UNIVERSIDAD DE CONCEPCIÓN
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QUANTUM BACKREACTIONS IN ADS3 FOR MASSIVE GRAVITY THEORIES

BACKREACTIONS CUÁNTICAS EN ADS3 PARA TEORÍAS DE GRAVEDAD MASIVA

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To me, from 10 years back

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Resumen

En el marco de la Gravedad Semi-Clásica, estamos interesados en estudiar como interactuan fluctuaciones cuánticas con teorías de gravedad masiva, construidas con potencias en la curvatura, para espacios asintóticamente (A)dS$_3$. Como primer paso, analizamos los efectos de una backreaction cuántica en la teoría cuadrática en la curvatura New Massive Gravity, en el punto especial del espacio de parametros donde sus dos vacíos coinciden. Mostraremos que la backreaction induce una corrección logarítmica a la geometría de vacío, pero manteniendo su estructura asintótica. Para el caso de espacios con constante cosmológica efectiva negativa $\Lambda < 0$, esto induce una relajación de las condiciones asintóticas de Brown-Henneaux, resultando en la adición de dos generadores de simetría adicionales a los asociados al álgebra infinito-dimensional de Virasoro. Finalmente, mostraremos que estos resultados pueden ser extendidos a teorías de gravedad con potencias arbitrariamente altas en la curvatura, estudiando también la masa, entropía y temperatura de estas teorías de alto orden generalizadas y su cumplimiento con la primera ley de la termodinámica.

Abstract

In the context of semiclassical gravity, we are interested in studying how quantum fluctuations interact with higher curvature massive gravity theories for asymptotically (A)dS$_3$ spacetimes. As a first step, we analized the effects of the backreaction in the quadratic curvature theory New Massive Gravity, at the special point where its two vacua coincide. We show that the backreaction induces a logarithmic correction to the geometry while keeping its asymptotic behavior. In the case of $\Lambda < 0$ this induces a relaxation of the standard Brown-Henneaux asymptotia, yielding the addition of two symmetry generators to the well known infinite-dimensional Virasoro algebra. Finally, we show that these results can be extended to arbitrarily higher curvature theories, as well as studying their mass, entropy and temperature of these generalized higher order theories, showing that they obey thermodynamics first law.

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Chapter 1

Introduction

Developing a fully consistent theory for Quantum Gravity (QG) is perhaps one of the most known challenges for current high-energy physics. This stems from the fact that, in order to construct a QG theory, both matter and metric fields of the spacetime containing such matter, must be quantum by nature. Even though protocols for quantizing matter fields have been developed, as we will soon see, there is still no fully consistent way to quantize gravity fields.

Given the difficulties of developing a theory of QG, an efficient, more modest approach comes from taking a semiclassical approximation; This consists on keeping all gravitational fields classical, while quantizing every other type of field on the spacetime, and thus avoiding the complications associated with QG while providing a framework to study the interaction of quantum matter with its ambient geometry [3].

However, even though the semiclassical approximation poses a much more approachable set-up for exploring quantum effects interacting with gravity, there are still some caveats to keep in mind. Quantum matter sources are associated with the expectation value of a renormalized stress-energy momentum tensor (RSET) $\langle \hat{T}_{\mu\nu} \rangle$, which implies (i) A full computation for the RSET requires having control over the background spacetime solving the semiclassical equations, which is hardly able to be done analytically, and (ii) The renormalization of the expectation value involves substracting divergent terms from an infinite sum mode, which complexity increase with the spacetime dimension d (see e.g. [4–7]).

The first concern can be managed by taking a perturbative approach, which is usually enough if working with weak matter sources, while the second may be dealt with by lowering the dimension, which has been shown to be a great test ground for computing the RSET of quantum matter in both $1+1$ and $2+1$ dimensions [8]. However, for Semiclassical General Relativity, this latter simplification comes at a price: There are no local, gravitational degrees of freedom in the form of a graviton, for $d < 4$ dimensions. Therefore, if one is interested in studying the dynamics of a propagating graviton field for lower dimensions, one is forced to go beyond General Relativity.

In this work, we task ourselves with exploring how the aforementioned quantum matter sources interact with the dynamics of a graviton for three-dimensional gravity theories. Specifically, we chose to study how these quantum effects interact with gravity theories admitting the propagation of massive graviton fields. These usually arise from higher curvature gravity theories, which is the case for the quadratic theory New Massive Gravity (NMG) [9, 10]. We will consider NMG as our $(2+1)$-dimensional toy-model for studying the interplay between these quantum fluctuations and the higher-curvature terms, aiming to answer the following questions:

- How are black hole solutions deformed by the presence of both quantum fluctuations and higher-curvature terms steming from the theory?
- Do the quantum fluctuations modify the asymptotia proper of the higher-curvature theory?, and if so, how does this change in asymptotia compare to the standard Brown-Henneaux conditions for asymptotically (A)dS$_3$ spaces?
- What is the asymptotic isometry algebra generating the group of transformation which preserve our quantum corrected space's asymptotia?

After looking into these questions, we will focus on extending the analysis to higher-curvature, three-dimensional gravity theories. This will allow us to generalize our result for a complete family of three-dimensional, massive gravity theories, whose general properties will be also studied separately of the quantum fluctuations.

Chapter 2

General Relativity

2.1 Action principle

General Relativity (GR) proposes that the geometry dynamics for a gravitationally curved spacetime will be governed by the Einstein-Hilbert action, namely

$$I = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} \, (R - 2\lambda) + I_{GH} + I_{CT}. \tag{2.1.1}$$

Here, the gravitational coupling constant is set as $\kappa = \sqrt{8\pi G}$ while λ represents the dynamic contributions of a cosmological constant. The counterterms I_{GH} and I_{CT} are added to have well-defined boundary conditions and finite conserved quantities.

The field equations coming from (2.1.1) are known as Einstein's Field Equations

$$G_{\mu\nu} + \lambda g_{\mu\nu} = 0, \tag{2.1.2}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein Tensor.

Maximally symmetric solutions can be described by the following metric ansatz

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2_{D-2}, \tag{2.1.3}$$

where $d\Omega_{D-2}$ is the line element of a $(D-2)$-sphere. These are defined by having the maximum possible number of isometries, namely $D(D+1)/2$, and in consequence having constant curvature $R^{\alpha\beta}_{\mu\nu} = \Lambda\delta^{\alpha\beta}_{\mu\nu} \equiv \frac{\sigma}{L_*^2}\delta^{\alpha\beta}_{\mu\nu}$. The metrics for different values of the curvature are obtained as follows:

1. **Flat spacetime:** $f(r) = g(r) = 1$
   - $\sigma := 0 \Leftrightarrow R^{\alpha\beta}_{\mu\nu} = 0$.
2. **Anti-de Sitter spacetime (AdS):** $f(r) = g(r) = 1 + \frac{r^2}{L_*^2}$
   - $\sigma := -1 \Leftrightarrow R^{\alpha\beta}_{\mu\nu} = -\frac{1}{L_*^2}\delta^{\alpha\beta}_{\mu\nu}$.
3. **De Sitter spacetime (dS):** $f(r) = g(r) = 1 - \frac{r^2}{L_*^2}$
   - $\sigma := 1 \Leftrightarrow R^{\alpha\beta}_{\mu\nu} = \frac{1}{L_*^2}\delta^{\alpha\beta}_{\mu\nu}$.

Here, $\Lambda = \frac{\sigma}{L_*^2}$ takes the role of an effective cosmological constant. Redefining this constant in terms of an effective (A)dS radii $L_*^2 > 0$ will be useful later on. Inserting these solutions onto the Einstein equations, we get a relation between the effective constants relate and the bared cosmological constant λ, which reads

$$\lambda = \Lambda\frac{(D-1)(D-2)}{2} = \sigma\frac{(D-1)(D-2)}{2L_*^2}. \tag{2.1.4}$$

Maximally symmetric solutions are understood to be vacuum solutions, and thus, they are deformed in the presence of massive objects such as black holes. For instance, the Schwarzschild-AdS solution represents the deformation of AdS by a static black hole, while the Kerr-AdS solution yields the deformation of AdS by a rotating black hole.

2.2 Fierz Pauli Action

Curved geometry fields governed by (2.1.1), at a weak field regime, can be understood as coming from a propagating graviton field on constant curvature background. For simplicity, let us first focus on the case $\lambda = 0$ Thus, for a given spacetime $g_{\mu\nu}$, we can separate the background from the graviton perturbation as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\bar{h}_{\mu\nu} = \eta_{\mu\nu} + \kappa\left(h^{[1]}_{\mu\nu} + \kappa h^{[2]}_{\mu\nu} + \kappa^2 h^{[3]}_{\mu\nu} + \mathcal{O}\left(\kappa^3\right)\right). \tag{2.2.1}$$

Here, $\eta_{\mu\nu}$ is the flat, Minkowski metric. Redefining $h^{[1]}_{\mu\nu} \to h_{\mu\nu}$ and inserting the expansion on (2.1.1), we get

$$I = \frac{1}{2\kappa^2}\int d^D x \sqrt{-\eta_{\mu\nu}}\left[\kappa G^{[0]}_{\mu\nu}h^{\mu\nu} + \kappa^2\left(-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}\right.\right. \tag{2.2.2}$$

$$\left.\left.+\frac{1}{2}\partial_\lambda h\partial^\lambda h - \partial_\mu h\partial_\nu h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} + G^{[0]}_{\mu\nu}h^{[2]\mu\nu}\right) + \mathcal{O}\left(\kappa^3\right)\right], \tag{2.2.3}$$

where we have disregarded the boundary terms. We have used the notation $G^{[0]}_{\mu\nu}$ to represent the Einstein tensor evaluated on the constant curvature background. Since $\eta_{\mu\nu}$ is flat, we can immediately assume that $G^{[0]}_{\mu\nu} = 0$, and $\sqrt{-\det\eta} = 1$ in cartesian coordinates, leaving us with a leading second order contribution in the expansion

$$I_{FP} = \frac{1}{2}\int d^D x\left(-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \frac{1}{2}\partial_\lambda h\partial^\lambda h - \partial_\mu h\partial_\nu h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda}\right). \tag{2.2.4}$$

This is the action for a free, spin 2 graviton field $h_{\mu\nu}$, dubbed as Fierz-Pauli Action. Therefore, gravity can be understood through the dynamics of a graviton field $h_{\mu\nu}$ propagating on a flat background, given by

$$\Box h_{\mu\nu} = 0, \qquad \nabla^\mu h_{\mu\nu} = 0, \qquad h^\mu_\mu = 0. \tag{2.2.5}$$

where $\Box \equiv \eta_{\mu\nu}\nabla_\mu\nabla_\nu$ is the Laplace-Beltrami operator in the metric $g_{\mu\nu}$. These arise from both the field equations coming from the action while also taking into account the diffeomorphisms invariance of (2.2.4) on the field $h_{\mu\nu}$. Thus, we can count the number of free components as

$$\textbf{Physical, on shell, DOF} = \frac{D(D+1)}{2} - (\textbf{Constraints from: } \nabla^\mu h_{\mu\nu} = 0) \tag{2.2.6}$$

$$- (\textbf{Constraints from gauxe fixing } h_{\mu\nu}).$$

Taking into account that $h_{\mu\nu}$ is a symmetric tensor of $D(D+1)/2$ components, with D constraints from a particular choice of gauge and another D constraints coming from the field equations (see A), we find the number of physical degrees of freedom of our field as

$$\textbf{Physical, on shell, DOF} = \frac{D(D+1)}{2} - D - D = \frac{D(D-3)}{2}. \tag{2.2.7}$$

For $D = 4$, graviton fields will have 2 degrees of freedom, which we interpret as two transverse polarization states of helicity ± 2.

One quickly sees that for $D < 4$, the field $h_{\mu\nu}$ has no physical degrees of freedom. This means that, in absence of matter fields, pure GR has no gravitons. In other words, $2+1$-dimensional GR propagates no local degrees of freedom.

2.3 Matter fields

The homogeneous Einstein equations (2.1.2) describe the dynamics of spacetime in vacuum. Now, if one were to study how spacetime behaves on the presence of matter sources, the action gets modified as

$$I = I_G + I_M. \tag{2.3.1}$$

Here, $I_G = I_G\left[g_{\mu\nu}\right]$ is the purely gravitational action, which for GR would be Einstein-Hilbert, while $I_M = I_M[g_{\mu\nu}, \Psi]$ contains the dynamics of a particular matter field Ψ and their coupling to the geometry given by $g_{\mu\nu}$.

Varying the matter action respect to Ψ we obtain the field equation for the coupled matter

$$\frac{\delta I_M}{\delta \Psi} = 0. \tag{2.3.2}$$

On the other hand, varying it respect to the metric field $g_{\mu\nu}$ yields

$$T_{\mu\nu}\left[g_{\mu\nu}, \Psi\right] = \frac{2}{\sqrt{-g}} \frac{\delta I_M}{\delta g_{\mu\nu}}. \tag{2.3.3}$$

which we interpret as an stress energy momentum tensor (SET), acting as a matter source on the vacuum field equations described by I_G. For example, adding a matter action to General Relativity with a cosmological constant (2.1.1) results in

$$\frac{\delta\left(I_G + I_M\right)}{\delta g_{\mu\nu}} = 0 \qquad \rightarrow \qquad G_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} = \kappa^2 T_{\mu\nu}. \tag{2.3.4}$$

The general solution of these equations is in most cases too hard to solve in a closed manner, with a few exceptions. For example, restricting to highly symmetric set-ups or lowering the spacetime dimension. This is the case for the static, circularly symmetric, asymptotically AdS3 and non-perturbative solution for a conformally coupled scalar field found by Martinez and Zanelli on (1996) [11].

A more modest approach involves tackling the addition of a SET perturbatively. This is known as the perturbative backreaction problem, which implies computing the SET on a small perturbation of the vacuum geometry, so that

$$g_{\mu\nu} = g^{[0]}_{\mu\nu} + \kappa \delta g_{\mu\nu}, \tag{2.3.5}$$

$$G_{\mu\nu} = G^{[0]}_{\mu\nu} + \kappa G^{[1]}_{\mu\nu} + \mathcal{O}\left(\kappa^2\right), \tag{2.3.6}$$

$$T_{\mu\nu} = T^{[0]}_{\mu\nu} + \kappa T^{[1]}_{\mu\nu} + \mathcal{O}\left(\kappa^2\right), \tag{2.3.7}$$

where $G^{[n]}$ and $T^{[n]}$ are the tensors n-th order corrections on the perturbation. Since $g^{[0]}$ is the vacuum solution, we check that the leading contribution to our perturbed field equation goes as

$$G^{[1]}_{\mu\nu} + \lambda \delta g_{\mu\nu} = \kappa T^{[0]}_{\mu\nu} \tag{2.3.8}$$

Which allows us to find the first metric correction $\delta g_{\mu\nu}$ coming from the addition to a SET on a background geometry $g^{[0]}_{\mu\nu}$. The resulting field $g_{\mu\nu} = g^{[0]}_{\mu\nu} + \kappa \delta g_{\mu\nu}$ is called a backreacted metric field.

The consistency of this approach relies on the metric perturbation being small respect to the background space. If not, the higher order corrections would need to be considered up to an arbitrary order. Naturally, that would imply losing the calculation advantages of sticking to a perturbative method. For the rest of this thesis we will be dealing with matter sources in a perturbative manner, allowing us to find the backreaction corrections in a consistent manner.

2.3.1 The Conformally Coupled Scalar Field

Let us review a particular matter source, coming from a massless scalar field Φ, conformally coupled to a spacetime background with cosmological constant λ.

The action reads as

$$I = I_G + I_C, \tag{2.3.9}$$

$$I_G = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}\left[R - 2\lambda\right], \tag{2.3.10}$$

$$I_C = -\frac{1}{2}\int d^D x\sqrt{-g}\left(\nabla_\mu\Psi\nabla^\mu\Psi - \xi_D R\Psi^2\right). \tag{2.3.11}$$

where $\xi_D = \frac{1}{4}(D-2)/(D-1)$ ensures that I_C remains invariant under the conformal transformations

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}, \qquad \Psi \to \Omega^{1-\frac{D}{2}}(x)\Psi. \tag{2.3.12}$$

The field equations coming from (2.3.9) are

$$G_{\mu\nu} + \lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{2.3.13}$$

and

$$\Box\Psi - \frac{1}{4}\frac{(D-2)}{(D-1)}R\Psi = 0. \tag{2.3.14}$$

The matter stress tensor is given by

$$T_{\mu\nu} = \nabla_\mu\Psi\nabla_\nu\Psi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\Psi\nabla_\alpha\Psi + \frac{1}{4}\frac{(D-2)}{(D-1)}\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\right)\Psi^2. \tag{2.3.15}$$

As expected from the SET from a conformally coupled scalar field, by inserting the scalar field equation in our SET, we can check that it is traceless $T^\mu_\mu = 0$.

This matter source has plenty of interesting features, such as leading to exact, non-perturbative solutions for the sourced Einstein field equations [11] at a classical level. This is what motivates the focus of this thesis on studying the quantum effects produced by this scalar field.

Chapter 3

Massive Gravity

3.1 Massive Fierz-Pauli Action

In the previous section, we discussed how weak gravity fields can be understood as a massless spin 2 particle (graviton), with ± 2 helicity states, propagating on flat space-time, whose dynamic is encapsulated by the Fierz-Pauli action.

This $4D$ model for propagating gravity fields has been remarkably successful describing macroscopic phenomena, such as Gravitational Waves, the shift of Mercury's perihelium, and other standard tests of GR. Unfortunately, this has not been the case at short-distance scales, since the quantum field theory for interacting gravitons is well-known to be non-renormalizable [12, 13]. This has led many theorists to study lower-dimensional gravity theories, where one would expect to have a better behaved model at short scales.

However, since General Relativity does not propagate local degrees of freedom at $D < 4$ dimensions [14], in order to explore lower dimensional models with propagating gravitons, one must go beyond GR. This is the case for the so called 3-dimensional Massive Gravity theories, which encapsulates the dynamics of a massive graviton through the Massive Fierz-Pauli action

$$
\begin{aligned}
I_{MFP} = \int d^3x \Big(& -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h \\
& + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{m^2}{2}\left(h_{\mu\nu}h^{\mu\nu} - h^2\right)\Big) .
\end{aligned}
\tag{3.1.1}
$$

Variation of this action principle leads to the following field equations and constraints

$$\left(\Box - m^2\right) h_{\mu\nu} = 0, \;\; h = 0, \;\; \partial_\mu h^{\mu\nu} = 0. \tag{3.1.2}$$

This action (3.1.1) may also be formulated in D dimensions, for which we could count their physical degrees of freedom and compare them to those of GR.

Analog to what happens when one considers a massive spin-1 Proca Field in contrast to Maxwell's, the addition of mass breaks the gauge invariance proper to the massless theories, and thus losing the D constraints coming from gauge fixing. Therefore, a symmetric $h_{\mu\nu}$ massive field has D restrictions from null divergence and one more from the traceless condition. Thus, the total count goes as

$$\textbf{Physical DOF} = \frac{D(D+1)}{2} - D - 1 = \frac{(D+1)(D-2)}{2}. \tag{3.1.3}$$

For $D = 4$, massive graviton fields will have 5 degrees of freedom, interpreted as 5 polarization states of helicities $0, \pm 1, \pm 2$.

Notoriously, for $D = 3$, massive graviton fields will exhibit 2 degrees of freedom, interpreted as transverse and longitudinal polarization states, instead of none.

Therefore, we see that $2 + 1$-dimensional Massive Gravity theories that expand to a Massive Fierz-Pauli action may serve as a toy model for 3+1 General Relativity, at least in terms of local degrees of freedom and having the spacetime metric as the dynamical field.

3.2 New Massive Gravity

In 2009, Bergshoeff, Hohm and Townsend discovered a quadratic, 2+1 dimensional, ghost-free, massive gravity theory, which they dubbed as New Massive Gravity [9]. The coupling constants λ and m corresponds to the cosmological term in the action and the mass of a massive graviton exhibited by the theory, as we will soon discuss. The action principle is given by

$$I_{NMG} = \frac{1}{2\kappa^2}\int d^3x\sqrt{-g}\left(R - 2\lambda - \frac{1}{m^2}K\right), \tag{3.2.1}$$

where $K = R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2$.

Naturally, variation of this action yields 4-th order field equations

$$G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0. \tag{3.2.2}$$

The explicit form of the tensor $K_{\mu\nu}$ is given by

$$\begin{aligned} K_{\mu\nu} =& 2\nabla^2 R_{\mu\nu} - \frac{1}{2}(\nabla_\mu \nabla_\nu R + \nabla^2 R g_{\mu\nu}) + 4R_{\mu\alpha\nu\beta}R^{\alpha\beta} - \frac{3}{2}RR_{\mu\nu} \\ &+ \frac{3}{8}g_{\mu\nu}R^2 - g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}, \end{aligned} \tag{3.2.3}$$

and satisfies the crucial property

$$g^{\mu\nu}K_{\mu\nu} = K. \tag{3.2.4}$$

In the following subsections, we will study in-depth the main properties of this theory.

3.2.1 Linearization around constant curvature backgrounds

A remarkable property of NMG is that, despite being a quadratic theory in the Riemann tensor, its action leads to the Massive Fierz-Pauli action (3.1.1), as discovered by [9]. Let us proof this by first obtaining the quadratic field perturbation of NMG, then varying it respect to the perturbation field to obtain its linearized field equations, and finally, showing how these results are easily replicated from the Massive Fierz Pauli action.

For simplicity, for this discussion we will stick to a flat background, implying that $\lambda = 0$ in (3.2.1), and by the end we will show the generalization for generic values of λ. Taking a graviton perturbation $h_{\mu\nu}$ about Minkowski space $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ will result in the perturbed NMG action

$$I^{[L]}_{NMG} = \frac{1}{2\kappa^2}\int d^3x \left[\frac{1}{2}h^{\mu\nu}G^L_{\mu\nu}[h] - \frac{1}{4m^2}G^L_{\mu\nu}[h]\left(R^{L\mu\nu}[h] - \frac{1}{4}\eta^{\mu\nu}R^L[h]\right)\right]. \tag{3.2.5}$$

Here, $R^{L\mu\nu}[h]$ and $R^L[h]$ are the linearized Ricci tensors, while $G^L_{\mu\nu}$ is the linearized

Einstein tensor, whose general form is given by

$$G^L_{\mu\nu} = -\frac{1}{2}\nabla_\mu\nabla_\nu h + \nabla_\lambda\nabla_{(\mu}h^\lambda_{\nu)} + \eta_{\mu\nu}\left(h^{\lambda\rho}R_{\lambda\rho} - \nabla_\lambda\nabla_\rho h^{\lambda\rho} + \Box h\right) - \left(\frac{1}{2}R + \Box\right)h_{\mu\nu}, \tag{3.2.6}$$

where derivatives are covariant on the flat spacetime background.

Varying this action respect to the perturbation field $h_{\mu\nu}$ will yield us the 4-th order equations

$$\left(\Box - m^2\right)G^L_{\mu\nu}[h] = 0, \tag{3.2.7}$$

whose null divergence and trace constraints will set $R^L[h] = 0$. On the other hand, if we consider the Massive Fierz-Pauli action (3.1.1), whose field equations were defined by

$$\left(\Box - m^2\right)h_{\mu\nu} = 0, \quad \nabla_\mu h^{\mu\nu} = 0, \quad h^\mu_\mu = 0, \tag{3.2.8}$$

we can recover the exact same results by doing the simple identification (see [15])

$$h_{\mu\nu} \quad \to \quad G^L_{\mu\nu}[h], \tag{3.2.9}$$

in the field equations (3.2.8), so that it yields 4th-order equations as expected

$$\left(\Box - m^2\right)\Box h_{\mu\nu} = 0. \tag{3.2.10}$$

The null trace and divergence conditions will ensure that $R^L[h] = 0$, and thus, we have proved that NMG leads to a Massive Fierz-Pauli theory.

One may add a cosmological constant to the action, which would slightly modify the linearization equation, but let us first analize the flat-spacetime set up and then generalize it.

Another way to think of this comes from recognizing that, provided $\nabla_\mu h^{\mu\nu} = 0$, $h^\mu_\mu = 0$, it holds that $G^L_{\mu\nu}[h] = \Box h_{\mu\nu}$, and thus, $G^L_{\mu\nu} = m^2 h_{\mu\nu}$. This allows us to rewrite our linearized field equations as

$$G^L_{\mu\nu}\left[G^L[h]\right] = G^L_{\mu\nu}[m^2 h] = 0, \tag{3.2.11}$$

which we can interpret as

$$\mathcal{O}_2\left[\mathcal{O}_1\left[h_{\mu\nu}\right]\right] = \mathcal{O}_2\left[G^L_{\mu\nu}\left[h\right]\right] = 0, \tag{3.2.12}$$

Through the annihilators method, one sees two possible solutions for (3.2.13), which are

- Solving $\mathcal{O}_1\left[h_{\mu\nu}\right] = \Box h_{\mu\nu} = 0$. However, for $2+1$ dim, the operator $\mathcal{O}_1$ is invertible, since the equations of motion force $h_{\mu\nu}$ onto the transverse traceless sector of solutions, and, therefore, the equation solves to a trivial solution $h_{\mu\nu} = 0$.
- Solving $\mathcal{O}_2\left[\mathcal{O}_1\left[h_{\mu\nu}\right]\right] = 0$, which is equivalent to $\mathcal{O}_2\left[G^L_{\mu\nu}(h)\right] = \left(\Box - m^2\right) G^L_{\mu\nu} = 0$. In this case, the operator $\mathcal{O}_2$ is not invertible, which leads us to a well-defined eigenvalue equation for $G^L_{\mu\nu}$. Therefore, we see that the linearized Einstein tensor is arranged as a spin-2 massive graviton.

In the case of having a cosmological constant parameter λ in the action, the expansion must be done respect to a constant curvature solution of the form $R^{\alpha\beta}_{\mu\nu} = \Lambda\delta^{\alpha\beta}_{\mu\nu}$, resulting in the general linearized NMG equations

$$\left(\Box - \frac{5}{2}\Lambda - m^2\right)\left(\Box + 2\Lambda\right) h_{\mu\nu} = 0. \tag{3.2.13}$$

We will come back to this equations after discussing NMG's vacuum solutions.

3.2.2 Absence of ghost modes

Gravity theories constructed with higher powers in the curvature usually give rise to pathological degrees of freedom. These are characterized by either having a negative kinetic term in the action, or having unwanted dynamical fields.

For example, let us take the most general quadratic gravity action

$$I = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-g}\left(R - 2\lambda + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{3.2.14}$$

Let us study $D = 3$, so we can decompose the Riemann tensor as $R_{\mu\alpha\nu\beta} = 2R_{\nu[\nu}g_{\alpha]\beta} + 2R_{\beta[\alpha}g_{\mu]\nu} - Rg_{\nu[\mu}g_{\alpha]\beta}$, allowing us to reabsorb the third quadratic invariant into the others

$$I = \frac{1}{2\kappa^2}\int d^3x\sqrt{-g}\left(R - 2\lambda + \tilde{\alpha}R^2 + \tilde{\beta}R_{\mu\nu}R^{\mu\nu}\right). \tag{3.2.15}$$

Variation of this action leads to the following field equations

$$R_{\mu\nu} + 2\tilde{\alpha}R^{\sigma}_{\mu}R_{\nu\sigma} + 2\tilde{\beta}R_{\mu\nu}R - 2\tilde{\beta}\nabla_{\mu}\nabla_{\nu} - \tilde{\alpha}\nabla_{\sigma}\nabla_{\mu}R^{\sigma}_{\nu} - \tilde{\alpha}\nabla_{\sigma}\nabla_{\nu}R^{\sigma}_{\mu} + \tilde{\alpha}\Box R_{\mu\nu}$$
$$+\left(2\tilde{\beta} + \frac{\tilde{\alpha}}{2}\right)g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}\left(R + \tilde{\alpha}R_{\rho\sigma}R^{\rho\sigma} + \tilde{\beta}R^2\right) + \lambda g_{\mu\nu} = 0. \tag{3.2.16}$$

Let us define the following auxiliary tensors

$$\begin{aligned}\tilde{R}_{ab} =& R_{\mu\nu} + 2\tilde{\alpha}R^{\sigma}_{\mu}R_{\nu\sigma} + 2\tilde{\beta}R_{\mu\nu}R - 2\tilde{\beta}\nabla_{\mu}\nabla_{\nu} - \tilde{\alpha}\nabla_{\sigma}\nabla_{\mu}R^{\sigma}_{\nu} \quad (3.2.17)\\ &- \tilde{\alpha}\nabla_{\sigma}\nabla_{\nu}R^{\sigma}_{\mu} + \tilde{\alpha}\Box R_{\mu\nu} + \left(2\tilde{\beta} + \frac{\tilde{\alpha}}{2}\right)g_{\mu\nu}\Box R, \quad \text{and}\\ \tilde{R} =& R + \tilde{\alpha}R_{\mu\nu}R^{\mu\nu} + \tilde{\beta}R^2. \quad (3.2.18)\end{aligned}$$

The scalar $\tilde{R}$ is algebraic in the curvature, while the rank-2 tensor $\tilde{R}_{\mu\nu}$ also includes covariant derivatives, which could lead to trace pathologies. To check this, let us take the trace of $\tilde{R}_{\mu\nu}$

$$\tilde{R}_{\mu\nu}g^{\mu\nu} = R + 2\left(\tilde{\alpha}R_{\alpha\beta}R^{\alpha\beta} + \tilde{\beta}R^2\right) + \left(\frac{\tilde{\alpha}}{2}D - 2\tilde{\beta}(D-1)\right)\Box R. \tag{3.2.19}$$

Plugging this trace onto the trace of the field equations (3.2.16) gives rise to a dynamical curvature field, that is to say

$$\left(\frac{\tilde{\alpha}}{2}D - 2\tilde{\beta}(D-1)\right)\Box R + \left(2 - \frac{D}{2}\right)(\tilde{\alpha}R_{\mu\nu}R^{\mu\nu} + \tilde{\beta}R^2) + \left(1 - \frac{D}{2}\right) + D\lambda = 0. \tag{3.2.20}$$

Since we want to stick to theories that only possess degrees of freedom contained on gravitons or massive fields, we set a relation between $\tilde{\alpha}$ and $\tilde{\beta}$ such that

$$\frac{\tilde{\alpha}}{2}D - 2\tilde{\beta}(D-1) = 0 \quad \rightarrow \tilde{\beta} = \frac{\tilde{\alpha}D}{4(D-1)} \tag{3.2.21}$$

For $D = 3$, by setting $\tilde{\alpha} = -1/m^2$, we recover the global relation between the coefficients proper to NMG, namely

$$\tilde{\alpha} = -\frac{1}{m^2} \quad \rightarrow \quad \tilde{\beta} = \frac{3}{8m^2}. \tag{3.2.22}$$

And thus, obtaining that a ghost-free, three-dimensional and quadratic action theory corresponds exactly to NMG.

$$\mathcal{L}_{ghost-free} = R - 2\lambda - \frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right) = \mathcal{L}_{NMG} \tag{3.2.23}$$

3.3 Vacuum solutions

Vacuum solutions for NMG, in particular those of constant Riemann curvature $R^{\alpha\beta}_{\mu\nu} = \Lambda\delta^{\alpha\beta}_{\mu\nu}$, will be double degenerate.

$$\Lambda_{\pm} = 2\lambda \pm \sqrt{m^2 - \lambda}. \tag{3.3.1}$$

for generic values of the coupling constants. This is a natural consequence of the theory being quadratic in the curvature fields.

At the particular point of the parameter space where $\lambda = m^2$, the two maximally symmetric vacua coincide, namely

$$\Lambda = \Lambda_+ = \Lambda_- = 2\lambda. \tag{3.3.2}$$

In the following section, we will discuss the properties that NMG acquires at this single vacuum point.

3.3.1 Properties of NMG for $\lambda = m^2$

This special point in the parameter space has all sorts of nice properties

- It accomodates BH solutions for all values of the Effective Cosmological constant Λ, including asymptotically dS_3 black holes [1, 10].
- When expanding around a maximally symmetric background, the theory acquires an additional gauge invariance [16], associated to the conformal transformation

  $$\delta g_{\mu\nu} = \omega(x^\alpha)\gamma_{\mu\nu}. \tag{3.3.3}$$

  where $\gamma_{\mu\nu}$ is the maximally symmetric background and ω is an arbitrary infinitesimal function.

  Let us check how (3.2.1) varies for the conformal transformation(3.3.3) while at the special point $\lambda = m^2$

The conformal variation of (3.2.1) gives

$$\delta I = -\frac{1}{\kappa^2}\int d^3\sqrt{-g}\left[\frac{1}{2}\left(1-\frac{\Lambda}{2m^2}\right)(\Box h - \nabla_\mu\nabla_{nu}h^{\mu\nu}) + \left(\lambda - \frac{\Lambda^2}{4m^2}\right)h\right]\omega. \tag{3.3.4}$$

- It accommodates Hairy Black hole solutions of the type $f(r) = \Lambda r^2 + br - \mu$, which for $b = 0$ lands on BTZ black holes [17]. Having BTZ as a solution for NMG field equations will come in handy for reasons we will discuss when dealing with quantum backreactions.

Chapter 4

QFT in Curved Spacetimes

4.1 Semiclassical Gravity

As we have previously discussed, developing a fully consistent theory of quantum gravity is quite a hefty challenge, and some approximations must be made in order to tackle the subject more modestly. With this in mind we focus our attention now to the Semiclassical approximation for Gravity theories, which allows us to keep gravity fields classical, while studying their interaction with quantized matter.

This approximation regime has provided a successful framework to understand some new phenomena, such as Hawking Radiation and CMB anisotropies. For this, techniques developed to consider quantum fields in curved spacetimes will be needed.

Let us take for example Einstein equations in presence of classical matter

$$G_{\mu\nu} + \lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}(g_{\mu\nu}, \Psi). \tag{4.1.1}$$

Then, the semiclassical Einstein equations (SCEE) take the form

$$G_{\mu\nu} + \lambda g_{\mu\nu} = \kappa^2 \langle \hat{T}_{\mu\nu}(g_{\mu\nu}, \hat{\Psi}) \rangle. \tag{4.1.2}$$

However, as we already discussed in the first section, finding solutions for matter sourced field equations is by itself quite an ambitious challenge. Because of the complications involving both the quantization of matter fields and the addition of a general matter source onto the field equations, we restrict ourselves to stay in a perturbative regime. For the SCEE, this means computing the expectation value of the Renormalized Stress Energy Tensor (RSET) respect to the vacuum solution $g^0_{\mu\nu}$, namely

$$G^{QG}_{\mu\nu} + \lambda g^{QG}_{\mu\nu} = \langle 0|\hat{T}_{\mu\nu}(g^0_{\mu\nu}, \hat{\Psi})|0\rangle_R \; . \tag{4.1.3}$$

From now on, we will be referring to the background valued RSET simply as $\langle \hat{T}_{\mu\nu} \rangle$.

In order to solve (4.1.3), one must take into consideration the following aspects.

- Firstly, in order to have a self-consistent semiclassical treatment, one must provide a coherent reason to ignore the quantum corrections for the graviton.
- Secondly, in order to quantize the field Ψ on top of a curved background, we must make sure we posses a well defined Cauchy problem. For asymptotically anti-de Sitter spacetimes, this implies getting rid of any Closed Time-like Curves (CTC) while also setting boundary conditions at spatial infinity. The later can be done by mapping our spacetime to half of Einstein's Static Universe [18]. If done, then the field Ψ can be canonically quantized.
- Next, since the quantization of a specific SET requires having control of the field's two-point function, we will briefly review the mechanism leading to the Anti-de Sitter's two point function by summing over positive-frequency modes of the quantum field Ψ [18]. Furthermore, by expressing the resulting two point function using the ambient flat space coordinates, we can also obtain the two-point function of the BTZ spacetime through the method of images.
- Lastly, one must be able to compute the Renormalized Expectation Value of the SET respect to a specific background choice, which is by far the hardest task out of them.

This section will be structured by tackling each of these aspects separately.

4.2 Consistency of the Semiclassical Approximation

As we stated in chapter 2, we are interested in studying the quantization of a conformally coupled scalar field. For this, let us study the action for General Relativity conformally coupled to a scalar field Ψ (2.3.11).

$$I = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}\left[R - \kappa^2\left(\nabla^\mu\Psi\nabla_\mu\Psi + \frac{1}{8}\frac{D-2}{D-1}R\Psi^2\right)\right]. \tag{4.2.1}$$

Recall that we defined the coupling constant as $\kappa^2 = 8\pi G$. As we previously did for GR, let us now study the propagation of a graviton field $h_{\mu\nu}$ on a flat background $\eta_{\mu\nu}$, given by the weak-field $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. Schematically, the expansion will have the following leading terms

$$I = \int d^D x\left(-\frac{1}{2}(\partial h)^2 - \frac{1}{2}(\partial\Psi)^2 + \kappa h(\partial h)^2 - \kappa h(\partial\Psi)^2 + \mathcal{O}(\kappa^2)\right). \tag{4.2.2}$$

Notice how the graviton $h_{\mu\nu}$ couples to itself at the same strength as it couples with the matter field fluctuation Ψ. Now, if we are interested in accounting the quantum effects coming strictly from the scalar field Ψ, we must provide an argument in order to ignore the same order contributions that would arise from a quantum graviton.

As we have established many times, we are not currently interested in formulating a framework involving quantum gravitons, so to bypass this point, we will consider a large number N of conformally coupled scalar fields, so that the leading loop contributions coming from a quantum matter field dominate over the quantum effects produced by a graviton self-interaction.

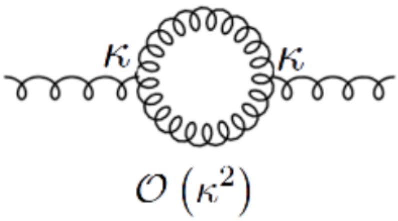

Figure 4.2.1: One loop contribution of a self-interacting graviton field $h_{\mu\nu}$

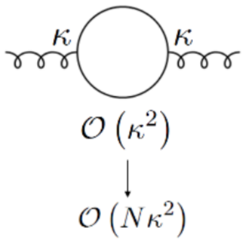

Figure 4.2.2: Leading loop contribution of graviton field $h_{\mu\nu}$ coupled to a matter field Ψ

With this in mind, we see that once we quantize our scalar field and find its RSET, the Semiclassical Einstein Equations, sourced by a conformally coupled scalar field, will read

$$G_{\mu\nu} + \lambda g_{\mu\nu} = \kappa^2 N \langle \hat{T}^{CCSF}_{\mu\nu} \rangle. \tag{4.2.3}$$

where the RSET $\langle \hat{T}^{CCSF}_{\mu\nu} \rangle$ comes from the quantization of the classic SET we studied on (2.3.15).

In what follows, we will go over the steps needed in order to have a proper quantization of Ψ and $\langle \hat{T}^{CCSF}_{\mu\nu} \rangle$ when considering a curved spacetime background.

4.3 Quantum field theory in anti-de Sitter spacetime

The quantization of a propagating field on a fixed, but curved background, requires having a well defined Cauchy problem, that is, being able to reconstruct the causal past of any event of the spacetime through the evolution of initial data laying on a given Cauchy hypersurface Σ. Spacetimes in which one can do so are known as globally hyperbolic. However, for anti-de Sitter we encounter two issues: The non-globally hyperbolicity of AdS and the existence of closed time-like curves.

As seen in B, closed time-like curves can be avoided by considering the covering space of anti-de Sitter (CAdS), and thus, avoiding any inconsistency in the time evolution of data.

On the other hand, the non-globally hyperbolicity of both AdS and CAdS may be dealt with by imposing boundary conditions at spatial infinity, as we also specified in B. However, if one is working with a conformally coupled scalar field, there is a much more clear way of dealing with the spatial boundary at CAdS: Taking advantage of the conformal relation between CAdS and half of Einstein's Static Universe (see Appendix B). This makes it so the spatial infinity of CAdS will be mapped to the equator of ESU, and allowing us to find the Ψ through its conformal scalar field equation, with the use of specific boundary condition on the field rather than on the spacetime.

In what follows, we will explore the details of this procedure, following the arguments of [18]. This will make the advantages of working with a conformal scalar field much more evident.

4.3.1 Quantum, conformally coupled scalar field

Let us start by constructing a quantum field $\hat{\Psi}(x)$ satisfying both the classical field equation and covariant commutation relation

$$(\Box + u(x))\,\Psi(x) = 0, \qquad \left[\hat{\Psi}(x), \hat{\Psi}(x')\right] = -i\hbar G^{AdS}(x,x'). \tag{4.3.1}$$

where $u(x)$ represents a c-number function, while the propagator $G^{AdS}(x,x')$ is the two-point function evolving the Cauchy data specified on a Cauchy hypersurface Σ by

$$\hat{\Psi}(x) = \int_{\Sigma} \left(G^{AdS}(x,x')\partial'_{\mu}\Psi(x') - \hat{\Psi}(x')\partial'_{\mu}G^{AdS}(x,x')\right) d\sigma^{\mu}(x'). \tag{4.3.2}$$

To reconstruct a classical scalar field $\Psi(x)$, we first start by finding a complete orthonormal set of positive-frequency solutions for (4.3.1), with respect to the time conserved Klein-Gordon norm [18].

$$\Psi^{AdS}_{lm} = e^{-i\omega_l t} h_{l,m}(\vec{x}), \qquad \omega_l \geq 0. \tag{4.3.3}$$

The time coordinate t is associated to the time-like Killing vector ∂_t, while $h_j(\vec{x})$ are a complete set of functions in the spatial coordinates. By having these mode functions, we can expand our field as

$$\hat{\Psi}^{AdS}(x) = \sum_{l,m} \left[\hat{a}_{l,m}\Psi^{AdS}_{l,m}(x) + \hat{a}^{\dagger}_{l,m}\Psi^{*AdS}_{l,m}(x)\right]. \tag{4.3.4}$$

Now, given the conformal mapping between CAdS, and half of ESU, since the later is globally hyperbolic, we can find the quantized field Ψ^{AdS} through the field equation for the conformal field Ψ^{ESU}, with the use of appropriate boundary conditions for this field.

Practically, this means that if the quantized field satisfies the field equation on AdS

$$\left(\Box^{AdS} - \frac{1}{6}R^{AdS}\right)\Psi^{AdS}(x) = 0, \tag{4.3.5}$$

then it is also satisfied in the conformal ESU space.

$$\left(\Box^{ESU} - \frac{1}{6}R^{ESU}\right)\Psi^{ESU}(x) = 0, \qquad \text{where} \quad \Psi^{ESU}(x) = \Omega^{1/2}(x)\Psi^{AdS}(x) \tag{4.3.6}$$

The positive frequency modes solving (4.3.6) are given by

$$\Psi^{ESU}_{lm} = N_{lm}e^{-i\omega_l\tau}Y^l_m(\rho,\theta), \qquad \omega_l > 0. \tag{4.3.7}$$

where Y_l^m are the spherical harmonics, $\omega = l + \frac{1}{2}$, m and l are integers with $l \geq 0$, $|m| \leq l$, and $N_{lm} = \frac{1}{\sqrt{2l+1}}$ [8].

And thus, we can reconstruct the field $\Psi^{AdS}(x)$ by multiplying it by the conformal factor relating AdS and ESU, $\Omega(x)$, found in B

$$\Psi^{AdS}(x) = (\cos\rho)^{1/2}\sum_l N_{lm}e^{-i\omega_l\tau}Y^l_m(\rho,\theta). \tag{4.3.8}$$

There are three natural set of conditions one can set for the field Ψ^{ESU}, as it was discussed in [18] and further explored in three dimensions by [8]:

- Dirichlet: $\Psi^{ESU}_{l,m}(\rho = \pi/2) = 0$, when $l + m = \text{odd}$.
- Neumann: $\frac{\partial}{\partial\rho}\Psi^{ESU}_{l,m}(\rho = \pi/2) = 0$, when $l + m = \text{even}$.
- Transparent: No boundary condition imposed at the equator. That is, letting the matter field extend its radial coordinate over the whole Einstein Universe.

The choice of boundary conditions will determine the two-point function $G^{AdS}(x, x')$ for the quantum field. In the next section, we will go over the details for computing the two-point function with the use of transparent boundary conditions.

4.4 Two-point function with Transparent Boundary conditions

The two-point function is defined by summing over the positive frequencies solutions, according to

$$\sum_{l,m}\left[\Psi^{AdS}_{lm}(x)\Psi^{*AdS}_{lm}(x') - \Psi^{*AdS}_{lm}(x)\Psi^{AdS}_{lm}(x')\right] = -iG^{AdS}(x,x'). \quad (4.4.1)$$

and if considering transparent boundary condition, this yields the following result

$$G^{AdS}(x,x') = \frac{1}{4\pi}\frac{1}{|x-x'|}. \quad (4.4.2)$$

As noted by [7], this Green function is equivalent to that of three-dimensional Minkowski spacetime, which makes sense since we are working with a conformally coupled scalar field, and AdS is a conformally flat spacetime. It should also be noted that the chordal distance $|x-x'| \equiv ((x-x')^a(x-x')_a)^{1/2}$ is determined in terms of the four-dimensional ambient space (see Appendix B), not with the distance in AdS_3.

We can also obtain the two-point function for a field Ψ conformally coupled to a BTZ background, by summing over the images identified by the $SO(2,2)$ element defined in B. This yields

$$G^{BTZ}(x,x') = \sum_{n=-\infty}^{\infty} G^{AdS}(x,\Lambda^n x') = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\frac{1}{|x-\Lambda^n x'|}. \quad (4.4.3)$$

4.5 Quantum Stress-Energy Tensor for Static BTZ

As stated in chapter 2, the Stress-Energy tensor for the matter source is obtained by varying the action (4.2.1) respect to the background $g_{\mu\nu}$

$$T_{\mu\nu} = \frac{3}{4}\nabla_\mu\Psi\nabla_\nu\Psi - \frac{1}{4}g_{\mu\nu}(\nabla\Psi)^2 - \frac{1}{4}\Psi\nabla_\mu\nabla_\nu\Psi + \frac{1}{4}g_{\mu\nu}\Psi\nabla^\lambda\nabla_\lambda\Psi + \frac{1}{8}G_{\mu\nu}\Psi^2. \quad (4.5.1)$$

By evaluating this SET on shell, we verify that is traceless $T^\mu_\mu = 0$, as expected. Now, in order to quantize this SET, we need to resort to point splitting techniques

[4], namely

$$\langle \hat{T}_{\mu\nu} \rangle = \lim_{x' \to x} \left(\frac{3}{4} \nabla^x_\mu \nabla^{x'}_\nu - \frac{1}{4} g_{\mu\nu} g^{\alpha\beta} \nabla^x_\alpha \nabla^{x'}_\beta - \frac{1}{4} \nabla^x_\mu \nabla^x_\nu - \frac{1}{16 l^2} g_{\mu\nu} \right) G(x, x'), \quad (4.5.2)$$

where symmetrization with respect to the coordinates x and x' must be prescribed. Following the procedure of [18], by using the projectors P^{ab}, taking us from the imbedding space η_{ab} to the AdS$_3$ space with metric $g_{\mu\nu}$, we can drastically simplify the propagators sum (4.4.3). These are given by

$$P^{ab} = e^a_\mu g^{\mu\nu} e^b_\nu = \eta^{ab} + x^a x^b / l^2, \qquad \text{with} \quad e^a_\mu = \partial_\mu x^a, \quad (4.5.3)$$

and satisfy

$$P^{ab} x_b = 0, \qquad P^a_b P^b_c = P^a_c. \quad (4.5.4)$$

Inserting this into our SET, we obtain the simplified expression

$$\langle \hat{T}_{\mu\nu} \rangle = \frac{3}{16\pi} \sum_{n \neq 0} \left(S^n_{\mu\nu} - \frac{1}{3} g_{\mu\nu} g^{\lambda\rho} S^n_{\lambda\rho} \right), \quad (4.5.5)$$

$$S^n_{\mu\nu} = e^a_\mu e^b_\nu S^n_{ab}, \quad S^n_{ab} = \frac{(\Lambda^n)_{ab}}{|x - \Lambda^n x|^3} + \frac{3(\Lambda^n)_{ac} x^c (\Lambda^{-n})_{bd} x^d - (\Lambda^n)_{ac} x^c (\Lambda^n)_{bd} x^d}{|x - \Lambda^n x|^5}. \quad (4.5.6)$$

This SET is easily renormalized by substracting the zero mode $n = 0$ from the sum. This is a remarkably simple renormalization, which results from having a three-dimensional geometry. In contrast, higher dimensions will require more intricate renormalization schemes. The resulting RSET is then given by

$$\langle \hat{T}^\nu_\mu \rangle = \frac{F(M)}{r^3} \text{diag}(1, 1, -2), \quad F(M) \equiv \frac{\sqrt{2}}{32\pi} \sum_{n=1}^\infty \frac{\cosh 2n\pi\sqrt{M} + 3}{(\cosh 2\pi\sqrt{M} - 1)^{3/2}}. \quad (4.5.7)$$

Here, the sum of function $F(M)$ converges and behaves as in the following figure Notice how the RSET is still traceless, and it is covariantely conserved on the BTZ background, namely

$$\nabla_\nu \langle \hat{T}^\nu_\mu \rangle \equiv 0. \quad (4.5.8)$$

On another note, since the density energy is negative, that is $\rho = \langle \hat{T}^{00} \rangle < 0$, we observe that this quantum RSET violates the weak energy condition [18, 19].

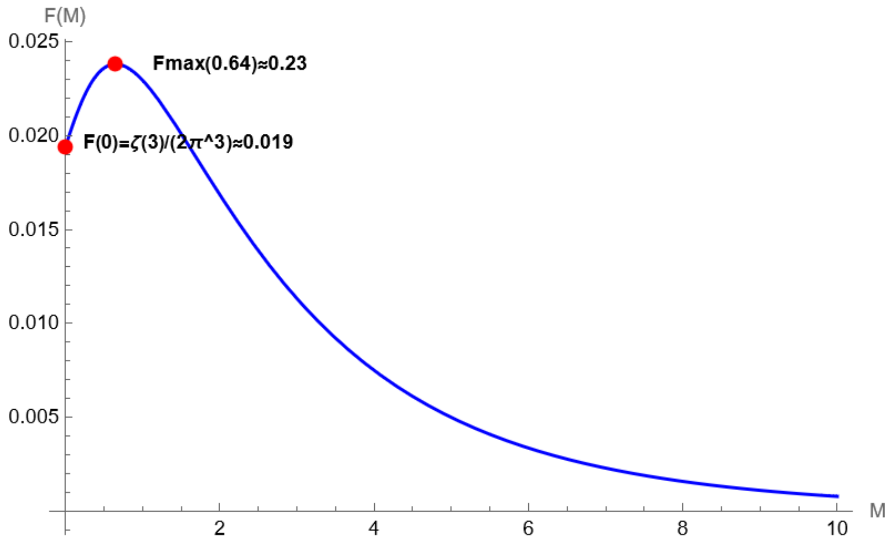

Figure 4.5.1: The function $F(M)$ from the RSET has a maximum at $M \approx 0.64$, where it takes the value $F_{max} = 0.023$. The zero mass limit goes as $F(0^+) = \zeta(3)/(2\pi^3) \approx 0.019$ and has an exponential decay for large values of M

Chapter 5

Quantum corrected Black Holes in Massive Gravity

In this section, we will jump ahead on studying the interaction of quantum matter and a massive graviton, which stands for the first original contribution of this thesis. For this, we will source our NMG action with a conformally coupled scalar field, which we defined for classical gravity in (2.3.9) and quantized for a BTZ background. Thus, the semiclassical NMG equations are given by

$$G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{m^2} K_{\mu\nu} = 8\pi G N \langle \hat{T}_{\mu\nu} \rangle. \tag{5.0.1}$$

where $K_{\mu\nu}$ was defined in (3.2) and the RSET was computed in (4.5.7).

We are interested in finding the most general circularly symmetric, static metric solution, given by the following ansatz

$$ds^2 = (1+\Omega(r)) \left[- \left(-\Lambda r^2 - M + f_1(r) \right) dt^2 + \frac{dr^2}{(-\Lambda r^2 - M + f_1(r))} + r^2 d\phi^2 \right]. \tag{5.0.2}$$

If our backreacted space allows such solutions, inserting this ansatz onto the field equations should allow us to integrate the two arbitrary functions $\Omega(r)$ and $f_1(r)$.

In what follows, we will plug this ansatz into our field equations, and study the vecinities of a BTZ solution by keeping only the first order terms in Ω and f_1, both for the special point $\lambda = m^2$, and for generic values of the couplings.

5.0.1 Semiclassical NMG at $\lambda = m^2$

Let us recall a key feature of NMG at $\lambda = m^2$: Additional conformal invariance at linear order around constant curvature backgrounds. This grants us freedom to set $\Omega = 0$ when expanding around BTZ in the field equations.

The integrated field equations lead to a solution of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \tag{5.0.3}$$

where

$$f(r) = -\Lambda r^2 - M - \frac{2\Lambda N l_p}{M} F(M) \left(r \ln r \sqrt{|\Lambda|} - r \right), \tag{5.0.4}$$

Given that NMG already had solutions including linear $\mathcal{O}(r)$ terms, the quantum backreaction only added a new correction, of order $\mathcal{O}(r \ln r)$. Just as in [20], the quantum backreaction is exponentially supressed for large black holes, given the dependence on $F(M)$.

In consequence, the event horizon experiences a quantum correction, given by

$$r_+ \approx \sqrt{M} + 2\frac{N l_p F(M)}{2Ml^2} \left(1 - \ln \sqrt{M}\right). \tag{5.0.5}$$

5.0.2 Semiclassical NMG at $\lambda \neq m^2$

Now, let us explore what happens for generic values of the coupling constants λ and m^2. Given the absence of the conformal invariance at linearized level, the ansatz suitable for studying backreactions is

$$ds^2 = -\left(\frac{r^2}{l^2} - M + f_1(r)\right) + \frac{dr^2}{\frac{r^2}{l^2} - M + g_1(r)} + r^2 d\phi^2, \tag{5.0.6}$$

where $f_1 \neq g_1$. Generic values of the couplings means that there are two possible BTZ geometries which we could expand around, which follows from (3.3.1) and by that, we no longer have a unique value for the AdS$_3$ radii L_* in terms of the bared cosmological constant λ and the graviton mass m. The lost of uniqueness make it so the linearized equations for f_1 and g_1, sourced by the quantum renormalized stress-energy tensor, cannot be solved in a closed manner.

Nonetheless, one can still explore the asymptotic behavior of the equations, by expanding around $r \to \infty$ and keeping the leading contributions. For this, we assume the form

$$f_1(r) = a_1 r^{-\delta_1}(1 + \mathcal{O}(r^{-1})), \text{ and } g_1(r) = b_1 e^{-\delta_2}(1 + \mathcal{O}(r^{-1})), \tag{5.0.7}$$

that when inserted into the field equations, yields the following tt component

$$4m^2l^2r - 6Ml^2a_1(\delta_1+2)^2r^{-\delta_1} + Ml^2b_1(9\delta_2^2 + 24 + \delta_2^3 + 8\delta_2)r^{-\delta_2} \tag{5.0.8}$$

$$+\delta_2 b_1(1+2m^2l^2)r^{2-\delta_2} - a_1\delta_1(\delta_1+2)(\delta_1+1)^2r^{2-\delta_1} = 0. \tag{5.0.9}$$

This equation leads to $\delta_1 = \delta_2 = 1$, without needing to tune any of the couplings. This condition is consistent with the large-r behavior of the remaining field equations as well. Consequently, the large distance behavior of the quantum backreacted metric reads

$$ds^2 = -\left(\frac{r^2}{l^2} - M + \frac{l_p\alpha_1}{r} + \mathcal{O}(r^{-2})\right)dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - M + \frac{l_p\beta_1}{r} + \mathcal{O}(r^{-2})} + r^2d\phi^2, \tag{5.0.10}$$

for some given constants α_1 and β_1. Therefore, for generic values of λ and m, the quantum backreaction of NMG leads to the same asymptotic behavior observed for quantum corrected BTZ black hole in Einstein gravity [20].

An intuitive way to explain why the backreaction for $\lambda = m^2$ leads to a logarithmic, slow-decaying correction with respect to the leading term r^2, is to think of this special point as a direct relation between a short-distance coupling parameter m, which weights corrections induced by the higher-curvature terms in the action, and a large-distance coupling parameter λ, relating to a cosmological constant of the geometry. This interaction is what ultimately results in a large range modification of the asymptotic structure. The latter is the subject of the next section.

Chapter 6

Asymptotic Structure

In this chapter, we will first do an overview on the definition of an asymptotically anti-de Sitter spacetime and the definition of asymptotic symmetry generators. For asymptotically AdS spacetimes, these are found to be two copies of Witt's algebra, whose canonical charges algebra allows for the emergence of a central charge [21]. Following that, we will review the asymptotic behavior proper to NMG, studied in [1] and its associated central charge and generators. Finally, we will study the asymptotic behavior of the quantum backreacted solutions found for NMG presented in chapter 5.0.1, where despite the relaxation of asymptotic conditions, the solutions are still invariant under two copies of Witt's algebra while also exhibiting additional symmetry generators.

6.1 Brown-Henneaux boundary conditions

Let us take a spacetime given by $g_{\mu\nu} = g^{AdS}_{\mu\nu} + \Delta g_{\mu\nu}$, where $\Delta g_{\mu\nu}$ corresponds to the deviation from an AdS metric $g^{AdS}_{\mu\nu}$ of the form

$$ds^2_{AdS} = -\left(\frac{r^2}{l^2} + 1\right) dt^2 + \left(\frac{r^2}{l^2} + 1\right)^{-1} dr^2 + r^2 d\phi^2. \tag{6.1.1}$$

Now, let us first make a convenient change of coordinates, so that $x^{\pm} = t \pm \phi$. The spacetime $g_{\mu\nu}$ is said to be asymptotically AdS if $\Delta g_{\mu\nu}$ has the following behavior

$$\Delta g_{rr} = \frac{f_{rr}}{r^4} + \mathcal{O}\left(r^{-5}\right), \tag{6.1.2}$$

$$\Delta g_{rm} = \frac{f_{rm}}{r^3} + \mathcal{O}\left(r^{-4}\right), \tag{6.1.3}$$

$$\Delta g_{mn} = f_{mn} + \mathcal{O}\left(r^{-1}\right). \tag{6.1.4}$$

Here, $f_{\mu\nu} = f_{\mu\nu}(t, \phi)$, and the indeces have been split as $\mu = (r, m = x^{\pm})$. These are the so-called Brown-Henneaux boundary conditions [21].

Spacetimes obeying these conditions are still preserved under the transformation produced by the symmetry generators of AdS_3, which are obtained by integrating the Killing equation.

$$\mathcal{L}_{\xi} g_{\mu\nu} \equiv \nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0. \tag{6.1.5}$$

However, we can also find sets of transformations which only preserve the Brown-Henneaux boundary conditions. The generators of these transformation are called asymptotic symmetry generators, and can be obtained by integrating the asymptotic Killing equation, namely

$$\mathcal{L}_{\eta} g_{\mu\nu} \equiv \nabla_{\mu}\eta_{\nu} + \nabla_{\nu}\eta_{\mu} = \mathcal{O}(\Delta g_{\mu\nu}). \tag{6.1.6}$$

By integrating these equations, we can construct an asymptotic Killing vector $\eta \equiv \eta^{\mu}\partial_{\mu}$, whose components are found to be

$$\begin{aligned}
\eta^{+} &= L^{+}(x^{+}) + \frac{l^2}{2r^2}\partial_{-}^2 L^{-}(x^{-}) + ..., \\
\eta^{-} &= L^{-}(x^{-}) + \frac{l^2}{2r^2}\partial_{+}^2 L^{+}(x^{+}) + ..., \\
\eta^{r} &= -\frac{r}{2}\left(\partial_{+}L^{+}(x^{+}) - \partial_{-}L^{-}(x^{-})\right) + ...
\end{aligned} \tag{6.1.7}$$

Here $L^{\pm}(x^{\pm})$ are arbitrary, periodic functions, and $L^{\pm}_n$ are its fourier modes, with $-\infty < n < +\infty$

This forms two families of infinite, discrete generators, which form two distinct copies of Witt's algebra, namely

$$\left[L_m^+, L_n^+\right] = (m-n)L_{m+n}^{,} \tag{6.1.8}$$

$$\left[L_m^-, L_n^-\right] = (m-n)L_{m+n}^-. \tag{6.1.9}$$

Naturally, each of these generators has an associated conserved charge, $Q_n^\pm$.

Remarkably, the Poisson brackets of these conserved charges reproduce the structure of Witt's algebra, but with the addition of a central extension

$$\left\{Q_m^+, Q_n^+\right\} = (m-n)Q_{m+n}^+ + \frac{c}{12}(m^3 - m)\delta_{m,-n}. \tag{6.1.10}$$

where c is the central charge. For General Relativity, the central charge was found to be [21]

$$c = \frac{3l}{2G}. \tag{6.1.11}$$

Now, if working with three-dimensional, higher-curvature gravity theories, of the form

$$I = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\;\mathcal{L}^{(n)}(R_{\mu\nu}, g_{\mu\nu}). \tag{6.1.12}$$

where $\mathcal{L}^{(n)}$ is the general higher order Lagrangian, we can find a generalization for the central charge for these theories by restricting ourselves to asymptotically AdS solutions (see e.g. [22, 23]), thus obtaining

$$c = g_{\mu\nu}\frac{l}{2G}\frac{\partial\mathcal{L}^{(n)}}{\partial R_{\mu\nu}}. \tag{6.1.13}$$

This is the formula we will be using further on, and was originally obtained in [24], and later on generalized to higher curvature set ups by [23, 25].

6.2 Logarithmically relaxed boundary conditions

Now, we are interested on studying the asymptotic structure of the quantum backreacted BTZ BH, which we obtained in chapter 5.0.1 and its accomodated by

$$g_{rr} = \frac{l^2}{r^2} + h_{rr}\left(\frac{\log r}{r^3}\right) + \frac{f_{rr}}{r^4} + ..., \tag{6.2.1}$$

$$g_{rm} = h_{rm}\left(\frac{\log r}{r^2}\right) + \frac{f_{rm}}{r^3} + ..., \tag{6.2.2}$$

$$g_{mn} = r^2 + h_{mn}\left(r \log r\right) + f_{mn} + ... \tag{6.2.3}$$

where again $f_{\mu\nu} = f_{\mu\nu}(t,\phi)$, $h_{\mu\nu} = h_{\mu\nu}(t,\phi)$, and the indices split as $\mu = (r, m = x^{\pm})$. This metric is now separated onto three segments, $g_{\mu\nu} = g^{AdS}_{\mu\nu} + \Delta g^{BR}_{\mu\nu} + \Delta g^{B-H}_{\mu\nu}$, where $\Delta g^{BR}_{\mu\nu}$ is the new subleading, relaxed deviation from $g^{AdS}_{\mu\nu}$, given by the functions $h_{\mu\nu}$, while $\Delta g^{B-H}_{\mu\nu}$ are the already studied Brown-Henneaux fall-off conditions given by $f_{\mu\nu}$. This family of solutions is still preserved by the action of the Killing vector $\eta = \eta^\mu \partial_\mu$ we found on (6.1.7).

This means that our solution, despite the relaxation of the asymptotic behavior, is still invariant under two copies of the Witt algebra.

In addition to these, two extra asymptotic Killing vectors preserving (6.2.2) emerge

$$\chi = A(x^+, x^-)\partial_r, \tag{6.2.4}$$

$$\xi = B(x^+, x^-)\log r \partial_r, \tag{6.2.5}$$

where $A(x^+, x^-)$ and $B(x^+, x^-)$ are also arbitrary functions.

These three Killing vectors follow the algebraic structure schematically

$$[\eta, \chi] = \xi, \tag{6.2.6}$$

$$[\eta, \xi] = \chi + \xi, \tag{6.2.7}$$

$$[\xi, \chi] = 0. \tag{6.2.8}$$

The generator χ was already observed in [1], and stands as a supertranslation proper to NMG.

On the other side, ξ stands for a new asymptotic symmetry generator due to the quantum backreaction, which we dubbed as a logarithmic supertranslation. Logarithmic symmetry operators have been previously seen emerging from spacetimes with a slowly asymptotic falling-off, yielding an additional charge contribution to that of the standard Virasoro generators, as seen in [26]. Nonetheless, spacetimes with slowly falling-off asymptotics have only been studied as produced by classical scalar fields. Thus, it would be interesting to revisit the interpretation of logarithmic supertranslation in this particular context, given that the asymptotic correction comes from the coupling to a quantized scalar field.

Chapter 7

Extension to Higher-Curvature Gravity

The results presented in the previous chapters can be extended to theories involving arbitrarily higher-curvature terms in the action. This defines a sort of universality for the backreaction on higher-curvature theories, given they obey two key conditions, namely

- Parameters of the theory must be set so the theory satisfies a holographic c-theorem.
- Global parameters of the theory must be set so the theory possesses a single vacuum solution.

These two conditions are analogous to what we studied for NMG in the previous chapters, and defines the second novel contribution of this thesis. Recall that a general quadratic extension to GR in three-dimensions can be written as

$$I = \frac{1}{2\kappa^2}\int d^3x\sqrt{-g}\left(R - 2\lambda + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right). \tag{7.0.1}$$

Demanding this theory to be absent of ghost modes fixes a relation between the parameters $\beta = -3\alpha/8$, and imposing a single-vacuum sets the global factor $\alpha \equiv -1/m^2$ in terms of the bared cosmological constant $m^2 = \lambda$.

With this idea in mind, we will separate this section as follows. First, we will go through the details and notations used to express an arbitrarily high curvature gravity theory satisfying both conditions. For this we will first make an overview on how to construct a theory obeying a holographic c-theorem, followed by the steps to impose a single-vacuum solution. This will lead to a Lagrangian density that satisfies both conditions. Afterwards we will show how this Lagrangian shares the same special properties of NMG when $\lambda = m^2$, namely: Having an additional conformal invariance at linear level around a constant curvature background, and admiting black hole solutions of the type $f(r) = ar^2 + br + c$, which includes the BTZ solution; remarkably, both of these properties will ensure that the quantum backreaction correct our higher-curvature theories in the same manner as for NMG.

Additionally, the end of this section will be focused on discussing thermodynamical properties related to these higher-curvature theories. We will study the mass, temperature and entropy of black hole solutions found for these theories in vacuum.

7.1 Higher-curvature theories

Let us define our generalized theory, constructed with contractions of the Ricci tensor and Ricci scalar curvatures. For this, we take into consideration Schouten identities, which restrict the number of independent invariants made with contractions of the Ricci tensor, that is

$$\delta^{a_1\ldots a_n}_{b_1\ldots b_n} R^{b_1}_{a_1} \cdots R^{b_n}_{a_n} \equiv 0, \qquad \text{for } n > 3. \tag{7.1.1}$$

In consequence, our most general three-dimensional gravity theory, involving the Ricci tensor and its contractions at orden n, can be written as follows

$$I = \frac{1}{2\kappa^2} \int d^3x \sqrt{-g}\mathcal{L}^{(n)}, \quad \mathcal{L}^{(n)} = R + \frac{2}{L^2} + \sum_n \mathcal{F}^{(n)}(R, \mathcal{R}_2, \mathcal{R}_3), \tag{7.1.2}$$

where we have set our bared cosmological constant to $\lambda = -1/L^2$, and defined the invariants as $\mathcal{R}_2 = R^\nu_\mu R^\mu_\nu$ and $\mathcal{R}_3 = R^\nu_\mu R^\lambda_\nu R^\mu_\lambda$. For later convenience, we will reconstruct these invariants with contractions of the traceless Ricci tensor $\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R$, defining

$$\mathcal{S}_2 \equiv \tilde{R}^\nu_\mu \tilde{R}^\mu_\nu, \quad \mathcal{S}_3 \equiv \tilde{R}^\nu_\mu \tilde{R}^\lambda_\nu \tilde{R}^\mu_\lambda, \tag{7.1.3}$$

so our most general Lagrangian density is redefined as follows

$$\mathcal{L}^{(n)} = R + \frac{2}{L^2} + \sum_n \mathcal{G}^{(n)}(R, \mathcal{S}_2, \mathcal{S}_3). \tag{7.1.4}$$

As we will see, this basis for our Lagrangian turns out to be more convenient to derive and present the results. We will focus on theories which can be constructed with polynomial expressions of the curvature invariants. For this, we define the functions $\mathcal{F}^{(n)}$ and $\mathcal{G}^{(n)}$, which read as

$$\mathcal{F}^{(n)}(R, \mathcal{R}_2, \mathcal{R}_3) = \sum_{i+2j+3k=n} \alpha_{ijk} L^{2(i+2j+3k-1)} R^i \mathcal{R}_2^j \mathcal{R}_3^k, \tag{7.1.5}$$

$$\mathcal{G}^{(n)}(R, \mathcal{S}_2, \mathcal{S}_3) = \sum_{i+2j+3k=n} \beta_{ijk} L^{2(i+2j+3k-1)} R^i \mathcal{S}_2^j \mathcal{S}_3^k. \tag{7.1.6}$$

Here, α_{ijk} and β_{ijk} represent dimensionless coupling constants for each order-n term. In what follows, we will specify the values for these coupling constants so the action satisfies both a simple c-theorem and possesses a single vacuum.

7.2 Theories satisfying a c-theorem with single vacuum

We say that a given three-dimensional gravity theory of order n,

$$I = \frac{1}{2\kappa^2} \int d^3x \sqrt{-g} \left(\mathcal{L} + \mathcal{L}_M\right), \tag{7.2.1}$$

satisfies a simple holographic c-theorem if its field equations are at most of second order when evaluated on the following ansatz

$$ds^2 = dr^2 + e^{2A(r)}(-dt^2 + dx^2). \tag{7.2.2}$$

The field equations involve a suitable stress-tensor $T_{\mu\nu}$ associated to the matter Lagrangian $\mathcal{L}_M$. $\mathcal{L}$ is the gravitational Lagrangian, which depends on the metric $g_{\mu\nu}$ and the Riemann tensor $R^{\alpha}_{\beta\mu\nu}$. This metric ansatz can interpolate between two asymptotically AdS$_3$ regions, representing the infrared (IR) and the ultraviolet (UV) fixed points in the dual description. The interpolation between these two regions is understood as a holographic realization of the renormalization group

(RG) flow; for further details see [27, 28].

For theories of this sort, we can construct a quantity $c(r)$, which decreases with the RG flow and coincides with the Virasoro central charge of the CFT, c, at the respective fixed point. Consequently, the function $c(r)$ whose derivative with respect to r obbeys

$$c'(r) \geq 0, \quad \forall r, \tag{7.2.3}$$

is a good candidate to be the c-function in the dual picture. More concretely, according to [25, 27, 29] we can consider

$$c'(r) = -\frac{\pi}{A'}(T_t^t - T_r^r), \tag{7.2.4}$$

and using the null energy condition (NEC), we have $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$, where ξ_μ is a null vector, $\xi_\mu\xi^\mu = 0$. For a perfect fluid of density ρ and pressure p, we have

$$T_{\mu\nu}\xi^\mu\xi^\nu = \left[(\rho + p)u_\mu u_\nu + pg_{\mu\nu}\right]\xi^\mu\xi^\nu = (\rho + p)(u_\mu\xi^\mu)^2 \geq 0. \tag{7.2.5}$$

Using $u^\mu = (1, 0, 0)$, with $u_\mu u^\mu = -1$ for a co-moving frame, we have $T_t^t = -\rho$ and $T_r^r = p$. Hence, the NEC implies

$$T_t^t - T_r^r \leq 0. \tag{7.2.6}$$

The explicit expression of the c-function in our family of theories, provided the NEC is fulfilled, can be obtained from the Wald-like formula [23, 24, 29]

$$c(r) = \frac{\pi}{2A'}\frac{\partial\mathcal{L}}{\partial R^{tr}_{tr}}, \tag{7.2.7}$$

where the derivative of the Lagrangian density $\mathcal{L}$ is to be evaluated on the interpolating ansatz (7.2.2). Below, we will see how this works for the quadratic example, and afterwards we will describe the general procedure for arbitrary order n.

For quadratic gravity, we can use the example introduced in chapter 3, where the field equations coming from a Lagrangian of the type

$$\mathcal{F}^{(2)} = L^2(\alpha R^2 + \beta\mathcal{R}_2), \tag{7.2.8}$$

read as

$$\frac{K^{(2)}_{\mu\nu}}{L^2} = 2\beta R_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}\left(\beta R^2 + \alpha\mathcal{R}_2\right) + 2\beta(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R + 2\alpha R^\lambda_\mu R_{\nu\lambda}$$
$$\frac{\alpha}{2}g_{\mu\nu}\Box R + \alpha\Box R_{\mu\nu} - 2\alpha\nabla_\lambda\nabla_{(\mu}R^\lambda_{\nu)}, \tag{7.2.9}$$

where L^2 is the length scale associated to the bared cosmological constant of AdS_3, added so that the coupling constants α and β are dimensionless. We also used the notation $A_{(a}B_{b)} = \frac{1}{2}\left(A_aB_b + A_bB_a\right)$ for tensor product symmetrization.

Considering this will be an extension to General Relativity, we can write our field equations coming from the quadratic lagrangian $\mathcal{F}^{(2)}$ as $K^{(2)}_{\mu\nu} = \kappa^2 T^{(2)}_{\mu\nu}$. With this in mind, we easily see that the combination $T^{(2)}{}^t_t - T^{(2)}{}^r_r$ reads

$$\frac{\kappa^2}{L^2}\left(T^{(2)}{}^t_t - T^{(2)}{}^r_r\right) = 4(3\beta+\alpha)(A')^2A'' - (8\beta+3\alpha)\left[4(A'')^2 + 2A'A''' + A''''\right]. \tag{7.2.10}$$

From here, we see that the choice $\alpha = -\frac{8}{3}\beta$ reduces the equation to a second order ODE. The resulting theory is, as expected, the already studied NMG. This goes to show how demanding the fulfillment of a simple c-theorem, corresponds to the mechanism used to get rid of ghost modes in NMG, as originally reported in [23].

After reviewing the procedure to obtain higher-curvature theories that a) satisfy a simple holographic c-theorem and b) possess a single vacuum for $n = 2$, we now move to arbitrary n. Going back to our general Lagrangian density (7.1.4), it was identified in [30] that Lagrangian densities obeying a simple c-theorem, denoted as $\mathcal{L}^{(n)}_{c-theorem}$, can have their polynomial Lagrangians to be decomposed as follows

$$\mathcal{G}^{(n)} = \gamma_n\mathcal{C}^{(n)} + \Omega^{(6)}\cdot\mathcal{G}^{(n-6)}_{general}, \tag{7.2.11}$$

where

$$\mathcal{C}^{(n)} = R^n - \sum_{p=1}^{\lfloor n/2\rfloor} 24^p\binom{n}{2p}R^{n-2p}\mathcal{S}_2^p + \sum_{p=1}^{\lfloor (n-1)/2\rfloor} 24^{p+1}p\binom{n}{2p+1}R^{n-2p-1}\mathcal{S}_2^{p-1}\mathcal{S}_3, \tag{7.2.12}$$

and $\Omega^{(6)} = 6\mathcal{S}_3^2 - \mathcal{S}_2^3$ is a density that vanishes identically when evaluated on the c-theorem metric ansatz (7.2.2). This means that, if we are interested in theories satisfying a simple c-theorem, the only non-trivial Lagrangian gets reduced to $\mathcal{C}^{(n)}$, since even though the Lagrangian $\mathcal{G}^{(n-6)}_{general}$ is constructed with arbitrary linear combinations of powers of the curvature of order $(n-6)$, given that it is proportional to $\Omega^{(6)}$, its contribution identically vanish for AdS_3.

Now, let us turn our attention to the second requirement: the theory must possess a single vacuum. The field equations coming from the general Lagrangian density

$$\mathcal{L} = R + \frac{2}{L^2} + \sum_n \beta_{(n)} \mathcal{C}^{(n)}, \tag{7.2.13}$$

with $\mathcal{C}^{(n)}$ defined by (7.2.12), are given by

$$\mathcal{E}^{(n)}_{\mu\nu} = G_{\mu\nu} - g_{\mu\nu}\frac{1}{L^2} + \sum_n \beta_{(n)} K^{(n)}_{\mu\nu} = 0, \tag{7.2.14}$$

where the generalized n-order invariant $K^{(n)}_{\mu\nu}$ reads

$$\begin{aligned} K^{(n)}_{\mu\nu} =& -\frac{1}{2} g_{\mu\nu}\mathcal{G}^{(n)} + 2\mathcal{G}^{(n)}_{\mathcal{S}_2}\tilde{R}^{\lambda}_{\mu}\tilde{R}_{\lambda\nu} + 3\mathcal{G}^{(n)}_{\mathcal{S}_3}\tilde{R}^{\lambda}_{\mu}\tilde{R}_{\lambda\sigma}\tilde{R}^{\sigma}_{\nu} + g_{\mu\nu}\nabla_\lambda\nabla_\sigma\left(\mathcal{G}^{(n)}_{\mathcal{S}_2}\tilde{R}^{\lambda\sigma} + \frac{3}{2}\mathcal{G}^{(n)}_{\mathcal{S}_3}\tilde{R}^{\lambda\rho}\tilde{R}^{\sigma}_{\rho}\right) \\ &+ \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + \tilde{R}_{\mu\nu} + \frac{1}{3}g_{\mu\nu}R\right)\left(\mathcal{G}^{(n)}_{R} - \mathcal{G}^{(n)}_{\mathcal{S}_3}\mathcal{S}_2\right) \\ &- 2\nabla_\lambda\nabla_{(\mu}\left(\tilde{R}^{\lambda}_{\nu)}\mathcal{G}^{(n)}_{\mathcal{S}_2} + \frac{3}{2}\tilde{R}^{\rho}_{\nu)}\tilde{R}^{\lambda}_{\rho}\mathcal{G}^{(n)}_{\mathcal{S}_3}\right) + \left(\Box + \frac{2}{3}R\right)\left(\mathcal{G}^{(n)}_{\mathcal{S}_2}\tilde{R}_{\mu\nu} + \frac{3}{2}\mathcal{G}^{(n)}_{\mathcal{S}_3}\tilde{R}^{\lambda}_{\mu}\tilde{R}_{\lambda\nu}\right) = 0. \end{aligned} \tag{7.2.15}$$

here, $\mathcal{G}^{(n)}_{\chi}$ represents the partial derivative of $\mathcal{G}^{(n)}$ with respect to the invariant χ, i.e. $\mathcal{G}_\chi = \frac{\partial \mathcal{G}}{\partial \chi}$.

Now, let us evaluate these results on AdS_3 spacetime with curvature radius L_*. The curvature invariants then take the form

$$\bar{R} = -\frac{6}{L_*^2}, \qquad \bar{\mathcal{S}}_2 = 0, \qquad \bar{\mathcal{S}}_3 = 0. \tag{7.2.16}$$

Here, we used the bar notation when evaluating on the AdS_3 spacetime. It is clear now why the use of traceless invariants was more convenient. In fact, by inserting this into the full field equations $\bar{\mathcal{E}}^{(n)}_{\mu\nu}$ in (7.2.14), we obtain a simpler relation, given by

$$\frac{2}{L^2} - \frac{2}{L_*^2} + \sum_n \left(\bar{\mathcal{G}}^{(n)} + \frac{4}{L_*^2}\bar{\mathcal{G}}^{(n)}_{R}\right) n = 0. \tag{7.2.17}$$

Now, recall that we are focusing on theories constructed with polynomials of our invariants (7.1.6). When evaluated on AdS_3, this reduces to

$$\bar{\mathcal{G}}^{(n)} = (-1)^n \beta_{n00} L^{2n-2} \left(\frac{6}{L_*^2}\right)^n , \tag{7.2.18}$$

while its derivative with respect to R simplifies to

$$\bar{\mathcal{G}}_R^{(n)} = (-1)^n \beta_{n00} L^{2n-2} \left(\frac{6}{L_*^2}\right)^{n-1} n. \tag{7.2.19}$$

This allows us to rewrite our field equation (7.2.17) as the following characteristic polynomial

$$1 - \chi_0 + \sum_n b_n \chi_0^n = 0. \tag{7.2.20}$$

where $b_n = (-1)^n 6^{n-1}(3-2n)\beta_{n00}$ and $\chi_0 = L^2/L_*^2$.

For single-vacuum theories, there is an order-n degeneration of the solution to the characteristic polynomial. For them, the value of χ_0 is given by

$$\chi_0 = n \qquad \text{which means} \quad 1/L_*^2 = n/L^2. \tag{7.2.21}$$

Based on this observation, it was shown in [30] that a Lagrangian density of the form

$$\mathcal{L}^{(n)}_{singlevac.} = \frac{2}{L^2} + R + \sum_{i=2}^{n} \binom{n}{i} \frac{L^{2(i-1)}}{n^i 6^{i-1}(3-2i)} R^i + \mathcal{S}_2 h_2(R, \mathcal{S}_2, \mathcal{S}_3) + \mathcal{S}_3 h_3(R, \mathcal{S}_2, \mathcal{S}_3), \tag{7.2.22}$$

where h_2 and h_3 are any analytic function of their arguments, possesses a single AdS_3 vacuum. Notice that, in these theories, the coupling constant of the order-n curvature term scales as $\mathcal{O}(L^{2n-2})$, with the AdS radius being $1/L^2$. This implies that the theory is strongly coupled, with the UV and the IR regimes being related. This is precisely the reason why, while higher-derivative terms are generically expected to yield short-distance modifications to general relativity, for these theories we still expect long-range modifications.

With this in mind, we can combine expressions (7.2.12) and (7.2.22) to find a theory satisfying both requirements: a) possesing a holographic c-theorem and b) a single vacuum. The single vacuum requirement imposes a particular value of the coupling constant for the term R^2, however, it leaves unspecified what h_2 and h_3 should be, as they do not modify the characteristic polynomial. As a consequence, we can include the same coupling constant for the additional two terms appearing in (7.2.12), so that we do not spoil the relative factors that guarantee the existence of a holographic c-theorem at order n. In turn, we obtain the following Lagrangian

$$\mathcal{D}^{(n)} = \frac{2}{L^2} + R + \sum_{i=2}^{n} \binom{n}{i} \frac{L^{2(i-1)}}{n^i 6^{i-1}(3-2i)} [R^i - \sum_{p=1}^{\lfloor i/2 \rfloor} 24^p (2p-1) \binom{i}{2p} R^{i-2p} S_2^p + \sum_{p=1}^{\lfloor (i-1)/2 \rfloor} 24^{p+1} p \binom{i}{2p+1} R^{i-2p-1} S_2^{p-1} S_3] \qquad n \geq 1. \tag{7.2.23}$$

Let us now illustrate how this generalized Lagrangian, possessing conformal invariance at linear level and a single vacuum, reproduces the already studied NMG theory in (3.2.1), for $n = 2$

$$\mathcal{D}^{(2)} = \frac{2}{L^2} + R - \frac{L^2}{24}(R^2 - 24\mathcal{S}_2). \tag{7.2.24}$$

Recall that $\mathcal{S}_2 = \tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} = (R_{\mu\nu} - \frac{1}{3}g_{\mu\nu}R)(R^{\mu\nu} - \frac{1}{3}g^{\mu\nu}R)$ while redefining $\lambda = -\frac{1}{L^2}$, so that (7.2.25) gives

$$\mathcal{D}^{(2)} = R - 2\lambda - \frac{1}{\lambda}(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2) \ , \tag{7.2.25}$$

which reproduces the Lagrangian density of NMG when $\lambda = m^2$.

Naturally, the expressions for higher n densities become larger, since the larger the order, the more possible ways to construct curvature invariants there will be.

To illustrate this, let us also work out some examples up to order $n = 6$

$$\mathcal{D}^{(3)} = \frac{2}{L^2} + R - \frac{L^2}{18}(R^2 - 24\mathcal{S}_2) - \frac{L^4}{2916}(R^3 - 72R\mathcal{S}_2 + 576\mathcal{S}_3), \tag{7.2.26}$$

$$\begin{aligned}\mathcal{D}^{(4)} = &\frac{2}{L^2} + R - \frac{L^2}{16}(R^2 - 24\mathcal{S}_2) - \frac{L^4}{1728}(R^3 - 72R\mathcal{S}_2 + 576\mathcal{S}_3) \\ &+ \frac{L^6}{276480}(R^4 - 144R^2\mathcal{S}_2 - 144R^2\mathcal{S}_2 - 1728\mathcal{S}_2^2 + 2304R\mathcal{S}_3),\end{aligned} \tag{7.2.27}$$

$$\begin{aligned}\mathcal{D}^{(5)} = &\frac{2}{L^2} + R - \frac{L^2}{15}(R^2 - 24\mathcal{S}_2) - \frac{L^4}{1350}(R^3 - 72R\mathcal{S}_2 + 576\mathcal{S}_3) \\ &+ \frac{L^6}{135000}(R^4 - 144R^2\mathcal{S}_2 - 144R^2\mathcal{S}_2 - 1728\mathcal{S}_2^2 + 2304R\mathcal{S}_3) \\ &\frac{L^8}{28350000}(R^5 - 240R^3\mathcal{S}_2 - 8640R\mathcal{S}_2^2 + 5760R^2\mathcal{S}_3 + 27648\mathcal{S}_2\mathcal{S}_3),\end{aligned} \tag{7.2.28}$$

$$\begin{aligned}\mathcal{D}^{(6)} = &\frac{2}{L^2} + R - \frac{5L^2}{72}(R^2 - 24\mathcal{S}_2) - \frac{5L^4}{5832}(R^3 - 72R\mathcal{S}_2 + 576\mathcal{S}_3) \\ &+ \frac{L^6}{93312}(R^4 - 144R^2\mathcal{S}_2 - 144R^2\mathcal{S}_2 - 1728\mathcal{S}_2^2 + 2304R\mathcal{S}_3) \\ &\frac{L^8}{11757312}(R^5 - 240R^3\mathcal{S}_2 - 8640R\mathcal{S}_2^2 + 5760R^2\mathcal{S}_3 + 27648\mathcal{S}_2\mathcal{S}_3) \\ &\frac{L^{10}}{3265173504}\left(R^6 - 360R^4\mathcal{S}_2 - 25920R^2\mathcal{S}_2^2 - 69120\mathcal{S}_2^3 + 11520R^3\mathcal{S}_3 + 165888R\mathcal{S}_2\mathcal{S}_3\right).\end{aligned} \tag{7.2.29}$$

In spite of the intricate form of these Lagrangians, let us emphasize that (7.2.23) provides a compact, comprehensive form for each term in these theories.

7.3 Linearized Field Equations

In this section, we will show that all the theories defined by the Lagrangian densities $\mathcal{D}^{(n)}$, given in (7.2.23), acquire an additional gauge invariance at a linearized level. For this, we start by linearizing (7.1.4) around a maximally symmetric AdS vacuum of the theory, with radius L_*, described by [30]

$$\frac{1}{4}\mathcal{E}^{(n)L}_{\mu\nu} = \left[e + c\left(\bar{\Box} + \frac{2}{L_*^2}\right)\right] G^L_{\mu\nu} + (2b+c)(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu)R^L - \frac{1}{L_*^2}(4b+c)\bar{g}_{\mu\nu}R^L = \frac{\kappa^2}{4}T^L_{\mu\nu}, \tag{7.3.1}$$

where $T^L_{\mu\nu}$ is the linearized stress-energy tensor and the linearized Einstein tensor, Ricci tensor and Ricci scalar are given by

$$G^L_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L + \frac{2}{L_*^2}h_{\mu\nu}, \tag{7.3.2}$$

$$R^L_{\mu\nu} = \bar{\nabla}_{(\mu|}\bar{\nabla}_\lambda h^\lambda_{|\nu)} - \frac{1}{2}\bar{\Box}h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu h - \frac{2}{L_*^2}h_{\mu\nu} + \frac{1}{L_*^2}h\bar{g}_{\mu\nu}, \tag{7.3.3}$$

$$R^L = \bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu} - \bar{\Box}h + \frac{2}{L_*^2}h, \tag{7.3.4}$$

respectively. The parameters e, c and b are to be interpreted as the effective Planck length κ^2_{eff} and the masses (squared) of gravitational modes, which we denote m_g^2 and m_s^2, as

$$\kappa^2_{\text{eff}} = \frac{1}{4e}, \qquad m_g^2 = -\frac{e}{c}, \qquad m_s^2 = \frac{e + \frac{8}{L_*^2}(3b+c)}{3c+8b}. \tag{7.3.5}$$

They are obtained from the relations

$$e = \frac{1}{4\kappa^2}[1+\bar{\mathcal{G}}_R], \qquad b = \frac{1}{4\kappa^2}\left[\frac{1}{2}\bar{\mathcal{G}}_{R,R} - \frac{1}{3}\bar{\mathcal{G}}_{\mathcal{S}_2}\right], \qquad c = \frac{1}{4\kappa^2}\bar{\mathcal{G}}_{\mathcal{S}_2}, \tag{7.3.6}$$

where we used again the notation $\mathcal{G}_\chi = \partial\mathcal{G}/\partial\chi$, $\mathcal{G}_{\chi,\chi} = \partial^2\mathcal{G}/\partial\chi^2$. Taking this into account, we can compute the specific values of the parameters e, b and c for the theory (7.2.23), namely

$$e = \frac{1}{4\kappa^2}\left[1 + \sum_{i=1}^{n}\binom{n}{i}\frac{(-1)^{i-2}i}{n(3-2i)}\right] = \frac{1}{4\kappa^2}\frac{\sqrt{\pi}\Gamma(n)}{\Gamma(n-1/2)}, \quad c = -L_*^2 e, \quad b = -\frac{3}{8}c. \tag{7.3.7}$$

In turn, this implies that the effective Planck length is given by

$$\kappa^2_{\text{eff}} = \kappa^2\frac{\Gamma(n-1/2)}{\sqrt{\pi}\Gamma(n)}, \tag{7.3.8}$$

while the other physical parameters read

$$m_g^2 = -\frac{1}{L_*^2} = -\frac{n}{L^2}, \quad m_s^2 \to \infty. \tag{7.3.9}$$

The latter shows that the ghost scalar modes decouples, generalizing what occurs in NMG.

Without imposing any particular value for e, b and c yet, the trace of the linearized field equations read

$$\bar{g}^{\mu\nu}\mathcal{E}^L_{\mu\nu} = (8b+3c)\bar{\Box}R^L - \left(\frac{24b}{L_*^2} + \frac{8c}{L_*^2+e}\right)R^L. \tag{7.3.10}$$

The term proportional to $\Box R^L$ vanishes as a consequence of imposing a c-theorem. Using the other relations, which implement the single-vacuum condition, we see that the coefficient of R^L in the second term of (7.3.10) vanishes too, and in that case we obtain

$$\mathcal{E}^L = 0. \tag{7.3.11}$$

This means that the theories we are considering have an extra conformal invariance at linearized level, around constant curvature backgrounds, which corresponds to a Weyl symmetry transformation. This symmetry is associated to the existence of a family of conformally flat solutions that generalize the BTZ black hole [16] to black holes with a gravitational hair. We will study these solutions in the next section.

7.4 Hairy black holes

Remarkably, it can be checked that the whole family $\mathcal{D}^{(n)}$, defined in (7.2.23), admit Hairy Black Hole (HBH) solutions of the type

$$ds^2_{HBH} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \;\; f(r) = \frac{r^2}{L_*^2} + br - \mu \;, \tag{7.4.1}$$

which reduces to the BTZ geometry when $b = 0$. This, added to the fact that these theories possess a conformal symmetry at the linearized level, suggests that they can be backreacted by a quantum conformally coupled scalar field in the same manner as NMG.

Furthermore, notice that if $b < 0$, this family of solutions will have two horizons

$$r_\pm = \frac{L_*}{2}\left(-bL_* \pm \sqrt{b^2L_*^2 + 4\mu}\right), \quad r_+ - r_- = L_*\sqrt{b^2L_*^2+4\mu}. \tag{7.4.2}$$

When $-b^2L_*^2 = 4\mu$, the solution becomes extremal. There is a rich family of possible causal structures captured by these black holes, as shown by the following figure

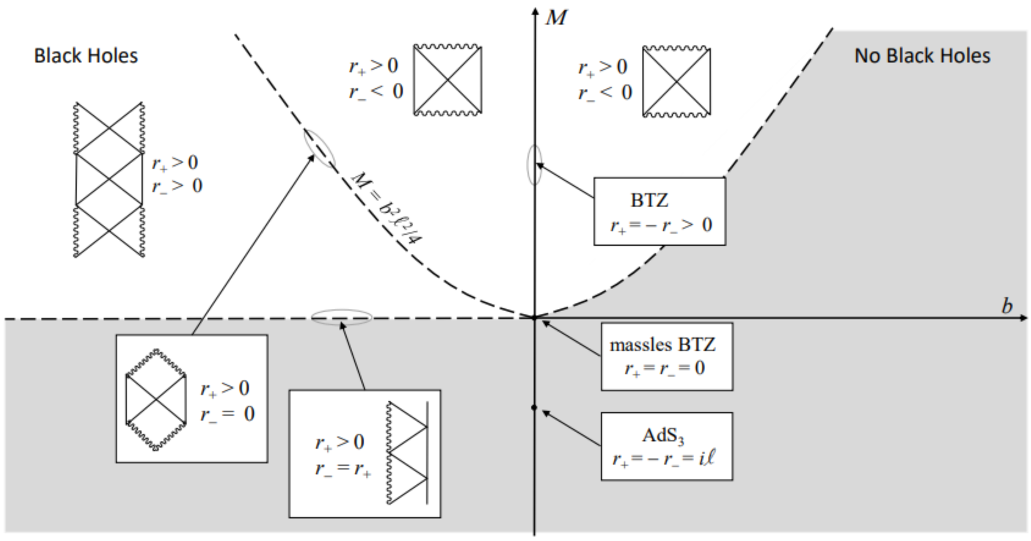

Figure 7.4.1: Spectrum of hairy black holes for NMG, found and studied in detail in [1]. Figure taken from [2]

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7.5 Conserved charges and Thermodynamics

This final section will be focused on studying thermodynamical quantities associated to the hairy black hole solutions (7.4.1) for the Higher-Curvature theories governed by (7.2.23).

The Hawking temperature is given by

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi L_*}\sqrt{b^2L_*^2 + 4\mu} \quad , \tag{7.5.1}$$

which naturally vanishes for the extremal solution $r_+ = r_-$.

Now, in order to compute the mass, we do the following change of coordinates $r' \to r + \frac{bL_*^2}{2}$.

Then, the metric function takes the form $f(r(r')) = \frac{r'^2}{L_*^2} - \left(\mu + \frac{b^2L_*^2}{4}\right)$, which is the usual form of the BTZ metric, with a shifted mass parameter $\mu \to \mu + \frac{b^2L_*^2}{4}$. Thus, the mass for this theories will be given by

$$M^{BTZ}(\mu \to \mu + \frac{b^2 L_*^2}{4}) = \frac{\pi}{4\kappa_{\text{eff}}^2}(b^2 L_*^2 + 4\mu). \tag{7.5.2}$$

This result was confirmed by the explicit calculation of the mass using the Abbot-Deser-Tekin method up to $n = 6$ (see [31] and references therein).

On the other hand, since we constructed our Lagrangian using Ricci tensors and its contractions, we can now use the Wald's formula to find the Bekenstein-Hawking entropy for this family of theories [24], namely

$$S = -\frac{\pi}{\kappa^2} \int_{\Sigma_h} \frac{\partial \mathcal{D}^{(n)}}{\partial R_{\mu\nu}} \epsilon_{\mu\lambda} \epsilon_\nu{}^\lambda \omega, \tag{7.5.3}$$

where ω represents a volume 1-form of the induced metric Σ_h and ϵ are binormal vectors to the induced space-like surfaces. We can calculate this derivative explicitly as

$$\begin{aligned}\frac{\partial \mathcal{D}^{(n)}}{\partial R_{\mu\nu}} =& g^{\mu\nu} + \sum_{i=2}^{n} \binom{n}{i} \frac{L^{2(i-1)}}{n^i 6^{i-1}(3-2i)} \Bigg\{ i R^{i-1} g^{\mu\nu} \\ & - \sum_{p=1}^{\lfloor i/2 \rfloor} 24^p (2p-1) \binom{i}{2p} \left[(i-2p) R^{i-2p-1} \mathcal{S}_2^p g^{\mu\nu} + 2p R^{i-2p} \mathcal{S}_2^{p-1} \tilde{R}^\mu_\lambda \tilde{R}^\nu \right] \\ & + \sum_{p=1}^{\lfloor (i-1)/2 \rfloor} 24^{p+1} p \binom{i}{2p+1} \Big[(i-2p-1) R^{i-2p-2} \mathcal{S}_2^{p-1} \mathcal{S}_3 g^{\mu\nu} \\ & (2p-1) R^{i-2p-1} \mathcal{S}_2^{p-2} \mathcal{S}_3 \tilde{R}^\mu_\lambda \tilde{R}^{\nu\lambda} + R^{i-2p-1} \mathcal{S}_2^{p-1} \mathcal{S}_3 \left(3\tilde{R}^\mu_\lambda \tilde{R}^{\lambda\nu} - \mathcal{S}_2 g^{\mu\nu} \right) \Big] \Bigg\}.\end{aligned} \tag{7.5.4}$$

After evaluating this expression on AdS$_3$ (i.e. $b = 0$ and $\mu = -1$) and replacing it back into (7.5.3), we obtain a final expression for the Bekenstein-Hawking entropy after integration, given by

$$S = \frac{2\pi^2 L_*}{\kappa^2} \frac{\sqrt{\pi}\Gamma(n)}{\Gamma(n-1/2)} \sqrt{b^2 L_*^2 + 4\mu}\ , \tag{7.5.5}$$

where n is the order in curvature of the theories.

Again, we see that for $b = 0$, we recover the entropy for the BTZ black hole in higher curvature theories [32].

Finally, we see that the combined results for the black hole's temperature, entropy and mass can be shown to satisfy the first law of black hole thermodynamics

$$dM = TdS, \quad \text{with} \quad d = d\mu \frac{\partial}{\partial \mu} + db \frac{\partial}{\partial b}. \tag{7.5.6}$$

By combining these expression, we can also construct the free energy of these black holes, namely

$$F = M - TS = -\frac{\pi (r_+ - r_-)^2}{2\kappa_{\text{eff}}^2 L_*^2}. \tag{7.5.7}$$

The free energy vanishes for the extremal solutions, and it is always negative.

Chapter 8

Conclusion

In this work, we have studied the interaction between higher-curvature terms, contained by three-dimensional massive gravity theories, and the backreaction on the geometry induced by quantum fluctuations. As a working example, we used the quadratic theory NMG coupled to N conformally coupled scalar fields in (A)dS$_3$ space. In the large N limit, we solved the semi-classical field equations analytically, at the first order approximation in GN.

We have shown that

- The backreaction only induces a relaxation of standard Brown-Henneaux asymptotia for (A)dS$_3$ if working at the point where NMG has a single vacuum, that is, where $\lambda = m^2$.
- The relaxation of the asymptotia yield the addition of two asymptotic symmetry generators. Thus, the isometry group of the special point backreacted solution has the structure of the semidirect sum of two copies of Witt's Algebra with supertranslations and logarithmic supertranslations.

Additionally, we showed that an extension to arbitrarily higher-curvature theories can be done, for which the entire analysis made for NMG holds. In relation to this family of theories, we have found that

- These solutions are constructed by imposing the fulfillment of a $c-$theorem and the property of having a single vacuum.
- We computed the masses, temperatures and entropy in a generalized

form. The computations were crosschecked by satisfying the first law of thermodynamics.

We have, in consequence, found a family of higher curvature gravity theories with universal properties both at the classical and quantum level, as well as holographically.

Appendix A

Diffeomorphism invariance of massless graviton

The massless graviton $h_{\mu\nu}$ is a rank-2, symmetric field, whose dynamics are given by the Fierz-Pauli action

$$I_{FP}\left[h_{\mu\nu}\right] = \int d^D x \left(\frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{2}\partial^\rho h^{\mu\nu}\partial_\mu h_{\rho\nu} + \frac{1}{2}\partial^\mu h \partial^\nu h_{\mu\nu} - \frac{1}{4}\partial^\mu h \partial_\mu h\right), \tag{A0.1}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. This action is quasi-invariant to the diffeomorphism transformation

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}. \tag{A0.2}$$

Here, ξ_μ is an arbitrary vector. We can take the Fourier transform of both $h_{\mu\nu}$ and ξ_μ in both time and space, namely

$$\xi_\mu = \int \frac{d^D k}{(2\pi)^D} e^{ik\cdot x}\tilde{\xi}_\mu(k), \tag{A0.3}$$

$$h_{\mu\nu} = \int \frac{d^D k}{(2\pi)^D} e^{ik\cdot x}\tilde{h}_{\mu\nu}(k). \tag{A0.4}$$

Now, let us consider the following vector basis $\left\{u^1_\mu, ..., u^D_\mu\right\}$, and define each of them in terms of longitudinal and transversal vectors, namely

$$u^1_\mu = k_\mu \qquad u^2_\mu = \bar{k}_\mu \qquad u^3_\mu = \epsilon^{(1)}_\mu \qquad ... \qquad u^D_\mu = \epsilon^{(D-2)}_\mu, \tag{A0.5}$$

where k^μ and $\bar{k}^\mu$ describe longitudinal polarizations, while $\epsilon^{(i)}_\mu$ are spacelike orthonormal fields. These satisfy

$$k^\mu k_\mu = \bar{k}^\mu \bar{k}_\mu \ , \qquad k^\mu \epsilon^{(i)}_\mu = \bar{k}^\mu \epsilon^{(i)}_\mu = 0, \quad \forall\ i = 1, ..., D-2. \tag{A0.6}$$

With these, we can construct a tensor base $\left\{ v^{(1)}_{\mu\nu}, ..., v^{(D(D+1)/2)}_{\mu\nu} \right\}$ for symmetric rank-2 fields, by

$$v^{(I)}_{\mu\nu} \equiv v^{\{i,j\}}_{\mu\nu} = \frac{1}{2}(u^i_\mu u^j_\nu + u^i_\nu u^j_\mu) = u^i_{(\mu} u^j_{\nu)}, \quad \text{where } I \in [1, D(D+1)/2]. \tag{A0.7}$$

For further clarity, we will be expanding the index I in the following way

$$v^{(1)}_{\mu\nu} = v^{\{1,1\}}_{\mu\nu} = k_\mu k_\nu \ , \tag{A0.8}$$
$$v^{(2)}_{\mu\nu} = v^{\{1,2\}}_{\mu\nu} = k_{(\mu} \bar{k}_{\nu)} \ , \tag{A0.9}$$
$$\left\{ v^{(3)}_{\mu\nu}, ..., v^{(D)}_{\mu\nu} \right\} = v^{\{1,i+2\}}_{\mu\nu} = k_{(\mu} \epsilon^{(i)}_{\nu)} \ , \tag{A0.10}$$
$$\tag{A0.11}$$
$$v^{(D+1)}_{\mu\nu} = v^{\{2,2\}}_{\mu\nu} = \bar{k}_\mu \bar{k}_\nu \ , \tag{A0.12}$$
$$\left\{ v^{(D+2)}_{\mu\nu}, ..., v^{(2D-1)}_{\mu\nu} \right\} = v^{\{2,i+2\}}_{\mu\nu} = \bar{k}_{(\mu} \epsilon^{(i)}_{\nu)} \ , \tag{A0.13}$$
$$\tag{A0.14}$$
$$\left\{ v^{(2D)}_{\mu\nu}, ..., v^{(D(D+1)/2)}_{\mu\nu} \right\} = v^{\{i+2,j+2\}}_{\mu\nu} = \epsilon^{(i)}_{(\mu} \epsilon^{(j)}_{\nu)}. \tag{A0.15}$$

Using this basis, we can expand our fields in Fourier space as

$$\tilde{\xi}_\mu = \sum_{i=1}^{D} \left(a^i u^i_\mu \right) \ , \tag{A0.16}$$
$$\tilde{h}_{\mu\nu} = \sum_{I=1}^{D(D+1)/2} \left(A^{(I)} v^{(I)}_{\mu\nu} \right). \tag{A0.17}$$

Since we defined $u^1_\mu = k_\mu$, we can factorize our field transformation as

$$\begin{aligned}
\tilde{h}'_{\mu\nu} &= \tilde{h}_{\mu\nu} + 2k_{(\mu}\tilde{\xi}_{\nu)} i && \text{(A0.18)}\\
&= \sum_{I=1}^{D(D+1)/2} \left(A^{(I)} v^{(I)}_{\mu\nu}\right) + 2\sum_{j=1}^{D} \left(a^j u^1_{(\mu} u^j_{\nu)}\right) i && \text{(A0.19)}\\
&= \sum_{i=1,j=1}^{\{i,j\}} \left(A^{\{i,j\}} v^{\{i,j\}}_{\mu\nu}\right) + 2\sum_{j=1}^{D} \left(a^j v^{\{1,j\}}_{\mu\nu}\right) i \\
&= \sum_{j=1}^{D} \left(A^{\{1,j\}} + 2ia^j\right) v^{\{1,j\}}_{\mu\nu} + \sum_{i=2,j=2}^{\{i,j\}} \left(A^{\{i,j\}} v^{\{i,j\}}_{\mu\nu}\right). && \text{(A0.20)}
\end{aligned}$$

Since the transformation is arbitrary, we can fix $a^j = -\frac{i}{2}A^{\{1,j\}}$. This automatically gets rid of D components of $\tilde{h}'_{\mu\nu}$, which let us safely set all original $A^{\{1,j\}}$ to zero. Thus, we have proven that diffeomorphism invariance gets rid of D degrees of freedom for a symmetric, rank-2 field $h_{\mu\nu}$, namely, all the components of $\tilde{h}_{\mu\nu} \sim k_{(\mu}u_{\nu)}$, for some u_ν of the vector basis can be gauged away.

Now, this diffeomorphism invariance will further eliminate degrees of freedom by analizing the field equations, which read

$$\Box h_{\mu\nu} + \partial_\mu\partial_\nu h + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} - \eta_{\mu\nu}\Box h - \partial^\rho\partial_\mu h_{\nu\rho} - \partial^\rho\partial_\nu h_{\mu\rho} = 0. \qquad \text{(A0.21)}$$

By going to Fourier space and inserting the gauged field $h_{\mu\nu} = \sum_{I=D+1}^{D(D+1)/2} A^{(I)} v^{(I)}_{\mu\nu}$ and vectors, we obtain

$$\sum_{I=D+1}^{D(D+1)/2} A^I \left[v^{(1)} v^{(I)}_{\mu\nu} + v^{(1)}_{\mu\nu} v^{(I)} + \eta_{\mu\nu}\left(v^{(1)}_{\alpha\beta} v^{(I)\ \alpha\beta} - v^{(1)} v^{(I)}\right) - v^{(1)\ \rho}_{\mu} v^{(I)}_{\nu\rho} - v^{(1)\ \rho}_{\nu} v^{(I)}_{\mu\rho}\right] = 0, \qquad \text{(A0.22)}$$

where we defined tensor traces as $v^{(I)} = v^{(I)}_{\alpha\beta}\eta^{\alpha\beta}$. Now, let us work out each of the contributions on the equations separately.

Tensor traces

$$v^{(1)} = k_\alpha k^\alpha = v^{(D+1)} \ , \tag{A0.23}$$

$$v^{(2)} = k_\alpha \bar{k}^\alpha \ , \tag{A0.24}$$

$$\left\{v^{(3)}, ..., v^{(D)}\right\} \sim k_\alpha \epsilon^{(i)\ \alpha} = 0 \ , \tag{A0.25}$$

$$\left\{v^{(D+2)}, ..., v^{(2D-1)}\right\} \sim \bar{k}_\alpha \epsilon^{(i)\ \alpha} = 0 \ , \tag{A0.26}$$

$$\left\{v^{(2D)}, ..., v^{(D(D+1)/2)}\right\} = \epsilon^{(i)}_\alpha \epsilon^{(j)\ \alpha} \ . \tag{A0.27}$$

Total contractions

$$v^{(1)}_{\alpha\beta} v^{(D+1)\ \alpha\beta} = k_\alpha k_\beta \bar{k}^\alpha \bar{k}^\beta, \tag{A0.28}$$

$$v^{(1)}_{\alpha\beta} \left\{v^{(D+2)\ \alpha\beta}, ..., v^{(2D-1)\ \alpha\beta}\right\} \sim k_\alpha k_\beta k^\alpha \epsilon^{(i)\ \beta} = 0, \tag{A0.29}$$

$$v^{(1)}_{\alpha\beta} \left\{v^{(2D)\ \alpha\beta}, ..., v^{D(D+1)/2\ \alpha\beta}\right\} \sim k_\alpha k_\beta \epsilon^{(i)\ \alpha} \epsilon^{(j)\ \beta} = 0. \tag{A0.30}$$

Partial contractions

$$v^{(1)\ \rho}_\mu v^{(D+1)}_{\nu\rho} = k^\rho \bar{k}_\rho k_\mu \bar{k}_\nu, \tag{A0.31}$$

$$v^{(1)\ \rho}_\nu v^{(D+1)}_{\mu\rho} = k^\rho \bar{k}_\rho k_\nu \bar{k}_\mu, \tag{A0.32}$$

$$v^{(1)\ \rho}_\mu \left\{v^{(D+2)}_{\nu\rho}, ..., v^{(2D-1)}_{\nu\rho}\right\} = k^\rho \bar{k}_\rho k_\mu \epsilon^{(i)}_\nu, \tag{A0.33}$$

$$v^{(1)\ \rho}_\nu \left\{v^{(D+2)}_{\mu\rho}, ..., v^{(2D-1)}_{\mu\rho}\right\} = k^\rho \bar{k}_\rho k_\nu \epsilon^{(i)}_\mu, \tag{A0.34}$$

$$v^{(1)\ \rho}_\nu \left\{v^{(2D)}_{\mu\rho}, ..., v^{(D(D+1)/2)}_{\mu\rho}\right\} \sim k^\rho \epsilon^i_\rho = 0. \tag{A0.35}$$

Summing (A0.31) with (A0.32) and (A0.33) with (A0.34) we see that the partial contractions reduce to

$$v^{(1)\ \rho}_\mu v^{(D+1)}_{\nu\rho} + v^{(1)\ \rho}_\nu v^{(D+1)}_{\mu\rho} = k_\rho \bar{k}^\rho \left(k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu\right) = k_\rho \bar{k}^\rho v^{(2)}_{\mu\nu} \ , \tag{A0.36}$$

$$\begin{aligned} &v^{(1)\ \rho}_\mu \left\{v^{(D+2)}_{\nu\rho}, ..., v^{(2D-1)}_{\nu\rho}\right\} + v^{(1)\ \rho}_\nu \left\{v^{(D+2)}_{\mu\rho}, ..., v^{(2D-1)}_{\mu\rho}\right\} \\ &= k_\rho \bar{k}^\rho \left(k_\mu \epsilon^{(i)}_\nu + k_\nu \epsilon^{(i)}_\mu\right) = k_\rho \bar{k}^\rho \left\{v^{(3)}_{\mu\nu}, ..., v^{(D)}_{\mu\nu}\right\} \end{aligned} \tag{A0.37}$$

Inserting these results back into the field equation, we obtain the following, factorized by base $v_{\mu\nu}^{(I)}$, expression

$$v_{\mu\nu}^{(1)}\left(\sum_{I=D+1}^{D(D+1)/2} A^{(I)}v^I\right)+\sum_{I=2}^{D} v_{\mu\nu}^{(I)}\left(-A^{(I+D-1)}k_\alpha\bar{k}^\alpha\right)+\sum_{I=D+1}^{D(D+1)/2} v_{\mu\nu}^{(I)}\left(A^{(I)}k_\alpha\bar{k}^\alpha\right)$$

$$+\,\eta_{\mu\nu}\left[A^{(D+1)}\left(k_\alpha\bar{k}^\alpha k_\beta\bar{k}^\beta-k_\alpha k^\alpha k_\beta k^\beta\right)-\sum_{I=2D}^{D(D+1)/2} v^{(I)}v^{(1)}\right]=0. \tag{A0.38}$$

Remember that our gauged field does not longer depend on the basis vectors $\left\{v_{\mu\nu}^{(1)},...,v_{\mu\nu}^{(D)}\right\}$, but these vectors reappered in the field equation due to the partial contractions. However, if a field does not depend on these components, its dynamics should also be independent of them. Thus, all of the components proportional to the already gauged coordinates in the fields must be set to zero, namely

(a) The component of $v_{\mu\nu}^{(1)}$ must vanish, so $\left(\sum_{I=D+1}^{D(D+1)/2} A^{(I)}v^I\right)=0.$ (A0.39)

(b) The components of $\left\{v_{\mu\nu}^{(2)},...,v_{\mu\nu}^{(D)}\right\}$ must vanish, so $\left(-A^{(I+D-1)}k_\alpha\bar{k}^\alpha\right)=0 \quad \forall I=2,...,D.$ (A0.40)

Let us review (b) first. Since $k_\alpha\bar{k}^\alpha\neq 0$, this automatically means

$A^{(I+D-1)}=0, \qquad \forall I=2,...,D-1$ which accounts for $D-1$ new constraints. (A0.41)

Now, by inserting (b) and our already calculated traces into (a), we obtain

$$\left(\sum_{I=D+1}^{D(D+1)/2} A^{(I)}v^I\right)=0 \quad \leftrightarrow \quad A^{(2D)}v^{(2D)}+...+A^{D(D+1)/2}v^{D(D+1)/2}=0. \tag{A0.42}$$

Allowing us to rewrite one coefficient $A^{(I)}$ in term of the others, namely

$A^{2D}=-\frac{1}{v^{2D}}\sum_{I=2D+1}^{D(D+1)/2} A^{(I)}v^{(I)}$ which accounts for 1 new constraint. (A0.43)

Finally, we see that diffeomorphism invariance gets rid of D components by the pure structure of the transformation, and by inserting the invariance on-shell, this transformation further restricts the components by D more constraints. Therefore, a symmetric, rank-2 tensor in D-dimensions has a total number of local degrees of freedom given by

$$\textbf{Physical, on shell, DOF} = \frac{D(D+1)}{2} - D - D = \frac{D(D-3)}{2}. \tag{A0.44}$$

Appendix B

Aspects of Anti-de Sitter

Three-dimensional Anti-de Sitter spacetime (AdS_3) is defined as the hypersurface defined by the constraint

$$-T_1^2 + X_1^2 - T_2^2 + X_2^2 = -L_*^2, \tag{A0.1}$$

embedded in four dimensional flat space with metric η_{ab}. From now on, Latin indices will be used for the ambient space with metric

$$ds^2 = -dT_1^2 + dX_1^2 - dT_2^2 + dX_2^2, \tag{A0.2}$$

and Greek indices for AdS3.

This pseudohyperbolic hypersurface has a radius vector defined as $x^a = (T_1, X_1, T_2, X_2)$, with norm $\sqrt{-x^a x_a} = L_*$, and constant curvature given by $R^{\alpha\beta}_{\mu\nu} = -\frac{1}{L_*^2}\delta^{\alpha\beta}_{\mu\nu}$, and therefore $R_{\mu\nu} = -\frac{2}{L_*^2} g_{\mu\nu}$ and $R = -6/L_*^2$.

In what follows, we will discuss both global and local properties of AdS relevant to the present work, with close reference to the arguments described by [7, 18].

A1 Global properties

A1.1 Closed time-like curves and Covering Anti-de Sitter spacetime

Let us start by addressing the existence of closed time-like curves.

There are plenty of parametrization to study AdS. Let us first review the parametrization used for studying the spacetime structure of AdS; the so-called global coordinate parametrization

$$T_1 = L_* \sec\rho \cos\tau, \qquad T_2 = L_* \sec\rho \sin\tau, \tag{A1.1}$$

$$X_1 = L_* \tan\rho \cos\theta, \qquad X_2 = L_* \tan\rho \sin\theta, \tag{A1.2}$$

leading to the AdS_3 metric in global coordinates

$$ds^2 = L_*^2 \sec^2\rho \left(-d\tau^2 + d\rho^2 + \sin^2\rho d\theta^2\right), \tag{A1.3}$$

where $0 \le \rho \le \pi/2$, $0 \le \phi < 2\pi$ and $-\pi < \tau \le \pi$, with $\tau = -\pi$ and $\tau = \pi$, identified. This periodicity in the time-coordinate leads to Closed Time-like Curves (CTC), as shown in Figure A1.1

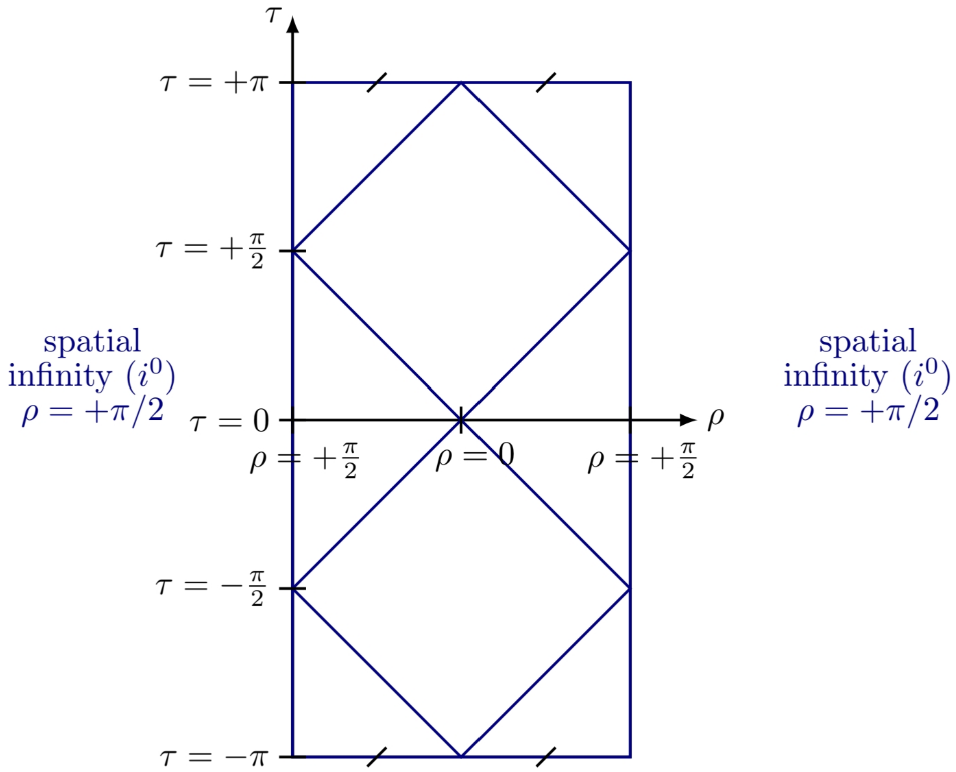

Figure A1.1: Penrose diagram for Global Anti-de Sitter space-time, with $\tau = \pm\pi$ identified

This causes a problem if we want to have a well defined Cauchy problem, since there is no distinction between cause and effect due to time periodicity. This issue is dealt with by 'unwrapping', or extending the domain of the time coordinate, so that $-\infty < \tau < \infty$, and thus getting rid of CTC.

This is known as the universal covering space of anti-de Sitter, due to the absence of non-contractible time cycles, whose structure is shown in Figure A1.2

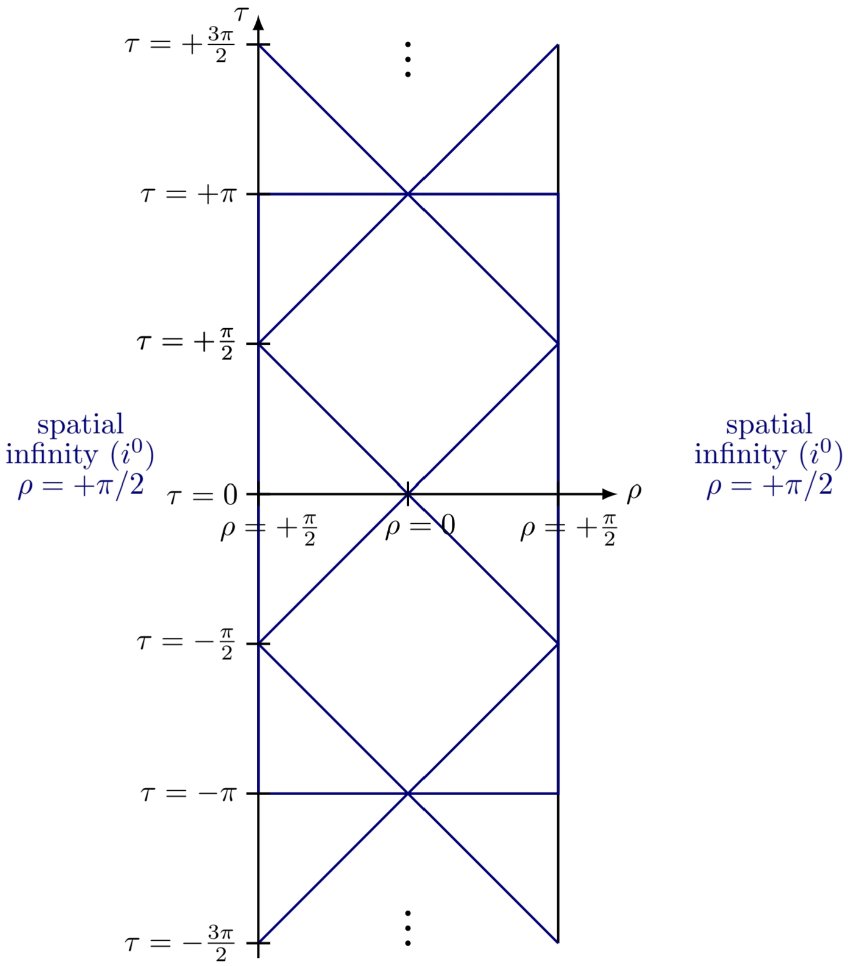

Figure A1.2: Penrose diagram for the universal covering spacetime of Anti-de Sitter.

A1.2 Non-global hyperbolicity

Having addressed the existence of CTC, there is still a key property of both AdS and CAdS that needs to be tackled, that is the existence of time-like boundary surfaces at spatial infinity. To understand this, let us work through some globally hyperbolic examples.

An event on Minkowski spacetime may be completely determined by the evolution of initial data laying on a Cauchy Hypersurface Σ. In other words, the causal past of an event in Minkowski spacetime completely intersects the space-like Cauchy surface, which is purely determined by initial conditions.

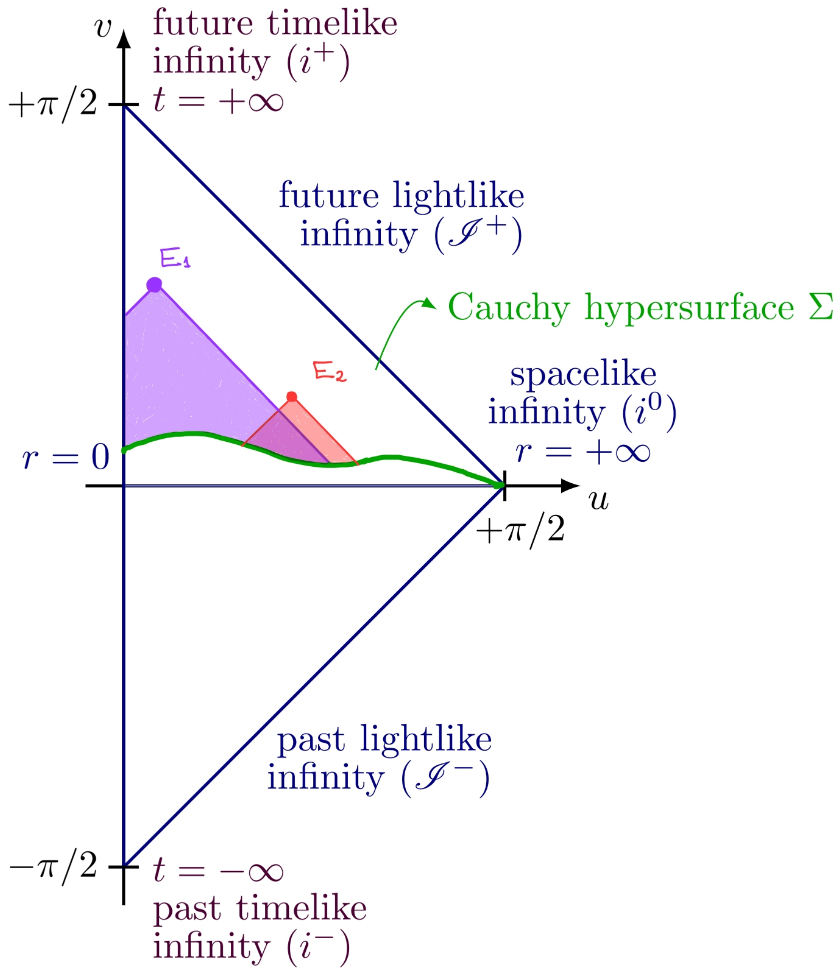

Figure A1.3: The causal past of events E_1 and E_2 completely intersect a Cauchy hypersurface on Minkowksi spacetime.

This is also the case for Schwarzschild spacetime, where data can always be evolved from the initial data defined on a Cauchy hypersurface.

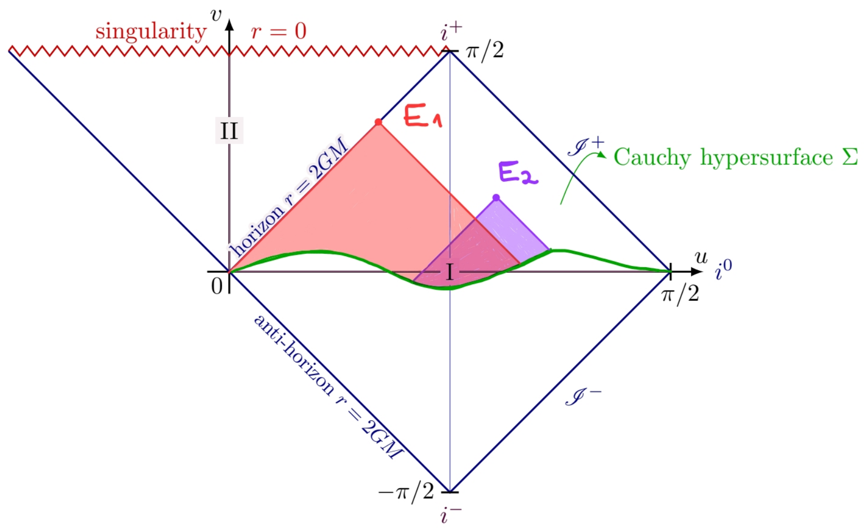

Figure A1.4: The causal past of events E_1 and E_2 completely intersect a Cauchy hypersurface on Schwarzschild spacetime.

However, the causal past of a given event on both AdS and CAdS spacetimes does not completely intersect a Cauchy hypersurface, but rather have intersections with the spatial boundary. This means that the initial data on Σ is not enough to have a well-defined causal past for a given event, but also, boundary conditions at spatial infinity must also be specified.

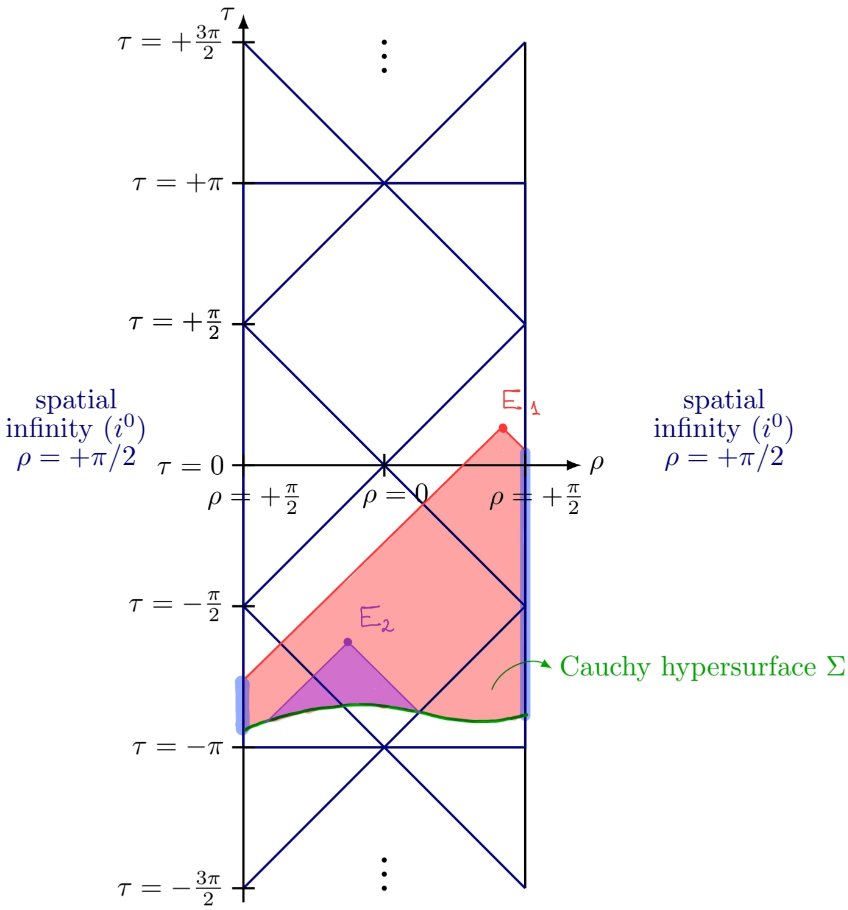

Figure A1.5: The causal past of the event E_2 fully intersects the Cauchy hypersurface Σ, However, the causal past of the event E_1 is partially determined by spatial infinity, shown by the blue highlights.

A more intuitive way to visualize this is that, if signals can reach infinity within a finite time, that means that signals can also come from infinity, and thus, can be part of the causal past for a given event. However, if boundary conditions are not specified, there is an ambiguous interpretation of what is really happening to signals when they do reach spatial infinity. Having reflective boundary conditions on CAdS is one common choice, implying that signals reaching infinity will bounce back into the bulk.

A1.3 Conformal relation to half of Einstein's Universe

A key feature of CAdS comes from being conformally equivalent to half of Einstein static universe (ESU), whose 3−dimensional global metric goes by

$$ds^2_{ESU} = -d\tau^2 + d\rho^2 + \sin^2\rho d\theta^2, \tag{A1.4}$$

where $0 \leq \tau < \infty$, $0 \leq \phi < 2\pi$ and $0 \leq \rho < \pi$.

As we showed in the previous subsection, the global coordinates of AdS$_3$ go as

$$ds^2_{AdS_3} = L_*^2 \sec^2\rho \left(-d\tau^2 + d\rho^2 + \sin^2\rho d\theta^2\right), \tag{A1.5}$$

where the only difference respect to ESU relies on a conformal factor we define as $\Omega^2(\rho) \equiv l^2 \sec^2\rho$ and only covering half of ESU's radial domain, namely $0 \leq \rho < \pi/2$. This means that AdS$_3$ is conformally equivalent to half of ESU.

A2 Local properties

A2.1 Isometry group

The isometry group of AdS$_3$ corresponds to the subgroup of the imbedding 4-dim space which leaves the constraint (A0.1) invariant, and it is given by $SO(2,2)$. This group is generated by boosts and rotations, and thus translating the imbedding space as two copies of 1+1 Minkowski space: M_1 with coordinates (T_1, X_1), and M_2 with coordinates (T_2, X_2) [7].

A2.2 Rindler coordinates

Having identified the isometry group of AdS$_3$ as $SO(2,2)$, we can further explore the local, causal properties of the space, by introducing the Rindler cordinates (t, r_i, ϕ), where $r_i = T_i^2 - X_i^2$ and thus, the constraint (A0.1) is now defined by $r_1 + r_2 = L_*^2$. This is given by

$$T_i = \sqrt{r_i}\cosh\chi_i, \quad X_i = \sqrt{r_i}\sinh\chi_i, \quad r > 0, \quad -\infty < \chi_i < \infty \quad , \tag{A2.1}$$

$$T_i = \sqrt{-r_i}\sinh\chi_i, \quad X_i = \sqrt{-r_i}\cosh\chi_i, \quad r < 0, \quad -\infty < \chi_i < \infty \quad . \tag{A2.2}$$

This parametrization is valid both in the lightcone interior ($r_i > 0$) and exterior ($r_i < 0$). This parametrization separates the space in three distinct regions:

1. $r_1 > L_*^2 \quad (r_2 < 0)$, where the Killing vectors ∂_t and ∂_ϕ are timelike and spacelike, respectively.

2. $0 < r_1, r_2 < L_*^2$, where both vectors ∂_t and ∂_ϕ are spacelike

3. $r_1 < 0 \quad (r_2 > 0)$, where the Killing vectors ∂_t and ∂_ϕ are spacelike and timelike, respectively.

Thus, if centering in the asymptotic region of our spacetime, we can restrict ourselves to the first region. By redefining the radial coordinates as $r \equiv r_1 = L_*^2 - r_2$, we obtain the metric

$$ds^2 = -\left(\frac{r^2}{L_*^2} - 1\right) dt^2 + \left(\frac{r^2}{L_*^2} - 1\right)^{-1} dr^2 + r^2 d\phi^2, \qquad t, \phi \in (-\infty, \infty). \quad \text{(A2.3)}$$

A3 Identifications of AdS$_3$: The BTZ black hole

The metric defined by (A2.3) can be regarded as a purely geometric spacetime, with no singularities produced by a black hole. If one were to deform the presence of a black hole of mass M and angular momentum J, we will arive at the spacetime containing a Rotating BTZ Black Hole, given by the metric

$$ds^2_{R-BTZ} = -\left(\frac{\bar{r}^2}{L_*^2} - M\right) d\bar{t}^2 - J d\bar{t} d\bar{\phi} + \left(\frac{\bar{r}^2}{L_*^2} - M + \frac{J^2}{4\bar{r}^2}\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\bar{\phi}^2, \quad \text{(A3.1)}$$

where the coordinates where transformed by taking

$$\begin{aligned} t &= \frac{1}{2\pi}\left(\alpha_+ \tilde{t} - \alpha_- L_* \bar{\phi}\right), \\ \phi &= \frac{1}{2\pi}\left(\alpha_+ \bar{\phi} - \alpha_- \bar{t}/L_*\right), \\ r^2 &= \frac{(2\pi\bar{r})^2 - \alpha_-^2 L_*^2}{\alpha_+^2 - \alpha_-^2}, \end{aligned} \quad \text{(A3.2)}$$

with

$$\alpha_\pm = \pi\left(\sqrt{M + J/L_*} \pm \sqrt{M - J/L_*}\right). \quad \text{(A3.3)}$$

Since this solution is a particular identification of AdS$_3$, is natural to see that points identified in the rotating black hole are still going to be related by an element of $SO(2,2)$, which in the ambient coordinates takes on the following form

$$\Lambda^a_b \equiv \begin{pmatrix} \cosh\alpha_+ & \sinh\alpha_+ & 0 & 0 \\ \sinh\alpha_+ & \cosh\alpha_+ & 0 & 0 \\ 0 & 0 & \cosh\alpha_- & -\sinh\alpha_- \\ 0 & 0 & -\sinh\alpha_- & \cosh\alpha_- \end{pmatrix}. \tag{A3.4}$$

Now, for static black holes, both the metric and transformation get reduced, since $J = \alpha_- = 0$. This yields the Static BTZ spacetime

$$ds^2_{BTZ} = -\left(\frac{\bar{r}^2}{L_*^2} - M\right) d\bar{t}^2 + \left(\frac{\bar{r}^2}{L_*^2} - M\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\bar{\phi}^2, \tag{A3.5}$$

and $SO(2,2)$ transformation

$$\begin{pmatrix} \cosh\alpha_+ & \sinh\alpha_+ & 0 & 0 \\ \sinh\alpha_+ & \cosh\alpha_+ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A3.6}$$

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