

Thus we obtained, considering the positive root of n^0, the following result

$$n^\mu = \begin{pmatrix} \frac{1}{N} \\ -\frac{N^a}{N} \end{pmatrix}, \tag{2.2.33}$$

$$n_\mu = \begin{pmatrix} -N & 0_a \end{pmatrix}. \tag{2.2.34}$$

2.2.1.2 Tangential covariant derivatives

Let us consider the case of a tensor defined entirely on the hypersurface, i.e.,

$$T^{\alpha_1\alpha_2\ldots\alpha_p}{}_{\beta_1\beta_2\ldots\beta_q} = e^{\alpha_1}{}_{a_1}e^{\alpha_2}{}_{a_2}\ldots e^{\alpha_p}{}_{a_p}e_{\beta_1}{}^{b_1}e_{\beta_2}{}^{b_2}\ldots e_{\beta_q}{}^{b_q}T^{a_1a_2\ldots a_p}{}_{b_1b_2\ldots b_q}\,. \tag{2.2.35}$$

Following [26], the projection of the covariant derivative of a rank-1 covariant tensor is given by

$$D_bT_a = e^\alpha{}_a e^\beta{}_b\nabla_\beta T_\alpha\,. \tag{2.2.36}$$

This is mainly because

$$e^\alpha{}_a e^\beta{}_b\nabla_\beta T_\alpha = e^\beta{}_b\nabla_\beta(e^\alpha{}_a T_\alpha) - e^\beta{}_b\nabla_\beta e^\alpha{}_a T_\alpha\,.$$

Considering the chain rule for the embedding (2.2.7) and

$$e_\alpha{}^c\nabla_\beta e^\alpha{}_a e^\beta{}_b = \Gamma^c{}_{ab} = \frac{1}{2}q^{cd}(\partial_a q_{db} + \partial_b q_{da} - \partial_d q_{ab})\,, \tag{2.2.37}$$

we arrive to the usual covariant derivative formula

$$D_bT_a = \partial_b T_a - \Gamma^c{}_{ab}T_c\,. \tag{2.2.38}$$

On the other hand, for a rank-1 contravariant tensor, one would expect that

$$D_bT^a = e_\alpha{}^a e^\beta{}_b\nabla_\beta T^\alpha\,. \tag{2.2.39}$$

Indeed, proceeding in the same way we arrive to

$$\begin{aligned} e_\alpha{}^a e^\beta{}_b \nabla_\beta T^\alpha &= e^\beta{}_b \nabla_\beta (e_\alpha{}^a T^\alpha) - e^\beta{}_b (\nabla_\beta e_\alpha{}^a) T^\alpha \\ &= \partial_b T^a - e^\beta{}_b (\nabla_\beta e_\alpha{}^a) e^\alpha{}_c T^c \\ &= \partial_b T^a + e^\beta{}_b e_\alpha{}^a (\nabla_\beta e^\alpha{}_c) T^c \\ &= \partial_b T^a + \Gamma^a{}_{cb} T^c \\ &= D_b T^a \, . \end{aligned}$$

With this, one might be tempted to generalize for a tensor with only tangential components

$$D_c T^{a_1 a_2 \ldots a_p}{}_{b_1 b_2 \ldots b_q} = e_{\alpha_1}{}^{a_1} \ldots e_{\alpha_p}{}^{a_p} e^{\beta_1}{}_{b_1} \ldots e^{\beta_q}{}_{b_q} e^\gamma{}_c \nabla_\gamma T^{\alpha_1 \alpha_2 \ldots \alpha_p}{}_{\beta_1 \beta_2 \ldots \beta_q} \, . \quad (2.2.40)$$

Although these are the tangential projections of the quantity $e^\beta{}_b \nabla_\beta T^\alpha$, it is natural to ask whether this quantity has a component along the normal direction. To see this, note that

$$\begin{aligned} e^\beta{}_b \nabla_\beta T^\alpha &= \delta^\alpha{}_\gamma e^\beta{}_b \nabla_\beta T^\gamma \\ &= (e^\alpha{}_c e_\gamma{}^c - n^\alpha n_\gamma) e^\beta{}_b \nabla_\beta T^\gamma \\ &= e^\alpha{}_c e_\gamma{}^c e^\beta{}_b \nabla_\beta T^\gamma - n^\alpha n_\gamma e^\beta{}_b \nabla_\beta T^\gamma \\ &= e^\alpha{}_c D_b T^c - n^\alpha e^\beta{}_b (\nabla_\beta n_\gamma) e^\gamma{}_c T^c \\ &= e^\alpha{}_c D_b T^c - n^\alpha K_{cb} T^c \, , \end{aligned}$$

where a new key ingredient appears, that also comes from the Gaussian geometrical description.

We define the extrinsic curvature as

$$K_{ab} := e^\alpha{}_a e^\beta{}_b \nabla_\beta n_\alpha . \quad (2.2.41)$$

that corresponds to the second fundamental form in Gaussian geometry. This is a symmetric tensor $K_{ab} = K_{ba}$ defined on the hypersurface σ. A conclusion from the identity shown above is that what measures how much the covariant derivative differs from its projection onto the hypersurface is the extrinsic curvature.

We can also note that

$$
\begin{aligned}
\nabla_\beta n_\alpha &= \delta^\gamma{}_\alpha \delta^\lambda{}_\beta \nabla_\lambda n_\gamma \\
&= \big(e^\gamma{}_a e_\alpha{}^a - n^\gamma n_\alpha\big)\big(e^\lambda{}_b e_\beta{}^b - n^\lambda n_\beta\big)\nabla_\lambda n_\gamma \\
&= e^\gamma{}_a e_\alpha{}^a e^\lambda{}_b e_\beta{}^b \nabla_\lambda n_\gamma - e^\gamma{}_a e_\alpha{}^a n^\lambda n_\beta \nabla_\lambda n_\gamma - n^\gamma n_\alpha e^\lambda{}_b e_\beta{}^b \nabla_\lambda n_\gamma + n^\gamma n_\alpha n^\lambda n_\beta \nabla_\lambda n_\gamma \\
&= e_\alpha{}^a e_\beta{}^b K_{ab} - e_\alpha{}^a n_\beta e^\gamma{}_a a_\gamma - n_\alpha e^\lambda{}_b e_\beta{}^b n^\gamma \nabla_\lambda n_\gamma + n_\alpha n_\beta n^\gamma n^\lambda \nabla_\lambda n_\gamma \\
&= e_\alpha{}^a e_\beta{}^b K_{ab} - n_\beta\big(\delta^\gamma{}_\alpha + n^\gamma n_\alpha\big)a_\gamma - \frac{1}{2} n_\alpha e^\lambda{}_b e_\beta{}^b \nabla_\lambda n^2 + \frac{1}{2} n_\alpha n_\beta n^\lambda \nabla_\lambda n^2 \\
&= e_\alpha{}^a e_\beta{}^b K_{ab} - a_\alpha n_\beta \,, \qquad (2.2.42)
\end{aligned}
$$

where a new quantity appears, known as the four-acceleration,

$$a_\mu := n^\lambda \nabla_\lambda n_\mu \,, \qquad (2.2.43)$$

which is purely tangential, i.e., $n^\mu a_\mu = 0$.

Using the definitions of Shift and Lapse (2.2.4) and considering that n^μ should be a normalized 4-vector, it is easy to see that writting n_μ in terms of N and N^a will allow us to rewrite the expression for the acceleration as follows

$$a_\mu = e_\mu{}^a D_a \ln N \,. \qquad (2.2.44)$$

Lastly, a very important identity is the Gauss-Weingarten equation. This relation can be obtained by finding the values for the projection of the covariant derivative of the embedding as follows:

$$
\begin{aligned}
\nabla_\lambda e^\mu{}_a &= \delta^\kappa{}_\lambda \delta^\mu{}_\nu \nabla_\kappa e^\nu{}_a \\
&= (e^\kappa{}_b e_\lambda{}^b - n^\kappa n_\lambda)(e^\mu{}_c e_\nu{}^c - n^\mu n_\nu)\nabla_\kappa e^\nu{}_a \\
&= e_\lambda{}^b e^\mu{}_c (e^\kappa{}_b e_\nu{}^c \nabla_\kappa e^\nu{}_a) - e^\mu{}_c n_\lambda (e_\nu{}^c n^\kappa \nabla_\kappa e^\nu{}_a) - n^\mu e_\lambda{}^b (n_\nu e^\kappa{}_b \nabla_\kappa e^\nu{}_a) \\
&\quad + n^\mu n_\lambda (n^\kappa n_\nu \nabla_\kappa e^\nu{}_a) \\
&= e_\lambda{}^b e^\mu{}_c (e^\kappa{}_b e_\nu{}^c \nabla_\kappa e^\nu{}_a) - e^\mu{}_c n_\lambda (e_\nu{}^c n^\kappa \nabla_\kappa e^\nu{}_a) + n^\mu e_\lambda{}^b (e^\nu{}_a e^\kappa{}_b \nabla_\kappa n_\nu) \\
&\quad - n^\mu n_\lambda (e^\nu{}_a n^\kappa \nabla_\kappa n_\nu) \\
&= e_\lambda{}^b e^\mu{}_c \Gamma^c{}_{ab} - e^\mu{}_c n_\lambda (e_\nu{}^c \mathcal{L}_n e^\nu{}_a + K^c{}_a) + n^\mu e_\lambda{}^b K_{ab} - n^\mu n_\lambda a_a \,,
\end{aligned}
$$

where we used that the Lie derivative of the tangential projector is

$$\mathcal{L}_n e^\nu{}_a \quad = \quad n^\kappa \nabla_\kappa e^\nu{}_a - e^{\nu b} K_{ba}\,.$$

Thus the projection onto the hypersurface (Gauss-Weingarten) and the projection along the normal are

$$\begin{aligned} e^\lambda{}_b \nabla_\lambda e^\mu{}_a &= e^\mu{}_c \Gamma^c{}_{ab} + n^\mu K_{ab}\,, \\ n^\lambda \nabla_\lambda e^\mu{}_a &= e^\mu{}_c (e_\nu{}^c \mathcal{L}_n e^\nu{}_a + K^c{}_a) + n^\mu a_a\,. \end{aligned}$$

2.2.1.3 Lie derivative as normal covariant derivative

The Lie derivative in differential geometry measures the change of a tensor field along the flow defined by a vector field. In other words, it drags tensorial objects along the streamlines of this flow and compares them to their values at the same point. In particular, we are interested in changes along the direction $m^\mu = N n^\mu$, which in ADM nomenclature is the normal direction to the spatial hypersurface including the separation between hypersurfaces, related to time evolution. With this we can find that

$$\mathcal{L}_m (e^\mu{}_a e_\nu{}^a) \quad = \quad m^\lambda \nabla_\lambda (e^\mu{}_a e_\nu{}^a) - (\nabla_\lambda m^\mu) e^\lambda{}_a e_\nu{}^a + (\nabla_\nu m^\lambda) e^\mu{}_a e_\lambda{}^a\,. \tag{2.2.45}$$

Using the definition of a_μ and n^μ we arrive to

$$\begin{aligned} \mathcal{L}_m \big(e^\mu{}_a e_\nu{}^a\big) &= N a^\mu n_\nu + N n^\mu a_\nu - N a_\nu n^\mu - N e_\nu{}^a e^\lambda{}_a (\nabla_\lambda n^\mu) + N (\nabla_\nu n^\lambda) e^\mu{}_a e_\lambda{}^a \\ &= N a^\mu n_\nu - N a^\mu n_\nu = 0\,, \end{aligned} \tag{2.2.46}$$

which means that the Lie derivative of a completely tangential tensor remains exclusively on the hypersurface.

Finally it is possible to generalize the result for purely tangential tensors as follows.

We take the Lie derivative of a purely tangential rank (p,q) tensor as usual:

$$\begin{aligned}
&\mathcal{L}_m(e^{\mu_1}{}_{a_1}\dots e^{\mu_p}{}_{a_p}e_{\nu_1}{}^{b_1}\dots e_{\nu_q}{}^{b_q}T^{a_1\dots a_p}{}_{b_1\dots b_q}) \\
= \; & m^\lambda\nabla_\lambda(e^{\mu_1}{}_{a_1}\dots e^{\mu_p}{}_{a_p}e_{\nu_1}{}^{b_1}\dots e_{\nu_q}{}^{b_q}T^{a_1\dots a_p}{}_{b_1\dots b_q}) \\
&-\nabla_\lambda m^{\mu_1}(e^{\lambda}{}_{a_1}\dots e^{\mu_p}{}_{a_p}e_{\nu_1}{}^{b_1}\dots e_{\nu_q}{}^{b_q}T^{a_1\dots a_p}{}_{b_1\dots b_q}) \\
&-\dots-\nabla_\lambda m^{\mu_p}(e^{\mu_1}{}_{a_1}\dots e^{\lambda}{}_{a_p}e_{\nu_1}{}^{b_1}\dots e_{\nu_q}{}^{b_q}T^{a_1\dots a_p}{}_{b_1\dots b_q}) \\
&+\nabla_{\nu_1} m^{\lambda}(e^{\mu_1}{}_{a_1}\dots e^{\mu_p}{}_{a_p}e_{\lambda}{}^{b_1}\dots e_{\nu_q}{}^{b_q}T^{a_1\dots a_p}{}_{b_1\dots b_q}) \\
&+\dots+\nabla_{\nu_q} m^{\lambda}(e^{\mu_1}{}_{a_1}\dots e^{\mu_p}{}_{a_p}e_{\nu_1}{}^{b_1}\dots e_{\lambda}{}^{b_q}T^{a_1\dots a_p}{}_{b_1\dots b_q})\,. \qquad (2.2.47)
\end{aligned}$$

But considering that

$$\begin{aligned}
(\nabla_\lambda m^\mu)e^\lambda{}_a &= (m^\mu\nabla_\lambda\ln N + N\nabla_\lambda n^\mu)e^\lambda{}_a \\
&= (m^\mu D_a\ln N + Ne^\lambda{}_a\nabla_\lambda n^\mu) \\
&= N(n^\mu a_a + e^\mu{}_c K^c{}_a)\,, \qquad (2.2.48)
\end{aligned}$$

and

$$\begin{aligned}
(\nabla_\nu m^\lambda)e_\lambda{}^b &= (m^\lambda\nabla_\nu\ln N + N\nabla_\nu n^\lambda)e_\lambda{}^b \\
&= Ne_\lambda{}^b\nabla_\nu n^\lambda \\
&= N(e_\nu{}^c K^b{}_c - n_\nu a^b)\,, \qquad (2.2.49)
\end{aligned}$$

we obtain

$$\begin{aligned}
&\mathcal{L}_m(e^{\mu_1}{}_{a_1}\dots e^{\mu_p}{}_{a_p}e_{\nu_1}{}^{b_1}\dots e_{\nu_q}{}^{b_q}T^{a_1\dots a_p}{}_{b_1\dots b_q}) \\
= \; & m^\lambda\nabla_\lambda(e^{\mu_1}{}_{a_1}\dots e^{\mu_p}{}_{a_p}e_{\nu_1}{}^{b_1}\dots e_{\nu_q}{}^{b_q}T^{a_1\dots a_p}{}_{b_1\dots b_q}) \\
&-N(n^{\mu_1}a_{a_1}+e^{\mu_1}{}_{c_1}K^{c_1}{}_{a_1})\dots e^{\mu_p}{}_{a_p}e_{\nu_1}{}^{b_1}\dots e_{\nu_q}{}^{b_q}T^{a_1\dots a_p}{}_{b_1\dots b_q} \\
&-\dots-Ne^{\mu_1}{}_{a_1}\dots(n^{\mu_p}a_{a_p}+e^{\mu_p}{}_{c_p}K^{c_p}{}_{a_p})e_{\nu_1}{}^{b_1}\dots e_{\nu_q}{}^{b_q}T^{a_1\dots a_p}{}_{b_1\dots b_q} \\
&+Ne^{\mu_1}{}_{a_1}\dots e^{\mu_p}{}_{a_p}(e_{\nu_1}{}^{c_1}K^{b_1}{}_{c_1}-n_{\nu_1}a^{b_1})\dots e_{\nu_q}{}^{b_q}T^{a_1\dots a_p}{}_{b_1\dots b_q}) \\
&+\dots+Ne^{\mu_1}{}_{a_1}\dots e^{\mu_p}{}_{a_p}e_{\nu_1}{}^{b_1}\dots(e_{\nu_q}{}^{c_q}K^{b_q}{}_{c_q}-n_{\nu_q}a^{b_q})T^{a_1\dots a_p}{}_{b_1\dots b_q} \qquad (2.2.50)
\end{aligned}$$

If we project onto the hypersurface it becomes

$$\begin{aligned}
&\frac{1}{N} e_{\mu_1}{}^{i_1} \dots e_{\mu_p}{}^{i_p} \dots e^{\nu_1}{}_{j_1} \dots e^{\nu_q}{}_{j_q} \mathcal{L}_m (e^{\mu_1}{}_{a_1} \dots e^{\mu_p}{}_{a_p} e_{\nu_1}{}^{b_1} \dots e_{\nu_q}{}^{b_q} T^{a_1 \dots a_p}{}_{b_1 \dots b_q}) \\
= \; & e_{\mu_1}{}^{i_1} \dots e_{\mu_p}{}^{i_p} \dots e^{\nu_1}{}_{j_1} \dots e^{\nu_q}{}_{j_q} n^\lambda \nabla_\lambda (e^{\mu_1}{}_{a_1} \dots e^{\mu_p}{}_{a_p} e_{\nu_1}{}^{b_1} \dots e_{\nu_q}{}^{b_q} T^{a_1 \dots a_p}{}_{b_1 \dots b_q}) \\
& - K^{i_1}{}_{c_1} T^{c_1 \dots i_p}{}_{j_1 \dots j_q} - \dots - K^{i_p}{}_{c_p} T^{i_1 \dots c_p}{}_{j_1 \dots j_q} \\
& + K^{c_1}{}_{j_1} T^{i_1 \dots i_p}{}_{c_1 \dots j_q} + \dots + K^{c_q}{}_{j_q} T^{i_1 \dots i_p}{}_{j_1 \dots c_q} \,.
\end{aligned} \tag{2.2.51}$$

To conclude, we must remember that time evolution in ADM is given by the direction t^μ in (2.2.4). Locally we can take the *Gaussian normal coordinates limit* by taking $N = 1$ and $N^a = 0$. The Lie derivative in the direction n^μ will be related to the time derivative, since

$$\mathcal{L}_t T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} := \dot{T}^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} \,. \tag{2.2.52}$$

2.2.1.4 Riemann tensor decomposition

The well-known projections of the Riemann tensor are given by the following expressions [27]:

$$e^\alpha{}_a e^\beta{}_b e^\gamma{}_c e^\delta{}_d \, {}^{(4)}R_{\alpha\beta\gamma\delta} = {}^{(3)}R_{abcd} - (K_{ad}K_{bc} - K_{ac}K_{bd}) \,, \tag{2.2.53}$$

$$n^\alpha e^\beta{}_b e^\gamma{}_c e^\delta{}_d \, {}^{(4)}R_{\alpha\beta\gamma\delta} = D_d K_{bc} - D_c K_{bd} \,, \tag{2.2.54}$$

$$n^\alpha e^\beta{}_b n^\gamma e^\delta{}_d \, {}^{(4)}R_{\alpha\beta\gamma\delta} = D_d a_b + a_d a_b + K^e{}_d K_{be} - \frac{1}{N} e^\rho{}_b e^\sigma{}_d \mathcal{L}_m (e_\rho{}^e e_\sigma{}^f K_{ef}) \,. \tag{2.2.55}$$

The previous result allow us to write the decomposition as:

$$\begin{aligned}
R_{\alpha\beta\gamma\delta} = \; & e_\alpha{}^a e_\beta{}^b e_\gamma{}^c e_\delta{}^d \big[{}^{(3)}R_{abcd} - (K_{ad}K_{bc} - K_{ac}K_{bd}) \big] \\
& - \big(n_\alpha e_\beta{}^b e_\gamma{}^c e_\delta{}^d - e_\alpha{}^b n_\beta e_\gamma{}^c e_\delta{}^d + e_\alpha{}^c e_\beta{}^d n_\gamma e_\delta{}^b - e_\alpha{}^c e_\beta{}^d e_\gamma{}^b n_\delta \big) \big[D_d K_{bc} - D_c K_{bd} \big] \\
& + \big(n_\alpha e_\beta{}^b n_\gamma e_\delta{}^d - n_\alpha e_\beta{}^b e_\gamma{}^d n_\delta - e_\alpha{}^b n_\beta n_\gamma e_\delta{}^d + e_\alpha{}^b n_\beta e_\gamma{}^d n_\delta \big) \Big[D_d a_b + a_d a_b \\
& + K^e{}_d K_{be} - \frac{1}{N} e^\rho{}_b e^\sigma{}_d \mathcal{L}_m (e_\rho{}^e e_\sigma{}^f K_{ef}) \Big] \,.
\end{aligned} \tag{2.2.56}$$

This result allows us to take contractions with unprojected quantities because the projectors appear outside the expression.

Now we are equipped with the projective technology of the ADM decomposition to study the cosmology in modified theories for gravity.

Chapter 3

Theoretical Framework II: Standard-Model Extension

The Standard Model of particle physics (SM) and General Relativity (GR) stand as the most successful frameworks describing the fundamental interactions of nature. Each describes physical phenomena within its own domain: the SM governs microscopic interactions, whereas GR accounts for macroscopic gravitational phenomena. Despite their remarkable success, they are not mutually compatible: the Standard Model is formulated on a fixed spacetime background, while General Relativity treats spacetime as a dynamical entity. It is widely expected that these theories are merely the low-energy limit of some more fundamental theory that would take over as the characteristic energies involved in experiments approach the Planck scale, $10^{19}GeV$. Experimental information to guide the development of a Planck-scale theory would, by conventional thinking, come from Planck-energy experiments, which are likely to remain infeasible far into the future. An alternative approach is to search for small deviations from known physics (the SM and GR) in present-day experiments, with the hope that small deviations, if found, would encode information about the underlying theory.

Lorentz symmetry, the idea that physical results are unchanged under rotations and boosts of the system, and CPT symmetry, the associated invariance of the system under the combination of discrete symmetries of charge conjugation, parity, and time reversal, are pillars of both the SM and GR. Hence violations of these symmetries, if found, would provide a novel signal of new physics. Moreover, the possibility of violations of these symmetries has been demonstrated in candidates for the underlying theory [8].

The systematic search for Lorentz and CPT violation using the comprehensive effective field theory based framework of the Standard-Model Extension (SME) provides a method of searching for Planck-suppressed effects in known physics in a complete and organized way.

3.1 Standard-Model Extension

In 1989, Alan Kostelecký and Stuart Samuel studied the possibility of spontaneous breakdown of Lorentz symmetry in string theory through covariant string field theory [8]. In order to describe a world with four flat dimensions, string theory require to break a 26- or 10-dimensional Poincaré symmetry through some mechanism. Kostelecký and Samuel proposed an alternative way to break this symmetry: a unstable vacuum of the string can lead to the Lorentz-symmetry breakdown through the spontaneous symmetry breaking mechanism. Unlike the conventional Standard Model, string theories typically involve interactions that could destabilize the naive vacuum and trigger the generation of nonzero expectation values for Lorentz tensors. If the breaking extends into the four macroscopic spacetime dimensions, apparent Lorentz violation could occur at the level of the Standard Model. This would represent a possible observable effect from the fundamental theory, originating outside the structure of conventional renormalizable gauge models. This motivated the exploration of Lorentz-violating phenomenology and stablished the foundations of the Standard-Model Extension [15].

The SME adds to known physics all Lorentz and CPT violating effects at the level of the action. The terms added to the action of the SM and GR to form the SME are generated from Lorentz and CPT violating operators acting on SM and GR fields along with coefficients for Lorentz and CPT violation that parameterize the amount of symmetry violation in the theory. The addition of Lorentz and CPT violating terms can be thought of as a series expansion about known physics in ever increasing mass dimension of the operators involved. The SME coefficients can then be sought in experiment. Over 1000 limits on SME coefficients have been set via experiment and observation [17], but much remains to be explored, particularly in the case of the so-called nonminimal operators of mass dimension greater than 4, where few constraints have been set by the direct analysis of experimental data. It should be emphasized that the SME is a test framework designed for a broad search for yet-unobserved symmetry violation, a philosophy that is quite different from model building. Though the SME is unique in providing a comprehensive test framework at the level of the action, other approaches to the study of Lorentz and CPT violation exist and the idea of a general test framework over specific models has philosophical resonance with efforts to parameterize deviations from GR.

3.1.1 Observer and particle transformations

It is convenient and useful to distinguish two notions of transformations, called particle and observer transformations [16]. Particle transformations change dynamical particles and fields, while observer transformations change the observer frame. In the absence of backgrounds, the component forms of the two transformations are inverses of each other and in that context are sometimes called active and passive. However, this equivalence fails in the presence of backgrounds.

A particle transformation affects dynamical particles and fields but leaves any backgrounds invariant, which can modify the physics associated with couplings between the dynamical variables and the background. In contrast, an observer transformation amounts to a coordinate transformation, which changes the components of fields and backgrounds but is assumed to leave the physics invariant.

A physical symmetry associated with a given particle transformation can therefore be violated in the presence of backgrounds, even though the physics remains invariant under the corresponding observer transformation. Mathematically, particle transformations involve mappings of the spacetime manifold and its tangent and cotangent bundles, whereas observer transformations are implemented on the atlas of the manifold. Since physics is independent of the coordinate frames used for the atlas but can depend on the manifold mappings, discussions of symmetry violations are best conducted in the language of particle transformations without invoking frame changes.

Physical Lorentz-violating effects are features of experimental configurations of particles and fields rather than features of the observer, so the general treatment of Lorentz violation cannot readily be described using modified observer transformations [7].

3.1.1.1 Toy model for Lorentz violations

Lorentz symmetry contains both rotations and boosts. Now we will appeal to the visual nature of rotation invariance and consider examples of rotation invariance and rotation-invariance violation as examples of the Lorentz violations [28].

Let us consider a classical non-relativistic theory for a particle in a magnetic field

$$L = \frac{1}{2}m|\dot{\vec{r}}|^2 + q\,\dot{\vec{r}}\cdot A \tag{3.1.1}$$

where the EOM are

$$\vec{a} = \frac{q}{m}\dot{\vec{r}} \times \vec{B} \tag{3.1.2}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A}\,. \tag{3.1.3}$$

We will first do what is known as an observer transformation on this system, in this case a rotation. This corresponds to the experimenter turning their head and taking their coordinates with them. This transformation is carried out by acting with the standard rotation matrix $R^i{}_j$ on all vector components such that the components of a generic vector $\vec{V}$ transforms as $V^i \to V'^i = R^i{}_j V^j$. Doing this transformation to all vector components in the Lagrangian reveals that it is form invariant. This is a signal that the theory possesses "observer-rotation invariance" as might have been expected. In other words, the outcome of the experiments does not depend on the coordinates used.

To see even more explicitly that the theory is invariant we can perform the following series of steps: we set up a system with some initial conditions, calculate the final configuration using the EOM, and then apply the symmetry transformation to the result. On the other hand we prepare the same initial conditions but now we apply the symmetry to the initial setup, to evolve the transformed initial conditions to the final configuration. As shown in 3.1.1, when these steps are applied for observer rotations the results of paths match, reflecting the obvious observer-rotation invariance of the system.

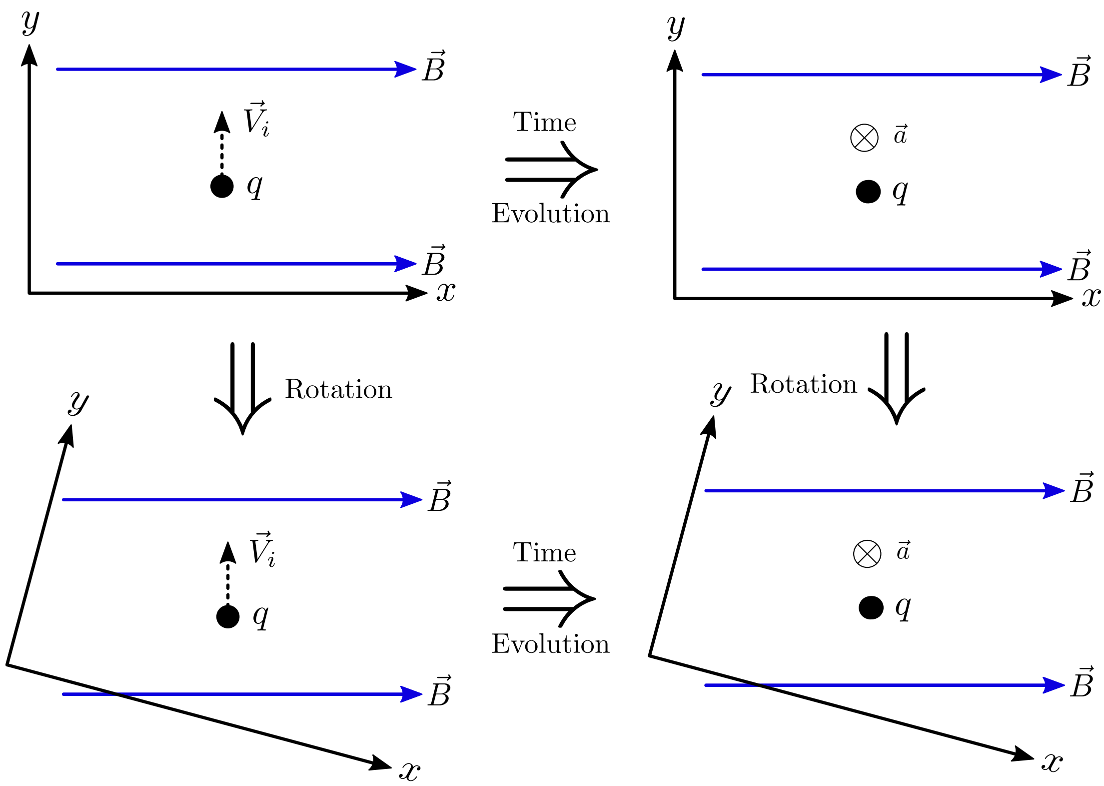


Figure 3.1.1: Illustration of observer symmetry in a sample system.

We can next apply the same procedure for a particle rotation. This procedure leaves the observer and the coordinates fixed but rotates all fields. Operationally, the procedure is carried out the same way on this rotation-invariant system, and as result of the symmetry the outcome will be identical. While we could draw this procedure in an identical way, observer and particle transformations will be distinct when spacetime symmetries are broken. To compare we notice figure 3.1.2 for the particle transformation analog to the previous observer transformation to highlight the difference between rotating the coordinates and rotating the physical system.

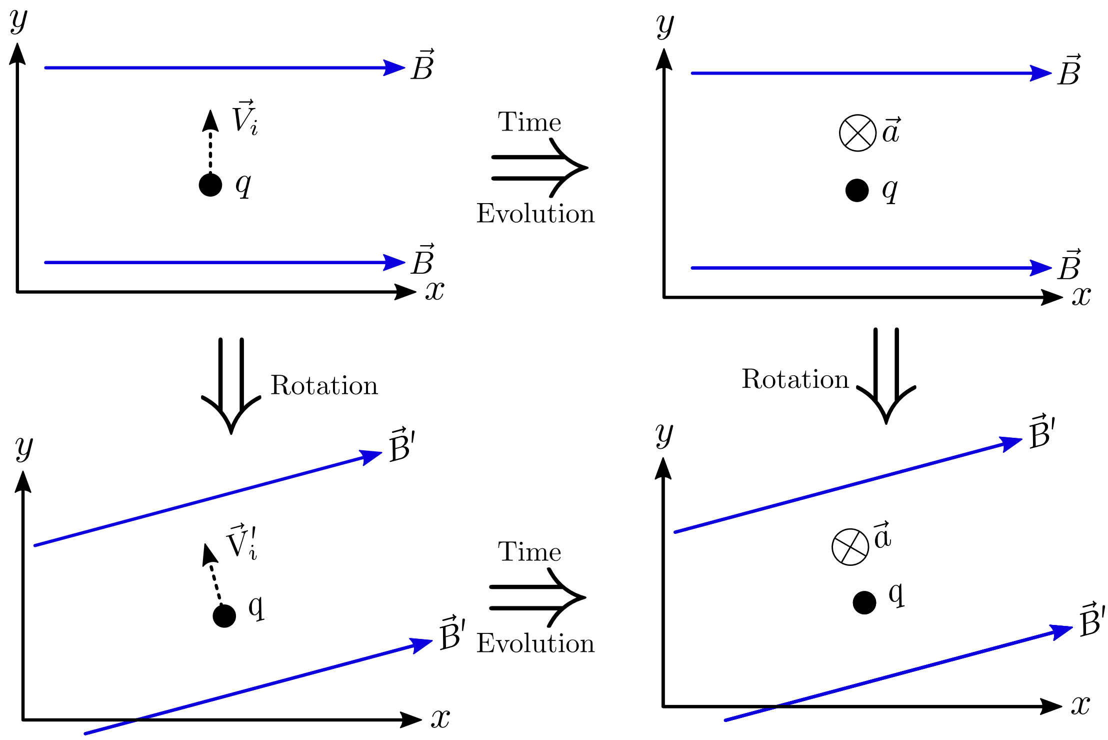


Figure 3.1.2: Illustration of particle-rotation symmetry in a sample system.

We can now use this system as a toy model for broken Lorentz invariance. Suppose that the experimenter was unaware of the physics generating the vector potential and hence the magnetic field. Perhaps it exists on much larger scale than their lab. If they now perform a particle rotation on the system they will rotate the apparatus in their lab, but not the magnetic field. Here the transformed theory will be

$$\begin{aligned} L &= \frac{1}{2}m|\dot{\vec{r}}'|^2 + q\,\dot{\vec{r}}' \cdot \vec{A} \\ &= \frac{1}{2}m|\dot{\vec{r}}|^2 + q\,(\dot{\vec{r}}\mathbf{R}^t) \cdot \vec{A} \end{aligned} \tag{3.1.4}$$

where $\mathbf{R}^t$ is the transposed rotation matrix. Note that the theory is no longer particle-rotation invariant as the form has changed. Applying this transformation to our toy example leads to the situation shown in 3.1.3 in which the acceleration found in the transformed system is different (acceleration is smaller under the conditions shown) than the original system.

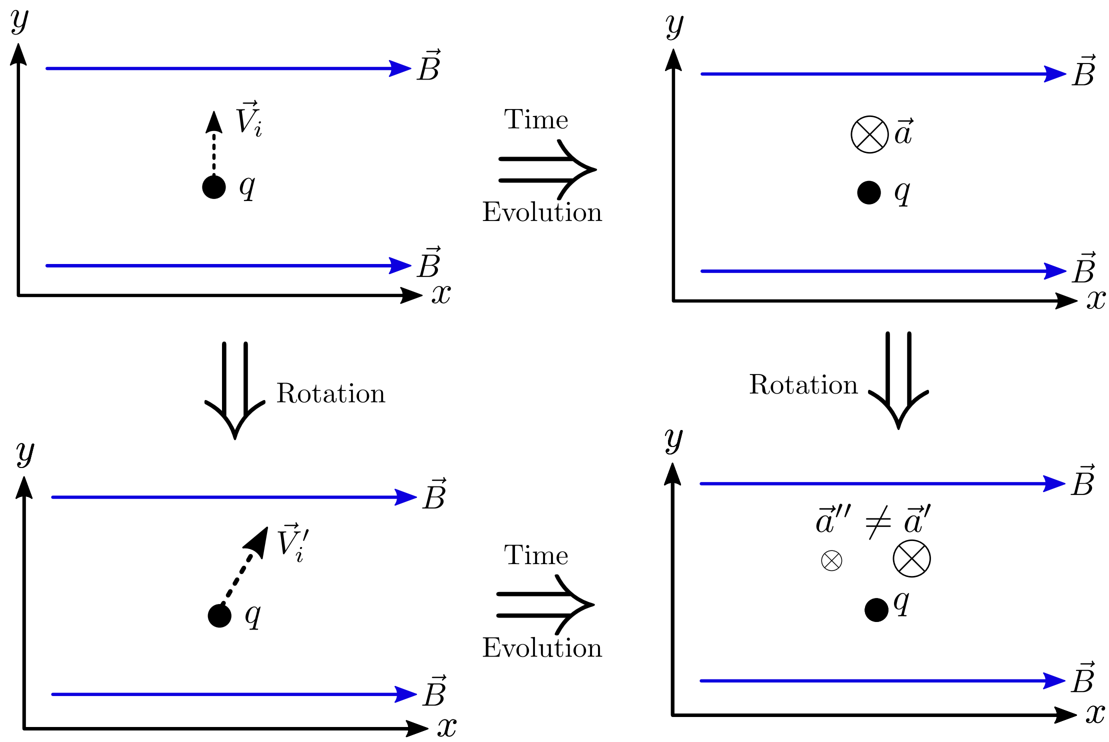


Figure 3.1.3: Illustration of effective particle-rotation symmetry violation in a sample system

Workers in the lab will then know if the system is particle-Lorentz invariant or not by performing their experiment, then rotating it, then comparing the results. If the acceleration is different in the rotated system, particle-Lorentz invariance is broken. In our discussion of the SME to follow, it will be large-scale fields called coefficients for Lorentz violation that will play the role of $\vec{B}$ in this example. We also note that if an undetected large-scale conventional field existed in the lab, it could also be detected in this way, an idea that has been applied to efforts to detect spacetime torsion [29] and gravitomagnetic effects in the lab [30].

3.1.2 Spacetime transformations

In Minkowski spacetime, the central spacetime transformations are global transformations that include spatial rotations, Lorentz boosts, and translations. The rotations and Lorentz boosts form the group of Lorentz transformations, which is enlarged by translations to the Poincare group. All these Minkowski-spacetime transformations are isometries of the Minkowski metric $\eta_{\mu\nu}$, and they move spacetime points. For example, a global rotation about a point P in the spacetime maps all points other than P into different points. In contrast, the metric $g_{\mu\nu}$ in a generic curved spacetime typically has no isometries, and so the usual notions of global Lorentz transformations and translations play no particular role. Instead, it is useful to study local Lorentz transformations and diffeomorphisms. Local Lorentz transformations are Lorentz transformations in the tangent space at each spacetime point, leaving the spacetime point unmoved. Under a local Lorentz transformation, the vierbein and metric transform as

$$\begin{aligned} \mathbf{e}_\mu{}^a(x) &\rightarrow \Lambda^a{}_b(x)\mathbf{e}_\mu{}^b(x)\,, \\ g_{\mu\nu}(x) &\rightarrow g_{\mu\nu}(x), \end{aligned} \tag{3.1.5}$$

where $\Lambda^a{}_b(x)$ are the components of the matrix $\mathbf{\Lambda}(x)$ for the local Lorentz transformation at the point x. Other dynamical boson fields transform similarly, with spacetime indices unchanged and local indices acted on by the components of $\mathbf{\Lambda}(x)$. Fermion fields are transformed by the corresponding matrices $S(\Lambda(x))$ in the appropriate spinor representation of the local Lorentz group.

Note that local Lorentz transformations at different spacetime points are typically different. However, an associated global transformation can be defined in any curved spacetime by requiring that the same local Lorentz transformation is performed simultaneously at every spacetime point. This can be termed a global local Lorentz transformation, and it is the analogue of a global gauge transformation constructed from a local gauge transformation in a gauge field theory. Global local Lorentz transformations leave spacetime points fixed, so they cannot be the analogues of global Lorentz transformations in Minkowski spacetime. Instead, the analogues can be taken to be certain types of Lorentz transformations defined in approximately flat spacetimes.

Diffeomorphisms in a curved spacetime capture the idea of moving spacetime points. Under a diffeomorphism, a spacetime point at position x is mapped to another point at x' according to

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x), \tag{3.1.6}$$

where $\xi^\mu(x)$ is smooth and the mapping is assumed invertible. Here x'^μ denotes the components of the new position in the original coordinates, which remain unchanged by the transformation. Dynamical fields on the manifold transform according to the pushforward or pullback induced by the diffeomorphism. For example, the vierbein and metric transform as

$$\mathbf{e}_\mu{}^a(x) \;\rightarrow\; \mathbf{e}'_\mu{}^a(x') = \frac{\partial x^\rho}{\partial x'^\mu}\mathbf{e}_\rho{}^a(x), \tag{3.1.7}$$

$$g_{\mu\nu}(x) \;\rightarrow\; g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x), \tag{3.1.8}$$

where $\mathbf{e}'_\mu{}^a(x)$ and $g'_{\mu\nu}(x')$ are the new vierbein and metric at the point x' after the diffeomorphism. In contrast, dynamical fields valued in local frames, including spinor fields, transform like scalar fields under a diffeomorphism.

Although the expressions appear similar to those for a general coordinate transformation, the physical interpretation is different. Only the coordinates change under general coordinate transformations, leaving physical particles and fields invariant. General coordinate transformations can thus be identified as observer diffeomorphisms. In contrast, the particle diffeomorphisms of interest here change physical particles and fields while leaving the coordinate system unchanged. Fields can be valued at any position on the manifold. When valued at the same position, dynamical fields undergoing a diffeomorphism with infinitesimal $\xi^\mu(x)$ change by the corresponding Lie derivative. For example, under an infinitesimal diffeomorphism the vierbein and metric transform as

$$\begin{aligned} \mathbf{e}'_\mu{}^a(x) &= \mathbf{e}_\mu{}^a(x) - \mathcal{L}_\xi \mathbf{e}_\mu{}^a(x) \\ &= \mathbf{e}_\mu{}^a - \mathbf{e}_\rho{}^a\partial_\mu\xi^\rho - \xi^\lambda\partial_\lambda\mathbf{e}_\mu{}^a, \end{aligned} \tag{3.1.9}$$

$$\begin{aligned} g'_{\mu\nu}(x) &= g_{\mu\nu}(x) - \mathcal{L}_\xi g_{\mu\nu}(x) \\ &= g_{\mu\nu} - g_{\rho\nu}\partial_\mu\xi^\rho - g_{\mu\sigma}\partial_\nu\xi^\sigma - \xi^\lambda\partial_\lambda g_{\mu\nu}\,. \end{aligned} \tag{3.1.10}$$

Equations where dynamical fields are valued at the same position are often used in calculations involving diffeomorphisms, and they are distinct from ones like (3.1.8) in which the fields are valued at different positions.

3.1.3 Backgrounds

A given term in the Lagrange density $\mathcal{L}$ of the general effective field theory extending GR coupled to the SM is the product of a field operator $\mathcal{O}(x)$ with a coupling coefficient $k(x)$ or its derivatives. Since it plays the role of a coupling, k can be viewed as a background in the theory or equivalently as a nonzero vacuum value of a field. This perspective holds irrespective of the detailed origin or nature of the coefficient in the context of the underlying theory.

Since the field operator $\mathcal{O}$ may behave nontrivially under spacetime transformations and since the Lagrange density is a scalar density under general coordinate transformations, the background k can carry spacetime and local indices. In effective field theory, the operator $\mathcal{O}$ is bosonic and so contains spinor fields only as combinations of fermion bilinears. The background k therefore carries no spinor indices. For definiteness and simplicity, we assume here that k carries no indices associated with any internal gauge degrees of freedom. We can therefore treat k as a tensor under general coordinate transformations and under observer local Lorentz transformations. Note that k must remain invariant under all particle transformations, including both diffeomorphisms and local Lorentz transformations, because it is nondynamical by construction.

Modulo possible derivatives acting on the background, the general structure of a term in the Lagrange density $\mathcal{L}$ can therefore be written in the form

$$\mathcal{L} \supset k^{\mu\ldots}{}_{\nu\ldots}{}^{a\ldots}(x)\mathcal{O}_{\mu\ldots}{}^{\nu\ldots}{}_{a\ldots}(x)\,, \tag{3.1.11}$$

where $\mathcal{O}$ contains all dynamical fields including any factors involving the vierbein $\mathbf{e}_\mu{}^a$ and metric $g_{\mu\nu}$. If derivatives acting on the background are present, their indices must also be contracted to insure that $\mathcal{L}$ remains a scalar density.

Two classes of backgrounds k can conveniently be identified, according to whether they are spontaneous or explicit. As the two classes have different physical implications, for clarity in much of what follows we denote spontaneous backgrounds by $\langle k\rangle$ and explicit ones by $\bar{k}$. Spontaneous backgrounds $\langle k\rangle$ arise as solutions of the equations of motion in the underlying theory and hence are vacuum expectation values of underlying fields. They satisfy the equations of motion and are thus on-shell quantities. Fluctuations of the underlying fields about $\langle k \rangle$ then exist and can represent additional modes in the effective theory, including Nambu-Goldstone and massive modes.

In contrast, explicit background fields $\bar{k}$ are nondynamical. They are unconstrained by equations of motion and hence can be off shell. Moreover, no dynamical fluctuations about them exist. Intuitively, a spontaneous background $\langle k \rangle$ can be viewed as a special nondynamical background $\bar{k}$ that must be on shell and that has accompanying dynamical fluctuations. The on-shell restriction and the presence of dynamical fluctuations imply that the backgrounds $\langle k \rangle$ and $\bar{k}$ are associated with distinct physics.

The presence of a background can violate spacetime symmetries because backgrounds behave differently from dynamical fields under particle spacetime transformations. Both backgrounds and dynamical fields behave covariantly under observer transformations, which ensures invariance of the physics under coordinate changes. For instance, physical invariance under general coordinate transformations, which are observer diffeomorphisms, is assumed to be a property of a realistic theory. However, backgrounds are invariant under particle transformations, while dynamical fields transform covariantly. This difference can lead to physical symmetry violations in observables that involve dynamical fields coupled to a background.

Consider, for example, a generic background $k^{a\cdots}$ in a local frame. This can be viewed as specifying an orientation in the frame, sometimes called a preferred direction, which is invariant under local Lorentz transformations. Unless $k^{a\cdots}$ happens to have no indices and is independent of position, or unless it is proportional to combinations of the Lorentz-group invariants η_{ab} and ϵ_{abcd}, the coupling of a dynamical field to $k^{a\cdots}$ can produce changes of physical observables under local rotations or local Lorentz boosts. These are violations of local Lorentz invariance, which can thus be traced to a direction-dependent background in a local frame. Note that even a scalar background $k(x)$ without indices but varying with spacetime position can introduce violations of local Lorentz invariance because the derivatives of $k(x)$ specify an orientation in a local frame. Similarly, a generic background $k^{\mu\cdots}{}_{\nu\cdots}(x)$ on the spacetime manifold can lead to violations of diffeomorphism invariance unless it has no indices and is independent of spacetime position. Only a background serving as a scalar coupling constant, such as the expectation value of the Higgs field in the SM, can preserve local Lorentz invariance and diffeomorphisms.

For explicit backgrounds, the above results hold without further subtleties. An explicit background $\bar{k}^{\mu\ldots}{}_{\nu\ldots}(x)$ defined both on the manifold and in local frames violates local Lorentz and diffeomorphism invariance in ways determined directly by its index structure and by its nonvanishing derivatives. Consequently a generic theory contains spontaneous local Lorentz violation if and only if it contains spontaneous diffeomorphism violation.

3.1.4 Effective field theory

The action of the effective field theory is defined as usual via integration over the spacetime manifold,

$$S = \int \mathrm{d}^4x \; e\mathcal{L}\,, \tag{3.1.12}$$

and is assumed invariant by construction under general coordinate transformations, which can be understood as observer diffeomorphisms. A generic term in $\mathcal{L}$ involving a background takes the form of the coupling previously described and its generalized by incorporating background derivatives. The properties of the term under local Lorentz transformations and diffeomorphisms are determined by the index structure and spacetime dependence of the background k.

We aim to construct a realistic effective field theory involving gravity and matter in the presence of arbitrary backgrounds. This enables the explicit derivation of all desired terms in the action, including ones in the puregravity sector and those involving matter-gravity couplings to gauge fields, fermions, and scalars. It also yields the terms describing the dynamics of the background.

We start by separating a general SME+Gravity action principle as follows,

$$S = \int \mathrm{d}^4x \; e\big(\mathcal{L}_g + \mathcal{L}_A + \mathcal{L}_\psi + \mathcal{L}_\phi\big), \tag{3.1.13}$$

where $\mathcal{L}_g$ contains pure-gravity terms and any background dynamics, $\mathcal{L}_A$ describes gauge fields and their gravity couplings, $\mathcal{L}_\psi$ involves fermions including their gravity and gauge couplings, and $\mathcal{L}_\phi$ contains all terms with scalars.

As we previously mentioned, any single effective term in the Lagrange density $\mathcal{L}$ takes the form of a contraction between a dynamical operator $\mathcal{O}$ and a background

k or its derivatives. A specific operator $\mathcal{O}$ may be contracted directly to one or more backgrounds k or their derivatives, or may be contracted instead via combinations of the vierbein, metric, and Levi-Civita tensor. It is convenient to adopt a compact notation for these various types of backgrounds and contractions, thereby simplifying expressions in the Lagrange density.

The idea is to introduce a quantity $\breve{k}^{\mu\dots}{}_{\nu\dots}{}^{a\dots}$ that is a linear combination of all terms formed from background fields, vierbeins, metrics, and the Levi-Civita tensor. Note that multiple vierbein and metric factors may appear in a given term, but at most one Levi-Civita factor is needed because products of the Levi-Civita tensor reduce to products of vierbeins or metrics. Contracting the combination $\breve{k}^{\mu\dots}{}_{\nu\dots}{}^{a\dots}$, with any specific operator $\mathcal{O}_{\mu\dots}{}^{\nu\dots}{}_{a\dots}$ then produces a single expression in the Lagrange density $\mathcal{L}$ of the form $\mathcal{L} \supset \breve{k}^{\mu\dots}{}_{\nu\dots}{}^{a\dots}(x)\mathcal{O}_{\mu\dots}{}^{\nu\dots}{}_{a\dots}(x)$. Terms involving contractions between dynamical operators and derivatives of backgrounds can also be combined in this way by using derivatives of $\breve{k}^{\mu\dots}{}_{\nu\dots}{}^{a\dots}$.

To construct terms in the Lagrange density $\mathcal{L}$, we require a procedure to build suitable dynamical operators. For effective field theory based on GR and gauge theory, the terms must be independent of observer general coordinate transformations and be locally gauge invariant.

For the gauge symmetry, consider first the scenario in Minkowski spacetime with a Dirac fermion ψ in a representation U of the gauge group. Then $\psi \to U\psi$ under a gauge transformation, while the Dirac conjugate transform as $\bar{\psi} \to \bar{\psi}U^\dagger$. The gauge-covariant derivative acting on ψ can be written as $D_\mu\psi = \partial_\mu\psi - igA_\mu\psi$, where g is the gauge coupling and A_μ is the gauge field in the U representation, and it transforms as $D_\mu \to UD_\mu U^\dagger$. The gauge field strenght $F_{\mu\nu}$ in the U representation is generated by the commutator $\left[D_\mu, D_\nu\right] = -igF_{\mu\nu}$. By definition, an operator $\mathcal{O}$ formed from gauge fields is called gauge covariant if $\mathcal{O} \to U\mathcal{O}U^\dagger$. Given gauge-covariant operators $\mathcal{O}$ and $\mathcal{O}'$, two kinds of gauge-invariant operators can be constructed, $\mathrm{tr}(\mathcal{O})$ and $(\bar{\mathcal{O}\psi})\Gamma\mathcal{O}'\psi$, where Γ represents the 16 matrices $\{1, i\gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}/2\}$ spaning the spinor space. These gauge-invariant operators are the desired objects from which to build terms in the Lagrange density for the effective field theory in Minkowski spacetime.

To generalize this construction to curved spacetime, we can work with spacetime-tensor fields and covariant derivatives extended to include an appropriate connection. Relevant spacetime-tensor fields include the metric $g_{\mu\nu}$, the curvature tensor $R_{\kappa\lambda\mu\nu}$, the gauge field strength $F_{\mu\nu}$ the spinor bilinears $\bar{\psi}\Gamma\psi$, scalars ϕ, and combinations (where now the Gamma matrices are defined in the local frame). In combinations, gauge-invariant operators are placed inside fermion bilinears. All these spacetime-tensor operators are also gauge covariant. The covariant derivative D_μ acting on ψ or on $DD\dots D\psi$ can be expressed as

$$D_\mu = \nabla_\mu + \frac{1}{4}\omega_\mu{}^{ab}\sigma_{ab} - igA_\mu\,, \tag{3.1.14}$$

where ∇_μ is the usual covariant derivative of GR containing the partial derivative ∂_μ and the appropiate connection term formed using Christoffel symbols, and where $\omega_\mu{}^{ab}$ is the spin connection. Direct calculation shows that any mixture of covariant derivatives and the spacetime-tensor operators also form a gauge-covariant spacetime-tensor operator.

Building the Lagrange density $\mathcal{L}$ using all possible gauge-covariant spacetime-tensor operators would introduce many redundancies due to relationships between various mixtures of operators. It is therefore useful to work instead with a standard basis set that has controlled redundancies or not. The key result here is that any mixture of g, R, F, Γ, and D can be written in the standard form

$$\big[(D_{(n_1)}R)\dots(D_{(n_t)}R)\big]\big[(D_{(m_1)}F)\dots(D_{(m_s)}F)\big]\Gamma D_{(l)}, \tag{3.1.15}$$

where all indices on spacetime-tensor fields are omitted for simplicity. In this expression, we introduce the notation

$$D_{(n)} \equiv \frac{1}{n!}D_{\alpha_1}D_{\alpha_2}\dots D_{\alpha_n}\,, \tag{3.1.16}$$

as a symmetric sum over the n indices. Next we consider the linear independence of the operators. Note that the operators $D_{(n_i)}R$ and $D_{(n_j)}R$ are linearly independent when $n_i \neq n_j$ because they have different mass dimensions. Also since $D_{(n_i)}R$ commutes with $D_{(n_j)}R$, we can impose $n_1 \leq \dots \leq n_t$ on the basis. In the case for $D_{(n_i)}F$ and $D_{(n_j)}F$ they are linearly independent when $n_i \neq n_j$, but they commute only for abelian gauge field theories. In this case we chose as basis

$$\Big\{\big[(D_{(n_1)}R)\dots(D_{(n_t)}R)\big]\big[(D_{(m_1)}F)\dots(D_{(m_s)}F)\big]\Gamma D_{(l)}|n_1\le\dots\le n_t, m_1\le\dots\le m_s\Big\}. \tag{3.1.17}$$

In a nonabelian gauge theory, we choose instead the basis

$$\Big\{\big[(D_{(n_1)}R)\dots(D_{(n_t)}R)\big]\big[(D_{(m_1)}F)\dots(D_{(m_s)}F)\big]\Gamma D_{(l)}|n_1\le\dots\le n_t\Big\}. \tag{3.1.18}$$

This basis is linearly independent in some cases and is almost linearly dependent in others, depending on the structure of the gauge group. We do not discuss here the particular choices required for Hermitian Lagrange densities. We will discuss how it appears in purely gravitational terms in section 3.2.

3.1.5 Explicit violations incompatibilities

In any model based on Riemann geometry or its extensions to include torsion and nonmetricity, the fields must satisfy the Bianchi identities, which are intrinsically imposed by the geometric structure. The Bianchi identities hold both on and off shell, and their compatibility with the variational principle imposes constraints that must be satisfied for consistency of the model. In GR, for example, the Bianchi identity implies the on-shell conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu}=0$, which is compatible with the dynamics and symmetries of the theory obtained by variation of the action. Similarly, in a model with spontaneous violation of one or more spacetime symmetries, compatibility with the Bianchi identities is maintained because the variational procedure is standard.

However, explicit violation of a spacetime symmetry requires the presence in the action of one or more nondynamical background fields $\bar{k}^{\mu\dots}{}_{\nu\dots}{}^{a\dots}(x)$, which behave unconventionally under variations. The variational results can then become incompatible with implications from the Bianchi identities and hence can render a problematic model containing an explicit violation. This can induce inconsistencies in the model or impose unnatural requirements such as fine tuning of the explicit background. The potential constraints on a model with explicit violation of spacetime symmetries are called no-go constraints. Their role has been the subject of extensive recent investigation by Bluhm and collaborators [31].

For explicit diffeomorphism violation, the no-go constraints can be identified with the Noether identities arising from the requirement of general coordinate invariance of the model. Consider first a model with an explicit background $\bar{k}_{\mu_1 \cdots \mu_n}$ carrying n covariant spacetime indices. The associated current can be defined as usual by variation of the action, $J^{\mu_1 \cdots \mu_n} \equiv \delta S/\delta \bar{k}_{\mu_1 \ldots \mu_n}$. Following the variation of the action principle and considering the variation as an particle diffeomorphism we will arrive to the following condition over the Bianchi identity:

$$\begin{aligned} 2\nabla_\mu T^\mu{}_\nu &= J^{\mu_1 \cdots \mu_n} \nabla_\nu \bar{k}_{\mu_1 \ldots \mu_n} - \nabla_{\mu_1}\left(J^{\mu_1 \cdots \mu_n} \bar{k}_{\nu \mu_2 \ldots \mu_n}\right) \\ &\quad - \cdots - \nabla_{\mu_n}\left(J^{\mu_1 \cdots \mu_n} \bar{k}_{\mu_1 \ldots \mu_{n-1} \nu}\right) \\ &= 0 \,. \end{aligned} \tag{3.1.19}$$

It represents four no-go constraints that must be obeyed by the model for internal consistency. Both results must hold at the same time to avoid discrepancies.

3.2 Gravitational SME

With the construction of generic gauge-invariant spacetime-tensor operators in hand, we can address specific sectors of the effective field theory in turn. In this subsection, we consider operators involving pure-gravity fields. For the pure-gravity sector, it is convenient to distinguish terms in the Lagrange density $\mathcal{L}_g$ according to mass dimension. We therefore write

$$\mathcal{L}_g = \frac{1}{2\kappa}\left(\mathcal{L}_{g0} + \mathcal{L}_g^{(2)} + \mathcal{L}_g^{(3)} + \mathcal{L}_g^{(4)} + \mathcal{L}_g^{(5)} + \mathcal{L}_g^{(6)} + \ldots\right), \tag{3.2.1}$$

where $1/2\kappa \equiv 1/16\pi G_N \simeq 3 \times 10^{36} GeV^2$ is the gravitational coupling constant formed from the Newton gravitational constant G_N. The term $\mathcal{L}_{g0} = R - 2\Lambda$ is the conventional Einstein-Hilbert expression with cosmological constant, while the terms $\mathcal{L}_g^{(d)}$ represent contributions to the effective field theory. Note that each individual component $\mathcal{L}_g^{(d)}$ has mass dimension two, but by convention the superscript d represents the mass dimension of the dynamical operator in $\mathcal{L}_g^{(d)}$ including the factor of the gravitational coupling constant. For example, $\mathcal{L}_g^{(4)}$

includes terms with the Riemann tensor as dynamical operator, which in this convention is of mass dimension two but $1/2\kappa$ is also of mass dimension two, thus we arrive to the (4) in the $\mathcal{L}_g^{(4)}$ notation.

Component	Expression
$\mathcal{L}_g^{(2)}$	$\breve{k}$
$\mathcal{L}_g^{(3)}$	$\mathcal{L}_{g,\delta}^{(3)} + \breve{k}(Dk)$
$\mathcal{L}_g^{(4)}$	$\breve{k}R + \breve{k}(Dk)(Dk)$
$\mathcal{L}_g^{(5)}$	$\breve{k}DR + \mathcal{L}_{g,\delta}^{(5)} + \breve{k}(Dk)R + \breve{k}(Dk)(Dk)(Dk) + \breve{k}(Dk)(D_{(2)}k)$
$\mathcal{L}_g^{(6)}$	$\breve{k}RR + \breve{k}D_{(2)}R + \breve{k}(Dk)(Dk)R + \breve{k}(D_{(2)}k)R$
	$+\breve{k}(Dk)(Dk)(Dk)(Dk) + \breve{k}(Dk)(Dk)(D_{(2)}k) + \breve{k}(D_{(2)}k)(D_{(2)}k)$

Table 3.2.1: All terms in $\mathcal{L}_g^{(d)}$ with $d \leq 6$ in schematic form. k can represent a distinct background even when occuring in a single term, and the various quantities $\breve{k}$ may also be distinct. All terms are invariant under general coordinate transformations except for $\mathcal{L}_{g,\delta}^{(d)}$ with $d = 3, 5$, which transform into a total derivative.

The $\breve{k}$ in every term can have different mass dimension. We have the freedom to use surface term to relate the differential operators acting over k in terms of dynamical operators in the expansion. Taking this in consideration we write the decomposition using only $\breve{k}$ fields. This allow us to write the explicit counterpart of the previous table:

Component	Expression
$\mathcal{L}_{g0}$	$R - 2\Lambda$
$\mathcal{L}_g^{(2)}$	$\breve{k}^{(2)}$
$\mathcal{L}_g^{(3)}$	$(\breve{k}_\Gamma^{(3)})^\mu \Gamma^\alpha{}_{\mu\alpha}$
$\mathcal{L}_g^{(4)}$	$(\breve{k}_R^{(4)})^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$
$\mathcal{L}_g^{(5)}$	$(\breve{k}_D^{(5)})^{\alpha\beta\gamma\delta\kappa} D_\kappa R_{\alpha\beta\gamma\delta} + (\breve{k}_{CS,1}^{(5)})_\kappa \epsilon^{\kappa\lambda\mu\nu} \eta_{ac}\eta_{bd} \big(\omega_\lambda{}^{ab} \partial_\mu \omega_\nu{}^{cd} + \frac{2}{3} \omega_\lambda{}^{ab} \omega_\mu{}^{ce} \omega_{\nu e}{}^d\big)$
	$+(\breve{k}_{CS,2}^{(5)})_\kappa \epsilon^{\kappa\lambda\mu\nu} \epsilon_{abcd} \big(\omega_\lambda{}^{ab} \partial_\mu \omega_\nu{}^{cd} + \frac{2}{3} \omega_\lambda{}^{ab} \omega_\mu{}^{ce} \omega_{\nu e}{}^d\big)$
$\mathcal{L}_g^{(6)}$	$(\breve{k}_D^{(6)})^{\alpha\beta\gamma\delta\kappa\lambda} D_{(\kappa} D_{\lambda)} R_{\alpha\beta\gamma\delta} + (\breve{k}_R^{(6)})^{\alpha_1\beta_1\gamma_1\delta_1\alpha_2\beta_2\gamma_2\delta_2} R_{\alpha_1\beta_1\gamma_1\delta_1} R_{\alpha_2\beta_2\gamma_2\delta_2}$

Table 3.2.2: Explicit index structure for $\mathcal{L}_g^{(d)}$ with $d \leq 6$. The quantities $(\breve{k}_{CS,1}^{(5)})_\kappa$, $(\breve{k}_{CS,2}^{(5)})_\kappa$ are coupled with Chern-Simmons invariants.

More sectors of the SME can be found in [32].

When this extension is coupled to matter and Gauge fields through the energy-momentum tensor and the spin density tensor, the field equations becomes:

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = \kappa T_{\mathbf{e}}{}^{\mu\nu}, \tag{3.2.2}$$

$$\widehat{T}^{\lambda\mu\nu} = \kappa S_{\omega}{}^{\lambda\nu\mu}, \tag{3.2.3}$$

where the trace corrected torsion tensor $\widehat{T}^{\lambda\mu\nu}$ is given by

$$\widehat{T}^{\lambda\mu\nu} \equiv T^{\lambda\mu\nu} + T^{\alpha}_{\ \alpha\mu} g_{\lambda\nu} + T^{\alpha}_{\ \alpha\nu} g_{\lambda\mu}. \tag{3.2.4}$$

In terms of *second order formalism quantities* the Lorentz violating sector will be written as

$$\begin{aligned} \mathcal{L}^{LV}_{\mathbf{e},\omega} &= \mathbf{e}(k_T)^{\lambda\mu\nu} T_{\lambda\mu\nu} + \mathbf{e}(k_R)^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} + \mathbf{e}(k_{TT})^{\alpha\beta\gamma\lambda\mu\nu} T_{\alpha\beta\gamma} T_{\lambda\mu\nu} \\ &\quad + \mathbf{e}(k_{DT})^{\kappa\lambda\mu\nu} D_\kappa T_{\lambda\mu\nu} + \dots . \end{aligned} \tag{3.2.5}$$

3.2.1 Riemmanian limit

It the case of null torsion, the spin connection will be related to the Christoffel connection as usual

$$\omega_{\mu}{}^{ab} = \mathbf{e}_{\nu}{}^{a} \Gamma^{\nu}{}_{\sigma\mu} \mathbf{e}^{\sigma b} - \mathbf{e}^{\nu b} \partial_\mu \mathbf{e}_{\nu}{}^{a}, \tag{3.2.6}$$

then the metricity and paralelism concepts of the Riemann-Cartan gravity are not independent anymore. This reduces the system to the following table

Component	Expression
$\mathcal{L}_{g0}$	$R - 2\Lambda$
$\mathcal{L}_g^{(4)}$	$\left(k^{(4)}\right)_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$
$\mathcal{L}_g^{(5)}$	$\left(k^{(5)}\right)_{\alpha\beta\gamma\delta\kappa} \nabla^\kappa R^{\alpha\beta\gamma\delta}$
$\mathcal{L}_g^{(6)}$	$\frac{1}{2}\left(k_1^{(6)}\right)_{\alpha\beta\gamma\delta\kappa\lambda} \left\{\nabla^\kappa, \nabla^\lambda\right\} R^{\alpha\beta\gamma\delta} + \left(k_2^{(6)}\right)_{\alpha\beta\gamma\delta\kappa\lambda\mu\nu} R^{\kappa\lambda\mu\nu} R^{\alpha\beta\gamma\delta}$

Table 3.2.3: Torsion free coefficients of the gravitational SME.

The leading order of the model corresponds to the usual Einstein-Hilbert lagrangian with cosmological constant together with the Lorentz violating terms. Considering the minimal sector of the gravitational SME (considering $d \leq 4$) we expand the coefficient $(k_R)^{\kappa\lambda\mu\nu}$ and write the action in the following form:

$$S_{g,\Lambda} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\ \mathrm{d}^4x[(1-u)R - 2\Lambda + s^{\mu\nu}R_{\mu\nu} + t^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}]. \tag{3.2.7}$$

The introduction of the coefficient $u, s^{\mu\nu}, t^{\mu\nu\rho\sigma}$ explicitly distinguish between non-conventional effects related to the Ricci scalar, Ricci tensor and Riemann tensor respectively. This coefficients are real and dimensionless. Every sector inherits the index symmetries of the dynamical operator through contraction. We consider that $s^{\mu\nu}$ and $t^{\mu\nu\rho\sigma}$ are traceless ($g_{\mu\nu}s^{\mu\nu} = 0$ and $g_{\mu\nu}g_{\rho\sigma}t^{\mu\rho\nu\sigma} = 0$), because we can always rewrite the expression to move the traces to the scalar sector u. Also every partial trace (as $t^{\lambda\mu\kappa\nu}g_{\lambda\kappa}$) can be absorbed in $s^{\mu\nu}$.
Notice that considering this conditions is equivalent to write the contractions in terms of the irreducible decomposition of the Riemann curvature tensor:

$$R^T_{\kappa\lambda} = R_{\kappa\lambda} - \frac{1}{D}Rg_{\kappa\lambda} \tag{3.2.8}$$

$$S_{\mu\nu\rho\sigma} = \frac{R}{D(D-1)}\big(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}\big) \tag{3.2.9}$$

$$E_{\mu\nu\rho\sigma} = \frac{1}{D-2}\big(R^T_{\mu\sigma}g_{\nu\rho} - R^T_{\nu\sigma}g_{\mu\rho} - R^T_{\mu\rho}g_{\nu\sigma} + R^T_{\nu\rho}g_{\mu\sigma}\big) \tag{3.2.10}$$

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - S_{\mu\nu\rho\sigma} - E_{\mu\nu\rho\sigma}\ , \tag{3.2.11}$$

where $R^T_{\mu\nu}$ is the traceless Ricci tensor and $W_{\mu\nu\rho\sigma}$ is the Weyl tensor. Thus we can alternatively write the action principle as

$$S_{g,\Lambda} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\ \mathrm{d}^4x[(1-u)R - 2\Lambda + s^{\mu\nu}R^T_{\mu\nu} + t^{\kappa\lambda\mu\nu}W_{\kappa\lambda\mu\nu}]. \tag{3.2.12}$$

Taking the variation of (3.2.7), we obtain:

$$\delta S_{g,\Lambda} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\ \mathrm{d}^4x\Big[\big(-G^{\mu\nu} - \Lambda g^{\mu\nu} + (T^{Rstu})^{\mu\nu}\big)\delta g_{\mu\nu} - R\delta u + R_{\mu\nu}\delta s^{\mu\nu} + R_{\mu\nu\rho\sigma}\delta t^{\mu\nu\rho\sigma}\Big]\ , \tag{3.2.13}$$

where the result on the variation of the background will dependent on the nature of the variation, and we won't take in consideration the boundary terms (yet).

The quantity $(T^{Rst})^{\mu\nu}$ in (3.2.13) is defined as [33]:

$$
\begin{aligned}
(T^{Rstu})^{\mu\nu} \equiv & -\frac{1}{2}\nabla^\mu\nabla^\nu u - \frac{1}{2}\nabla^\nu\nabla^\mu u + g^{\mu\nu}\nabla^2 u + uG^{\mu\nu} \\
& + \frac{1}{2}s^{\alpha\beta}R_{\alpha\beta} + \frac{1}{2}\nabla_\alpha\nabla^\mu s^{\alpha\nu} + \frac{1}{2}\nabla_\alpha\nabla^\nu s^{\alpha\mu} - \frac{1}{2}\nabla^2 s^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\nabla_\alpha\nabla_\beta s^{\alpha\beta} \\
& + \frac{1}{2}t^{\alpha\beta\gamma\mu}R_{\alpha\beta\gamma}{}^{\nu} + \frac{1}{2}t^{\alpha\beta\gamma\nu}R_{\alpha\beta\gamma}{}^{\mu} + \frac{1}{2}t^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}g^{\mu\nu} \\
& - \nabla_\alpha\nabla_\beta t^{\mu\alpha\nu\beta} - \nabla_\alpha\nabla_\beta t^{\nu\alpha\mu\beta}. \qquad (3.2.14)
\end{aligned}
$$

3.2.2 Extended Gibbons-Hawking term

The action principle (3.2.7) requires several boundary-terms to ensure a well-posed variational principle. We start decomposing the action principles in ADM variables. Following the decomposition of the Riemann tensor given in Eq. (2.2.56), the contractions between the curvature tensors and the background fields gives

$$
\begin{aligned}
t^{\alpha\beta\gamma\delta}\,{}^{(4)}R_{\alpha\beta\gamma\delta} = & \; t^{abcd}\big[{}^{(3)}R_{abcd} - (K_{ad}K_{bc} - K_{ac}K_{bd})\big] - 4t^{\mathbf{n}bcd}\big[D_dK_{bc} - D_cK_{bd}\big] \\
& + 4t^{\mathbf{n}b\mathbf{n}d}\big[D_da_b + a_da_b + K^e{}_dK_{be}\big] \\
& + \frac{1}{N}(-4t^{\mathbf{n}b\mathbf{n}d})e^\rho{}_be^\sigma{}_d\mathcal{L}_m(e_\rho{}^ee_\sigma{}^fK_{ef})\,, \qquad (3.2.15) \\
s^{\beta\delta}\,{}^{(4)}R_{\beta\delta} = & \; s^{bd}\big[{}^{(3)}R_{bd} - (2K^c{}_dK_{bc} - KK_{bd}) - D_da_b - a_da_b\big] \\
& - 2s^{\mathbf{n}d}\big[D_dK - D^cK_{cd}\big] + s^{\mathbf{nn}}\big[D\cdot a + a^2 + K^c{}_dK^d{}_c\big] \\
& + \frac{1}{N}\big(s^{bd} - q^{bd}s^{\mathbf{nn}}\big)e^\rho{}_be^\sigma{}_d\mathcal{L}_m(e_\rho{}^ee_\sigma{}^fK_{ef})\,, \qquad (3.2.16) \\
(-u)\,{}^{(4)}R = & \; (-u)\big[{}^{(3)}R - 3K^c{}_dK^d{}_c + K^2 - 2D\cdot a - 2a^2\big] \\
& + \frac{1}{N}(-2u)q^{bd}e^\rho{}_be^\sigma{}_d\mathcal{L}_m(e_\rho{}^ee_\sigma{}^fK_{ef})\,. \qquad (3.2.17)
\end{aligned}
$$

Considering the identity

$$
\begin{aligned}
\frac{1}{N}M^{bd}e^\rho{}_be^\sigma{}_d\mathcal{L}_m\big(e_\rho{}^ee_\sigma{}^fK_{ef}\big) = & \; \nabla_\lambda\big(n^\lambda M^{bd}K_{bd}\big) - M^{bd}KK_{bd} \\
& - \frac{1}{N}e_\rho{}^ee_\sigma{}^fK_{ef}\mathcal{L}_m\big(M^{bd}e^\rho{}_be^\sigma{}_d\big) \qquad (3.2.18)
\end{aligned}
$$

we obtain three total derivatives contributions that are purely boundary terms:

$$
\nabla_\lambda\big(n^\lambda(-2u)q^{bd}K_{bd}\big)\,, \nabla_\lambda\big(n^\lambda(4t^{\mathbf{n}b\mathbf{n}d})K_{bd}\big)\,, \nabla_\lambda\big(n^\lambda(s^{bd} - q^{bd}s^{\mathbf{nn}})K_{bd}\big)\,. \qquad (3.2.19)
$$

Considering a more general value for $n_\mu n^\mu$ and using the Stokes theorem on the total covariant derivatives, we obtain the Extended Gibbons-Hawking boundary terms:

$$S^{(u)}_{\partial\mathcal{M}} = \frac{1}{2\kappa}\oint_{\partial\mathcal{M}} \mathrm{d}^3y\sqrt{q}\varepsilon(2(-u)K)\,, \tag{3.2.20}$$

$$S^{(s)}_{\partial\mathcal{M}} = \frac{1}{2\kappa}\oint_{\partial\mathcal{M}} \mathrm{d}^3y\sqrt{q}\varepsilon\big(s^{ab}K_{ab} - s^{\mathbf{nn}}K\big)\,, \tag{3.2.21}$$

$$S^{(t)}_{\partial\mathcal{M}} = \frac{1}{2\kappa}\oint_{\partial\mathcal{M}} \mathrm{d}^3y\sqrt{q}\varepsilon\big(4t^{\mathbf{n}a\mathbf{n}b}K_{ab}\big)\,, \tag{3.2.22}$$

Here, q_{ab} is the induced metric on the boundary hypersurface $\partial\mathcal{M}$, q its determinant, and $\varepsilon = n_\mu n^\mu$ where n_μ is the boundary normal evaluated at each point of $\partial\mathcal{M}$. This result for the t sector was part of our research, published in [34], as an extension of the results for the u and s sectors obtained in [35]–[37].

As we discussed, the Standard-Model Extension provides a comprehensive framework that includes coefficients for all possible higher-order derivative operators violating diffeomorphism, Lorentz, and CPT symmetries. A concrete example of a model integrated into this framework can be the Myers-Pospelov model for fermionic matter in QED, which is connected to the CPT-odd non-minimal sector of the QED extension in the SME framework. In this subject we published two articles related to the preservation of unitarity at tree level [18], and at one-loop level [19].

Our research will focus on the minimal sector of the gravitational Riemannian extension of the SME. We will investigate the compatibility of cosmological setups with both explicit and spontaneous diffeomorphism invariance violation. In the following chapter, we will begin by studying explicit violations in FLRW universes with bosonic matter.

Chapter 4

Explicit violations in gravity

The ΛCDM model, or concordance cosmology, has been constructed from a wide-ranging and extensive set of observational data. The model describes a hot Big Bang with a tiny positive cosmological constant, cold dark matter, and an initial period of rapid expansion known as inflation. It aligns extremely well with numerous observations, particularly the Hubble law describing the recession of galaxies, the cosmic microwave background (CMB), and the abundance of light elements as predicted by primordial nucleosynthesis [38]–[40]. Considering the observed galaxy distribution [41] and the measurements of CMB anisotropy [42], [43], the assumptions of spatial isotropy and homogeneity are good approximations for our universe on large scales.

Paradoxically, although dark matter constitutes the largest fraction of matter in the universe, it has not been directly detected [44]. Nevertheless, it plays a crucial role in the formation of large-scale structure [41], [45], [46] and in shaping the temperature fluctuations and polarization patterns in the CMB [47]–[51].

Despite its observational success, the ΛCDM model is considered an effective theory, requiring extensions at both the smallest distance scales, relevant to pre-inflationary times, and the largest distance scales. In the very early universe, when it was extremely dense and hot, quantum fluctuations of the gravitational field became significant. This regime necessitates a theory of quantum gravity, which remains an unsolved challenge. On the largest scales, the standard cosmological model poses profound questions, in particular why the cosmological constant, responsible for the current acceleration of the universe, has such a small value

[52]–[55].

To address some of these theoretical challenges, numerous modifications of gravity and effective theories have been proposed. Our hypothesis is that the accelerated expansion, often attributed to an unknown form of vacuum energy or dark energy [56], could be generated through the introduction of a diffeomorphism-violating background field within an effective field theory framework. This suggests that the small value of the cosmological constant is an artifact of symmetry breaking and the Planck length scale.

In this chapter, we will focus on explicit diffeomorphism breaking in a cosmological setting. Here, the assumptions about spacetime geometry are established from the outset by the cosmological principle, which incorporates spatial isotropy and homogeneity. As a result, the original generators of diffeomorphisms are reduced to six Killing vector fields describing rotations and translations. Given that models with explicit symmetry breaking in gravity are subject to no-go results, we will study the compatibility between these symmetry-breaking backgrounds and a homogeneous and isotropic universe. Our aim is to revisit the u and $s^{\mu\nu}$ sectors, previously studied in Ref. [57], from a new perspective. We will obtain the respective modified Friedmann equations and derive the conditions for satisfying the Bianchi identity for the background fields. Furthermore, we explore the t sector in a cosmological setup for the first time and discuss the viability of an accelerated expansion of the universe produced exclusively by the background field.

4.1 Cosmology in the presence of backgrounds

In this thesis, we adopt the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model, which describes a homogeneous and isotropic universe. The matter content is modeled as a perfect fluid, characterized by its energy density and pressure. By specifying an equation of state for this fluid, we can analyze the distinct dynamical epochs of cosmic expansion, particularly the phases dominated by matter and radiation.

We are assuming a foliation of spacetime $\mathcal{M}$ into spacelike hypersurfaces Σ which are homogeneous and isotropic. The directions of the isometries are characterized by six Killing vectors fields which are also Noether symmetries, i.e., three generators of rotations

$$\eta^a = \epsilon^{abc}\bar{y}_b\partial_c\,, \tag{4.1.1a}$$

and three generators of translations

$$\xi^a = \left(\frac{k\bar{y}^a\bar{y}^b}{2} + \delta^{ab}\left(1 - \frac{k\bar{y}^2}{4}\right)\right)\partial_b\,, \tag{4.1.1b}$$

with $\partial_i = \frac{\partial}{\partial\bar{y}^a}$ and $a = 1, 2, 3$, and where k represents the curvature of Σ that describes a closed ($k = 1$), flat ($k = 0$) and open ($k = -1$) universe, and $\bar{y}^a$ are flat coordinates.

This set of Killing vectors generates the symmetry group for FLRW metric, given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2_{S^2}\right], \tag{4.1.2}$$

with coordinates (t, r, θ, ϕ), and the angular part is given by

$$\mathrm{d}\Omega^2_{S^2} = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\,. \tag{4.1.3}$$

The motivation for using the ADM formalism becomes clear when we consider the Gaussian normal coordinates limit, $N \to 1$, $N^a \to 0^a$. In this limit, the ADM line element (2.2.28) reduces to

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + q_{ab}\mathrm{d}y^a\mathrm{d}y^b\,, \tag{4.1.4}$$

which is compatible with the FLRW metric. This formalism projects four-dimensional spacetime quantities onto the spatial hypersurfaces, revealing explicitly how the spatial metric (with its own covariant derivative) contributes. Simply taking components is insufficient to see how the different spacetime directions contribute. Therefore, the ADM approach provides the necessary technique to gain a deeper insight into the interaction with spatial geometrical quantities.

To complete our cosmological setup we consider a energy-momentum tensor associated to the matter source, considering a perfect fluid

$$(T_m)^{\mu\nu} := (\rho + P)U^\mu U^\nu + P g^{\mu\nu}\,, \tag{4.1.5}$$

where ρ is the fluid density, P its pressure, and U^μ its four-velocity. At the rest frame, we consider $U^\mu = (1, 0, 0, 0)$. In ADM approach we can write instead

$$(T_m)^{\mu\nu} = e^\mu{}_a e^\nu{}_b \big[P q^{ab}\big] + n^\mu n^\nu \big[\rho\big]\,. \tag{4.1.6}$$

that will be the source of matter from now on. Also we will impose separately the conservation of the matter content

$$\nabla_\mu (T_m)^{\mu\nu} = 0\,. \tag{4.1.7}$$

The proposed setup will yield modified Friedmann equations for the expansion of the universe in the presence of background fields, but only if we can resolve the incompatibilities between the geometry and the dynamics.

4.1.1 Isotropy and homogeneity

In standard cosmology, three time-dependent parameters: the scale factor, pressure, and energy density of matter, enter into the Friedmann equations. These equations remain consistent as long as certain conditions are imposed on the energy-momentum tensor. For the matter content one demands that the perfect fluid is form-invariant in the directions of the Killing vector fields. In our modified cosmological model in order to preserve isotropy and homogeneity we impose form invariance of the like energy-momentum tensor $\widetilde{T}^{\mu\nu}$ associated to the background fields, i.e.,

$$\mathcal{L}_\chi \widetilde{T}^{\mu\nu} = 0\,, \tag{4.1.8}$$

where χ^a corresponds to spatial (or purely tangential) Killing vectors given by (4.1.1a) and (4.1.1b) .

One can show after some calculation by decomposing the Lie derivative acting on the like energy-momentum tensor $\widetilde{T}^{\mu\nu}$ along the Killing directions χ that the correct conditions to be imposed are

$$\mathcal{L}_\chi \widetilde{T}^{ab} = 0\,, \tag{4.1.9a}$$

$$\mathcal{L}_\chi \widetilde{T}^{\mathbf{nn}} = 0\,, \tag{4.1.9b}$$

$$\mathcal{L}_\chi \widetilde{T}^{a\mathbf{n}} = 0\,, \tag{4.1.9c}$$

where $\widetilde{T}^{ab}$ represents the spatially like energy-momentum tensor associated to the backgrounds fields, $\widetilde{T}^{\mathbf{nn}} = n_\mu n_\nu T^{\mu\nu}$, and the Lie derivative is taken in the direction of the six Killing fields. It follows that $\widetilde{T}^{a\mathbf{n}} = 0$, since there is no maximal symmetric vector except for the trivial one [58].

The general solution of the system of Eqs. (4.1.9) is found to be

$$\widetilde{T}^{\mu\nu} = \begin{pmatrix} \mathcal{E}_1(t) & 0 \\ 0 & \mathcal{E}_2(t) q^{ab} \end{pmatrix}\,, \tag{4.1.10}$$

where $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ are arbitrary functions of time. Therefore, in order to preserve isotropy and homogeneity one is forced to select a like energy-momentum tensor for the background in the form (4.1.9).

In general, the background fields for both explicit and spontaneous breaking, $\bar{k}^{\mu_1\cdots\mu_n}$ and $\widetilde{k}^{\nu_1\cdots\nu_m}$, do not necessarily yield a like energy-momentum tensor given in the form of (4.1.9). To address this issue which involve to apply extra conditions, we shall elaborate a little bit deeper.

Consider the irreducible decomposition of the spatial tensor $\widetilde{T}_{ab}$ in terms of an antisymmetric $\widetilde{T}_{[ab]}$, a symmetric traceless $\widetilde{T}_{\langle ab\rangle}$ and a trace part proportional to the three metric $\frac{q_{ab}}{3}\widetilde{T}$

$$\widetilde{T}_{ab} = \widetilde{T}_{[ab]} + \widetilde{T}_{\langle ab\rangle} + \frac{q_{ab}}{3}\widetilde{T}\,. \tag{4.1.11}$$

In many cases we will have $\widetilde{T}_{[ab]} = 0$, and we can write the symmetric traceless part as

$$\widetilde{T}_{\langle ab\rangle} = \widetilde{T}_{(ab)} - \frac{q_{ab}}{3}\widetilde{T}\,, \tag{4.1.12}$$

where we have defined the spatial trace $\widetilde{T} = q^{ab}\widetilde{T}_{ab}$. Indeed, one can count the independent components: 3 for the antisymmetric part, 5 for the symmetric trace free part and 1 for the trace, which sum up to 9 independent components of $\widetilde{T}_{ab}$.

In order to preserve the symmetries of the model the like energy-momentum tensor has to be proportional to the induced three metric times an arbitrary function of time. This amounts to impose

$$\widetilde{T}_{\langle ab \rangle} = 0\,, \tag{4.1.13}$$

and also

$$f(t) = \frac{\widetilde{T}}{3}\,. \tag{4.1.14}$$

We will show in the current chapter that these conditions lead to a form-invariant background satisfying $\mathcal{L}_\chi \bar{k}^{\mu_1 \cdots \mu_n} = 0$ for explicit breaking of the symmetry, but it won't be necessary for an spontaneous symmetry breaking as we will show in Chapter 5.

4.2 Gravitational SME Cosmology: the u sector

We begin by considering the scalar background field, coupled to the Ricci scalar as a dynamical operator. According to Eqs. (3.2.13), (4.1.6), and (3.2.14), and in the case where $s^{\mu\nu} = t^{\mu\rho\nu\sigma} = 0$, the modified Einstein field equations in the absence of a cosmological constant are:

$$G^{\mu\nu} = (T^{Ru})^{\mu\nu} + \kappa (T_m)^{\mu\nu}\,, \tag{4.2.1}$$

where the energy-momentum (like) tensor asociated to the background becomes

$$(T^{Ru})^{\mu\nu} \quad = \quad -\frac{1}{2}\big(\nabla^\mu \nabla^\nu u + \nabla^\nu \nabla^\mu u\big) + g^{\mu\nu}\nabla^2 u + u G^{\mu\nu}\,. \tag{4.2.2}$$

The decomposed energy-momentum tensor for the u-background can be written in the compact form

$$(T^{Ru})^{\mu\nu} \quad = \quad e^\mu{}_{} e^\nu{}_b T_1^{ab} - \big(e^\mu{}_a n^\nu + n^\mu e^\nu{}_a\big) T_2^a + n^\mu n^\nu T_3\,, \tag{4.2.3}$$

with components given by

$$
\begin{aligned}
T_1^{ab} &= -\frac{1}{2}\Big(D^a D^b u + D^b D^a u\Big) \\
&\quad + \left[D^2 u - 2H\dot{u} - \ddot{u} - u\left(\frac{k}{a(t)^2} + 2\dot{H} + 3H^2\right)\right] q^{ab}\,, \qquad (4.2.4) \\
T_2^a &= -D^a\big(\mathcal{L}_t u - Hu\big)\,, \qquad (4.2.5) \\
T_1 &= -D^2 u + 3H\mathcal{L}_t u + 3u\left(\frac{k}{a(t)^2} + H^2\right), \qquad (4.2.6)
\end{aligned}
$$

where $H(t)$ corresponds to the Hubble parameter given by

$$
H(t) := \frac{\dot{a}(t)}{a(t)}\,, \qquad (4.2.7)
$$

We must remark that T_2^a is related to the momentum constrain in the Hamiltonian formalism of the theory [35].

Considering $(T_1)^{ab}$ we construct the perpendicular contribution to the metric tensor $(T_\perp)^{ab}$ as detailed in subsection 4.1.1, obtaining

$$
(T_\perp)^{ab} = -q^{ac}q^{bd}\left(D_c D_d u - \frac{1}{3}q_{cd}D^2 u\right). \qquad (4.2.8)
$$

The general solution to the condition $(T_\perp)^{ab} = 0$ corresponds to

$$
\begin{aligned}
u(t,r,\theta,\phi) &= r\big(f_1(t)\sin\phi + f_2(t)\cos\phi\big)\sin\theta + f_3(t) r\cos\theta \\
&\quad + f_4(t)\sqrt{1-kr^2} + f_5(t)\,. \qquad (4.2.9)
\end{aligned}
$$

On the other hand, considering that the modified Einstein equations have only purely tangential and purely normal non-zero components for the Einstein tensor and the matter source, the so-called momentum constraint (4.2.5) must vanish. Substituting solution (4.2.9) into (4.2.5), we obtain

$$
\begin{aligned}
&\Big(\big(\dot{f}_1(t) - Hf_1(t)\big)r\sin\phi + \big(\dot{f}_2(t) - Hf_2(t)\big)r\cos\phi\Big)\sin\theta \\
&+\big(\dot{f}_3(t) - Hf_3(t)\big)r\cos\theta + \big(\dot{f}_4(t) - Hf_4(t)\big)\sqrt{1-kr^2} \\
&\qquad + \dot{f}_5(t) - Hf_5(t) = f_6(t)\,. \qquad (4.2.10)
\end{aligned}
$$

Taking this in consideration we need to solve the following system

$$\begin{aligned}
\dot{f}_1(t) - Hf_1(t) &= 0\,, && (4.2.11)\\
\dot{f}_2(t) - Hf_2(t) &= 0\,, && (4.2.12)\\
\dot{f}_3(t) - Hf_3(t) &= 0\,, && (4.2.13)\\
\dot{f}_4(t) - Hf_4(t) &= 0\,, && (4.2.14)\\
\dot{f}_5(t) - Hf_5(t) &= f_6(t)\,. && (4.2.15)
\end{aligned}$$

If the equations are

$$\dot{f}_i(t) = Hf_i(t), \qquad (i = 1, \dots, 4)\,, \tag{4.2.16}$$

then the system has the solution

$$f_i(t) = c^{(i)}a(t), \qquad (i = 1, \dots, 4)\,, \tag{4.2.17}$$

while

$$\dot{f}_5(t) - Hf_5(t) = f_6(t) \implies f_5(t) = \left(c^{(5)} + \int \frac{f_6(t)}{a(t)}\mathrm{d}t\right)a(t)\,. \tag{4.2.18}$$

Thus the general solution becomes

$$\begin{aligned}
u(t,r,\theta,\phi) &= a(t)\Bigg(\big(c^{(1)}\sin\phi + c^{(2)}\cos\phi\big)r\sin\theta + c^{(3)}r\cos\theta \\
&\quad + c^{(4)}\sqrt{1-kr^2} + \int \frac{f_6(t)}{a(t)}\mathrm{d}t + c^{(5)}\Bigg)\,.
\end{aligned} \tag{4.2.19}$$

We have the freedom to redefine the f_6 function as

$$f_6(t) \to \frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{f}_6(t))a(t)\,, \tag{4.2.20}$$

that is also an arbitrary function of time, having a more compact solution

$$\begin{aligned}
u(t,r,\theta,\phi) =& a(t)\Bigg(\big(c^{(1)}\sin\phi + c^{(2)}\cos\phi\big)r\sin\theta + c^{(3)}r\cos\theta \\
&+ c^{(4)}\sqrt{1-kr^2} + c^{(5)} + f_6(t)\Bigg)
\end{aligned} \tag{4.2.21}$$

After satisfying the momentum constraint, and the condition $(T_\perp)^{\mu\nu} = 0$ the tensor energy-momentum tensor (4.2.3) reduces to

$$\begin{aligned}(T^{Ru})^{\mu\nu} &= e^{\mu}{}_{a}e^{\nu}{}_{b}\left[\frac{2}{3}\big(D^2u-3H\mathcal{L}_tu\big)-\mathcal{L}_t^2u-\left(\frac{k}{a(t)^2}+2(\dot{H}+H^2)+H^2\right)u\right]q^{ab}\\ &\quad +n^{\mu}n^{\nu}\left[-\big(D^2u-3H\mathcal{L}_tu\big)+3\left(\frac{k}{a(t)^2}+H^2\right)u\right] \qquad (4.2.22)\end{aligned}$$

We are looking for solutions that keeps the energy-momentum tensor as (4.1.10). In order to obtain $(T_3)=\mathcal{E}_1(t)$ and $(T_1)^{ab}=\mathcal{E}_2(t)q^{ab}$, we need to satisfy

$$c^{(1)}=c^{(2)}=c^{(3)}=c^{(4)}=0 \qquad (4.2.23)$$

Thus, $u(t,y^a)$ becomes a function of time only, which corresponds to the scalar invariant for the FLRW metric. We thus obtain the invariant energy-momentum tensor associated with this invariant scalar background u, given by

$$\begin{aligned}(T^{Ru})^{\mu\nu} &= e^{\mu}{}_{a}e^{\nu}{}_{b}\left[-\ddot{u}(t)-2H\dot{u}(t)-\left(\frac{k}{a(t)^2}+2(\dot{H}+H^2)+H^2\right)u(t)\right]q^{ab}\\ &\quad +n^{\mu}n^{\nu}\left[3H\dot{u}(t)+3\left(\frac{k}{a(t)^2}+H^2\right)u(t)\right]. \qquad (4.2.24)\end{aligned}$$

The modified Einstein field equations now consist of two independent equations, which are often rewritten in terms of the evolution of the Hubble parameter,$H(t)$. The first is the purely temporal component, while the second is obtained by taking the purely spatial part, multiplying it by three, and adding the temporal component. The resulting set of equations are the so-called modified Friedmann equations

$$\dot{H}+H^2 = \frac{1}{-6\big(1-u(t)\big)}\Big[\kappa(3P+\rho)-3H\dot{u}(t)-3\ddot{u}(t)\Big], \qquad (4.2.25)$$

$$H^2 = \frac{1}{3\big(1-u(t)\big)}\Big[\kappa\rho+3H\dot{u}(t)\Big]-\frac{k}{a(t)^2}. \qquad (4.2.26)$$

This set of equations describes how the presence of the u background field influences the expansion of the universe. However the consistency condition required by the dynamics is to satisfy also the so-called Bianchi identity for the background's energy-momentum tensor. This identity demands that the covariant divergence of the energy-momentum tensor associated with the background field must be zero.

4.2.1 Bianchi identity for the u sector

As we previously discussed, every gravitational configuration in presence of background fields need to satisfy the null covariant divergence condition (or Bianchi in some cases). In particular we need to impose the following set of equations:

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{4.2.27}$$

In order to obtain the correct Bianchi identity we project the identity in their tangential and normal components as follows: we start considering an energy-momentum tensor

$$T^{\mu\nu} \;=\; e^\mu{}_a e^\nu{}_b \big[f_1(x) q^{ab}\big] + n^\mu n^\nu \big[f_2(x)\big]\,. \tag{4.2.28}$$

Taking the covariant divergence, after some calculations, we arrive to

$$\begin{aligned}\nabla_\mu T^{\mu\nu} \;&=\; e^\nu{}_b \big[D^b f_1(x) + a^b f_1(x) + a^b f_2(x)\big] \\ &\quad -n^\nu \Big[-\frac{1}{N}\mathcal{L}_m f_2(x) - K\big(f_1(x)+f_2(x)\big)\Big]\,.\end{aligned} \tag{4.2.29}$$

Taking the normal coordinates limit and considering the Friedmann metric we finally obtain

$$\nabla_\mu T^{\mu\nu} = e^\nu{}_b \big[D^b f_1(t,y)\big] - n^\nu\big[-\dot f_2(t,y) - 3H\big(f_1(t,y)+f_2(t,y)\big)\big]\,. \tag{4.2.30}$$

Considering $f_1(t,y) = f_1(t), f_2(t,y) = f_2(t)$, as in a maximally invariant tensor for the FLRW metric, the system reduces to the condition

$$\dot f_2(t) + 3H\big(f_1(t)+f_2(t)\big) = 0\,, \tag{4.2.31}$$

In our case, it becomes

$$\left(\frac{k}{a(t)^2} + \dot H + 2H^2\right)\dot u(t) = 0 \tag{4.2.32}$$

So, if we work without source of matter, the condition to obtain an isotropic and homogeneous theory with background need to be a constant function but only *apparently.*

4.2.2 u sector solutions

Let us consider the continuity equation for a perfect fluid. For Eq. (4.1.7), by using the FLRW metric we can obtain:

$$\kappa\dot{\rho}(t) + 3H\kappa\big(\rho(t) + P(t)\big)\,. \tag{4.2.33}$$

Together with the modified Friedmann equations (4.2.25),(4.2.26):

$$\begin{aligned} \dot{H} + H^2 &= \frac{1}{-6\big(1-u(t)\big)}\Big[\kappa(3P(t)+\rho(t)) - 3H\dot{u}(t) - 3\ddot{u}(t)\Big] \qquad (4.2.34)\\ H^2 &= \frac{1}{3\big(1-u(t)\big)}\Big[\kappa\rho(t) + 3H\dot{u}(t)\Big] - \frac{k}{a(t)^2} \qquad (4.2.35) \end{aligned}$$

and the Bianchi identity equation

$$\left(\frac{k}{a(t)^2} + \dot{H} + 2H^2\right)\dot{u}(t) = 0\,. \tag{4.2.36}$$

Consider the state equation $P(t) = w\rho(t)$ where $w = 0$ is for non-relativistic matter and $w = 1/3$ for radiation. We solve the continuity equation Eq. (4.2.33) obtaining

$$\rho(t) = \rho(0)\frac{a(0)^{3(1+w)}}{a(t)^{3(1+w)}}\,. \tag{4.2.37}$$

At this point we have two choices:

- Constant u background:

Having a constant background results in a Bianchi identity satisfied, and the modified Friedmann equations becomes

$$\begin{aligned} \dot{H} + H^2 &= \frac{\kappa(1+3w)\rho(t)}{-6\big(1-u\big)}\,, \qquad (4.2.38)\\ H^2 &= \frac{\kappa\rho(t)}{3\big(1-u\big)} - \frac{k}{a(t)^2}\,, \qquad (4.2.39) \end{aligned}$$

that corresponds to a rescaling of the quantities $\rho(t), P(t)$ of the matter content:

$$\rho(t) \to \frac{\rho(t)}{1-u}\,, \tag{4.2.40}$$

$$P(t) \to \frac{P(t)}{1-u}\,. \tag{4.2.41}$$

- Non-constant u background:

For the non-constant background we need to solve the Bianchi identity for the scale factor:

$$\frac{k}{a(t)^2} + \dot{H} + 2H^2 = 0\,, \tag{4.2.42}$$

obtaining

$$a(t)^2 = -kt^2 + a(0)^2(1 + 2H(0)t)\,. \tag{4.2.43}$$

Here we notice an important feature of this solution. We call it the *Ricci flat* solution, because we can identify the Bianchi identity to be proportional to the Ricci scalar of $\mathcal{M}$:

$$R = 6\left(\frac{k}{a(t)^2} + \dot{H} + 2H^2\right). \tag{4.2.44}$$

A question emerges: how can we obtain a vanishing Ricci scalar through a background field. If we recall the original Einstein equations Eq. (4.2.1):

$$G^{\mu\nu} = -\frac{1}{2}\big(\nabla^\mu\nabla^\nu u + \nabla^\nu\nabla^\mu u\big) + g^{\mu\nu}\nabla^2 u + uG^{\mu\nu} + \kappa(T_m)^{\mu\nu}\,. \tag{4.2.45}$$

Taking the trace, we obtain

$$-(1-u)R = 3\nabla^2 u + \kappa(T_m)\,, \tag{4.2.46}$$

where the quantities can be written more explicitly as

$$\nabla^2 u = -3H\dot{u}(t) - \ddot{u}(t) \tag{4.2.47}$$

$$(T_m) = \rho(0)(3w-1)\frac{a(0)^{3(1+w)}}{a(t)^{3(1+w)}}\,. \tag{4.2.48}$$

Thus, a Ricci flat solution will produce the following condition the be fulfilled:

$$3\nabla^2 u + \kappa(T_m) = 0 \tag{4.2.49}$$

The setup will admit the following solutions:

For non-relativistic matter ($w = 0$) we have

$$\begin{aligned} u(t) &= \frac{\big(\kappa\rho(0) + 3\dot{u}(0)H(0)\big)a(0)^2}{3\big(k + H(0)^2 a(0)^2\big)}\left(1 - \frac{a(0)}{a(t)}\right) \\ &\quad - \frac{\big(\kappa\rho(0)a(0)^2 H(0) - 3\dot{u}(0)k\big)a(0)}{3\big(k + H(0)^2 a(0)^2\big)}\frac{t}{a(t)} + u(0)\,. \end{aligned} \tag{4.2.50}$$

On the other hand, for radiation ($w = 1/3$) we have

$$u(t) = \frac{\dot{u}(0)a(0)^2 H(0)}{(H(0)^2 a(0)^2 + k)}\left(1 - \frac{a(0)}{a(t)}\right) + \frac{k a(0)\dot{u}(0)}{(H(0)^2 a(0)^2 + k)}\frac{t}{a(t)} + u(0)\,. \tag{4.2.51}$$

The choice of this particular background field preserves the functional dependence on time, satisfies the complete set of equations, and, more importantly, satisfy the no-go relation associated with the Bianchi identity in the cosmological setup. Even though a Ricci-flat spacetime can lead to some oversimplification of the problem, this choice remains valid. The scalar sector of the minimal gravitational SME lacks functional dependence: it consists of just one scalar function that depends on four coordinates, yet there are many equations to satisfy. We will now explore a less trivial sector: the 2-tensor sector.

4.3 Gravitational SME Cosmology: the s sector

We now continue by considering the rank-2 symmetric background field, using the Ricci tensor as a dynamical operator for the coupling. According to Eq. (3.2.13), Eq. (4.1.6), and Eq. (3.2.14) in the case $u = t^{\mu\rho\nu\sigma} = 0$ the modified Einstein equations in absence of cosmological constant are

$$G^{\mu\nu} = (T^{Rs})^{\mu\nu} + \kappa(T_m)^{\mu\nu}\,, \tag{4.3.1}$$

where

$$(T^{Rs})^{\mu\nu} = \frac{1}{2}s^{\alpha\beta}R_{\alpha\beta}g^{\mu\nu} + \frac{1}{2}\big(\nabla_\alpha\nabla^\mu s^{\alpha\nu} + \nabla_\alpha\nabla^\nu s^{\alpha\mu}\big) - \frac{1}{2}\nabla^2 s^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\nabla_\alpha\nabla_\beta s^{\alpha\beta}\,. \tag{4.3.2}$$

The decomposed energy-momentum tensor for the s-background can be written in the compact form

$$(T^{Rs})^{\mu\nu} = e^\mu{}_a e^\nu{}_b (T_1)^{ab} - \big(e^\mu{}_a n^\nu + n^\mu e^\nu{}_a\big)(T_2)^a + n^\mu n^\nu (T_3)\,, \tag{4.3.3}$$

where

$$\begin{aligned}(T_1)^{ab} = \frac{1}{2}\Bigg[& D_c D^a s^{cb} + D_c D^b s^{ca} - D^c D_c s^{ab} + 2(2\dot{H} + 8H^2)s^{ab} + 7H\dot{s}^{ab} + \ddot{s}^{ab} \\ & + \left(2\left(\frac{k}{a(t)^2} - H^2\right)s^{cd}q_{cd} - D_c D_d s^{cd} - H q_{cd}\dot{s}^{cd}\right)q^{ab} \\ & - 3HD^a s^{\mathbf{n}b} - D^a \dot{s}^{b\mathbf{n}} - 3HD^b s^{\mathbf{n}a} - D^b \dot{s}^{a\mathbf{n}} + \Big(2D_c \dot{s}^{c\mathbf{n}} + 6HD_c s^{c\mathbf{n}}\Big)q^{ab} \\ & - \Big(2(2\dot{H} + 3H^2)s^{\mathbf{nn}} + 4H\dot{s}^{\mathbf{nn}} + \ddot{s}^{\mathbf{nn}}\Big)q^{ab}\Bigg]\,,\end{aligned} \tag{4.3.4}$$

$$\begin{aligned}(T_2)^a = \frac{1}{2}\Bigg[& - Hq_{cd}D^a s^{cd} + D_c \dot{s}^{ca} + 2HD_c s^{ca} + D_c D^a s^{c\mathbf{n}} - D^c D_c s^{a\mathbf{n}} \\ & + 2(\dot{H} + 3H^2)s^{a\mathbf{n}} - HD^a s^{\mathbf{nn}} - D^a \dot{s}^{\mathbf{nn}}\Bigg]\,,\end{aligned} \tag{4.3.5}$$

$$\begin{aligned}(T_3) = \frac{1}{2}\Bigg[& - 2\left(\frac{k}{a(t)^2} + H^2\right)s^{cd}q_{cd} + D_c D_d s^{cd} - Hq_{cd}\dot{s}^{cd} \\ & + D_c \dot{s}^{c\mathbf{n}} - 5HD_c s^{c\mathbf{n}} + 6(\dot{H} + 2H^2)s^{\mathbf{nn}} - D^c D_c s^{\mathbf{nn}} + 3H\dot{s}^{\mathbf{nn}}\Bigg]\,.\end{aligned} \tag{4.3.6}$$

Considering $(T_1)^{ab}$ we construct $(T_\perp)^{ab}$ obtaining

$$\begin{aligned}(T_\perp)^{ab} = \frac{1}{2}\Bigg[& D_c D^a s^{cb} + D_c D^b s^{ca} - D^c D_c s^{ab} - \frac{1}{3}q^{ab}\left(2D_c D_d s^{cd} - q_{cd}D^2 s^{cd}\right) \\ & + 2(2\dot{H} + 8H^2)\left(s^{ab} - \frac{1}{3}q^{ab}q_{cd}s^{cd}\right) + 7H\left(\dot{s}^{ab} - \frac{1}{3}q^{ab}q_{cd}\dot{s}^{cd}\right) \\ & + \left(\ddot{s}^{ab} - \frac{1}{3}q^{ab}q_{cd}\ddot{s}^{cd}\right) - 3HD^a s^{\mathbf{n}b} - 3HD^b s^{\mathbf{n}a} - D^a \dot{s}^{b\mathbf{n}} - D^b \dot{s}^{a\mathbf{n}} \\ & + \frac{1}{3}q^{ab}\left(6HD_c s^{c\mathbf{n}} + 2D_c \dot{s}^{c\mathbf{n}}\right)\Bigg]\,.\end{aligned} \tag{4.3.7}$$

Now we want solutions for $s^{\mu\nu}$ that satisfies $(T_2)^a = 0$, and $(T_\perp)^{ab} = 0$. We start by studying one sub-sector of $s^{\mu\nu}$ in the ADM decomposition. Let us consider a purely tangential background field,

$$s^{\mu\nu} = e^\mu{}_a e^\nu{}_b s^{ab} \,. \tag{4.3.8}$$

The system of equations reduces to

$$\begin{aligned}(T_2)^a &= \frac{1}{2}\Big[- H q_{cd} D^a s^{cd} + D_c \dot{s}^{ca} + 2H D_c s^{ca} \Big] \,, &(4.3.9)\\ (T_\perp)^{ab} &= \frac{1}{2}\Big[D_c D^a s^{cb} + D_c D^b s^{ca} - D^c D_c s^{ab} - \frac{1}{3} q^{ab}\Big(2D_c D_d s^{cd} - q_{cd} D^2 s^{cd}\Big)\\ &\quad +2(2\dot{H} + 8H^2)\Big(s^{ab} - \frac{1}{3}q^{ab}q_{cd}s^{cd}\Big) + 7H\Big(\dot{s}^{ab} - \frac{1}{3}q^{ab}q_{cd}\dot{s}^{cd}\Big)\\ &\quad + \ddot{s}^{ab} - \frac{1}{3}q^{ab}q_{cd}\ddot{s}^{cd}\Big] \,. &(4.3.10)\end{aligned}$$

Here we notice that one valid ansatz to consider is

$$s^{ab} - \frac{1}{3}q^{ab}q_{cd}s^{cd} = 0 \,, \tag{4.3.11}$$

that appears multiple times in Eq. (4.3.10). This ansatz will produce the following set of relations as consequence

$$\dot{s}^{ab} = \frac{1}{3}q^{ab}q_{cd}\dot{s}^{cd} \tag{4.3.12}$$

$$\ddot{s}^{ab} = \frac{1}{3}q^{ab}q_{cd}\ddot{s}^{cd} \tag{4.3.13}$$

$$D_c D^a s^{cb} = \frac{1}{3} D^b D^a s \tag{4.3.14}$$

$$D^2 s^{ab} = \frac{1}{3} q^{ab} D^2 s \tag{4.3.15}$$

$$D_c D_d s^{cd} = \frac{1}{3} D^2 s \tag{4.3.16}$$

Reducing $(T_\perp)^{ab}$ and $(T_2)^a$ to the following expressions:

$$(T_\perp)^{ab} = \frac{1}{6}\Big[D^b D^a s + D^a D^b s - \frac{2}{3} q^{ab} D^2 s \Big] \,, \tag{4.3.17}$$

$$(T_2)^a = \frac{1}{6} D^a \big[q_{cd}\dot{s}^{cd} - H q_{cd} s^{cd} \big] \,. \tag{4.3.18}$$

The general solution to the ansatz equation corresponds to

$$s^{ab} = s_1(t, y^a) q^{ab} . \tag{4.3.19}$$

By using Eq. (4.3.19) in Eq. (4.3.18) we arrive to

$$(T_2)^a = \frac{1}{2} D^a \big[\dot{s}_1(t, y^a) - 3H s_1(t, y^a) \big] . \tag{4.3.20}$$

Thus the momentum constraint suggest that the functional dependence of Eq. (4.3.19) should be distributed as

$$s_1(t, y^a) = a(t)^3 \big(F(t) + G(y^a) \big) . \tag{4.3.21}$$

Considering $(T_\perp)^{ab}$ we obtain

$$\Big(D^a D^b - \frac{1}{3} q^{ab} D^2 \Big) G(y^a) = 0 , \tag{4.3.22}$$

with general solution

$$G(y^a) = \sqrt{1 - kr^2} C_1 + r \sin\theta (C_2 \cos\phi + C_3 \sin\phi) + C_4 r \cos\theta + C_5 . \tag{4.3.23}$$

Now we want an invariant energy-momentum tensor associated to the background, thus we are looking to extra conditions that produces $(T_1)^{ab} = \mathcal{E}_1(t) q^{ab}$ and $(T_3) = \mathcal{E}_2(t)$. Replacing the values obtained for s^{ab} in Eq. (4.3.6) we obtain

$$\begin{aligned} (T_3) = \frac{a(t)^3}{2} \Big[& - \frac{6k}{a(t)^2} F(t) - 9H^2 F(t) - 3H \dot{F}(t) \\ & - \frac{6k}{a(t)^2} G(y^a) + D^2 G(y^a) - 9H^2 G(y^a) \Big] = \mathcal{E}_2(t) \end{aligned} \tag{4.3.24}$$

The only possible outcome becomes $G(y^a) = 0$, arriving the result

$$s^{\mu\nu} = e^\mu{}_a e^\nu{}_b s^{ab}, \qquad s^{ab} = s_1(t) q^{ab} , \tag{4.3.25}$$

that satisfies $\mathcal{L}_{\xi_{(i)}} (T^{Rs})^{\mu\nu} = 0$ for all the Killing directions of the FLRW metric.

The solution obtained is the maximally invariant tensor of rank-2 symmetric for Homogeneity and Isotropy. We found that, in order to obtain an energy-momentum tensor invariant under the Killing transformations we found always at least one solution: an invariant tensor of the isometry group. We already obtained information related in the u-sector, because the invariant rank-0 tensor for the FLRW metric is precisely an function of time. Considering that there is no invariant vector for homogeneity and isotropy, we arrive to a background tensor $s^{\mu\nu}$ that should satisfy all the condition imposed.

For a more general case, now we will consider an maximally symmetric background $s^{\mu\nu}$ given by

$$s^{\mu\nu} = e^{\mu}{}_{a} e^{\nu}{}_{b} \big[s_1(t) q^{ab} \big] + n^{\mu} n^{\nu} \big[s_2(t) \big] \,. \tag{4.3.26}$$

The energy-momentum tensor becomes immediately an invariant tensor of homogeneity and isotropy:

$$\begin{aligned}(T^{Rs})^{\mu\nu} = e^{\mu}{}_{a} e^{\nu}{}_{b} \Bigg[&\bigg(\frac{3k}{a(t)^2} + \dot{H} + 3H^2 \bigg) s_1(t) - (2\dot{H} + 3H^2) s_2(t) \\ &- 2H\dot{s}_2(t) + \frac{1}{2}(\ddot{s}_1(t) - \ddot{s}_2(t)) \Bigg] q^{ab} \\ &+ n^{\mu} n^{\nu} \Bigg[- \frac{3k}{a(t)^2} s_1(t) + 3(\dot{H} + 2H^2) s_2(t) - \frac{3}{2} H(\dot{s}_1(t) - \dot{s}_2(t)) \Bigg] \,.\end{aligned} \tag{4.3.27}$$

After some algebraic manipulations we arrive to the respective modified Friedmann equations

$$\begin{aligned} H^2 &= \frac{1}{3\big(1 + \frac{s_1(t)}{2} - \frac{s_2(t)}{2}\big)} \Bigg[\kappa\rho - \frac{3k}{a(t)^2}\big(1 + s_1(t)\big) + \frac{3}{2} H^2 (s_1(t) - s_2(t)) \\ &\quad + 3(\dot{H} + 2H^2) s_2(t) - \frac{3}{2} H(\dot{s}_1(t) - \dot{s}_2(t)) \Bigg] \,, \end{aligned} \tag{4.3.28}$$

$$\begin{aligned} \dot{H} + H^2 &= \frac{1}{(-6)\big(1 + \frac{s_1(t)}{2} - \frac{s_2(t)}{2}\big)} \Bigg[\kappa(\rho + 3P) + 6\bigg(\frac{k}{a(t)^2} + H^2 \bigg) s_1(t) \\ &\quad - \frac{3}{2} H(\dot{s}_1(t) + 3\dot{s}_2(t)) + \frac{3}{2}(\ddot{s}_1(t) - \ddot{s}_2(t)) \Bigg] \,. \end{aligned} \tag{4.3.29}$$

4.3.1 Bianchi identity for the s sector

Now we need to impose the Bianchi identity

$$\nabla_\mu (T^{Rs})^{\mu\nu} = 0\,. \tag{4.3.30}$$

By using Eq. (4.2.31) for the energy-momentum tensor given by Eq. (4.3.27) we arrive to

$$\left(\frac{2k}{a(t)^2} + \dot{H} + 3H^2\right)\big(2Hs_1(t) - \dot{s}_1(t)\big) + 2(\ddot{H} + 2H\dot{H})s_2(t)$$
$$+3(\dot{H} + H^2)\big(2Hs_2(t) + \dot{s}_2(t)\big) = 0\,, \tag{4.3.31}$$

or equivalently

$$\frac{R}{3}\big(2Hs_1(t) - \dot{s}_1(t)\big) + \partial_t\big[2s_2(t)(\dot{H} + H^2)\big]$$
$$+(\dot{H} + H^2)\big(6Hs_2(t) + \dot{s}_2(t) - 2Hs_1(t) + \dot{s}_1(t)\big) = 0\,, \tag{4.3.32}$$

where R is the Ricci scalar for the spacial part of the FLRW metric.

4.3.2 s sector solutions

We study a configuration for the background field with different properties. An observation from Eq. (4.3.31) is that each term containing the time derivatives of $s_1(t)$ and $s_2(t)$, respectively, appears with a global function scaled by H. This suggests choosing a background configuration satisfying

$$\dot{s}_1(t) = \alpha H s_1(t)\,, \tag{4.3.33a}$$

$$\dot{s}_2(t) = \beta H s_2(t)\,, \tag{4.3.33b}$$

with constant parameters α and β.

By inserting the latter into Eq. (4.3.31), we arrive at the algebraic equation

$$\frac{R}{3}\big(2-\alpha\big)Hs_1(t) - (\dot{H} + H^2)\big(2-\alpha\big)Hs_1(t) + 2(\ddot{H} + 2H\dot{H})s_2(t)$$
$$+3H(\dot{H} + H^2)\big(2+\beta\big)s_2(t) = 0\,, \tag{4.3.34}$$

with solution

$$(2-\alpha)s_1(t) = \frac{2(\ddot{H}+2H\dot{H})+3H(\dot{H}+H^2)\big(2+\beta\big)}{\frac{R}{3}H-(\dot{H}+H^2)H}s_2(t)\,. \tag{4.3.35}$$

Here we can obtain multiple solutions depending of the values for α and β.

- Ricci flat solution

If $R = 0$ we obtain

$$(2-\alpha)\alpha s_1(t) = (2-3\beta)\beta s_2(t) \tag{4.3.36}$$

Taking the time derivative and using the ansatz again we notice that it must satisfy also

$$(2-\alpha)\alpha^n s_1(t) = (2-3\beta)\beta^n s_2(t) \tag{4.3.37}$$

for an arbitrary value of n. This consideration leads to the choice of the values for $\alpha = 2$ and $\beta = \frac{2}{3}$ as the general solution to this setup

$$s_1(t) = s_1(0)\frac{a(t)^2}{a(0)^2} \tag{4.3.38}$$

$$s_2(t) = s_2(0)\frac{a(t)^{2/3}}{a(0)^{2/3}} \tag{4.3.39}$$

for the scale factor Eq. (4.2.43).

- Constant expansion solution

We can also take other combinations of values for α and β, for example the intriguing choices $\alpha = 2$ and $\beta = -4/3$ lead to a quite simple equation, which is independent of the functions composing the background field:

$$\frac{\partial}{\partial t}\big[a(t)(\dot{H}+H^2)\big] = 0\,. \tag{4.3.40}$$

The latter has the solution $\ddot{a} =$ const., which is to be considered. By solving

Eqs. (4.3.33a) and (4.3.33b), we are able to state an alternative solution:

$$s_1(t) = s_1(0)\frac{a(t)^2}{a(0)^2}\,, \tag{4.3.41}$$

$$s_2(t) = s_2(0)\frac{a(0)^{4/3}}{a(t)^{4/3}}\,, \tag{4.3.42}$$

with the scale factor being

$$a(t) = \frac{1}{2}\ddot{a}(0)t^2 + \dot{a}(0)t + a(0)\,. \tag{4.3.43}$$

Again, the latter is a nontrivial background field that is in accordance with the symmetries demanded. Consequently, it does not provide a discrepancy with the no-go result for gravitational models.

4.4 Gravitational SME Cosmology: the t sector

Finally we consider the rank-4 Riemann-like background field, using the Riemann tensor as a dynamical operator for the coupling. According to Eq. (3.2.13), Eq. (4.1.6), and Eq. (3.2.14) in the case $u = s^{\mu\nu} = 0$ the modified Einstein equations in absence of cosmological constant are

$$G^{\mu\nu} = \left(T^{Rt}\right)^{\mu\nu} + \kappa(T_m)^{\mu\nu}\,, \tag{4.4.1}$$

where

$$\begin{aligned}(T^{Rt})^{\mu\nu} = {}& \frac{1}{2}t^{\alpha\beta\gamma\mu}\,{}^{(4)}R_{\alpha\beta\gamma}{}^{\nu} + \frac{1}{2}t^{\alpha\beta\gamma\nu}\,{}^{(4)}R_{\alpha\beta\gamma}{}^{\mu} + \frac{1}{2}g^{\mu\nu}t^{\alpha\beta\gamma\delta}\,{}^{(4)}R_{\alpha\beta\gamma\delta} \\ & - \nabla_\alpha\nabla_\beta t^{\mu\alpha\nu\beta} - \nabla_\alpha\nabla_\beta t^{\nu\alpha\mu\beta}\,. \end{aligned} \tag{4.4.2}$$

The decomposed energy-momentum tensor for the t background can be written in the compact form

$$(T^{Rt})^{\mu\nu} = e^\mu{}_a e^\nu{}_b T_1^{ab} + \left(e^\mu{}_a n^\nu + n^\mu e^\nu{}_a\right) T_2^a + n^\mu n^\nu T_3\,. \tag{4.4.3}$$

where

$$
\begin{aligned}
T_1^{ab} \;=\;& 2\left(\frac{k}{a^2(t)}+H^2\right)q_{cd}t^{acbd}+q^{ab}\left[\left(\frac{k}{a^2(t)}+H^2\right)q_{cd}q_{rs}t^{crds}-2\big(H^2+\dot{H}\big)q_{cd}t^{\mathbf{n}c\mathbf{n}d}\right]\\
&-\Big(D_cD_dt^{acbd}+D_cD_dt^{bcad}+2\ddot{t}^{a\mathbf{n}b\mathbf{n}}-2D_c\dot{t}^{acb\mathbf{n}}-2D_c\dot{t}^{bca\mathbf{n}}+2Hq_{cd}\dot{t}^{acbd}\\
&+16H\dot{t}^{a\mathbf{n}b\mathbf{n}}-10HD_ct^{a\mathbf{n}bc}-10HD_ct^{b\mathbf{n}ac}+2(\dot{H}+6H^2)q_{cd}t^{acbd}\\
&+2(5\dot{H}+17H^2)t^{a\mathbf{n}b\mathbf{n}}\Big)\,, \qquad (4.4.4)\\
T_2^{a} \;=\;& D_cD_dt^{ac\mathbf{n}d}+D_cD_dt^{\mathbf{n}cad}-D_c\dot{t}^{\mathbf{n}ca\mathbf{n}}-2Hq_{cd}D_et^{acde}-5HD_ct^{a\mathbf{n}\mathbf{n}c}\\
&+Hq_{cd}\dot{t}^{acd\mathbf{n}}-\left(\dot{H}+11H^2+\frac{k}{a^2(t)}\right)q_{cd}t^{cad\mathbf{n}}\,, \qquad (4.4.5)\\
T_3 \;=\;& \left(\frac{k}{a^2(t)}+H^2\right)q_{ab}q_{cd}t^{acbd}-2\Big(D_cD_dt^{\mathbf{n}c\mathbf{n}d}+Hq_{cd}\dot{t}^{\mathbf{n}cd\mathbf{n}}-2Hq_{cd}D_at^{ca\mathbf{n}d}\\
&+H^2h_{cb}q_{ad}t^{acbd}-2(\dot{H}+3H^2)q_{cd}t^{\mathbf{n}c\mathbf{n}d}\Big)\,. \qquad (4.4.6)
\end{aligned}
$$

Again $(T_2)^a$ is related to the momentum constraint. Considering $(T_1)^{ab}$ we construct the traceless symmetric part of the like energy-momentum tensor associated to the background

$$
\begin{aligned}
(T_\perp)^{ab} =& -2\left(\frac{k}{a(t)^2}-(\dot{H}+5H^2)\right)\left(q_{cd}t^{acbd}-\frac{1}{3}q^{ab}q_{cd}q_{ef}t^{cedf}\right)\\
&+2(5\dot{H}+17H^2)\left(t^{a\mathbf{n}b\mathbf{n}}-\frac{1}{3}q^{ab}q_{cd}t^{c\mathbf{n}d\mathbf{n}}\right)+2\Big(\big(\ddot{t}^{a\mathbf{n}b\mathbf{n}}+Hq_{cd}\dot{t}^{acbd}+8H\dot{t}^{a\mathbf{n}b\mathbf{n}}\big)\\
&-\frac{1}{3}q^{ab}q_{ef}\big(\ddot{t}^{e\mathbf{n}f\mathbf{n}}+Hq_{cd}\dot{t}^{cedf}+8H\dot{t}^{e\mathbf{n}f\mathbf{n}}\big)\Big)+\Big(D_cD_dt^{acbd}+D_cD_dt^{bcad}\\
&-2D_c\dot{t}^{acb\mathbf{n}}-2D_c\dot{t}^{bca\mathbf{n}}-10HD_ct^{a\mathbf{n}bc}-10HD_ct^{acb\mathbf{n}}\Big)\\
&-\frac{1}{3}q^{ab}q_{ef}\Big(2D_cD_dt^{ecfd}-4D_c\dot{t}^{ecf\mathbf{n}}-20HD_ct^{e\mathbf{n}fc}\Big)\,. \qquad (4.4.7)
\end{aligned}
$$

We introduce a simplification by considering only the purely spatial sector t^{abcd}, i.e.

$$t^{\mu\nu\rho\sigma}={e^\mu}_a{e^\nu}_b{e^\rho}_c{e^\sigma}_dt^{abcd}\,, \qquad (4.4.8)$$

setting all other sectors to zero. Our aim is to determine if this specific example can account for the accelerated expansion of the universe using only standard matter and radiation. For this particular case, the tensor in equation Eq. (4.4.3) simplifies to

$$(T^{Rt}_{\text{t- spatial}})^{\mu\nu} = e^{\mu}{}_{a}e^{\nu}{}_{b}\bigg[2\bigg(\frac{k}{a(t)^2}+H^2\bigg)q_{cd}t^{acbd}+\bigg(\frac{k}{a(t)^2}+H^2\bigg)q^{ab}q_{cd}q_{ef}t^{cedf} - \Big(D_cD_dt^{acbd}+D_cD_dt^{bcad}+2Hq_{cd}\dot{t}^{acbd}+2(\dot{H}+6H^2)q_{cd}t^{acbd}\Big)\bigg] - e^{\mu}{}_{a}n^{\nu}\Big[2Hq_{bc}D_dt^{acbd}\Big] - n^{\mu}e^{\nu}{}_{b}\Big[2Hq_{ad}D_ct^{acbd}\Big] + n^{\mu}n^{\nu}\bigg[\bigg(\frac{k}{a(t)^2}+3H^2\bigg)q_{ab}q_{cd}t^{acbd}\bigg]. \tag{4.4.9}$$

The traceless symmetric tensor in Eq. (4.4.7) reduces to

$$(T_{\perp})^{ab} = 2\bigg(\frac{k}{a(t)^2}-(\dot{H}+5H^2)\bigg)\bigg(q_{cd}t^{acbd}-\frac{1}{3}q^{ab}q_{ef}q_{cd}t^{ecfd}\bigg) - 2H\bigg(q_{cd}\dot{t}^{acbd}-\frac{1}{3}q^{ab}q_{ef}q_{cd}\dot{t}^{ecfd}\bigg) - \bigg(\Big(D_cD_dt^{acbd}+D_cD_dt^{bcad}\Big) - \frac{1}{3}q^{ab}q_{ef}\Big(D_cD_dt^{ecfd}+D_cD_dt^{fced}\Big)\bigg). \tag{4.4.10}$$

Since we are interested in the preservation of isotropy and homogeneity we impose the condition Eq. (4.1.9a), which consequently leads to $(T_{\perp})^{ab}=0$. For this, we consider the ansatz

$$q_{cd}t^{acbd}-\frac{1}{3}q^{ab}q_{ef}q_{cd}t^{ecfd} \;=\; 0\,. \tag{4.4.11}$$

This choice results to be consistent and it also determines the form of t^{abcd}. The components of t^{abcd} can be written in a simplified form as solution to the ansatz

$$t^{abcd} = a(t)^4\eta(t)(q^{ac}q^{bd}-q^{bc}q^{ad})\,, \tag{4.4.12}$$

with $\eta(t)$ being an arbitrary function of time. Taking advantage of the expression (4.4.12), we can write (4.4.9) as a maximally invariant tensor of rank two

$$(T^{Rt}_{\text{t-spatial}})^{\mu\nu} = e^{\mu}{}_{a}e^{\nu}{}_{b}\bigg[2\bigg(\frac{5k}{a(t)^2}-(2\dot{H}+7H^2)\bigg)a(t)^4\eta(t)-4Ha(t)^4\dot{\eta}(t)\bigg]q^{ab} + n^{\mu}n^{\nu}\bigg[6\bigg(\frac{k}{a(t)^2}+3H^2\bigg)a(t)^4\eta(t)\bigg]. \tag{4.4.13}$$

After algebraic manipulations we arrive to the modified Friedmann equations for the purely tangential t sector:

$$H^2 = \frac{1}{3\big(1-2a(t)^4\eta(t)\big)}\left(\kappa\rho - \frac{3k}{a(t)^2}\big(1+2a(t)^4\eta(t)\big)\right), \tag{4.4.14}$$

$$\dot{H}+H^2 = \frac{1}{(-6)\big(1-2a(t)^4\eta(t)\big)}\left[\kappa(\rho+3P)+24a(t)^4\left(\frac{k}{a(t)^2}+2(\dot{H}+3H^2)\right)\eta(t)\right]. \tag{4.4.15}$$

4.4.1 Bianchi identity for the spatial t sector

We have found a maximally invariant solution that provides an maximally invariant energy-momentum tensor. Now we need to impose the Bianchi identity. By using Eq. (4.4.13) on (4.2.31) we arrive to

$$6H\Big(2\big(2\dot{H}+7H^2\big)\eta(t)+H\dot{\eta}(t)\Big)a^4(t)=0\,. \tag{4.4.16}$$

4.4.2 Spatial t sector solution and accelerated expansion

Considering the Bianchi identity Eq. (4.4.16) we notice that the system admits an exact solution for $\eta(t)$ given by:

$$\eta(t)=\eta(0)\frac{H(0)^4a(0)^{14}}{H(t)^4a(t)^{14}}\,. \tag{4.4.17}$$

Now we are equipped with solutions for the background field that satisfy the no-go result.

At this point, we have gained better insight into how the compatibility between the background fields and the isotropy helps to satisfy the Einstein field equations. We have also developed a mechanism to solve most of the equations in the no-go result. After dealing with the compatibility conditions, we now want to know if certain sectors can produce an accelerated expansion of the universe.

For the u sector we obtained that the background field can be a constant function, producing the same expansion as the usual cosmology, but for Ricci flat solution we arrive to a different kind of expansion, where the scalar background produce the effect of suppressing changes in the curvature.

For the s sector we obtained that the background field can generate Ricci flat solutions and constant expansion solutions for the expansion of the universe. Some of this results were already obtained in [57] for the u and s sector, but taking as priority satisfying different relations.

Now we study the possibility of accelerated expansion for the t sector. We consider the state equation for the matter content as in previous sectors. From the first modified Friedmann equation we obtain the following limit:

$$(1 - 2a(t)^4\eta(t)) > 0\,, \tag{4.4.18}$$

that exhibits the regime of expansion of the modification. On the other hand we use the definition of the deceleration parameter

$$q(t) = -\frac{\ddot{a}(t)a(t)}{\dot{a}(t)^2}\,, \tag{4.4.19}$$

to rewrite the second modified Friedmann equation as

$$H^2 = \frac{1}{6q\big(1 - 2a(t)^4\eta(t)\big)}\left[(1+3w)\kappa\rho + 24a(t)^4\left(\frac{k}{a(t)^2} + 2(2-q)H^2\right)\eta(t)\right] \tag{4.4.20}$$

Comparing with the first modified Friedmann equation, we arrive to

$$q = \frac{(1+3w)\kappa\rho + 24a^2\left(k + 4a^2H^2\right)\eta(t)}{2\left[\kappa\rho - \frac{3k}{a^2} + 6a^2\left(4a^2H^2 - k\right)\eta(t)\right]}\,, \tag{4.4.21}$$

Replacing the solution for the continuity equation Eq. (4.2.37), and the solution for the Bianchi identity for the spatial t sector Eq. (4.4.17) in Eq. (4.4.14) we obtain an algebraic equation for $H(t)$ in terms of $a(t)$:

$$(H^2)^3 - \left(\frac{\kappa\rho}{3} - \frac{k}{a(t)^2}\right)(H^2)^2 - \frac{2\eta(0)H(0)^4a(0)^{14}}{a(t)^{10}}\left(H^2 - \frac{k}{a(t)^2}\right) = 0\,. \tag{4.4.22}$$

If we can solve this algebraic equation, we can obtain a differential equation to explore the solution for $a(t)$. Considering the current observational data [51], we will consider $k = 0$, reducing the equation to a quadratic polynomial in H^2, with solution

$$H^2 = \frac{\kappa\rho}{6} + \sqrt{\left(\frac{\kappa\rho}{6}\right)^2 + \frac{2\eta(0)H(0)^4a(0)^{14}}{a(t)^{10}}}\,. \tag{4.4.23}$$

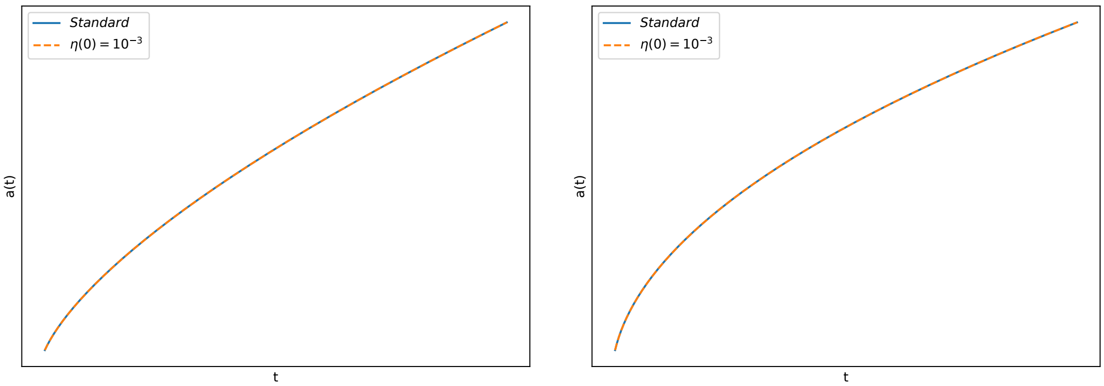
Taking $t = 0$ we can obtain the value for $H(0)$ in terms of the other set of initial conditions ($a(0), \rho(0)$, and $\eta(0)$), given by

$$H(0)^2 = \frac{\kappa\rho(0)}{3}\frac{1}{1 - 2\eta(0)a(0)^4}\,. \tag{4.4.24}$$

Replacing, we can write explicitly the differential equations as

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{\kappa\rho(0)}{6}\left[\frac{a(0)^{3(1+w)}}{a(t)^{3(1+w)}} + \sqrt{\frac{a(0)^{6(1+w)}}{a(t)^{6(1+w)}} + \frac{8\eta(0)a(0)^{14}}{(1-2\eta(0)a(0)^4)^2}\frac{1}{a(t)^{10}}}\right]\,. \tag{4.4.25}$$

The numerical solutions to Eq. (4.4.25) are shown in Fig. 4.4.1 comparing respect to the standard case ($\eta(t) = 0$). The plot shows no appreciable difference between the presence and absence of the background field. Moreover, the deceleration parameter obtained in Eq. (4.4.21) does not become negative in the validity regime given by $(1 - 2a(t)^4\eta(t)) > 0$. This implies that the field dilutes before it can affect the expansion rate of the universe, whether by acceleration or deceleration:



(a) $w = 0$ **(b)** $w = 1/3$

Figure 4.4.1: Plot of the scale factor as a function of time for $w = 0$ and $w = 1/3$. For both cases, the standard solution is shown as a blue solid line, and the result from solving Eq. (4.4.25) with initial condition $\eta(0) = 10^{-3}$. No region of accelerated expansion is visible in either case.

In contrast to the result published in [34] where merging the Bianchi identity with matter conservation produced accelerated expansion, we conclude that considering them separately for the t sector does not lead to accelerated expansion.

Our study of explicit diffeomorphism violation in a cosmological setting revealed that the dynamics compel the background field to become form-invariant under the spacetime isometries. Although the Bianchi identity must still be resolved for the background energy-momentum tensor, the restrictive form-invariance condition simplifies this task considerably, reducing it from a tensorial equation to a scalar one. In the following chapter, we will explore the minimal sector of the gSME, now considering a spontaneous breaking of diffeomorphism symmetry.

Chapter 5

Spontaneous violations in gravity

The consideration of spontaneous backgrounds in cosmology dates back to the late eighties [59], [60], where this framework was used to study a string-theory-inspired model featuring two expansion scales. The particular spontaneous background employed, known as the bumblebee field, gained rapid popularity due to both its simplicity and its non-trivial role in spontaneous symmetry breaking. Owing to its interpretation as a dynamical preferred frame, the bumblebee model has recently been applied to various contexts, including radiative corrections [61]–[63], time-like cosmology [64], [65], Gödel-type universes [66], [67], asymptotic flatness [68], and connections to the Kalb-Ramond field [69] and Einstein-aether theories [70], [71].

In this chapter, we further investigate the compatibility conditions between dynamical background fields and spacetime isometries. We focus specifically on spontaneous diffeomorphism breaking through an examination of the bumblebee model, which involves a dynamical vector field acquiring a nonzero vacuum expectation value. Since homogeneous and isotropic vectors cannot exist, we derive the non-trivial conditions emerging from requiring the bumblebee energy-momentum tensor to be form-invariant under FLRW isometries. We also derive the modified Friedmann equations to facilitate future investigations of cosmic expansion effects driven by the background field.

5.1 The bumblebee model

The Bumblebee model was first introduced in 1989 by Kostelecky and Samuel [60]. This is the simplest case of a theory with spontaneous Lorentz symmetry breaking produced through the introduction of a vectorial background field and a potential that fix the background to a value that breaks the symmetry. Consider the action:

$$\tilde{S} = S_B + S_m \,, \tag{5.1.1}$$

with

$$S_B = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2\kappa}\left({}^{(4)}R + \xi B^\mu B^\nu\, {}^{(4)}R_{\mu\nu}\right) - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} - V(B_\mu B^\mu \pm b^2)\right], \tag{5.1.2}$$

where we have included the EH term without cosmological constant and a perfect fluid source S_m. Also B^μ is the bumblebee field with mass dimension one which in contrast to the nondynamical background has its own Euler-Lagrange equation of motion. We define the field strength tensor of the bumblebee field by $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and we are considering $b^2 = b_\mu b^\mu$ where b_μ is the VEV of the bumblebee field with constant b^2, that is to say, $\langle B_\mu \rangle = b_\mu$. Furthermore, ξ is a coupling constant of the interaction term between the background and gravity having inverse mass dimension two. We will take the potential to be

$$V(B_\mu B^\mu \pm b^2) = \frac{\lambda}{4}(B_\mu B^\mu \pm b^2)^2 \,, \tag{5.1.3}$$

where λ is a coupling constant. Alternatively, we may have obtained the action (5.1.2) by replacing $u \to \frac{1}{4}\xi B^\mu B_\mu$, $s^{\mu\nu} \to \xi\left(B^\mu B^\nu - \frac{1}{4}g^{\mu\nu}B^\alpha B_\alpha\right)$ and $t^{\mu\nu\rho\sigma} \to 0$ in (3.2.7), together with installing kinetic and potential terms by hand.

We find the modified Einstein equation

$$G^{\mu\nu} = (T_B)^{\mu\nu} + \kappa(T_m)^{\mu\nu} \,, \tag{5.1.4}$$

where $(T_m)^{\mu\nu}$ is the energy-momentum tensor of the perfect fluid (4.1.5) and

$$(T_B)^{\mu\nu} = \kappa\Big[2V'B^\mu B^\nu + B^\mu{}_\kappa B^{\nu\kappa} - \Big(V + \frac{1}{4}B^{\lambda\kappa}B_{\lambda\kappa}\Big)g^{\mu\nu}\Big] + \frac{\xi}{2}\Big[B^\lambda B^\kappa R_{\lambda\kappa}g^{\mu\nu} - 2\big(g^{\mu\rho}B^\nu + g^{\nu\rho}B^\mu\big)B^\sigma R_{\rho\sigma} + \nabla_\lambda\nabla^\mu\big(B^\lambda B^\nu\big) + \nabla_\lambda\nabla^\nu\big(B^\lambda B^\mu\big) - \nabla_\kappa\nabla_\lambda\big(B^\lambda B^\kappa\big)g^{\mu\nu} - \nabla_\lambda\nabla^\lambda\big(B^\mu B^\nu\big)\Big]. \tag{5.1.5}$$

The equation of motion of the bumblebee field is

$$\nabla_\mu B^{\mu\nu} = 2V'B^\nu - \frac{\xi}{\kappa}B_\mu R^{\mu\nu}\,, \tag{5.1.6}$$

where the prime denotes differentiation with respect to the argument of V.

5.1.1 Boundary term for bumblebee gravity

As part of our research we derive the boundary term for the bumblebee model. The boundary terms arise from the first two terms in (5.1.2), which we decompose as follows.

We start by decomposing the field B^α into their normal and tangential projections

$$B^\alpha = e^\alpha{}_a B^a - n^\alpha B^{\mathbf{n}}\,. \tag{5.1.7}$$

Thus the decomposition for $B^\alpha B^\beta$ becomes

$$B^\alpha B^\beta = e^\alpha{}_b e^\beta{}_b B^a B^b - e^\alpha{}_a n^\beta B^a B^{\mathbf{n}} - n^\alpha e^\beta{}_b B^{\mathbf{n}} B^b + n^\alpha n^\beta\big(B^{\mathbf{n}}\big)^2\,. \tag{5.1.8}$$

From Eq. (2.2.56), the decomposition of the Ricci tensor is easily obtained, yielding:

$$\begin{aligned} R_{\beta\delta} \;=\; & e_\beta{}^b e_\delta{}^d\Big(R_{bd} - 2K_{be}K^e{}_d + KK_{bd} - D_d a_b - a_d a_b + \frac{1}{N}e^\rho{}_b e^\sigma{}_d \mathcal{L}_m(e_\rho{}^e e_\sigma{}^f K_{ef})\Big) \\ & - e_\beta{}^b n_\delta\Big(D^e K_{eb} - D_b K\Big) - n_\beta e_\delta{}^d\Big(D^e K_{ed} - D_d K\Big) \\ & + n_\beta n_\delta\Big(D_e a^e + a_e a^e + K^{ef}K_{ef} - \frac{1}{N}q^{bd}e^\rho{}_b e^\sigma{}_d \mathcal{L}_m(e_\rho{}^e e_\sigma{}^f K_{ef})\Big)\Big)\,. \end{aligned} \tag{5.1.9}$$

With these elements, we have

$$\begin{aligned} R_{\beta\delta}B^\beta B^\delta &= B^a B^b \Big(R_{ab} - 2K_{ac}K^c{}_b + KK_{ab} - D_b a_a - a_b a_a + \frac{1}{N} e^\rho{}_a e^\sigma{}_b \mathcal{L}_m (e_\rho{}^e e_\sigma{}^f K_{ef}) \Big) \\ &\quad - B^a B^{\mathbf{n}} \Big(2\big(D^c K_{ca} - D_a K \big) \Big) + \left(B^{\mathbf{n}}\right)^2 \Big(D_a a^a + a_a a^a + K^{ab}K_{ab} \\ &\quad - \frac{1}{N} q^{ab} e^\rho{}_a e^\sigma{}_b \mathcal{L}_m (e_\rho{}^e e_\sigma{}^f K_{ef}) \Big) , \end{aligned} \tag{5.1.10}$$

and recalling Eq. (5.1.2), the four curvature leads to the standard boundary term which we include in the final expression.

We use the identity

$$\begin{aligned} & e^\rho{}_a e^\sigma{}_b \Big[B^a B^b - \left(B^{\mathbf{n}}\right)^2 q^{ab} \Big] \frac{1}{N} \mathcal{L}_m (e_\rho{}^e e_\sigma{}^f K_{ef}) = \nabla_\lambda \Big[n^\lambda \Big(B^a B^b - \left(B^{\mathbf{n}}\right)^2 q^{ab} \Big) K_{ab} \Big] \\ & \quad - \Big[B^a B^b - \left(B^{\mathbf{n}}\right)^2 q^{ab} \Big] K K_{ab} - \frac{1}{N} e_\rho{}^e e_\sigma{}^f K_{ef} \mathcal{L}_m \Big[e^\rho{}_a e^\sigma{}_b \Big(B^a B^b - \left(B^{\mathbf{n}}\right)^2 q^{ab} \Big) \Big] , \end{aligned} \tag{5.1.11}$$

in the previous relation (5.1.10), to find the extended boundary term for the bumblebee model

$$S^{(\mathrm{B})}_{\partial\mathcal{M}} = \frac{1}{2\kappa} \oint_{\partial\mathcal{M}} \mathrm{d}^3 y \, \varepsilon \sqrt{q} \left(2K + \xi \Big(B^a B^b - \left(B^{\mathbf{n}}\right)^2 q^{ab} \Big) K_{ab} \right) . \tag{5.1.12}$$

We observe that when the interaction term vanishes, the usual Gibbons-Hawking-York (GHY) boundary term is recovered. Furthermore, this term is intimately related to the boundary terms appearing in the u and $s^{\mu\nu}$ sectors (see Eq. (3.2.20)), which follows from the connection between the bumblebee model and these sectors.

5.1.2 Modified Friedmann equations for $B_\mu = (0, B_a(y,t))$

Here we decompose the modified Einstein equation, by using our strategy in order to find the modified Friedmann equations. It is important to emphasize that, unlike the explicit case, the bumblebee model has its own equations of motion since it involves spontaneous symmetry breaking. This allows the field to fluctuate and not necessarily be a maximally invariant tensor under the symmetries of the theory. This is why the choice of the bumblebee vector field can be extended, even though a maximally form-invariant vector does not exist in four dimensions [58]. One of the main goals in this chapter, is to extend the standard treatment beyond a purely timelike bumblebee field ([64], [65]) to incorporate a bumblebee field with spatial components, providing a foundation for exploring alternative mechanisms in early-universe cosmology.

In this way, we consider a purely tangential bumblebee field

$$B^{\alpha} = e^{\alpha}{}_{a}B^{a}\,, \tag{5.1.13}$$

which produces the projection for the bumblebee field strength

$$B_{\mu\nu} = e_{\mu}{}^{a}e_{\nu}{}^{b}B_{ab} - n_{\mu}e_{\nu}{}^{b}\Xi_{b} + e_{\mu}{}^{a}n_{\nu}\Xi_{a}\,, \tag{5.1.14}$$

where we introduced the following quantities:

$$B_{ab} = D_aB_b - D_bB_a\,, \tag{5.1.15}$$
$$\Xi_c = \frac{1}{N}e^{\kappa}{}_{c}\mathcal{L}_m\big(e_{\kappa}{}^{d}B_d\big)\,. \tag{5.1.16}$$

By using the decomposition for the second covariant derivative of the product of bumblebee fields, see the appendix A, we can decompose the bumblebee energy-momentum tensor in their normal and tangential projections as

$$(T_B)^{\mu\nu} = e^{\mu}{}_{a}e^{\nu}{}_{b}(T_1^B)^{ab} - (e^{\mu}{}_{a}n^{\nu} + n^{\mu}e^{\nu}{}_{a})(T_2^B)^{a} + n^{\mu}n^{\nu}(T_3^B)\,, \tag{5.1.17}$$

with spatial components

$$\begin{aligned}(T_1^B)^{ab} = {}& \kappa\Big[2V'B^aB^b + B^{ac}B^{b}{}_{c} - q^{ac}q^{bd}\mathcal{L}_tB_c\mathcal{L}_tB_d - \Big(V + \frac{1}{4}\big(B_{cd}B^{cd} - 2h^{cd}\mathcal{L}_tB_c\mathcal{L}_tB_d\big)\Big)q^{ab}\Big] \\ & + \frac{\xi}{2}\Big[q^{ab}\Big(\frac{2k}{a(t)^2}B^cB_c - D_cD_d(B^cB^d) - H\mathcal{L}_t(B_cB^c)\Big) + D_cD^a(B^cB^b) + D_cD^b(B^cB^a) \\ & - D^2(B^aB^b) + q^{bc}\Big(\mathcal{L}_t\mathcal{L}_t(B^aB_c) + 3H\mathcal{L}_t(B^aB_c) - 2\Big(\frac{4k}{a(t)^2} + (\dot{H} + 3H^2)\Big)B^aB_c\Big]\,,\end{aligned} \tag{5.1.18}$$

mixed components

$$(T_2^B)^{a} = \kappa\Big[B^{ac}\mathcal{L}_tB_c\Big] + \frac{\xi}{2}\Big[D_c\big(\mathcal{L}_t(B^cB^a)\big) - HD^a(B^cB_c) + 2HD_c(B^cB^a)\Big]\,, \tag{5.1.19}$$

and doubly-normal components

$$(T_3^B) = \kappa\left[V + \frac{1}{4}\left(B_{cd}B^{cd} + 2q^{cd}\mathcal{L}_tB_c\mathcal{L}_tB_d\right)\right] - \frac{\xi}{2}\left[\frac{2k}{a(t)^2}B^cB_c + H\mathcal{L}_t(B^cB_c) - D_cD_d(B^cB^d)\right]. \tag{5.1.20}$$

Considering Eq. (4.1.6) and the projections for the Einstein tensor, we obtain the modified Einstein equation

$$\begin{aligned} G^{ab} = \kappa\Big[&2V'B^aB^b + B^{ac}B^b{}_c - q^{ac}q^{bd}\mathcal{L}_tB_c\mathcal{L}_tB_d - \left(V + \frac{1}{4}\left(B_{cd}B^{cd} - 2q^{cd}\mathcal{L}_tB_c\mathcal{L}_tB_d\right)\right)q^{ab}\Big] \\ &+ \frac{\xi}{2}\Big[q^{ab}\left(\frac{2k}{a(t)^2}B^cB_c - D_cD_d(B^cB^d) - H\mathcal{L}_t(B_cB^c)\right) + D_cD^a(B^cB^b) + D_cD^b(B^cB^a) \\ &- D^2(B^aB^b) + \mathcal{L}_t^2(B^aB^b) + 7H\mathcal{L}_t(B^aB^b) - 4\left(\frac{2k}{a(t)^2} - H^2\right)B^aB^b\Big] + \kappa(T_m)^{ab}, \end{aligned} \tag{5.1.21}$$

and

$$G^{\mathbf{nn}} = \kappa\left[V + \frac{1}{4}\left(B_{cd}B^{cd} + 2q^{cd}\mathcal{L}_tB_c\mathcal{L}_tB_d\right)\right] - \frac{\xi}{2}\left[\frac{2k}{a(t)^2}B^cB_c + H\mathcal{L}_t(B^cB_c) - D_cD_d(B^cB^d)\right] + \kappa(T_m)^{\mathbf{nn}}. \tag{5.1.22}$$

Also, the bumblebee field equations (5.1.6) can be decomposed in their spatial and normal projections as

$$e_b^\nu\left[D_a(B^{ab}) - Hq^{ba}\mathcal{L}_tB_a - q^{ba}\mathcal{L}_t^2B_a - 2\left[V' - \frac{\xi}{2\kappa}\left(\frac{2k}{a(t)^2} + (\dot{H} + 3H^2)\right)\right]B^b\right] - n^\nu\left[-D^c\left(\mathcal{L}_tB_c\right)\right] = 0. \tag{5.1.23}$$

At this point we make another simplification. By looking the normal projection of the bumblebee field equation we take as ansatz the condition $D_mB_n = 0$ meaning that the bumblebee field will only depends on time $B_a = B_a(t)$.

Under this consideration we obtain the first modified Friedmann equations

$$H^2 = \frac{1}{3\Big(1-\frac{\xi}{3}B^cB_c\Big)}\Bigg(\kappa\rho + \kappa\Big[V + \frac{1}{2}q^{cd}\dot{B}_c\dot{B}_d\Big] - \frac{\xi}{2}\Big[\frac{2k}{a(t)^2}B^cB_c + 2HB^c\dot{B}_c\Big] - \frac{3k}{a(t)^2}\Bigg)\,, \tag{5.1.24}$$

the second Friedmann equation

$$\begin{aligned}\dot{H} + H^2 = \frac{1}{(-6)\Big(1-\frac{\xi}{3}B^cB_c\Big)}\Bigg(&\kappa(3P+\rho) + \kappa\Big[2V'B^cB_c - 2V + q^{cd}\dot{B}_c\dot{B}_d\Big] \\ &+ \frac{\xi}{2}\Big[-\frac{4k}{a(t)^2}B^cB_c + 4H^2B^cB_c - 10Hq^{cd}\dot{B}_cB_d + 2q^{cd}\ddot{B}_cB_d + 2q^{cd}\dot{B}_c\dot{B}_d\Big]\Bigg)\,,\end{aligned} \tag{5.1.25}$$

and the perpendicular extra condition from the previous method becomes

$$\begin{aligned}(T_\perp)^{ab} = \kappa&\Big[2V'B^aB^b - q^{ac}q^{bd}\dot{B}_c\dot{B}_d - \frac{1}{3}q^{ab}\Big(2V'B^cB_c - q^{cd}\dot{B}_c\dot{B}_d\Big)\Big]\,, \\ &+ \frac{\xi}{2}\Big[q^{bc}\Big(\mathcal{L}_t^2(B^aB_c) + 3H\mathcal{L}_t(B^aB_c) - 2\Big(\frac{4k}{a(t)^2} + (\dot{H}+3H^2)\Big)B^aB_c\Big) \\ &- \frac{1}{3}q^{ab}\Big(\mathcal{L}_t^2(B^cB_c) + 3H\mathcal{L}_t(B^cB_c) - 2\Big(\frac{4k}{a(t)^2} + (\dot{H}+3H^2)\Big)B^cB_c\Big)\Big] = 0\,.\end{aligned} \tag{5.1.26}$$

Together with the bumblebee field equations

$$\ddot{B}_a + H\dot{B}_a + 2\Big[V' - \frac{\xi}{2\kappa}\Big(\frac{2k}{a(t)^2} + (\dot{H}+3H^2)\Big)\Big]B_a = 0\,. \tag{5.1.27}$$

We provide the details of the decomposition in the appendix A.

5.1.2.1 The case $\xi = 0$

We turn off the interaction term with $\xi = 0$. Thus the bumblebee field equation of motion simplifies to

$$\ddot{B}_a + H\dot{B}_a + 2V'B_a = 0\,. \tag{5.1.28}$$

We have the first modified Friedmann equation

$$H^2 = \frac{1}{3}\left\{\kappa\rho + \kappa\left[V + \frac{1}{2}q^{ab}\dot{B}_a\dot{B}_b\right] - \frac{3k}{a(t)^2}\right\}, \quad (5.1.29)$$

and the second modified Friedmann equation

$$\dot{H} + H^2 = \frac{1}{(-6)}\left\{\kappa(3P+\rho) + \kappa\left[2V'q^{ab}B_aB_b - 2V + q^{ab}\dot{B}_a\dot{B}_b\right]\right\}, \quad (5.1.30)$$

and the condition Eq. (5.1.26) becomes

$$0 = \kappa\left[\left(2V'B_aB_b - \dot{B}_a\dot{B}_b\right) - \frac{1}{3}q_{ab}q^{cd}\left(2V'B_cB_d - \dot{B}_c\dot{B}_d\right)\right]. \quad (5.1.31)$$

For $V' > 0$ the condition Eq. (5.1.31) has the general solution

$$\dot{B}_a = \pm\sqrt{2V'}B_a\,. \quad (5.1.32)$$

By taking the time derivative, we obtain an expression for $\ddot{B}_a$ as follows

$$\begin{aligned}\ddot{B}_a &= \pm\sqrt{2}\left(\frac{1}{2}\frac{1}{\sqrt{V'}}\frac{\mathrm{d}V'}{\mathrm{d}(B^cB_c)}\frac{\partial(B^bB_b)}{\partial t}B_a + \sqrt{V'}\dot{B}_a\right)\\ &= \pm\sqrt{2}\left(\frac{1}{2}\frac{V''}{\sqrt{V'}}\frac{\partial(B^bB_b)}{\partial t}B_a + \sqrt{V'}\dot{B}_a\right)\\ &= \pm 2\left(\frac{V''}{\sqrt{2V'}}\big(-H \pm \sqrt{2V'}\big)h^{bc}B_bB_c \pm V'\right)B_a\,.\end{aligned} \quad (5.1.33)$$

Using this expression in the bumblebee field equation we obtain

$$\left(2V''\big(\mp H + (2V')^{1/2}\big)q^{bc}B_bB_c \pm H(2V') + 2(2V')^{3/2}\right)B_a = 0\,. \quad (5.1.34)$$

Thus the final equation to solve becomes an scalar equation

$$2V''\big(\mp H + (2V')^{1/2}\big)B^2 \pm H(2V') + 2(2V')^{3/2} = 0\,. \quad (5.1.35)$$

Lets recall the quartic potential

$$V(B^2) = \frac{\lambda}{4}(B^2 - b^2)^2, \quad (5.1.36)$$

$$B^2 = B^aB_a\,. \quad (5.1.37)$$

We obtain

$$\mp H\lambda b^2 + (\lambda(B^2 - b^2))^{1/2}(\lambda(3B^2 - 2b^2)) = 0\,, \tag{5.1.38}$$

which lead to

$$(B^2 - b^2)^{1/2}(3B^2 - 2b^2) = \pm\frac{H}{\sqrt{\lambda}}b^2\,. \tag{5.1.39}$$

Defining the quantity $A^2 = B^2 - b^2$ we obtain

$$A^2|A| + \frac{b^2}{3}|A| \mp \frac{H}{3\sqrt{\lambda}}b^2 = 0\,. \tag{5.1.40}$$

We recognize this equation as the depressed cubic equation

$$A^3 + pA + q = 0\,, \tag{5.1.41}$$

with

$$p = \frac{b^2}{3}\,, \tag{5.1.42}$$

$$q = \mp\frac{H}{3\sqrt{\lambda}}b^2\,. \tag{5.1.43}$$

The equation is well know for having two types of solutions: for three real solutions we have the Viette formula, and for one real solution and two complex solutions we have the Cardano formula. In this case the discriminant becomes

$$\Delta = -(4p^3 + 27q^2) = -b^4\left(\frac{4}{27}b^2 + 3\frac{H^2}{\lambda}\right) < 0\,. \tag{5.1.44}$$

Thus we have one real root and two complex roots for the cubic polynomial. The Cardano formula gives the real solution

$$\begin{aligned} A &= \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} \\ &= \left(\pm\frac{H}{6\sqrt{\lambda}}b^2 + \frac{b^2}{3}\sqrt{\frac{H^2}{4\lambda} + \frac{b^2}{81}}\right)^{1/3} + \left(\pm\frac{H}{6\sqrt{\lambda}}b^2 - \frac{b^2}{3}\sqrt{\frac{H^2}{4\lambda} + \frac{b^2}{81}}\right)^{1/3}\,. \end{aligned} \tag{5.1.45}$$

Obtaining the time dependent bumblebee field solution

$$
\begin{aligned}
B^2(t) &= \left(\frac{b^2}{6\sqrt{\lambda}}\right)^{2/3}\left[\left(\pm H+\sqrt{H^2+\frac{4\lambda b^2}{81}}\right)^{2/3}+\left(\pm H-\sqrt{H^2+\frac{4\lambda b^2}{81}}\right)^{2/3}\right] \\
&\quad -\frac{7}{9}b^2\,. \qquad (5.1.46)
\end{aligned}
$$

Therefore, the first modified Friedmann equations can be written as

$$
H^2 = \frac{1}{3}\left(\kappa\rho+\frac{\lambda\kappa}{4}(3B^dB_d-b^2)(B^cB_c-b^2)-\frac{3k}{a(t)^2}\right), \qquad (5.1.47)
$$

and the second

$$
\dot{H}+H^2 = \frac{1}{(-6)}\left[\kappa(3P+\rho)+\frac{\lambda\kappa}{2}(3B^dB_d+b^2)(B^cB_c-b^2)\right]. \qquad (5.1.48)
$$

Here we notice an important result: even if we consider a tangential contribution from the bumblebee field, its field equations can be solved for $B(t)^2$ without fixing a direction. This suggests that spontaneous symmetry breaking does not impose the background field to be invariant under the isometries, as is the case in explicit symmetry breaking.

5.1.2.2 Nonminimal coupling extended case $\xi \neq 0$

Inspired by the advances made in stating the differential equations in the previous part we extend to consider $\xi \neq 0$.

Lets consider the tangential bumblebee field equation

$$
\ddot{B}_a = -\left(H\dot{B}_a+2\left[V'-\frac{\xi}{2\kappa}\left(\frac{2k}{a(t)^2}+(\dot{H}+3H^2)\right)\right]B_a\right). \qquad (5.1.49)
$$

We replace in the perpendicular equation Eq. (5.1.26),

$$\kappa\Big[2V'B_aB_b-\dot B_a\dot B_b-\frac{1}{3}q_{ab}\Big(2V'B^cB_c-q^{cd}\dot B_c\dot B_d\Big)\Big]+\frac{\xi}{2}\Big[2\dot B_a\dot B_b-2H\dot B_aB_b-2HB_a\dot B_b$$
$$-4\Big[V'-\frac{\xi}{2\kappa}\Big(\frac{2k}{a(t)^2}+(\dot H+3H^2)\Big)+\Big(\frac{2k}{a(t)^2}+(\dot H+2H^2)\Big)\Big]B_aB_b-\frac{1}{3}q_{ab}\Big(2q^{cd}\dot B_c\dot B_d$$
$$-4Hq^{cd}\dot B_cB_d-4\Big[V'-\frac{\xi}{2\kappa}\Big(\frac{2k}{a(t)^2}+(\dot H+3H^2)\Big)+\Big(\frac{2k}{a(t)^2}+(\dot H+2H^2)\Big)\Big]$$
$$\times B^cB_c\Big)\Big]=0\,. \tag{5.1.50}$$

obtaining a tensorial equation with lower degrees in time derivatives. We consider the tangential bumblebee field equation for $\xi=0$ (5.1.32) as an ansatz at this point. Hence, we obtain a major simplification for the isotropy-homogeneity equation

$$-2\xi\Big[H\sqrt{2V'}-\frac{\xi}{2\kappa}H^2+\Big(\frac{2k}{a(t)^2}+(\dot H+2H^2)\Big)\Big(1-\frac{\xi}{2\kappa}\Big)\Big]$$
$$\Big(B_aB_b-\frac{1}{3}q_{ab}B^cB_c\Big)=0\,. \tag{5.1.51}$$

Notice that in the limit $\xi\to 0$, we do not recover the previous case, leading to a discontinuity. Considering the relation (5.1.10) and the fact that, in the maximally symmetric case $R_{ab}\sim q_{ab}$, the term terms B_aB^a acquires a mass contribution, with ξ contributing to a term we can identify as an effective mass parameter. This discontinuity closely resembles the vDVZ discontinuity observed in massive gravity [72]–[74].

The solution for the second parenthesis correspond to a null bumblebee field. The only possibility is to solve the scalar equation

$$H\sqrt{2V'}+\Big(1-\frac{\xi}{2\kappa}\Big)\Big(\frac{2k}{a(t)^2}+(\dot H+2H^2)\Big)-\frac{\xi}{2\kappa}H^2=0\,. \tag{5.1.52}$$

Considering again the quartic potential and solving the term with the derivative of the potential we obtain a new equation for the bumblebee field

$$B^2=\frac{1}{H^2}\Big[\Big(1-\frac{\xi}{2\kappa}\Big)\Big(\frac{2k}{a(t)^2}+(\dot H+2H^2)\Big)-\frac{\xi}{2\kappa}H^2\Big]^2+b^2\,, \tag{5.1.53}$$

and a new set of modified Friedmann equations

$$H^2 = \frac{1}{3\Big(1 - \frac{\xi}{3}\big(\frac{1}{H^2}\big[(\dot{H} + 2H^2) - \frac{\xi}{2\kappa}(\dot{H} + 3H^2)\big]^2 \mp b^2\big)\Big)} \times \Bigg(\kappa\rho + \frac{\lambda\kappa}{4H^2}\left[(\dot{H} + 2H^2) - \frac{\xi}{2\kappa}(\dot{H} + 3H^2)\right]^2 \left[\frac{3}{H^2}\left[(\dot{H} + 2H^2) - \frac{\xi}{2\kappa}(\dot{H} + 3H^2)\right]^2 \mp 2b^2\right] + \xi\left[(\dot{H} + 2H^2) - \frac{\xi}{2\kappa}(\dot{H} + 3H^2)\right]\left(\frac{1}{H^2}\left[(\dot{H} + 2H^2) - \frac{\xi}{2\kappa}(\dot{H} + 3H^2)\right]^2 \mp b^2\right)\Bigg), \tag{5.1.54}$$

and

$$\dot{H} + H^2 = \frac{1}{(-6)\Big(1 - \frac{\xi}{3}\big(\frac{1}{H^2}\big[(\dot{H} + 2H^2) - \frac{\xi}{2\kappa}(\dot{H} + 3H^2)\big]^2 \mp b^2\big)\Big)} \times \Bigg(\kappa(3P + \rho) + \frac{\lambda\kappa}{2H^2}\left[(\dot{H} + 2H^2) - \frac{\xi}{2\kappa}(\dot{H} + 3H^2)\right]^2 \left[\frac{3}{H^2}\left[(\dot{H} + 2H^2) - \frac{\xi}{2\kappa}(\dot{H} + 3H^2)\right]^2 \mp 4b^2\right] + 2\xi\left[(3\dot{H} + 7H^2) - \frac{\xi}{\kappa}(\dot{H} + 3H^2)\right] \left(\frac{1}{H^2}\left[(\dot{H} + 2H^2) - \frac{\xi}{2\kappa}(\dot{H} + 3H^2)\right]^2 \mp b^2\right)\Bigg). \tag{5.1.55}$$

We again leave a detailed discussion of direct contributions to the universe's expansion for a future work, which will focus on general solutions in bumblebee cosmology. An interesting result emerges: for gravitational setups with explicit symmetry breaking, the background must be invariant under the isometry group. In contrast, spontaneous symmetry breaking does not require an invariant background. Instead, in the case of spontaneous breaking, the background interacts with the other equations of motion through invariants constructed from the background fields, with this case being the modulus $B^a B_a$.

In the following chapter, we will discuss how the isometries play a role not only in satisfying the Einstein field equations but also in predicting additional configurations compatible with the Bianchi identity for the energy-momentum tensor of the background fields (or the no-go result).

Chapter 6

The method with Isometries

The intrinsic nonlinearity of General Relativity makes finding solutions a challenging task, which is why it is often valuable to impose symmetries before solving the field equations. Building on this principle, in previous chapters we saw that the Einstein field equations themselves require background fields to respect spacetime isometries in cases of explicit violation. Alternatively, in spontaneous violation, restrictions can be imposed through invariants constructed from these isometries. While background fields are constrained by the isometries, we observe that the Bianchi identity equations for their energy-momentum tensor are more likely to be satisfied, thus preventing discrepancies between geometry and dynamics. At this point, it is not clear how isometries play a role in avoiding the no-go results, and this discussion allows us to develop an alternative method to the well-known treatments for such discrepancies, published in Ref. [75], which we will present in this chapter.

In general, there are two mechanisms for symmetry violation: spontaneous or explicit. The former occurs when tensor-valued fields dynamically acquire symmetry-violating vacuum expectation values. Spontaneous diffeomorphism violation has been the preferred mechanism for spacetime symmetry breaking in the gravitational Standard-Model Extension (SME) [16], [32], particularly since it was shown to be dynamically consistent with Riemannian geometry [76], [77] through the use of the background field equations.

In contrast, explicit diffeomorphism violation is plagued by conflicts between the dynamics and Riemannian geometry. Reconciling the modified Einstein equations with the contracted second Bianchi identities leads to a set of challenging, coupled partial differential equations for the SME background coefficients. This important finding, known as the no-go result in the contemporary literature [16], [31], [32], [78], has led to two primary interpretations. To maintain Riemannian geometry, a nondynamical SME background field must be severely restricted; alternatively, one is forced to consider a beyond-Riemannian framework such as Finsler geometry [79]–[82].

6.1 Strategies to avoid the no-go result

To date, two main approaches have been proposed in the literature to address this problem while maintaining Riemannian geometry. The first employs a procedure known as the Stückelberg trick. Originally, Stückelberg introduced an auxiliary scalar field into Proca theory [83], [84] to restore gauge symmetry, thereby ensuring a smooth limit as the Proca mass approaches zero. This idea was later adopted in massive gravity [85], [86] and in the gravitational SME with explicit diffeomorphism breaking [87], where it operates at the level of linearized gravity. In this context, the technique relies on several auxiliary fields that emulate the Nambu-Goldstone modes arising from spontaneous spacetime symmetry breaking. Thus, the minimal number of excitations is reintroduced to restore the broken symmetries. This procedure recovers some essential properties of spontaneous symmetry breaking, even though the background field remains nondynamical. However, the approach does not reintroduce the Higgs-like modes, which also emerge naturally in the spontaneous symmetry-breaking picture.

The second method involves restricting the spacetime geometry to dynamically suppress diffeomorphism violation. This approach was notably applied by Jackiw and Pi to the gravitational Chern-Simons (CS) term in four spacetime dimensions [88]. This term can be expressed as the divergence of the CS topological current. After suitable integration by parts, it is identified with the Chern-Pontryagin scalar density $2\,{}^*R^{\sigma}{}_{\tau}{}^{\mu\nu}R^{\tau}{}_{\sigma\mu\nu} =: 2\,{}^*RR$, where $R^{\tau}{}_{\sigma\mu\nu}$ is the Riemann curvature tensor and ${}^*R^{\tau}{}_{\sigma}{}^{\mu\nu} := (1/2)\varepsilon^{\mu\nu\alpha\beta}R^{\tau}{}_{\sigma\alpha\beta}$ is its dual. With a suitable normalization, the integral of this density over spacetime yields a $\mathbb{Z}_2$ topological quantity known as the second Chern number or the gravitational instanton number [89]. For the CS-like term to be consistent with the contracted second Bianchi identities, the condition ${}^*RR = 0$

must be imposed. This requirement restricts the space of possible geometries to those that satisfy it. Consequently, the topological properties characterized by the second Chern number must be trivial for spacetimes to be dynamically consistent within this framework.

Our approach, on the other hand, has been applied to a cosmological model modified by the t sector of the minimal gravitational SME with explicit symmetry violation [34], and is extended to the u and s sectors in this thesis. The modified-gravity theory considered here incorporates the isotropy and homogeneity that underpin many cosmological models. This was achieved by ensuring that the purely spacelike part of the second-rank tensor background field—which modifies the Einstein equations—satisfies an essential property: it must be form-invariant under the six isometries generated by the corresponding spatial Killing vector fields. As a result, the spacetime geometry is effectively restricted to remain dynamically consistent.

We will now describe the basic ideas behind a third approach to reconciling a framework of explicit spacetime symmetry violation with the dynamical field equations. This method involves restricting the spacetime geometry by imposing symmetries on the metric that are dictated by the specific properties of the gravitational system under study.

6.2 Isometries in explicit breaking

Let us consider the following generic gravitational action, which is composed of the Einstein-Hilbert action plus a Lagrangian density that violates diffeomorphism invariance.:

$$S = \int \mathrm{d}^4x \, \frac{\sqrt{-g}}{2\kappa} \left[R + \mathcal{L}'(g_{\mu\nu}, \bar{k}^{\alpha\beta\ldots\omega}) \right] + S_B \,. \tag{6.2.1}$$

The Lagrange density $\sqrt{-g}\mathcal{L}'$ involves the coefficients $\bar{k}^{\alpha\beta\ldots\omega}$ of a generic nondynamical background field, which makes the action noninvariant under particle diffeomorphisms. Moreover, for consistency with the stationary-action principle, we include a suitable extension of the Gibbons-Hawking-York (GHY) boundary term [35]–[37], [90], [91] described by S_B. The action is constructed to be invariant under observer diffeomorphisms. Then, the variation of S with respect to such transformations reads:

$$\delta S_{\text{obs}} = \frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\Big[\big(-G^{\mu\nu} + T'^{\mu\nu}\big)\delta g_{\mu\nu} + J_{\alpha\beta\ldots\omega}\delta\bar{k}^{\alpha\beta\ldots\omega}\Big] = 0\,, \tag{6.2.2}$$

where we defined

$$T'^{\mu\nu} := \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}')}{\delta g_{\mu\nu}}\,, \quad J_{\alpha\beta\ldots\omega} := \frac{\delta\mathcal{L}'}{\delta\bar{k}^{\alpha\beta\ldots\omega}}\,, \tag{6.2.3}$$

and $G_{\mu\nu} = R_{\mu\nu} - (R/2)g_{\mu\nu}$ is the Einstein tensor with the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R. Since the presence of the background field violates particle diffeomorphism invariance, there is a mismatch between the variations of the action with respect to general coordinate transformations and particle diffeomorphisms. The latter can be expressed via the following integral equation:

$$\delta S_{\text{obs}} - \delta S_{\text{part}} = \int \mathrm{d}^4x\,\frac{\sqrt{-g}}{2\kappa}J_{\alpha\beta\ldots\omega}\delta\bar{k}^{\alpha\beta\ldots\omega}\,. \tag{6.2.4}$$

Thus, Eq. (6.2.4) describes the origin of the clash between nondynamical background fields and Riemannian geometry in a explicit manner.

For a diffeomorphism being an isometry with Killing vector field χ, by definition we have $\delta g_{\mu\nu} = \mathcal{L}_\chi g_{\mu\nu} = 0$, where $\mathcal{L}_\chi$ denotes the Lie derivative [92], [93] along χ. Moreover, we specifically use $\delta\bar{k}^{\alpha\beta\ldots\omega} = \mathcal{L}_\chi\bar{k}^{\alpha\beta\ldots\omega}$. Then, Eq. (6.2.2) implies

$$\delta S_{\text{obs}} = 0 = \int \mathrm{d}^4x\,\frac{\sqrt{-g}}{2\kappa}J_{\alpha\beta\ldots\omega}\mathcal{L}_\chi\bar{k}^{\alpha\beta\ldots\omega}\,. \tag{6.2.5}$$

Hence, there only remains the Lie derivative part of the background field. This leads to the requirement

$$\mathcal{L}_\chi\bar{k}^{\alpha\beta\ldots\omega} = 0\,, \tag{6.2.6}$$

such that the isometry is imposed on the background field. For a diffeomorphism that is not an isometry, it holds that $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$ and $\delta\bar{k}^{\alpha\beta\ldots\omega} = \mathcal{L}_\xi\bar{k}^{\alpha\beta\ldots\omega}$, with the Lie derivative $\mathcal{L}_\xi$ along the generator ξ of the diffeomorphism. Performing several integrations by parts in Eq. (6.2.2) and employing the contracted second Bianchi identities $\nabla^\mu G_{\mu\nu} = 0$, we arrive at the identity

$$2\nabla_\mu T'^\mu{}_\nu = J_{\alpha\beta\ldots\omega}\nabla_\nu \bar{k}^{\alpha\beta\ldots\omega} + \nabla_\lambda(J_{\nu\beta\ldots\omega}\bar{k}^{\lambda\beta\ldots\omega}) \\ + \nabla_\lambda(J_{\alpha\nu\ldots\omega}\bar{k}^{\alpha\lambda\ldots\omega}) + \cdots + \nabla_\lambda(J_{\alpha\beta\ldots\nu}\bar{k}^{\alpha\beta\ldots\lambda}) \,, \tag{6.2.7}$$

mentioned previously in Eq. (3.1.19), but we now know that this corresponds to an identity and not to a condition to be fulfilled. Basically, the right hand side of Eq. (6.2.7) appear due to how background fields transform under general coordinate transformation.

For the theory to be dynamically consistent, a critical requirement is that the Bianchi identity for the energy-momentum tensor of the backgrounds is satisfied [16], [32]. This equation is obtained by following the same procedure used for the observer transformation without considering background variations. The resulting differential equation is:

$$\nabla_\mu T'^\mu{}_{,,\nu} = 0, . \tag{6.2.8}$$

In what follows, we offer an alternative perspective on this problem. The preceding equations elucidate how the conflict between explicit spacetime symmetry breaking and dynamics can be resolved, at least for specific systems possessing isometries. In particular, Equation (6.2.4) clearly illustrates this conflict by contrasting the variation of the action under observer and particle diffeomorphisms in the presence of a nondynamical background field.

First, consider a gravitational system that does not exhibit any isometries identifiable with diffeomorphisms. In this case, we must resort to Eq. (6.2.8). For a generic metric of complicated form, this set of coupled, nonlinear partial differential equations is likely to heavily restrict the background field, potentially to the point where all coefficients vanish identically. However, a possible resolution may exist for gravitational systems that exhibit at least one diffeomorphism corresponding to an isometry, as we will argue below.

By following this line of reasoning, suppose that the gravitational system is characterized by its invariance under a certain diffeomorphism. For example, the Schwarzschild *ansatz*, which describes a static metric, is invariant under infinitesimal time diffeomorphisms. Such a diffeomorphism is then an isometry of the metric, which allows us to identify the generator of this diffeomorphism with a Killing vector field χ. Now, Eq. (6.2.6) applies, whereupon the nondynamical background field must be compatible with the symmetry of the metric. Consistency between dynamics and Riemannian geometry demands that Eq. (6.2.8) also be valid. The latter is a consistency requirement, whereas the former takes the role of a physical statement on a symmetry that the gravitational system possesses.

The role of the isometry is to restrict the degrees of freedom and the functional dependence on the coordinates for both the metric $g_{\mu\nu}$ and the background field $\bar{k}^{\alpha\beta\dots\omega}$. By doing so, the differential equations of Eq. (6.2.8) are supposed to simplify, which contributes to finding nontrivial solutions $\bar{k}^{\alpha\beta\dots\omega}$ compatible with the dynamics. The presence of each further isometry that can be identified with a diffeomorphism reduces the complexity of $g_{\mu\nu}$ and $\bar{k}^{\alpha\beta\dots\omega}$ even more. Consequently, the system of differential equations to be solved for the coefficients of the background field is expected to be ever more manageable.

For simple background fields such as the coordinate scalar u, these conditions are likely to permit only backgrounds that are either trivial or have a simple dependence on the spacetime coordinates. However, as the number of Lorentz indices in $\bar{k}^{\alpha\beta\dots\omega}$ increases, particularly in the nonminimal SME, the background field possesses a greater number of independent components, making it probable that some nonzero coefficients survive these restrictions. Thus, isometries prove to be critical in the search for nondynamical background fields that yield a consistent theory.

For a spacetime with n isometries described by Killing vector fields $\chi^{(i)}$, $i = 1 \dots n$ the following holds: (i) each Killing vector field generates a diffeomorphism that satisfies Eq. (6.2.6) on its own and (ii) the second term in the left-hand side of Eq. (6.2.4), when expressed in terms of the Lie derivative, vanishes identically along these symmetry directions. This implies that both variations are equal, $\delta S_{\text{part}} = \delta S_{\text{obs}}$, and so, diffeomorphism invariance is partially restored along the Killing directions. For generic diffeomorphisms, which are not necessarily isometries, the identity (6.2.7) should be consulted. Then, the consistency condition of Eq. (6.2.8) must hold on-shell, imposing additional dynamical constraints on the backgrounds.

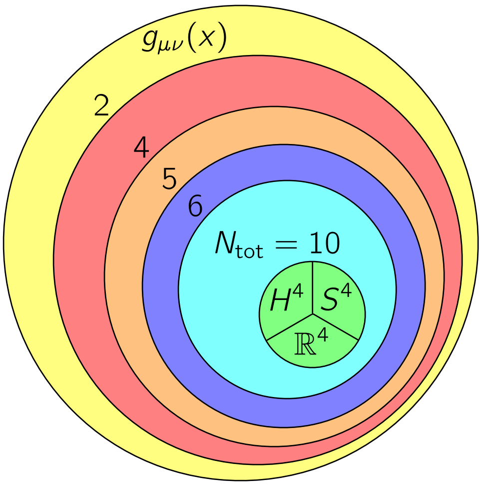


Figure 6.2.1: Space of all possible spacetime metrics containing metric *ansätze* as well as solutions of the dynamical field equations. This space involves subspaces of metrics with an ever-increasing number of isometries described by $N_{\rm tot} \in [1, 10]$ Killing vector fields. The innermost subspace comprises maximally symmetric spacetimes with $N_{\rm tot} = 10$, where examples are the de-Sitter (S^4), Minkowski ($\mathbb{R}^4$), and anti-de-Sitter (H^4) metrics. Examples for $N_{\rm tot} = 6$ are the FLRW-type metrics, for $N_{\rm tot} = 5$ the metric of the Gödel Universe, for $N_{\rm tot} = 4$ the Schwarzschild metric, and for $N_{\rm tot} = 2$ the Kerr metric.

Maximally symmetric spacetimes with 10 isometries constitute a special case. Their metrics exhibit a high degree of symmetry, leading to a simple functional dependence on the spacetime coordinates. The three well-known classes of these geometries are de Sitter (dS_4), Minkowski ($\mathbb{M}^4$), and anti-de Sitter (AdS_4) spacetimes.

Consider, for example, $g_{\mu\nu} = \eta_{\mu\nu}$, the Minkowski metric. This metric possesses 10 isometries, corresponding to the 4 translations and 6 Lorentz transformations (boosts and rotations) of the Poincaré group. When working in Cartesian coordinates, the Killing vector fields associated with the translations, $\chi^{(i)}$ for $i = 1 \dots 4$, can be chosen to have constant components. Under these conditions, Eq. (6.2.6) is automatically satisfied for constant background field coefficients. This explains a key convenience of the nongravitational SME: it is naturally free of the geometrical inconsistencies with dynamics that arise in curved spacetime.

Moreover, the presence of isometries can be interpreted as a restriction of the spacetime geometry, analogous to what occurs for the gravitational CS term [88] and in the treatment of the minimal gravitational SME in Ref. [94]. To understand this, we start from the space of all possible spacetime metrics $g_{\mu\nu}$ without any imposed symmetries, which comprises all real, symmetric (4×4) matrices that are functions of the spacetime coordinates. Let $\chi^{(i)}$ be a set of Killing vector fields for $i=1\dots N_{\rm tot}$, where $1\leq N_{\rm tot}\leq 10$ and $N_{\rm tot}=10$ corresponds to the maximum number of isometries possible in $d=4$ dimensions.

Each existing isometry reduces the space of all possible metrics to a subspace comprising only those metrics consistent with the required symmetry. As the number of Killing vectors is increased incrementally, the symmetry of the metric is progressively enhanced, thereby reducing the number of its independent degrees of freedom (see Fig. 6.2.1).

From Chapter 4 we learned that in highly symmetric spacetimes, the dynamics constrain background fields to be form-invariant under the isometry group. This constraint automatically satisfies three of the four Bianchi identity equations while still allowing non-trivial background configurations. This understanding enables us to adopt a more efficient approach: starting with form-invariant backgrounds ensures the energy-momentum tensor respects the symmetries present in the Einstein field equations and reduces the number of Bianchi identity equations to solve.

For example, we list the previous results obtained in Chapter. 4 concerning the FLRW universes. For homogeneity and isotropy, we obtained the following set of form-invariant background fields

$$\mathcal{L}_{\xi^{(i)}}u(x^\lambda) \implies u(x^\lambda)=u(t)\,, \tag{6.2.9}$$
$$\mathcal{L}_{\xi^{(i)}}s^{\mu\nu}(x^\lambda) \implies s^{\mu\nu}(x^\lambda)=e^\mu{}_a e^\nu{}_b\big[s_1(t)q^{ab}\big]+n^\mu n^\nu\big[s_2(t)\big]\,, \tag{6.2.10}$$
$$\mathcal{L}_{\xi^{(i)}}t^{\mu\nu\rho\sigma}_{\rm spatial}(x^\lambda) \implies t^{\mu\nu\rho\sigma}_{\rm spatial}(x^\lambda)(x^\lambda)=e^\mu{}_a e^\nu{}_b e^\rho{}_c e^\sigma{}_d\big[\eta(t)a(t)^4\big(q^{ac}q^{bd}-q^{ad}q^{bc}\big)\big]\,. \tag{6.2.11}$$

All the previous results can be related to the form-invariant tensors $R, R^{\mu\nu}$, and $R^{\mu\nu\rho\sigma}$ respectively, that can be constructed with the isometries. By considering form-invariant background fields, we reduce dynamic inconsistencies with the geometry at the level of the equations of motion. More importantly, this approach establishes an identification of spacetime directions where field variations do not require the Bianchi identity to be explicitly satisfied. Simultaneously, it reduces both the number of degrees of freedom and the number of equations that must satisfy the Bianchi identity in the remaining directions.

In the following chapter, we employ this knowledge to identify background fields that exhibit greater compatibility with both geometric and dynamic constraints, applying this approach to study causality violations in a Gödel-type universe.

Chapter 7

Examples: the Gödel spacetime

The explicit violation of diffeomorphism invariance introduces tensions between the gravitational dynamics and the underlying Riemannian geometry. In particular, reconciling the modified Einstein field equations with the contracted Bianchi identities of the Riemann tensor leads to a set of coupled partial differential equations involving the Standard-Model Extension (SME) background coefficients. These equations are, in general, highly nontrivial to solve. This fundamental difficulty is commonly referred to in the literature as the "no-go" theorem, highlighting the challenges in constructing consistent gravitational models with explicit symmetry breaking.

In the previous chapter, we developed an alternative approach to satisfy the no-go result, which relies on exploiting the isometries of a given gravitational configuration. This framework requires that the energy-momentum tensor associated with the background fields remains invariant under the transformations generated by the system's Killing vectors. This condition introduces additional symmetry directions along which the Bianchi identities can be satisfied, thereby helping to circumvent the restrictions imposed by general diffeomorphism invariance. Although this strategy imposes constraints that reduce the system's degrees of freedom, it enhances the likelihood of achieving compatibility between the Bianchi identities and the modified equations of motion. A direct consequence of this requirement is that the background fields themselves must also be invariant under the spacetime isometries. Therefore, any consistent formulation of explicit diffeomorphism violation must ensure that both the Bianchi identities and the equations of motion are satisfied simultaneously.

In this context, the Gödel metric is studied as a concrete example to explore the consistency of solutions within the gravitational sector of the SME under explicit symmetry breaking.

7.1 The Gödel metric

The Gödel metric, introduced by Kurt Gödel in 1949 [95]–[97], is an exact solution to Einstein's field equations that describes a stationary, rotating universe. It is particularly notable for admitting closed timelike curves (CTCs), which theoretically allow for travel into one's own past, raising profound questions about causality. Importantly, this violation is a consequence of the global spacetime structure, while locally, general relativity remains consistent with special relativity, preserving causality in small regions. However, CTCs are not unique to the Gödel solution. They appear in a range of other solutions, including the Kerr black hole [98], the van Stockum spacetime [99], and cosmic string models [100], among others. These examples indicate that the emergence of CTCs is a broader feature of general relativity under specific conditions, rather than an anomaly exclusive to Gödel's universe.

In recent years, various extensions of general relativity have been explored to study how these exotic structures behave in modified gravitational frameworks. Notably, the Gödel solution has been investigated in the context of Bumblebee gravity [66], [67].

The line element of the Gödel metric is given by

$$\mathrm{d}s^2 = \frac{1}{2\omega^2}\left(-\mathrm{d}t^2 - 2e^x\mathrm{d}t\mathrm{d}y + \mathrm{d}x^2 - \frac{1}{2}e^{2x}\mathrm{d}y^2 + \mathrm{d}z^2 \right), \tag{7.1.1}$$

where ω is the angular velocity of the cosmic rotation [96].

By applying this metric to the Einstein field equations with a cosmological constant and assuming that the matter content is a pressureless dust, the following condition is obtained in order to satisfy the equations

$$\Lambda = -\omega^2 = -\frac{\kappa\rho}{2}. \tag{7.1.2}$$

In order to investigate the Gödel invariants that will serve as background fields, it is essential to first examine the symmetries of the Gödel metric through its Killing vectors. The Gödel spacetime is a highly symmetric solution, admitting five independent Killing vectors. These vectors correspond to the spacetime's isometries and reflect its homogeneity and stationarity. Specifically, three of the Killing vectors are associated with translations in time and space, while the remaining two correspond to rotational and Lorentz-boost-like symmetries in the spatial sections. These symmetries play a crucial role in the physical interpretation of the Gödel universe, particularly in relation to its global structure. The explicit form of the Killing vectors provides insight into the conserved quantities and invariant properties of fields propagating in this background. These Killing vectors are given by:

$$\xi_1^\alpha(t,r,\phi,z) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_2^\alpha(t,r,\phi,z) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \xi_3^\alpha(t,r,\phi,z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{7.1.3}$$

$$\xi_4^\alpha(t,r,\phi,z) = \begin{pmatrix} 0 \\ 1 \\ -y \\ 0 \end{pmatrix}, \quad \xi_5^\alpha(t,r,\phi,z) = \begin{pmatrix} -2e^{-x} \\ y \\ e^{-2x} - \frac{y^2}{2} \\ 0 \end{pmatrix}. \tag{7.1.4}$$

Using these Killing vectors, we will construct the Gödel invariants that are to act as background fields for the explicit breaking of particle diffeomorphism invariance.

7.2 Gödel invariants

Here, in order to incorporate explicit violations of diffeomorphism invariance without introducing inconsistencies with the Bianchi identity, the invariant tensors of the Gödel metric are considered as fixed background fields. These include a scalar (rank-0), a symmetric rank-2 tensor, and a Riemann-like rank-4 tensor.

The analysis is started with the scalar background. An arbitrary function $f(x^\mu)$

is considered, and the Lie derivative along the Killing directions is given by

$$\mathcal{L}_{\xi_{(i)}} f = \xi^{\lambda}_{(i)} \nabla_{\lambda} f = \xi^{\lambda}_{(i)} \partial_{\lambda} f \tag{7.2.1}$$

for $i = 1, \cdots, 5$. It is straightforward to verify that the only Gödel-invariant function is a constant function, given by $f(x^{\mu}) = f$.

An arbitrary symmetric rank-2 tensor $T^{\mu\nu}$ is now considered. The Lie derivative along the Killing directions is given by

$$\mathcal{L}_{\xi_{(i)}} T^{\mu\nu} = \xi^{\lambda}_{(i)} \nabla_{\lambda} T^{\mu\nu} - \nabla_{\lambda} \xi^{\mu}_{(i)} T^{\lambda\nu} - \nabla_{\lambda} \xi^{\nu}_{(i)} T^{\mu\lambda} \tag{7.2.2}$$

for $i = 1, \cdots, 5$. By considering $i = 1, 2, 3$, it is concluded that $T^{\mu\nu}$ does not depend on the coordinates t, y, or z. Then

$$\mathcal{L}_{\xi_{(1)}} T^{\mu\nu} = 0 \implies \partial_t T^{\mu\nu} = 0, \tag{7.2.3}$$

$$\mathcal{L}_{\xi_{(2)}} T^{\mu\nu} = 0 \implies \partial_y T^{\mu\nu} = 0, \tag{7.2.4}$$

$$\mathcal{L}_{\xi_{(3)}} T^{\mu\nu} = 0 \implies \partial_z T^{\mu\nu} = 0. \tag{7.2.5}$$

If $i = 4, 5$ is considered, it is found that the final form of a rank-2 Gödel-invariant tensor is

$$T^{\mu\nu} = \begin{pmatrix} T_1 & 0 & -2T_2 e^{-x} & T_3 \\ 0 & T_2 & 0 & 0 \\ -2T_2 e^{-x} & 0 & 2T_2 e^{-2x} & 0 \\ T_3 & 0 & 0 & T_4 \end{pmatrix}, \tag{7.2.6}$$

where $T_{(j)}$, with $j = 1, \ldots, 4$, are arbitrary constants.

Finally, a rank-4 tensor $t^{\mu\nu\rho\sigma}$ with the symmetries of a Riemann tensor is considered. The Lie derivative along the Killing directions is given by

$$\mathcal{L}_{\xi_{(i)}} t^{\mu\nu\rho\sigma} = \xi^{\lambda}_{(i)} \nabla_{\lambda} t^{\mu\nu\rho\sigma} - \nabla_{\lambda} \xi^{\mu}_{(i)} t^{\lambda\nu\rho\sigma} - \nabla_{\lambda} \xi^{\nu}_{(i)} t^{\mu\lambda\rho\sigma} - \nabla_{\lambda} \xi^{\rho}_{(i)} t^{\mu\nu\lambda\sigma} - \nabla_{\lambda} \xi^{\sigma}_{(i)} t^{\mu\nu\rho\lambda} \tag{7.2.7}$$

for $i = 1, \cdots, 5$. By considering $i = 1, 2, 3$, it is concluded that $t^{\mu\nu\rho\sigma}$ does not depend on the coordinates t, y, or z. Then

$$\mathcal{L}_{\xi_{(1)}} t^{\mu\nu\rho\sigma} = 0 \implies \partial_t t^{\mu\nu\rho\sigma} = 0, \tag{7.2.8}$$

$$\mathcal{L}_{\xi_{(2)}} t^{\mu\nu\rho\sigma} = 0 \implies \partial_y t^{\mu\nu\rho\sigma} = 0, \tag{7.2.9}$$

$$\mathcal{L}_{\xi_{(3)}} t^{\mu\nu\rho\sigma} = 0 \implies \partial_z t^{\mu\nu\rho\sigma} = 0. \tag{7.2.10}$$

If $i = 4, 5$ is considered, it is found that the final form of a rank-4 Gödel-invariant tensor possesses non-vanishing components

$$\begin{aligned}
t^{txtx} &= T_1, & t^{txtz} &= T_2, \\
t^{txxy} &= T_3 e^{-x}, & t^{txxz} &= T_4, \\
t^{txyz} &= T_5 e^{-x}, & t^{tyty} &= 2(T_1 - T_3)e^{-2x}, \\
t^{tytz} &= -2T_4 e^{-x}, & t^{tyxz} &= -T_5 e^{-x}, \\
t^{tyyz} &= 2T_4 e^{-2x}, & t^{tztz} &= T_6, \\
t^{tzxy} &= (T_5 + T_2)e^{-x}, & t^{tzyz} &= T_7 e^{-x}, \\
t^{xyxy} &= T_3 e^{-2x}, & t^{xzxz} &= -\frac{1}{2}T_7, \\
t^{yzyz} &= -T_7 e^{-2x}.
\end{aligned} \tag{7.2.11}$$

where $T_{(i)}$, with $i = 1, \cdots, 7$ are arbitrary constants.

These invariants are associated with the u, $s^{\mu\nu}$, and $t^{\mu\nu\rho\sigma}$ coefficients that characterize Lorentz-violating effects in the gravitational sector of the SME. In the next section, these quantities will be introduced as fixed background fields that explicitly modify the Einstein field equations. We will then investigate the consistency of the Gödel metric within this extended theoretical framework, paying particular attention to whether the presence of Lorentz-violating terms permits the metric to remain a valid solution.

7.3 The Gödel Solution in the Gravitational Sector of the SME

In accordance with the approach used to ensure compatibility in the presence of explicit diffeomorphism violation, a set of Killing vectors $\xi^{\mu}_{(i)}$ associated with the given metric is identified. It is then required that both the energy-momentum tensor of the background fields and the background fields themselves remain invariant along these Killing directions, i.e.,

$$\mathcal{L}_{\xi_{(i)}} T_{\mu\nu} = 0\,, \tag{7.3.1}$$

and

$$\mathcal{L}_{\xi_{(i)}} u = \mathcal{L}_{\xi_{(i)}} s^{\mu\nu} = \mathcal{L}_{\xi_{(i)}} t^{\mu\nu\rho\sigma} = 0. \tag{7.3.2}$$

Under these conditions, the Killing vectors associated with the Gödel metric are examined in the following sections. Subsequently, the Gödel-invariant background fields are constructed, providing the necessary ingredients for analyzing the consistency of the metric within the gravitational SME framework. We aim to determine whether the Gödel solution remains viable under Lorentz-violating corrections and to analyze the constraints imposed by the SME coefficients. As shown in Eq. (3.2.13), the modified gravitational field equations are given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = (T^{Rstu})_{\mu\nu} + \kappa (T_m)_{\mu\nu}\,. \tag{7.3.3}$$

Here dust is considered as the matter content, whose energy-momentum tensor is defined as

$$(T_m)_{\mu\nu} = \rho u_\mu u_\nu, \tag{7.3.4}$$

where ρ is the energy density and u_μ is the four-velocity. Considering the fluid rest frame, the four-velocity is defined as

$$u^\mu = (u^0, 0, 0, 0). \tag{7.3.5}$$

Using the normalization condition $g_{\mu\nu} u^\mu u^\nu = -1$ and the metric (7.1.1), we find

$$u^\mu = (\sqrt{2}\,\omega, 0, 0, 0). \tag{7.3.6}$$

Thus, the covariant components become

$$u_\mu = \left(-\frac{1}{\sqrt{2}\,\omega}, 0, -\frac{e^x}{\sqrt{2}\,\omega}, 0 \right). \tag{7.3.7}$$

With these elements, the matter source in the matrix form becomes

$$(T_m)_{\mu\nu} = \frac{\rho}{2\omega^2}\begin{pmatrix} 1 & 0 & e^x & 0 \\ 0 & 0 & 0 & 0 \\ e^x & 0 & e^{2x} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{\rho}{2\omega^2}R_{\mu\nu}\,. \tag{7.3.8}$$

Now, let us analyze Eq. (7.3.3) by examining the energy-momentum tensor $(T^{Rstu})_{\mu\nu}$ in terms of its individual sector contributions. Specifically, we consider each sector separately: for the u-sector, the tensor becomes $(T^{Ru})_{\mu\nu}$; for the s-sector, $(T^{Rs})_{\mu\nu}$; and for the t-sector, $(T^{Rt})_{\mu\nu}$.

7.3.1 u sector

It is found that the energy-momentum tensor associated with the u-sector, for Gödel-invariant functions (which are essentially constant), is given by

$$(T^{Ru})_{\mu\nu} = \frac{u}{2}\begin{pmatrix} 1 & 0 & e^x & 0 \\ 0 & 1 & 0 & 0 \\ e^x & 0 & \frac{3}{2}e^{2x} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = uG_{\mu\nu}. \tag{7.3.9}$$

Thus, the Einstein field equations become

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = (T^{Ru})_{\mu\nu} + \kappa(T_m)_{\mu\nu}. \tag{7.3.10}$$

Using the metric (7.1.1), the energy-momentum tensor for the dust (7.3.8), and the energy-momentum tensor for the u-sector (7.3.9), the field equations are given by

$$\frac{1}{2}(1-u) - \frac{\Lambda}{2\omega^2} = \frac{\kappa\rho}{2\omega^2}, \tag{7.3.11}$$

$$\frac{1}{2}(1-u) + \frac{\Lambda}{2\omega^2} = 0, \tag{7.3.12}$$

$$\frac{3}{4}(1-u) - \frac{\Lambda}{4\omega^2} = \frac{\kappa\rho}{2\omega^2}. \tag{7.3.13}$$

Solving this system of equations we obtain for the energy density

$$\kappa\rho = 2\omega^2(1-u) \tag{7.3.14}$$

and for the cosmological constant

$$\Lambda = -\omega^2(1-u) = -\frac{\kappa\rho}{2}. \tag{7.3.15}$$

It is important to note that the Gödel-invariant background field u acts as a scaling factor for the quantities

$$\Lambda \quad \to \quad \Lambda' = \frac{\Lambda}{1-u}, \tag{7.3.16}$$

$$\rho \quad \to \quad \rho' = \frac{\rho}{1-u}. \tag{7.3.17}$$

Then, it is possible to write

$$\Lambda' = -\omega^2 = -\frac{\kappa\rho'}{2}. \tag{7.3.18}$$

This equation shares the same structure as its counterpart in general relativity, with the Gödel universe remaining a valid solution within this Lorentz-violating sector. Consequently, the standard results of general relativity for the Gödel solution are recovered in the limit $u \to 0$.

7.3.2 s sector

By requiring that the background field $s^{\mu\nu}$ be invariant under transformations along the Killing directions of the Gödel metric, it is found that $s^{\mu\nu}$ must take the following form

$$s^{\mu\nu} = \begin{pmatrix} s_1 & 0 & -2s_2e^{-x} & s_3 \\ 0 & s_2 & 0 & 0 \\ -2s_2e^{-x} & 0 & 2s_2e^{-2x} & 0 \\ s_3 & 0 & 0 & s_4 \end{pmatrix}, \tag{7.3.19}$$

where s_i with $i = 1, \cdots, 4$ are arbitrary constants.

Using this result, the energy-momentum tensor contribution arising from the

s-sector, associated with the Gödel-invariant background field, is given by

$$(T^{Rs})_{\mu\nu} = \frac{1}{2\omega^2}\begin{pmatrix} \frac{1}{2}\big(6s_2 - 5s_1\big) & 0 & \frac{e^x}{2}\big(6s_2 - 5s_1\big) & s_3 \\ 0 & -\frac{s_1}{2} & 0 & 0 \\ \frac{e^x}{2}\big(6s_2 - 5s_1\big) & 0 & \frac{e^{2x}}{4}\big(12s_2 - 11s_1\big) & s_3 e^x \\ s_3 & 0 & s_3 e^x & -\frac{1}{2}\big(2s_2 - s_1\big) \end{pmatrix} \tag{7.3.20}$$

In this case, considering $(T^{Rs})_{\mu\nu}$, the Einstein field equations take the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = (T^{Rs})_{\mu\nu} + \kappa (T_m)_{\mu\nu}. \tag{7.3.21}$$

In terms of components, the field equations are given by the following set

$$\frac{1}{2} - \frac{\Lambda}{2\omega^2} = \frac{1}{4\omega^2}(6s_2 - 5s_1) + \frac{\kappa\rho}{2\omega^2}, \tag{7.3.22}$$

$$0 = s_3, \tag{7.3.23}$$

$$\frac{1}{2} + \frac{\Lambda}{2\omega^2} = -\frac{s_1}{4\omega^2}, \tag{7.3.24}$$

$$\frac{3}{4} - \frac{\Lambda}{4\omega^2} = \frac{1}{8\omega^2}(12s_2 - 11s_1) + \frac{\kappa\rho}{2\omega^2}, \tag{7.3.25}$$

$$\frac{1}{2} + \frac{\Lambda}{2\omega^2} = -\frac{1}{4\omega^2}(2s_2 - s_1). \tag{7.3.26}$$

From Eq. (7.3.23), we know that $s_3 = 0$. By considering Eqs. (7.3.24) and (7.3.26), we conclude that

$$s_1 = s_2 = s, \tag{7.3.27}$$

where s is a constant. It follows that the equations reduce to

$$\frac{1}{2} - \frac{\Lambda}{2\omega^2} = \frac{s}{4\omega^2} + \frac{\kappa\rho}{2\omega^2}, \tag{7.3.28}$$

$$\frac{1}{2} + \frac{\Lambda}{2\omega^2} = -\frac{s}{4\omega^2}, \tag{7.3.29}$$

$$\frac{3}{4} - \frac{\Lambda}{4\omega^2} = \frac{s}{8\omega^2} + \frac{\kappa\rho}{2\omega^2}. \tag{7.3.30}$$

From Eq. (7.3.29), the following relation is obtained

$$\omega^2 = -\Lambda - \frac{s}{2}. \tag{7.3.31}$$

By using Eq. (7.3.29) to eliminate Λ from Eqs. (7.3.28) and (7.3.30), the following expression is find

$$\omega^2 = \frac{\kappa\rho}{2}. \tag{7.3.32}$$

We observe that the tensor $s^{\mu\nu}$ contributes to the cosmological constant while leaving the energy density unchanged. This occurs because, under the conditions $s_3 = 0$ and $s_1 = s_2 = s$, we obtain

$$(T^{Rs})_{\mu\nu} = -\frac{s}{4\omega^2}\begin{pmatrix} -1 & 0 & -e^x & 0 \\ 0 & 1 & 0 & 0 \\ -e^x & 0 & -\frac{1}{2}e^{2x} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -\frac{s}{2}g_{\mu\nu}. \tag{7.3.33}$$

Under this assumption, the system of equations given in Eq. (7.3.21) admits an equivalent representation, which can be expressed as

$$G_{\mu\nu} + \left(\Lambda + \frac{s}{2}\right)g_{\mu\nu} = \kappa(T_m)_{\mu\nu}. \tag{7.3.34}$$

This leads to the relation

$$-\left(\Lambda + \frac{s}{2}\right) = \omega^2 = \frac{\kappa\rho}{2}, \tag{7.3.35}$$

which connects the cosmological constant Λ, the parameters s and ω^2, and the energy density ρ. This result demonstrates that, although the cosmological constant is modified by the Lorentz-violating background, the Gödel metric remains a consistent and exact solution within the s-sector of the gravitational sector of the SME, just as it is in general relativity.

7.3.3 t sector

Here, the proposal is to investigate the consistency of the Gödel metric within the t-sector. Considering the rank-4 Gödel-invariant background tensor found in (7.2.11), it yields an energy-momentum tensor that takes the form

$$(T^{Rt})_{\mu\nu} = \frac{1}{2\omega^4}\begin{pmatrix} -5T_1+\frac{13}{4}T_3 & 0 & \big(-5T_1+\frac{13}{4}T_3\big)e^x & -2T_4 \\ 0 & -\frac{1}{4}T_3 & 0 & 0 \\ \big(-5T_1+\frac{13}{4}T_3\big)e^x & 0 & \big(-5T_1+\frac{25}{8}T_3\big)e^{2x} & -2T_4e^x \\ -2T_4 & 0 & -2T_4e^x & T_1-\frac{3}{4}T_3 \end{pmatrix}, \quad (7.3.36)$$

where $T_{(i)}$, with $i = 1,\ldots,7$, are arbitrary constants. Thus, the Einstein field equations become

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = (T^{Rt})_{\mu\nu} + \kappa(T_m)_{\mu\nu}. \quad (7.3.37)$$

When expressed in terms of its independent components, the following is obtained

$$\frac{1}{2} - \frac{\Lambda}{2\omega^2} = \frac{1}{2\omega^4}(-5T_1 + \frac{13}{4}T_3) + \frac{\kappa\rho}{2\omega^2}, \quad (7.3.38)$$

$$0 = -2T_4, \quad (7.3.39)$$

$$\frac{1}{2} + \frac{\Lambda}{2\omega^2} = -\frac{1}{2\omega^4}\frac{T_3}{4}, \quad (7.3.40)$$

$$\frac{3}{4} - \frac{\Lambda}{4\omega^2} = \frac{1}{2\omega^4}(-5T_1 + \frac{25}{8}T_3) + \frac{\kappa\rho}{2\omega^2}, \quad (7.3.41)$$

$$\frac{1}{2} + \frac{\Lambda}{2\omega^2} = \frac{1}{2\omega^4}(T_1 - \frac{3}{4}T_3). \quad (7.3.42)$$

Here, we observe from Eq. (7.3.39) that $T_4 = 0$. By applying Eqs. (7.3.40) and (7.3.42), we obtain the following additional constraints

$$2T_1 = T_3. \quad (7.3.43)$$

Using this result, the remaining equations become

$$\frac{1}{2} - \frac{\Lambda}{2\omega^2} = \frac{3}{4}\frac{T_1}{\omega^4} + \frac{\kappa\rho}{2\omega^2}, \quad (7.3.44)$$

$$1 + \frac{\Lambda}{\omega^2} = -\frac{T_1}{2\omega^4}, \quad (7.3.45)$$

$$\frac{3}{4} - \frac{\Lambda}{4\omega^2} = \frac{5}{8}\frac{T_1}{\omega^4} + \frac{\kappa\rho}{2\omega^2}. \quad (7.3.46)$$

From this set of equations, a system can be formulated to express ω^2 in terms of Λ or ρ, i.e.,

$$\omega^4 + \Lambda\omega^2 + \frac{T_1}{2} = 0, \tag{7.3.47}$$

$$\omega^4 - \frac{\kappa\rho}{2}\omega^2 - \frac{T_1}{2} = 0. \tag{7.3.48}$$

Here, it is recognized that setting $T_1 = 0$ leads to the standard result from general relativity, namely the relation $\omega^2 = -\Lambda = \frac{\kappa\rho}{2}$. When this expression is solved for ω^2, the following is obtained

$$-\frac{\Lambda}{2} \pm \frac{1}{2}\sqrt{\Lambda^2 - 2T_1} = \omega^2 = \frac{\kappa\rho}{4} \pm \frac{1}{2}\sqrt{\left(\frac{\kappa\rho}{2}\right)^2 + 2T_1}. \tag{7.3.49}$$

It is important to note that, in order to connect this result with the case $T_1 = 0$ - that is, the standard result from general relativity - we must choose the positive square root in the equation above. Then,

$$-\frac{\Lambda}{2} - \frac{1}{2}\sqrt{\Lambda^2 - 2T_1} = \omega^2 = \frac{\kappa\rho}{4} + \frac{1}{2}\sqrt{\left(\frac{\kappa\rho}{2}\right)^2 + 2T_1}. \tag{7.3.50}$$

Therefore, our result demonstrates that the t-sector admits the Gödel metric as an exact solution, although the relation between the cosmological constant and the energy density is modified by the presence of the background field.

Its important to notice that the energy-momentum tensor obtained in Eqs. (**??**) satisfy identically the conditions

$$\nabla^\mu (T^{Ru})_{\mu\nu} \equiv 0\,, \tag{7.3.51}$$

$$\nabla^\mu (T^{Rs})_{\mu\nu} \equiv 0\,, \tag{7.3.52}$$

$$\nabla^\mu (T^{Rt})_{\mu\nu} \equiv 0\,, \tag{7.3.53}$$

even before the Einstein equations were completely solved.

Up to this point, the consistency between the Gödel solution and all sectors describing explicit diffeomorphism violation has been established.

7.4 On Causality and its violation

It was shown that the Gödel metric constitutes a consistent solution within a gravitational theory that includes background fields leading to explicit violations of diffeomorphism invariance and Lorentz symmetry. Now we focus in the existence of closed timelike curves (CTCs), which imply the possibility of causality violation. Within this context, the computation of the critical radius associated with the solution is essential, as causality is violated beyond this radius.

To determine the critical radius, the metric given in Eq. (7.1.1) is expressed in cylindrical coordinates, and a condition is imposed on the $g_{\phi\phi}$ component. A circular curve defined by $C = \{(t, r, \phi, z); t, r, z = \mathit{const}; \phi \in [0, 2\pi]\}$ represents a closed timelike curve (CTC) if $g_{\phi\phi}$ becomes negative for certain values of r [95], [97]. This condition indicates the existence of a noncausal region for $r > r_c$, where the critical radius is given by

$$r_c = \frac{\sqrt{2}}{\omega}\sinh^{-1}(1). \tag{7.4.1}$$

It is important to emphasize that the determination of the critical radius is based solely on the properties of the metric and is therefore independent of the specific gravitational theory under consideration. However, it does depend on the parameter ω, which characterizes the rotation of the Gödel universe. While the general structure of the condition for the critical radius remains the same across different theories, the value of ω can be modified by the presence of additional terms or background fields specific to each framework. As a result, the critical radius may vary depending on the underlying theory. In this context, we now investigate how the background fields introduced in each sector of the gravitational theory discussed in the previous sections influence the value of ω, and consequently, the critical radius.

Here, each sector will be considered separately in order to determine the corresponding critical radius.

In the u sector, subsection 7.3.1, after the set of field equations was solved, the following result was obtained

$$\omega^2 = \frac{\kappa\rho}{2(1-u)}. \tag{7.4.2}$$

This leads to the critical radius

$$r_c = \sinh^{-1}(1)\sqrt{\frac{4(1-u)}{\kappa\rho}}. \tag{7.4.3}$$

It is shown that the background field u acts to reduce the critical radius. Consequently, the critical radius remains finite, implying that causality violation persists.

In the s sector, subsection 7.3.2, has been obtained that

$$\omega^2 = -\left(\Lambda + \frac{s}{2}\right), \tag{7.4.4}$$

as a consequence the critical radius becomes

$$r_c = \sinh^{-1}(1)\sqrt{\frac{4}{-(2\Lambda + s)}}. \tag{7.4.5}$$

In order to obtain a finite and real value, a new condition is imposed on the parameter s and the cosmological constant, namely $2\Lambda + s < 0$. If this condition is satisfied, causality violation is permitted in this sector. Another important observation arises at this point: if $s = -2\Lambda$, the critical radius becomes infinite, and causality violation is avoided, resulting in a fully causal region. Therefore, in the s sector, both causal and non-causal regions may emerge, depending on the relationship between the parameter s and the cosmological constant Λ.

Now, the t sector – studied in subsection 7.3.3 – is analyzed, where the following result has been found

$$\omega^2 = \frac{\kappa\rho}{4} + \frac{1}{2}\sqrt{\left(\frac{\kappa\rho}{2}\right)^2 + 2T_1}. \tag{7.4.6}$$

In this case, the critical radius is given by the expression

$$r_c = \sinh^{-1}(1)\sqrt{\frac{4}{\frac{\kappa\rho}{2} + \left[\left(\frac{\kappa\rho}{2}\right)^2 + 2T_1\right]^{1/2}}}. \tag{7.4.7}$$

It is important to note that the presence of the parameter T_1 increases the denominator of the expression for the critical radius. As a result, the value of the critical radius is reduced. Therefore, in this sector, a finite critical radius is obtained, indicating that a non-causal region is permitted and causality violation remains possible.

Chapter 8

Conclusion and Discussion

The preservation of fundamental symmetries in high-energy physics is not guaranteed. As energy scales increase and experimental techniques diversify, effective field theories provide the tools to systematically constrain and rule out models based on empirical measurements. However, each field theory offers distinct configurations where such deviations may manifest. Therefore, studying the family of Standard Model extensions and gravity modifications allows for theoretical progress in parallel with experiments, guiding the quest for a more fundamental theory.

My initial contribution in this direction included verifying the preservation of unitarity at tree-level and one-loop order in Myers-Pospelov QED [18], [19]. This work served as an introduction to the field during my doctoral studies and laid the groundwork for the research contained in this thesis.

This thesis has revisited essential aspects of Lorentz and diffeomorphism symmetry violation in gravity, establishing a consistent framework for incorporating both non-dynamical (explicit breaking) and dynamical (spontaneous breaking) background fields into cosmological models. A central achievement has been advancing the understanding of explicit symmetry breaking within the gravitational sectors of the Standard-Model Extension (SME). We conducted a systematic study on the u and s sectors, where the specific background field configurations consistent with the homogeneity and isotropy of the universe were identified. For these sectors, the corresponding modified Friedmann equations were rigorously derived, revealing non-trivial background configurations that can actively influence cosmic expansion.

A principal and novel contribution of this work is the first comprehensive analysis of the t sector in a cosmological context. The compatibility conditions with an isotropic and homogeneous universe were established, leading to the identification of a viable, purely spacelike background configuration. The modified Friedmann equations for this sector were derived, and their numerical analysis demonstrated that such a spacelike configuration does not yield appreciable accelerated expansion within the bounds of the effective theory. This key finding indicates that more general t-sector configurations must be explored in future work to fully assess the phenomenological potential of this minimal sector.

A fundamental conclusion that emerged consistently across all sectors is that maintaining dynamical consistency necessitates the background fields to be form-invariant tensors under the spacetime isometry group. This requirement, far from being a mere mathematical convenience, played a critical physical role by drastically reducing the number of independent background components. It thereby guaranteed that the resulting modified Einstein equations remain form-invariant under the isometries group, ensuring compatibility with the foundational principles of the cosmological background.

Regarding spontaneous symmetry breaking, we focused on the Bumblebee model, which is intimately related to the minimal gravitational sector of the SME. In this model, we successfully enforced isotropy and homogeneity on the Einstein equations for a purely tangential bumblebee field, while simultaneously satisfying the bumblebee field equations. The required condition constrained the magnitude of the bumblebee field, but not its direction. This indicates that as long as the bumblebee field satisfies its own equations of motion it is not required to be form-invariant, as the form-invariant vector in this context would be the null vector. We successfully derived the extended GHY boundary term that enable a proper Dirichlet variational formulation, and the modified Friedmann equations, a contribution to the literature on the bumblebee model that will facilitate the study of different sectors across various epochs of the universe's expansion and generalize the purely normal (time-like) bumblebee field used in the literature for cosmological setups [64], [65]. The non-linear dynamics of the bumblebee model are set aside for future study.

We have clearly reviewed the potential conflict between dynamics and spacetime geometry for explicit diffeomorphism symmetry breaking, demonstrating its resolution through complementary methods. These include restricting the geometry of spacetime itself and utilizing the symmetries (isometries) of a gravitational system. Having recognized the critical role of isometries in the dynamics, we developed a general strategy based on a key insight: for a diffeomorphism along a Killing direction, a form-invariant background field ensures no discrepancy between particle and observer transformations. This form-invariance condition constrains the background and its energy-momentum tensor, ensuring compatibility with the equations of motion and the Bianchi identities.

The exhaustive application of this method to all available isometries in a gravitational setup leads us to a more compatible framework. Therefore, we conclude by advocating for a priori compatibility as a guiding principle, which is realized by selecting backgrounds that are invariant under the spacetime symmetries. The technique developed here, relying on the full non-linear theory and the power of isometries, form a robust theoretical foundation for future phenomenological studies, from gravitational waves where explicit breaking could imply additional, non-suppressed propagating modes, to other symmetric systems like black holes that are already being studied.

Finally, to test our approach, we investigated causality violations in the Gödel universe using invariants constructed from its isometries. We demonstrated that these invariants do not reveal any discrepancies between the Gödel metric and the dynamical field equations, where we considered a pressureless dust source and a cosmological constant as is standard Gödel analysis. In addition, the question of causality and its potential violation was addressed through the analysis of closed timelike curves (CTCs). We demonstrated that modifications introduced by the background fields alter the critical radius that determines the boundary between causal and non-causal regions, revealing that different SME sectors can either preserve causality or exacerbate its violation for the same matter content. These findings highlight the sensitivity of Gödel solution to Lorentz and diffeomorphism-violating effects and provide further insight into the rich phenomenology of the gravitational SME.

Future work is needed to clarify several key aspects of the framework. This includes: investigating a second Noether theorem for variations that do not vanish at the boundary in cases of explicit violations; understanding how ADM projectors ensure the background field remains non-dynamical; searching for accelerated expansion in more general t sectors and bumblebee field models; and studying causality violations in Gödel-like metrics for the minimal sector.

The main conclusion is that this thesis lays the foundation for solving background configurations that are consistent with dynamics through geometry.

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Appendix A

Background fields projections

This first appendix outlines the strategy for obtaining the optimal projections of tensors onto an arbitrary spacetime foliation.

A1 Projection using ADM variables

We stablish the following strategy to decompose derivatives an arbitrary tensor $G^{\mu_1\mu_2\cdots\mu_n}$:

- As the first step we decompose the tensor $G^{\mu_1\mu_2\cdots\mu_n}$ in its tangential and normal projections using Dirac deltas (2.2.17).
- We apply the covariant derivative on the decomposed tensor focusing on the derivative terms of tangential components. Additionally, terms like $\nabla_\mu n^\nu$ can be decomposed into the extrinsic curvature K_{ab} and the acceleration vector a^a.
- In cases where the free indices of an expression are inside a covariant derivative, meaning they are not factored out as projectors outside the expression, we will introduce Kronecker deltas and the completeness relation to extract these indices outside of the covariant derivative.
- In terms like $n_\mu \nabla_\nu (e^\mu{}_a \ldots)$ we complete the derivative using the orthogonality of the projectors, allowing us to rewrite it as $e^\mu{}_a \nabla_\nu n_\mu \ldots$. This form can be identified with geometric quantities and has the projectors with free indices on the outside.

- Given that we have previously obtained the normal and tangential projections of the covariant derivative of tangential tensors, we use these expressions to identify Lie derivatives and covariant derivatives on the hypersurface. This allows us to derive an expression that contains well-defined quantities on the hypersurface, accompanied by the corresponding projectors for each free index in their normal and tangential directions.

We can show the steps by computing the covariant derivative of a contravariant vector. After using the completeness relation Eq. (2.2.17), a vector is decomposed as

$$G^\alpha = e^\alpha{}_a G^a - n^\alpha G^{\mathbf{n}} \,. \tag{A1.1}$$

Taking the covariant derivative of G^α we obtain

$$\nabla_\beta G^\alpha = \nabla_\beta\big(e^\alpha{}_a G^a\big) - (\nabla_\beta n^\alpha) G^{\mathbf{n}} - n^\alpha \nabla_\beta G^{\mathbf{n}} \,, \tag{A1.2}$$

and using the definition of the extrinsic curvature

$$\nabla_\mu n_\nu = e_\mu{}^a e_\nu{}^b K_{ab} - n_\mu a_\nu \,, \tag{A1.3}$$

we have

$$\nabla_\beta G^\alpha = \nabla_\beta\big(e^\alpha{}_a G^a\big) - \big(e^\alpha{}_a K^a_{\ b} e_\beta{}^b - n_\beta a^\alpha\big) G^{\mathbf{n}} - n^\alpha \nabla_\beta G^{\mathbf{n}} \,. \tag{A1.4}$$

Now we introduce two Kronecker deltas in the first term due to two unprojected indices and in the last term we introduce one delta. As described we have as many deltas as unprojected indices has the term, yielding

$$\nabla_\beta G^\alpha = \delta^\alpha_\gamma \delta^\delta_\beta \nabla_\delta\big(e^\gamma_a G^a\big) - (e^\alpha{}_a K^a_{\ b} e_\beta{}^b - n_\beta a^\alpha) G^{\mathbf{n}} - n^\alpha \delta^\delta_\beta \nabla_\delta G^{\mathbf{n}} \,, \tag{A1.5}$$

and by using the completeness relation we obtain

$$\begin{aligned}\nabla_\beta G^\alpha = {} & (e^\alpha{}_c e_\gamma{}^c - n^\alpha n_\gamma)(e^\delta{}_d e_\beta{}^d - n^\delta n_\beta)\nabla_\delta\big(e^\gamma{}_a G^a\big) - (e^\alpha{}_a K^a_{\ b} e_\beta{}^b - n_\beta a^\alpha) G^{\mathbf{n}} \\ & - n^\alpha (e^\delta{}_d e_\beta{}^d - n^\delta n_\beta)\nabla_\delta G^{\mathbf{n}} \,. \end{aligned} \tag{A1.6}$$

We have

$$\begin{aligned}\nabla_\beta G^\alpha =& e^\alpha{}_c e_\gamma{}^c e^\delta{}_d e_\beta{}^d \nabla_\delta\big(e^\gamma{}_a G^a\big) - e^\alpha{}_c e_\gamma{}^c n^\delta n_\beta \nabla_\delta\big(e^\gamma{}_a G^a\big) - n^\alpha n_\gamma e^\delta{}_d e_\beta{}^d \nabla_\delta\big(e^\gamma{}_a G^a\big)\\ &+ n^\alpha n_\gamma n^\delta n_\beta \nabla_\delta\big(e^\gamma{}_a G^a\big) - (e^\alpha{}_a K^a{}_b e_\beta{}^b - n_\beta a^\alpha) G^{\mathbf{n}} - n^\alpha e^\delta{}_d e_\beta{}^d \nabla_\delta G^{\mathbf{n}}\\ &- n^\alpha n^\delta n_\beta \nabla_\delta G^{\mathbf{n}}\,. \end{aligned} \tag{A1.7}$$

Integrating by parts the third and fourth terms, and recognizing geometrical quantities, we arrive at

$$\begin{aligned}\nabla_\beta G^\alpha =& e^\alpha{}_a e_\beta{}^b \left(D_b G^a - K^a_b G^{\mathbf{n}}\right) - e^\alpha{}_a n_\beta \left(e_\gamma{}^a n^\delta \nabla_\delta\big(e^\gamma{}_c G^c\big) - a^a G^{\mathbf{n}}\right)\\ &- n^\alpha e_\beta{}^b \big(D_b G^{\mathbf{n}} - K_{cb} G^c\big) + n^\alpha n_\beta \big(n^\delta \nabla_\delta G^{\mathbf{n}} - a_c G^c\big)\,. \end{aligned} \tag{A1.8}$$

We can go further by using the expression (2.2.51) to write the projections as

$$\begin{aligned}\nabla_\beta G^\alpha =& e^\alpha{}_a e_\beta{}^b \big(D_b G^a - K^a_b G^{\mathbf{n}}\big) - e^\alpha{}_a n_\beta \left(\frac{1}{N} e_\gamma{}^a \mathcal{L}_m\big(e^\gamma{}_c G^c\big) + K^a_c G^c - a^a G^{\mathbf{n}}\right)\\ &- n^\alpha e_\beta{}^b \big(D_b G^{\mathbf{n}} - K_{cb} G^c\big) + n^\alpha n_\beta \left(\frac{1}{N}\mathcal{L}_m G^{\mathbf{n}} - a_c G^c\right)\,. \end{aligned} \tag{A1.9}$$

Taking this example as reference, we start the projections for the relevant terms in the like energy-momentum tensor (3.2.14).

A2 Projections in the u sector

Lets start by decomposing the covariant derivative of u

$$\begin{aligned}\nabla_\mu u &= \delta^\alpha_\mu \nabla_\alpha u\\ &= (e^\alpha{}_a e_\mu{}^a - n^\alpha n_\mu)\nabla_\alpha u\\ &= e_\mu{}^a \big[D_a u\big] - n_\mu \big[n^\alpha \nabla_\alpha u\big]\\ &= e_\mu{}^a \big[D_a u\big] - n_\mu \left[\frac{1}{N}\mathcal{L}_m u\right]. \end{aligned} \tag{A2.1}$$

Taking a second covariant derivative

$$\begin{aligned}
\nabla_\mu\nabla_\nu u &= \nabla_\mu\Big(e_\nu{}^b\big[D_b u\big] - n_\nu\Big[\frac{1}{N}\mathcal{L}_m u\Big]\Big) \\
&= \nabla_\mu\Big(e_\nu{}^b\big[D_b u\big]\Big) - \nabla_\mu n_\nu\Big[\frac{1}{N}\mathcal{L}_m u\Big] - n_\nu\nabla_\mu\Big[\frac{1}{N}\mathcal{L}_m u\Big] \\
&= (e^\alpha{}_a e_\mu{}^a - n^\alpha n_\mu)(e^\beta{}_b e_\nu{}^b - n^\beta n_\nu)\nabla_\alpha\Big(e_\beta{}^c\big[D_c u\big]\Big) \\
&\quad -\big(e_\mu{}^a K_{ab} - n_\mu a_b\big)e_\nu{}^b\Big[\frac{1}{N}\mathcal{L}_m u\Big] - n_\nu(e^\alpha{}_a e_\mu{}^a - n^\alpha n_\mu)\nabla_\alpha\Big[\frac{1}{N}\mathcal{L}_m u\Big] \\
&= e_\mu{}^a e_\nu{}^b\Big[D_a D_b u - K_{ab}\frac{1}{N}\mathcal{L}_m u\Big] - n_\mu e_\nu{}^b\Big[\frac{1}{N}e^\beta{}_b\mathcal{L}_m\big(e_\beta{}^c\big[D_c u\big]\big) \\
&\quad -K^c{}_b D_c u - a_b\frac{1}{N}\mathcal{L}_m u\Big] - e_\mu{}^a n_\nu\Big[D_a\Big(\frac{1}{N}\mathcal{L}_m u\Big) - K^c{}_a D_c u\Big] \\
&\quad +n_\mu n_\nu\Big[\frac{1}{N}\mathcal{L}_m\Big(\frac{1}{N}\mathcal{L}_m u\Big) - a^b D_b u\Big]. \qquad \text{(A2.2)}
\end{aligned}$$

Thus

$$\begin{aligned}
&\nabla^\mu\nabla^\nu u + \nabla^\nu\nabla^\mu u \\
= \;& g^{\mu\alpha}g^{\nu\beta}\big(\nabla_\alpha\nabla_\beta u + \nabla_\beta\nabla_\alpha u\big) \\
= \;& g^{\mu\alpha}g^{\nu\beta}\Bigg(e_\alpha{}^a e_\beta{}^b\Big[D_a D_b u + D_b D_a u - 2K_{ab}\frac{1}{N}\mathcal{L}_m u\Big] \\
& -n_\alpha e_\beta{}^b\Big[\frac{1}{N}e^\delta{}_b\mathcal{L}_m\Big(e_\delta{}^c\big[D_c u\big]\Big) - K^c{}_b D_c u - a_b\frac{1}{N}\mathcal{L}_m u + D_b\Big(\frac{1}{N}\mathcal{L}_m u\Big) - K^c{}_b D_c u\Big] \\
& -e_\alpha{}^a n_\beta\Big[\frac{1}{N}e^\gamma{}_a\mathcal{L}_m\Big(e_\gamma{}^c\big[D_c u\big]\Big) - K^c{}_a D_c u - a_a\frac{1}{N}\mathcal{L}_m u + D_a\Big(\frac{1}{N}\mathcal{L}_m u\Big) - K^c{}_a D_c u\Big] \\
& +n_\alpha n_\beta\Big[\frac{2}{N}\mathcal{L}_m\Big(\frac{1}{N}\mathcal{L}_m u\Big) - 2a^b D_b u\Big]\Bigg) \\
= \;& e^\mu{}_a e^\nu{}_b\Big[D^a D^b u + D^b D^a u - 2K^{ab}\frac{1}{N}\mathcal{L}_m u\Big] \\
& -n^\mu e^\nu{}_b\Big[q^{bd}\frac{1}{N}e^\delta{}_d\mathcal{L}_m\Big(e_\delta{}^c\big[D_c u\big]\Big) - 2K^{cb}D_c u - a^b\frac{1}{N}\mathcal{L}_m u + D^b\Big(\frac{1}{N}\mathcal{L}_m u\Big)\Big] \\
& -e^\mu{}_a n^\nu\Big[q^{ac}\frac{1}{N}e^\gamma{}_c\mathcal{L}_m\Big(e_\gamma{}^d\big[D_d u\big]\Big) - 2K^{da}D_d u - a^a\frac{1}{N}\mathcal{L}_m u + D^a\Big(\frac{1}{N}\mathcal{L}_m u\Big)\Big] \\
& +n^\mu n^\nu\Big[\frac{2}{N}\mathcal{L}_m\Big(\frac{1}{N}\mathcal{L}_m u\Big) - 2a^c D_c u\Big]. \qquad \text{(A2.3)}
\end{aligned}$$

On the other hand, we can contract the covariant derivatives an obtain

$$\begin{aligned} g^{\mu\nu}\nabla^2 u &= e^{\mu}{}_{a}e^{\nu}{}_{b}\Big[q^{ab}\Big(D^2u - K\frac{1}{N}\mathcal{L}_m u - \frac{1}{N}\mathcal{L}_m\Big(\frac{1}{N}\mathcal{L}_m u\Big) + a^c D_c u\Big)\Big] \\ &\quad + n^{\mu}n^{\nu}\Big[-\Big(D^2u - K\frac{1}{N}\mathcal{L}_m u - \frac{1}{N}\mathcal{L}_m\Big(\frac{1}{N}\mathcal{L}_m u\Big) + a^c D_c u\Big)\Big]. \end{aligned} \tag{A2.4}$$

Together with the projections of the Einstein tensor

$$\begin{aligned} G^{\mu\nu} &= {}^{(4)}R^{\mu\nu} - \frac{1}{2}{}^{(4)}Rg^{\mu\nu} \\ &= e^{\mu}{}_{a}e^{\nu}{}_{b}\Big[{}^{(3)}R^{ab} - 2K^{a}{}_{c}K^{cb} + K^{ab}K - D^b a^a - a^a a^b + \frac{1}{N}q^{ac}q^{bd}e^{\rho}{}_{c}e^{\sigma}{}_{d}\mathcal{L}_m(e_{\rho}{}^{e}e_{\sigma}{}^{f}K_{ef}) \\ &\quad -\frac{1}{2}q^{ab}\Big({}^{(3)}R + K_{cd}K^{cd} + K^2 - 2D\cdot a - 2a^2 + \frac{2}{N}\mathcal{L}_m K\Big)\Big] \\ &\quad -n^{\mu}e^{\nu}{}_{b}\Big[D_a K^{ab} - D^b K\Big] - e^{\mu}{}_{a}n^{\nu}\Big[D_b K^{ab} - D^a K\Big] \\ &\quad +n^{\mu}n^{\nu}\Big[\frac{1}{2}\Big({}^{(3)}R - K_{ab}K^{ab} + K^2\Big)\Big], \end{aligned} \tag{A2.5}$$

we are equipped to take the Gaussian coordinates limit and use the FLRW metric to obtain the like energy-momentum tensor for the u background field.

A3 Projections in the s sector

In order to construct the projections of the energy-momentum tensor we need to compute the projections of the second covariant derivative of $s^{\mu\nu}$ with free indexes. We start by taking the first covariant derivative: We consider the decomposition

$$s^{\mu\nu} = e^{\mu}{}_{a}e^{\nu}{}_{b}s^{ab} - e^{\mu}{}_{a}n^{\nu}s^{a\mathbf{n}} - n^{\mu}e^{\nu}{}_{b}s^{\mathbf{n}b} + n^{\mu}n^{\nu}s^{\mathbf{nn}}. \tag{A3.1}$$

Taking the first covariant derivative we obtain

$$\begin{aligned} \nabla_{\beta}s^{\mu\nu} &= \nabla_{\beta}\big(e^{\mu}{}_{a}e^{\nu}{}_{b}s^{ab} - e^{\mu}{}_{a}n^{\nu}s^{a\mathbf{n}} - n^{\mu}e^{\nu}{}_{b}s^{\mathbf{n}b} + n^{\mu}n^{\nu}s^{\mathbf{nn}}\big) \\ &= \nabla_{\beta}\big(e^{\mu}{}_{a}e^{\nu}{}_{b}s^{ab}\big) - n^{\nu}\nabla_{\beta}\big(e^{\mu}{}_{a}s^{a\mathbf{n}}\big) - \nabla_{\beta}n^{\nu}\big(e^{\mu}{}_{a}s^{a\mathbf{n}}\big) - n^{\mu}\nabla_{\beta}\big(e^{\nu}{}_{b}s^{\mathbf{n}b}\big) - \nabla_{\beta}n^{\mu}\big(e^{\nu}{}_{b}s^{\mathbf{n}b}\big) \\ &\quad +\nabla_{\beta}n^{\mu}\big(n^{\nu}s^{\mathbf{nn}}\big) + \nabla_{\beta}n^{\nu}\big(n^{\mu}s^{\mathbf{nn}}\big) + n^{\mu}n^{\nu}\nabla_{\beta}\big(s^{\mathbf{nn}}\big) \\ &= \nabla_{\beta}\big(e^{\mu}{}_{a}e^{\nu}{}_{b}s^{ab}\big) - n^{\nu}\nabla_{\beta}\big(e^{\mu}{}_{a}s^{a\mathbf{n}}\big) - \big(e_{\beta}{}^{b}K^{d}{}_{b} - n_{\beta}a^{d}\big)e^{\nu}{}_{d}\big(e^{\mu}{}_{a}s^{a\mathbf{n}}\big) - n^{\mu}\nabla_{\beta}\big(e^{\nu}{}_{b}s^{\mathbf{n}b}\big) \\ &\quad -\big(e_{\beta}{}^{b}K^{c}{}_{b} - n_{\beta}a^{c}\big)e^{\mu}{}_{c}\big(e^{\nu}{}_{b}s^{\mathbf{n}b}\big) + \big(e_{\beta}{}^{b}K^{c}{}_{b} - n_{\beta}a^{c}\big)e^{\mu}{}_{c}\big(n^{\nu}s^{\mathbf{nn}}\big) \\ &\quad +\big(e_{\beta}{}^{b}K^{d}{}_{b} - n_{\beta}a^{d}\big)e^{\nu}{}_{d}\big(n^{\mu}s^{\mathbf{nn}}\big) + n^{\mu}n^{\nu}\nabla_{\beta}\big(s^{\mathbf{nn}}\big). \end{aligned} \tag{A3.2}$$

Here we compute every unprojected term as follows. For the first term we have

$$\begin{aligned}
\nabla_\beta\big(e^\mu{}_a e^\nu{}_b s^{ab}\big) &= \delta^\delta{}_\beta \delta^\mu{}_\rho \delta^\nu{}_\sigma \nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big)\\
&= \big(e^\delta{}_d e_\beta{}^d - n^\delta n_\beta\big)\big(e^\mu{}_a e_\rho{}^a - n^\mu n_\rho\big)\big(e^\nu{}_b e_\sigma{}^b - n^\nu n_\sigma\big)\nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big)\\
&= e^\delta{}_d e_\beta{}^d e^\mu{}_a e_\rho{}^a e^\nu{}_b e_\sigma{}^b \nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big) - n^\delta n_\beta e^\mu{}_a e_\rho{}^a e^\nu{}_b e_\sigma{}^b \nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big)\\
&\quad - e^\delta{}_d e_\beta{}^d n^\mu n_\rho e^\nu{}_b e_\sigma{}^b \nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big) - e^\delta{}_d e_\beta{}^d e^\mu{}_a e_\rho{}^a n^\nu n_\sigma \nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big)\\
&\quad + n^\delta n_\beta n^\mu n_\rho e^\nu{}_b e_\sigma{}^b \nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big) + n^\delta n_\beta e^\mu{}_a e_\rho{}^a n^\nu n_\sigma \nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big)\\
&\quad + e^\delta{}_d e_\beta{}^d n^\mu n_\rho n^\nu n_\sigma \nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big) - n^\delta n_\beta n^\mu n_\rho n^\nu n_\sigma \nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big)\\
&= e_\beta{}^d e^\mu{}_a e^\nu{}_b \big[D_d s^{ab}\big] - n_\beta e^\mu{}_a e^\nu{}_b \big[e_\rho{}^a e_\sigma{}^b n^\delta \nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big)\big] - e_\beta{}^d n^\mu e^\nu{}_b \big[-K_{da} s^{ab}\big]\\
&\quad - e_\beta{}^d e^\mu{}_a n^\nu \big[-K_{db} s^{ab}\big] + n_\beta n^\mu e^\nu{}_b \big[-a_a s^{ab}\big] + n_\beta e^\mu{}_a n^\nu \big[-a_b s^{ab}\big]\,. \qquad \text{(A3.3)}
\end{aligned}$$

For the second and fourth term we have

$$\begin{aligned}
\nabla_\beta\big(e^\mu{}_a s^{a\mathbf{n}}\big) &= \delta^\delta{}_\beta \delta^\mu{}_\rho \nabla_\delta\big(e^\rho{}_e s^{e\mathbf{n}}\big)\\
&= \big(e^\delta{}_d e_\beta{}^d - n^\delta n_\beta\big)\big(e^\mu{}_a e_\rho{}^a - n^\mu n_\rho\big)\nabla_\delta\big(e^\rho{}_e s^{e\mathbf{n}}\big)\\
&= e^\delta{}_d e_\beta{}^d e^\mu{}_a e_\rho{}^a \nabla_\delta\big(e^\rho{}_e s^{e\mathbf{n}}\big) - n^\delta n_\beta e^\mu{}_a e_\rho{}^a \nabla_\delta\big(e^\rho{}_e s^{e\mathbf{n}}\big) - e^\delta{}_d e_\beta{}^d n^\mu n_\rho \nabla_\delta\big(e^\rho{}_e s^{e\mathbf{n}}\big)\\
&\quad + n^\delta n_\beta n^\mu n_\rho \nabla_\delta\big(e^\rho{}_e s^{e\mathbf{n}}\big)\\
&= e_\beta{}^d e^\mu{}_a \big[D_d s^{a\mathbf{n}}\big] - n_\beta e^\mu{}_a \big[e_\rho{}^a n^\delta \nabla_\delta\big(e^\rho{}_e s^{e\mathbf{n}}\big)\big] - e_\beta{}^d n^\mu \big[-K_{da} s^{a\mathbf{n}}\big]\\
&\quad + n_\beta n^\mu \big[-a_a s^{a\mathbf{n}}\big]\,, \qquad \text{(A3.4)}
\end{aligned}$$

and

$$\begin{aligned}
\nabla_\beta\big(e^\nu{}_b s^{\mathbf{n}b}\big) &= e_\beta{}^d e^\nu{}_b \big[D_d s^{\mathbf{n}b}\big] - n_\beta e^\nu{}_b \big[e_\sigma{}^b n^\delta \nabla_\delta\big(e^\sigma{}_f s^{\mathbf{n}f}\big)\big] - e_\beta{}^d n^\nu \big[-K_{db} s^{\mathbf{n}b}\big]\\
&\quad + n_\beta n^\nu \big[-a_b s^{\mathbf{n}b}\big]\,. \qquad \text{(A3.5)}
\end{aligned}$$

Finally, for the last term we have

$$\begin{aligned}
\nabla_\beta\big(s^{\mathbf{nn}}\big) &= \delta^\delta{}_\beta \nabla_\delta\big(s^{\mathbf{nn}}\big)\\
&= \big(e^\delta{}_d e_\beta{}^d - n^\delta n_\beta\big)\nabla_\delta\big(s^{\mathbf{nn}}\big)\\
&= e^\delta{}_d e_\beta{}^d \nabla_\delta\big(s^{\mathbf{nn}}\big) - n^\delta n_\beta \nabla_\delta\big(s^{\mathbf{nn}}\big)\\
&= e_\beta{}^d \big[D_d\big(s^{\mathbf{nn}}\big)\big] - n_\beta \big[n^\delta \nabla_\delta\big(s^{\mathbf{nn}}\big)\big]\,. \qquad \text{(A3.6)}
\end{aligned}$$

Replacing on Eq. (A3.2) we arrive to

$$\begin{aligned}\nabla_\beta s^{\mu\nu} \;=\;& e_\beta{}^d e^\mu{}_a e^\nu{}_b \big[D_d s^{ab} - K^b{}_d s^{a\mathbf{n}} - K^a{}_d s^{\mathbf{n}b}\big] \\ & - n_\beta e^\mu{}_a e^\nu{}_b \big[e_\rho{}^a e_\sigma{}^b n^\delta \nabla_\delta\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big) - a^b s^{a\mathbf{n}} - a^a s^{\mathbf{n}b}\big] \\ & - e_\beta{}^d n^\mu e^\nu{}_b \big[D_d s^{\mathbf{n}b} - K_{da} s^{ab} - K^b{}_d s^{\mathbf{nn}}\big] \\ & - e_\beta{}^d e^\mu{}_a n^\nu \big[D_d s^{a\mathbf{n}} - K_{db} s^{ab} - K^a{}_d s^{\mathbf{nn}}\big] \\ & + n_\beta n^\mu e^\nu{}_b \big[e_\sigma{}^b n^\delta \nabla_\delta\big(e^\sigma{}_f s^{\mathbf{n}f}\big) - a_a s^{ab} - a^b s^{\mathbf{nn}}\big] \\ & + n_\beta e^\mu{}_a n^\nu \big[e_\rho{}^a n^\delta \nabla_\delta\big(e^\rho{}_e s^{e\mathbf{n}}\big) - a_b s^{ab} - a^a s^{\mathbf{nn}}\big] \\ & + e_\beta{}^d n^\mu n^\nu \big[D_d\big(s^{\mathbf{nn}}\big) - K_{db} s^{\mathbf{n}b} - K_{da} s^{a\mathbf{n}}\big] \\ & - n_\beta n^\mu n^\nu \big[n^\delta \nabla_\delta\big(s^{\mathbf{nn}}\big) - a_a s^{a\mathbf{n}} - a_b s^{\mathbf{n}b}\big] \,. \end{aligned} \tag{A3.7}$$

By using Eq. (2.2.51) we obtain

$$\begin{aligned}\nabla_\beta s^{\mu\nu} \;=\;& e_\beta{}^d e^\mu{}_a e^\nu{}_b \Big[D_d s^{ab} - K^b{}_d s^{a\mathbf{n}} - K^a{}_d s^{\mathbf{n}b}\Big] \\ & - n_\beta e^\mu{}_a e^\nu{}_b \left[\frac{1}{N} e_\rho{}^a e_\sigma{}^b \mathcal{L}_m\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big) + K^a{}_e s^{eb} + K^b{}_f s^{af} - a^b s^{a\mathbf{n}} - a^a s^{\mathbf{n}b}\right] \\ & - e_\beta{}^d n^\mu e^\nu{}_b \Big[D_d s^{\mathbf{n}b} - K_{da} s^{ab} - K^b{}_d s^{\mathbf{nn}}\Big] \\ & - e_\beta{}^d e^\mu{}_a n^\nu \Big[D_d s^{a\mathbf{n}} - K_{db} s^{ab} - K^a{}_d s^{\mathbf{nn}}\Big] \\ & + n_\beta n^\mu e^\nu{}_b \left[\frac{1}{N} e_\sigma{}^b \mathcal{L}_m\big(e^\sigma{}_f s^{\mathbf{n}f}\big) + K^b{}_f s^{\mathbf{n}f} - a_a s^{ab} - a^b s^{\mathbf{nn}}\right] \\ & + n_\beta e^\mu{}_a n^\nu \left[\frac{1}{N} e_\rho{}^a \mathcal{L}_m\big(e^\rho{}_e s^{e\mathbf{n}}\big) + K^a{}_e s^{e\mathbf{n}} - a_b s^{ab} - a^a s^{\mathbf{nn}}\right] \\ & + e_\beta{}^d n^\mu n^\nu \Big[D_d\big(s^{\mathbf{nn}}\big) - K_{db} s^{\mathbf{n}b} - K_{da} s^{a\mathbf{n}}\Big] \\ & - n_\beta n^\mu n^\nu \left[\frac{1}{N} \mathcal{L}_m\big(s^{\mathbf{nn}}\big) - a_a s^{a\mathbf{n}} - a_b s^{\mathbf{n}b}\right] . \end{aligned} \tag{A3.8}$$

In order to simplify the expressions we define the following quantities:

$$\Lambda^{ab}{}_d \;:=\; D_d s^{ab} - K^b{}_d s^{a\mathbf{n}} - K^a{}_d s^{\mathbf{n}b} \,, \tag{A3.9}$$

$$\Sigma^{ab} \;:=\; \frac{1}{N} e_\rho{}^a e_\sigma{}^b \mathcal{L}_m\big(e^\rho{}_e e^\sigma{}_f s^{ef}\big) + K^a{}_e s^{eb} + K^b{}_f s^{af} - a^a s^{\mathbf{n}b} - a^b s^{a\mathbf{n}} \,, \tag{A3.10}$$

$$\Omega^a{}_d \;:=\; D_d s^{a\mathbf{n}} - K_{db} s^{ab} - K^a{}_d s^{\mathbf{nn}} \,, \tag{A3.11}$$

$$\Psi^a \;:=\; \frac{1}{N} e_\rho{}^a \mathcal{L}_m\big(e^\rho{}_e s^{e\mathbf{n}}\big) + K^a{}_e s^{e\mathbf{n}} - a_b s^{ab} - a^a s^{\mathbf{nn}} \,, \tag{A3.12}$$

$$\Pi_d \;:=\; D_d s^{\mathbf{nn}} - K_{da} s^{a\mathbf{n}} - K_{db} s^{\mathbf{n}b} \,, \tag{A3.13}$$

$$\Phi \;:=\; \frac{1}{N} \mathcal{L}_m\big(s^{\mathbf{nn}}\big) - a_a s^{a\mathbf{n}} - a_b s^{\mathbf{n}b} \,. \tag{A3.14}$$

Thus we write instead

$$\begin{aligned}\nabla_\beta s^{\mu\nu} &= e_\beta{}^d e^\mu{}_a e^\nu{}_b \Lambda^{ab}{}_d - n_\beta e^\mu{}_a e^\nu{}_b \Sigma^{ab} - e_\beta{}^d n^\mu e^\nu{}_b \Omega^b{}_d - e_\beta{}^d e^\mu{}_a n^\nu \Omega^a{}_d \\ &\quad + n_\beta n^\mu e^\nu{}_b \Psi^b + n_\beta e^\mu{}_a n^\nu \Psi^a + e_\beta{}^d n^\mu n^\nu \Pi_d - n_\beta n^\mu n^\nu \Phi\,. \qquad \text{(A3.15)}\end{aligned}$$

Now we take the second covariant derivative:

$$\begin{aligned}\nabla_\alpha\nabla_\beta s^{\mu\nu} &= \nabla_\alpha\big(e_\beta{}^d e^\mu{}_a e^\nu{}_b \Lambda^{ab}{}_d\big) - \nabla_\alpha n_\beta e^\mu{}_a e^\nu{}_b \Sigma^{ab} - n_\beta \nabla_\alpha\big(e^\mu{}_a e^\nu{}_b \Sigma^{ab}\big) - e_\beta{}^d \nabla_\alpha n^\mu e^\nu{}_b \Omega^b{}_d \\ &\quad - n^\mu \nabla_\alpha\big(e_\beta{}^d e^\nu{}_b \Omega^b{}_d\big) - e_\beta{}^d e^\mu{}_a \nabla_\alpha n^\nu \Omega^a{}_d - n^\nu \nabla_\alpha\big(e_\beta{}^d e^\mu{}_a \Omega^a{}_d\big) \\ &\quad + \big(\nabla_\alpha n_\beta n^\mu + n_\beta \nabla_\alpha n^\mu\big) e^\nu{}_b \Psi^b + n_\beta n^\mu \nabla_\alpha\big(e^\nu{}_b \Psi^b\big) + \big(\nabla_\alpha n_\beta n^\nu + n_\beta \nabla_\alpha n^\nu\big) e^\mu{}_a \Psi^a \\ &\quad + n_\beta n^\nu \nabla_\alpha\big(e^\mu{}_a \Psi^a\big) + \big(\nabla_\alpha n^\mu n^\nu + n^\mu \nabla_\alpha n^\nu\big) e_\beta{}^d \Pi_d + n^\mu n^\nu \nabla_\alpha\big(e_\beta{}^d \Pi_d\big) \\ &\quad - \big(\nabla_\alpha n_\beta n^\mu n^\nu + n_\beta \nabla_\alpha n^\mu n^\nu + n_\beta n^\mu \nabla_\alpha n^\nu\big)\Phi - n_\beta n^\mu n^\nu \nabla_\alpha \Phi \\ &= \nabla_\alpha\big(e_\beta{}^d e^\mu{}_a e^\nu{}_b \Lambda^{ab}{}_d\big) - \big(e_\alpha{}^c K_{cd} - n_\alpha a_d\big) e_\beta{}^d e^\mu{}_a e^\nu{}_b \Sigma^{ab} - n_\beta \nabla_\alpha\big(e^\mu{}_a e^\nu{}_b \Sigma^{ab}\big) \\ &\quad - e_\beta{}^d \big(e_\alpha{}^c K^a{}_c - n_\alpha a^a\big) e^\mu{}_a e^\nu{}_b \Omega^b{}_d - n^\mu \nabla_\alpha\big(e_\beta{}^d e^\nu{}_b \Omega^b{}_d\big) \\ &\quad - e_\beta{}^d e^\mu{}_a \big(e^c_\alpha K^b{}_c - n_\alpha a^b\big) e^\nu{}_b \Omega^a{}_d - n^\nu \nabla_\alpha\big(e_\beta{}^d e^\mu{}_a \Omega^a{}_d\big) \\ &\quad + \Big(\big(e_\alpha{}^c K_{cd} - n_\alpha a_d\big) e_\beta{}^d n^\mu + n_\beta \big(e_\alpha{}^c K^a{}_c - n_\alpha a^a\big) e^\mu{}_a\Big) e^\nu{}_b \Psi^b + n_\beta n^\mu \nabla_\alpha\big(e^\nu{}_b \Psi^b\big) \\ &\quad + \Big(\big(e_\alpha{}^c K_{cd} - n_\alpha a_d\big) e_\beta{}^d n^\nu + n_\beta \big(e^c_\alpha K^b{}_c - n_\alpha a^b\big) e^\nu{}_b\Big) e^\mu{}_a \Psi^a + n_\beta n^\nu \nabla_\alpha\big(e^\mu{}_a \Psi^a\big) \\ &\quad + \Big(\big(e_\alpha{}^c K^a{}_c - n_\alpha a^a\big) e^\mu{}_a n^\nu + n^\mu \big(e^c_\alpha K^b{}_c - n_\alpha a^b\big) e^\nu{}_b\Big) e_\beta{}^d \Pi_d + n^\mu n^\nu \nabla_\alpha\big(e_\beta{}^d \Pi_d\big) \\ &\quad - \Big(\big(e_\alpha{}^c K_{cd} - n_\alpha a_d\big) e_\beta{}^d n^\mu n^\nu + \big(e_\alpha{}^c K^a{}_c - n_\alpha a^a\big) n_\beta e^\mu{}_a n^\nu \\ &\quad + \big(e_\alpha{}^c K^b{}_c - n_\alpha a^b\big) n_\beta n^\mu e^\nu{}_b\Big)\Phi - n_\beta n^\mu n^\nu \nabla_\alpha \Phi\,. \qquad \text{(A3.16)}\end{aligned}$$

We extract the free indexes using the delta-strategy obtaining for the first term

$$\begin{aligned}\nabla_\alpha\big(e_\beta{}^d e^\mu{}_a e^\nu{}_b \Lambda^{ab}{}_d\big) &= e_\alpha{}^c e_\beta{}^d e^\mu{}_a e^\nu{}_b \big(D_c \Lambda^{ab}{}_d\big) - e_\alpha{}^c e_\beta{}^d e^\mu{}_a n^\nu \big(-K_{cb} \Lambda^{ab}{}_d\big) \\ &\quad - e_\alpha{}^c e_\beta{}^d n^\mu e^\nu{}_b \big(-K_{ca} \Lambda^{ab}{}_d\big) - e_\alpha{}^c n_\beta e^\mu{}_a e^\nu{}_b \big(-K^d{}_c \Lambda^{ab}{}_d\big) \\ &\quad - n_\alpha e_\beta{}^d e^\mu{}_a e^\nu{}_b \big[e^\delta{}_d e_\rho{}^a e_\sigma{}^b n^\gamma \nabla_\gamma \big(e_\delta{}^h e^\rho{}_e e^\sigma{}_f \Lambda^{ef}{}_h\big)\big] \\ &\quad + n_\alpha e_\beta{}^d e^\mu{}_a n^\nu \big(-a_b \Lambda^{ab}{}_d\big) + n_\alpha e_\beta{}^d n^\mu e^\nu{}_b \big(-a_a \Lambda^{ab}{}_d\big) \\ &\quad + n_\alpha n_\beta e^\mu{}_a e^\nu{}_b \big(-a^d \Lambda^{ab}{}_d\big)\,, \qquad \text{(A3.17)}\end{aligned}$$

for the terms with two free indexes

$$\begin{aligned}
\nabla_\alpha\big(e^\mu{}_a e^\nu{}_b \Sigma^{ab}\big) &= e_\alpha{}^c e^\mu{}_a e^\nu{}_b\big(D_c\Sigma^{ab}\big) - e_\alpha{}^c e^\mu{}_a n^\nu\big(-K_{cb}\Sigma^{ab}\big) - e_\alpha{}^c n^\mu e^\nu{}_b\big(-K_{ca}\Sigma^{ab}\big) \\
&\quad -n_\alpha e^\mu{}_a e^\nu{}_b\big[e_\rho{}^a e_\sigma{}^b n^\gamma\nabla_\gamma\big(e^\rho{}_e e^\sigma{}_f\Sigma^{ef}\big)\big] + n_\alpha e^\mu{}_a n^\nu\big(-a_b\Sigma^{ab}\big) \\
&\quad +n_\alpha n^\mu e^\nu{}_b\big(-a_a\Sigma^{ab}\big)\,, \qquad (A3.18) \\
\nabla_\alpha\big(e_\beta{}^d e^\mu{}_a \Omega^a{}_d\big) &= e_\alpha{}^c e_\beta{}^d e^\mu{}_a\big(D_c\Omega^a{}_d\big) - e_\alpha{}^c e_\beta{}^d n^\mu\big(-K_{ca}\Omega^a{}_d\big) - e_\alpha{}^c n_\beta e^\mu{}_a\big(-K^d{}_c\Omega^a{}_d\big) \\
&\quad -n_\alpha e_\beta{}^d e^\mu{}_a\big[e^\delta{}_d e_\rho{}^a n^\gamma\nabla_\gamma\big(e_\delta{}^h e^\rho{}_e\Omega^e{}_h\big)\big] + n_\alpha e_\beta{}^d n^\mu\big(-a_a\Omega^a{}_d\big) \\
&\quad +n_\alpha n_\beta e^\mu{}_a\big(-a^d\Omega^a{}_d\big)\,, \qquad (A3.19)
\end{aligned}$$

for the terms with one free index

$$\begin{aligned}
\nabla_\alpha\big(e^\mu{}_a\Psi^a\big) &= e_\alpha{}^c e^\mu{}_a\big(D_c\Psi^a\big) - e_\alpha{}^c n^\mu\big(-K_{ca}\Psi^a\big) - n_\alpha e^\mu{}_a\big[e_\rho{}^a n^\gamma\nabla_\gamma\big(e^\rho{}_e\Psi^e\big)\big] \\
&\quad +n_\alpha n^\mu\big(-a_a\Psi^a\big)\,, \qquad (A3.20) \\
\nabla_\alpha\big(e_\beta{}^d\Pi_d\big) &= e_\alpha{}^c e_\beta{}^d\big(D_c\Pi_d\big) - e_\alpha{}^c n_\beta\big(-K^d{}_c\Pi_d\big) - n_\alpha e_\beta{}^d\big[e^\delta{}_d n^\gamma\nabla_\gamma\big(e_\delta{}^h\Pi_h\big)\big] \\
&\quad +n_\alpha n_\beta\big(-a^d\Pi_d\big)\,, \qquad (A3.21)
\end{aligned}$$

and finally for the last term

$$\nabla_\alpha\Phi = e_\alpha{}^c\big(D_c\Phi\big) - n_\alpha\big(n^\gamma\nabla_\gamma\Phi\big)\,. \qquad (A3.22)$$

Replacing this results we obtain

$$
\begin{aligned}
\nabla_\alpha\nabla_\beta s^{\mu\nu} \;=\;& e_\alpha{}^c e_\beta{}^d e^\mu{}_a e^\nu{}_b\Big[D_c\Lambda^{ab}{}_d - K_{cd}\Sigma^{ab} - K^a{}_c\Omega^b{}_d - K^b{}_c\Omega^a{}_d\Big] \\
& - n_\alpha e_\beta{}^d e^\mu{}_a e^\nu{}_b\Big[e^\delta{}_d e_\rho{}^a e_\sigma{}^b n^\gamma\nabla_\gamma\big(e_\delta{}^h e^\rho{}_e e^\sigma{}_f\Lambda^{ef}{}_h\big) - a_d\Sigma^{ab} - a^a\Omega^b{}_d - a^b\Omega^a{}_d\Big] \\
& - e_\alpha{}^c n_\beta e^\mu{}_a e^\nu{}_b\Big[D_c\Sigma^{ab} - K^d{}_c\Lambda^{ab}{}_d - K^a{}_c\Psi^b - K^b{}_c\Psi^a\Big] \\
& - e_\alpha{}^c e_\beta{}^d n^\mu e^\nu{}_b\Big[D_c\Omega^b{}_d - K_{ca}\Lambda^{ab}{}_d - K_{cd}\Psi^b - K^b{}_c\Pi_d\Big] \\
& - e_\alpha{}^c e_\beta{}^d e^\mu{}_a n^\nu\Big[D_c\Omega^a{}_d - K_{cb}\Lambda^{ab}{}_d - K_{cd}\Psi^a - K^a{}_c\Pi_d\Big] \\
& + n_\alpha n_\beta e^\mu{}_a e^\nu{}_b\Big[e_\rho{}^a e_\sigma{}^b n^\gamma\nabla_\gamma\big(e^\rho{}_e e^\sigma{}_f\Sigma^{ef}\big) - a^d\Lambda^{ab}{}_d - a^a\Psi^b - a^b\Psi^a\Big] \\
& + n_\alpha e_\beta{}^d n^\mu e^\nu{}_b\Big[e^\delta{}_d e_\sigma{}^b n^\gamma\nabla_\gamma\big(e_\delta{}^h e^\sigma{}_f\Omega^f{}_h\big) - a_a\Lambda^{ab}{}_d - a_d\Psi^b - a^b\Pi_d\Big] \\
& + n_\alpha e_\beta{}^d e^\mu{}_a n^\nu\Big[e^\delta{}_d e_\rho{}^a n^\gamma\nabla_\gamma\big(e_\delta{}^h e^\rho{}_e\Omega^e{}_h\big) - a_b\Lambda^{ab}{}_d - a_d\Psi^a - a^a\Pi_d\Big] \\
& + e_\alpha{}^c n_\beta n^\mu e^\nu{}_b\Big[D_c\Psi^b - K_{ca}\Sigma^{ab} - K^d{}_c\Omega^b{}_d - K^b{}_c\Phi\Big] \\
& + e_\alpha{}^c n_\beta e^\mu{}_a n^\nu\Big[D_c\Psi^a - K_{cb}\Sigma^{ab} - K^d{}_c\Omega^a{}_d - K^a{}_c\Phi\Big] \\
& + e_\alpha{}^c e_\beta{}^d n^\mu n^\nu\Big[D_c\Pi_d - K_{ca}\Omega^a{}_d - K_{ca}\Omega^a{}_d - K_{cd}\Phi\Big] \\
& - n_\alpha n_\beta e^\mu{}_a n^\nu\Big[e_\rho{}^a n^\gamma\nabla_\gamma\big(e^\rho{}_e\Psi^e\big) - a_b\Sigma^{ab} - a^d\Omega^a{}_d - a^a\Phi\Big] \\
& - n_\alpha n_\beta n^\mu e^\nu{}_b\Big[e_\sigma{}^b n^\gamma\nabla_\gamma\big(e^\sigma{}_f\Psi^f\big) - a_a\Sigma^{ab} - a^d\Omega^b{}_d - a^b\Phi\Big] \\
& - n_\alpha e_\beta{}^d n^\mu n^\nu\Big[e^\delta{}_d n^\gamma\nabla_\gamma\big(e_\delta{}^h\Pi_h\big) - a_a\Omega^a{}_d - a_a\Omega^a{}_d - a_d\Phi\Big] \\
& - e_\alpha{}^c n_\beta n^\mu n^\nu\Big[D_c\Phi - K_{ca}\Psi^a - K_{ca}\Psi^a - K^d{}_c\Pi_d\Big] \\
& + n_\alpha n_\beta n^\mu n^\nu\Big[n^\gamma\nabla_\gamma\Phi - a_a\Psi^a - a_a\Psi^a - a^d\Pi_d\Big] \qquad \text{(A3.23)}
\end{aligned}
$$

and using Eq. (2.2.51) we obtain

$$
\begin{aligned}
\nabla_\alpha \nabla_\beta s^{\mu\nu} \;=\;& e_\alpha{}^c e_\beta{}^d e^\mu{}_a e^\nu{}_b \Big[D_c \Lambda^{ab}{}_d - K_{cd}\Sigma^{ab} - K^a{}_c \Omega^b{}_d - K^b{}_c \Omega^a{}_d \Big] \\
& - n_\alpha e_\beta{}^d e^\mu{}_a e^\nu{}_b \Big[\frac{1}{N} e^\delta{}_d e_\rho{}^a e_\sigma{}^b \mathcal{L}_m \big(e_\delta{}^h e^\rho{}_e e^\sigma{}_f \Lambda^{ef}{}_h \big) + K^a{}_e \Lambda^{eb}{}_d + K^b{}_f \Lambda^{af}{}_d \\
& - K^h{}_d \Lambda^{ab}{}_h - a_d \Sigma^{ab} - a^a \Omega^b{}_d - a^b \Omega^a{}_d \Big] \\
& - e_\alpha{}^c n_\beta e^\mu{}_a e^\nu{}_b \Big[D_c \Sigma^{ab} - K^d{}_c \Lambda^{ab}{}_d - K^a{}_c \Psi^b - K^b{}_c \Psi^a \Big] \\
& - e_\alpha{}^c e_\beta{}^d n^\mu e^\nu{}_b \Big[D_c \Omega^b{}_d - K_{ca} \Lambda^{ab}{}_d - K_{cd} \Psi^b - K^b{}_c \Pi_d \Big] \\
& - e_\alpha{}^c e_\beta{}^d e^\mu{}_a n^\nu \Big[D_c \Omega^a{}_d - K_{cb} \Lambda^{ab}{}_d - K_{cd} \Psi^a - K^a{}_c \Pi_d \Big] \\
& + n_\alpha n_\beta e^\mu{}_a e^\nu{}_b \Big[\frac{1}{N} e_\rho{}^a e_\sigma{}^b \mathcal{L}_m \big(e^\rho{}_e e^\sigma{}_f \Sigma^{ef} \big) + K^a{}_e \Sigma^{eb} + K^b{}_f \Sigma^{af} \\
& - a^d \Lambda^{ab}{}_d - a^a \Psi^b - a^b \Psi^a \Big] \\
& + n_\alpha e_\beta{}^d n^\mu e^\nu{}_b \Big[\frac{1}{N} e^\delta{}_d e_\sigma{}^b \mathcal{L}_m \big(e_\delta{}^h e^\sigma{}_f \Omega^f{}_h \big) + K^b{}_f \Omega^f{}_d \\
& - K^h{}_d \Omega^b{}_h - a_a \Lambda^{ab}{}_d - a_d \Psi^b - a^b \Pi_d \Big] \\
& + n_\alpha e_\beta{}^d e^\mu{}_a n^\nu \Big[\frac{1}{N} e^\delta{}_d e_\rho{}^a \mathcal{L}_m \big(e_\delta{}^h e^\rho{}_e \Omega^e{}_h \big) + K^a{}_e \Omega^e{}_d - K^h{}_d \Omega^a{}_h \\
& - a_b \Lambda^{ab}{}_d - a_d \Psi^a - a^a \Pi_d \Big] \\
& + e_\alpha{}^c n_\beta n^\mu e^\nu{}_b \Big[D_c \Psi^b - K_{ca} \Sigma^{ab} - K^d{}_c \Omega^b{}_d - K^b{}_c \Phi \Big] \\
& + e_\alpha{}^c n_\beta e^\mu{}_a n^\nu \Big[D_c \Psi^a - K_{cb} \Sigma^{ab} - K^d{}_c \Omega^a{}_d - K^a{}_c \Phi \Big] \\
& + e_\alpha{}^c e_\beta{}^d n^\mu n^\nu \Big[D_c \Pi_d - K_{ca} \Omega^a{}_d - K_{ca} \Omega^a{}_d - K_{cd} \Phi \Big] \\
& - n_\alpha n_\beta e^\mu{}_a n^\nu \Big[\frac{1}{N} e_\rho{}^a \mathcal{L}_m \big(e^\rho{}_e \Psi^e \big) + K^a{}_e \Psi^e - a_b \Sigma^{ab} - a^d \Omega^a{}_d - a^a \Phi \Big] \\
& - n_\alpha n_\beta n^\mu e^\nu{}_b \Big[\frac{1}{N} e_\sigma{}^b \mathcal{L}_m \big(e^\sigma{}_f \Psi^f \big) + K^b{}_f \Psi^f - a_a \Sigma^{ab} - a^d \Omega^b{}_d - a^b \Phi \Big] \\
& - n_\alpha e_\beta{}^d n^\mu n^\nu \Big[\frac{1}{N} e^\delta{}_d \mathcal{L}_m \big(e_\delta{}^h \Pi_h \big) - K^h{}_d \Pi_h - a_a \Omega^a{}_d - a_a \Omega^a{}_d - a_d \Phi \Big] \\
& - e_\alpha{}^c n_\beta n^\mu n^\nu \Big[D_c \Phi - K_{ca} \Psi^a - K_{ca} \Psi^a - K^d{}_c \Pi_d \Big] \\
& + n_\alpha n_\beta n^\mu n^\nu \Big[\frac{1}{N} \mathcal{L}_m \Phi - a_a \Psi^a - a_a \Psi^a - a^d \Pi_d \Big],
\end{aligned}
\tag{A3.24}
$$

A3.0.1 Covariant derivative contractions

We need three contractions:

$$\nabla_\alpha \nabla^\mu s^{\alpha\nu},\ \nabla^2 s^{\mu\nu},\ \nabla_\alpha \nabla_\beta s^{\alpha\beta}$$

We start with the first one: We contract α and μ in the $\nabla_\alpha \nabla_\beta s^{\mu\nu}$ expression

$$\begin{aligned}
\nabla_\alpha \nabla_\beta s^{\alpha\nu} &= e_\beta{}^d e^\nu{}_b \Big[D_c \Lambda^{cb}{}_d - K_{cd}\Sigma^{cb} - K\Omega^b{}_d - 2K^b{}_c\Omega^c{}_d + K^h{}_d\Omega^b{}_h \\
&\quad - \frac{1}{N} e^\delta{}_d e_\sigma{}^b \mathcal{L}_m\big(e_\delta{}^h e^\sigma{}_f \Omega^f{}_h\big) + a_a\Lambda^{ab}{}_d + a_d\Psi^b + a^b\Pi_d \Big] \\
&\quad - n_\beta e^\nu{}_b \Big[D_c\Sigma^{cb} - K^d{}_c\Lambda^{cb}{}_d - K\Psi^b - 2K^b{}_c\Psi^c - \frac{1}{N} e_\sigma{}^b \mathcal{L}_m\big(e^\sigma{}_f \Psi^f\big) \\
&\quad + a_a\Sigma^{ab} + a^d\Omega^b{}_d + a^b\Phi \Big] \\
&\quad - e_\beta{}^d n^\nu \Big[D_c\Omega^c{}_d - K_{cb}\Lambda^{cb}{}_d - K_{cd}\Psi^c - K\Pi_d - \frac{1}{N} e^\delta{}_d \mathcal{L}_m\big(e_\delta{}^h \Pi_h\big) \\
&\quad + K^h{}_d\Pi_h + 2a_a\Omega^a{}_d + a_d\Phi \Big] \\
&\quad + n_\beta n^\nu \Big[D_c\Psi^c - K_{cb}\Sigma^{cb} - K^d{}_c\Omega^c{}_d - K\Phi - \frac{1}{N}\mathcal{L}_m\Phi + 2a_a\Psi^a + a^d\Pi_d \Big]. \qquad \text{(A3.25)}
\end{aligned}$$

Now we raise the index β with $g^{\mu\beta}$, obtaining:

$$\begin{aligned}
\nabla_\alpha \nabla^\mu s^{\alpha\nu} &= e^\mu{}_a e^\nu{}_b \Big[q^{ad}\Big(D_c \Lambda^{cb}{}_d - K_{cd}\Sigma^{cb} - K\Omega^b{}_d - 2K^b{}_c\Omega^c{}_d + K^h{}_d\Omega^b{}_h \\
&\quad - \frac{1}{N} e^\delta{}_d e_\sigma{}^b \mathcal{L}_m\big(e_\delta{}^h e^\sigma{}_f \Omega^f{}_h\big) + a_a\Lambda^{ab}{}_d + a_d\Psi^b + a^b\Pi_d \Big)\Big] \\
&\quad - n^\mu e^\nu{}_b \Big[D_c\Sigma^{cb} - K^d{}_c\Lambda^{cb}{}_d - K\Psi^b - 2K^b{}_c\Psi^c - \frac{1}{N} e_\sigma{}^b \mathcal{L}_m\big(e^\sigma{}_f \Psi^f\big) \\
&\quad + a_a\Sigma^{ab} + a^d\Omega^b{}_d + a^b\Phi \Big] \\
&\quad - e^\mu{}_a n^\nu \Big[q^{ad}\Big(D_c\Omega^c{}_d - K_{cb}\Lambda^{cb}{}_d - K_{cd}\Psi^c - K\Pi_d - \frac{1}{N} e^\delta{}_d \mathcal{L}_m\big(e_\delta{}^h \Pi_h\big) + K^h{}_d\Pi_h \\
&\quad + 2a_a\Omega^a{}_d + a_d\Phi \Big)\Big] \\
&\quad + n^\mu n^\nu \Big[D_c\Psi^c - K_{cb}\Sigma^{cb} - K^d{}_c\Omega^c{}_d - K\Phi - \frac{1}{N}\mathcal{L}_m\Phi + 2a_a\Psi^a + a^d\Pi_d \Big]. \qquad \text{(A3.26)}
\end{aligned}$$

And we can symmetrize in the index μ and ν to obtain the exact contribution:

$$\begin{aligned}
\nabla_\alpha\nabla^\mu s^{\alpha\nu} + \nabla_\alpha\nabla^\nu s^{\alpha\mu} = {}& e^\mu{}_a e^\nu{}_b\Big[q^{ad}\Big(D_c\Lambda^{cb}{}_d - K_{cd}\Sigma^{cb} - K\Omega^b{}_d - 2K^b{}_c\Omega^c{}_d + K^h{}_d\Omega^b{}_h \\
& - \frac{1}{N}e^\delta{}_d e_\sigma{}^b\mathcal{L}_m\big(e_\delta{}^h e^\sigma{}_f\Omega^f{}_h\big) + a_c\Lambda^{cb}{}_d + a_d\Psi^b + a^b\Pi_d\Big) \\
& + q^{bd}\Big(D_c\Lambda^{ca}{}_d - K_{cd}\Sigma^{ca} - K\Omega^a{}_d - 2K^a{}_c\Omega^c{}_d + K^h{}_d\Omega^a{}_h \\
& - \frac{1}{N}e^\delta{}_d e_\sigma{}^a\mathcal{L}_m\big(e_\delta{}^h e^\sigma{}_f\Omega^f{}_h\big) + a_c\Lambda^{ca}{}_d + a_d\Psi^a + a^a\Pi_d\Big)\Big] \\
& - n^\mu e^\nu{}_b\Big[D_c\Sigma^{cb} - K^d{}_c\Lambda^{cb}{}_d - K\Psi^b - 2K^b{}_c\Psi^c - \frac{1}{N}e_\sigma{}^b\mathcal{L}_m\big(e^\sigma{}_f\Psi^f\big) \\
& + a_c\Sigma^{cb} + a^d\Omega^b{}_d + a^b\Phi + q^{bd}\Big(D_c\Omega^c{}_d - K_{ca}\Lambda^{ca}{}_d - K_{cd}\Psi^c - K\Pi_d \\
& - \frac{1}{N}e^\delta{}_d\mathcal{L}_m\big(e_\delta{}^h\Pi_h\big) + K^h{}_d\Pi_h + 2a_c\Omega^c{}_d + a_d\Phi\Big)\Big] \\
& - e^\mu{}_a n^\nu\Big[q^{ad}\Big(D_c\Omega^c{}_d - K_{cb}\Lambda^{cb}{}_d - K_{cd}\Psi^c - K\Pi_d - \frac{1}{N}e^\delta{}_d\mathcal{L}_m\big(e_\delta{}^h\Pi_h\big) \\
& + K^h{}_d\Pi_h + 2a_c\Omega^c{}_d + a_d\Phi\Big) + D_c\Sigma^{ca} - K^d{}_c\Lambda^{ca}{}_d - K\Psi^a - 2K^a{}_c\Psi^c \\
& - \frac{1}{N}e_\sigma{}^a\mathcal{L}_m\big(e^\sigma{}_f\Psi^f\big) + a_c\Sigma^{ca} + a^d\Omega^a{}_d + a^a\Phi\Big] \\
& + n^\mu n^\nu\Big[2\Big(D_c\Psi^c - K_{cb}\Sigma^{cb} - K^d{}_c\Omega^c{}_d - K\Phi - \frac{1}{N}\mathcal{L}_m\Phi \\
& + 2a_a\Psi^a + a^d\Pi_d\Big)\Big]\,. \qquad \text{(A3.27)}
\end{aligned}$$

Now we compute the second contraction needed: we contract with $g^{\alpha\beta}$ the expression for $\nabla_\alpha\nabla_\beta s^{\mu\nu}$ as follows

$$\begin{aligned}
\nabla^2 s^{\mu\nu} \quad = \quad & e^\mu{}_a e^\nu{}_b\Big[q^{cd}\Big(D_c\Lambda^{ab}{}_d - K_{cd}\Sigma^{ab} - K^a{}_c\Omega^b{}_d - K^b{}_c\Omega^a{}_d\Big) \\
& -\frac{1}{N}e_\rho{}^a e_\sigma{}^b\mathcal{L}_m\big(e^\rho{}_e e^\sigma{}_f\Sigma^{ef}\big) - K^a{}_e\Sigma^{eb} - K^b{}_f\Sigma^{af} + a^d\Lambda^{ab}{}_d + a^a\Psi^b + a^b\Psi^a\Big] \\
& -n^\mu e^\nu{}_b\Big[q^{cd}\Big(D_c\Omega^b{}_d - K_{ca}\Lambda^{ab}{}_d - K_{cd}\Psi^b - K^b{}_c\Pi_d\Big) - \frac{1}{N}e_\sigma{}^b\mathcal{L}_m\big(e^\sigma{}_f\Psi^f\big) \\
& -K^b{}_f\Psi^f + a_a\Sigma^{ab} + a^d\Omega^b{}_d + a^b\Phi\Big] \\
& -e^\mu{}_a n^\nu\Big[q^{cd}\Big(D_c\Omega^a{}_d - K_{cb}\Lambda^{ab}{}_d - K_{cd}\Psi^a - K^a{}_c\Pi_d\Big) - \frac{1}{N}e_\rho{}^a\mathcal{L}_m\big(e^\rho{}_e\Psi^e\big) \\
& -K^a{}_e\Psi^e + a_b\Sigma^{ab} + a^d\Omega^a{}_d + a^a\Phi\Big] \\
& +n^\mu n^\nu\Big[q^{cd}\Big(D_c\Pi_d - K_{ca}\Omega^a{}_d - K_{ca}\Omega^a{}_d - K_{cd}\Phi\Big) - \frac{1}{N}\mathcal{L}_m\Phi \\
& +2a_a\Psi^a + a^d\Pi_d\Big]\,. \qquad \text{(A3.28)}
\end{aligned}$$

Finally the last contraction can be obtained contracting the indexes β and ν in the $\nabla_\alpha \nabla_\beta s^{\alpha\beta}$ expression, obtaining

$$\begin{aligned}\nabla_\alpha \nabla_\beta s^{\mu\nu} &= D_c \Lambda^{cd}{}_d - K\Omega^d{}_d - \frac{1}{N} e^\delta{}_d e_\sigma{}^d \mathcal{L}_m \big(e_\delta{}^h e^\sigma{}_f \Omega^f{}_h\big) + a_c \Lambda^{cd}{}_d - a_c \Psi^c - D_c \Psi^c \\ &\quad + K\Phi + \frac{1}{N}\mathcal{L}_m \Phi \,. \end{aligned} \tag{A3.29}$$

A3.0.2 Ricci contractions

We need to compute the $R_{\alpha\beta} s^{\alpha\beta}$. The Riemann tensor projections are well know in the ADM formulation. In particular, the Ricci tensor can be decompose as

$$\begin{aligned} R_{\alpha\beta} &= e_\alpha{}^c e_\beta{}^d \left(R_{cd} - 2K_{ce}K^e{}_d + K_{cd}K - D_d a_c - a_c a_d + \frac{1}{N} e^\rho{}_c e^\sigma{}_d \mathcal{L}_m (e_\rho{}^e e_\sigma{}^f K_{ef}) \right) \\ &\quad - n_\alpha e_\beta{}^d \big(D_c K^c{}_d - D_d K\big) - e_\alpha{}^c n_\beta \big(D_d K^d{}_c - D_c K\big) \\ &\quad + n_\alpha n_\beta \left(D\cdot a + a^2 + K^{cd}K_{cd} - \frac{1}{N} q^{cd} e^\rho{}_c e^\sigma{}_d \mathcal{L}_m (e_\rho{}^e e_\sigma{}^f K_{ef}) \right) . \end{aligned} \tag{A3.30}$$

Thus, contracting with $s^{\alpha\beta}$, we obtain

$$\begin{aligned} R_{\alpha\beta} s^{\alpha\beta} &= s^{cd} \left(R_{cd} - 2K_{ce}K^e{}_d + K_{cd}K - D_d a_c - a_c a_d + \frac{1}{N} e^\rho{}_c e^\sigma{}_d \mathcal{L}_m (e_\rho{}^e e_\sigma{}^f K_{ef}) \right) \\ &\quad - 2 s^{\mathbf{cn}} \big(D_d K^d{}_c - D_c K\big) + s^{\mathbf{nn}} \bigg(D\cdot a + a^2 + K^{cd}K_{cd} \\ &\quad - \frac{1}{N} q^{cd} e^\rho{}_c e^\sigma{}_d \mathcal{L}_m (e_\rho{}^e e_\sigma{}^f K_{ef}) \bigg) . \end{aligned} \tag{A3.31}$$

The previous results are enough to construct the energy-momentum tensor for the $s^{\mu\nu}$ background field.

A4 Projections in the t sector

We start decomposing the background tensor $t^{\mu\alpha\nu\beta}$ in its normal and tangential projections

$$\begin{aligned} t^{\mu\alpha\nu\beta} &= e^\mu{}_c e^\alpha{}_a e^\nu{}_d e^\beta{}_b t^{cadb} - n^\mu e^\alpha{}_a e^\nu{}_d e^\beta{}_b t^{\mathbf{n}adb} - e^\mu{}_c n^\alpha e^\nu{}_d e^\beta{}_b t^{c\mathbf{n}db} \\ &\quad - e^\mu{}_c e^\alpha{}_a n^\nu e^\beta{}_b t^{ca\mathbf{n}b} - e^\mu{}_c e^\alpha{}_a e^\nu{}_d n^\beta t^{cad\mathbf{n}} + n^\mu e^\alpha{}_a n^\nu e^\beta{}_b t^{\mathbf{n}a\mathbf{n}b} \\ &\quad + n^\mu e^\alpha{}_a e^\nu{}_d n^\beta t^{\mathbf{n}ad\mathbf{n}} + e^\mu{}_c n^\alpha n^\nu e^\beta{}_b t^{c\mathbf{n}\mathbf{n}b} + e^\mu{}_c n^\alpha e^\nu{}_d n^\beta t^{c\mathbf{n}d\mathbf{n}} \,. \end{aligned} \tag{A4.1}$$

We apply the covariant derivative

$$
\begin{aligned}
\nabla_\lambda t^{\mu\alpha\nu\beta} &= \nabla_\lambda\Big(e^\mu{}_c e^\alpha{}_a e^\nu{}_d e^\beta{}_b t^{cadb} - n^\mu e^\alpha{}_a e^\nu{}_d e^\beta{}_b t^{\mathbf{n}adb} - e^\mu{}_c n^\alpha e^\nu{}_d e^\beta{}_b t^{c\mathbf{n}db} - e^\mu{}_c e^\alpha{}_a n^\nu e^\beta{}_b t^{ca\mathbf{n}b} \\
&\quad - e^\mu{}_c e^\alpha{}_a e^\nu{}_d n^\beta t^{cad\mathbf{n}} + n^\mu e^\alpha{}_a n^\nu e^\beta{}_b t^{\mathbf{n}a\mathbf{n}b} + n^\mu e^\alpha{}_a e^\nu{}_d n^\beta t^{\mathbf{n}ad\mathbf{n}} + e^\mu{}_c n^\alpha n^\nu e^\beta{}_b t^{c\mathbf{n}\mathbf{n}b} \\
&\quad + e^\mu{}_c n^\alpha e^\nu{}_d n^\beta t^{c\mathbf{n}d\mathbf{n}}\Big) \\
&= \nabla_\lambda\big(e^\mu{}_c e^\alpha{}_a e^\nu{}_d e^\beta{}_b t^{cadb}\big) - (\nabla_\lambda n^\mu) e^\alpha{}_a e^\nu{}_d e^\beta{}_b t^{\mathbf{n}adb} - n^\mu \nabla_\lambda\big(e^\alpha{}_a e^\nu{}_d e^\beta{}_b t^{\mathbf{n}adb}\big) \\
&\quad - (\nabla_\lambda n^\alpha) e^\mu{}_c e^\nu{}_d e^\beta{}_b t^{c\mathbf{n}db} - n^\alpha \nabla_\lambda\big(e^\mu{}_c e^\nu{}_d e^\beta{}_b t^{c\mathbf{n}db}\big) - (\nabla_\lambda n^\nu) e^\mu{}_c e^\alpha{}_a e^\beta{}_b t^{ca\mathbf{n}b} \\
&\quad - n^\nu \nabla_\lambda\big(e^\mu{}_c e^\alpha{}_a e^\beta{}_b t^{ca\mathbf{n}b}\big) - (\nabla_\lambda n^\beta) e^\mu{}_c e^\alpha{}_a e^\nu{}_d t^{cad\mathbf{n}} - n^\beta \nabla_\lambda\big(e^\mu{}_c e^\alpha{}_a e^\nu{}_d t^{cad\mathbf{n}}\big) \\
&\quad + (\nabla_\lambda n^\mu n^\nu + n^\mu \nabla_\lambda n^\nu) e^\alpha{}_a e^\beta{}_b t^{\mathbf{n}a\mathbf{n}b} + n^\mu n^\nu \nabla_\lambda\big(e^\alpha{}_a e^\beta{}_b t^{\mathbf{n}a\mathbf{n}b}\big) \\
&\quad + (\nabla_\lambda n^\mu n^\beta + n^\mu \nabla_\lambda n^\beta) e^\alpha{}_a e^\nu{}_d t^{\mathbf{n}ad\mathbf{n}} + n^\mu n^\beta \nabla_\lambda\big(e^\alpha{}_a e^\nu{}_d t^{\mathbf{n}ad\mathbf{n}}\big) \\
&\quad + (\nabla_\lambda n^\alpha n^\nu + n^\alpha \nabla_\lambda n^\nu) e^\mu{}_c e^\beta{}_b t^{c\mathbf{n}\mathbf{n}b} + n^\alpha n^\nu \nabla_\lambda\big(e^\mu{}_c e^\beta{}_b t^{c\mathbf{n}\mathbf{n}b}\big) \\
&\quad + (\nabla_\lambda n^\alpha n^\beta + n^\alpha \nabla_\lambda n^\beta) e^\mu{}_c e^\nu{}_d t^{c\mathbf{n}d\mathbf{n}} + n^\alpha n^\beta \nabla_\lambda\big(e^\mu{}_c e^\nu{}_d t^{c\mathbf{n}d\mathbf{n}}\big)\,. \qquad \text{(A4.2)}
\end{aligned}
$$

Taking the contraction in the β and λ indices produces

$$
\begin{aligned}
\nabla_\beta t^{\mu\alpha\nu\beta} &= \nabla_\beta\big(e^\mu{}_c e^\alpha{}_a e^\nu{}_d e^\beta{}_b t^{cadb}\big) - e^\mu{}_c e^\alpha{}_a e^\nu{}_d K^c{}_b t^{\mathbf{n}adb} - n^\mu \nabla_\beta\big(e^\alpha{}_a e^\nu{}_d e^\beta{}_b t^{\mathbf{n}adb}\big) \\
&\quad - e^\alpha{}_a e^\mu{}_c e^\nu{}_d K^a{}_b t^{c\mathbf{n}db} - n^\alpha \nabla_\beta\big(e^\mu{}_c e^\nu{}_d e^\beta{}_b t^{c\mathbf{n}db}\big) - e^\nu{}_d e^\mu{}_c e^\alpha{}_a K^d{}_b t^{ca\mathbf{n}b} \\
&\quad - n^\nu \nabla_\beta\big(e^\mu{}_c e^\alpha{}_a e^\beta{}_b t^{ca\mathbf{n}b}\big) - e^\mu{}_c e^\alpha{}_a e^\nu{}_d K t^{cad\mathbf{n}} - n^\beta \nabla_\beta\big(e^\mu{}_c e^\alpha{}_a e^\nu{}_d t^{cad\mathbf{n}}\big) \\
&\quad + (e^\mu{}_c K^c_b n^\nu + n^\mu e^\nu{}_d K^d_b) e^\alpha{}_a t^{\mathbf{n}a\mathbf{n}b} + n^\mu n^\nu \nabla_\beta\big(e^\alpha{}_a e^\beta{}_b t^{\mathbf{n}a\mathbf{n}b}\big) \\
&\quad + (e^\mu{}_c a^c + n^\mu K) e^\alpha{}_a e^\nu{}_d t^{\mathbf{n}ad\mathbf{n}} + n^\mu n^\beta \nabla_\beta\big(e^\alpha{}_a e^\nu{}_d t^{\mathbf{n}ad\mathbf{n}}\big) \\
&\quad + (e^\alpha{}_a K^a_b n^\nu + n^\alpha e^\nu{}_d K^d_b) e^\mu{}_c t^{c\mathbf{n}\mathbf{n}b} + n^\alpha n^\nu \nabla_\beta\big(e^\mu{}_c e^\beta{}_b t^{c\mathbf{n}\mathbf{n}b}\big) \\
&\quad + (e^\alpha{}_a a^a + n^\alpha K) e^\mu{}_c e^\nu{}_d t^{c\mathbf{n}d\mathbf{n}} + n^\alpha n^\beta \nabla_\beta\big(e^\mu{}_c e^\nu{}_d t^{c\mathbf{n}d\mathbf{n}}\big)\,. \qquad \text{(A4.3)}
\end{aligned}
$$

We introduce Kronecker deltas using Eq. (2.2.17) for every unprojected term. The terms with three free indexes gives

$$
\begin{aligned}
\nabla_\beta\big(e^\mu{}_c e^\alpha{}_a e^\nu{}_d e^\beta{}_b t^{cadb}\big) &= e^\mu{}_a e^\nu{}_b e^\alpha{}_c\big(D_d t^{acbd} + a_d t^{acbd}\big) + n^\mu e^\nu{}_b e^\alpha{}_c\big(K_{ad} t^{acbd}\big) \\
&\quad + e^\mu{}_a e^\nu{}_b n^\alpha\big(K_{cd} t^{acbd}\big)\,, \qquad \text{(A4.4)} \\
n^\beta \nabla_\beta\big(e^\mu{}_c e^\alpha{}_a e^\nu{}_d t^{cad\mathbf{n}}\big) &= e^\mu{}_a e^\nu{}_b e^\alpha{}_c\big(e_\rho{}^a e_\sigma{}^b e_\lambda{}^c n^\beta \nabla_\beta\big(e^\rho{}_i e^\lambda{}_k e^\sigma{}_j t^{ikj\mathbf{n}}\big)\big) + n^\mu e^\nu{}_b e^\alpha{}_c\big(a_a t^{acb\mathbf{n}}\big) \\
&\quad + e^\mu{}_a n^\nu e^\alpha{}_c\big(a_b t^{acb\mathbf{n}}\big) + e^\mu{}_a e^\nu{}_b n^\alpha\big(a_c t^{acb\mathbf{n}}\big)\,, \qquad \text{(A4.5)}
\end{aligned}
$$

the ones with two free indexes gives

$$\nabla_\beta\big(e^\alpha{}_a e^\nu{}_d e^\beta{}_b t^{\mathbf{n}adb}\big) = e^\nu{}_b e^\alpha{}_c\big(D_d t^{\mathbf{n}cbd} + a_d t^{\mathbf{n}cbd}\big) + e^\nu{}_b n^\alpha\big(K_{cd} t^{\mathbf{n}cbd}\big)\,, \tag{A4.6}$$

$$n^\beta\nabla_\beta\big(e^\mu{}_c e^\nu{}_d t^{c\mathbf{n}d\mathbf{n}}\big) = e^\mu{}_a e^\nu{}_b\big(e_\rho{}^a e_\sigma{}^b n^\beta\nabla_\beta\big(e^\rho{}_i e^\sigma{}_j t^{i\mathbf{n}j\mathbf{n}}\big)\big) + n^\mu e^\nu{}_b\big(a_a t^{a\mathbf{n}b\mathbf{n}}\big) + e^\mu{}_a n^\nu\big(a_b t^{a\mathbf{n}b\mathbf{n}}\big)\,. \tag{A4.7}$$

and the one with one free index gives

$$\nabla_\beta\big(e^\alpha{}_a e^\beta{}_b t^{\mathbf{n}a\mathbf{n}b}\big) \;\; = e^\alpha{}_c\big(D_d t^{\mathbf{n}c\mathbf{n}d} + a_d t^{\mathbf{n}c\mathbf{n}d}\big) + n^\alpha\big(K_{kd} t^{\mathbf{n}k\mathbf{n}d}\big)\,. \tag{A4.8}$$

Replacing the projections in the covariant divergence of the t tensor we obtain a final expression for their projection

$$\nabla_\beta t^{\mu\alpha\nu\beta} = e^\mu{}_a e^\alpha{}_c e^\nu{}_b \Theta^{acb} + \big(-n^\mu e^\alpha{}_a e^\nu{}_b + n^\alpha e^\mu{}_a e^\nu{}_b\big)\,\Sigma^{ab} - e^\mu{}_a e^\alpha{}_c n^\nu \Phi^{ac} + (n^\mu e^\alpha{}_a - n^\alpha e^\mu{}_a) n^\nu \Lambda^a\,, \tag{A4.9}$$

where we defined the projections as

$$\Theta^{acb} := D_d t^{acbd} - K^a_d t^{\mathbf{n}cbd} - K^c_d t^{a\mathbf{n}bd} - K^b_d t^{ac\mathbf{n}d} - K t^{acb\mathbf{n}} - e_\rho{}^a e_\lambda{}^c e_\sigma{}^b n^\kappa \nabla_\kappa (e^\rho{}_i e^\lambda{}_k e^\sigma{}_j t^{ikj\mathbf{n}}) + a_d t^{acbd} + a^a t^{\mathbf{n}cb\mathbf{n}} + a^c t^{a\mathbf{n}b\mathbf{n}}\,, \tag{A4.10}$$

$$\Sigma^{cb} := D_d t^{\mathbf{n}cbd} - K_{ad} t^{acbd} - K^b_d t^{\mathbf{n}c\mathbf{n}d} - K t^{\mathbf{n}cb\mathbf{n}} - e_\lambda{}^c e_\sigma{}^b n^\kappa \nabla_\kappa (e^\lambda{}_k e^\sigma{}_j t^{\mathbf{n}kj\mathbf{n}}) + a_d t^{\mathbf{n}cbd} + a_a t^{acb\mathbf{n}}\,, \tag{A4.11}$$

$$\Phi^{ac} := D_d t^{ac\mathbf{n}d} - K^a_d t^{\mathbf{n}c\mathbf{n}d} - K^c_d t^{a\mathbf{n}\mathbf{n}d}\,, \tag{A4.12}$$

$$\Lambda^c := D_d t^{\mathbf{n}c\mathbf{n}d} - K_{ad} t^{ac\mathbf{n}d}\,. \tag{A4.13}$$

We continue by acting with an second covariant derivative

$$\nabla_\lambda\nabla_\beta t^{\mu\alpha\nu\beta} = \nabla_\lambda\big(e^\mu{}_a e^\alpha{}_c e^\nu{}_b \Theta^{acb}\big) - (\nabla_\lambda n^\mu) e^\alpha{}_c e^\nu{}_b \Sigma^{cb} - n^\mu \nabla_\lambda\big(e^\alpha{}_c e^\nu{}_b \Sigma^{cb}\big) + (\nabla_\lambda n^\alpha) e^\mu{}_a e^\nu{}_b \Sigma^{ab} + n^\alpha \nabla_\lambda\big(e^\mu{}_a e^\nu{}_b \Sigma^{ab}\big) - (\nabla_\lambda n^\nu) e^\mu{}_a e^\alpha{}_c \Phi^{ac} - n^\nu \nabla_\lambda\big(e^\mu{}_a e^\alpha{}_c \Phi^{ac}\big) + \big(n^\mu \nabla_\lambda n^\nu + n^\nu \nabla_\lambda n^\mu\big) e^\alpha{}_c \Lambda^c + n^\mu n^\nu \nabla_\lambda\big(e^\alpha{}_c \Lambda^c\big) - \big(n^\alpha \nabla_\lambda n^\nu + n^\nu \nabla_\lambda n^\alpha\big) e^\mu{}_a \Lambda^a - n^\alpha n^\nu \nabla_\lambda\big(e^\mu{}_a \Lambda^a\big)\,. \tag{A4.14}$$

Contracting in the λ and α indices we obtain

$$\begin{aligned}\nabla_\alpha\nabla_\beta t^{\mu\alpha\nu\beta} &= \nabla_\alpha\big(e^\mu{}_a e^\alpha{}_c e^\nu{}_b \Theta^{acb}\big) - n^\mu\nabla_\alpha\big(e^\alpha{}_c e^\nu{}_b\Sigma^{cb}\big) + n^\alpha\nabla_\alpha\big(e^\mu{}_a e^\nu{}_b \Sigma^{ab}\big) \\ &- e^\mu{}_a e^\nu{}_b\big(K^a{}_c\Sigma^{cb} - K\Sigma^{ab} + K^b{}_c\Phi^{ac} + a^b\Lambda^a\big) - n^\nu\nabla_\alpha\big(e^\mu{}_a e^\alpha{}_c\Phi^{ac}\big) \\ &+ n^\mu e^\nu{}_b K^b{}_c\Lambda^c + e^\mu{}_a n^\nu\big(K^a{}_c\Lambda^c - K\Lambda^a\big) + n^\mu n^\nu\nabla_\alpha\big(e^\alpha{}_c\Lambda^c\big) \\ &- n^\nu n^\alpha\nabla_\alpha\big(e^\mu{}_a\Lambda^a\big)\,. \end{aligned} \tag{A4.15}$$

We proceed as usual introducing Kronecker deltas in the unprojected terms, integrating by parts and recognizing geometrical quantities. We list the projections obtained with three free indexes

$$\begin{aligned}\nabla_\alpha\big(e^\mu{}_a e^\alpha{}_c e^\nu{}_b\Theta^{acb}\big) &= e^\mu{}_a e^\nu{}_b\big[D_c\Theta^{acb} + a_c\Theta^{acb}\big] + n^\mu e^\nu{}_b\big[K_{ac}\Theta^{acb}\big] \\ &+ e^\mu{}_a n^\nu\big[K_{bc}\Theta^{acb}\big]\,, \end{aligned} \tag{A4.16}$$

with two free indexes

$$\begin{aligned} n^\alpha\nabla_\alpha\big(e^\mu_a e^\nu_b\Sigma^{ab}\big) &= e^\mu{}_a e^\nu{}_b\big[e_\rho{}^a e_\sigma{}^b n^\alpha\nabla_\alpha\big(e^\rho{}_i e^\sigma{}_j\Sigma^{ij}\big)\big] + n^\mu e^\nu{}_b\big[a_a\Sigma^{ab}\big] \\ &+ e^\mu{}_a n^\nu\big[a_b\Sigma^{ab}\big]\,, \end{aligned} \tag{A4.17}$$

with one free index

$$\nabla_\alpha\big(e^\alpha{}_c e^\nu{}_b\Sigma^{cb}\big) = e^\nu{}_b\big[D_c\Sigma^{cb} + a_c\Sigma^{cb}\big] + n^\nu\big[K_{bc}\Sigma^{cb}\big]\,, \tag{A4.18}$$

$$\nabla_\alpha\big(e^\mu{}_a e^\alpha{}_c\Phi^{ac}\big) = e^\mu{}_a\big[D_c\Phi^{ac} + a_c\Phi^{ac}\big] + n^\mu\big[K_{ac}\Phi^{ac}\big]\,, \tag{A4.19}$$

$$n^\alpha\nabla_\alpha\big(e^\mu_a\Lambda^a\big) = e^\mu{}_a\big[e_\rho{}^a n^\alpha\nabla_\alpha\big(e^\rho{}_i\Lambda^i\big)\big] + n^\mu\big[a_a\Lambda^a\big]\,. \tag{A4.20}$$

and finally with no index

$$\nabla_\alpha\big(e^\alpha{}_c\Lambda^c\big) = D_c\Lambda^c + a_c\Lambda^c\,. \tag{A4.21}$$

Replacing the previous result in the expression for the second derivative we obtain

$$\begin{aligned}\nabla_\alpha\nabla_\beta t^{\mu\alpha\nu\beta} &= e^\mu{}_a e^\nu{}_b\big[D_c\Theta^{acb} + a_c\Theta^{acb} + e_\rho{}^a e_\sigma{}^b n^\alpha\nabla_\alpha\big(e^\rho_c e^\sigma_d\Sigma^{cd}\big) - K^a{}_c\Sigma^{cb} \\ &+ K\Sigma^{ab} - K^b{}_c\Phi^{ac} - a^b\Lambda^a\big] - n^\mu e^\nu{}_b\big[D_c\Sigma^{cb} - K_{ac}\Theta^{acb} - K^b{}_a\Lambda^a\big] \\ &- e^\mu{}_a n^\nu\big[D_b\Phi^{ab} - K_{bc}\Theta^{acb} - a_b\Sigma^{ab} + a_b\Phi^{ab} - K^a_b\Lambda^b + K\Lambda^a \\ &+ e_\rho{}^a n^\alpha\nabla_\alpha\big(e^\rho{}_c\Lambda^c\big)\big] + n^\mu n^\nu\big[D_a\Lambda^a - K_{ab}\Sigma^{ab} - K_{ab}\Phi^{ab}\big]\,. \end{aligned} \tag{A4.22}$$

Using the definitions of $\Theta^{acb}, \Sigma^{cb}, \Phi^{ac}$ and Λ^c we obtain an explicit expression for the projection of the double covariant divergence of t

$$\nabla_\alpha \nabla_\beta t^{\mu\alpha\nu\beta} = e^\mu{}_a e^\nu{}_b \mathcal{S}^{ab} - e^\mu{}_a n^\nu \mathcal{S}^{a\mathbf{n}} - n^\mu e^\nu{}_b \mathcal{S}^{\mathbf{n}b} + n^\mu n^\nu \mathcal{S}^{\mathbf{nn}}\,, \tag{A4.23}$$

where

$$\begin{aligned}
\mathcal{S}^{ab} = {} & D_c\Big(D_d t^{acbd} - K^a{}_d t^{\mathbf{n}cbd} - K^c{}_d t^{a\mathbf{n}bd} - K^b{}_d t^{ac\mathbf{n}d} - K t^{acb\mathbf{n}} \\
& - e_\mu{}^a e_\alpha{}^c e_\nu{}^b n^\beta \nabla_\beta (e^\mu{}_i e^\alpha{}_k e^\nu{}_j t^{ikj\mathbf{n}}) + a_d t^{acbd} + a^a t^{\mathbf{n}cb\mathbf{n}} + a^c t^{a\mathbf{n}b\mathbf{n}}\Big) \\
& + a_c\Big(D_d t^{acbd} - K^a{}_d t^{\mathbf{n}cbd} - K^c{}_d t^{a\mathbf{n}bd} - K^b{}_d t^{ac\mathbf{n}d} - K t^{acb\mathbf{n}} \\
& - e_\mu{}^a e_\alpha{}^c e_\nu{}^b n^\beta \nabla_\beta (e^\mu{}_i e^\alpha{}_k e^\nu{}_j t^{ikj\mathbf{n}}) + a_d t^{acbd} + a^a t^{\mathbf{n}cb\mathbf{n}} + a^c t^{a\mathbf{n}b\mathbf{n}}\Big) \\
& - e_\mu{}^a e_\nu{}^b n^\alpha \nabla_\alpha \Big(e^\mu{}_i e^\nu{}_j \big(D_l t^{i\mathbf{n}jl} - K_{kl} t^{ikjl} - K^j{}_l t^{i\mathbf{nn}l} - K t^{i\mathbf{n}j\mathbf{n}} \\
& - e_\rho{}^i e_\sigma{}^j n^\beta \nabla_\beta (e^\rho{}_c e^\sigma{}_d t^{c\mathbf{n}d\mathbf{n}}) + a_l t^{i\mathbf{n}jl} + a_k t^{ikj\mathbf{n}}\big)\Big) - K^a{}_c \Big(D_d t^{\mathbf{n}cbd} - K_{ed} t^{ecbd} \\
& - K^b{}_d t^{\mathbf{n}c\mathbf{n}d} - K t^{\mathbf{n}cb\mathbf{n}} - e_\alpha{}^c e_\nu{}^b n^\beta \nabla_\beta (e^\alpha{}_k e^\nu{}_j t^{\mathbf{n}kj\mathbf{n}}) + a_d t^{\mathbf{n}cbd} + a_a t^{acb\mathbf{n}}\Big) \\
& - K\Big(D_d t^{a\mathbf{n}bd} - K_{cd} t^{acbd} - K^b{}_d t^{a\mathbf{nn}d} - K t^{a\mathbf{n}b\mathbf{n}} - e_\alpha{}^a e_\nu{}^b n^\beta \nabla_\beta (e^\alpha{}_i e^\nu{}_j t^{i\mathbf{n}j\mathbf{n}}) \\
& + a_d t^{a\mathbf{n}bd} + a_c t^{acb\mathbf{n}}\Big) - K^b{}_c \Big(D_d t^{ac\mathbf{n}d} - K^a{}_d t^{\mathbf{n}c\mathbf{n}d} - K^c{}_d t^{a\mathbf{nn}d}\Big) \\
& + a^b\Big(D_d t^{\mathbf{n}ad\mathbf{n}} - K_{cd} t^{cad\mathbf{n}}\Big)\,,
\end{aligned} \tag{A4.24}$$

$$\begin{aligned}
\mathcal{S}^{a\mathbf{n}} = {} & D_c\Big(D_d t^{ac\mathbf{n}d} - K^a_d t^{\mathbf{n}c\mathbf{n}d} - K^c_d t^{a\mathbf{nn}d}\Big) - K_{bc}\Big(D_d t^{acbd} - K^a_d t^{\mathbf{n}cbd} - K^c_d t^{a\mathbf{n}bd} \\
& - K^b_d t^{ac\mathbf{n}d} - K t^{acb\mathbf{n}} - e_\mu{}^a e_\alpha{}^c e_\nu{}^b n^\beta \nabla_\beta (e^\mu{}_i e^\alpha{}_k e^\nu{}_j t^{ikj\mathbf{n}}) \\
& + a_d t^{acbd} + a^a t^{\mathbf{n}cb\mathbf{n}} + a^c t^{a\mathbf{n}b\mathbf{n}}\Big) \\
& - K^a_c\big(D_d t^{\mathbf{n}c\mathbf{n}d} - K_{id} t^{ic\mathbf{n}d}\big) - K\big(D_d t^{a\mathbf{nn}d} - K_{cd} t^{ac\mathbf{n}d}\big) \\
& + a_b\Big(D_d t^{a\mathbf{n}bd} - K_{cd} t^{acbd} - K^b_d t^{a\mathbf{nn}d} - K t^{a\mathbf{n}b\mathbf{n}} - e_\mu{}^a e_\nu{}^b n^\alpha \nabla_\alpha (e^\mu_i e^\nu_j t^{i\mathbf{n}j\mathbf{n}}) \\
& + a_d t^{a\mathbf{n}bd} + a_c t^{acb\mathbf{n}}\Big) + a_c\big(D_d t^{ac\mathbf{n}d} - K^a_d t^{\mathbf{n}c\mathbf{n}d} - K^c_d t^{a\mathbf{nn}d}\big) \\
& - e_\mu{}^a n^\alpha \nabla_\alpha \Big(e^\mu{}_i \big(D_d t^{i\mathbf{nn}d} - K_{bd} t^{i\mathbf{n}bd} - K_{cd} t^{ic\mathbf{n}d}\big)\Big)\,,
\end{aligned} \tag{A4.25}$$

$$\begin{aligned}
\mathcal{S}^{\mathbf{n}b} = {} & D_c\Big(D_d t^{\mathbf{n}cbd} - K_{ad} t^{acbd} - K^b_d t^{\mathbf{n}c\mathbf{n}d} - K t^{\mathbf{n}cb\mathbf{n}} - e_\alpha{}^c e_\nu{}^b n^\beta \nabla_\beta (e^\alpha{}_k e^\nu{}_j t^{\mathbf{n}kj\mathbf{n}}) \\
& + a_d t^{\mathbf{n}cbd} + a_a t^{acb\mathbf{n}}\Big) - K^b_c\big(D_d t^{\mathbf{n}c\mathbf{n}d} - K_{ad} t^{ac\mathbf{n}d}\big)\,,
\end{aligned} \tag{A4.26}$$

$$\mathcal{S}^{\mathbf{nn}} = D_c\Big(D_d t^{\mathbf{nc}\mathbf{n}d} - K_{ad}t^{ac\mathbf{n}d}\Big) - K_{cb}\Big(D_d t^{\mathbf{n}cbd} - K_{ad}t^{acbd} - K^b_d t^{\mathbf{nc}\mathbf{n}d} - K t^{\mathbf{n}cb\mathbf{n}}$$
$$- e_\alpha{}^c e_\nu{}^b n^\beta \nabla_\beta(e^\alpha{}_k e^\nu{}_j t^{\mathbf{n}kj\mathbf{n}}) + a_d t^{\mathbf{n}cbd} + a_a t^{acb\mathbf{n}}\Big)\,. \quad \text{(A4.27)}$$

with this result we are equipped to construct the energy-momentum tensor for the t sector easily.

A5 Projections in the bumblebee model

In this section we decompose the second covariant derivative of the bumblebee field by providing a general expression. We note that the key projection to compute is the second covariant derivative of a product of two bumblebee fields with all indices free. Below, we compute this term, from which all other terms involving second covariant derivatives can be derived.

We start decomposing the bumblebee field in their normal and tangential projections

$$B^\alpha = e^\alpha{}_a B^a - n^\alpha B^{\mathbf{n}}\,. \quad \text{(A5.1)}$$

With this result the product of B^α fields have the following projections

$$B^\alpha B^\beta = e^\alpha{}_a e^\beta{}_b B^a B^b - n^\alpha e^\beta{}_b B^{\mathbf{n}} B^b - e^\alpha{}_a n^\beta B^a B^{\mathbf{n}} + n^\alpha n^\beta \big(B^{\mathbf{n}}\big)^2\,. \quad \text{(A5.2)}$$

Taking the first covariant derivative of the product

$$\nabla_\nu\big(B^\alpha B^\beta\big) = \nabla_\nu\big(e^\alpha{}_a e^\beta{}_b B^a B^b\big) - e^\alpha{}_a e^\beta{}_b\big(e_\nu{}^d K^a{}_d - n_\nu a^a\big)B^{\mathbf{n}} B^b - n^\alpha \nabla_\nu\big(e^\beta{}_b B^{\mathbf{n}} B^b\big)$$
$$- e^\alpha{}_a e^\beta{}_b\big(e_\nu{}^d K^b{}_d - n_\nu a^b\big)B^a B^{\mathbf{n}} - n^\beta \nabla_\nu\big(e^\alpha{}_a B^a B^{\mathbf{n}}\big) + e^\alpha{}_a n^\beta\big(e_\nu{}^d K^a{}_d - n_\nu a^a\big)\big(B^{\mathbf{n}}\big)^2$$
$$+ n^\alpha e^\beta{}_b\big(e_\nu{}^d K^b{}_d - n_\nu a^b\big)\big(B^{\mathbf{n}}\big)^2 + n^\alpha n^\beta \nabla_\nu\Big(\big(B^{\mathbf{n}}\big)^2\Big)\,. \quad \text{(A5.3)}$$

Introducing Kronecker deltas and identifying the geometric quantities we obtain the following projections: The term with three free indexes become

$$\nabla_\nu\big(e^\alpha{}_a e^\beta{}_b B^a B^b\big) = e^\alpha{}_a e^\beta{}_b e_\nu{}^d D_d\big(B^a B^b\big) + e^\alpha{}_a n^\beta e_\nu{}^d K_{db} B^a B^b + n^\alpha e^\beta{}_b e_\nu{}^d K_{da} B^a B^b$$
$$- e^\alpha{}_a e^\beta{}_b n_\nu\Big(e_\gamma{}^a e_\delta{}^b n^\lambda \nabla_\lambda\big(e^\gamma{}_c e^\delta{}_d B^c B^d\big)\Big) - e^\alpha{}_a n^\beta n_\nu a_b B^a B^b$$
$$- n^\alpha e^\beta{}_b n_\nu a_a B^a B^b\,, \quad \text{(A5.4)}$$

the terms with two free indexes become

$$\nabla_\nu\big(e^\beta{}_b B^{\mathbf{n}} B^b\big) = e^\beta{}_b e_\nu{}^d D_d\big(B^{\mathbf{n}} B^b\big) + n^\beta e_\nu{}^d K_{db}\big(B^{\mathbf{n}} B^b\big) - e^\beta{}_b n_\nu \Big(e_\delta{}^b n^\lambda \nabla_\lambda\big(e^\delta{}_d B^{\mathbf{n}} B^d\big)\Big) - n^\beta n_\nu a_b B^{\mathbf{n}} B^b\,, \tag{A5.5}$$

and for the term with one free index we have

$$\nabla_\nu\big(B^{\mathbf{n}}\big)^2 = e_\nu{}^d D_d\big(B^{\mathbf{n}}\big)^2 - n_\nu n^\lambda \nabla_\lambda\big(B^{\mathbf{n}}\big)^2\,. \tag{A5.6}$$

With this results we obtain the projection for the first covariant derivative

$$\begin{aligned}\nabla_\nu\big(B^\alpha B^\beta\big) &= e^\alpha{}_a e^\beta{}_b e_\nu{}^d\Big[D_d\big(B^a B^b\big) - K^a{}_d B^{\mathbf{n}} B^b - K^b{}_d B^a B^{\mathbf{n}}\Big] - e^\alpha{}_a n^\beta e_\nu{}^d\Big[D_d\big(B^a B^{\mathbf{n}}\big)\\
&- K_{db} B^a B^b - K^a{}_d\big(B^{\mathbf{n}}\big)^2\Big] - n^\alpha e^\beta{}_b e_\nu{}^d\Big[D_d\big(B^{\mathbf{n}} B^b\big) - K_{da} B^a B^b - K^b{}_d\big(B^{\mathbf{n}}\big)^2\Big]\\
&- e^\alpha{}_a e^\beta{}_b n_\nu\Big[e_\gamma{}^a e_\delta{}^b n^\lambda \nabla_\lambda\big(e^\gamma{}_c e^\delta{}_d B^c B^d\big) - a^a B^{\mathbf{n}} B^b - a^b B^a B^{\mathbf{n}}\Big]\\
&+ e^\alpha{}_a n^\beta n_\nu\Big[e_\gamma{}^a n^\lambda \nabla_\lambda\big(e^\gamma{}_c B^c B^{\mathbf{n}}\big) - a_b B^a B^b - a^a\big(B^{\mathbf{n}}\big)^2\Big]\\
&+ n^\alpha e^\beta{}_b n_\nu\Big[e_\delta{}^b n^\lambda \nabla_\lambda\big(e^\delta{}_d B^{\mathbf{n}} B^d\big) - a_a B^a B^b - a^b\big(B^{\mathbf{n}}\big)^2\Big]\\
&+ n^\alpha n^\beta e_\nu{}^d\Big[D_d\Big(\big(B^{\mathbf{n}}\big)^2\Big) - K_{db} B^{\mathbf{n}} B^b - K_{da} B^a B^{\mathbf{n}}\Big]\\
&- n^\alpha n^\beta n_\nu\Big[n^\lambda \nabla_\lambda\Big(\big(B^{\mathbf{n}}\big)^2\Big) - a_b B^{\mathbf{n}} B^b - a_a B^a B^{\mathbf{n}}\Big]\end{aligned} \tag{A5.7}$$

We define the following quantities in order to simplify the second derivative expression

$$\Phi^{ab}{}_d := D_d\big(B^a B^b\big) - K^a{}_d B^{\mathbf{n}} B^b - K^b{}_d B^a B^{\mathbf{n}}\,, \tag{A5.8}$$

$$\Theta^a{}_d := D_d\big(B^a B^{\mathbf{n}}\big) - K_{db} B^a B^b - K^a{}_d\big(B^{\mathbf{n}}\big)^2\,, \tag{A5.9}$$

$$\Psi^{ab} := e_\gamma{}^a e_\delta{}^b n^\lambda \nabla_\lambda\big(e^\gamma{}_c e^\delta{}_d B^c B^d\big) - a^a B^{\mathbf{n}} B^b - a^b B^a B^{\mathbf{n}}\,, \tag{A5.10}$$

$$\Pi^a := e_\gamma{}^a n^\lambda \nabla_\lambda\big(e^\gamma{}_c B^c B^{\mathbf{n}}\big) - a_b B^a B^b - a^a\big(B^{\mathbf{n}}\big)^2\,, \tag{A5.11}$$

$$\Sigma_d := D_d\Big(\big(B^{\mathbf{n}}\big)^2\Big) - K_{db} B^{\mathbf{n}} B^b - K_{da} B^a B^{\mathbf{n}}\,, \tag{A5.12}$$

$$\Omega := n^\lambda \nabla_\lambda\Big(\big(B^{\mathbf{n}}\big)^2\Big) - a_b B^{\mathbf{n}} B^b - a_a B^a B^{\mathbf{n}}\,, \tag{A5.13}$$

obtaining

$$\begin{aligned}\nabla_\nu\big(B^\alpha B^\beta\big) &= e^\alpha{}_a e^\beta{}_b e_\nu{}^d \Phi^{ab}{}_d - e^\alpha{}_a n^\beta e_\nu{}^d \Theta^a{}_d - n^\alpha e^\beta{}_b e_\nu{}^d \Theta^b{}_d \\ &- e^\alpha{}_a e^\beta{}_b n_\nu \Psi^{ab} + e^\alpha{}_a n^\beta n_\nu \Pi^a + n^\alpha e^\beta{}_b n_\nu \Pi^b \\ &+ n^\alpha n^\beta e_\nu{}^d \Sigma_d - n^\alpha n^\beta n_\nu \Omega\,. \end{aligned} \tag{A5.14}$$

We continue taking the second covariant derivative

$$\begin{aligned}\nabla_\mu\nabla_\nu\big(B^\alpha B^\beta\big) &= \nabla_\mu\big(e^\alpha{}_a e^\beta{}_b e_\nu{}^d \Phi^{ab}{}_d\big) - \big(e_\mu{}^c K^b{}_c - a^b n_\mu\big) e^\alpha{}_a e^\beta{}_b e_\nu{}^d \Theta^a{}_d \\ &- n^\beta \nabla_\mu\big(e^\alpha{}_a e_\nu{}^d \Theta^a{}_d\big) - \big(e_\mu{}^c K^a{}_c - a^a n_\mu\big) e^\alpha{}_a e^\beta{}_b e_\nu{}^d \Theta^b{}_d \\ &- n^\alpha \nabla_\mu\big(e^\beta{}_b e_\nu{}^d \Theta^b{}_d\big) - \big(e_\mu{}^c K_{cd} - a_d n_\mu\big) e^\alpha{}_a e^\beta{}_b e_\nu{}^d \Psi^{ab} - n_\nu \nabla_\mu\big(e^\alpha{}_a e^\beta{}_b \Psi^{ab}\big) \\ &+ \Big[e^\beta{}_b\big(e_\mu{}^c K^b{}_c - a^b n_\mu\big) n_\nu + n^\beta e_\nu{}^d\big(e_\mu{}^c K_{cd} - a_d n_\mu\big)\Big] e^\alpha{}_a \Pi^a + n^\beta n_\nu \nabla_\mu\big(e^\alpha{}_a \Pi^a\big) \\ &+ \Big[e^\alpha{}_a\big(e_\mu{}^c K^a{}_c - a^a n_\mu\big) n_\nu + n^\alpha e_\nu{}^d\big(e_\mu{}^c K_{cd} - a_d n_\mu\big)\Big] e^\beta{}_b \Pi^b + n^\alpha n_\nu \nabla_\mu\big(e^\beta{}_b \Pi^b\big) \\ &+ \Big[\big(e_\mu{}^c K^a{}_c - a^a n_\mu\big) e^\alpha{}_a n^\beta + n^\alpha e^\beta{}_b\big(e_\mu{}^c K^b{}_c - a^b n_\mu\big)\Big] e_\nu{}^d \Sigma_d + n^\alpha n^\beta \nabla_\mu\big(e_\nu{}^d \Sigma_d\big) \\ &- \Big[e^\alpha{}_a\big(e_\mu{}^c K^a{}_c - a^a n_\mu\big) n^\beta n_\nu + n^\alpha e^\beta{}_b\big(e_\mu{}^c K^b{}_c - a^b n_\mu\big) n_\nu\Big]\Omega \\ &- n^\alpha n^\beta e_\nu{}^d\big(e_\mu{}^c K_{cd} - a_d n_\mu\big)\Omega - n^\alpha n^\beta n_\nu \nabla_\mu \Omega\,. \end{aligned} \tag{A5.15}$$

As usual we introduce the deltas in the non-explicit terms and we recognize the geometrical quantities obtaining for the term with four free indexes

$$\begin{aligned}\nabla_\mu\big(e^\alpha{}_a e^\beta{}_b e_\nu{}^d \Phi^{ab}{}_d\big) &= e_\mu{}^c e^\alpha{}_a e^\beta{}_b e_\nu{}^d D_c \Phi^{ab}{}_d - n_\mu e^\alpha{}_a e^\beta{}_b e_\nu{}^d \Big(e_\gamma{}^a e_\delta{}^b e^\lambda{}_d n^\kappa \nabla_\kappa\big(e^\gamma{}_c e^\delta{}_e e_\lambda{}^l \Phi^{ce}{}_l\big)\Big) \\ &+ e_\mu{}^c n^\alpha e^\beta{}_b e_\nu{}^d K_{ca} \Phi^{ab}{}_d + e_\mu{}^c e^\alpha{}_a n^\beta e_\nu{}^d K_{cb} \Phi^{ab}{}_d + e_\mu{}^c e^\alpha{}_a e^\beta{}_b n_\nu K^d{}_c \Phi^{ab}{}_d \\ &- n_\mu n^\alpha e^\beta{}_b e_\nu{}^d a_a \Phi^{ab}{}_d - n_\mu e^\alpha{}_a n^\beta e_\nu{}^d a_b \Phi^{ab}{}_d - n_\mu e^\alpha{}_a e^\beta{}_b n_\nu a^d \Phi^{ab}{}_d\,, \end{aligned} \tag{A5.16}$$

for the term with three indexes

$$\begin{aligned}\nabla_\mu\big(e^\alpha{}_a e_\nu{}^d \Theta^a{}_d\big) &= e_\mu{}^c e_\nu{}^d e^\alpha{}_a D_c \Theta^a{}_d - n_\mu e_\nu{}^d e^\alpha{}_a \Big(e^\lambda{}_d e_\gamma{}^a n^\kappa \nabla_\kappa\big(e^\gamma{}_c e_\lambda{}^l \Theta^c{}_l\big)\Big) \qquad \text{(A5.17)} \\ &+ e_\mu{}^c n_\nu e^\alpha{}_a K^d{}_c \Theta^a{}_d + e_\mu{}^c e_\nu{}^d n^\alpha K_{ca} \Theta^a{}_d - n_\mu n_\nu e^\alpha{}_a a^d \Theta^a{}_d \\ &- n_\mu e_\nu{}^d n^\alpha a_a \Theta^a{}_d\,, \end{aligned}$$

for the term with two free indexes

$$\nabla_\mu\big(e^\alpha{}_a e^\beta{}_b \Psi^{ab}\big) = e_\mu{}^c e^\alpha{}_a e^\beta{}_b D_c \Psi^{ab} - n_\mu e^\alpha{}_a e^\beta{}_b \Big(e_\gamma{}^a e_\delta{}^b n^\kappa \nabla_\kappa \big(e^\gamma{}_c e^\delta{}_d \Psi^{cd}\big)\Big) + e_\mu{}^c n^\alpha e^\beta{}_b K_{ca} \Psi^{ab} + e_\mu{}^c e^\alpha{}_a n^\beta K_{cb} \Psi^{ab} - n_\mu e^\alpha{}_a n^\beta a_b \Psi^{ab} - n_\mu n^\alpha e^\beta{}_b a_a \Psi^{ab}\,, \tag{A5.18}$$

for the term with one free index

$$\nabla_\mu\big(e^\alpha{}_a \Pi^a\big) = e_\mu{}^c e^\alpha{}_a D_c \Pi^a - n_\mu e^\alpha{}_a e_\gamma{}^a n^\kappa \nabla_\kappa \big(e^\gamma{}_c \Pi^c\big) + e_\mu{}^c n^\alpha K_{ca} \Pi^a - n_\mu n^\alpha a_a \Pi^a\,, \tag{A5.19}$$

$$\nabla_\mu\big(e_\nu{}^d \Sigma_d\big) = e_\mu{}^c e_\nu{}^d D_c \Sigma_d - n_\mu e_\nu{}^d n^\kappa e^\lambda{}_d \nabla_\kappa \big(e_\lambda{}^l \Sigma_l\big) + e_\mu{}^c n_\nu K^d{}_c \Sigma_d - n_\mu n_\nu a^d \Sigma_d\,, \tag{A5.20}$$

and lastly for the term with no free indexes

$$\nabla_\mu \Omega = \widetilde{E}^c_\mu D_c \Omega - n_\mu n^\kappa \nabla_\kappa \Omega\,. \tag{A5.21}$$

Replacing we obtain a general expression for the projections of the second covariant derivatives of a product of bumblebee fields

$$
\begin{aligned}
\nabla_\mu \nabla_\nu \big(B^\alpha B^\beta\big) \;=\;& e_\mu{}^c e^\alpha{}_a e^\beta{}_b e_\nu{}^d \Big[D_c \Phi^{ab}{}_d - K^b{}_c \Theta^a{}_d - K^a{}_c \Theta^b{}_d - K_{cd}\Psi^{ab}\Big] \\
& - n_\mu e^\alpha{}_a e^\beta{}_b e_\nu{}^d \Big[e_\gamma{}^a e_\delta{}^b e^\lambda{}_d n^\kappa \nabla_\kappa\big(e^\gamma{}_c e^\delta{}_e e_\lambda{}^l \Phi^{ce}{}_l\big) - a^b \Theta^a{}_d - a^a \Theta^b{}_d - a_d \Psi^{ab}\Big] \\
& - e_\mu{}^c n^\alpha e^\beta{}_b e_\nu{}^d \Big[D_c \Theta^b{}_d - K_{ca}\Phi^{ab}{}_d - K_{cd}\Pi^b - K^b{}_c \Sigma_d\Big] \\
& - e_\mu{}^c e^\alpha{}_a n^\beta e_\nu{}^d \Big[D_c \Theta^a{}_d - K_{cb}\Phi^{ab}{}_d - K_{cd}\Pi^a - K^a{}_c \Sigma_d\Big] \\
& - e_\mu{}^c e^\alpha{}_a e^\beta{}_b n_\nu \Big[D_c \Psi^{ab} - K^d{}_c \Phi^{ab}{}_d - K^b{}_c \Pi^a - K^a{}_c \Pi^b\Big] \\
& + n_\mu n^\alpha e^\beta{}_b e_\nu{}^d \Big[e^\lambda{}_d e_\delta{}^b n^\kappa \nabla_\kappa\big(e^\delta{}_e e_\lambda{}^l \Theta^e{}_l\big) - a_a \Phi^{ab}{}_d - a_d \Pi^b - a^b \Sigma_d\Big] \\
& + n_\mu e^\alpha{}_a n^\beta e_\nu{}^d \Big[e^\lambda{}_d e_\gamma{}^a n^\kappa \nabla_\kappa\big(e^\gamma{}_c e_\lambda{}^l \Theta^c{}_l\big) - a_b \Phi^{ab}{}_d - a_d \Pi^a - a^a \Sigma_d\Big] \\
& + n_\mu e^\alpha{}_a e^\beta{}_b n_\nu \Big[e_\gamma{}^a e_\delta{}^b n^\kappa \nabla_\kappa\big(e^\gamma{}_c e^\delta{}_d \Psi^{cd}\big) - a^d \Phi^{ab}{}_d - a^b \Pi^a - a^a \Pi^b\Big] \\
& + e_\mu{}^c n^\alpha n^\beta e_\nu{}^d \Big[D_c \Sigma_d - K_{ca}\Theta^a{}_d - K_{cb}\Theta^b{}_d - K_{mn}\Omega\Big] \\
& + e_\mu{}^c n^\alpha e^\beta{}_b n_\nu \Big[D_c \Pi^b - K^d{}_c \Theta^b{}_d - K_{ca}\Psi^{ab} - K^b{}_c \Omega\Big] \\
& + e_\mu{}^c e^\alpha{}_a n^\beta n_\nu \Big[D_c \Pi^a - K^d{}_c \Theta^a{}_d - K_{cb}\Psi^{ab} - K^a{}_c \Omega\Big] \\
& - n_\mu n^\alpha n^\beta e_\nu{}^d \Big[n^\kappa e^\lambda{}_d \nabla_\kappa\big(e_\lambda{}^l \Sigma_l\big) - a_a \Theta^a{}_d - a_b \Theta^b{}_d - a_d \Omega\Big] \\
& - n_\mu n^\alpha e^\beta{}_b n_\nu \Big[e_\delta{}^b n^\kappa \nabla_\kappa\big(e^\delta{}_d \Pi^d\big) - a^d \Theta^b{}_d - a_a \Psi^{ab} - a^b \Omega\Big] \\
& - n_\mu e^\alpha{}_a n^\beta n_\nu \Big[e_\gamma{}^a n^\kappa \nabla_\kappa\big(e^\gamma{}_c \Pi^c\big) - a^d \Theta^a{}_d - a_b \Psi^{ab} - a^a \Omega\Big] \\
& - e_\mu{}^c n^\alpha n^\beta n_\nu \Big[D_c \Omega - K_{ca}\Pi^a - K_{cb}\Pi^b - K^d{}_c \Sigma_d\Big] \\
& + n_\mu n^\alpha n^\beta n_\nu \Big[n^\kappa \nabla_\kappa \Omega - a_a \Pi^a - a_b \Pi^b - a^d \Sigma_d\Big] . \qquad \text{(A5.22)}
\end{aligned}
$$

In this way, we have successfully projected the entire tensor $\nabla_\mu \nabla_\nu (B^\alpha B^\beta)$, which serves as the common core for constructing the bumblebee energy-momentum tensor. Each term involving second covariant derivatives can be obtained through the contractions of this tensor. This calculation considers a general bumblebee field with functional dependence on both time and hypersurface coordinates. For the specific case under study, we have simplified to a tangential bumblebee field function of time. The remaining contractions and those with the Riemann tensor are left for the reader.