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Classical perturbations of AdS spacetimes

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Resumen

En esta tesis estudiaremos las perturbaciones gravitacionales de espaciotiempos asintóticamente anti-de Sitter (AdS) en cuatro dimensiones, usando la maquinaria desarrollada por Chandrasekhar, la cual se basa en el uso de una métrica general, axisimétrica y que puede depender del tiempo, escrita en el formalismo de tétradas. Este formalismo desacopla naturalmente las perturbaciones de los modos axiales (impares) y los modos polares (pares), y permite expresas sus ecuaciones como ecuaciones tipo Shroedinger.

Usaremos este método en dos backgrounds distintos: Schwarzschild AdS y el solitón AdS. En el primer caso obtenemos potenciales efectivos para ambos modos, axial y polar, y verificamos que coinciden con los potenciales encontrados en la literatura, también enfatizamos que la isoespectralidad se rompe cuando el espaciotiempo es asintóticamente AdS. En el segundo caso hacemos un análisis similar, logrando obtener una ecuación para las perturbaciones axiales la cual pudimos resolver numericaménte para determinar los modos normales del solitón AdS, mostrando que coinciden con los resultados encontrados en la literatura.

Concluímos que el método de Chandrasekhar que fue originalmente planteado para espaciotiempos asintóticamente planos puede ser extendido exitosamente a espaciotiempos con constante cosmológica negativa, incluyendo espaciotiempos que son regulares en todas partes, como el solitón AdS.

Keywords – Agujeros negros, solitones, espaciotiempos AdS, perturbación axial, perturbación polar.

Abstract

In this thesis we study the gravitational perturbations in four dimensional asymptotically anti-de Sitter (AdS) backgrounds using the approach developed by Chandrasekhar, which is based in a general time-dependent axisymmetric metric worked in the tetrad frame. This formalism allows to naturally decouple the perturbation of axial (odd) modes and polar (even) modes, and to express their equations as second order Schroedinger-like ones.

We apply the method to two different backgrounds: Schwarzschild-AdS and the AdS soliton. In the first case, we obtain effective potentials for both modes, axial and polar, and we verify that they correspond to the potentials found in literature by other methods and we emphasise how the isospectrality is broken by the presence of a negative cosmological constant. In the second case we perform a similar analysis, obtaining the differential equation for the axial perturbation. We were able to solve it numerically to obtain the normal modes of the AdS soliton, showing that they are in agreement with the results obtained by Constable and Myers in [1] .

We conclude that Chandrasekhar's approach originally formulated for asymptotically flat spacetimes, can be successfully extended to spacetimes with a negative cosmological constant, including spacetimes which are regular everywhere, such as the AdS soliton.

Keywords – Black holes, solitons, AdS spacetimes, axial perturbations, polar perturbations.

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Chapter 1

Introduction

The Einstein solutions with a negative cosmological constant have been of great interest since the early days of the AdS/CFT correspondence, a tool that proposes a duality between gravitational theories in asymptotically Anti-de Sitter (AdS) spacetimes and conformal field theories (CFT) defined in the boundary of said spacetimes. This duality has motivated the study of diverse asymptotically AdS gravitational backgrounds, in particular, the study of normal and quasinormal modes gives direct information about the dynamic of the strongly coupled dual system [2], [3], [4].

Regarding solutions without horizon, the AdS soliton is a particularly relevant background since it was proposed originally as a configuration with less energy than pure AdS, this is, a new positive energy conjecture at the time [5]. The AdS soliton is a regular spacetime, without singularities nor horizons, but with a non-trivial causal structure. In the context of the AdS/CFT correspondence, its normal modes are interpreted as the spectrum of spin-two excitations in the dual field theory, namely of glueballs, due to the absence of gravitons in the boundary.

On the other hand, AdS black holes (BHs) have been studied thoroughly given their role in the analysis of the thermal dual theories. The quasinormal modes (QNM) that describe the ringdown of BHs in AdS, also allow to describe the relaxation time of the thermal perturbations of the theory at the boundary.

In this thesis, we worked on a systematic analysis of the gravitational perturbations of different stationary and axisymmetric solutions in the presence of a negative cosmological constant, applying the machinery developed by Chandrasekhar but without cosmological constant. This approach, based on the use of the tetrad frame and the general structure of a stationary and axisymmetric metric in four dimensions, allows to study the axial and polar modes, and to formulate master Schroedinger-like equations for the perturbations.

In particular, we analyse in detail the stability of the Schwarzschild-AdS BH, recovering the known effective potentials for the axial and polar perturbations, and establishing the breaking of isospectrality between this two parities in contrast with the Schwarzschild BH. Subsequently, we try to extend this approach to the AdS soliton case, and we partially achieve to show that it is applicable to backgrounds without horizons, succeeding only at the level of vector-like perturbations.

The results we obtained make evident the utility of the tetrad frame in this kind of problems, and support previous results obtained using other methods. It is left for future work extensions of the same analysis for more complex backgrounds, such as rotating or charged ones, and the utilisation of the Newman-Penrose formalism, via Teukolski equations, which automatically gives decoupled equations for the linearised Penrose scalars, in the case of Petrov type D backgrounds.

This thesis is organized as follows: in Chapter 2, the backgrounds we will work with are presented. In Chapter 3 we present in detail the study of the gravitational perturbations of Schwarzschild BH, including the first approach of Regge and Wheeler, and Zerilli, and then the analysis of Chandrasekhar. In Chapter 4, we extend this machinery to two asymptotically AdS backgrounds, Schwarzschild AdS BH and the AdS soliton, and finally, we present our conclusions in Chapter 5.

In order to explore these ideas concretely, we have developed new numerical codes in Mathematica, attached in Appendix B.

Chapter 2

Gravity and AdS spacetimes

In 1686, Sir Isaac Newton presented his three laws of motion in the *Principia Mathematica Philosophiae Naturalis*, and they seemed to solve planet motion and explained gravity well enough for these to be the laws used for over two centuries, however, they presented problems at certain regimes and were incompatible with some data and experiments.

Newton assumes that time and space are absolute, independent of the speed of the observer. This is consistent with the fact that Newton's mechanics was based on Galilean transformations, but these transformations are not accurate for systems moving at velocities near speed of light, so in this regime Newton's laws are not longer valid.

There were also data that was not consistent with the predictions of Newtonian gravity, such as the anomalous precession of Mercury's perihelion, or the Michelson-Morley experiment that suggested that the speed of light was the same in any inertial frame, proving that there was not an absolute inertial frame, i.e., there was not something like ether that allowed bodies with mass to affect the motion of other bodies instantaneously.

Such problems motivated the development of a new theory of gravity, achieved by Einstein in 1915. First, in 1905, he addressed the speed of light issue in its Special Relativity theory, implementing two principles: the speed of light is the same in any inertial frame, and all inertial frames observe the same physics laws. The theory also established speed of light as a maximum speed for any signal to travel, including gravity; anything travelling faster than that would violate causality [6]. Special Relativity, however, does not describe gravity, since it is valid only for inertial frames in flat space. It is necessary to include accelerated systems of reference to describe how massive bodies affect each other. It is evident that Special Relativity does not include Newton's laws, but they were valid at low speeds, so the solution to integrate both results arrived 10 years later in the form of General Relativity [7].

Let's recall Newton's laws. The first law of inertia establishes that an object at rest, remains at rest, and an object moving at constant speed remains at constant speed unless acted on by a force. This law sets as inertial frames those for which free particles move along straight lines at constant speed. The key observation of Einstein was that objects in free fall will not notice that they are accelerating, so locally, they are also inertial frames. This is called the *strong equivalence principle*, and establishes that, at any point in spacetime, in a sufficiently small region, the physics laws of a free falling system are the same as those in absence of a gravitation field.

The strong equivalence principle implies that gravity is not a force in the classical sense, there is nothing pushing us down to the Earth surface. Instead, what Einstein proposed was that gravity is a consequence of the geometry of spacetime, specifically, he relates the presence of matter to the curvature of spacetime, and this curvature is the responsible of the acceleration of bodies and signals through spacetime, defining their trajectories. The preferred trajectories are called geodesics and they correspond to straight lines in flat space since flat space means no matter, so nothing is affecting the path of the object.

All these ideas are formalised in the following equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.0.1}$$

The left hand side is the Einstein tensor, $G_{\mu\nu}$, and contains the Ricci tensor, $R_{\mu\nu}$, the metric, $g_{\mu\nu}$, and the Ricci scalar, R. The Einstein tensor contains all the information about geometrical aspects of spacetime, and since originally the theory was formulated without torsion, it can be calculated only with the metric. The right hand side has the energy momentum tensor, $T_{\mu\nu}$, that contains all the information about density of energy and momentum in spacetime, i.e. all the information about matter content. It is important to notice that Einstein imposed conservation of energy and momentum, meaning that the energy momentum tensor needed to satisfy, $\nabla_\mu T^\mu{}_\nu = 0$. Consequently, the Einstein tensor is built such that $\nabla_\mu G^\mu{}_\nu = 0$.

In this manner, equation (2.0.1) successfully relates geometry of spacetime and matter. Energy "tells" spacetime how to curve, and the curvature "tells" the particles how to move [8].

As always, new theories bring new problems. In this case, the problem arose once Einstein realised that his theory did not admit solutions for a static universe. At the time, and as someone that believed in "the cosmic sense of religion", impulsed by the ideas of Baruch Spinoza, the concept of a dynamic universe was inadmissible, it needed to be static, eternal and immutable.

The Einstein field equations as presented previously predicted that any matter and energy distribution made the universe collapse under gravity or indefinitely expand. To solve this problem, Einstein, in 1917, introduced a new term in his equations to counteract the expansion, allowing a static universe to exist. The new equation took the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.0.2}$$

where Λ is so-called the *cosmological constant* [9].

In 1929, Hubble's observations showed that the universe was, in fact, expanding [10]; the static universe was a mistake. This expansion was consistent with the dynamic solution of Einstein, so he determined that the cosmological constant was not meant to exist, and he erased it from his equations, referring to it as the biggest blunder he had made in his entire life [11]

It was believed that the expansion velocity was diminishing due to the gravitational attraction between the matter in the universe, but in 1998, observations proved the contrary and the cosmological constant and the solutions presented with it took again the spotlight, but this time, instead of counteract an acceleration, it was introduced to add to it ([12], [13]).

In the followings subsections we will describe the spacetime solutions of the Einstein equations, with and without cosmological constant, that we will be interested in perturbing.

2.1 Spacetime solutions

We will consider vacuum solutions, meaning $T_{\mu\nu} = 0$, in $d = 4$. If we get the trace of Einstein's field equations (2.0.2) in vacuum, we obtain

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} &= 0 \qquad /g^{\mu\nu} \\ R - \frac{d}{2} R + d\Lambda &= 0 \\ R &= \frac{2d}{d-2} \Lambda = 4\Lambda. \end{aligned} \tag{2.1.1}$$

With this, we can conclude that in vacuum and without cosmological constant, $R = 0$ and Einstein field equations reduce to

$$R_{\mu\nu} = 0. \tag{2.1.2}$$

2.1.1 Schwarzschild black hole

The first solution to Einstein field equations without cosmological constant ($\Lambda = 0$) was found in 1916 by Karl Schwarzschild. This solution is the simplest non-trivial one; it describes the spacetime generated by a spherically symmetric body, without rotation nor charge.

When using units such that $G = c = 1$, Schwarzschild metric in spherical coordinates is written as:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{2.1.3}$$

where M is the mass of the spherically symmetric body, that generates the gravitational field (2.1.3).

The metric is singular for $r = 2M$ and $r = 0$, the former represents a singularity of the coordinates, meaning that it can be removed applying the right coordinate change, and the later is a singularity of spacetime. The coordinate singularity, $r = 2M = r_S$ is known as the Schwarschild radius, and when no matter is present and (2.1.3) is considered as a global solution, it describes a black hole.

This simple solution was fundamental in the development of black hole physics in

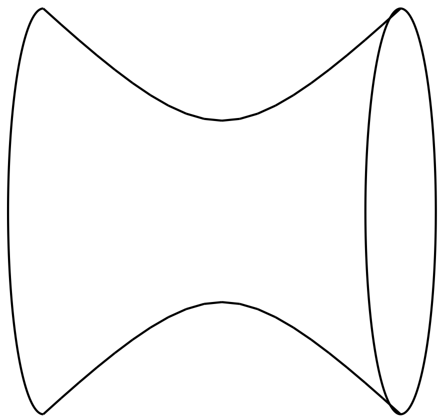

Figure 2.1.1: Two dimensional AdS spacetime

general relativity.

2.1.2 Anti-de Sitter (AdS) spacetime

Anti-de Sitter spacetime in d dimensions (AdS_d) is defined as the quadric

$$X_1^2 + \cdots + X_{d-1}^2 - U^2 - V^2 = -L^2 \tag{2.1.4}$$

embedded in a flat $(d-1)+2$ dimensional space with metric

$$ds^2 = dX_1^2 + \cdots + dX_{d-1}^2 - dU^2 - dV^2. \tag{2.1.5}$$

L is a natural length scale, proper of curved spaces, and we will call it the *AdS radius*, just as for a sphere $x^2 + y^2 + z^2 = R^2$, R is its radius and a natural length scale in that case.

In Figure 2.1.1 is represented a two dimensional AdS space as an hyperboloid of one sheet embedded in a three dimensional Minkowski space, where it is clear that corresponds to a hyperbolic geometry, i.e., negative curvature. AdS has topology $S^1(\text{time}) \times R^{d-1}(\text{space})$, meaning that it has closed timelike curves, e.g. the waist of the hyperboloid, and this translates to having periodic time, but this is not usually a problem since we can always "unwrap" the S^1 and go to the covering space (CAdS), which has the topology of R^4, hence, no periodic time.

AdS spacetime arises as a solution of Einstein's field equations with a negative cosmological constant ($\Lambda < 0$) and with no matter content ($T_{\mu\nu} = 0$). Remembering eq.(2.1.1), a negative cosmological constant is equivalent to a negative curvature, consistent with hyperbolic geometry. Equation (2.0.2) takes then the form

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{2.1.6}$$

and to hold the quadratic (2.1.4), it is necessary that $\Lambda = -\frac{3}{L^2}$.

The metric of AdS in spherical coordinates can be written as

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{L^2}\right)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.1.7}$$

with $-\infty < t < \infty$, $0 \leq r < \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

This is a stationary and spherically symmetric form of the metric; in fact, it has been proven that AdS spacetime is the only strictly globally stationary and asymptotically AdS solution of Einstein's field equations with $\Lambda < 0$ and $T_{\mu\nu} = 0$ ([14, 15]). Also note that there is no event horizon since $1 + \frac{r^2}{L^2} > 0$ everywhere. Pedagogical and extensive lectures notes about AdS spaces can be found in [16].

Note that this spacetime can be generalized for higher dimensions and its uniqueness is also proved [15], but this is of no interest to us, since we are interested in computing QNM using Chandrasekhar's approach, which is made for four dimensional axisymetric spaces.

This spacetime took relevance in the second half of last century, as a negative Λ arose in the maximally supersymmetric phase of gauged extended supergravity theories [17], or as people began noticing its relation with conformal field theory in the frontier of AdS, resulting in the famous Maldacena's paper [2] that founded the AdS/CFT correspondence in 1997. For a more recent article about uses of AdS spacetimes see [18]. Previous historical, seminal references of physics in AdS are [19–24].

2.1.3 Schwarzschild AdS

If we place a spherically symmetric body of mass M in a spacetime we want it to be asymptotically anti-de Sitter spacetime, we obtain Schwarzschild AdS spacetime [25]. The metric that describes the exterior field of the body in four dimensions can be written as

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{r^2}{L^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{r^2}{L^2}\right)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (2.1.8)$$

It is easy to see that in absence of the body, $M = 0$, AdS is recovered, and if $\Lambda = 0$ ($L \to \infty$), Schwarzschild spacetime (2.1.3) is also recovered.

For $M > 0$ and $0 \leq r < \infty$, the metric function $1 - \frac{2M}{r} + \frac{r^2}{L^2}$ has only one root for $r > 0$ and it represents a black hole horizon.

This metric can be generalized to higher dimensions, just as for AdS, but we will only work with the four dimensional case because of the same reason given above.

2.1.4 AdS soliton

The AdS soliton is a regular solution to Einstein equations with a negative cosmological constant, i.e., it does not contain singularities nor event horizons. It was first mentioned in 1998 by Horowitz and Myers in [5] in the context of a nonsupersymmetric version of the AdS/CFT correspondence. It was constructed in p+2-dimensions via a double analytic continuation of the near-extremal p-brane solution, which is an asymptotically AdS metric.

In four dimensions, the metric of the AdS soliton can be written as

$$ds^2 = -\frac{r^2}{L^2}dt^2 + \frac{dr^2}{f(r)} + f(r)d\phi^2 + \frac{r^2}{L^2}dz^2, \quad (2.1.9)$$

with $f(r) = \dfrac{r^2}{L^2} - \dfrac{2M}{r}$. Note that $r \geq r_0$, with r_0 such that $f(r_0) = 0$, and ϕ must be identified in order to avoid a conical singularity at $r = r_0$ with period $\beta = \dfrac{3\pi L^2}{3r_0}$.

In the same paper it was proposed that the AdS soliton was the lowest energy solution when the boundary conditions are non-supersymmetryc using holographic arguments, and it was further analysed in [26–28].

Chapter 3

Gravitational perturbations

3.1 Regge-Wheeler and Zerilli's equations

The perturbation analysis of the Schwarszchild BH was first addressed by Regge and Wheeler in [29]. In their article they wonder if small departures from Schwarszchild's metric affect its stability, and they proceed as follows:

Consider the corresponding metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \tag{3.1.1}$$

and small departures from it $h_{\mu\nu}$. The coordinate indices will be $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$.

The Ricci tensor computed from $g_{\mu\nu}$ is $R_{\mu\nu}$ and the one computed from $g_{\mu\nu} + h_{\mu\nu}$ will be $R_{\mu\nu} + \delta R_{\mu\nu}$. With this, it is easy to arrive to the expression

$$\delta R_{\mu\nu} = \left[-\frac{1}{2} g^{\alpha\beta} \left(h_{\mu\beta;\nu} + h_{\nu\beta;\mu} - h_{\mu\nu;\beta}\right)\right]_{;\alpha} + \left[\frac{1}{2} g^{\alpha\beta} \left(h_{\mu\beta;\alpha} + h_{\alpha\beta;\mu} - h_{\mu\alpha;\beta}\right)\right]_{;\nu}, \tag{3.1.2}$$

where semicolons denote covariant derivatives. As the spacetime is given by (3.1.1), where $R_{\mu\nu} = 0$, then the equation to be solved is

$$\delta R_{\mu\nu} = 0. \tag{3.1.3}$$

This equation is a second order partial differential equation for $h_{\mu\nu}$ and to try and solve it the separation of variables method is used. For the dependency on θ and ϕ it is suitable to use spherical harmonics given the spherical symmetry of Schwarszchild's background. By studying how $h_{\mu\nu}$ transforms under rotations, it is concluded that its components transforms like three scalars (h_{00}, h_{01}, h_{11}), two vectors $\left(\begin{bmatrix} h_{02} \\ h_{03} \end{bmatrix}, \begin{bmatrix} h_{12} \\ h_{13} \end{bmatrix}\right)$, and a second order tensor $h_{\lambda\sigma}$ $(\lambda, \sigma = 2, 3)$.

Each of the corresponding spherical harmonics belongs to a wave parity, odd $(-1)^{l+1}$ or even $(-1)^l$, where l stands for angular momentum. Given the spherical symmetry of the background, parities cannot be mixed in equation (3.1.2). Considering the perturbation to have a frequency ω so the time dependence of $h_{\mu\nu}$ is of the form $e^{-i\omega t}$ for each component, and defining $h_{\mu\nu} = e^{-i\omega t}\tilde{h}_{\mu\nu}$ results in the following form for **odd perturbations**, also referred as vector perturbations,

$$\tilde{h}_{\mu\nu} = \begin{bmatrix} 0 & 0 & -\dfrac{h_0}{\sin\theta}\dfrac{\partial}{\partial\phi} & h_0 \sin\theta \dfrac{\partial}{\partial\phi} \\ 0 & 0 & -\dfrac{h_1}{\sin\theta}\dfrac{\partial}{\partial\phi} & h_1 \sin\theta \dfrac{\partial}{\partial\phi} \\ h_{20} & h_{21} & h_2 \dfrac{\partial}{\partial\theta}\left(\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\phi}\right) & \dfrac{h_2}{2}\left(\dfrac{1}{\sin\theta}\dfrac{\partial^2}{\partial\phi^2} - \dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right)\right) \\ h_{30} & h_{31} & h_{32} & -h_2 \dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\phi}\right) \end{bmatrix} Y_{lm}, \tag{3.1.4}$$

and for **even perturbations** or scalar perturbations,

$$\tilde{h}_{\mu\nu} = \begin{bmatrix} fH_0 & H_1 & h_0 \dfrac{\partial}{\partial\theta} & h_0 \dfrac{\partial}{\partial\phi} \\ h_{10} & \dfrac{H_2}{f} & h_1 \dfrac{\partial}{\partial\phi} & h_1 \dfrac{\partial}{\partial\phi} \\ h_{20} & h_{21} & r^2\left(K + G\dfrac{\partial^2}{\partial\theta^2}\right) & r^2 G \dfrac{\partial}{\partial\phi}\left(\dfrac{\partial}{\partial\theta} - \dfrac{\cos\theta}{\sin\theta}\right) \\ h_{30} & h_{31} & h_{32} & r^2\left(K\sin^2\theta + G\left(\dfrac{\partial^2}{\partial\phi^2} + \dfrac{\sin 2\theta}{2}\dfrac{\partial}{\partial\theta}\right)\right) \end{bmatrix} Y_{lm}. \tag{3.1.5}$$

Note that due to the symmetry on the indices of $h_{\mu\nu}$, some components have been written symbolically and that the dependencies on r have been omitted, but $h_0, h_1, h_2, H_0, H_1, K, G$ and the metric function f are all dependent on the radial coordinate, r.

This matrices can be massively simplified considering that under an infinitesimal coordinate transformation $x'^{\alpha} = x^{\alpha} + \chi^{\alpha}$, the perturbation $h_{\mu\nu}$ changes as

$$h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}. \tag{3.1.6}$$

In [29] a detailed explanation about how to use gauge transformations to simplify the gravitational perturbations of Schwarszchild spacetime is presented (see also [30]). For odd perturbations, the gauge that fixes $h_2 = 0$ so the higher derivatives of the spherical harmonics can be eliminated from the equations, is called the *Regge-Wheeler gauge*, and the same name is given to the gauge that for even parities setS $h_0 = h_1 = G = 0$.

Using Regge-Wheeler gauges, $h_{\mu\nu}$ takes the following simpler forms:

$$\tilde{h}^{\text{odd}}_{\mu\nu} = e^{-i\omega t} \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{bmatrix} \sin\theta \frac{\partial}{\partial\theta} P_L(\cos\theta), \tag{3.1.7}$$

and

$$h^{\text{even}}_{\mu\nu} = e^{-i\omega t} \begin{bmatrix} f(r)H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & \dfrac{H_2(r)}{f(r)} & 0 & 0 \\ 0 & 0 & r^2 K(r) & 0 \\ 0 & 0 & 0 & r^2 \sin\theta K(r) \end{bmatrix} P_L(\cos\theta). \tag{3.1.8}$$

When h_0 and h_1 are set to 0 by a gauge transformation on the even case, the independence of odd and even parity perturbations becomes evident. The odd perturbation depends only of $h_0(r)$ and $h_1(r)$, and the even one, on $H_0(r), H_1(r), H_2(r)$ and $K(r)$.

From the 10 Einstein equations, for the odd case just 3 are non-trivial and just 2 are independent, and for the even case, 7 are non-trivial, making the former a simpler problem to study. Regge and Wheeler were able to write this case in a one dimensional Schroedinger like equation (for details see [29]).

$$\frac{d^2 Q}{dr*^2} + V^2_{\text{eff,odd}}(r) Q = 0, \tag{3.1.9}$$

with $Q = \dfrac{f(r)h_1(r)}{r}$, $dr^* = \dfrac{dr}{f(r)}$ and effective potential

$$V^2_{\text{eff,odd}}(r) = \omega^2 - L(L+1)\frac{f(r)}{r^2} + 6M\frac{f(r)}{r^3}. \tag{3.1.10}$$

Equation (3.1.9) is known as the **Regge-Wheeler equation**.

In their article they also worked with the even parity pertubations, but in 1968 Vishveshwara presented corrections to this work in his PhD thesis [31] and later published them in [32]. Although the seven equations for even parity perturbations were put in a single equation, this was not a Schroedinger like one. This was achieved by Zerilli in 1970 in [33]. This article is mainly algebraic and adresses the problem in a clever way.

By rewritting the equations for the perturbations as three independent equations plus an algebraic identity, Zerilli was able to write the following Schroedinger-like equation for the function K:

$$\frac{d^2K}{dr*^2} + V^2_{\text{eff,even}}(r)K = 0, \tag{3.1.11}$$

with

$$V_{\text{eff,even}} = f(r)\frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2 mr^2 + 18\lambda m^2 r + 18m^3}{r^3(\lambda r + 3m)^2}, \tag{3.1.12}$$

and $\lambda = \dfrac{1}{2}(L-1)(L+2)$. This equation is known as the **Zerilli equation**.

3.2 Chandrasekhar's approach to gravitational perturbations

Chandrasekhar studied black holes stability exhaustively and has a number of articles on this topic regarding different spacetimes. Main results for Schwarszchild BH are in [34], [35] and [36], later compiled with more results in the book *The Mathematical Theory of Black Holes* [37]. In this section we will study his approach in detail, given that this is the approach used later in this thesis.

It all begins with the consideration of the general form of a time-dependent axisymmetric spacetime

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}\left(d\varphi - q_2 dx^2 - q_3 dx^3 - \omega dt\right)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2, \tag{3.2.1}$$

where ν, ψ, q_2, q_3, ω, μ_2 and μ_3 can be functions of the four variables t, φ, x^2 and x^3. Notice that in the book [37] the mostly minus signature is used.

Given the symmetries of Schwarzschild spacetime, it is natural to assume that the functions depend only on t, x^2 and x^3, one is therefore considering axially symmetric spacetimes.

A general perturbation when considering (3.2.1) consists in each function experimenting small increments, this is, a χ function perturbed is $\chi + \delta\chi$. Notice that since we are interested on the stability of the background we only consider time-dependent perturbations, and do not focus on the existence of zero modes.

If the metric functions do not depend on φ as in this case, the general perturbation will naturally decouple into two sets of differential equations when second order terms on the perturbation are dismissed. The first set is related to the functions q_2, q_3 y ω, namely, the functions outside of the metric's diagonal. These functions must change sign when considering $\varphi \to -\varphi$, they induce a dragging effect of the inertial frames and rotation to the BH, this is the reason why they are called **polar perturbations** and are the ones that Regge and Wheeler called *even perturbations*. The second set of equations relate to the functions in the metric's diagonal ν, ψ, μ_2 y μ_3 and they do not impart such effects on the BH. Chandrasekhar calls these by **axial perturbations** and are the same as the *odd perturbations*. We will have to work with each kind of perturbations, separately.

It is necessary to stablish a relation between the metric (3.2.1) and the metric of Schwarzschild BH (3.1.1). Comparing these expresions, we identify $x_2 = r$, $x_3 = \theta$ and $\varphi = \phi$. For this choice, the metric functions are

$$\begin{aligned} e^{2\nu} &= f(r), & e^{2\mu_2} &= \frac{1}{f(r)}, & e^{2\mu_3} &= r^2, & e^{2\psi} &= r^2\sin^2(\theta), \\ \omega &= 0, & q_2 &= 0, & q_3 &= 0. \end{aligned} \tag{3.2.2}$$

Again, the equations to study the perturbations relate to the Ricci tensor, but we will also use the Einstein tensor. The explicit expressions of these tensors for the metric (3.2.1) can be found on Apendix A, where number indices refers to the tetrad frame.

3.2.1 Tetrad formalism

In general relativity, it is a good idea to not use a local coordinate basis since there are many options and not all of them are good choices for the problem at hand. Instead, we will use an orthonormal basis that maps the metric into a constant one.

Inverse vielbeins are basis of d linearly independent contravariant vectors, where d is the dimension of the spacetime. When $d = 4$, it is called a *tetrad basis.* This basis is represented by $e_a{}^\mu$, where Latin indices $a = 0, 1, 2, 3$ represent tetrad components and Greek indices $\mu = t, r, \theta, \phi$ represent components in the coordinate basis.

The associated covariant vectors are,

$$e_{a\mu} = g_{\mu\nu} e_a{}^\nu, \tag{3.2.3}$$

where $g_{\mu\nu}$ is the metric tensor, and the vielbeins $e^b{}_\nu$ fulfil

$$e_a{}^\mu e^b{}_\mu = \delta_a{}^b \qquad \text{and} \qquad e_a{}^\mu e^a{}_\nu = \delta^\mu{}_\nu. \tag{3.2.4}$$

We will also assume that

$$e_a{}^\mu e_{b\mu} = \eta_{ab}, \tag{3.2.5}$$

where η_{ab} is a constant symmetric matrix, but when the basis vectors $e_a{}^\mu$ are orthonormal, this matrix corresponds to the Minkowski metric.

With these definitions, we can prove that the spacetime metric is related to the vielbeins through the relation

$$g_{\mu\nu} = e_{a\mu} e^a{}_\nu = e^a{}_\mu e^b{}_\nu \eta_{ab}, \tag{3.2.6}$$

meaning that the vectors $e_a{}^\mu$ transform global quantities in a curved background into quantities defined on a locally flat spacetime.

Finally, the tetrad components of any tensor field in a coordinate basis can be obtained as

$$T_{ab} = e_a{}^{\mu} e_b{}^{\nu} T_{\mu\nu}, \tag{3.2.7}$$

where T_{ab} are the tensor components in the local Minkowski spacetime.

And the coordinate components of any tensor field in a tetrad basis can be obtained as

$$T_{\mu\nu} = e^a{}_{\mu} e^b{}_{\nu} T_{ab}. \tag{3.2.8}$$

It was mentioned before that the equations governing the geometry of Schwarzschild BH is $R_{\mu\nu} = 0$, but note that for this section, we will use its tetrad components

$$R_{ab} = e_a{}^{\mu} e_b{}^{\nu} R_{\mu\nu}, \tag{3.2.9}$$

and these are the components listed in Apendix A.

Using vielbeins will simplify the computations when considering AdS spacetimes, as we can work with them as locally flat spacetimes. We will see this in detail in Chapter 4.

3.2.2 Axial perturbations

As stated before, axial perturbations are related to the perturbation of the functions q_2, q_3 y ω. We will define Q functions as

$$Q_{ab} = \frac{\partial}{\partial x^b} q_a - \frac{\partial}{\partial x^a} q_b \quad \text{and} \quad Q_{0a} = \frac{\partial}{\partial x^a} \omega - \frac{\partial}{\partial t} q_a,$$

and they represent the only combinations in which the functions ω, q_2 and q_3 appear in the Ricci and Einstein tensors.

Only R_{01}, R_{12} and R_{13} are linear in the Q functions, so they are the only components that we can consider to work with in the context of axial perturbations. Among them, only two of the equations $R_{ab} = 0$ for this components will be independent, so we choose the following two equations to work with

$$R_{12} = 0 \quad \text{and} \quad R_{13} = 0, \tag{3.2.10}$$

where

$$R_{12} = \frac{1}{2} e^{\psi - 2\nu - \mu_2} \left[\frac{\partial}{\partial t} \left(3\psi - \nu - \mu_2 + \mu_3\right) Q_{02} + \frac{\partial}{\partial t} Q_{02} \right.$$
$$\left. - e^{-\mu_3} \left(\frac{\partial}{\partial x^3} \left(3\psi - \nu - \mu_2 - \mu_3\right) Q_{32} + \frac{\partial}{\partial x^3} Q_{32} \right) \right]. \qquad (3.2.11)$$

R_{13} is obtained by interchanging the 2 and 3 index in the expression for R_{12}.

Naturally, the equations for the perturbations will be

$$\delta R_{12} = 0 \quad \text{and} \quad \delta R_{13} = 0, \qquad (3.2.12)$$

where δR_{ab} is defined in the same way as in the Regge and Wheeler approach.

Given that axial and polar perturbations decouple, it will be sufficient to replace the unperturbed values of ψ, ν, μ_2 y μ_3 and the perturbed values of q_2, q_3 and ω. Considering that the unperturbed part of this three functions is null, we arrive to the following equations:

$$\delta R_{12} = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial x^3} \left(e^{3\psi + \nu - \mu_2 - \mu_3} \delta Q_{23} \right) + e^{3\psi - \nu - \mu_2 + \mu_3} \frac{\partial}{\partial t} \delta Q_{02} = 0, \qquad (3.2.13)$$

$$\delta R_{13} = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial x^2} \left(e^{3\psi + \nu - \mu_2 - \mu_3} \delta Q_{23} \right) - e^{3\psi - \nu - \mu_2 + \mu_3} \frac{\partial}{\partial t} \delta Q_{03} = 0, \qquad (3.2.14)$$

where the δQ functions are

$$\delta Q_{ab} = \frac{\partial}{\partial x^b} \delta q_a - \frac{\partial}{\partial x^a} \delta q_b \quad \text{and} \quad \delta Q_{0a} = \frac{\partial}{\partial x^a} \delta \omega - \frac{\partial}{\partial t} \delta q_a. \qquad (3.2.15)$$

Replacing the variables and functions with the identification made in (3.2.2) and defining $\delta Q = r^2 f(r) \sin^3(\theta) \delta Q_{23}$, the equations (3.2.13) and (3.2.14) can be written as

$$-\frac{\partial^2}{\partial t^2} \delta q_2 + \frac{\partial^2}{\partial t \partial r} \delta \omega + \frac{1}{r^4 \sin^3(\theta)} \frac{\partial}{\partial \theta} \delta Q = 0, \qquad (3.2.16)$$

$$-\frac{\partial^2}{\partial t^2} \delta q_3 + \frac{\partial^2}{\partial t \partial \theta} \delta \omega - \frac{f(r)}{r^2 \sin^3(\theta)} \frac{\partial}{\partial r} \delta Q = 0. \qquad (3.2.17)$$

In what follows we assume that all functions depend on time in the same manner, $e^{-i\sigma t}$, with σ a complex value being the frequency of the quasinormal modes of the BH. Notice that the real part of σ represents oscillations and the imaginary part represents damping if negative, and an instability if positive. The dependence

$e^{-i\sigma t}$ will be used in following chapters since its use is well argued: it is a Fourier transformation in t and we use that the system is linear.

In consequence one obtains:

$$\sigma^2 \delta q_2 - i\sigma \frac{\partial}{\partial r}\delta\omega + \frac{1}{r^4 \sin^3(\theta)} \frac{\partial}{\partial \theta}\delta Q = 0, \tag{3.2.18}$$

$$\sigma^2 \delta q_3 - i\sigma \frac{\partial}{\partial \theta}\delta\omega - \frac{f(r)}{r^2 \sin^3(\theta)} \frac{\partial}{\partial r}\delta Q = 0. \tag{3.2.19}$$

Notice that the same symbol was used to represent $\delta Q(t, r, \theta)$ and $\delta Q(r, \theta)$.

It is easy to delete ω from the equations deriving (3.2.18) by θ and (3.2.19) by r, and then substracting both equations. We are left with a single first order PDE for $\delta Q = \delta Q(r, \theta)$.

$$\frac{\partial}{\partial r}\left(\frac{f(r)}{r^2 \sin^3(\theta)} \frac{\partial}{\partial r}\delta Q\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r^4 \sin^3(\theta)} \frac{\partial}{\partial \theta}\delta Q\right) + \frac{\sigma^2}{r^2 f(r) \sin^3(\theta)}\delta Q = 0. \tag{3.2.20}$$

When using the ansatz $\delta Q(r, \theta) = R(r)C(\theta)$ in (3.2.20) we see that, in fact, this equation is separable.

$$f(r)\frac{d^2R(r)}{dr^2}\frac{r^2}{R(r)} - \left(r^2\frac{df(r)}{dr} - 2rf(r)\right)\frac{1}{R(r)}\frac{dR(r)}{dr} + \frac{\sigma^2 r^2}{f(r)} + \frac{1}{C(\theta)}\frac{d^2C(\theta)}{d\theta^2} - \frac{3\cos(\theta)}{C(\theta)\sin(\theta)}\frac{dC(\theta)}{d\theta} = 0. \tag{3.2.21}$$

On one hand, the equation for θ is

$$C''(\theta) - \frac{3\cos(\theta)}{\sin(\theta)}C'(\theta) + n^2 C(\theta) = 0, \tag{3.2.22}$$

and has an analytical solution in terms of associated Legendre polinomials,

$$C(\theta) = A\sin^2(\theta)P_\ell^2(\cos(\theta)) + B\sin^2(\theta)Q_\ell^2(\cos(\theta)), \tag{3.2.23}$$

where $n^2 = (l+2)(l-1)$ emerges as a separation constant and the precise values are required for regularity.

On the other hand, the radial equation is

$$r^2 f(r)R''(r) - \left(r^2\frac{df(r)}{dr} - 2rf(r)\right)R'(r) + \frac{\sigma^2 r^2}{f(r)}R(r) - n^2 R(r) = 0, \tag{3.2.24}$$

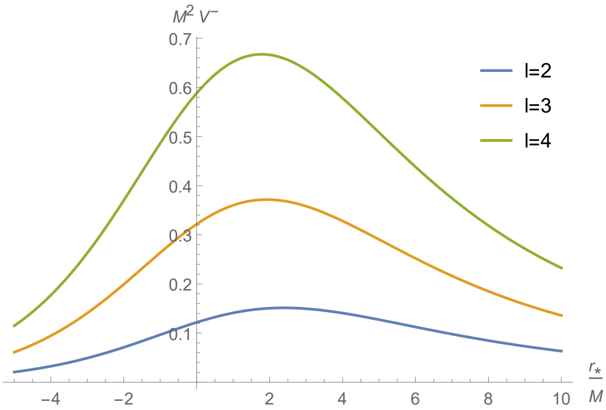

Figure 3.2.1: Potential barriers for axial perturbations of Schwarzschild BH.

but this one needs to be solved numerically.

Remarkably, Chandrasekhar was able to write this equation as a Schroedinger type by making $R(r) = rZ^{(-)}(r)$,

$$f^2(r)Z''^{(-)} + f(r)f'(r)Z'^{(-)} + \sigma^2 Z^{(-)} = f(r)\left(\frac{2f(r) - n^2}{r^2} - \frac{f'(r)}{r}\right)Z^{(-)}, \quad (3.2.25)$$

and recognizing the second derivative with respect to the tortoise coordinate r_* in the first two terms. Remember that the derivative with respect to the tortoise coordinate acts as

$$\frac{d}{dr_*} = f(r)\frac{d}{dr} \quad \Rightarrow \quad \frac{d^2}{dr_*^2} = f^2(r)\frac{d^2}{dr^2} + f(r)f'(r)\frac{d}{dr}. \quad (3.2.26)$$

This enable us to write the radial equations as the following Schroedinger-like equation:

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right)Z^{(-)} = V^{(-)}Z^{(-)}, \quad (3.2.27)$$

where the effective potential is

$$V^{(-)} = f(r)\left(\frac{2f(r) + n^2}{r^2} - \frac{f'(r)}{r}\right) = \frac{f(r)}{r^3}\left[(2 + n^2)r - 6M\right], \quad (3.2.28)$$

and it is shown on Figure 3.2.1. Notice that at this last step we have used the value of $f(r)$.

3.2.3 Polar perturbations

As previously stablished, polar perturbations are related to the functions ν, ψ, μ_2 and μ_3. To study these functions we can use any of the Ricci and Einstein tensor components except R_{01}, R_{12} and R_{13}, this is, any of the components quadratic in the Q functions, so the decoupling of axial and polar perturbations becomes evident.

In this case, we will use

$$\delta R_{02} = 0, \qquad \delta R_{03} = 0, \qquad \delta R_{23} = 0, \quad \text{and} \quad \delta G_{22} = 0. \tag{3.2.29}$$

After replacing the values in (3.2.2), linearizing and assuming equal time dependencies for every function, $e^{-i\sigma t}$, so, for example $\delta\rho(t, r, \theta) = e^{-i\sigma t}\delta\rho(r, \theta) = e^{-i\sigma t}\delta\rho$, the equations (3.2.29) take the form

$$\cot(\theta)\left(\delta\psi - \delta\mu_3\right) + \frac{\partial}{\partial\theta}\left(\delta\mu_2 + \delta\psi\right) = 0, \tag{3.2.30}$$

$$\frac{\partial}{\partial r}\left(\delta\mu_3 + \delta\psi\right) + \left(\frac{1}{r} - \frac{1}{2f(r)}\frac{\partial f(r)}{\partial r}\right)\left(\delta\mu_3 + \delta\psi\right) - 2\frac{\delta\mu_2}{r} = 0, \tag{3.2.31}$$

$$\begin{aligned}\frac{\partial^2}{\partial r\partial\theta}\left(\delta\psi + \delta\nu\right) &+ \frac{\partial}{\partial r}\left(\delta\psi - \delta\mu_3\right)\cot(\theta) \\ &+ \frac{1}{2f(r)}\frac{\partial f(r)}{\partial r}\frac{\partial}{\partial\theta}\left(\delta\nu - \delta\mu_2\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\delta\nu + \delta\mu_2\right) = 0,\end{aligned} \tag{3.2.32}$$

$$\begin{aligned}\left[\frac{\sigma^2}{f(r)} + \left(\frac{1}{2}\frac{\partial f(r)}{\partial r} + \frac{f(r)}{r}\right)\frac{\partial}{\partial r}\right]\left(\delta\mu_3 + \delta\psi\right) + \frac{2f(r)}{r}\frac{\partial\delta\nu}{\partial r} \\ - \frac{2}{r}\left(\frac{\partial f(r)}{\partial r} + \frac{f(r)}{r}\right)\delta\mu_2 + \frac{2\delta\mu_3}{r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\delta\psi + \delta\nu\right) \\ + \frac{\cot(\theta)}{r^2}\frac{\partial}{\partial\theta}\left(2\delta\psi + \delta\nu - \delta\mu_3\right) = 0.\end{aligned} \tag{3.2.33}$$

It is clear that only three of the four functions are independent, since, for example, it is possible to solve $\delta\mu_3$ from (3.2.30) or $\delta\mu_2$ from (3.2.31). To solve this equations, first we separate variables to eliminate the dependence on θ. This is accomplished by a decomposition into spherical harmonics and it was specified by Friedman in [38]. It reads as follows:

$$\delta\nu = N(r)P_l(\cos\theta), \tag{3.2.34}$$

$$\delta\mu_2 = L(r)P_l(\cos\theta), \tag{3.2.35}$$

$$\delta\mu_3 = T(r)P_l(\cos\theta) + \frac{2X(r)}{(l-1)(l+2)}\frac{d^2}{d\theta^2}P_l(\cos(\theta)), \tag{3.2.36}$$

$$\delta\psi = T(r)P_l(\cos\theta) + \frac{2X(r)}{(l-1)(l+2)}\cot\theta\frac{d}{d\theta}P_l(\cos(\theta)). \tag{3.2.37}$$

This, in the equation (3.2.30) gives

$$T(r) - \frac{2X(r)}{(l-1)(l+2)} + L(r) = 0, \tag{3.2.38}$$

so the fact that there are three independent functions is still explicit for this variable separation. We will solve for $T(r)$ and use it in equations (3.2.31), (3.2.32) and (3.2.33). The radial dependency of functions L, N and X will be omitted so we writte

$$\left[2f(r)\left(r\frac{\partial}{\partial r}+1\right) - r\frac{\partial f(r)}{\partial r}\right](L+X) + 2f(r)L = 0, \tag{3.2.39}$$

$$\left[2f(r)\left(r\frac{\partial}{\partial r}+1\right) + r\frac{\partial f(r)}{\partial r}\right](L-N) + 4f(r)N = 0, \tag{3.2.40}$$

$$\begin{aligned} rf(r)\left(r\frac{\partial f(r)}{\partial r} + 2f(r)\right)\frac{\partial}{\partial r}(L+X) - 2f(r)\left(N + X + rf(r)\frac{\partial N}{\partial r}\right) \\ + 2f(r)\left(r\frac{\partial f(r)}{\partial r} + f(r)\right)L - f(r)(l^2+l-2)(L-N) \\ + 2r^2\sigma^2(L+X) = 0. \end{aligned} \tag{3.2.41}$$

After manipulating these equations it is possible to write the system in a much simpler form as a system of first order differential equations, where the derivative of the functions is equal to a linear combination of the functions without derivatives.

$$L'(r) = \left(a - \frac{2}{r}\right) L + bN + \left(c - \frac{1}{r}\right) X, \tag{3.2.42}$$

$$X'(r) = -\left(a + \frac{f'(r)}{2f(r)}\right) L - bN - \left(c + \frac{f'(r)}{2f(r)}\right) X, \tag{3.2.43}$$

$$N'(r) = \left(a + \frac{f'(r)}{2f(r)} - \frac{1}{r}\right) L + \frac{m+1}{rf(r)}N + \left(c - \frac{1}{r}\right) X, \tag{3.2.44}$$

with $m = \frac{1}{2}(l+2)(l-1)$ and

$$a = \frac{1}{4rf(r)^2}\left(r^2 f'(r)^2 + 4r^2\sigma^2 - 4mf(r)\right),$$

$$b = \frac{1}{2rf(r)}(rf'(r) - 2f(r) + 2m + 2),$$

$$c = \frac{1}{4rf(r)^2}\left(r^2 f'(r)^2 + 4r^2\sigma^2 + 4f(r)\right).$$

To this point, although $f(r)$ is fixed to the function of the solution, its explicit expression has yet not been used, but with the goal of writing this system as a Schroedinger-like equation this will be necessary.

First, we define the function

$$Z^{(+)} = \frac{r^2}{mr + 3M}\left(\frac{3M}{mr}X - L\right). \tag{3.2.45}$$

This function is key to write the second order equation as wanted. It was first derived empirically but later its existence was justified via the algorithm developed by Xanthopoulos in [39].

In contrast to the previous case, where we defined a function to replace in the equations and later recognized the second derivative with respect to the tortoise coordinate, this case is worked the other way around. We define the $Z^{(+)}$ function, and now we will compute its second derivative with respect to the tortoise coordinate. After this, we will analyse the conditions to obtain a Schroedinger-like equation.

$$\begin{aligned}\frac{d^2Z^{(+)}}{dr_*^2} =&\frac{1}{r^4m(mr+3M)^2}\left[r^3\left(mr+3M\right)^2\left(2M-r\right)^2X''(r)\right.\\&+r^2\left(6M^2-r^2m-6rM\right)\left(mr+3M\right)\left(2M-r\right)X'(r)\\&-r\left(mr+3M\right)\left(mr^3-9r^2Mm+3M^2(5m-4)r+27M^3\right)X(r)\\&+m\left(2m^2r^4-14Mm^2r^3+3(7m-8)M^2mr^2+18M^3(3m-1)r\right.\\&\left.\left.+45M^4\right)Z^{(+)}\right].\end{aligned} \quad (3.2.46)$$

$Z^{(+)}$ is reconstructed and only terms with $X(r)$ an its derivatives are keeping us away from the correct structure. To deal with these terms is convenient to inspect the equation $\delta R_{11}=0$.

After linearizing and making the same assumptions as for the equations (3.2.29), including the separations of variables (3.2.34) - (3.2.37), we obtain an equation of the form

$$A_1(r)P_l(\cos(\theta))+A_2(r)P_{l+1}(\cos(\theta))=0.$$

$A_1(r)$ and $A_2(r)$ must vanish. This is consistently verified as follows:

$A_2(r)$ is zero if the following equation holds

$$f(r)X''+\left(f'(r)+\frac{2f(r)}{r}\right)X'+\frac{m}{r^2}(L+N)+\frac{\sigma^2}{f(r)}X=0. \quad (3.2.47)$$

And $A_1(r)$ is identically zero when replacing: $X''(r)$ solved from the previous expression, the first derivatives from (3.2.42) - (3.2.44), and $f(r)=1-\dfrac{2M}{r}$.

Finally, using X'' from (3.2.47) and X' from (3.2.43) in (3.2.46), we obtain

$$\left(\frac{d^2}{dr_*^2}+\sigma^2\right)Z^{(+)}=V^{(+)}Z^{(+)}, \quad (3.2.48)$$

where the effective potential takes the value

$$V^{(+)}=\frac{2f(r)}{r^3(mr+3M)^2}(m^2(m+1)r^3+3Mm^2r^2+9M^2mr+9M^3) \quad (3.2.49)$$

and it is shown in Figure 3.2.2.

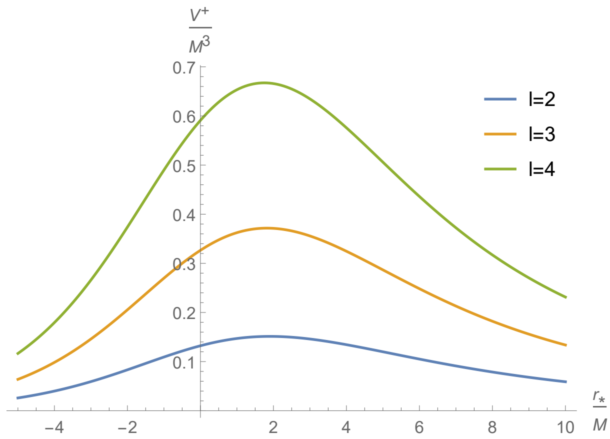

Figure 3.2.2: Potential barrier for polar perturbations of Schwarzschild BH.

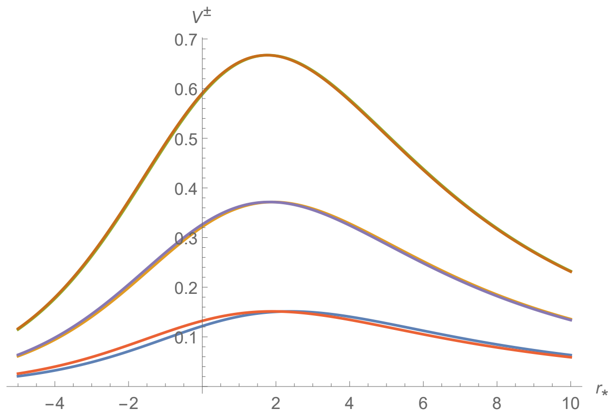

Figure 3.2.3: Potentials V^+ and V^-, for $M = 1$

3.2.4 Relations between $V^{(+)}$ and $V^{(-)}$

In Figure 3.2.3 are potentials $V^{(-)}$ and $V^{(+)}$ simultaneously. Notice that they are extremely similar and this is more evident for larger values of l. In fact, Chandrasekhar was able to relate them in a very simple manner through a generating function, $W(r)$, and then relate the functions $Z^{(+)}$ and $Z^{(-)}$. It is direct to verify that we can write both potentials as

$$V^{(\pm)} = \pm\beta\frac{dW(r)}{dr_*} + \beta^2 W^2(r) + \kappa W(r), \tag{3.2.50}$$

where $\beta = 3M$, $\kappa = 2m(m+1)$ and

$$W(r) = \frac{2f(r)}{r(mr+3M)} = \frac{r-2M}{r^2(mr+3M)}. \tag{3.2.51}$$

To relate functions $Z^{(+)}$ and $Z^{(-)}$ we write

$$Z''_{(\pm)} + \sigma_{(\pm)} Z_{(\pm)} = V^{(\pm)} Z^{(\pm)} \Rightarrow Z''_{(\pm)} = (V^{(\pm)} - \sigma_{(\pm)}) Z^{(\pm)} \tag{3.2.52}$$

where W' denotes the derivative of $W(r)$ with respect to r_*, and although we know that the potentials lead to the same spectra, here we make a distinction for $\sigma_{(+)}$ and $\sigma_{(-)}$ for generality.

We can assume that $Z_{(+)}$ and $Z_{(-)}$ are related in the following form:

$$Z_{(+)} = p(r) Z_{(-)} + q(r) Z'_{(-)} \tag{3.2.53}$$

where $p(r)$ and $q(r)$ are functions that need to be defined. To achieve that, we derive with respect to r_* twice, replacing $Z''_{(-)}$ from (3.2.52) when it appears, so we obtain

$$Z''_{(+)} = \left(p'' + 2\left(V_{(-)} - \sigma_{(-)}\right) q' + q V'_{(-)} - jp\right) Z_{(-)} + \left(q'' + 2p' - jq\right) Z'_{(-)}, \tag{3.2.54}$$

with $j = j(r) = V_{(+)} - V_{(-)} + \sigma_{(-)} - \sigma_{(+)}$. This must be equal to $Z''_{(+)}$ from (3.2.52), from where we can write as $Z''_{(+)} = (V^{(+)} - \sigma_{(+)})(pZ_{(-)} + qZ'_{(-)})$. We arrive to the following two equations for the coefficients of $Z_{(-)}$ and $Z'_{(-)}$:

$$jp = 2\left(V_{(-)} - \sigma_{(-)}\right) q' + q V'_{(-)} + p'' \tag{3.2.55}$$

$$jq = 2p' + q'' \tag{3.2.56}$$

Of course, we can eliminate j by multiplying (3.2.55) by q and (3.2.56) by p, then subtracting the equations. By doing this we arrive to an expression that can be integrated, resulting in

$$p^2 + q'p - qp' - (V_{(-)} - \sigma_{(-)}) q^2 = C^2, \tag{3.2.57}$$

with C^2 is the constant from the integration.

The system of equations (3.2.55) and (3.2.57) for the functions $p(r)$ and $q(r)$ (and the constant C^2), so $Z_{(+)}$ and $Z_{(-)}$ are related in the form (3.2.53), cannot be solved for arbitrary $V_{(+)}$ and $V_{(-)}$, but it can be solved for potentials with structure

(3.2.50) and if $\sigma_{(+)} = \sigma_{(-)} = \sigma$. The solution reads:

$$q = 2\beta \qquad \text{and} \qquad p = \kappa + 2\beta^2 W(r), \tag{3.2.58}$$

with $C^2 = \kappa + 4\beta^2\sigma^2$

The results up to here apply to general relativity with vanishing cosmological constant.

In what follows, we extend the framework to the realm on general relativity with non-vanishing Λ, with special focus on the $\Lambda < 0$ case, which is relevant, for example, in holography [2]. Gravitational perturbations allow to compute transport properties of the dual system in the strong coupling regime (see e.g. [40], [41])

Chapter 4

Gravitational perturbations of AdS spacetimes using Chandrasekhar's approach

AdS spaces need the presence of a cosmological constant, so if we use the Regge-Wheeler approach we will encounter that the variation of the Ricci tensor is no longer equal to zero. Luckily, Chandrasekhar did work in the tetrad frame, so first let us prove that with his approach, the variation of the Ricci tensor is still equal to zero for solutions of the Einstein equations in presence of a negative cosmological constant.

The value of Λ in d dimensions in terms of the AdS radius of the maximally symmetric solution is $\Lambda = -\frac{(d-1)(d-2)}{2L^2}$, and using the expression in (2.1.1) directly in Einstein equations one has that:

$$R_{\mu\nu} = -\frac{2}{d-2}\frac{(d-1)(d-2)}{2L^2}g_{\mu\nu} = \frac{d-1}{L^2}g_{\mu\nu} \quad \Rightarrow \quad \delta R_{\mu\nu} = \frac{d-1}{L^2}\delta g_{\mu\nu}.$$

However, if we use vielbeins we have

$$R_{ab} = e_a{}^{\mu} e_b{}^{\nu} R_{\mu\nu} = \frac{d-1}{L^2} e_a{}^{\mu} e_b{}^{\nu} g_{\mu\nu} = \frac{d-1}{L^2}\eta_{ab}.$$

Since η_{ab} is a constant metric, we conclude that

$$\delta R_{ab} = 0.$$

This is valid for any solution of $G_{ab} + \Lambda\eta_{ab} = 0$, namely for the Einstein equation in terms of tetrads. Since the equation to solve is the same one as for Schwarzchild BH case, we can use the same results if we make appropiate choices for the indentification of coordinates between Schwarzschild AdS and the general metric (3.2.1) .

4.1 Gravitational perturbations of Schwarschild AdS black hole

The Schwarzschild-AdS BH metric is

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2, \tag{4.1.1}$$

with $f(r) = 1 - \frac{2M}{r} + \frac{r^2}{L^2}$, where M is the mass and L is the AdS radius.

Since the structure of the metric is the same as the $\Lambda = 0$ case, we make the same identification as before, this is, $x_2 = r$, $x_3 = \theta$ and $\varphi = \phi$ and using the functions as in (3.2.2), but this time $f(r)$ takes a different value.

As we saw in Section 3.2.2, for **axial perturbations** the results were valid for any $f(r)$ until equation (3.2.27), which gives us immediately a Schroedinger type equation for the axial perturbation of Schwarzschild-AdS BH [1]:

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right) Z^{(-)} = V^{(-)} Z^{(-)}, \tag{4.1.2}$$

but this time the potential takes the value

$$V^{(-)} = f(r)\left(\frac{2f(r) + n^2}{r^2} - \frac{f'(r)}{r}\right) = \frac{f(r)}{r^3}\left[(2+n^2)r - 6M\right], \tag{4.1.3}$$

with $n^2 = (l+2)(l-1)$.

Notice that besides the potential having the same structure as before, it has also the same factor in brackets after replacing $f(r)$. This potential is the same as the one obtained using Regge and Wheeler's formalism [42], so it is safe to say that

[1] Of course the unperturbed equation solved by the background metric do contain Λ, and therefore is sensitive to the precise form of $f(r)$.

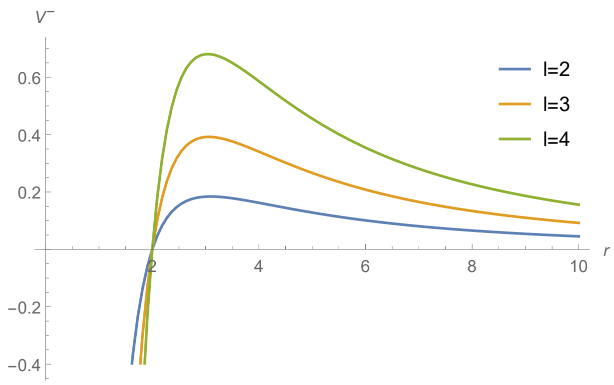

Figure 4.1.1: Effective potential for gravitational perturbations of Schwarzschild-AdS BH.

working with Chandrasekhar's approach in AdS spacetimes is valid, even when using the locally flat frame.

In Figure 4.1.1 the form of the potential is shown with the goal of later comparing it with the effective potential for polar perturbations. Note that this time the plot is $V^{(-)}$ vs r instead of $V^{(-)}$ vs r_*. Also, the values of the BH mass and the AdS radius are fixed equal to 1.

Now, for **polar perturbation**, we refer to section 3.2.3 where we worked polar perturbations for the asymptotically flat case and the computations where valid for an arbitrary $f(r)$ until equations (3.2.42), (3.2.43) and (3.2.44). The step we need to specialize for this case is finding a master variable such that it is possible to obtain a Schroedinger-like equation. Amazingly, this is possible by using the same variable as in Section 3.2.3, namely

$$Z^{(+)} = \frac{r^2}{mr + 3M}\left(\frac{3M}{mr}X - L\right). \tag{4.1.4}$$

In this case, the second derivative with respect to the tortoise coordinate has a similar structure as in the previous section, this is

$$\frac{d^2 Z^{(+)}}{dr_*^2} = A(r)X'' + B(r)X' + C(r)X + E(r)Z^{(+)}. \tag{4.1.5}$$

Let's remember that from the inspection of equation $\delta R_{11} = 0$ we have to deal with two factors that have to be zero separately. The equation takes the form

$$A_1(r)P_l(\cos(\theta)) + A_2(r)P_{l+1}(\cos(\theta)) = 0.$$

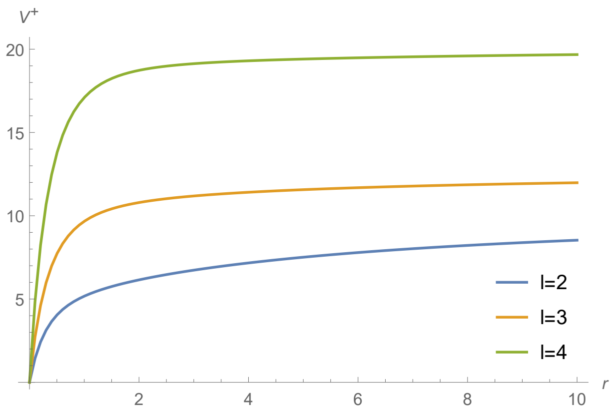

Figure 4.1.2: Effective potential for polar perturbations of Schwarzschild AdS BH, for $M = 1$ and $L = 1$.

From $A_2(r)$ the same equation (3.2.47) holds and, and surprisingly, we found that again $A_1(r)$ identically vanishes when using X'' from (3.2.47), replacing first derivatives from (3.2.42), (3.2.43) and (3.2.44) and using $f(r) = 1 - \frac{2M}{r} + \frac{r^2}{L^2}$.

Finally, replacing X'' in the expression for $\frac{d^2 Z^{(+)}}{dr_*^2}$, we obtain

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right) Z^{(+)} = V^{(+)} Z^{(+)}, \tag{4.1.6}$$

where the effective potential takes the following form and it is shown in Figure 4.1.2

$$V^{(+)} = \frac{2f(r)((m^3 + m^2 + 9M^2/L^2)r^3 + 3M(m^2r^2 + 3Mmr + 3M^2))}{(mr + 3M)^2 r^3}. \tag{4.1.7}$$

Again, this potential is the same as the even parity potential found in [42] using the Regge and Wheeler formalism, so we will not write the numerical analysis for the Schwarzschild-AdS BH since it is already done in the paper. However, it is worth noticing that although this time the potentials are different (see Figures 4.1.2 and 4.1.1) and lead to different spectra, it is possible to relate them through a generating function, just as for the potentials for the Schwarzschild BH. We were able to replicate this result from [42] but it has not the same structure as (3.2.50), so we did find a result with this structure on our own and it reads as follows:

$$V^{(\pm)} = \pm\beta\frac{dW(r)}{dr_*} + \beta^2 W^2(r) + \kappa W(r), \tag{4.1.8}$$

with $\kappa = 2m(m+1)$, $\beta = 3M$, and

$$W(r) = \frac{f(r)}{r^2(mr+3M)} = \frac{r^3 - 2L^2M + L^2r}{L^2r^2(mr+3M)}. \tag{4.1.9}$$

From here, we understand that the reason why the isospectrality between axial and polar quasinormal modes is broken for the Schwarzschild AdS BH is because of the completely different behaviour of the generating function at infinity.

4.2 Gravitational perturbations of AdS soliton

A natural extension of this work is to use the same strategy to study the stability of solitonic backgrounds in four dimensions. We will restric this work to the AdS soliton presented in section 2.1.4, and since its normal modes are already studied in [1] using both, a scalar probe and a different method of gravitational perturbations, it serves as a good comparison to prove if Chandrasekhar's aproach to gravitational perturbations also works for solitonic backgrounds.

First we need to map the AdS soliton metric (2.1.9) to the general stationary and axisymmetric metric (3.2.1). Given the signature, it is mandatory to make maintain the coordinate for time, and given the periodicity of the ϕ coordinate in the AdS soliton metric, it is also mandatory to identify $\varphi = \phi$. We chose to keep $x^2 = r$ and therefore $x^3 = z$.

Then, comparing the metrics we conclude that the mapping between them is the following:

$$\begin{aligned} &e^{2\nu} = r^2, \qquad e^{2\mu_2} = \frac{1}{f(r)}, \qquad e^{2\mu_3} = f(r), \qquad e^{2\psi} = r^2, \\ &\omega = 0, \qquad\quad q_2 = 0, \qquad\qquad q_3 = 0. \end{aligned} \tag{4.2.1}$$

4.2.1 Axial perturbations

As established previously, axial perturbations correspond to those related with the functions ω, q_2 and q_3, and the equations governing these functions are $R_{12} = 0$ and $R_{13} = 0$. Given that in this case the unperturbed values of the functions ω, q_2 and q_3 are zero, just as in the Schwarzschild BH case, we conclude that the results on section 3.2.2 can be used up to the point at which the values of the functions

ψ, ν, μ_2 and μ_3 are replaced, since those values do change for this background, so we can start the analysis in equations (3.2.13) and (3.2.14).

When replacing the variables and functions as per (4.2.1) in equations (3.2.13) and (3.2.14), defining $\delta Q = r^4 \delta Q_{23}$ and assuming that the dependence on time for every function is $e^{-i\sigma t}$, we get

$$\sigma^2 \delta q_2 + i\sigma \frac{\partial}{\partial r}\delta\omega + \frac{1}{r^2 f(r)}\frac{\partial}{\partial z}\delta Q = 0, \tag{4.2.2}$$

$$\sigma^2 \delta q_3 + i\sigma \frac{\partial}{\partial z}\delta\omega - \frac{f(r)}{r^2}\frac{\partial}{\partial r}\delta Q = 0. \tag{4.2.3}$$

At this point, it is easy to note the similarities between these equations and the analogues in the Schwarzschild BH, (3.2.18) and (3.2.19), so we get rid of $\delta\omega$ in the same fashion, differentiating and subtracting the equations to obtain

$$\frac{f(r)}{r^2}\frac{\partial^2 \delta Q}{\partial r^2} + \frac{1}{r^2 f(r)}\frac{\partial^2 \delta Q}{\partial z^2} + \left(\frac{f'(r)}{r^2} - \frac{2f(r)}{r^3}\right)\frac{\partial \delta Q}{\partial r} + \frac{\sigma^2}{r^4}\delta Q = 0. \tag{4.2.4}$$

Making $\delta Q = R(r)Z(z)$ it is seen that this is actually separable. For the dependence on z we obtain the harmonic oscillator equation

$$Z''(z) + l^2 Z(z) = 0, \tag{4.2.5}$$

with l a linear momentum. And for the radial dependence the equation is

$$f(r)R''(r) + f'(r)R'(r) + \frac{1}{r}\left(f'(r) + \frac{\sigma^2 - 2f(r)}{r} - \frac{l^2 r^5}{f(r)^3}\right)R(r) = 0. \tag{4.2.6}$$

Notice that, although we can recognize the second derivative with respect to the tortoise coordinate in the first two terms when multiplying by $f(r)$, this is not a Schroedinger-like equation because there is no term of the form $\sigma^2 R(r)$.

It is interesting to see that the allowed asymptotic behaviours when $r \to \infty$ are

$$R(r \to \infty) = \frac{D_1}{r^{\Delta_+}}\left(1 + \mathcal{O}\left(\frac{1}{r}\right)\right) + \frac{D_2}{r^{\Delta_-}}\left(1 + \mathcal{O}\left(\frac{1}{r}\right)\right), \tag{4.2.7}$$

with $\Delta_+ = 1$ and $\Delta_- = 0$.

Again, it is not possible to solve this equation analytically, but we will solve it numerically with the purpose of comparing the normal modes that this method provides with those computed in [1].

Notice that in [1] the authors use a completely different method to study the gravitational perturbations of the AdS soliton in $(p+2)$ dimensions, with the goal of getting the mass spectrum for the graviton and consequently the mass spectrum of the dual glueballs. Using the WKB approximation following [43], they were able to get an analytic expression for it in each studied case. For axial (vector) perturbations, the spectrum is

$$M^2(p) = n\left(n + \frac{p+5}{4}\right)\frac{16\pi^3}{\beta^2}\left(\frac{\Gamma\left(\frac{3+p}{2(p+1)}\right)}{\Gamma\left(\frac{1}{p+1}\right)}\right)^2 + O(n^0), \tag{4.2.8}$$

where n is the harmonic, β is the period of the ϕ coordinate, and $L = r_0 = 1$. In four dimensions ($p = 2$), this reduces to

$$M^2(2) = 9\pi n\left(n + \frac{7}{4}\right)\left(\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}\right)^2. \tag{4.2.9}$$

For the numerical procedure described below, we have set $l = 0$ for simplicity and also because in [1] the authors work in the rest frame. It is important to mention that the analysis below was also made for $l \neq 0$, and for greater l values our results differ more from those in [1].

Our numerical solution works as follows:

A variable change is made such that the new variable, x, can take values from 0 to 1, where 0 is identified with the origin of the soliton and 1 corresponds to infinity. We chose to make $r = r_0/(1-x)$, so the new equation (when $L = r_0 = 1$) is

$$x(x^2 - 3x + 3)R''(x) + 3(x-1)^2R'(x) + (\sigma^2 - 3z + 3)R(x) = 0, \tag{4.2.10}$$

Then, we observe the asymptotic behaviour of (4.2.10) in this new variable, near $x = 0$. The two branches in this case are (with $l = 0$)

$$\{1 + O(x), \ln(x) + O(x)\} \tag{4.2.11}$$

It is clear that the second branch diverges near the origin, so we choose to work with the first branch. If necessary, we rewrite $R(x)$ as $k(x)F(x)$, with an appropriate

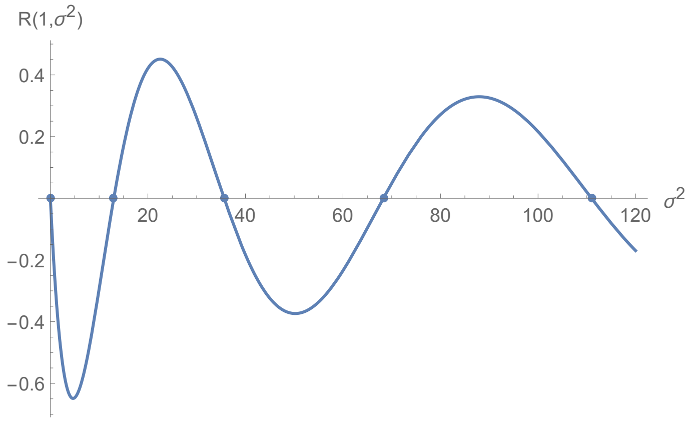

Figure 4.2.1: Numeric result for $R(1,\sigma^2)$.

Harmonic	$M^2(p)$	σ^2
0	0	2.94×10^{-5}
1	13.80	13.01
2	37.65	35.66
3	71.53	68.31

Table 4.2.1: Comparison of the normal modes obtained by Constable and Myers ($M^2(p)$) and by using Chandrasekhar's approach (σ^2).

choice for $k(x)$ in order to obtain an equation for $F(x)$ with asymptotic behaviour $1+O(x)$. This time this is not necessary, so we immediately look the second term in the expansion in order to set boundary conditions at $x=0$, namely

$$R(x \to 0) = 1 - \left(\frac{\sigma^2}{3} + 1\right) x + O(x^2) \tag{4.2.12}$$

We set the boundary conditions $R(0) = 1$ and $R'(0) = \left(\frac{\sigma^2}{3} + 1\right)$, and with them we are able to solve (4.2.10) numerically. This solution has still an unknown value of σ^2, so we will call it $R(x, \sigma^2)$.

At infinity, $x = 1$, we expect $R(x,\sigma^2)$ to vanish, so imposing it as a boundary condition implies that the values of σ^2 that satisfy $R(1,\sigma^2) = 0$ are the normal modes of the AdS soliton. Figure 4.2.1 shows $R(1,\sigma^2)$ vs σ^2, and we are able to extract the values of σ^2 that intersect the horizontal axis. To compare with the results in [1] we identify $M^2(p=2) = \sigma^2$ and the values are shown in Table 4.2.1.

It is evident that the values start close to each other and they differ more as the harmonic is larger. This is expected as we are using different equations obtained from different methods, and also in [1] the WKB approximation is used. Despite this, it is safe to say that Chandrasekhar's approach that uses a general static axisymmetric metric form as its starting point, used to analyse the gravitational perturbations of different black holes, also works for solitonic backgrounds.

4.2.2 Polar perturbations

Now we move to the analysis of polar perturbations on the soliton matric. Just as in section 3.2.3, we start analysing equations $\delta R_{02} = 0$, $\delta R_{03} = 0$, $\delta R_{23} = 0$ and $\delta G_{22} = 0$. After linearizing the equations and asumming each functions depends on time in the form $e^{i\sigma t}$, we obtain:

$$\frac{\partial}{\partial r}\left(\delta\mu_3 + \delta\psi\right) - \frac{f'(r)}{2f(r)}\left(\delta\mu_2 - \delta\mu_3\right) + \frac{1}{r}\left(\delta\mu_2 + \delta\mu_3\right) = 0, \tag{4.2.13}$$

$$\frac{\partial}{\partial y}\left(\delta\mu_2 + \delta\psi\right) = 0, \tag{4.2.14}$$

$$\frac{\partial^2}{\partial r\partial y}\left(\delta\nu + \delta\psi\right) + \left(\frac{1}{r} - \frac{f'(r)}{2f(r)}\right)\frac{\partial}{\partial y}\left(\delta\nu + \delta\psi\right) - \frac{2}{r}\frac{\partial\delta\mu_2}{\partial y} = 0, \tag{4.2.15}$$

$$\frac{1}{f(r)}\frac{\partial^2}{\partial y^2}\left(\delta\nu + \delta\psi\right) + \frac{f(r)}{r}\frac{\partial}{\partial r}\left(\delta\nu + \delta\psi + 2\delta\mu_3\right) + \frac{f'(r)}{2}\frac{\partial}{\partial y}\left(\delta\nu + \delta\psi\right) - 2\left(f'(r) + \frac{f(r)}{r^2}\right)\delta\mu_2 + \frac{\sigma^2}{r^2}\left(\delta\mu_3 + \delta\psi\right) = 0, \tag{4.2.16}$$

where each function depends on r and z.

Similar to the previous cases, at this point one proofs that these equations are separable. This time we need a different variables separation since we no longer have spherical symmetry, so instead of spherical harmonics we use planar harmonic functions:

$$\delta\nu = T(r)e^{-ilz}, \quad \delta\mu_2 = L(r)e^{-ilz}, \quad \delta\mu_3 = N(r)e^{-ilz}, \quad \delta\psi = Q(r)e^{-ilz}. \tag{4.2.17}$$

Replacing in (4.2.14), we obtain

$$L(r) + Q(r) = 0. \tag{4.2.18}$$

Again, only three of the four radial functions are independent.

Using $Q(r) = -L(r)$ in the rest of the equations, we achieve to write the following coupled system of first order ordinary differential equations:

$$L'(r) = \left(a - \frac{1}{r}\right) L(r) - \left(a - \frac{\sigma^2}{2rf(r)}\right) T(r) + bN(r), \tag{4.2.19}$$

$$T'(r) = \left(a + \frac{2}{r} - \frac{f'(r)}{2f(r)}\right) L(r) - \left(a - b - \frac{\sigma^2}{rf(r)}\right) T(r) + bN(r), \tag{4.2.20}$$

$$N'(r) = \left(a + \frac{f'(r)}{2f(r)}\right) L(r) - \left(a + \frac{\sigma^2}{2rf(r)}\right) T(r) - \frac{\sigma^2}{2rf(r)} N(r), \tag{4.2.21}$$

with

$$a = \frac{r^2 f'(r)^2 - 4l^2r^2 + 4\sigma^2 f(r) - 4f(r)^2}{8rf(r)^2},$$

$$b = \frac{rf'(r) - \sigma^2}{2f(r)r} - \frac{1}{r}.$$

Regretfully, we were not able to write this system as a single second order differential equation, despite trying different elections of (4.2.17), so no further analysis can be done in this case. Based on the good results that this approach gave for the axial perturbations of the AdS soliton in the previous section, we are hopeful on the existence of the combination of radial functions such that the single second order equation exist, and that it depends on the choice of separation (4.2.17) and/or how the ansatz for the function that decouples the equations is constructed, but the correct choice is still eluding us.

Chapter 5

Conclusion

In this work we have showed that the approach developed by Chandrasekhar to study gravitational perturbations of stationary and axisymmetric geometries allow to obtain the master equations to compute the quasinormal modes for Schwarzschild AdS spacetime and the axial normal modes for the AdS soliton, with consistent results in comparison with those obtained by alternative methods.

We have proven the applicability of this formalism in the context of gravity with negative cosmological constant and we have partially extended its applicability to spacetimes that are everywhere regular. In the case of Schwarzschild AdS, we recover the known effective potentials for both, axial and polar perturbations, and with it, we were able to construct the Schroedinger-like equations as in the Schwarzschild case. Besides, we explicitly showed how the isospectrality is broken in the asymptotically AdS case, originated in the generating function of the potential. For the AdS soliton, we were not able construct an equation to work on the polar modes, but we did find it for the axial modes and we computed those normal modes numerically, helping us to verify its concordance with previous results.

There are still relevant extensions to make to this work. For example, we could extend this analysis to more general solutions, such as rotating ones (like Kerr AdS) or charged ones (Reissner-Nordström AdS). We could also add matter fields, which has already been done for asymptotically flat spacetimes, and study how its coupling to gravity affects the gravitational perturbations. Finally, we could also use the Newman-Penrose formalism as it would make the computation more brief since it gives automatically decoupled equations, this could solve the problem for the polar perturbations of the AdS soliton and we could continue its analysis.

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Appendix A

Components of the Ricci and Einstein tensor

Given the metric

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}\left(d\varphi - q_2 dx^2 - q_3 dx^3 - \omega dt\right)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2,$$

with ν, ψ, q_2, q_3, ω, μ_2 and μ_3 dependent on t, x^2 y x^3, the components of the Ricci and Einstein tensor are:

$$\begin{aligned} R_{00} = &-e^{-2\nu}\left[\frac{\partial^2}{\partial t^2}(\psi+\mu_2+\mu_3) + \frac{\partial}{\partial t}\psi\frac{\partial}{\partial t}(\psi-\nu) + \frac{\partial}{\partial t}\mu_2\frac{\partial}{\partial t}(\mu_2-\nu)\right. \\ &\left.+\frac{\partial}{\partial t}\mu_3\frac{\partial}{\partial t}(\mu_3-\nu)\right] + e^{-2\mu_2}\left[\frac{\partial^2}{\partial (x^2)^2}(\nu) + \frac{\partial}{\partial x^2}\nu\frac{\partial}{\partial x^2}(\psi+\nu-\mu_2+\mu_3))\right] \\ &+ e^{-2\mu_3}\left[\frac{\partial^2}{\partial (x^3)^2}\nu + \frac{\partial}{\partial x^3}\nu\frac{\partial}{\partial x^3}(\psi+\nu+\mu_2-\mu_3)\right] \\ &+ \frac{1}{2}e^{2\psi-2\nu}\left[e^{-2\mu_2}Q_{20}^2 + e^{-2\mu_3}Q_{30}^2\right], \end{aligned} \tag{A0.1}$$

$$\begin{aligned} R_{11} = &-e^{-2\mu_2}\left[\frac{\partial^2}{\partial (x^2)^2}\psi + \frac{\partial}{\partial x^2}\psi\frac{\partial}{\partial x^2}(\psi+\nu+\mu_3-\mu_2)\right] - e^{-2\mu_3}\left[\frac{\partial^2}{\partial (x^3)^2}\psi\right. \\ &\left.+\frac{\partial}{\partial x^3}\psi\frac{\partial}{\partial x^3}(\psi+\nu+\mu_2-\mu_3)\right] + e^{-2\nu}\left[\frac{\partial^2}{\partial t^2}\psi + \frac{\partial}{\partial t}\psi\frac{\partial}{\partial t}(\psi-\nu+\mu_2+\mu_3)\right] \\ &- \frac{1}{2}e^{2\psi-2\mu_2-2\mu_3}Q_{23}^2 + \frac{1}{2}e^{2\psi-2\nu}\left[e^{-2\mu_3}Q_{30}^2 + e^{-2\mu_2}Q_{20}^2\right], \end{aligned} \tag{A0.2}$$

$$R_{22} = -e^{-2\mu_2}\left[\frac{\partial^2}{\partial (x^2)^2}(\psi+\nu+\mu_3) + \frac{\partial}{\partial x^2}\psi\frac{\partial}{\partial x^2}(\psi-\mu_2) + \frac{\partial}{\partial x^2}\mu_3\frac{\partial}{\partial x^2}(\mu_3-\mu_2)\right.$$
$$\left.+\frac{\partial}{\partial x^2}\nu\frac{\partial}{\partial x^2}(\nu-\mu_2)\right] - e^{-2\mu_3}\left[\frac{\partial^2}{\partial (x^3)^2}\mu_2 + \frac{\partial}{\partial x^3}\mu_2\frac{\partial}{\partial x^3}(\psi+\nu+\mu_2-\mu_3)\right]$$
$$+ e^{-2\nu}\left[\frac{\partial^2}{\partial t^2}\mu_2 + \frac{\partial}{\partial t}\mu_2\frac{\partial}{\partial t}(\psi-\nu+\mu_2+\mu_3)\right] - \frac{1}{2}e^{2\psi-2\mu_2}\left[e^{-2\mu_3}Q_{23}^2\right.$$
$$\left.-e^{-2\nu}Q_{20}^2\right], \tag{A0.3}$$

$$R_{01} = -\frac{1}{2}e^{-2\psi-\mu_2-\mu_3}\left[\frac{\partial}{\partial x^2}\left(e^{3\psi-\nu-\mu_2+\mu_3}Q_{20}\right) + \frac{\partial}{\partial x^3}\left(e^{3\psi-\nu-\mu_3+\mu_2}Q_{30}\right)\right], \tag{A0.4}$$

$$R_{12} = -\frac{1}{2}e^{-2\psi-\nu-\mu_3}\left[\frac{\partial}{\partial x^3}\left(e^{3\psi+\nu-\mu_2-\mu_3}Q_{32}\right) - \frac{\partial}{\partial t}\left(e^{3\psi-\nu+\mu_3-\mu_2}Q_{02}\right)\right], \tag{A0.5}$$

$$R_{02} = -e^{-\mu_2-\nu}\left[\frac{\partial^2}{\partial x^2\partial t}(\psi+\mu_3) + \frac{\partial}{\partial x^2}\psi\frac{\partial}{\partial t}(\psi-\mu_2) + \frac{\partial}{\partial x^2}\mu_3\frac{\partial}{\partial t}(\mu_3-\mu_2)\right.$$
$$\left.-\frac{\partial}{\partial t}(\psi+\mu_3)\frac{\partial}{\partial x^2}\nu\right] + \frac{1}{2}e^{2\psi-\nu-2\mu_3-\mu_2}Q_{23}Q_{30}, \tag{A0.6}$$

$$R_{23} = -e^{-\mu_2-\mu_3}\left[\frac{\partial^2}{\partial x^2\partial x^3}(\nu+\psi) - \frac{\partial}{\partial x^2}(\psi+\nu)\frac{\partial}{\partial x^3}\mu_2 - \frac{\partial}{\partial x^3}(\psi+\nu)\frac{\partial}{\partial x^2}\mu_3\right.$$
$$\left.+\frac{\partial}{\partial x^2}\psi\frac{\partial}{\partial x^3}\psi + \frac{\partial}{\partial x^2}\nu\frac{\partial}{\partial x^3}\nu\right] + \frac{1}{2}e^{2\psi-2\nu-\mu_2-\mu_3}Q_{20}Q_{30}, \tag{A0.7}$$

$$G_{00} = -e^{-2\mu_2}\left[\frac{\partial^2}{\partial (x^2)^2}(\psi+\mu_3) + \frac{\partial}{\partial x^2}\psi\frac{\partial}{\partial x^2}(\psi-\mu_2+\mu_3) + \frac{\partial}{\partial x^2}\mu_3\frac{\partial}{\partial x^2}(\mu_3-\mu_2)\right]$$
$$- e^{-2\mu_3}\left[\frac{\partial^2}{\partial (x^3)^2}(\psi+\mu_2) + \frac{\partial}{\partial x^3}\psi\frac{\partial}{\partial x^3}(\psi-\mu_3+\mu_2) + \frac{\partial}{\partial x^3}\mu_2\frac{\partial}{\partial x^3}(\mu_2-\mu_3)\right]$$
$$+ e^{-2\nu}\left[\frac{\partial}{\partial t}\psi\frac{\partial}{\partial t}(\mu_2+\mu_3) + \frac{\partial}{\partial t}\mu_3\frac{\partial}{\partial t}\mu_2\right] - \frac{1}{4}e^{2\psi-2\nu}\left[e^{-2\mu_2}Q_{20}^2 + e^{-2\mu_3}Q_{30}^2\right]$$
$$- \frac{1}{4}e^{2\psi-2\mu_2-2\mu_3}Q_{23}^2, \tag{A0.8}$$

$$\begin{aligned}
G_{11} = e^{-2\mu_2} & \left[\frac{\partial^2}{\partial(x^2)^2}(\nu+\mu_3) + \frac{\partial}{\partial x^2}\nu\frac{\partial}{\partial x^2}(\nu-\mu_2+\mu_3) + \frac{\partial}{\partial x^2}\mu_3\frac{\partial}{\partial x^2}(\mu_3-\mu_2)\right] \\
& + e^{-2\mu_3}\left[\frac{\partial^2}{\partial(x^3)^2}(\nu+\mu_2) + \frac{\partial}{\partial x^3}\nu\frac{\partial}{\partial x^3}(\nu-\mu_3+\mu_2) + \frac{\partial}{\partial x^3}\mu_2\frac{\partial}{\partial x^3}(\mu_2-\mu_3)\right] \\
& - e^{-2\nu}\left[\frac{\partial^2}{\partial t^2}(\mu_2+\mu_3) + \frac{\partial}{\partial t}\mu_2\frac{\partial}{\partial t}(\mu_2-\nu) + \frac{\partial}{\partial t}\mu_3\frac{\partial}{\partial t}(\mu_3-\nu) + \frac{\partial}{\partial t}\mu_2\frac{\partial}{\partial t}\mu_3\right] \\
& + \frac{3}{4}e^{2\psi}\left[e^{-2\mu_2-2\mu_3}Q_{23}^2 - e^{-2\mu_2-2\nu}Q_{20}^2 - e^{-2\mu_3-2\nu}Q_{30}^2\right], \qquad \text{(A0.9)}
\end{aligned}$$

$$\begin{aligned}
G_{22} = e^{-2\mu_3} & \left[\frac{\partial^2}{\partial(x^3)^2}(\nu+\psi) + \frac{\partial}{\partial x^3}(\psi+\nu)\frac{\partial}{\partial x^3}(\nu-\mu_3) + \frac{\partial}{\partial x^3}\psi\frac{\partial}{\partial x^3}\psi\right] \\
& + e^{-2\mu_2}\left[\frac{\partial}{\partial x^2}\nu\frac{\partial}{\partial x^2}(\psi+\mu_3) + \frac{\partial}{\partial x^2}\psi\frac{\partial}{\partial x^2}\mu_3\right] \\
& - e^{-2\nu}\left[\frac{\partial^2}{\partial t^2}(\psi+\mu_3) + \frac{\partial}{\partial t}(\psi+\mu_3)\frac{\partial}{\partial t}(\mu_3-\nu) + \frac{\partial}{\partial t}\psi\frac{\partial}{\partial t}\psi\right] \\
& - \frac{1}{4}e^{2\psi}\left[e^{-2\mu_2-2\mu_3}Q_{23}^2 - e^{-2\mu_2-2\nu}Q_{20}^2 + e^{-2\mu_3-2\nu}Q_{30}^2\right]. \qquad \text{(A0.10)}
\end{aligned}$$

The Q functions are defined as

$$Q_{ab} = \frac{\partial}{\partial x^b}q_a - \frac{\partial}{\partial x^a}q_b \quad \text{and} \quad Q_{0a} = \frac{\partial}{\partial x^a}\omega - \frac{\partial}{\partial t}q_a.$$

The components R_{33}, R_{13}, R_{03} and G_{33} are obtained interchanging index 2 and 3 in R_{22}, R_{12}, R_{02} and G_{22}, and the components not mentioned are identically zero.

Appendix B

Mathematica code for the numerical analysis

Here we show the code we used to make all the numerical analysis. This specific example uses the equation for the axial perturbation of the AdS soliton (4.2.10).

```
1 Needs["NumericalCalculus`"]
2 Equ[w_] = (z^3 - 3*z^2 + 3*z)*(D[F[z], z, z]) +
3    3*(-1 + z)^2*(D[F[z], z]) - (3*(-(1/3)*w + z - 1))*F[z];
4 bcsolu[w_] = 1;
5 Dbcsolu[w_] = 1/3 (-3 - w);
6 SOL[w_] := Module[{bc, sys},
7    bc = {F[0.001] == bcsolu[w], ND[F[z], z, 0.001] == Dbcsolu[w]};
8    sys = {Equ[w] == 0, bc[[1]], bc[[2]]};
9    NDSolve[ sys, F, {z, 1/1000, 1}]];
10 SOL1[w_] := Evaluate[F[z] /. SOL[w]][[1]] /. z -> 1
11 plot = Plot[SOL1[w], {w, 0, 120}, Mesh -> {{0}}]
12 D1 = Sort@Cases[Normal@plot, Point[{x_, y_}] -> x, Infinity]
```

Line 11 plots Figure 4.2.1 and line 12 gives the list of values of σ^2 presented in Table 4.2.1 in page 34.

For solitons, this analysis is simpler since normal modes are real. When working with the quasinormal modes of black holes, lines 11 and 12 are no longer useful because the values of the frequencies are complex. In this case, we use ComplexPlot to identify approximately the poles of the function and then we use FindRoot to find the precise values of the frequencies.