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Consequences of Torsion in Four Dimensions

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PERLA SOLEDAD MEDINA MEDINA

Thesis Advisor : Andrés Anabalón Dupuy
Departamento de Física,
Universidad de Concepción (UDEC)

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Concepción, Chile

Thesis Advisor : Dr. Andrés Anabalón Dupuy
Departamento de Física,
Universidad de Concepción (UDEC),

Assessment Committee : Dr. Jorge Zanelli Iglesias
Laboratorio de Física Teórica,
Centro de Estudios Científicos (CECs),
Valdivia, Chile.
Concepción, Chile.

Dr. Fernando Izaurieta Aranda
Facultad de Ingeniería, Arquitectura y Diseño,
Universidad San Sebastián (USS),
Concepción, Chile.

Dr. Luis Roa Oplingger
Departamento de Físia,
Universidad de Concepción (UDEC),
Concepción, Chile.

A mi familia.

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Abstract

This thesis addresses two fundamental topics in the field of gravity. First, a reformulation of the Mimetic theory of gravity in first-order formalism for differential forms, i.e., the mimetic version of Einstein-Cartan-Kibble-Sciama (ECKS) gravity. Here we consider different possibilities on how torsion is affected by conformal transformations and discuss how this translates into the interpolation between two different conformal transformations of the spin connection, parameterized with a zero-form parameter λ. We prove that regardless of the type of transformation one chooses, in this setting torsion remains as a non-propagating field. We also discuss the conservation of the mimetic stress-energy tensor and show that the trace of the total stress-energy tensor is not null but depends on both, the value of λ and spacetime torsion. Second, the Einstein-Hilbert action is studied in the first-order formalism, considering the coupling of a scalar field to different topological invariants. The implications of this coupling on the dynamics of the gravitational field and the evolution of the early universe are analyzed. The results obtained contribute to a better understanding of gravity and its relationship with the topology of spacetime. Second, the Einstein-Hilbert action is studied in the first-order formalism, considering the coupling of a scalar field to different topological invariants. The implications of these couplings on the field equations are analyzed. The results obtained shed light on a theoretical framework where it could be easier

to introduce couplings of scalar fields to the geometry.

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Introduction

General Relativity (GR) is a classical field theory describing the gravitational interaction through the Einstein field equations. Remarkably, it has proven successful in a wide range of phenomena [1] including black-holes as realistic astrophysical objects [2] and the existence of gravitational waves [3–5]. Another significant development of GR lies in the context of cosmology, in which an extension of Einstein's field equations, by the inclusion of an early inflationary stage and a cold dark matter contribution, is in good agreement with observational data [6]. This fact is somehow dramatic since we know very little about the nature of dark matter. For this reason, the problem of identifying dark matter candidates attracts attention from both cosmology and particle physics. There are several dark matter candidates, as weakly interacting massive particles, sterile neutrinos, axions, cold massive halo objects, and primordial black holes [7–9].

Given the difficulties for the standard cosmology models to describe the nature of dark matter (See, for instance, [10]), there has been a popular trend for considering modified gravity models [11–16]. However, observational constraints make it hard to create a consistent model of modified gravity still compatible with the principles of GR.

Another exciting direction is to consider GR beyond the limits of Riemannian geometry. The canonical model generalizing GR is the Einstein-Cartan-Kibble-Sciama (ECKS) gravity. This modification came firstly with the works of Elie Cartan in 1922, before the discovery of spin. However, Cartan's model did not bring much attention until the late 1950s, when Sciama and Kibble rediscovered Cartan's results [17, 18]. The main feature of ECKS gravity is that it accounts for spacetime torsion (See [19]). In ECKS, quantum mechanical spin acts as the source of torsion[1], in the same way as energy is a source of curvature [21, 22]. For this reason, torsion could have been relevant at the extremely high fermion densities in the early Universe [23–30]. However, in standard ECKS, the torsion two-form does not propagate in a vacuum, and it interacts very weakly with Standard Model fermions (See Chap. 8.4 of [31] and [32–34]). Thus, torsion could be a component of dark matter [35–37, 37–40].

More recently, in [41, 42], Chamseddine and Mukhanov have considered a different approach for addressing dark matter with the Mimetic gravity theory. In this model the conformal degree of freedom of the gravitational field becomes dynamical even in the absence of matter. This extra degree of freedom corresponds to the mimetic field's energy density, which mimics the stress-energy tensor of extra pressureless dust without needing dark matter particles. Moreover, [43] discusses whether mimetic cosmology can create late-time acceleration and the inflationary stage of the universe. Besides, many authors have considered different aspects of mimetic gravity during the last years with exciting results. For instance, in black holes [44–54], black strings [55], brane-world scenario [56–61], among others. For a more exhaustive survey, see [62–103] and references therein.

In this thesis, we construct a mimetic theory of gravity in first-order formalism for differential forms. For a sufficiently general family of conformal transformations of the affine connection, the resulting theory is equivalent to "mimicking" the ECKS model. For the construction, we assume that the vierbein one-form $e^a(x)$ and the spin connection one-form $\omega^{ab}(x)$ describe independent metric and affine properties of the spacetime. We consider diverse ways of generalizing conformal transformations for a Riemann-Cartan geometry with independent vierbein and spin connection, following a similar approach to [104]. Remarkably, none of these generalizations generate a propagating torsion field. Therefore, in this setting, only a non-vanishing spin tensor can be a source of torsion. When using

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[1] It is important to stress that with spin, we refer only to intrinsic quantum mechanical spin, and we should not confuse this with the angular momentum density. This natural confusion has already led to some mistakes in the literature, see [20]

scalar fields, non-minimal couplings and higher derivatives terms seem to be the way of producing a propagating torsion (See, for instance, [105, 106]). When considering generalized conformal invariance on Riemann-Cartan geometry, the stress-energy tensor's trace, which usually vanishes for conformally invariant theories of gravity, has a non-zero value depending on torsion and on a parameter which characterizes the conformal transformations for the spin connection.

On the other hand, the study of scalar fields coupled to topological invariants can have significant implications in cosmology, such as the universe undergoing an accelerated expansion without requiring either dark energy or negative pressure fluids [107]. If we couple a scalar field to the Euler or Pontryagin topological invariant, torsion is not explicitly included in the action, but is introduced indirectly through the non-minimal coupling of the scalar field, which is a small modification to general relativity that can have important consequences.On the other hand, the study of scalar fields coupled to topological invariants can have significant implications in cosmology, such as the universe undergoing an accelerated expansion without requiring either dark energy or negative pressure fluids. If we couple a scalar field to the Euler or Pontryagin topological invariant, torsion is not explicitly included in the action, but is introduced indirectly through the non-minimal coupling of the scalar field, which is a small modification to general relativity that can have important consequences.

Thesis plan

In the first chapter, we delve into the magnetic monopole, introducing fiber bundle theory as a motivating framework for its study. This approach is favored due to its ability to circumvent the evaluation of multiple integrals and its elegant description of the monopole using $U(1)$. Subsequently, the first-order formalism is elucidated, wherein the fiber at each point is associated with the Lorentz group $SO(3,1)$. Later, the mimetic theory of gravity ,is recast within this formalism, and its implications are examined. Finally, the

Einstein-Hilbert Lagrangian is studied in this formalism with a non-minimal coupling of a scalar field to different topological invariants in 4 dimensions. Five appendices with some technical details are also included.

Chapter 1

Fiber Bundle Theory: An Introduction and the Cartan Formalism

In the following chapter, we will address the study of the magnetic monopole, first through the study of a regularized potential and then with principal fiber bundles. Both formulations are valid for describing magnetic monopoles, but the principal fiber bundle approach offers significant advantages in terms of mathematical elegance and ease of calculation, especially by avoiding the need to evaluate multiple integrals.

1.1 Motivation: Magnetic Monopole

From Maxwell's equations, we know that

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{1.1}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho_e. \tag{1.2}$$

Where (1.1) implies we can write the magnetic field as follow $\vec{B} = \vec{\nabla} \times \vec{A}$. Besides, $\rho_e = e\delta^3(\vec{r})$, for a point charge e placed at the origin. That is, there are sources of electric charge, but there are no sources of magnetic charge. Furthermore, we can write the electric field as

$$\vec{E} = e\frac{\vec{r}}{r^3}.$$

On the other hand, let us recall that Maxwell's equations without sources admit a symmetry under the transformations $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$. This leads us to the question: Can the theory be made symmetric even with sources? Indeed, Dirac [108] noticed an asymmetry in Maxwell's equations: the equation $\vec{\nabla} \cdot \vec{B} = 0$ denies the existence of magnetic charges. He introduced the magnetic monopole to make the theory symmetric. If we assume the existence of magnetic charge and current densities, in addition to electric ones, Maxwell's equations in Gaussian units become:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{B} &= 4\pi\rho_m, \\
\vec{\nabla} \cdot \vec{E} &= 4\pi\rho_e, \\
\vec{\nabla} \times \vec{B} &= \frac{4\pi}{c}\vec{J}_e + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \\
-\vec{\nabla} \times \vec{E} &= \frac{4\pi}{c}\vec{J}_m + \frac{1}{c}\frac{\partial \vec{B}}{\partial t}.
\end{aligned}$$

Now we consider what the magnetic field of a point charge would look like. Analogous to the electric case, let's consider a static Coulomb-type magnetic field and define a magnetic charge g (centered at the origin) as its source. Then,

$$\vec{B} = g\frac{\vec{r}}{r^3}. \tag{1.3}$$

Using $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$ we get,

$$\vec{B} = -g\vec{\nabla}\left(\frac{1}{r}\right). \tag{1.4}$$

Therefore, since $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})$,

$$\vec{\nabla}\cdot\vec{B} = 4\pi\delta^3(\vec{r}). \tag{1.5}$$

If this is true, then we cannot write $\vec{B} = \vec{\nabla}\times\vec{A}$ all over the space. Furthermore, we can see that a point in $\mathbb{R}^3 - origin : \vec{\nabla}\cdot\vec{B} = 0$, this means $\vec{A}$ exist everywhere except at the origin.

However, the magnetic flux will be not zero (analogy to Gauss's Law)

$$\Phi = \oint_{S^2_R} \vec{B}\cdot d\vec{S} = 4\pi g. \tag{1.6}$$

The most crucial aspect is that the integral is taken over a closed surface that encloses the monopole but with an arbitrary shape and size. However, this conflicts with the assumption of a monopole's existence. If we can express the magnetic field as the curl of a potential vector $\vec{A}$, then the aforementioned integral should be zero. Let's assume there exists a potential $\vec{A}$, such that $\vec{B} = \vec{\nabla}\times\vec{A}$. If we now integrate $\vec{B}$ over a sphere of radius R minus a "disk" of radius r, that is,

$$\int_{S^2_R - disco} \vec{B}\cdot d\vec{S} = \oint_{S^2_R - disco}\left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S} = \oint_{disco} \vec{A}\cdot d\vec{l}. \tag{1.7}$$

In the limit $r \to 0$, this integral vanishes, which conflicts with (1.6), unless the potential

$\vec{A}$ is singular at at least one point.

Now we are going to address the question: What is the potential that satisfies (1.3)? Since the magnetic field is spherically symmetric, we can write the vector potential as:

$$\vec{A}(\vec{r}) = A(\theta)\vec{\nabla}\varphi$$

with $A(\theta) = -g(1+\cos\theta)$.

Then,

$$\vec{A}(\vec{r}) = \left(g\frac{(1+\cos\theta)\sin\varphi}{r\sin\theta}, -g\frac{(1+\cos\theta)\cos\varphi}{r\sin\theta}, 0 \right) \tag{1.8}$$

Note also that we can write this potential as:

$$\begin{aligned} \vec{A}(\vec{r}) &= \left(\frac{gy}{r[r-z]}, \frac{-gx}{r[r-z]}, 0 \right) && (1.9) \\ &= \frac{g}{r}\frac{\vec{r}\times\hat{z}}{[r-z]} && (1.10) \end{aligned}$$

Now, if we calculate $\vec{\nabla}\times\vec{A}$, it is easy to see that it gives us a magnetic field

$$\vec{B}(\vec{r}) = g\frac{\vec{r}}{r^3}.$$

The issue here is that this potential has two singularities: one at $r = 0$ and another at $z = r$. Therefore, if we want to calculate the magnetic flux, we will encounter problems as it will be inconsistent with (1.6). However, then there is no contradiction because we cannot use Gauss's law for a potential like the one we proposed, since it does not exist at every point in the space $\mathbb{R}^3$.

A first attempt to solve this problem is to make the following assumption: there exists a vector potential $\vec{A}$ such that the equation $\vec{\nabla} \times \vec{A} = \vec{B}$ is valid in all of space $\mathbb{R}^3$. Mathematically, this means that the potential must be a non-singular, smooth, and differentiable function.

The detailed calculation of this regularized potential can be found in Appendix A. This calculation has been performed in detail and is not present in the literature, however, it is based on [109], [110]. Where it is finally obtained that the magnetic field produced by a magnetic charge is given by

$$\vec{B}_R(\vec{r}) = g\frac{\hat{r}}{r^2} - 4\pi g \delta(x)\delta(y)\Theta(z)\hat{z}. \tag{1.11}$$

This means that all the flux goes to the positive semi-axis z for Gauss's law to work. That is, an infinitely thin flux tube appears along the positive z-axis. Now, since a zero-thin tube is a string, this object is called the Dirac string (see Figure 1).

This string carries the flux that exactly cancels the flux of $\vec{B}_g$. Now the question is, are there other solutions for the vector potential that lead us to the same magnetic field? The answer is yes and will be addressed below.

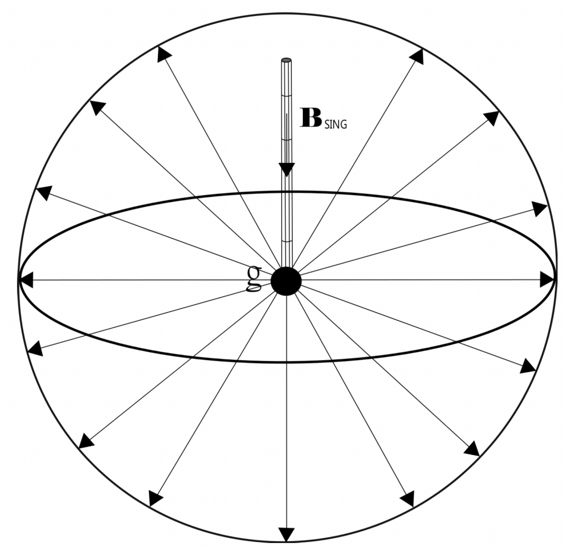

Figure 1.1: Dirac's String

1.1.1 Wu Yang Monopole

Wu and Yang (1975) noted that the geometric and topological structure underlying the Dirac monopole was best described by Fiber Bundles. First, we outline the idea of Wu and Yang without introducing the fiber bundle. Wu and Yang noted that we may employ more than one vector potential to describe a monopole [111].

If we define another vector potential

$$\vec{A}^N(\vec{r}) = g(1-\cos\theta)\vec{\nabla}\varphi \tag{1.12}$$

$$= g\frac{(1-\cos\theta)}{r\sin\theta}\hat{e}_\varphi, \tag{1.13}$$

and perform the same procedure as before, we will have $\vec{\nabla}\times\vec{A}^N = \vec{B}$, except along the negative z-axis ($\theta = \pi$ or equivalently $r = -z$). We use the notation $\vec{A}^S$ for the potential that diverges in the northern hemisphere and $\vec{A}^N$ for the potential that diverges in the southern hemisphere. Therefore, outside the z-axis, these two potentials will necessarily lead to the same magnetic field $\vec{B}$. Thus, we can say that both potentials are related by a gauge transformation such that

$$\vec{A}^N - \vec{A}^S = \vec{\nabla}\lambda \tag{1.14}$$

In order to achieve a magnetic field without any singularities, it is necessary to use two non-singular potentials to cover the entire sphere surrounding the monopole. One potential should cover the northern hemisphere and the other should cover the southern hemisphere. This approach results in a magnetic field that is non-singular at every point. A consequence of this singularity-free construction is that we must abandon the traditional parametrization of $\mathbb{R}^3$ surrounding the monopole by a single set of coordinates. Instead, we must divide $\mathbb{R}^3 - 0$ into two overlapping hemispheres, the northern hemisphere R^N and the southern hemisphere R^S. Their intersection $R^N \cap R^S$ will correspond to the equator, and the entire space surrounding the monopole must now be parametrized by separate sets of coordinates.

As a result, we can write two potentials $\vec{A}^N$ and $\vec{A}^S$ which are singularity-free everywhere in their domain of definition:

$$\begin{cases} \vec{A}^N = g\frac{1-\cos\theta}{r\sin\theta}\hat{e}_\varphi & \text{si} \quad 0 \leq \theta < \frac{\pi}{2} + \frac{\varepsilon}{2} : \quad R^N \\ \vec{A}^S = -g\frac{1+\cos\theta}{r\sin\theta}\hat{e}_\varphi & \text{si} \quad \frac{\pi}{2} - \frac{\varepsilon}{2} < \theta \leq \pi : \quad R^S \end{cases} \tag{1.15}$$

Furthermore, in the intersection region $R^N \cap R^S$, both potentials are well-defined and there is a gauge transformation connecting the two potentials, of the form (1.15),

$$\begin{aligned} \vec{A}^N - \vec{A}^S &= g\frac{1-\cos\theta}{r\sin\theta}\hat{e}_\varphi + g\frac{1+\cos\theta}{r\sin\theta}\hat{e}_\varphi \\ &= \frac{2g}{r\sin\theta}\hat{e}_\varphi \\ &= \vec{\nabla}(2g\varphi) \end{aligned}$$

Since both potentials are regular in their domain of definition, we can write the total flux as

$$\Phi = \oint_S \vec{\nabla} \times \vec{A} \cdot \overrightarrow{dS^2} = \int_{R^N} \vec{\nabla} \times \vec{A}^N \cdot d\vec{S} + \int_{R^U} \vec{\nabla} \times \vec{A}^S \cdot d\vec{S}.$$

And Stokes' theorem yields

$$\Phi = \oint_{S^2} \vec{B} \cdot d\vec{S} \tag{1.16}$$

$$= \oint_{S^2} [\vec{\nabla} \times \vec{A}] \cdot d\vec{S} \tag{1.17}$$

$$= \oint_{R^N} \left[\vec{\nabla} \times \vec{A}^N\right] \cdot d\vec{S} + \oint_{R^S} \left[\vec{\nabla} \times \vec{A}^S\right] \cdot d\vec{S} \tag{1.18}$$

$$= \oint_{\text{ecuador}} \vec{A}^N \cdot d\vec{r} + \oint_{\text{ecuador}} \vec{A}^S \cdot d\vec{r}' \tag{1.19}$$

$$= \oint_{\text{ecuador}} \vec{A}^N \cdot d\vec{r} - \oint_{\text{ecuador}} \vec{A}^S \cdot d\vec{r} \tag{1.20}$$

$$= \oint_{\text{ecuador}} \left(\vec{A}^N - \vec{A}^S\right) \cdot d\vec{r} \tag{1.21}$$

$$= \oint_{\text{ecuador}} \vec{\nabla}(2g\varphi) \cdot d\vec{r} \tag{1.22}$$

$$= 2g \int_0^{2\pi} d\varphi \tag{1.23}$$

$$= 4\pi g \tag{1.24}$$

where the minus sign is due to the orientation of the surface (see figure 1.2. Note that the value of the flux is the same (but more difficult to evaluate explicitly) if it is calculated over another surface that has the same topology as the sphere.

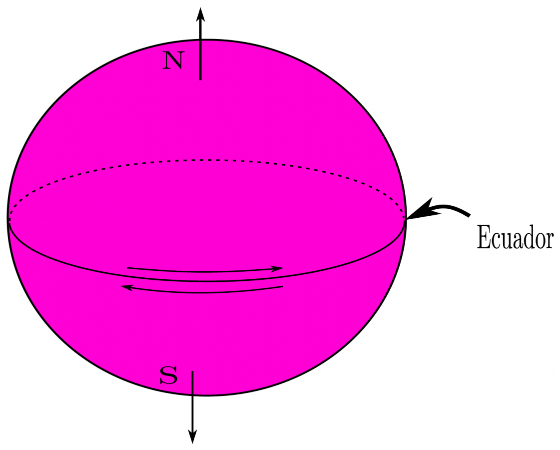

Figure 1.2: Dirac's Duality

In conclusion, if we choose to work with only one potential, we must live with the Dirac string. However, if we choose two well-defined potentials in different regions, the string does not appear and we observe that both potentials are related at the equator by a gauge transformation, whose effect is to take the Dirac string and move it to the other hemisphere.

In the following, we will study Fiber Bundles to analyze the Dirac monopole from this perspective.

1.2 Fiber Bundles

In this section, we are going to do a short review of fibre bundles based on [112].

A fiber bundle (E, π, M, F, G) consists of the following elements:

1. A differentiable manifold E called the total space.
2. A differentiable manifold M called the base space.
3. A differentiable manifold F called the fiber.
4. A surjective map $\pi : E \to M$ called the projection. The inverse image $\pi^{-1}(p) = F_p \cong F$ is called the fiber at the point p, where p is a point in an open set of M.
5. A Lie group G called the structure group, which acts on the left on F.
6. A collection of open covers $\{U_i\}$ of M with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$. The mapping ϕ_i is called the local trivialization since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ surjectively onto the product $U_i \times F$.
7. If we write $\phi_i(p, f) = \phi_{i,p}(f)$, the mapping $\phi_{i,p}(f) : F \to F_p$ is a diffeomorphism. In $U_i \cap U_j \neq \emptyset$, we require that $t_{ij} := \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$ be an element of G. Thus, ϕ_i and ϕ_j are related by a continuous mapping $t_{ij} : U_i \cap U_j \to G$ as

$$\phi_j(p,f) = \phi_i\left(p, t_{ij}(p)f\right)$$

t_{ij} is called the transition function.

The diffeomorphism ϕ_i sends elements of the product space $U_i \times F$ to a sector of the total space E, $\pi^{-1}(U_i)$.

- The projection $\pi : E \to M$ collapses a collection of points in the total space E onto M.
- The inverse π^{-1} acting on a point p reveals the fiber over that point, that is, it maps the point p surjectively onto the space E.
- F_p is the fiber over the point p.
- F is the entire fiber over U_i.
- It can 'smoothly unroll' that space into a product space. It guarantees smoothness in that process.

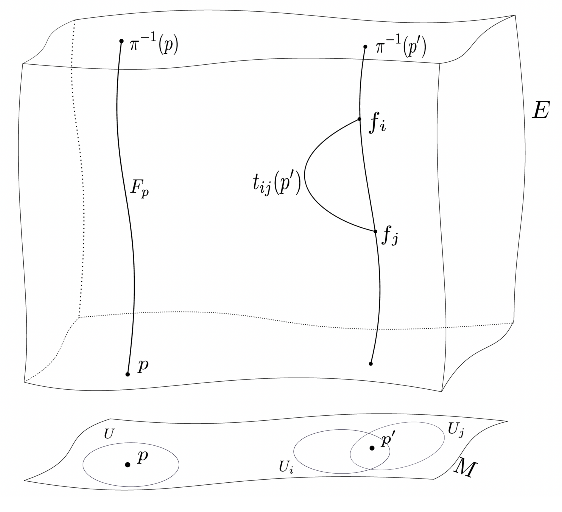

Figure 1.3: Fibre bundle

The diffeomorphism ϕ_i "unrolls" the collection of fibers over U_i, making it locally resemble a product $U_i \times F$.

Since ϕ_i is a diffeomorphism, over each subset $\pi^{-1}(U_i)$ of E, the inverse $\phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F$ (i.e., it takes elements of the fiber to the product space) ”unrolls” the fiber bundle. The existence of the diffeomorphism (bijective function with a smooth inverse) $U_i \times F \to \pi^{-1}(U_i)$ allows for this smooth unrolling of the space into a product space. To achieve this, it associates each point $u \in \pi^{-1}(p) \subset E$ with an element $(p, f_i) \in U_i \times F$. The element of the fiber f_i is unique for each u, and the subscript i indicates that the elements of the fiber are referred to a particular U_i.

1.2.1 Conditions on the Transition Functions

Consider a chart of the base space M. The mapping $\pi^{-1}(U_i)$ is diffeomorphic to a product space $U_i \times F$, under the mapping $\phi^{-1} : \pi^{-1}(U_i) \to U_i \times F$. If the intersection of the open sets is non-empty, $U_i \cap U_j \neq \emptyset$, we will have two mappings in $U_i \cap U_j$: ϕ_i and ϕ_j. Let's take a point in the total space $u \in E$ such that $\pi(u) = p \in U_i \cap U_j$. We then assign two elements of the fiber F_p, f_i and f_j, via two diffeomorphisms ϕ_i^{-1} and ϕ_j^{-1} such that $(u \in \phi^{-1})$

$$\phi_i^{-1}(u) = (p, f_i)\,,$$
$$\phi_j^{-1}(u) = (p, f_j)\,.$$

Then there exists a mapping $t_{ij} : U_i \cap U_j \to G$ which relates f_i and f_j as $f_i = t_{ij}(p) f_j$, where

$$t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p}$$

with $\phi_{i,p} := \phi_i(p, f), f \in F$.

Let's consider the case where the fiber bundle is the set of all tangent vector spaces over an m-dimensional manifold M. Indeed, the set

$$TM = \sum_p T_p M$$

satisfies the axioms of a fiber bundle, where the base space is locally homeomorphic (locally respects the structure of the spaces) to a subset of $\mathbb{R}^m$, and where T_pM is also homeomorphic to $\mathbb{R}^m$. Consider two charts U_i and U_j such that their intersection is non-empty. Then, if we take a vector $V \in T_pM$ with $p \in U_i \cap U_j$, we have that the vector has two coordinate representations, namely,

$$\begin{aligned} V &= V^\mu \frac{\partial}{\partial x^\mu}\bigg|_p \\ &= \bar{V}^\mu \frac{\partial}{\partial y^\mu}\bigg|_p \end{aligned}$$

The components $\bar{V}^\mu$ and V^μ are related by

$$\bar{V}^\nu = \frac{\partial y^\nu}{\partial x^\mu}(p) V^\mu$$

Therefore, if we want the set of coordinates x and y to be well-behaved and not give rise to indeterminacies, it is necessary that the matrix $\frac{\partial y}{\partial x}$ be non-singular, that is, that $R^\nu{}_\mu = \frac{\partial y^\nu}{\partial x^\mu} \in GL(m, \mathbb{R})$, the general linear group of invertible $m \times m$ matrices. This element of the general linear group corresponds precisely to the transition function, and the elements of the fiber correspond to the components of the vector V. The components V^μ and $\bar{V}^\mu$ are related by a transformation of $GL(m, \mathbb{R})$ generated by $R^\mu{}_\nu$ and refer to the same vector V, which is why it is required that the rotation be well-behaved.

The idea of the fiber bundle is that local pieces of it, determined by the covers U_i, are 'glued' together consistently, and the way to achieve this is by adding structure to the transition functions. This was accomplished in the previous example by stating that the change-of-basis matrix corresponds to an element of the group $GL(m, \mathbb{R})$. Thus, we require that the transition functions satisfy the following consistency conditions:

$$t_{ii} = id_F \quad (p \in U_i) \tag{14}$$

$$t_{ij} = t_{ji}(p)^{-1} \quad (p \in U_i \cap U_j) \tag{15}$$

$$t_{ij}(p)t_{jk}(p) = t_{ik}(p) \quad (p \in U_i \cap U_j \cap U_k) \tag{16}$$

Here, id_F denotes the identity element of the fiber F.

As we can observe, the above conditions (existence of the identity element, inverse, closure under composition) are equivalent to saying that the t_{ij} are elements of a group. An interesting case is when the transition functions can be mapped from the identity of the group. In that case, we have what is called a trivial bundle, since when passing through each local piece there is no need to twist the fiber, so that the bundle will look globally like a product space $M \times F$. This tells us that the information of the topology of the group is contained in the transition functions.

Observations: The transition function being the identity means that we are either moving within the same open set or that the entire bundle is trivial. For the latter to occur, it is necessary that the transition function be the identity in all intersections that can be formed with the collection of open sets. Therefore, if a transition function is not the identity in at least one intersection, it can make the bundle non-trivial.

1.2.2 Cross-sections of Fiber Bundles

- Definition: Let $E \xrightarrow{\pi} M$ be a fiber bundle. A cross-section $\sigma : M \to E$ is a smooth map that satisfies $\pi \circ \sigma = id_M$.

It is important to note that although the definition might lead one to think that σ corresponds to the inverse of the projection π, they are distinct mappings. The section σ passes through a unique point of F_p, $\sigma(p) \in F_p = \pi^{-1}(p)$. Since π^{-1} maps p surjectively onto F_p, and by the definition of the section, it is guaranteed that the composition $\pi \circ \sigma$ will map p to itself.

Geometrically, σ can be thought of as the continuous image of E to M that intersects each individual fiber at a single point. In the case of a trivial bundle, i.e., where we globally have $M \times F$, cross-sections can be taken as continuous functions defined on the base space M that take values on the space F, similar to how an ordinary function $f : \mathbb{R} \to \mathbb{R}$ takes elements in $\mathbb{R}$ and maps them to $\mathbb{R}$. Given the section σ, we can consider the set of all points through which this function passes in the total space. We denote the set of sections in M by $\Gamma(M, F)$.

Example 1: Let E be the fiber bundle $E \xrightarrow{\pi} S^1$ with fiber $F = [-1, 1]$. Let $U_1 = (0, 2\pi)$ and $U_2 = (-\pi, \pi)$ be two open sets that cover S^1, with $A = (0, \pi)$ and $B = (\pi, 2\pi)$ being the intersections in $U_1 \cap U_2$, see 1.4.

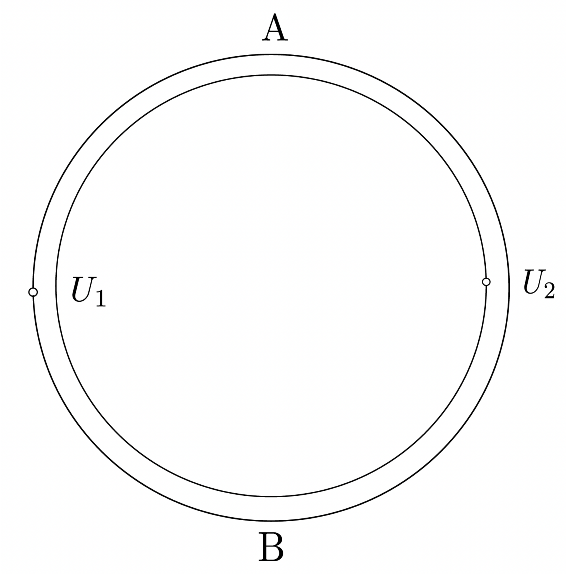

Figure 1.4: The base space S^1 and the two charts U_1 and U_2.

That is, U_1 and U_2 are open covers of S^1. U_1 covers the entire circle except $\theta = 2\pi$, and U_2 covers the entire circle except $\theta = \pi$. Recall that the local trivializations have $\pi^{-1}(U_i)$ as their domain and map it to the product space $\phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F$. Thus, the local trivializations ϕ_1^{-1} and ϕ_2^{-1} are given by

$$\phi_1^{-1}(u) = (\theta, t), \quad \phi_2^{-1}(u) = (\theta, t)$$

for $\theta \in A$ and $t \in F$. That is, we choose ϕ^{-1}, which has $U_i \times F$ and $U_j \times F$ as its codomain, to unroll the fiber as (θ, t) in both cases. On the other hand, the local trivialization ϕ_i is a bijection in $U_i \times F$, $\phi_i : U_i \times F \to \pi^{-1}(U_i)$.

We need this and its inverse to find the transition functions. We know that it's inverse $\phi^{-1}(u)$ takes a point $u \in \pi^{-1}$ and sends it to the point (θ, t), which is a point in the Cartesian space $U_i \times F$ (it unrolls a small piece of the fiber bundle into a product space). In this case, it is clearer to use ϕ^{-1} because we want to see explicitly how the fiber is unrolled into (θ, t), meaning we want to see what a point in $U_i \times F$ looks like.

The transition functions $t_{ij}(p) : U_i \cap U_j \to G$, with $p \in U_i \cap U_j$, are elements of a group G whose action on the elements of a fiber takes us to another element of the fiber: $F \to F$, and they are constructed from the local trivializations. The idea is that the transition functions map F to F bijectively. We need to relate a point in the fiber referred to one open set with another point in the fiber referred to as another open set. If we previously required that nothing happens in the intersection A, then in the intersection B there are two possibilities:

$$\text{I) } \phi_1^{-1}(u) = (\theta, t), \phi_2^{-1}(u) = (\theta, t)$$

$$\text{II) } \phi_1^{-1}(u) = (\theta, t), \phi_2^{-1}(u) = (\theta, -t)$$

That is, there are two ways to map the fiber onto itself via a bijection: Case I: $t \to t$ or Case II: $t \to -t$. This essentially means acting with $\mathbb{Z}_2$ on each element of the fiber.

For Case I, the transition function $t_{12}(\theta)$ is the identity, and two local pieces of the fiber bundle are glued together to form a cylinder (see Figure 1.5). On the other hand, we know that when the transition functions are the identity, the fiber bundle is trivial, i.e., globally it is a product space $S^1 \times F$. Furthermore, each point on the circle is assigned a unique value on the fiber, so that the set of all images of σ, $\Gamma(S^1, F)$, can be taken as a continuous curve on the surface of the cylinder represented by the dotted line. In this case, the total space E corresponds to the cylinder, and the fiber F is the width of the cylinder.

For Case II, we have $t_{12}(\theta) = t \to -t$, where $\theta \in B$, resulting in the Möbius strip (see Figure 1.6). This mapping "twists" the fiber as we move from one open set to another.

Recall that the elements of the group $g \in G$ act on the fiber F. Thus, the bijection $t \to t$ is equivalent to a left action $e \triangleleft t$, with e being the identity of the group G. Whereas

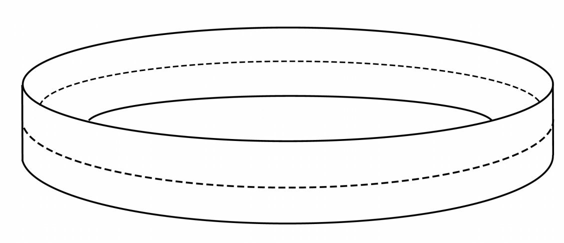

Figure 1.5: Cylinder.

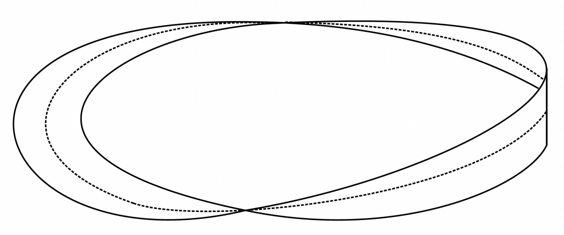

Figure 1.6: Möbius strip.

$g \triangleleft t = -t$, with $g \in G$, corresponds to the function $t \to -t$. Clearly, $g^2 = e$, that is, if we compose $g \triangleleft (g \triangleleft t) = e \triangleleft t = t$, we recover the identity. To see the definition of actions of a Lie group on a manifold, see Appendix B.

1.2.3 Principal Fiber Bundles

The motivation for defining a principal bundle arises from the desire to work with the concept of a connection. This type of bundle is special because each fiber corresponds to the same structure group G, allowing us to use concepts from group theory more freely and providing the setting for defining connections and their applications in physics.

- Definition: A principal fiber bundle is a fiber bundle $P \xrightarrow{\pi} M$ where the fiber F is identical to the structure group G. A principal fiber bundle is denoted by $P(M, G)$.

This type of bundle retains all the properties of the fiber bundles seen previously, with the difference that now $F = G$. Thus, the local trivializations ϕ_i locally unroll the bundle P as a product space $U_i \times G$. Principal bundles are very important because they allow us to understand any bundle with a fiber F on which the algebra G acts. For example, in General

Relativity, the fiber corresponds to $G = SO(3,1)$. Given a point u in a general fiber bundle F, the local trivializations associated with a certain cover $\{U_i\}$, $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ unroll the fiber locally. Thus, for $u \in \pi^{-1}(U_i) \subset E$ and $p \in U_i$, it follows that

$$u = \phi_i\left(p, f_i\right)$$

However, in the case of a principal bundle, we denote the elements on the fiber referred to an open set $\{U_i\}$ by g_i, because the element of the fiber is an element of a Lie group G,

$$u = \phi_i\left(U_i, g_i\right).$$

1.2.4 Dirac monopole

The Dirac monopole is defined in $\mathbb{R}^3$ with the origin O removed. $\mathbb{R}^3 - \{0\}$ and S^2 are of the same homotopy type and the relevant bundle is a U(1) bundle $P\left(S^2, \mathrm{U}(1)\right)$. S^2 is covered by two charts (see 1.2)

$$U_{\mathrm{N}} \equiv \left\{(\theta, \phi) \middle| 0 \leq \theta \leq \frac{1}{2}\pi + \epsilon\right\} \quad U_{\mathrm{S}} \equiv \left\{(\theta, \phi) \middle| \frac{1}{2}\pi - \epsilon \leq \theta \leq \pi\right\}$$

where θ and ϕ are polar coordinates. Take a local section σ_{N} (σ_{S}) on U_{N} (U_{S}) and define the local gauge potentials $\mathcal{A}_{\mathrm{N}}$ and $\mathcal{A}_{\mathrm{S}}$ to be of the Wu– Yang form (1.15)

$$\mathcal{A}_{\mathrm{N}} = \mathrm{i}g(1 - \cos\theta)\mathrm{d}\phi \quad \mathcal{A}_{\mathrm{S}} = -\mathrm{i}g(1 + \cos\theta)\mathrm{d}\phi$$

where g is the strength of the monopole.

Let t_{NS} be the transition function defined on the equator $U_{\mathrm{N}} \cap U_{\mathrm{S}}$. t_{NS} defines a map from S^1 (equator) to U(1) (structure group), which is classified by $\pi_1(U(1)) = \mathbb{Z}$. Let us write

$$t_{\mathrm{NS}}(\phi) = \exp[\mathrm{i}\varphi(\phi)] \quad \left(\varphi : S^1 \to \mathbb{R}\right)$$

The gauge potentials $\mathcal{A}_{\mathrm{N}}$ and $\mathcal{A}_{\mathrm{S}}$ are related on $U_{\mathrm{N}} \cap U_{\mathrm{S}}$ by

$$\mathcal{A}_{\mathrm{N}} = t_{\mathrm{NS}}^{-1} \mathcal{A}_{\mathrm{S}} t_{\mathrm{NS}} + t_{\mathrm{NS}}^{-1}\, \mathrm{d} t_{\mathrm{NS}} = \mathcal{A}_{\mathrm{S}} + \mathrm{i} \mathrm{d}\varphi$$

For the gauge potentials (1.15), we find

$$\mathrm{d}\varphi = -\mathrm{i}\left(\mathcal{A}_{\mathrm{N}} - \mathcal{A}_{\mathrm{S}}\right) = 2g\ \mathrm{d}\phi$$

While ϕ runs from 0 to 2π around the equator, $\varphi(\phi)$ takes the range

$$\Delta\varphi \equiv \int \mathrm{d}\varphi = \int_0^{2\pi} 2g\ \mathrm{d}\phi = 4\pi g$$

1.3 First Order Gravitation Theory

The following chapter is based on [113]- [114]. In mathematical terms, the equivalence principle translates to the spacetime being endowed with a smooth, differentiable D-dimensional manifold M with isolated singularities, such that at each point $x \in M$ we can define a D-dimensional tangent space T_x with Lorentzian signature $(-, +, ..., +)$. The connection to the equivalence principle lies in the fact that the transition from an open set in M to an open set in T_x corresponds to a change of reference frame to that of a freely falling observer. This isomorphism is precisely a way to relate tensors in the space M to tensors in T_x, and viceversa. This isomorphism is represented by a linear (bijective) mapping e, called the *vielbein*.

In General Relativity, the metric and affine properties are not independent, that is, the spacetime metric $g_{\mu\nu}$ should be the only dynamically independent field in the theory, while the affine connection (Christoffel symbol) $\Gamma^{\lambda}_{\mu\nu}$ is derived from it and will be more economical in terms of assumptions.

Cartan's point of view is more general; he advocated economy of assumptions and argued that metricity and parallelism could be considered as independent. Cartan emphasizes the distinction between the spacetime manifold M as the base of a fiber bundle, and the tangent space at every point T_x, where Lorentz vectors, tensors and spinors live, while the fiber at each point is the Lorentz group itself $G = SO(3,1)$.

Given that, from now on, we will only be working in this formalism where Geometry has two ingredients:

- Metric structure (length/area/volume, scale) e^a.
- Affine structure (parallel transport, congruence) $\longrightarrow \omega^a{}_b$.

1.3.1 Metric Structure

Each tangent space T_x is a copy of Minkowski space. Since Minkowski space is endowed with the Lorentzian metric, the diffeomorphism induces a metric structure on M :

$$dz^a = e^a_\mu(x)dx^\mu \equiv e^a$$

(e^a : vielbein, soldering form, local orthonormal frame).

$$\begin{aligned} ds^2 &= \eta_{ab}dz^a dz^b \\ &= \eta_{ab}e^a_\mu e^b_v dx^\mu dx^v \equiv g_{\mu v}dx^\mu dx^v \end{aligned}$$

Metric on M :

$$g_{\mu v}(x) = \eta_{ab}e^a_\mu(x)e^b_v(x)$$

- The vielbein can be viewed as the Jacobian matrix that relates differential forms in T_x and M.

$$dz^a = e^a_\mu(x)dx^\mu$$

- M admits a D-dimensional tangent space T_x at each point.
- An open set around any point x is diffeomorphic to an open set in the tangent.

1.3.2 Equivalence Principle

A sufficiently small vicinity of any point of M can be accurately approximated by T_x.

- In a sufficiently small region of spacetime a reference frame can always be found in which the laws of physics are those of special relativity (free fall).
- The laws of physics are invariant under local Lorentz transformations.
- General Reativity is a nonabelian gauge theory for the group $SO(3,1)$ in $4D$.

This means that it is possible to rotate the vielbein by a Lorentz transformation, and this should be undetectable from the perspective of the manifold M. However, recall that this isomorphism is valid locally, that is, for a sufficiently small laboratory and for a sufficiently short duration of the experiments. In effect, a Poincaré transformation would indeed be detectable.

1.3.3 Affine Structure

The differential operations appropriate for the theory should be Lorentz covariant, e.g.,

$$D_\mu u^a = \partial_\mu u^a + \omega^a{}_{b\mu} u^b$$

where u^a is a Lorentz vector and $\omega^a{}_{b\mu}$ is the connection for the Lorentz group. This means that under a Lorentz transformations, u^a and $D_\mu u^a$ must transform in the same manner:

$$\begin{aligned} u^a &\to u'^a = \Lambda^a_b u^b, \Lambda^a_b \in SO(D-1,1) \\ Du^a &\to (Du^a)' = \Lambda^a_b \left(Du^b\right) \end{aligned}$$

Geometric interpretation: The parallel transport of a vector $u^a(x+dx)$ from T_{x+dx} to the tangent space T_x is a vector $u^a_{\|}(x+dx)$,

$$
\begin{aligned}
& u^a{}_{\|}(x+dx) - u^a(x) \\
& = u^a(x+dx) + \omega^a{}_{b\mu} dx^\mu u^a - u^a \\
& = dx^\mu \left[\partial_\mu u^a + \omega^a{}_{b\mu} u^a\right] = dx^\mu D_\mu u^a.
\end{aligned}
$$

In this formalism the covariant derivative and the curvature 2-form are defined uniquely by the Lorentz connection 1-form

$$\omega^a_b = \omega^a_{b\mu} dx^\mu$$

The **2-form torsion** T^a is defined as the covariant exterior derivative of the 1-form vielbein,

$$T^a \equiv Du^a = du^a + \omega^a{}_b e^b,$$

which, due to the definition of the covariant derivative, involves both the vielbein and the spin connection.

The **2-form curvature** R^a_b is defined

$$DDu^a = R^a_b u^b, \quad R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b,$$

which satisfies Bianchi identity:

$$DR^a_b u^b = R^a_b Du^b$$

where $DR^a{}_b = dR^a{}_b + \omega^a{}_c R^c{}_b - \omega^a{}_c R^c{}_b \equiv 0$.

1.3.4 Building Blocks

So far, we have defined the fundamental elements to construct a dynamical theory of the spacetime geometry:

e^a	vector, 1-form
$\omega^a{}_b$	connection 1-form
$R^a{}_b = d\omega^a{}_b + \omega^a{}_c\omega^c{}_b$	curvature 2-form
$T^a = de^a + \omega^a{}_c e^c$	torsion 2-form

together with the invariant tensors 0-forms $\varepsilon_{a_1a_2\cdots a_D}$, and η_{ab}, the most general D-dimensional Lorentz invariant Lagrangian built with these ingredients is a polynomial in e^a, ω^a, and their exterior derivatives. If no further ingredients or assumptions are added, there is a finite family of Lagrangians that can be constructed with these elements in each dimension.

Chapter 2

Mimetic Einstein-Cartan-Kibble-Sciama (ECKS) gravity

2.1 Review of Mimetic gravity

Mimetic Gravity was first introduced by A. Chamseddine and V. Mukhanov as a theory of gravity which naturally exhibits conformal symmetry as internal degree of freedom [41]. Let M_4 be a four dimensional spacetime and let us consider a physical metric $g_{\mu\nu}$, with Lorentz signature $(-,+,+,+)$, depending on an auxiliary metric $\bar{g}_{\mu\nu}$ and a scalar field ϕ, namely

$$g_{\mu\nu} = -\bar{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\bar{g}_{\mu\nu}\,. \tag{2.1}$$

The metric $g_{\mu\nu}$ is invariant with respect to conformal transformations of the auxiliary metric $\bar{g}_{\mu\nu}$, i.e., it remains unchanged after rescaling

$$\bar{g}_{\mu\nu} \to \Omega^2\left(x\right)\bar{g}_{\mu\nu}. \tag{2.2}$$

Additionally, it follows from (2.1) that

$$g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = -1\,. \tag{2.3}$$

The resultant new degree of freedom associated with the transformation (2.1) represents the longitudinal mode of gravity which is excited even in the absence of any matter field configurations.

The canonical action of GR is rewritten by considering the physical metric $g_{\mu\nu}$ as function of the scalar field ϕ and the auxiliary metric $\bar{g}_{\mu\nu}$

$$S = \frac{1}{c}\int \mathrm{d}^4x\sqrt{-g(\bar{g}_{\mu\nu},\phi)}\left[\frac{1}{\kappa_4}\left(\frac{1}{2}R\left(\bar{g}_{\mu\nu},\phi\right) - \Lambda\right) + L_m\right], \tag{2.4}$$

where $\kappa_4 = \frac{8\pi G}{c^4}$, R is the Ricci scalar constructed from $g_{\mu\nu}$ and L_m stands for the matter Lagrangian. The action (2.4) is invariant under conformal transformation because it only depends on $g_{\mu\nu}$ which is conformally invariant under (2.2). The resulting dynamics can be directly obtained by starting from the variation of (2.4) with respect to the physical metric $g_{\mu\nu}$, then expressing $\delta g_{\mu\nu}$ in terms of $\delta\bar{g}_{\mu\nu}$ and $\delta\phi$ and assuming that the last two are independent. Thus,

$$G^{\mu\nu} - \kappa_4\mathcal{T}^{\mu\nu} + \left(G - \kappa_4\mathcal{T}\right)g^{\lambda\mu}g^{\sigma\nu}\partial_\lambda\phi\partial_\sigma\phi = 0\,, \tag{2.5}$$

$$\nabla_\mu\left[\left(G - \kappa_4\mathcal{T}\right)\partial^\mu\phi\right] = 0\,, \tag{2.6}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}$ is the Einstein tensor, $\mathcal{T}_{\mu\nu}$ the energy momentum tensor, and G, $\mathcal{T}$ denote their respective traces. The dynamics given in (2.5) and (2.6) departs from pure GR. In [115], an equivalent formulation of Mimetic Gravity has been proposed where, instead of introducing ϕ through the reparametrization (2.4), the physical metric $g_{\mu\nu}$ is directly used together with a constrained scalar field, enforcing (2.3) through a Lagrange multiplier.

Taking the trace in (2.5), direct calculation shows

$$(G - \kappa_4 \mathcal{T}) \left(1 + g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right) = 0\,. \tag{2.7}$$

This last equation is automatically satisfied by the constraint (2.3) even for $G \neq \kappa_4\mathcal{T}$. From this point of view, even in absence of matter, the gravitational field equations have non-trivial solutions for the conformal mode. To understand this extra degree of freedom, let us rewrite Eq.(2.5) as

$$G^{\mu\nu} = \kappa_4 \left(\mathcal{T}^{\mu\nu} + \bar{\mathcal{T}}^{\mu\nu}\right)\,, \tag{2.8}$$

where

$$\bar{\mathcal{T}}^{\mu\nu} = \left(\mathcal{T} - \frac{G}{\kappa_4}\right) g^{\mu\alpha} g^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi\,. \tag{2.9}$$

Now compare this expression with the energy momentum tensor for a perfect fluid

$$\mathcal{T}^{\mu\nu} = \frac{1}{c^2}\left(\varepsilon + p\right) u^\mu u^\nu - p g^{\mu\nu}\,, \tag{2.10}$$

where ε is the energy density, p is the pressure and u^μ is the four-velocity which satisfies $\frac{1}{c^2}u^\lambda u_\lambda = -1$. Setting $p = 0$ and making the following identification

$$\varepsilon = \left(\mathcal{T} - \frac{G}{\kappa_4}\right)\,, \tag{2.11}$$

$$u^\mu = c\, g^{\mu\alpha}\partial_\alpha\phi\,, \tag{2.12}$$

the energy momentum tensor (2.10) becomes equivalent to $\bar{\mathcal{T}}^{\mu\nu}$. Thus, the extra degree of freedom mimics the potential motions of dust with energy density $\left(\mathcal{T} - \frac{G}{\kappa_4}\right)$, and the scalar field plays the role of a velocity potential. In absence of matter this energy density is proportional to $G = 4\Lambda - R$, which does not vanish for generic solutions. The normalization condition for the four velocity u^μ and the conservation law for $\bar{\mathcal{T}}^{\mu\nu}$, are equivalent to (2.3) and (2.6), respectively.

2.2 ECKS gravity and First order formalism

So far we have used Greek indices $\mu, \nu, \ldots$ to denote tensor components in the coordinate basis. From now on, we use lower case Latin indices $a, b, \ldots$ for tensors defined in Lorentz (orthonormal) basis. We denote by $\Omega^p(M_4)$ to the set of differential p-forms defined over M_4.

At a particular point $P \in M_4$, the vierbein components $e^a{}_\mu(x)$ are determined through the relation

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \,, \tag{2.13}$$

where η_{ab} is the Minkowski metric. In terms of $e^a{}_\mu(x)$ we define the vierbein $e^a = e^a{}_\mu(x)\mathrm{d}x^\mu$ as the set of one-forms $\Omega^1(M_4) \in T^*_x(M_4)$. The vierbein contains all the information on the metric. The one-form spin connection describes the affine properties of the geometry $\omega^{ab} = \omega^{ab}{}_\mu(x)\mathrm{d}x^\mu$. The spin connection ω^{ab} relates to $\Gamma^\lambda{}_{\mu\nu}$ through the vierbein postulate

$$\partial_\mu e^a{}_\nu + \omega^a{}_{b\mu} e^b{}_\nu - \Gamma^\lambda{}_{\mu\nu} e^a{}_\lambda = 0 \,. \tag{2.14}$$

The covariant derivative of the vierbein is defined as the two-form torsion $T^a = \mathrm{D}e^a$ where

$$\mathrm{D}e^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b \,. \tag{2.15}$$

Unlike $\mathrm{d}^2 = 0$, higher order covariant derivatives of the vierbein does not vanish. In fact, direct calculation shows $\mathrm{D}T^a = R^a{}_b \wedge e^b$ where

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \,, \tag{2.16}$$

is the Lorentz two-form curvature which transforms covariantly under local Lorentz transformations.

The spin connection can also be decomposed in a torsion free part $\mathring{\omega}^{ab}$ satisfying

$$\mathrm{d}e^a + \mathring{\omega}^a{}_b \wedge e^b = 0\,, \tag{2.17}$$

and a second rank anti-symmetric one-form κ^{ab}, usually called the contorsion or contortion. An important observation is that $\mathring{\omega}^{ab}$ is completely determined in terms of the vierbein. Therefore, all the affine degree of freedom are encoded into the contorsion

$$\kappa^{ab} = \omega^{ab} - \mathring{\omega}^{ab}\,, \tag{2.18}$$

and consequently $T^a = \kappa^a{}_b \wedge e^b$. In terms of this splitting, the Lorentz curvature is given by

$$R^{ab} = \mathring{R}^{ab} + \mathring{\mathrm{D}}\kappa^{ab} + \kappa^a{}_c \wedge \kappa^{cb}\,, \tag{2.19}$$

where $\mathring{R}^{ab} = \mathrm{d}\mathring{\omega}^{ab} + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^{cb}$ is the Riemann curvature two-form and $\mathring{\mathrm{D}}$ stands for the covariant derivative with respect to the torsion free part of the connection $\mathring{\omega}^{ab}$.

There are many theories leading to non-vanishing torsion; see for instance [33, 36–39, 105, 116–123], and for the most general case, Poincaré Gauge Theory, see [21, 124–128]). However, the closest to standard General Relativity is Einstein-Cartan-Kibble-Sciama (ECKS) gravity [129–135]. In this framework, geometry also depends of the spin tensor of matter, but its physical effects would be noticeable only for densities much higher than nuclear density [136].

Moreover, it has been shown that torsion prevents cosmological singularities [137–142], gives rise to a new universe from a collapse [23, 143, 144], and introduce an effective ultraviolet cutoff in a quantum field theory for fermions [136]. In differential form language, the ECKS four-form Lagrangian is given by

$$\mathcal{L}_{\mathrm{ECKS}} = \mathcal{L}_{\mathrm{G}}\left(e, \omega\right) + \mathcal{L}_{\mathrm{M}}\left(e, \omega, \varphi\right)\,, \tag{2.20}$$

where

$$\mathcal{L}_{\mathrm{G}} = \frac{1}{4\kappa_4}\epsilon_{abcd}\left(R^{ab} - \frac{\Lambda}{3!}e^a \wedge e^b\right) \wedge e^c \wedge e^d\,, \tag{2.21}$$

is the four-form Lagrangian for geometry and $\mathcal{L}_{\mathrm{M}}$ is the four-form Lagrangian for matter fields. Up to boundary terms, variation of the action functional $S = \frac{1}{c}\int_{M_4}\mathcal{L}_{\mathrm{ECKS}}$ reads

$$\delta S = \frac{1}{c}\int_{M_4}\frac{1}{\kappa_4}\left(\frac{1}{2}\delta\omega^{ab} \wedge \mathcal{W}_{ab} + \mathcal{E}_d \wedge \delta e^d\right)\,, \tag{2.22}$$

with

$$\mathcal{E}_d = \epsilon_{abcd}\left(\frac{1}{2}R^{ab} - \frac{\Lambda}{3!}e^a \wedge e^b\right) \wedge e^c - \kappa_4 * \mathcal{T}_d\,, \tag{2.23}$$

$$\mathcal{W}_{ab} = \epsilon_{abcd}T^c \wedge e^d - \kappa_4 * \sigma_{ab}\,. \tag{2.24}$$

Here, the relations

$$\delta_e\mathcal{L}_{\mathrm{M}}^{(4)} = - * \mathcal{T}_d \wedge \delta e^d\,, \tag{2.25}$$

$$\delta_\omega\mathcal{L}_{\mathrm{M}}^{(4)} = -\frac{1}{2}\delta\omega^{ab} \wedge * \sigma_{ab}\,. \tag{2.26}$$

with $* : \Omega^p(M_d) \to \Omega^{d-p}(M_d)$ denoting the Hodge dual operator implicitly define the stress-energy one-form $\mathcal{T}^a = \mathcal{T}^a{}_\mu \mathrm{d}x^\mu$ and the spin tensor 1-form $\sigma^{ab} = \sigma^{ab}{}_\mu \mathrm{d}x^\mu$.

A usual practice when studying this system's phenomenology is to pack all the torsional terms (coming from the Lorentz curvature (2.19)) to create an extra stress-energy tensor in (2.23).

2.3 Conformal Riemann-Cartan structure

To characterize conformal structures in differential forms language, let us introduce an operator $\mathrm{I}_{a_1\ldots a_q} : \Omega^p(M_d) \to \Omega^{p-q}(M_d)$ defined in four dimensional spacetime $(-,+,+,+)$

by [105]

$$\mathrm{I}_{a_1 \ldots a_q} = -\left(-1\right)^{p(p-q)} * e_{a_1} \wedge \ldots \wedge e_{a_q} \wedge * \,. \tag{2.27}$$

The case $q = 1$ that gives

$$\mathrm{I}_a = - * e_a \wedge * \,, \tag{2.28}$$

is of particular relevance because it satisfies useful properties. It satisfies the Leibniz rule for differential forms and, together with D, the operator I_a defines another important object $\mathcal{D}_a : \Omega^p\left(M_d\right) \rightarrow \Omega^p\left(M_d\right)$ via the anti-commutator

$$\mathcal{D}_a = \left\{\mathrm{I}_a, \mathrm{D}\right\} = \mathrm{I}_a\mathrm{D} + \mathrm{D}\mathrm{I}_a \,, \tag{2.29}$$

The operators $\mathcal{D}$, I_a and D, form an open superalgebra where the two-forms curvature and torsion play the role of structure parameters (See [106]).

The conformal transformation

$$g_{\mu\nu} = \exp\left(2\sigma\right) \bar{g}_{\mu\nu} \,, \tag{2.30}$$

that relates the spacetime and the auxiliary manifolds supposes implicitly that a local mapping $\gamma : M_4 \rightarrow \bar{M}_4$ has been chosen in such a way that the same coordinates x^μ can be used for $P \in M_4$ and $\bar{P} = \gamma\left(P\right) \in \bar{M}_4$. This means that a coordinate transformation $x'^\mu = x'^\mu\left(x^\nu\right)$ in M_4 induces the same transformation in $\bar{M}_4$ and thus, tensors or forms defined on these manifolds transforms with the same Jacobian matrices. This fact allows us to find the relation between the vielbeins associated with these metrics, which by definition satisfy

$$g_{\mu\nu} = e^a_\mu e^a_\nu \eta_{ab} \,, \tag{2.31}$$

$$\bar{g}_{\mu\nu} = \bar{e}^a_\mu \bar{e}^b_\nu \eta_{ab} \,. \tag{2.32}$$

Indeed, mixing these expressions together with (2.30), it is direct to see that

$$e^{a} = \exp\left(\sigma\right)\bar{e}^{a}\,. \tag{2.33}$$

From the vierbein $e^{a}(x)$, it is possible to define "structure parameters" $\mathcal{C}_{ab}{}^{c}(x)$ satisfying a generalized Maurer-Cartan equation.

$$\mathrm{d}e^{c} = -\frac{1}{2}\mathcal{C}_{ab}{}^{c}e^{a}\wedge e^{b}\,. \tag{2.34}$$

This parameter allow us to solve the torsion-free part of the spin connection

$$\mathring{\omega}_{ab} = \frac{1}{2}\left(\mathcal{C}_{abc} + \mathcal{C}_{cba} - \mathcal{C}_{cab}\right)e^{c}\,. \tag{2.35}$$

Using eqs.(2.33), (2.34) and (2.35), we find

$$\mathring{\omega}_{ab} = \overline{\mathring{\omega}}_{ab} + \bar{e}_{a}\xi_{b} - \xi_{a}\bar{e}_{b}\,, \tag{2.36}$$

where

$$\xi^{a} = \bar{\mathrm{I}}^{a}\mathrm{d}\sigma\,, \tag{2.37}$$

and

$$\bar{\mathrm{I}}_{a} = -\bar{*}\left(\bar{e}_{a}\wedge\bar{*}\right. \tag{2.38}$$

with the bar in the Hodge dual denoting $\bar{e}^{a}$-vierbein dependence. In this way, Eq.(2.36) characterizes the conformal transformation associated to the torsion free part of the spin connection.

Notice that we have no information yet on the conformal transformation of the contorsion κ^{ab}. This is due to the fact that in the context of Riemann-Cartan geometry, e^{a} and κ^{ab} are completely independent degrees of freedom. Therefore, there are multiple possible choices on how κ^{ab} should transform under a Weyl dilatation. An important family of choices can be parameterized as

$$\bar{e}^{a} \to e^{a} = \exp\left(\sigma\right) \bar{e}^{a} \,, \tag{2.39}$$

$$\bar{\kappa}_{ab} \to \kappa_{ab} = \bar{\kappa}_{ab} + \left(\lambda - 1\right) \theta_{ab} \,, \tag{2.40}$$

$$\bar{\omega}_{ab} \to \omega_{ab} = \bar{\omega}_{ab} + \lambda \theta_{ab} \,, \tag{2.41}$$

where λ is a parameter $0 \leq \lambda \leq 1$ and $\theta^{ab} = -\theta^{ba}$ corresponds to the 1-form

$$\theta^{ab} = \bar{e}^{a} \xi^{b} - \xi^{a} \bar{e}^{b} \,. \tag{2.42}$$

The case $\lambda = 1$ implies,

$$\kappa_{ab} = \bar{\kappa}_{ab} \,, \tag{2.43}$$

$$\omega_{ab} = \bar{\omega}_{ab} + \theta_{ab} \,, \tag{2.44}$$

which is the "canonical case": the full spin connection changes as the torsionless case, and the contorsion is left untouched by the dilatation. The most "exotic" case corresponds to $\lambda = 0$

$$\kappa_{ab} = \bar{\kappa}_{ab} - \theta_{ab} \,, \tag{2.45}$$

$$\omega_{ab} = \bar{\omega}_{ab} \,, \tag{2.46}$$

where the spin connection is left untouched by the dilatation and the contorsion absorbs the transformation.

It is clear that the torsionless condition is preserved only for the $\lambda = 1$ case. In fact, the

Lorentz curvature and torsion change under the generalized Weyl dilatation (2.39-2.41) as

$$\bar{T}_a \to T_a = \exp(\sigma)\,\bar{T}_a + (\lambda - 1)\,\bar{e}_a \wedge \mathrm{d}\exp(\sigma)\,, \tag{2.47}$$

$$\bar{R}^{ab} \to R^{ab} = \bar{R}^{ab} + \lambda \bar{\mathrm{D}}\theta^{ab} + \lambda^2 \theta^a{}_c \wedge \theta^{cb}. \tag{2.48}$$

2.4 Mimetic ECKS gravity

Mimetic transformations are a particular choice of Weyl dilatations. Let us consider the auxiliary vierbein $\bar{e}^a$ and spin connection $\bar{\omega}^{ab}$ 1-forms, and a scalar field $\phi(x)$. In terms of operators $\bar{\mathrm{I}}_a$ and ϕ, let us define a zero-form Lorentz vector

$$\bar{Z}_a = \bar{\mathrm{I}}_a \mathrm{d}\phi\,, \tag{2.49}$$

and the scalar

$$\bar{Z}^2 = -\eta_{ab}\bar{Z}^a\bar{Z}^b\,. \tag{2.50}$$

The generalized mimetic vierbein e^a, contorsion κ^{ab}, and spin connection ω^{ab} are defined by

$$\bar{e}^a \to e^a = \bar{Z}\bar{e}^a\,, \tag{2.51}$$

$$\bar{\kappa}_{ab} \to \kappa_{ab} = \bar{\kappa}_{ab} + (\lambda - 1)\,\theta_{ab}\,, \tag{2.52}$$

$$\bar{\omega}_{ab} \to \omega_{ab} = \bar{\omega}_{ab} + \lambda\theta_{ab}\,, \tag{2.53}$$

where the one-form θ^{ab} is given in (2.42) with

$$\xi^a = \frac{1}{\bar{Z}}\bar{\mathrm{I}}^a \mathrm{d}\bar{Z}\,. \tag{2.54}$$

Notice that I_a and $\bar{\mathrm{I}}^a$ operator relate each other by

$$\mathrm{I}_a = \frac{1}{\bar{Z}}\bar{\mathrm{I}}_a\,, \tag{2.55}$$

and consequently

$$Z_a = \frac{1}{\bar{Z}} \bar{Z}_a \,, \tag{2.56}$$

so the constraint (2.3) reads

$$Z^2 = -\eta_{ab} Z^a Z^b = 1 \,. \tag{2.57}$$

2.4.1 Mimetic field equations

To construct the mimetic version of ECKS theory, let us consider the Lagrangian (2.20), with the vierbein e^a and ω^{ab} in terms of the auxiliary variables $\bar{e}^a$, $\bar{\omega}^{ab}$ and ϕ as in Eqs. (2.51)-(2.53). A priori, it would seem that different choices of λ would lead us to different dynamics. In particular, the canonical and exotic choices $\lambda = 1$ and $\lambda = 0$ seem to lead to completely different theories. However, nothing is further from truth. The dynamics of the generalized mimetic theory is the same regardless of the choice of λ. Since

$$e^a = \bar{Z} \bar{e}^a, \tag{2.58}$$

$$\omega_{ab} = \bar{\omega}_{ab} + \lambda \left(\bar{e}_a \xi_b - \xi_a \bar{e}_b \right) , \tag{2.59}$$

we have that the functional variations of the vierbein and the spin connection are given by

$$\delta_{\bar{\omega}} e^d = 0 \,, \tag{2.60}$$

$$\delta_{\bar{e}} e^d = \delta_{\bar{e}} \bar{Z} \bar{e}^d + \bar{Z} \delta \bar{e}^d \,, \tag{2.61}$$

$$\delta_{\phi} e^d = \delta_{\phi} \bar{Z} \bar{e}^d \,, \tag{2.62}$$

and

$$\delta_{\bar{\omega}} \omega_{ab} = \delta \bar{\omega}_{ab} \,, \tag{2.63}$$

$$\delta_{\bar{e}} \omega_{ab} = \lambda \left(\delta \bar{e}_a \xi_b - \xi_a \delta \bar{e}_b \right) + \lambda \left(\bar{e}_a \delta_{\bar{e}} \xi_b - \delta_{\bar{e}} \xi_a \bar{e}_b \right) \,, \tag{2.64}$$

$$\delta_{\phi} \omega_{ab} = \lambda \left(\bar{e}_a \delta_{\phi} \xi_b - \delta_{\phi} \xi_a \bar{e}_b \right) \,. \tag{2.65}$$

Notice that we need special care when performing the functional variation $\delta\bar{Z}^a$. In fact, from definition (2.49), it is clear that $\bar{Z} = \bar{Z}\left(\bar{e}, \partial\phi\right)$. Therefore, we have to consider independent variations of $\bar{Z}^a$ with respect to both, the vierbein $\bar{e}^a(x)$ and the scalar field $\phi(x)$. Since

$$\delta\bar{Z} = -\frac{1}{\bar{Z}}\bar{Z}^a\delta\bar{Z}_a\,, \tag{2.66}$$

it is possible to prove that

$$\delta_{\bar{e}}\bar{Z} = \bar{Z}^2 Z_a Z_b \mathrm{I}^a\left(\delta\bar{e}^b\right)\,, \tag{2.67}$$

$$\delta_\phi\bar{Z} = -\bar{Z}Z^a\mathrm{I}_a\mathrm{d}\delta\phi\,. \tag{2.68}$$

Replacing (2.67)-(2.68) into (2.61)-(2.62) we get the expressions

$$\delta_{\bar{\omega}}e^d = 0\,, \tag{2.69}$$

$$\delta_{\bar{e}}e^d = \bar{Z}\left[Z_a Z_b \mathrm{I}^a\left(\delta\bar{e}^b\right)e^d + \delta\bar{e}^d\right]\,, \tag{2.70}$$

$$\delta_\phi e^d = -e^d Z^a\mathrm{I}_a\mathrm{d}\delta\phi\,. \tag{2.71}$$

Up to boundary terms, variation of (2.20) reads

$$\delta_{\bar{e}}\mathcal{L}^{(4)}_{\mathrm{ECKS}} = \frac{1}{\kappa_4}\left[\frac{1}{2}\delta_{\bar{e}}\omega^{ab}\wedge\mathcal{W}_{ab} + \mathcal{E}_d\wedge\delta_{\bar{e}}e^d\right] = 0\,, \tag{2.72}$$

$$\delta_\phi\mathcal{L}^{(4)}_{\mathrm{ECKS}} = \frac{1}{\kappa_4}\left[\frac{1}{2}\delta_\phi\omega^{ab}\wedge\mathcal{W}_{ab} + \mathcal{E}_d\wedge\delta_\phi e^d\right] = 0\,, \tag{2.73}$$

$$\delta_{\bar{\omega}}\mathcal{L}_{\mathrm{ECKS}} = \frac{1}{2}\frac{1}{\kappa_4}\delta_{\bar{\omega}}\omega^{ab}\wedge\mathcal{W}_{ab} = 0\,, \tag{2.74}$$

where the three-forms $\mathcal{E}_a$ and $\mathcal{W}_{ab}$ are given in (2.23) and (2.24) respectively. Since $\delta_{\bar{\omega}}\omega_{ab} = \delta\bar{\omega}_{ab}$, Eq.(2.74) implies $\delta\bar{\omega}^{ab}\wedge\mathcal{W}_{ab} = 0$ and consequently

$$\mathcal{W}_{ab} = 0\,, \tag{2.75}$$

just as in the standard ECKS model. Inserting $\mathcal{W}_{ab} = 0$ in the equations of motion, we are left with

$$\delta_{\bar{e}}\mathcal{L}_{\mathrm{G}}^{(4)} = \frac{1}{\kappa_4}\mathcal{E}_d \wedge \delta_{\bar{e}} e^d = 0\,, \tag{2.76}$$

$$\delta_{\phi}\mathcal{L}_{\mathrm{G}}^{(4)} = \frac{1}{\kappa_4}\mathcal{E}_d \wedge \delta_{\phi} e^d = 0\,. \tag{2.77}$$

From here, using the expressions (2.70) and (2.71), and integrating by parts in I^a and d, we get the set of mimetic ECKS field equations

$$\mathcal{E}_d - Z_a Z_d \mathrm{I}^a \left(\mathcal{E}_m \wedge e^m\right) = 0\,, \tag{2.78}$$

$$\mathrm{d}\left[Z^a \mathrm{I}_a \left(\mathcal{E}_d \wedge e^d\right)\right] = 0\,, \tag{2.79}$$

$$\mathcal{W}_{ab} = 0\,. \tag{2.80}$$

It is remarkable that they do not depend on the choice of λ. For the mimetic theory, all the choices of conformal transformations for the contorsion lead to the same dynamics.

In order to study the equivalence of these equations written using tensors, it is useful to consider Hodge duality between p-forms and $(d-p)$-forms. For a three-form $\mathcal{E}_d$ in four dimensions, we have

$$\mathcal{E}_d = \mathcal{E}_{md} * e^m. \tag{2.81}$$

It is straightforward to prove that

$$\mathcal{E}_q \wedge e^q = -\mathcal{E}^p{}_p v_{(4)}, \tag{2.82}$$

$$\mathrm{I}^m v_{(4)} = *e^m, \tag{2.83}$$

where $v_{(4)}$ denotes the volume form in four-dimensions. Replacing these relations into the field equations (2.78)-(2.80) it is possible to write them as

$$\mathcal{E}_d - *\left(Z_d \mathcal{E}^p{}_p \mathrm{d}\phi\right) = 0\,, \tag{2.84}$$

$$-\mathrm{d} * \left[\mathcal{E}^p{}_p \mathrm{d}\phi\right] = 0\,, \tag{2.85}$$

$$\mathcal{W}_{ab} = 0\,. \tag{2.86}$$

Remarkably, Eqs. (2.84)-(2.85) have the same form as Eqs. (2.5)-(2.6) but in terms of the full Lorentz curvature (2.19) instead of just the Riemannian piece. Note that Eq. (2.86) is the standard field equation for torsion in terms of the spin tensor of matter.

2.4.2 Conservation laws

From (2.84), it is clear that one can define a mimetic energy-momentum one-form

$$\bar{\mathcal{T}}_d = \frac{1}{\kappa_4} Z_d \mathcal{E}^p{}_p \mathrm{d}\phi\,. \tag{2.87}$$

The conservation law of $\bar{\mathcal{T}}_d$ in terms of the torsionless covariant derivative implies

$$\begin{aligned}
\mathring{\mathrm{D}} * \bar{\mathcal{T}}_d &= \frac{1}{\kappa_4} \mathring{\mathrm{D}}\left(Z_d * \left[\mathcal{E}^p{}_p \mathrm{d}\phi\right]\right) \\
&= \frac{1}{\kappa_4}\left(\mathring{\mathrm{D}} Z_d \wedge * \left[\mathcal{E}^p{}_p \mathrm{d}\phi\right] + Z_d \wedge \mathring{\mathrm{D}} * \left[\mathcal{E}^p{}_p \mathrm{d}\phi\right]\right) \\
&= \frac{1}{\kappa_4}\left(\mathcal{E}^p{}_p \mathring{\mathrm{D}} Z_d \wedge * \mathrm{d}\phi + Z_d \wedge \mathrm{d} * \left[\mathcal{E}^p{}_p \mathrm{d}\phi\right]\right)\,.
\end{aligned} \tag{2.88}$$

Using (2.85), we have

$$\mathring{\mathrm{D}} * \bar{\mathcal{T}}_d = \frac{1}{\kappa_4} \mathcal{E}^p{}_p \mathring{\mathrm{D}} Z_d \wedge * \mathrm{d}\phi\,. \tag{2.89}$$

Notice that

$$\mathring{\mathrm{D}} Z_d \wedge * \mathrm{d}\phi = Z^a \mathring{\mathcal{D}}_a Z_d v_{(4)}\,, \tag{2.90}$$

and therefore, since $\mathring{\mathcal{D}}_a Z_d = \mathring{\mathcal{D}}_d Z_a$, one obtains

$$\mathring{\mathrm{D}} Z_d \wedge *\mathrm{d}\phi = Z^a \mathring{\mathrm{D}}_d Z_a v_{(4)} = \frac{1}{2} \mathrm{I}_d \mathrm{d}(Z^a Z_a) v_{(4)} = 0\,, \tag{2.91}$$

where we have used (2.57). Consequently,

$$\mathring{\mathrm{D}} * \bar{\mathcal{T}}_d = 0\,, \tag{2.92}$$

which is the conservation law for the effective mimetic stress-energy tensor.

2.5 The trace of the stress-energy tensor, torsion and λ

For the mimetic theory dynamics, the choice of the parameter λ for the conformal transformations (2.51)-(2.53) seems to be irrelevant. However, it does not mean that the parameter is meaningless. In fact, it is related with the value of the trace of stress-energy tensor of matter when its Lagrangian has conformal symmetry.

Let us consider a matter Lagrangian $\mathcal{L}_{\mathrm{M}}$ obeying conformal symmetry by itself

$$\mathcal{L}_{\mathrm{M}}(e, \omega, \phi) = \mathcal{L}_{\mathrm{M}}(\Omega e^a,\, \omega^{ab} + \frac{\lambda}{\Omega} \left[e^a, \mathrm{I}^b\right] \mathrm{d}\Omega,\, \frac{1}{\Omega^\alpha} \phi)\,. \tag{2.93}$$

where $\left[e^a, \mathrm{I}^b\right] = e^a \mathrm{I}^b - e^b \mathrm{I}^a$. In the standard torsionless case, it would lead to a traceless on-shell stress-energy tensor. This is no longer true in the current context of non-vanishing torsion. In fact, under an infinitesimal dilatation $\Omega = 1 + \varepsilon$, the field content of the matter Lagrangian changes according to

$$\delta_\varepsilon e^a = \varepsilon\, e^a\,, \tag{2.94}$$

$$\delta_\varepsilon \omega^{ab} = \lambda \left[e^a, \mathrm{I}^a\right] \mathrm{d}\varepsilon\,, \tag{2.95}$$

$$\delta_\varepsilon \phi = -\alpha \varepsilon \phi\,. \tag{2.96}$$

Moreover, an arbitrary variation of $\mathcal{L}_{\mathrm{M}}(e,\omega,\phi)$ is given by

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{M}} = & - *\mathcal{T}_d \wedge \delta e^d + \frac{1}{2} * \sigma_{ab} \wedge \delta\omega^{ab} + \Phi\delta\phi \\ & + \mathrm{d}\left(\mathcal{B}^{(2)}_a \wedge \delta e^a + \frac{1}{2}\mathcal{B}^{(2)}_{ab} \wedge \delta\omega^{ab} + \mathcal{B}^{(3)}\delta\phi\right),\end{aligned} \tag{2.97}$$

where Φ denotes the field equation for ϕ and $\mathcal{B}_a$, $\mathcal{B}_{ab}$, and $\mathcal{B}$ are boundary terms. Therefore, demanding invariance of $\mathcal{L}_{\mathrm{M}}$ under the infinitesimal conformal transformations (2.94)-(2.96) we obtain

$$\begin{aligned}& \varepsilon\, e^d \wedge *\mathcal{T}_d + \lambda * \sigma_{ab} \wedge e^a \mathrm{I}^b \mathrm{d}\varepsilon - \alpha\varepsilon\Phi\phi \\ & + \mathrm{d}\left(\mathcal{B}^{(2)}_a \wedge \varepsilon e^a + \lambda\mathcal{B}^{(2)}_{ab} \wedge e^a \mathrm{I}^b \mathrm{d}\varepsilon - \alpha\varepsilon\mathcal{B}^{(3)}\phi\right) = 0\,.\end{aligned} \tag{2.98}$$

It is straightforward to prove the identity

$$*\sigma_{ab} \wedge e^a \mathrm{I}^b \mathrm{d}\varepsilon = -\mathrm{d}\left[\varepsilon \mathrm{I}_a \sigma^a{}_b * e^b\right] + \varepsilon \mathrm{d}\left(\mathrm{I}_a \sigma^a{}_b * e^b\right), \tag{2.99}$$

and therefore to conclude that

$$\int_{M_4} \mathcal{H}^{(4)} + \int_{\partial M_4} \mathcal{U}^{(3)} = 0\,. \tag{2.100}$$

where

$$\mathcal{H}^{(4)} = \varepsilon\left[e^d \wedge *\mathcal{T}_d + \lambda \mathrm{d}\left(\mathrm{I}_a \sigma^a{}_b * e^b\right) - \alpha\Phi\phi\right] \tag{2.101}$$

and

$$\mathcal{U}^{(3)} = \mathcal{B}^{(2)}_a \wedge \varepsilon e^a + \lambda\left[\mathcal{B}^{(2)}_{ab} \wedge e^a \mathrm{I}^b \mathrm{d}\varepsilon - \varepsilon \mathrm{I}_a \sigma^a{}_b * e^b\right] - \alpha\varepsilon\mathcal{B}^{(3)}\phi \tag{2.102}$$

Since ε is arbitrary, the integrals over the bulk and over the boundary must vanish independently. Even more, on the bulk we must have

$$e^d \wedge *\mathcal{T}_d + \lambda \mathrm{d}\left(\mathrm{I}_a \sigma^a{}_b * e^b\right) - \alpha\Phi\phi = 0\,. \tag{2.103}$$

Since the trace of the stress-energy tensor corresponds to $\mathcal{T} = -*\left(e^d \wedge *\mathcal{T}_d\right)$, we have that

$$\mathcal{T} - \lambda * \mathrm{d}\left(\mathrm{I}_a \sigma^a{}_b * e^b\right) + \alpha * \Phi\phi = 0\,. \tag{2.104}$$

In a theory as ECKS or its current mimetic version, we have the on-shell relationships

$$\epsilon_{abcd} T^c \wedge e^d = \kappa_4 * \sigma_{ab}\,, \tag{2.105}$$

$$\Phi = 0\,, \tag{2.106}$$

and replacing them in Eq. (2.104), it lead us to the trace value

$$\mathcal{T} = \frac{2\lambda}{\kappa_4} \mathrm{d}^\dagger \mathrm{I}_a T^a\,, \tag{2.107}$$

with the four-dimensional coderivative operator $\mathrm{d}^\dagger$ given by $\mathrm{d}^\dagger = *\mathrm{d}*$. This way, we can see that the stress-energy tensor for a dilatation invariant Lagrangian is not always traceless. It is traceless only when the torsion vanishes or when the dilatation invariance is associated to the case $\lambda = 0$, i.e. when the spin connection remains untouched by the transformation.

Chapter 3

Einstein-Cartan Theory and Scalar Fields Couplet to Topological Invariants

Geometry can be constructed using two fundamental elements: the compass and the straightedge. The former allows for the comparison of lengths and the drawing of circles, while the latter is used to draw straight lines, which is a basic affine operation. Parallelism can be reduced to metricity, as we can construct a parallel line to a given line using a compass and an unmarked straightedge. In other words, given a way to measure distances and straight lines in space, one can define parallel transport. This is the case in Riemannian geometry, where all geometric objects defined on the manifold can be constructed from the metric. However, it is convenient to distinguish between the metric and affine properties of spacetime. On one hand, the metric refers to measurements of lengths, angles, areas, or volumes of objects that are defined locally in spacetime, while affinity refers to properties that remain invariant under affine transformations (e.g., translations), such as parallelism. A completely affine construction makes it possible to define the notion of parallel transport along a curve without the need for a metric structure, but in this case, straight lines would not necessarily be the shortest. Here, the connection can be conceived as a geometric object not necessarily related to a metric. This idea of considering the metric and the connection as independent geometric fields was developed by Elie Cartan. Whereas in Riemannian geometry, the affine connection must be a function of the metric given by the Christoffel symbols.

In 1915, Einstein presented the final version of his theory of General Relativity within the mathematical framework of Riemannian theory. The first solution to these equations describing the exterior of an isolated spherically symmetric object is the so-called Schwarzschild solution. This solution was found in 1916, just a couple of months after the derivation of Einstein's equations. In the case of a rotating object, the problem is more difficult. But now we know the exact solution that describes a rotating, stationary black hole with axial symmetry, which is the Kerr solution, derived in 1963 by Roy Kerr.

The equations of motion for the Einstein-Hilbert Lagrangian plus the coupling of a scalar field to different topological invariants in the first-order formalism will be presented next, and their field equations will be studied.

3.1 Euler topological invariant

Let us consider the Lagrangian, where ϕ is a scalar field coupled to the Euler topological invariant and a_1, a_2, and a_3 are constants

$$\begin{aligned}\mathcal{L} &= a_1\epsilon_{abcd}R^{ab}\wedge e^c\wedge e^d + a_2\epsilon_{abcd}e^a\wedge e^b\wedge e^c\wedge e^d \\ &\quad + a_3\phi\epsilon_{abcd}R^{ab}\wedge R^{cd}. \end{aligned} \tag{3.1}$$

3.1.1 Field Equations

Its field equations are given by

$$\delta e \ : \epsilon_{abcd}\left(a_1\mathring{R}^{ab}+2a_2e^a\wedge e^b\right)\wedge e^c=-\epsilon_{abcd}a_1\left(\mathring{D}\kappa^{ab}+\kappa^a{}_f\wedge\kappa^{fb}\right)\wedge e^c, \tag{3.2}$$

$$\delta\omega \ : -a_1\epsilon_{abcd}T^c\wedge e^d+a_3\epsilon_{abcd}R^{cd}d\phi=0, \tag{3.3}$$

$$\delta\phi \ : \epsilon_{abcd}R^{ab}\wedge R^{cd}=0. \tag{3.4}$$

3.2 Pontryagin topological invariant

Similarly, we now couple the scalar field ϕ to the Pontryagin topological invariant, namely,

$$\begin{aligned}\mathcal{L}&=a_1\epsilon_{abcd}R^{ab}\wedge e^c\wedge e^d+a_2\epsilon_{abcd}e^a\wedge e^b\wedge e^c\wedge e^d\\&\quad+a_3\phi R^a_{\ b}\wedge R^b_{\ a}.\end{aligned} \tag{3.5}$$

3.2.1 Field Equations

In an analogous manner,

$$\delta e \ : \epsilon_{abcd}\left(a_1\mathring{R}^{ab}+2a_2e^a\wedge e^b\right)\wedge e^c=-\epsilon_{abcd}a_1\left(\mathring{D}\kappa^{ab}+\kappa^a{}_f\wedge\kappa^{fb}\right)\wedge e^c, \tag{3.6}$$

$$\delta\omega \ : -a_1\epsilon_{abcd}T^c\wedge e^d+a_3d\phi R_{ab}=0, \tag{3.7}$$

$$\delta\phi \ : R^a_{\ b}\wedge R^b_{\ a}=0. \tag{3.8}$$

From equations (3.3) and (3.7), we note that in both cases, if the scalar field ϕ is constant, then the torsion will automatically vanish.

3.3 Symmetries

We seek symmetric solutions, that is, solutions in which the geometric properties of spacetime remain invariant under certain transformations. Formally, this translates into requiring that the Lie derivative of all independent fields of the theory, along the generating vectors of the symmetry, vanish. From a mathematical standpoint, the use of symmetries makes the Einstein field equations more manageable. However, since we are working with non-zero torsion, it is no longer sufficient to impose this condition solely on the metric, as there are more independent fields in the theory.

Therefore, the conditions to be satisfied will be:

$$\mathcal{L}_\xi g_{\mu\nu} = \mathring{\nabla}_\mu \xi_\nu + \mathring{\nabla}_\nu \xi_\mu = 0, \tag{3.9}$$

$$\mathcal{L}_\xi T^\alpha{}_{\mu\nu} = \xi^\lambda \partial_\lambda T^\alpha{}_{\mu\nu} - T^\lambda{}_{\mu\nu} \partial_\lambda \xi^\alpha + T^\alpha{}_{\lambda\nu} \partial_\mu \xi^\lambda + T^\alpha{}_{\mu\lambda} \partial_\nu \xi^\lambda = 0, \tag{3.10}$$

$$\mathcal{L}_\xi \phi = \xi^\lambda \partial_\lambda \phi = 0. \tag{3.11}$$

Due to the increased number of independent fields in the theory, even under the imposition of an isotropic and homogeneous spacetime, we are able to obtain the metric

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right). \tag{3.12}$$

the torsion can have a non-trivial form, and we will have to contend with 8 additional functions [145], namely,

$$
\begin{aligned}
T^0 &= G_1(r)e^0\wedge e^1 + G_2(r)e^2\wedge e^3,\\
T^1 &= G_3(r)e^0\wedge e^1 + G_4(r)e^2\wedge e^3,\\
T^2 &= G_5(r)e^0\wedge e^2 + G_6(r)e^1\wedge e^2 + G_7(r)e^0\wedge e^3 + G_8(r)e^1\wedge e^3,\\
T^3 &= G_5e^0\wedge e^3 + G_6(r)e^1\wedge e^3 - G_7(r)e^0\wedge e^2 - G_8(r)e^1\wedge e^2.
\end{aligned}
\tag{3.13}
$$

This is quite good, as we have 8 additional functions. However, we know that this can still make the field equations overly complex. So the question is: can we do better?

3.4 Nieh-Yan topological invariant

Finally, let us consider the Lagrangian

$$
\begin{aligned}
\mathcal{L} = a_1\epsilon_{abcd}R^{ab}\wedge e^c\wedge e^d &+ a_2\epsilon_{abcd}e^a\wedge e^b\wedge e^c\wedge e^d\\
&+ a_3\phi\left[T^a\wedge T_a - R^{ab}\wedge e_a\wedge e_b\right].
\end{aligned}
\tag{3.14}
$$

3.4.1 Field equations

$$
\delta e\ : \epsilon_{abcd}\left(a_1\mathring{R}^{ab} + 2a_2e^a\wedge e^b\right)\wedge e^c = -\epsilon_{abcd}a_1\left(\mathring{D}\kappa^{ab} + \kappa^a{}_f\wedge\kappa^{fb}\right)\wedge e^c - a_3d\phi\wedge T_d,
\tag{3.15}
$$

$$
\delta\omega\ : 2a_1\epsilon_{abcd}T^c\wedge e^d + a_3d\phi\wedge e_a\wedge e_b = 0,
\tag{3.16}
$$

$$
\delta\phi\ : T^a\wedge T_a - R^{ab}\wedge e_a\wedge e_b = 0.
\tag{3.17}
$$

3.4.2 Analysis of the Field Equations

In this case, equation (3.16) is a rather strong constraint, as it indicates that only certain components of the torsion contribute to the field equations. In the language of exterior forms, if $\phi = \phi(r)$, which means that $\phi =$ constant defines a spacelike surface, then $d\phi = \xi(r)e^1$ and

$$T^1 = 0 \, , \tag{3.18}$$

$$T^0 = \alpha \, \xi(r) \, e^2 \wedge e^3 \, , \tag{3.19}$$

$$T^2 = \alpha \, \xi(r) \, e^0 \wedge e^3 \, , \tag{3.20}$$

$$T^3 = \alpha \, \xi(r) \, e^2 \wedge e^0 \, . \tag{3.21}$$

where α is a constant. This result can be verified in two ways: by inserting this ansatz for the scalar field directly into equations (3.16) and solving in the language of forms (in this case, we only require the scalar field to have radial dependence, which is quite powerful as it also allows us to study the case with axial symmetry and only one component of torsion turned on). If one wishes to carry out the procedure seen in [145] to compare results, one can review Appendix C, where the Killing vectors for an isotropic and stationary spacetime are applied, and the constraint of equation (3.16) is imposed in a coordinate basis. However, recall that it is not necessary to impose all of these Killing vectors to obtain (3.19), (3.18), (3.20), (3.21).

For simplicity, and to facilitate a subsequent analogy with the Schwarzschild equations, we will focus our study on the case of a spherically symmetric and stationary metric, namely,

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right) . \tag{3.22}$$

3.5 Final Equations

Appendix D contains an outline of the derivation leading to the final equations employed here considering the spherically symmetric and stationary metric (3.22)

$$-4a_1\left[h'(r)r + h(r)^2 - h(r)\right] - 24a_2 h(r)^2 r^2 = \frac{5a_3^2}{4a_1} h(r)\phi'(r)^2 r^2. \tag{3.23}$$

$$-4a_1\left[f'(r)r - f(r)h(r) + f(r)\right] + 24a_2 f(r)h(r)r^2 = -\frac{3a_3^2}{4a_1} f(r)\phi'(r)^2 r^2 \tag{3.24}$$

$$\begin{aligned} a_1\sqrt{h(r)}&\left[2f''(r)h(r)f(r)r - f'(r)^2 h(r)r - f'(r)h'(r)f(r)r + 2f'(r)f(r)h(r) - 2h'(r)f(r)^2\right] \\ &-24a_2\sqrt{h(r)}f(r)^2 h(r)^2 r = \frac{5a_3^2}{4a_1}\phi'(r)^2 f(r)^2 h(r)^{3/2} r. \end{aligned} \tag{3.25}$$

where Eq. (3.25) is a linear combination of the other two equations.

Additionally, from the equation for the scalar field, we have that

$$\phi(r) = C_1 \int^r \frac{\sqrt{h(r')}}{r'^2\sqrt{f(r')}}\,dr' + C_2 \tag{3.26}$$

Dividing (3.23) by $h(r)/4$, we obtain

$$-a_1\left[\frac{h'(r)r}{h(r)} + h(r) - 1\right] - 6a_2 h(r)r^2 = \frac{5a_3^2}{16a_1}\phi'(r)^2 r^2. \tag{3.27}$$

Similarly, dividing (3.24) by $f(r)/4$, we have

$$-a_1\left[\frac{f'(r)r}{f(r)} - h(r) + 1\right] + 6a_2 h(r)r^2 = -\frac{3a_3^2}{16a_1}\phi'(r)^2 r^2 \tag{3.28}$$

Adding (3.27)+$\frac{5}{3}$(3.28), we finally arrive at

$$a_1\left[\frac{h'(r)}{h(r)}+\frac{5}{3}\frac{f'(r)}{f(r)}\right]r-2h(r)\left[2a_2r^2+\frac{a_1}{3}\right]+\frac{2}{3}a_1=0. \tag{3.29}$$

Whereas, if we add equations (3.27) and (3.28), we obtain

$$-a_1\left[\frac{f'(r)}{f(r)}+\frac{h'(r)}{h(r)}\right]=\frac{a_3^2}{8a_1}\phi'(r)^2r^2 \tag{3.30}$$

$$\frac{d\log\left(f(r)h(r)\right)}{dr}=-\frac{a_3^2}{8a_1^2} \tag{3.31}$$

3.5.1 Analogy with the Schwarzschild Solution

We can write eq. (3.30) in the following way

$$f'(r)h(r)+h'(r)f(r)=-\frac{a_3^2}{8a_1^2}\phi'(r)^2r^2f(r)h(r) \tag{3.32}$$

Also we had from (3.26) → $\phi'(r)^2=C_1\frac{h(r)}{r^4f(r)}$. Then we get

$$\partial_r\left[f(r)h(r)\right]=-\frac{a_3^2C_1}{8a_1^2r^2}h(r)^2 \tag{3.33}$$

defining $\beta:=\frac{a_3^2C_1}{8a_1^2}$, then

$$\partial_r\left[f(r)h(r)\right]=-\beta\left(\frac{h(r)}{r}\right)^2 \tag{3.34}$$

Additionally, if we subtract equation (3.27) from (3.28), we obtain

$$\left[\frac{f'(r)r}{f(r)} - \frac{h'(r)}{h(r)}r - 2h(r) + 2\right] - 12a_2 h(r) r^2 = 4\beta \left(\frac{h(r)}{r}\right)^2 \tag{3.35}$$

we can see that if the constant a_3 is zero, that is, no coupling of the scalar field with the Nieh-Yan topological invariant, then we recover the Schwarzschild solution E. The implications of this analysis extend to the study of black hole solutions and cosmological scenarios within this framework, which will be discussed in the conclusions.

Conclusions

In summary, we have developed the closest version of mimetic gravity in first-order formalism, i.e., the mimetic version of ECKS gravity theory. Dynamics in this theory is described by the following equations of motion

$$\mathcal{E}_d - \kappa_4 * \bar{\mathcal{T}}_d = 0\,, \tag{3.36}$$

$$\mathrm{d} * [\mathcal{E}^p{}_p \mathrm{d}\phi] = 0\,, \tag{3.37}$$

$$\mathcal{W}_{ab} = 0\,, \tag{3.38}$$

where $\mathcal{E}_a$, $\mathcal{W}_{ab}$, and $\bar{\mathcal{T}}_d$ are given in eqs. (2.23), (2.24), and (2.87) respectively. By construction, the system obeys the following conditions:

$$\mathring{\mathrm{D}} * \bar{\mathcal{T}}_d = 0\,, \tag{3.39}$$

$$Z^2 = 1\,. \tag{3.40}$$

These equations reduce to the standard mimetic gravity equation when torsion T^a vanishes.

We have considered different possibilities of how torsion is affected by conformal transformation (2.1), all of them mapped by the parameter λ. The torsionless part of the spin connection $\mathring{\omega}^{ab}$ has a definite conformal transformation (2.35), obtained from purely metric properties. However, we have the freedom to choose the contorsion κ^{ab} transformation. The possibilities (2.39)-(2.41) range from a non-changing contorsion, to a contorsion that changes in such a way that it leaves the full spin connection invariant. Regardless of the type of transformation under consideration, dynamics enforce torsion to remain a non-propagating field, as in standard ECKS.

This model may have non-trivial consequences in cosmology [1]. Let us consider an explicit solution of equation (3.37). For the sake of simplicity, during this section, we take $c = 1$ and $\Lambda = 0$. Additionally, it is convenient to work in synchronous coordinates where the metric adopts the form

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \,, \tag{3.41}$$

with γ_{ij} the spatial section of the metric $g_{\mu\nu}$ (See [147, Chap.11]). As discussed in [41], we take the scalar field to be the same as the hypersurfaces of constant time, namely

$$\phi\left(x^\mu\right) = \tau \,, \tag{3.42}$$

which naturally satisfies (3.40). In this coordinates, Eq.(3.37) reads

$$\partial_0\left(\sqrt{\gamma}\left(G - \mathcal{T}\right)\right) = 0 \,, \tag{3.43}$$

and consequently

$$G - \mathcal{T} = \frac{\mathcal{C}\left(x^i\right)}{\sqrt{\gamma}} \,. \tag{3.44}$$

Here $\mathcal{C}$ is an integration τ-constant, depending only on the spatial coordinates x^i. For a flat Friedmann Universe, the metric γ_{ij} corresponds to

$$\gamma_{ij} = a^2\left(\tau\right)\delta_{ij} \,, \tag{3.45}$$

and therefore (3.43) leads to

$$G - \mathcal{T} = \frac{\mathcal{C}\left(x^i\right)}{a^3\left(\tau\right)} \,. \tag{3.46}$$

Therefore, the scalar field coming from the conformal degree of freedom mimics a dark

---

[1] By the way, almost simultaneous with this article, another interesting cosmological application of mimetic theories of gravity involving non-vanishing torsion has been presented in [146]

matter source. However, torsion is also present, and it behaves as an additional dark matter source in $G = -R$. Let us split the Ricci scalar as

$$R = \mathring{R} + R\left(\kappa\right) \,, \tag{3.47}$$

$$R\left(\kappa\right) = 2\mathring{\nabla}_{\mu}\kappa^{\mu\nu}{}_{\nu} + \kappa^{\mu\gamma}{}_{\mu}\kappa_{\gamma\nu}{}^{\nu} - \kappa^{\mu}{}_{\gamma\nu}\kappa^{\gamma\nu}{}_{\mu} \,, \tag{3.48}$$

where $\kappa^{\mu\nu}{}_{\lambda}$ is the contortion, $\mathring{R}$ is the torsionless Ricci scalar and $\mathring{\nabla}$ is the covariant derivative associated with the Christoffel symbol. Thus, using $\mathring{G} = -\mathring{R}$, we get

$$G = \mathring{G} - R\left(\kappa\right) \,. \tag{3.49}$$

In order to evaluate Eq.(3.48) explicitly, let us consider a spin tensor distribution σ_{ab} which may be relevant at cosmological scales. Such spin tensor distribution has been considered, for instance, in [40] where the Ansatz for the torsion tensor reads

$$T_{\lambda\mu\nu} = \left[X(\tau)(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\mu}g_{\nu\rho}) - 2\sqrt{g}Y(\tau)\epsilon_{\lambda\mu\nu\rho}\right]u^{\rho} \,, \tag{3.50}$$

where u^{ρ} is the co-moving four-velocity which in synchronous coordinates, and X and Y are arbitrary functions of time. These configurations can arise from a dark matter candidates, such as dark/Elko spinors, see for instance [148–154]. The non-vanishing components of (3.50) are

$$T_{ij0} = X\gamma_{ij}u^{0} = a^{2}X\delta_{ij} \,, \tag{3.51}$$

$$T_{kij} = -2\sqrt{\gamma}\,Y\epsilon_{kij}u^{0} = -2a^{3}Y\epsilon_{kij} \,. \tag{3.52}$$

Since

$$\kappa_{\mu\nu\lambda} = \frac{1}{2}\left(T_{\nu\mu\lambda} - T_{\mu\nu\lambda} + T_{\lambda\mu\nu}\right) \,, \tag{3.53}$$

we can evaluate Eq.(3.48) and Eqs.(3.46)-(3.49) to arrive to

$$\mathring{G} - \mathcal{T} = \frac{\mathcal{C}}{a^3} + 3! \left[X \left(\mathcal{S} + 3\mathcal{H} \right) + X^2 + Y^2 \right] , \tag{3.54}$$

where $\mathcal{C} = \mathcal{C}\left(x^i\right)$, $\mathcal{S}\left(\tau\right) = \frac{\dot{X}}{X}$ and $\mathcal{H}\left(\tau\right) = \frac{\dot{a}}{a}$. From here the role of the spin-tensor as dark matter becomes evident.

In the mimetic ECKS theory, there are three dark matter species. The first is the dark matter by itself, with a non-vanishing spin tensor. The second arises from isolating the conformal mode in a covariant way, and the last comes from the torsional degrees of freedom.

On the other hand, the finding that coupling a scalar field to the Nieh-Yan topological invariant results in only one non-vanishing torsion component is an intriguing result with potential advantages for studying the consequences of torsion in 4 dimensions. The resulting field equations are considerably simpler than those obtained when coupling to the Euler or Pontryagin topological invariants, and they even allow for the study of field equations under other kinds of symmetries, for example, axial symmetry. Finding cosmological solutions within this theoretical framework would also be of great interest. This will be part of a future research project which we aim to pursue. However, despite the simplicity of the torsion in this case, the field equations do not yield an analytical solution. Therefore, our next step will involve analyzing the perturbation theory to investigate the existence of black hole solutions and employing numerical methods to solve the equations.

Appendix A

Magnetic Monopole

Let's then study a regularized way of the potential

$$\vec{A}_R = \frac{x\hat{y} - y\hat{x}}{R(R-z)} \tag{A.1}$$

where $R = \left(r^2 + \varepsilon^2\right)^{1/2} = \left(x^2 + y^2 + z^2 + \varepsilon^2\right)^{1/2}$.

We see that the magnetic field will have components

$$\vec{B}_R = (-\partial_z A_{R,y}, \partial_z A_{R,x}, \partial_x A_{R,y} - \partial_y A_{R,x})$$

Since $A_{R,y} = \frac{-gx}{R^2 - Rz}$ y $A_{R,x} = \frac{gy}{R^2 - Rz}$, it is trivial to see that

$$B_{R,x} = g\frac{x}{R^3}$$
$$B_{R,y} = g\frac{y}{R^3}$$

Let's calculate the B_z component of the magnetic field,

$$\begin{aligned}
B_{R,z} &= \frac{g}{\left[R^2 - Rz\right]^3}\left\{-2\left[R^2 - Rz\right] + 2\left(R^2 - z^2 - \varepsilon^2\right)\left[R^2 - Rz\right] - \frac{z}{R}\left(R^2 - z^2 - z\varepsilon\right)\left[R^2 - Rz\right]\right\} \\
&= \frac{g\left[R^2 - Rz\right]}{\left[R^2 - Rz\right]^3}\left\{-2\left[R^2 - Rz\right] + 2\left(R^2 - z^2 - \varepsilon^2\right) - \frac{z}{R}\left(R^2 - z^2 - \varepsilon^2\right)\right\} \\
&= \frac{g\left[R^2 - Rz\right]}{\left[R^2 - Rz\right]^3}\left\{-2R^2 + 2Rz + 2R^2 - 2z^2 - 2\varepsilon^2 - zR + \frac{z^3}{R} + \frac{z\varepsilon^2}{R}\right\} \\
&= \frac{g\left[R^2 - Rz\right]}{\left[R^2 - Rz\right]^3}\left\{Rz - 2z^2 - 2\varepsilon^2 + \frac{z^3}{R} + \frac{\varepsilon^2}{R}\right\} \\
&= \frac{g\left[R^2 - Rz\right]}{\left[R^2 - Rz\right]^3}\left\{\frac{z}{R^3}\left[R^4 - 2zR^3 + z^2R^2\right] - \varepsilon^2\left[2 - \frac{z}{R}\right]\right\}.
\end{aligned}$$

From here, we can see that we can separate the z component of the field into three terms: the first will be analogous to B_x and B_y, while the second and third will be proportional to ε^2:

$$\begin{aligned}
B_{R,z} &= \frac{gz}{R^3} - \frac{g}{\left[R^2 - Rz\right]^3}\varepsilon^2\left[2 - \frac{z}{R}\right]\left[R^2 - Rz\right] \\
&= \frac{gz}{R^3} - \frac{g}{R^3[R - z]^3}\varepsilon^2\left[2R^2 - 2Rz - zR + z^2\right] \\
&= \frac{gz}{R^3} - \frac{g}{R[R - z]^3}\varepsilon^2\left[2 - 3\frac{z}{R} + \frac{z^2}{R^2}\right] \\
&= \frac{gz}{R^3} - \frac{g}{R[R - z]^3}\varepsilon^2\left[2 - 3\frac{z}{R} + \frac{z^2}{R^2}\right] \\
&= \frac{gz}{R^3} - \frac{g}{R[R - z]^3}\varepsilon^2\left[\left(1 - 2\frac{z}{R} + \frac{z^2}{R^2}\right) + \left(1 - \frac{z}{R}\right)\right] \\
&= \frac{gz}{R^3} - \frac{g}{R[R - z]^3}\varepsilon^2\left[\frac{1}{R^2}\left(R^2 - 2zR + z^2\right) + \frac{1}{R}(R - z)\right] \\
&= \frac{gz}{R^3} - g\varepsilon^2\left[\frac{1}{R^3[R - z]} + \frac{1}{R^2[R - z]^2}\right]
\end{aligned}$$

Therefore, the regularized magnetic field is

$$\vec{B}_R(\vec{r}, \varepsilon) = g\frac{\vec{r}}{R^3} - g\varepsilon^2\left[\frac{\hat{z}}{R^3[R - z]} + \frac{\hat{z}}{R^2[R - z]^2}\right]. \tag{A.2}$$

Now that we have the regularized magnetic field, let's see what happens in the limit $\varepsilon \to 0$ and integrate to obtain the flux. Note that we expect this flux to be zero due to Gauss's theorem, and we should also note that this magnetic field $\vec{B}_R$ is not equal to the field $\vec{B}_g$, whose flux is given by (1.6).

Observe that the second and third terms in (A.2) are proportional to

$$A = \frac{\varepsilon^2}{R - z},$$

with

$$\begin{aligned} R &= \sqrt{r^2 + \varepsilon^2} \\ &= r\sqrt{1 + \frac{\varepsilon^2}{r^2}} \\ &= r\left(1 + \frac{1}{2}\frac{\varepsilon^2}{r^2} + \mathcal{O}\left(\varepsilon^4\right)\right) \end{aligned}$$

Moreover,

$$\begin{aligned} R^2 &= x^2 + y^2 + z^2 + \varepsilon^2 \\ \Rightarrow x^2 + y^2 + \varepsilon^2 &= R^2 - z^2 \\ &= (R + z)(R - z). \end{aligned}$$

Then,

$$\begin{aligned} R - z &= \frac{x^2 + y^2 + \varepsilon^2}{R + z}, \\ \Rightarrow \frac{1}{R - z} &= \frac{R + z}{x^2 + y^2 + \varepsilon^2} \\ &= \frac{r + z + \frac{\varepsilon^2}{2r}}{x^2 + y^2 + \varepsilon^2}. \end{aligned}$$

Therefore

$$\lim_{\varepsilon \to 0} A = \lim_{\varepsilon \to 0} \frac{\varepsilon^2}{x^2 + y^2 + \varepsilon^2}(r + z).$$

Casos:

1. For points outside the z axis: $x^2+y^2 \neq 0$, then $\lim\limits_{\varepsilon\to 0} A = 0$.
2. For points on the negative z axis: $x^2+y^2 = 0$ and $z=-r$, then $\lim\limits_{\varepsilon\to 0} A = 0$.
3. For points on the positive z axis: $x^2+y^2=0$ and $r=z$, we have that this will be the only case in which the limit will have a non-zero value

$$\lim_{\varepsilon\to 0} A = \lim_{\varepsilon\to 0} \frac{2r\varepsilon^2}{x^2+y^2+\varepsilon^2}. \tag{A.3}$$

In such a way that we can summarize the three previous cases by incorporating the Heaviside step function $\Theta(z)$, which finally leads me to

$$\lim_{\varepsilon\to 0} A = \lim_{\varepsilon\to 0} \frac{2r\varepsilon^2}{x^2+y^2+\varepsilon^2}\Theta(z) \tag{A.4}$$

Recall that the original terms are

$$\frac{\varepsilon^2}{R^3(R-z)} + \frac{\varepsilon^2}{R^2(R-z)^2}.$$

Therefore, in the limit $\varepsilon \to 0$, we see that the regularized magnetic field $\vec{B}_R$ defined in (A.2) approaches

$$\begin{aligned}\vec{B}_R(\vec{r},\varepsilon) &\overset{\varepsilon\to 0}{\sim} g\left[\frac{\vec{r}}{r^3} - \lim_{\varepsilon\to 0}\frac{2\varepsilon^2\Theta(z)\hat{z}}{r^2\left(x^2+y^2+\varepsilon^2\right)} - \lim_{\varepsilon\to 0}\frac{4\varepsilon^2\Theta(z)\hat{z}}{\left(x^2+y^2+\varepsilon^2\right)^2}\right] \\ &\overset{\varepsilon\to 0}{\sim} \vec{B}_g(\vec{r}) + \vec{B}_{\text{sing}}\left(\vec{r},\varepsilon\right),\end{aligned}$$

where

$$\vec{B}_g(\vec{r}) = g\frac{\vec{r}}{r^3} = g\frac{\hat{r}}{r^2} \tag{A.5}$$

and

$$\vec{B}_{\text{sing}}\left(\vec{r},\varepsilon\right) = -g\lim_{\varepsilon\to 0}\left[\frac{2\varepsilon^2\Theta(z)\hat{z}}{r^2\left(x^2+y^2+\varepsilon^2\right)} + \frac{4\varepsilon^2\Theta(z)\hat{z}}{\left(x^2+y^2+\varepsilon^2\right)^2}\right]. \tag{A.6}$$

Here, we have called the singular magnetic field the field given in (A.6), which vanishes throughout the space $\mathbb{R}^3$, except on the positive z axis, as long as $\varepsilon \to 0$. To calculate the magnetic flux Φ, we integrate (A.6) over a surface element orthogonal to the z axis, and then take the limit $\varepsilon \to 0$. That is (we must integrate first since if we take the limit directly the second term is divergent):

$$\begin{aligned}
\Phi &= \iint_S \vec{B}_g(\vec{r}) \cdot d\vec{S} + \iint_S \vec{B}_{\text{sing}}(\vec{r},\varepsilon) \cdot d\vec{S} \\
&= \iint_S \vec{B}_g(\vec{r}) \cdot d\vec{S} + \iint_S \vec{B}_{1,\text{sing}}(\vec{r},\varepsilon) \cdot d\vec{S} + \iint_S \vec{B}_{2,\text{sing}}(\vec{r},\varepsilon) \cdot d\vec{S} \\
&= \Phi_g + \Phi_1 + \Phi_2 \qquad \text{(A.7)}
\end{aligned}$$

where

$$\vec{B}_{1,\text{sing}}(\vec{r},\varepsilon) := \lim_{\varepsilon\to 0} -g\frac{2\varepsilon^2\Theta(z)\hat{z}}{r^2(x^2+y^2+\varepsilon^2)}, \quad \vec{B}_{2,\text{sing}}(\vec{r},\varepsilon) := \lim_{\varepsilon\to 0} -g\frac{4\varepsilon^2\Theta(z)\hat{z}}{(x^2+y^2+\varepsilon^2)^2}. \qquad \text{(A.8)}$$

So,

$$\Phi_g := \iint_S \vec{B}_g(\vec{r}) \cdot d\vec{S}, \quad \Phi_1 := \lim_{\varepsilon\to 0}\iint_S \vec{B}_{1,\text{sing}}(\vec{r},\varepsilon) \cdot d\vec{S}, \quad \Phi_2 := \lim_{\varepsilon\to 0}\iint_S \vec{B}_{2,\text{sing}}(\vec{r},\varepsilon) \cdot d\vec{S} \qquad \text{(A.9)}$$

From (A.8) and (A.9), we have that

$$\Phi_1 = \lim_{\varepsilon\to 0}\iint_S \frac{2\varepsilon^2(-g)\Theta(z)\hat{z}}{r^2\,(x^2+y^2+\varepsilon^2)} \cdot d\vec{S}. \qquad \text{(A.10)}$$

As we mentioned earlier, the singular term $\vec{B}_{\text{sing}}$ differs from zero only on the positive z semi-axis when $\varepsilon \to 0$. We must calculate the flux $\Phi_{\text{sing}} = \Phi_1 + \Phi_2$ over a closed surface that includes the singularity. The shape of the surface is irrelevant, since the flux over any closed surface that does not include the singularity is zero. Let's start by calculating Φ_1. Due to the symmetry of the field $\vec{B}_{1,\text{sing}}(\vec{r},\varepsilon)$, the surface that best fits these conditions is a cylinder, where the top cap is orthogonal to the positive z semi-axis. In this case, the surface element in integral (A.10) is given by $d\vec{S} = dxdy\hat{z}$. Furthermore, since we are only considering the region $z > 0$, we have that $\Theta(z) = 1$, then

$$\Phi_1 = \lim_{\varepsilon \to 0} \iint_{\mathbb{R}^2} \frac{-2\varepsilon^2 g}{r^2 \left(x^2 + y^2 + \varepsilon^2\right)} dxdy. \tag{A.11}$$

As we are considering a cylindrical surface and the symmetry of the field $\vec{B}^R_{1,\text{sing}}(\vec{r}, \varepsilon)$ is spherical, we have that the relationship between spherical and cylindrical coordinates is given by

$$r^2 = \rho^2 + z^2, \quad \rho^2 = x^2 + y^2, \quad dS = \rho d\rho d\varphi \tag{A.12}$$

where

$$\rho \in [0, +\infty[, \quad \varphi \in [0, 2\pi[$$

Substituting the transformations given in (A.12) into (A.11), we obtain

$$\begin{aligned} \Phi_1 &= \lim_{\varepsilon \to 0} \int_0^{+\infty} \int_0^{2\pi} \frac{2\varepsilon(-g)\hat{z}}{r^2(x^2 + y^2 + \varepsilon^2)} \rho d\rho d\varphi \\ &= -4\pi g \lim_{\varepsilon \to 0} \int_0^{+\infty} \frac{\varepsilon^2 \rho}{(\rho^2 + z^2)(\rho^2 + \varepsilon^2)} d\rho \end{aligned} \tag{A.13}$$

Making the change of variable

$$\rho = \varepsilon t, \quad d\rho = \varepsilon dt$$

we obtain that (A.13) takes the form

$$\begin{aligned} \Phi_1 &= -4\pi g \lim_{\varepsilon \to 0} \int_0^{+\infty} \frac{\varepsilon^4 t}{\left(\varepsilon^2 t^2 + z^2\right)\left(\varepsilon^2 t^2 + \varepsilon^2\right)} dt \\ &= -4\pi g \lim_{\varepsilon \to 0} \int_0^{+\infty} \frac{\varepsilon^2 t}{\left(\varepsilon^2 t^2 + z^2\right)\left(t^2 + 1\right)} dt. \end{aligned} \tag{A.14}$$

For convenience, we make the change of variable

$$u = \varepsilon^2 t^2 + z^2, \quad \frac{1}{t^2+1} = \frac{\varepsilon^2}{u - z^2 + \varepsilon^2}, \quad \frac{du}{2} = \varepsilon^2 t dt$$

then (A.14) takes the form

$$\Phi_1 = -2\pi g \lim_{\varepsilon \to 0} \int_{z^2}^{+\infty} \frac{\varepsilon^2}{u - z^2 + \varepsilon^2} \frac{1}{u} du \tag{A.15}$$

To further facilitate the calculations, we can again make the change of variable

$$\alpha = u - z^2 + \varepsilon^2, \quad d\alpha = du$$

then (A.15) takes the form

$$\Phi_1 = -2\pi g \lim_{\varepsilon \to 0} \int_{\varepsilon^2}^{+\infty} \frac{\varepsilon^2}{\alpha} \frac{1}{\alpha + z^2 - \varepsilon^2} d\alpha. \tag{A.16}$$

To solve the integral given in (A.16), we will use the method of partial fractions, that is,

$$\begin{aligned} \frac{\varepsilon^2}{\alpha\left(\alpha + z^2 - \varepsilon^2\right)} &= \frac{A}{\alpha} + \frac{B}{\alpha + z^2 - \varepsilon^2} \\ &= \frac{A\left(\alpha + z^2 - \varepsilon^2\right) + B\alpha}{\alpha\left(\alpha + z^2 - \varepsilon^2\right)} \end{aligned} \tag{A.17}$$

From (A.17) we have that

$$\varepsilon^2 = A(\alpha + z^2 - \varepsilon^2) + B\alpha.$$

If $\alpha = 0$, then

$$\begin{aligned}\varepsilon^2 &= A\left(z^2-\varepsilon^2\right)\\ A &= \frac{\varepsilon^2}{z^2-\varepsilon^2}\end{aligned}\tag{A.18}$$

If $\alpha = \varepsilon^2 - z^2$, then

$$\begin{aligned}\varepsilon^2 &= B(\varepsilon^2-z^2)\\ B &= -\frac{\varepsilon^2}{z^2-\varepsilon^2}\end{aligned}\tag{A.19}$$

Substituting (A.18) and (A.19) into (A.17), we obtain

$$\frac{\varepsilon^2}{\alpha\left(\alpha+z^2-\varepsilon^2\right)} = \frac{\varepsilon^2}{z^2-\varepsilon^2}\frac{1}{\alpha} - \frac{\varepsilon^2}{z^2-\varepsilon^2}\frac{1}{\alpha+z^2-\varepsilon^2}\tag{A.20}$$

Substituting (A.20) into (A.16), we get

$$\begin{aligned}
\Phi_1 &= -2\pi g \lim_{\varepsilon\to 0}\int_{\varepsilon^2}^{+\infty}\frac{\varepsilon^2}{z^2-\varepsilon^2}\left(\frac{1}{\alpha}-\frac{1}{\alpha+z^2-\varepsilon^2}\right)d\alpha\\
&= -2\pi g \lim_{\varepsilon\to 0}\frac{\varepsilon^2}{z^2-\varepsilon^2}\lim_{x\to+\infty}\left(\int_{\varepsilon^2}^{x}\frac{d\alpha}{\alpha}-\int_{\varepsilon^2}^{x}\frac{d\alpha}{\alpha+z^2-\varepsilon^2}\right)\\
&= -2\pi g \lim_{\varepsilon\to 0}\frac{\varepsilon^2}{z^2-\varepsilon^2}\lim_{x\to+\infty}\left(\ln(\alpha)\big|_{\varepsilon^2}^{x}-\ln\left(\alpha+z^2-\varepsilon^2\right)\big|_{\varepsilon^2}^{x}\right)\\
&= -2\pi g \lim_{\varepsilon\to 0}\frac{\varepsilon^2}{z^2-\varepsilon^2}\lim_{x\to+\infty}\left(\ln\left(\frac{x}{\varepsilon^2}\right)-\ln\left(\frac{x+z^2-\varepsilon^2}{z^2}\right)\right)\\
&= -2\pi g \lim_{\varepsilon\to 0}\frac{\varepsilon^2}{z^2-\varepsilon^2}\lim_{x\to+\infty}\left(\ln\left(\frac{x}{\varepsilon^2}\right)+\ln\left(\frac{z^2}{x+z^2-\varepsilon^2}\right)\right)\\
&= -2\pi g \lim_{\varepsilon\to 0}\frac{\varepsilon^2}{z^2-\varepsilon^2}\lim_{x\to+\infty}\ln\left(\frac{xz^2}{x\varepsilon^2+z^2\varepsilon^2-\varepsilon^4}\right)\\
&= -2\pi g \lim_{\varepsilon\to 0}\frac{\varepsilon^2}{z^2-\varepsilon^2}\ln\left(\lim_{x\to+\infty}\frac{xz^2}{x\varepsilon^2+z^2\varepsilon^2-\varepsilon^4}\right)\\
&= -2\pi g \lim_{\varepsilon\to 0}\frac{\varepsilon^2}{z^2-\varepsilon^2}\ln\left(\frac{z^2}{\varepsilon^2}\right)\\
&= -2\pi g\left(\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{z^2-\varepsilon^2}\ln\left(z^2\right)-\lim_{\varepsilon\to 0}\frac{2\varepsilon^2}{z^2-\varepsilon^2}\ln(\varepsilon)\right)\\
&= 0.
\end{aligned}\tag{A.21}$$

On the other hand, from (A.8) and (A.9) we have that

$$\Phi_2=\lim_{\varepsilon\to 0}\iint_S\frac{4\varepsilon^2(-g)\Theta(z)\hat{z}}{\left(x^2+y^2+\varepsilon^2\right)^2}\cdot d\vec{S}\tag{A.22}$$

We have that the magnetic field $\vec{B}^{R}_{2,\text{ sing}}$ has the same symmetry as the magnetic field $\vec{B}^{R}_{1,\text{ sing}}$. Due to this, we can use the coordinate transformations given in (A.12) to calculate the flux Φ_2, obtaining

$$\begin{aligned}
\Phi_2 &= -4g\lim_{\varepsilon\to 0}\int_0^{+\infty}\int_0^{2\pi}\frac{\varepsilon^2\rho}{\left(\rho^2+\varepsilon^2\right)^2}d\rho d\varphi\\
&= -8\pi g\lim_{\varepsilon\to 0}\varepsilon^2\int_0^{+\infty}\frac{\rho}{\left(\rho^2+\varepsilon^2\right)^2}d\rho.
\end{aligned}\tag{A.23}$$

Again making the change of variable

$$u = \rho^2 + \varepsilon^2, \quad \frac{du}{2} = \rho d\rho,$$

we see that (A.23) takes the form

$$\begin{aligned}
\Phi_2 &= -4\pi g \lim_{\varepsilon\to 0} \varepsilon^2 \lim_{x\to\infty} \int_{\varepsilon^2}^{x} \frac{du}{u^2} \\
&= -\ 4\pi g \lim_{\varepsilon\to 0} \varepsilon^2 \lim_{x\to\infty} \left. \frac{u^{-1}}{-1} \right|_{\varepsilon^2}^{x} \\
&= -\ 4\pi g \lim_{\varepsilon\to 0} \varepsilon^2 \lim_{x\to\infty} \left. \frac{1}{u} \right|_{x}^{\varepsilon^2} \\
&= -4\pi g \lim_{\varepsilon\to 0} \varepsilon^2 \lim_{x\to\infty} \left(\frac{1}{\varepsilon^2} - \frac{1}{x} \right) \\
&= -4\pi g. \qquad (A.24)
\end{aligned}$$

We have that the total magnetic flux produced by the magnetic field of the string, that is $\vec{B}_{\text{sing}}$, is given by

$$\Phi_S = \Phi_1 + \Phi_2 \qquad (A.25)$$

Substituting (A.21) and (A.24) into (A.25), we obtain

$$\Phi_S = -4\pi g \qquad (A.26)$$

On the other hand, from (A.5) and (A.9) we have that

$$\Phi_g = \iint_S g \frac{\hat{r}}{r^2} \cdot d\vec{S} \qquad (A.27)$$

In spherical coordinates, the surface element $d\vec{S}$ is given by $d\vec{S} = r^2 d\Omega \hat{r}$, where $d\Omega = \sin(\theta) d\theta d\varphi$ is the solid angle element. Then, the flux (A.27) takes the form

$$\begin{aligned}\Phi_g &= g\int_0^{2\pi}\int_0^{\pi} d\Omega \\ &= 4\pi g. \end{aligned} \tag{A.28}$$

Finally, recall that the volumetric density for a point (magnetic) charge q_m located at $\vec{r} = (0, 0, 0)$ is given by

$$\rho(\vec{r}) = q_m \delta(x)\delta(y)\delta(z). \tag{A.29}$$

Gauss's law states that the flux of certain fields through a closed surface is proportional to the magnitude of the sources of said field inside the same surface. To calculate the fluxes (A.26) and (A.28) we have considered a cylindrical and spherical surface, respectively.

Therefore, as expected, we have that

$$\oint \vec{B}_R \cdot d\vec{S} = 4\pi g - 4\pi g = 0. \tag{A.30}$$

Recall that the field B_{sing} was proportional to

$$\vec{B}_{\text{sing}} = \hat{z}\left[\frac{1}{r^2\left(x^2+y^2+\varepsilon^2\right)} + \frac{1}{\left(x^2+y^2+\varepsilon^2\right)^2}\right]\Theta(z), \tag{A.31}$$

where the Heaviside function summarized the non-zero cases to $x^2 + y^2 = 0$, so that the previous field only contributes to the flux on the positive z-axis for $\delta(x)\delta(y)$. Thus

$$\vec{B}_{\text{sing}}(\vec{r}) = -4\pi g \delta(x)\delta(y)\Theta(z)\hat{z}. \tag{A.32}$$

Por lo tanto, de (A.5) y (A.32) se tiene que el campo magnético producido por la carga magnética es dado por

$$\vec{B}_R(\vec{r}) = g\frac{\hat{r}}{r^2} - 4\pi g \delta(x)\delta(y)\Theta(z)\hat{z}. \tag{A.33}$$

Appendix B

Actions of a Lie Group on a Manifold

Definition: Let $(G, \cdot)$ be a Lie group, and M a smooth manifold. A **left action** of G on M is a smooth map

$$\triangleright : G \times M \to M$$

where e is the identity element of the Lie group, satisfying the following properties:

i) $e \triangleright p = p, \quad \forall p \in M$,

ii) $g_2 \triangleright (g_1 \triangleright p) = (g_2 \cdot g_1) \triangleright p$.

Analogously, a **right action** of G on M is defined as a smooth map

$$\triangleleft : M \times G \to M$$

such that

i) $p \triangleleft e = p \quad \forall p \in M$,

ii) $(p \triangleleft g_1) \triangleleft g_2 = p \triangleleft (g_1 \cdot g_2)$.

Observation: Let $\triangleright$ be a left action. From the definition of a right action,

$$\triangleleft : M \times G \to M$$

We can define a right action from the left action as

$$p \triangleleft g := g^{-1} \triangleright p.$$

Proofs:

i) Let e be the identity element of the Lie group, then

$$p \triangleleft e = e^{-1} \triangleright p = p$$

Since e is the identity, then $e^{-1} = e$, so that

$$p \triangleleft e = e \triangleright p = p$$

therefore, the right action is also smooth since taking the inverse is a smooth operation because we are in a Lie group.

ii) Let $g_1, g_2 \in G$

$$\begin{aligned}
(p \triangleleft g_1) \triangleleft g_2 &= \left(g_1^{-1} \triangleright p\right) \triangleleft g_2 \\
&= g_2^{-1} \triangleright \left(g_1^{-1} \triangleright p\right) \\
&= \left(g_2^{-1} \cdot g_1^{-1}\right) \triangleright p \\
&= (g_1 \cdot g_2)^{-1} \triangleright p \\
&= p \triangleright (g_1 \cdot g_2) .
\end{aligned}$$

Appendix C

Isotropic and Stationary Spacetime

Writing Eq. (3.16) in a coordinate basis, we obtain

$$-2a_1\left[\delta^{\lambda}_{\sigma}T^{\mu}{}_{\gamma\mu}-\delta^{\lambda}_{\gamma}T^{\mu}{}_{\sigma\mu}+\delta^{\lambda}_{\mu}T^{\mu}{}_{\sigma\gamma}\right]=a_3\partial^{\mu}\phi\epsilon_{\mu\sigma\gamma}{}^{\lambda},$$

Tracing over $\gamma=\lambda$, we see that $T^{\mu}{}_{\sigma\mu}=0$. Therefore, the contributing terms must satisfy:

$$T^{\lambda}{}_{\sigma\gamma}=\alpha\partial^{\mu}\phi\epsilon_{\mu\sigma\gamma}{}^{\lambda}, \tag{C.1}$$

where we have defined $\alpha\equiv -a_3/2a_1$.

C.1 Spherical Symmetry

In this case, the Killing vectors are the generators of the rotation group $SO(3):[\xi_i,\xi_j]=\epsilon_{ijk}\xi_k$.

$$
\begin{aligned}
\xi_1 &= \sin\varphi\partial_\theta + \cos\varphi\cot\theta\partial_\varphi \\
\xi_2 &= -\cos\varphi\partial_\theta + \sin\varphi\cot\theta\partial_\varphi \\
\xi_3 &= \partial\varphi.
\end{aligned}
$$

The scalar field ϕ satisfying condition (C.1) for these Killing vectors has the form $\phi = \phi(r,t)$. This is advantageous because it implies that only some components of the torsion will contribute, namely

$$T^{\lambda}{}_{\sigma\gamma} = \alpha\partial^t\phi\epsilon_{t\sigma\gamma}{}^{\lambda} + \alpha\partial^r\phi\epsilon_{r\sigma\gamma}{}^{\lambda},$$

we then observe that the only components that will contribute are

$$
\begin{cases}
T^{\varphi}{}_{r\theta} = \alpha\partial^t\phi\epsilon_{tr\theta}{}^{\varphi}, \\
T^{r}{}_{\theta\varphi} = \alpha\partial^t\phi\epsilon_{t\theta\varphi}{}^{r}, \\
T^{\theta}{}_{\varphi r} = \alpha\partial^t\phi\epsilon_{t\varphi r}{}^{\theta}, \\
T^{\varphi}{}_{t\theta} = \alpha\partial^r\phi\epsilon_{rt\theta}{}^{\varphi}, \\
T^{t}{}_{\theta\varphi} = \alpha\partial^r\phi\epsilon_{r\theta\varphi}{}^{t}, \\
T^{\theta}{}_{\varphi t} = \alpha\partial^r\phi\epsilon_{r\varphi t}{}^{\theta}.
\end{cases}
\tag{C.2}
$$

This restricts the components of the torsion as follows

$$T^{\varphi}{}_{r\theta} + T^{\varphi}{}_{t\theta} = \alpha\epsilon_{tr\theta}{}^{\varphi}\left(\dot{\phi} - \phi'\right), \tag{C.3}$$

$$T^{\theta}{}_{r\varphi} + T^{\theta}{}_{t\varphi} = \alpha\epsilon_{tr\varphi}{}^{\theta}\left(\dot{\phi} - \phi'\right). \tag{C.4}$$

where the components of the torsion, such that (3.10), satisfy the following equations:

$$\partial_\theta T^t{}_{tr} = \partial_\theta T^r{}_{tr} = \partial_\theta T^\theta{}_{t\theta} = \partial_\theta T^\theta{}_{r\theta} = \partial_\theta T^\varphi{}_{t\varphi} = \partial_\theta T^\varphi{}_{t\varphi} = 0, \tag{C.5}$$

$$T^t{}_{t\theta} = T^t{}_{t\varphi} = T^t{}_{r\theta} = T^t{}_{r\varphi} = T^r{}_{t\theta} = T^r{}_{t\varphi} = T^r{}_{r\theta} = T^\theta{}_{tr} = T^\varphi{}_{tr} = T^r{}_{r\varphi} = T^\theta{}_{\theta\varphi} = T^\varphi{}_{\theta\varphi} = 0. \tag{C.6}$$

Whereas the components $T^t{}_{\theta\varphi}, T^r{}_{\theta\varphi}, T^\theta{}_{t\varphi}, T^\theta{}_{r\varphi}$ satisfy the equation

$$\partial_\theta T^\alpha_{\mu\nu} - \cot\theta T^\alpha{}_{\mu\nu} = 0 \tag{C.7}$$

and the components $T^\varphi{}_{t\theta}, T^\varphi{}_{r\theta}$

$$\partial_\theta T^\alpha_{\mu\nu} + \cot\theta T^\alpha{}_{\mu\nu} = 0. \tag{C.8}$$

In addition to the following relations

$$\begin{aligned}
T^\theta{}_{t\theta} &= T^\varphi{}_{t\varphi},\\
T^\theta{}_{r\theta} &= T^\varphi{}_{r\varphi},\\
T^\varphi{}_{t\theta} &= -\csc^2\theta T^\theta{}_{t\varphi},\\
T^\varphi{}_{r\theta} &= -\csc^2\theta T^\theta{}_{r\varphi}.
\end{aligned}$$

Finally,

$$\begin{cases} T^{\varphi}{}_{r\theta} = -N_1(t,r)\dfrac{1}{\sin\theta}, \\ T^{r}{}_{\theta\varphi} = M_1(t,r)\sin\theta, \\ T^{\theta}{}_{r\varphi} = N_1(t,r)\sin\theta, \\ T^{\varphi}{}_{t\theta} = -N_0(t,r)\dfrac{1}{\sin\theta}, \\ T^{t}{}_{\theta\varphi} = M_0(t,r)\sin\theta, \\ T^{\theta}{}_{t\varphi} = N_0(t,r)\sin\theta, \\ T^{t}{}_{tr} = A_0(t,r), \\ T^{\theta}{}_{r\theta} = T^{\varphi}{}_{r\varphi} = B_1(t,r), \\ T^{r}{}_{tr} = A_1(t,r), \\ T^{\theta}{}_{t\theta} = T^{\varphi}{}_{t\varphi} = B_0(t,r). \end{cases} \tag{C.9}$$

With the condition $T^{\mu}{}_{\sigma\mu} = 0$, which now reads

$$T^{r}{}_{tr} + T^{\theta}{}_{t\theta} + T^{\varphi}{}_{t\varphi} = 0,$$

$$\Rightarrow A_1(t,r) = -2B_0(t,r), \tag{C.10}$$

$$-T^{t}{}_{tr} + T^{\theta}{}_{r\vartheta} + T^{\varphi}{}_{r\varphi} = 0,$$

$$\Rightarrow A_0(t,r) = 2B_1(t,r). \tag{C.11}$$

Using these relations, the components (C.9) in the vielbein basis can be written

$$
\begin{aligned}
&T^0 = 2B_1(r,t)e^0 \wedge e^1 + M_0(r,t)e^2 \wedge e^3, \\
&T^1 = -2B_0(r,t)e^0 \wedge e^1 + M_1(r,t)e^2 \wedge e^3, \\
&T^2 = B_0(r,t)e^0 \wedge e^2 + N_0(r,t)e^0 \wedge e^3 + B_1(r,t)e^1 \wedge e^2 + N_1(r,t)e^1 \wedge e^3, \\
&T^3 = -N_0(r,t)e^0 \wedge e^2 + B_0(r,t)e^0 \wedge e^3 - N_1(r,t)e^1 \wedge e^2 + B_1(r,t)e^1 \wedge e^3.
\end{aligned}
$$

C.2 Isotropic and stationary spacetime

This type of space, in addition to possessing spherical symmetry, has the property of being time-independent. Mathematically, this implies the existence of a new Killing vector ∂_t. With these assumptions, the metric becomes

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{C.12}$$

In this case, we see that $N_1(r)$ and $M_1(r)$ are trivially zero, as the field $\phi(r)$ has no temporal dependence. The components of the torsion reduces to

$$
\begin{aligned}
&T^0 = 2B_1(r)e^0 \wedge e^1 + M_0(r)e^2 \wedge e^3, \\
&T^1 = -2B_0(r)e^0 \wedge e^1, \\
&T^2 = B_0(r)e^0 \wedge e^2 + N_0(r)e^0 \wedge e^3 + B_1(r)e^1 \wedge e^2, \\
&T^3 = -N_0(r)e^0 \wedge e^2 + B_0(r)e^0 \wedge e^3 + B_1(r)e^1 \wedge e^3.
\end{aligned}
$$

Introducing this into Eq. (3.16), we have that

$$B_0(r) = 0, \tag{C.13}$$

$$B_1(r) = 0, \tag{C.14}$$

along with

$$N_0(r) = M_0(r) = -\frac{a_3}{2a_1}\partial_r\phi(r)\frac{1}{\sqrt{h(r)}}. \tag{C.15}$$

That is, the problem is reduced from 24 to only 1 component, namely,

$$\begin{aligned}
T^0 &= M_0(r)e^2 \wedge e^3, \\
T^1 &= 0, \\
T^2 &= M_0(r)e^0 \wedge e^3, \\
T^3 &= -M_0(r)e^0 \wedge e^2.
\end{aligned}$$

Appendix D

Final Equations

D.1 Variation with respect to the Vierbein

For $d = 0$, equation (3.15) takes the form (with $u, v = 2, 3$ and $d\phi = \partial_a \phi e^a = e_a{}^{\mu} \partial_\mu \phi e^a$):

$$-\epsilon_{1uv}\left(a_1 R^{uv} \wedge e^1 + 2a_1 R^{1u} \wedge e^v + 6a_2 e^1 \wedge e^u \wedge e^v\right) + a_3\left(\partial_1 \phi e^1 + \partial_2 \phi e^2\right) \wedge T^0 = 0$$

$$\Rightarrow -4a_1\left[h'(r)r + h(r)^2 - h(r)\right] - 24a_2 h(r)^2 r^2 = -2a_3 h(r)^{3/2} M_0(r)\phi'(r)r^2 + a_1 h(r)^2 M_0(r)^2 r^2.$$

For the index $d = 1$ we have

$$\epsilon_{1uv}\left(a_1 R^{uv} \wedge e^0 + 2a_2 R^{0u} \wedge e^v + 6a_2 e^0 \wedge e^u \wedge e^v\right) - a_3\left(\partial_1 \phi e^1 + \partial_2 \phi e^2\right) \wedge T^1 = 0$$

$$\Rightarrow -4a_1\left[f'(r)r - f(r)h(r) + f(r)\right] + 24a_2 f(r)h(r)r^2 = -\,3a_1 f(r)h(r)M_0(r)^2 r^2.$$

When the index is valued at $d = u = 2, 3$, we have

$$\epsilon_{1uv}\left(2a_1R^{01}\wedge e^u+2a_1R^{1u}\wedge e^0-2a_1R^{0u}\wedge e^1+12a_2e^0\wedge e^1\wedge e^u\right)-a_3\left(\partial_1\phi e^1+\partial_2\phi e^2\right)\wedge T^v=0$$

$$\begin{aligned}\Rightarrow a_1\sqrt{h(r)}\left[2f''(r)h(r)f(r)r-f'(r)^2h(r)r-f'(r)h'(r)f(r)r+2f'(r)f(r)h(r)-2h'(r)f(r)^2\right]\\ -24a_2\sqrt{h(r)}f(r)^2h(r)^2r=a_1\sqrt{h(r)}f(r)^2h(r)^2M_0(r)^2r-2a_3\phi'(r)h(r)^2f(r)^2M_0(r)r.\end{aligned}$$

D.2 Variation with respect to the Spin Connection

This analysis has already been carried out, but since we are using the function M_0 defined in the appendix **??** in the field equations, it is convenient to draw the analogy

$$\begin{aligned}d\phi&=-\frac{2a_1}{a_3}M_0(r)e^1,\\&=\xi(r)e^1,\end{aligned}$$

Where clearly $\xi(r)=(-2a_1/a_3)M_0(r)$, recalling that we defined $\alpha=-a_3/2a_1$. Thus, the torsion can be consistently written as

$$\begin{aligned}T^0&=\alpha\,\xi(r)\,e^2\wedge e^3\ ,\\T^2&=\alpha\,\xi(r)\,e^0\wedge e^3\ ,\\T^3&=-\alpha\,\xi(r)\,e^0\wedge e^2\ .\end{aligned}$$

D.3 Variation with respect to the Scalar Field

Since the Nieh-Yan invariant is written as

$$T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b = d(T^a \wedge e_a) = d\left[-(\alpha/2)\xi(r)\epsilon_{abc}e^a \wedge e^b \wedge e^c\right] \ . \tag{D.1}$$

then the field equation 3.17 tells us that this must be equal to 0.

Using the ansatz of an isotropic and static spacetime,

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{D.2}$$

then $\xi = \phi' h^{-1/2}$, and equation (D.1) reduces to

$$\phi' r^2 f^{1/2} h^{-1/2} = C_1 \ . \tag{D.3}$$

Or, equivalently,

$$\phi(r) = C_1 \int^r \frac{h^{1/2}(r')}{r'^2 f^{1/2}(r')} dr' + C_2, \tag{D.4}$$

where C_1 and C_2 are arbitrary constants.

Another way to see this, using the same method that we described for the other fields, is

Thus, we see that the following is satisfied:

$$\left[\log(\phi')\right]' = -\left[\log(r^2 f^{1/2} h^{-1/2})\right]'$$

Integrating, we obtain

$$\phi(r) = C_1 \int^r \frac{\sqrt{h(r')}}{r'^2\sqrt{f(r')}} dr' + C_2. \tag{D.5}$$

Appendix E

Matching with the Schwarzschild Solution

Considering equations (3.27) and (3.28), namely

$$
\begin{aligned}
-a_1\left[\frac{h'(r)r}{h(r)}+h(r)-1\right]-6a_2h(r)r^2&=\frac{5a_3^2}{16a_1}\phi'(r)^2r^2,\\
-a_1\left[\frac{f'(r)r}{f(r)}-h(r)+1\right]+6a_2h(r)r^2&=-\frac{3a_3^2}{16a_1}\phi'(r)^2r^2.
\end{aligned}
$$

For the spherically symmetric solution (torsion free) without cosmological constant (in our case, the term with a_2), we have

$$-a_1\left[\frac{h'(r)r}{h(r)}+h(r)-1\right]=0, \tag{E.1}$$

$$-a_1\left[\frac{f'(r)r}{f(r)}-h(r)+1\right]=0 \tag{E.2}$$

Adding equations (E.1) and (E.2), we get

$$\frac{h'(r)r}{h(r)} + \frac{f'(r)r}{f(r)} = 0 \tag{E.3}$$

Here we get

$$\boxed{h(r)f'(r) + f(r)h'(r) = 0}, \tag{E.4}$$

this implies

$$\begin{aligned} &\partial_r \left[f(r)h(r) \right] = 0, \\ \Longrightarrow\ &f(r)h(r) = K, \quad \text{here} \quad K = cte. \end{aligned}$$

Then

$$h(r) = \frac{K}{f(r)}.$$

Hence,

$$g_{\mu\nu} = \begin{pmatrix} -f(r) & 0 & 0 & 0 \\ 0 & h(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

in the limit $r \to \infty$ it will approach the flat Minkowski (in spherical coordinates)

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

So, in this limit $r \to \infty$

$$\begin{aligned} f(r) &\to 1, \\ h(r) &\to 1, \\ \implies (1)(1) &= K \end{aligned}$$

Since K is the same constant for all r values

$$\boxed{K = 1}, \quad \text{for all} \quad r.$$

This means that

$$\boxed{h(r) = \frac{1}{f(r)}}, \tag{E.5}$$

for all r. And

$$h'(r) = \partial_r(f^{-1}(r)) = -\frac{f'(r)}{f(r)^2} \tag{E.6}$$

On the other hand, if we subtract equation (E.1) from (E.2),

$$-a_1 \frac{h'(r)r}{h(r)} - a_1 h(r) + a_1 + a_1 \frac{f'(r)r}{f(r)} - a_1 h(r) + a_1 = 0$$

Equivalent to

$$h'(r)f(r)r - f'(r)h(r)r + 2h^2(r)f(r) - 2h(r)f(r) = 0 \tag{E.7}$$

Substituting (E.5) and (E.6) into (E.7), we obtain

$$-\frac{f'(r)}{f(r)}r - \frac{f'(r)}{f(r)}r + \frac{2}{f(r)} - 2 = 0,$$

which leads us to the differential equation

$$a - f(r) = f'(r)r \tag{E.8}$$

has the solution

$$f(r) = 1 - \frac{k}{r}, \tag{E.9}$$

thus,

$$h(r) = -\left[1 - \frac{k}{r}\right]^{-1}, \tag{E.10}$$

where k is a constant whose value can be determined by matching with Newtonian gravity in the low velocity and weak gravity limit

$$k = \frac{2GM}{c^2}.$$

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